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Heterotic string data and theta functions

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Abstract

We use the language of differential cohomology to give an analytic description of the moduli space of classical vacua for heterotic string theory in eight dimensions. The complex structure of this moduli space is then related to the character of the appropriate Kac–Moody algebra.

1 Introduction

The general framework of heterotic string theory [10, 23, 24] involves, as defining geometric data, a space-time in the form of a smooth spin manifold X and a principal G-bundle $P \to X$. In this context, the background fields are given by a triplet (g, A, B) consisting of a Riemannian metric g on X, a connection A on P and a B-field B, which is represented, at least locally, by a 2-form on X.

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The first conditions that one has to impose, as required by the cancellation of anomalies, are topological. The space-time X has to be 10 dimensional. The Lie group G must be $(E_8 \times E_8) \rtimes \mathbb{Z}_2$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$. Moreover, the three background fields are to be considered up to gauge equivalence. This means that the connection is considered up to bundle isomorphism, that the metric is considered up to diffeomorphism equivalence and that the two-form B is considered up to equivalence by adding exact 2-forms.

One of the major problems at this point has been to understand exactly the mathematical nature of the B-field. In the original physics interpretation [24], the B-field B is manipulated as a globally defined two-form on X with its contribution to the world-sheet path integral being given by the phase:

$$\exp\left(i \int_\Sigma x^* B\right). \tag{1.1}$$

However, this definition is not fully satisfactory. First, B in this formulation lacks a gauge invariance. Secondly, the B-field interpretation in heterotic string theory must obey the world-sheet anomaly cancellation mechanism, as explained, for instance, by Witten [19]. This requires, among other conditions, that the path-integral contribution (1.1) of B is not a complex number of unit length, but rather an element in the total space of a non-trivial circle bundle. In addition, the field strength H_B, which is a globally defined three form, has to satisfy the equation:

$$dH_B \ = \ \frac{1}{4\pi c_G}\ \mathrm{Tr}(F_A \wedge F_A) - \frac{1}{4\pi}\ \mathrm{Tr}(F_g \wedge F_g). \tag{1.2}$$

Here, F_A and F_g represent the curvature forms of the connection A and the Levi-Civita connection associated to the metric g in TX, respectively. The number c_G is the dual Coxeter number of G. In the naive interpretation of B as a globally defined 2-forms, $H_B = dB$. In particular, H_B is a closed three form and cannot satisfy (1.2).

The earlier issues suggest that the B-field is not, in fact, a globally defined 2-form, but rather must be given by a more subtle object, which locally resembles a 2-form but has a non-trivial global structure. In the recent years, it has been noted by a number of authors (see, for instance, [19, 5]) that in order to satisfy these conditions, the B-field has to be understood within a gerbe-like formalism. In this article, we follow ideas developed by Freed [5] and Hopkins and Singer [21] and interpret the B-field as a non-flat two-cochain in differential cohomology.[1] These cochains appear as a

[1]In the general heterotic string setting, this approach introduces the B-field as an object in differential KO theory. However, when one compactifies over the 2-torus, as is the case we shall be dealing with here, the topological type of the space-time is sufficiently

mixture of Cech cocycles and local differential forms and can be regarded, in an open covering U_a of X, as multiplets:

$$B = (H, \omega^2_a, \omega^1_{ab}, \omega^0_{abc}, \omega^{-1}_{abcd})$$

with H being a globally defined 3-form (the field strength H_B) and ω^i local i-forms (constant functions to $2\pi\mathbb{Z}$ if $i = -1$). The anomaly cancellation condition appears in this formulation as

$$\check{d}B = \mathcal{CS}_A - \mathcal{CS}_g, \tag{1.3}$$

where $\check{d}$ represents the differentiation operator in differential cohomology and $\mathcal{CS}_A$ and $\mathcal{CS}_g$ are Chern–Simons terms regarded as non-flat 3-cocycles. One can introduce then a concept of gauge invariance for B-fields. Two B-fields are said to be gauge equivalent if the two differential 2-cochains underlying them differ by a coboundary of a flat differential 1-cochain. It follows that the globally defined 3-form H_B, the field strength of B, is gauge invariant. However, it is not necessarily true that H_B is closed. In fact, one recovers easily the expected Eq. (1.2) by just writing the anomaly cancellation condition (1.3) at the field strength level.

Moreover, apart from their rather technical construction, the B-fields as differential 2-cochains carry a very nice geometrical interpretation. They represent sections in a two gerbe [11] and carry holonomy denoted by

$$\exp\left(i\int_\Sigma x^*B\right) \tag{1.4}$$

along any map $x\colon \Sigma \to X$ in the same way circle bundle connections carry holonomy along loops. This is the proper meaning of (1.1), the B-field contribution to the world-sheet path integral. However, the quantity (1.4) is not an unitary complex number, but rather a point in the total space of a circle fibration, which fits exactly the picture required anomaly cancellation (see [19] for details).

In this framework, the heterotic classical vacua are obtained by taking the triplets (A, g, B) satisfying the cancellation condition (1.3) and imposing over them the classical equations of motion. These equations can be written as

$$F_A = 0, \quad \mathrm{Ric}(g) = 0, \quad H_B = 0. \tag{1.5}$$

The moduli space of heterotic classical vacua $\mathcal{M}^G_{\mathrm{het}}$ is then constructed out of all solutions to (1.5) modulo gauge equivalence.

The goal of this paper is to work out a precise mathematical description of the moduli space of classical vacua for heterotic string theory when the

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low dimensional that one can ignore the differences between KO theory and standard cohomology.

space-time is compactified along a 2-torus. Namely, $X = \mathbb{R}^8 \times E$ with E a 2-torus and the fields are considered up to Euclidean equivalence on the $\mathbb{R}^8$ factor. The moduli space of quantum vacua associated to this case has been described in [23, 24] from a purely physical perspective. In that formulation, the B-field appears as a globally defined two-form B and the metric and B fit together to form the imaginary and, the real part of the so-called complexified Kahler class respectively. The physical quantum features of the theory then depend on a certain lattice of momenta $\mathcal{L}_{(A,g,B)}$ described by Narain in [23]. This is a rank 20 lattice that lives inside a fixed ambient real space $\mathbb{R}^{2,18}$, is well-defined up to $O(2) \times O(18)$ rotations and is even and unimodular. The real group $O(2, 18)$ acts transitively on the set of all $\mathcal{L}_{(A,g,B)}$ and, in this regard, one can consider the physical momenta as being parameterized by the 36-dimensional real homogeneous space:

$$\frac{O(2,18)}{O(2)} \times O(18). \tag{1.6}$$

One identifies then the points in (1.6) corresponding to equivalent quantum theories. This amounts to factoring out the left action of the group Γ of integral isometries of the lattice. The quantum (Narain) moduli space of distinct heterotic string theories compactified on the two torus appears as

$$\mathcal{M}_{\text{het}}^{\text{quantum}} = \frac{\Gamma \backslash O(2,18)}{O(2)} \times O(18). \tag{1.7}$$

From a geometric interest point of view, this physics-inspired Narain construction has a couple of disadvantages. First, since (1.7) provides a description of quantum states, not all the identifications are accounted by classical geometry. As explained, for example, in [25], part of the Γ action models the so-called quantum corrections and results in identifications of momenta for pairs of triplets (A, g, B), which are not isomorphic. Secondly, the Narain construction does not provide a holomorphic description. Technically, one can endow the homogeneous quotient (1.7) with a complex structure, but holomorphic families of elliptic curves and flat connections do not embed as holomorphic subvarieties in $\mathcal{M}_{\text{het}}^{\text{quantum}}$ (see the appendix of [20] for an outline of this issue).

In this paper, we take a completely geometrical approach and use the interpretation of the B-fields as differential cochains in order to give an analytic description for the moduli space of classical vacua $\mathcal{M}_{\text{het}}^{G}$. In Section 4, we prove the following Theorem 1.1

Theorem 1.1. *The moduli space of classical solutions $\mathcal{M}_{\text{het}}^{G}$ can be given the structure of a 18-dimensional complex variety with orbifold singularities.*

Moreover, $\mathcal{M}^G_{\text{het}}$ represents the total space of a holomorphic $\mathbb{C}^*$-fibration

$$\mathcal{M}^G_{\text{het}} \to \mathcal{M}_{E,G} \tag{1.8}$$

over the moduli space $\mathcal{M}_{E,G}$ of isomorphism classes of pairs of elliptic curves and flat G-bundles, where $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$ in the case of the $E_8 \times E_8$ heterotic theory and $G = \mathrm{Spin}(32)/\mathbb{Z}_2$ in the case of $\mathrm{Spin}(32)/\mathbb{Z}_2$ theory.

In this context, it is important then to analyze the holomorphic type of fibration (1.8). The structure of the base space $\mathcal{M}_{E,G}$ is well known [9]. Given a fixed elliptic curve E, the family of equivalence classes of flat G-bundles (for G simply connected) can be identified with the complex quotient:

$$W \backslash (\mathrm{Pic}^o(E) \otimes_{\mathbb{Z}} \Lambda_G),$$

where Λ_G represents the coroot lattice of G and W is the Weyl group. Using a coordinate oriented model to track down the variation of the elliptic curve, one can regard $\mathcal{M}_{E,G}$ as a complex orbifold:

$$\Pi_G \backslash V_G, \tag{1.9}$$

where $V_G = \mathcal{H} \times (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_G)$. $\mathcal{H}$ represents the complex upper half-plane and Π_G is a modular group acting on V_G by mixing as a semi-direct product the $\mathrm{SL}(2,\mathbb{Z})$ action on $\mathcal{H}$ and the affine Weyl group action on $\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_G$.

The model (1.9) allows us to analyze easily holomorphic $\mathbb{C}^*$-fibrations over $\mathcal{M}_{E,G}$. Such fibrations over a complex orbifold are best described in terms of equivariant line bundles over the universal cover. Those are holomorphic line bundles $L \to V_G$, where the action of the modular group Π_G on the base is given a lift to the fibers. All holomorphic line bundles over V_G are trivializable and a lift of the action of Π_G to fibers can be obtained through a set of automorphy factors $(\varphi_a)_{a \in \Pi_G}$ with $\varphi_a \in H^0(V_G, \mathcal{O}^*_{V_G})$ satisfying

$$\varphi_{ab}(x) = \varphi_a(bx) \cdot \varphi_b(x).$$

Such a set generates a class in the group cohomology $H^1(\Pi_G, H^0(V_G, \mathcal{O}^*_{V_G}))$. Two automorphy factor sets provide isomorphic fibrations if and only if they determine the same group cohomology class.

There is one particular important holomorphic $\mathbb{C}$-fibration over $\mathcal{M}_{E,G}$, the Λ_G-character fibration. This fibration can be defined using the model (1.9) as the fibration supporting the Λ_G-character function:

$$B_{\Lambda_G} \colon V_G \to \mathbb{C}, \quad B_{\Lambda_G}(\tau, z) = \frac{\Theta_{\Lambda_G}(\tau, z)}{\eta(\tau)^{16}}, \tag{1.10}$$

where $\Theta_{\Lambda_G}(\tau, z)$ represents the holomorphic theta function associated to the lattice Λ_G and η is Dedekind's eta function. The holomorphic function B_{Λ_G} has an important interpretation. It represents the zero character of the level $l = 1$ basic highest weight representation of the Kac–Moody algebra associated to G (see [12] for details). Under the action of the modular group, B_{Λ_G} descends to a section in a non-trivial $\mathbb{C}$-fibration:

$$\mathcal{Z} \to \mathcal{M}_{E,G}. \tag{1.11}$$

We call this the Λ_G-character fibration. Then, as we prove in Section 6, one has.

Theorem 1.2. *The heterotic $\mathbb{C}^*$-fibration* (1.8) *can be holomorphically identified with the $\mathbb{C}^*$-fibration induced by the Λ_G-character fibration* (1.11).

One can then conclude that in the light of Theorem 1.1, the moduli space $\mathcal{M}^G_{\text{het}}$ of heterotic classical vacua compactified along the 2-torus can be seen naturally as the total space of the $\mathbb{C}^*$-fibration associated to (1.11).

As a final note, we comment on the relation between the space $\mathcal{M}^G_{\text{het}}$ that we construct here and the Narain moduli space $\mathcal{M}^{\text{quantum}}_{\text{het}}$ of quantum vacua (1.7). It turns out that there exists a diffeomorphism between an open neighborhood along the zero section of the zero section in the $\mathbb{C}^*$-fibration of $\mathcal{M}^G_{\text{het}}$ and an open subset of $\mathcal{M}^{\text{quantum}}_{\text{het}}$ on which the volume of the elliptic curve is large. This is precisely the region where physics predicts that the quantum corrections became insignificant and the moduli spaces of classical and quantum vacua should coincide. This issue, as well as the place of $\mathcal{M}^G_{\text{het}}$ in the framework of the F-theory/heterotic string duality in eight dimensions, is the subject of a joint paper [20] with John Morgan.

2 *B*-field as a differential cochain

This section introduces the framework needed for giving a precise definition of the B-field. We shall treat this object in the language of differential integral cohomology following ideas of Freed [5] and Hopkins–Singer [21]. This approach mixes together differential forms and integral cohomology and has its roots in earlier works of Deligne [26] (see also Brylinski's exposition in [4]) and Cheeger–Simons [8]. However, as we shall see, the B-field is a cocycle rather than a class in this theory.

2.1 Differential cohomology

Let X be a smooth manifold. Choose $\mathcal{U} = (U_i)_{i\in I}$, an open covering of X indexed by an ordered set I. For $r, s \in \mathbb{Z}$, we set

$$\check{C}^{r,s}(\mathcal{U}) = \begin{cases} 0 & \text{if } s < -1 \text{ or } r < 0 \ ; \\ C^r(\mathcal{U}, \Omega^s_X) & \text{if } 0 \leq s; \\ C^r(\mathcal{U}, 2\pi\mathbb{Z}) & \text{if } s = -1, \end{cases}$$

where $C^r(X, \Omega^s_X)$ represents the set of Cech r-cochains with values on the sheaf of s-forms on X, Ω^s_X:

$$C^r(\mathcal{U}, \Omega^s_X) = \prod_{i_0, i_1, \ldots, i_r} \Omega^s_X(U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_r}).$$

Following [5], a **flat differential n-cochain** is defined as an element of the direct sum

$$\check{C}^n(\mathcal{U}) := \bigoplus_{r+s=n} \check{C}^{r,s}(\mathcal{U}).$$

In other words, one can see a flat differential n-cochain as a multiplet of differential Cech cochains

$$\omega = (\omega^n_a, \omega^{n-1}_{a_1 a_2}, \omega^{n-2}_{a_1 a_2 a_3}, \ldots, \omega^{-1}_{a_1 a_2 \cdots a_{n+2}}).$$

There are two derivation operators acting on these differential cochains. A vertical differentiation operator $d\colon \check{C}^{r,s}(\mathcal{U}) \to \check{C}^{r,s+1}(\mathcal{U})$ represents the usual differentiation on each form component (inclusion if $s = -1$). A second horizontal operator $\delta\colon \check{C}^{r,s}(\mathcal{U}) \to \check{C}^{r+1,s}(\mathcal{U})$ is the Cech coboundary operator

$$\delta(\omega^n)_{a_1, a_2, \ldots, a_{n+2}} = \sum_{j=1}^{n+2} (-1)^{j+1} \omega^n_{a_1 \cdots a_{j-1} a_{j+1} \cdots a_{n+2}}|_{U_{a_1} \cap \cdots \cap U_{a_{n+2}}}.$$

The flat cochains together with the two derivations build a double complex $(\check{C}^{r,s}(\mathcal{U}), d, \delta)$. As usual, in such situations, a total differentiation can be introduced:

$$\check{d}\colon \check{C}^n(\mathcal{U}) \to \check{C}^{n+1}(\mathcal{U}), \quad \check{d}|_{\check{C}^{r,s}} = \delta|_{\check{C}^{r,s}} + (-1)^{r+1} d|_{\check{C}^{r,s}}.$$

$(\check{C}^*(\mathcal{U}), \check{d})$ is a cochain complex, defining flat n-cocycles and flat n-coboundaries, which in turn generate $\mathcal{U}$-cohomology groups $\check{H}^n(\mathcal{U})$.

Let $\mathcal{V} = (V_j)_{j\in J}$ be a refinement of the open covering $\mathcal{U} = (U_i)_{i\in I}$ and let $\sigma\colon J \to I$ be a subordination map such that $V_j \subset U_{\sigma(j)}$ for any $j \in J$. One then has the restriction homomorphism

$$\sigma^*\colon \check{C}^n(\mathcal{U}) \to \check{C}^n(\mathcal{V})$$

commuting with the two differentiation operators d and δ. This defines a morphism of cochain complexes inducing a homomorphism at cohomology level

$$\sigma^{\mathcal{U}}_{\mathcal{V}}\colon \check{H}^n(\mathcal{U}) \to \check{H}^n(\mathcal{V}).$$

As in Cech theory:

Lemma 2.1. *The homomorphism $\sigma^{\mathcal{U}}_{\mathcal{V}}$ depends only on the open covering $\mathcal{U}$ and refinement $\mathcal{V}$, but not on the choice of subordination map $\sigma\colon J \to I$. Furthermore, $\sigma^{\mathcal{U}}_{\mathcal{U}}$ is identity, and if $\mathcal{W}$ is a refinement of $\mathcal{V}$, then $\sigma^{\mathcal{U}}_{\mathcal{W}} = \sigma^{\mathcal{V}}_{\mathcal{W}} \circ \sigma^{\mathcal{U}}_{\mathcal{V}}$.*

Proof. Assume $\sigma, \sigma'\colon J \to I$ are two subordination maps with $V_j \subset U_{\sigma(j)} \cap U_{\sigma'(j)}$. We define a family of homomorphisms

$$k^q\colon \check{C}^q(\mathcal{U}) \to \check{C}^{q-1}(\mathcal{V})$$

by the following pattern. If

$$\omega = (\omega^q_{i_1}, \omega^{q-1}_{i_1 i_2}, \omega^{q-2}_{i_1 i_2 i_3}, \ldots, \omega^{-1}_{i_1 i_2 \ldots i_{q+2}}) \in \check{C}^q(\mathcal{U}),$$

then $k^q(\omega) = \eta \in \check{C}^{q-1}(\mathcal{V})$, where

$$\eta = (\eta^{q-1}_{j_1}, \eta^{q-2}_{j_1 j_2}, \eta^{q-3}_{j_1 j_2 j_3}, \ldots, \eta^{-1}_{j_1 j_2 \cdots j_{n+1}}) \in \check{C}^q(\mathcal{U})$$

with components

$$\eta^{q-r}_{j_1 j_2 \cdots j_r} = \sum_{t=1}^{r} (-1)^{t+1} \omega^{q-r}_{\sigma(j_1)\cdots\sigma(j_t)\sigma'(j_t)\sigma'(j_{t+1})\cdots\sigma'(j_r)}|_{V_{j_1}\cap V_{j_1}\cap\cdots\cap V_{j_r}}. \tag{2.1}$$

We claim that

$$\check{d}k^q + k^{q+1}\check{d} = \sigma^* - (\sigma')^*. \tag{2.2}$$

Indeed, for a multi-index $(j) = (j_1 j_2 \cdots j_r)$, one can write

$$\begin{aligned}
\left(\check{d}k(\omega)\right)_{(j)} &= (-1)^r dk(\omega)_{(j)} + [\delta k(\omega)]_{(j)} \\
&= (-1)^r \sum_{t=1}^{r} (-1)^{t+1} d\omega_{\sigma(j_1)\cdots\sigma(j_t)\sigma'(j_t)\cdots\sigma'(j_r)} \\
&\quad + \sum_{l=1}^{r} (-1)^{l+1} (k\omega)_{j_1\cdots j_{l-1} j_{l+1}\cdots j_r} \\
&= (-1)^r \sum_{t=1}^{r} (-1)^{t+1} d\omega_{\sigma(j_1)\cdots\sigma(j_t)\sigma'(j_t)\cdots\sigma'(j_r)} \\
&\quad + \sum_{l=1}^{r} \sum_{t=1}^{l-1} (-1)^{t+l} \omega_{\sigma(j_1)\cdots\sigma(j_t)\sigma'(j_t)\cdots\sigma'(j_{l-1})\sigma'(j_{l+1})\cdots\sigma'(j_r))} \\
&\quad + \sum_{l=1}^{r} \sum_{t=l}^{r-1} (-1)^{t+l} \omega_{\sigma(j_1)\cdots\sigma(j_{l-1})\sigma(j_{l+1})\cdots\sigma(j_{t-1})\sigma'(j_{t-1})\cdots\sigma'(j_r))}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
[k(\check{d}\omega)]_{(j)} &= \sum_{t=1}^{r} (-1)^{t+1} \left(\check{d}\omega\right)_{\sigma(j_1)\cdots\sigma(j_t)\sigma'(j_t)\cdots\sigma'(j_r)} \\
&= (-1)^{r+1} \sum_{t=1}^{r} (-1)^{t+1} d\omega_{\sigma(j_1)\cdots\sigma(j_t)\sigma'(i_t)\cdots\sigma'(i_r)} \\
&\quad + \sum_{t=1}^{r} \sum_{l=1}^{t-1} (-1)^{t+l} \omega_{\sigma(j_1)\cdots\sigma(j_{l-1})\sigma(j_{l+1})\cdots\sigma(j_t)\sigma'(j_t)\cdots\sigma'(j_r)} \\
&\quad + \sum_{t=1}^{r} \left(-\omega_{\sigma(j_1)\cdots\sigma(j_{t-1})\sigma'(j_t)\cdots\sigma'(j_r)} + \omega_{\sigma(j_1)\cdots\sigma(j_t)\sigma'(j_{t+1})\cdots\sigma'(j_r)}\right) \\
&\quad + \sum_{t=1}^{r} \sum_{l=t+2}^{r+1} (-1)^{t+l} \omega_{\sigma(j_1)\cdots\sigma(j_t)\sigma'(j_t)\cdots\sigma'(j_{l-1})\sigma'(i_{l+1})\cdots\sigma'(j_r)}.
\end{aligned}$$

After canceling the terms with opposite signs, one obtains

$$(\check{d}k(\omega))_{(j)} + \left[k(\check{d}\omega)\right]_{(j)} = \omega_{\sigma(i_1)\sigma(i_2)\cdots\sigma(i_r)} - \omega_{\sigma'(i_1)\sigma'(i_2)\cdots\sigma'(i_r)}.$$

Relation (2.2) shows that k^q represents a homotopy operator between the 2-cochain complex morphisms $(\sigma')^*$ and σ^*. Hence, they must induce identical morphisms at cohomology level. This proves the first part of lemma. The second part follows immediately. $\square$

Lemma 2.1 shows that $(\check{H}^r(\mathcal{U}), \sigma^{\mathcal{U}}_{\mathcal{V}})$ forms a direct system. One then takes the direct limit over all possible coverings, defining

$$\check{H}^r(X) = \lim_{\substack{\rightarrow \\ \mathcal{U}}} \left(\check{H}^r(\mathcal{U}), \sigma^{\mathcal{U}}_{\mathcal{V}} \right).$$

These **flat** differential cohomology groups are known to form the smooth Deligne cohomology of X [4].

The differential objects we wish to study and make use of are non-flat extensions of the above cochains. By definition, a **non-flat** differential cochain is a pair (H, ω) consisting of a global $(n+1)$-form $H \in \Omega^{n+1}(X)$ and a flat n-cochain ω. One represents such an object, in an open covering, as a multi-plet of form-valued Cech cochains:

$$\omega = (H, \omega^n_a, \omega^{n-1}_{a_1 a_2}, \omega^{n-2}_{a_1 a_2 a_3}, \ldots, \omega^{-1}_{a_1 a_2 \cdots a_{n+2}}).$$

The top global form H represents the **field strength** of ω. Let us denote the set of non-flat differential n-cochains defined over an open covering $\mathcal{U}$ by

$$N\check{C}^n(\mathcal{U}) = \check{C}^n(\mathcal{U}) \times \Omega^{n+1}(X).$$

The differentiation operator $\check{d}$ extends naturally to $\check{C}^n(\mathcal{U})$. As in the flat case, $(N\check{C}^*(\mathcal{U}), \check{d})$ determine a cochain complex defining cocycles $N\check{Z}^n(\mathcal{U})$ and coboundaries $N\check{Z}^n(\mathcal{U})$.

The discussion in Lemma 2.1 can be immediately reformulated to the new context. For a refinement $\mathcal{V}$ of $\mathcal{U}$ with two distinct subordination maps $\sigma, \sigma' \colon J \to I$, the two induced restriction homomorphisms

$$\sigma^*, (\sigma')^* \colon \check{C}^n(\mathcal{U}) \to \check{C}^n(\mathcal{V})$$

differ by

$$\check{d} k^q + k^{q+1} \check{d} = \sigma^* - (\sigma')^*. \tag{2.3}$$

In above expression, k^* is the homotopy operator

$$k^q \colon N\check{C}^q(\mathcal{U}) \to N\check{C}^{q-1}(\mathcal{V})$$

defined by

$$\begin{aligned} k^q & \left(H, \omega^q_{i_1}, \omega^{q-1}_{i_1 i_2}, \omega^{q-2}_{i_1 i_2 i_3}, \ldots, \omega^{-1}_{i_1 i_2 \cdots i_{q+2}} \right) \\ & = \left(0, \eta^{q-1}_{j_1}, \eta^{q-2}_{j_1 j_2}, \eta^{q-3}_{j_1 j_2 j_3}, \ldots, \eta^{-1}_{j_1 j_2 \cdots j_{n+1}} \right) \end{aligned}$$

with components $\eta^{q-r}_{j_1 j_2 \ldots . j_r}, 1 \leq r \leq q$ given by formula (2.1). In a similar manner to the flat case, projection

$$\sigma^{\mathcal{U}}_{\mathcal{V}} \colon N\check{H}^q(\mathcal{U}) \to N\check{H}^q(\mathcal{V})$$

does not depend on the choice of subordination assignment. Differential non-flat cohomology groups can then be defined by

$$N\check{H}^*(X) = \lim_{\overrightarrow{\mathcal{U}}} \left(N\check{H}^r(\mathcal{U}), \sigma^{\mathcal{U}}_{\mathcal{V}}\right). \tag{2.4}$$

However, we are more interested here in non-flat cochains as explicit differential objects rather than as a framework for the earlier non-flat cohomology groups. In fact, one can show that the non-flat cohomology (2.4) recovers the Cech cohomology of X with coefficients in the sheaf of smooth functions to the circle, $N\check{H}^n(X) \simeq H^n_{\mathrm{Cech}}(X, S^1)$.

Let us denote the set of non-flat differential cochains on X by

$$N\check{C}^n(X) = \bigcup_{\mathcal{U}} N\check{C}^n(\mathcal{U}).$$

Among the cochains in $N\check{C}^n(X)$, we give special consideration to those associated to global differential forms. Let $T \in \Omega^n(X)$ be such a n-form. In a given open covering, $\mathcal{U} = (U_i)_{i\in I}$, one can consider a differential cocycle with only two non-vanishing components as follows:

$$(dT, T|_{U_i}, 0, 0, \ldots). \tag{2.5}$$

This makes a non-flat differential n-cocycle. In future considerations, we shall use differential forms in cocycle-like computations, always interpreting the form as in (2.5).

There exists a natural equivalence relation on the set of non-flat differential n-cocycles.

Definition 2.2. Two cocycles $\omega_1 \in N\check{Z}^n(\mathcal{U}_1)$ and $\omega_2 \in N\check{Z}^n(\mathcal{U}_2)$ are said to be equivalent (denoted $\omega_1 \sim \omega_2$) if there exists a common refinement $\mathcal{U}$ and corresponding subordination maps such that the difference of the two restrictions of ω_1 and ω_2 on $N\check{Z}^n(\mathcal{U})$ is a coboundary of a differential flat $(n-1)$-cochain on $\mathcal{U}$.

Clearly, two flat n-cocycles are equivalent if and only if they determine the same class in $\check{H}^n(X)$. The relation "$\sim$" does not extend as equivalence relation for general non-flat n-cochains, because there exist n-cochains $\omega \in N\check{C}^n(\mathcal{U})$, which restricted on a refinement $\mathcal{U}'$ through two distinct subordination maps give restrictions $\omega', \omega'' \in N\check{C}^n(\mathcal{U}')$ such that the difference

$$\omega' - \omega'' = \check{d}k^n\omega + k^{n+1}\check{d}\omega$$

is not necessarily a coboundary of a flat $(n-1)$ cochain. However, relation "$\sim$" is still an equivalence relation on the subset $\mathcal{B}^n(X) \subset N\check{C}^n(X)$

defined as

$$\mathcal{B}^n(X) = \left\{ \omega \in N\check{C}^n(\mathcal{U}) | \mathcal{U} \text{ open covering, } \check{d}\omega \in \Omega^{n+1}(X) \right\}. \tag{2.6}$$

2.2 Geometrical interpretation

The differential cochains introduced earlier have a nice geometrical interpretation. Roughly speaking, they represent connections on higher dimensional analogues of circle bundles, n-gerbes. Let us start by looking at the low-dimensional models.

In a given open covering $\mathcal{U} = (U_a)_{a \in \mathcal{A}}$ of X, a non-flat differential zero cochain appears as a triplet $\sigma = (T, f_a, t_{ab})$ with T being a global 1-form, f_a local functions and t_{ab} $2\pi\mathbb{Z}$-valued assignments. If σ is a cocycle, relations $dT = 0$, $T_a = df_a$ and $f_a - f_b = t_{ab}$ assure us that the local functions $q_a = \exp(if_a)$ glue together to a global function $q\colon X \to S^1$. Moreover, the global 1-form T is obtained by just pulling back through q the Maurer–Cartan form of S^1. Two zero-cocycles are equivalent if and only if they determine the same global map $q\colon X \to S^1$.

In a similar open neighborhood, a non-flat 1-cochain appears as

$$\omega = (H, \theta_a, h_{ab}, r_{abc}),$$

where H is a global 2-form, θ_a local 1-forms, h_{ab} local functions and r_{abc} $2\pi\mathbb{Z}$-valued integral assignments. Again, if ω is a cocycle, then the local ingredients mentioned earlier are related by $dH = 0$, $H_a = d\theta_a$, $\theta_a - \theta_b = h_{ab}$, $h_{bc} - h_{ac} + h_{ab} = r_{abc}$ and $r_{bcd} - r_{acd} + r_{abd} - r_{abc} = 0$. Let $g_{ab}\colon U_a \cap U_b \to S^1$ be the local functions $g_{ab} = \exp(ih_{ab})$. They satisfy the relation $g_{ab} g_{ac}^{-1} g_{bc} = 1$ and therefore make a Cech 1-cocycle with a cohomology class in $H^1(X, S^1)$. Such an object is known to determine a circle bundle $L \to X$. The rest of the cocycle conditions tell that local 1-forms θ_a glue together to form a circle connection on L. The global 2-form H is naturally the curvature of the connection. One can read the connection holonomy along 1-loops directly from the cocycle ω. Let us assume $\gamma\colon S^1 \to X$ is a smooth loop in X. The open covering $(U_a)_{a \in \mathcal{A}}$ will, by restriction, cover γ. We choose a triangulation of the loop γ subordinated to the covering. Such a feature is determined by a union

$$\gamma = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_n$$

with $\Delta_i\colon [0,1] \to \gamma$, $\Delta_i(1) = \Delta_{i+1}(0)$ for $1 \leq i \leq n-1$ and $\Delta_n(1) = \Delta_1(0)$. The triangulation γ is subordinated to the covering $(U_a)_{a \in \mathcal{A}}$ in the sense that a subordination map $\rho\colon \{1, \cdots n\} \to \mathcal{A}$ is chosen such that $\Delta_i \subset U_{\rho(i)}$.

The holonomy of the connection around the loop γ can then be computed as the $\exp(i \cdot \mathrm{hol}_\gamma)$, where

$$\mathrm{hol}_\gamma = \sum_i \int_{\Delta_i} \theta_{\rho(i)} - \sum_i \int_{\Delta_i(0)} h_{\rho(i)\rho(i-1)}. \tag{2.7}$$

Here, $\rho(-1) = \rho(n)$. So far, it seems that this formula for hol_γ depends on the choice of triangulation and subordination map. However, one can show that under variations of triangulation of the loop or subordination map ρ, the expression hol_γ gets modified by an element in $2\pi\mathbb{Z}$. The exponential $\exp(i \cdot \mathrm{hol}_\gamma)$ remains therefore unchanged under such modifications.

Geometrically, a non-flat differential 1-cocycle is therefore just a circle bundle connection. One verifies immediately that two 1-cocycles are equivalent if and only if the corresponding connections are gauge equivalent.

Assume now that under the earlier conditions, one introduces a non-flat 1-cochain σ satisfying $\check{d}\sigma = \omega$. Using the earlier notation, one obtains $H = dT,\ \theta_a = T_a - df_a,\ h_{ab} = -f_a + f_b + t_{ab}$ and $r_{abc} = t_{bc} - t_{ac} + t_{ab}$. From here, one deduces that $g_{ab} = q_a^{-1} q_b$ and therefore the local functions q_a glue together to form a global section trivializing the circle bundle L. Moreover, the holonomy of the connection is described in this trivialization by just integrals of the global 1-form T. One can say, therefore, that relation $\check{d}\sigma = \omega$ realizes σ as a "geometrical" section in L, in the sense that the information provides a section together with its behavior under holonomy of the existing S^1-connection.

These above arguments can be generalized to fit higher dimensional differential cochains. Let $\omega \in N\check{Z}^n(X)$ be a differential non-flat n-cocycle described in an open covering $(U_a)_{a\in\mathcal{A}}$ as a multiplet:

$$\omega = (H, \omega_a^n, \omega_{a_1a_2}^{n-1}, \omega_{a_1a_2a_3}^{n-2}, \ldots, \omega_{a_1a_2\cdots a_{n+1}}^0, \omega_{a_1a_2\cdots a_{n+2}}^{-1}). \tag{2.8}$$

The upper index represents the degree of the corresponding local form. The -1 index corresponds to locally constant functions taking values in $2\pi\mathbb{Z}$. One defines local S^1-valued functions

$$g_{a_1a_2\cdots a_{n+1}} : U_{a_1} \cap U_{a_2} \cap \cdots \cap U_{a_{n+1}} \to S^1,$$
$$g_{a_1a_2\cdots a_{n+1}} = \exp(i \cdot \omega^0_{a_1a_2\cdots a_{n+1}}).$$

It is a straightforward computation that $(\delta g)_{a_1a_2\cdots a_{n+2}} = 1$ and therefore g defines a Cech n-cocycle. This data defines (see [11] for details) a $(n-1)$-gerbe. These are objects which, due to their geometrical features, can be considered higher dimensional analogues of circle bundles. The cocycle ω defines then a connection on such a $(n-1)$-gerbe. Such a connection carries, as we shall explain shortly, a holonomy along any embedded closed n-manifold. Two connections are said to be gauge equivalent if they determine similar holonomies. Moreover an equality of type $\check{d}\sigma = \omega$ with $\sigma \in N\check{C}^{n-1}(X)$ can also be explained geometrically. Similar to the circle bundle case, the $(n-1)$-cochain σ determines a section trivializing the n-gerbe underlying ω.

Let us explain how the holonomy associated to a gerbe connection of type ω is defined. Suppose Y is a n-dimensional closed submanifold embedded in X. In order to define the holonomy of ω along Y, one needs two additional ingredients.

1) *A dual cell decomposition for* Y. This is a decomposition dual to a triangulation. (We need each vertex to be adjacent to n edges, each edge to be adjacent to $n-1$ 2-cells, and so on.) Let us describe the top cells as $(\Delta_i)_{i\in I}$. They inherit orientation from the orientation of Y. We denote by $\Delta_{i_1 i_2,\dots,i_k}$ the $n+1-k$ dimensional cell obtained by intersecting $\Delta_{i_1}, \Delta_{i_2}, \dots, \Delta_{i_k}$ (if such a cell exist). We assume the following orientation convention. A cell $\Delta_{i_1 i_2 \cdots i_k}$ receives orientation as boundary component in $\Delta_{i_1 i_2 \cdots i_{k-1}}$. That means $\Delta_{i_1 i_2}$ is oriented as boundary component in Δ_{i_1}, which is oriented a priori. $\Delta_{i_1 i_2 i_3}$ gets orientation as part of the boundary of $\Delta_{i_1 i_2}$, and so on. We will refer to a multi-index in the form $(i) = (i_1 i_2 \cdots i_k)$. $\Delta_{(i)}$ will then be a $(n+1-k)$-cell with a certain orientation. Permuting the elements in the index does not change the cell, but the orientation changes according to the signature of the permutation.
2) *A subordination map.* $\rho\colon I \to \mathcal{A}$ such that $\Delta_i \subset U_{\rho(i)}$.

These being settled, one defines the holonomy of ω along Y as $\exp(i \cdot \mathrm{hol}_Y(\omega))$, where

$$\mathrm{hol}_Y(\omega) = \sum_{k=1}^{n+1} (-1)^{k+1} \sum_{(i)=(i_1>i_2>\cdots>i_k)} \int_{\Delta_{(i)}} \omega^{n+1-k}_{\rho(i_1)\rho(i_2)\cdots\rho(i_k)} \tag{2.9}$$

Here, we just pick an order relation on I to make sure, that we do not use the same cell twice during summation. It can be seen that earlier formula generalizes the holonomy description for 1-cocycles presented earlier. The holonomy does not depend on the choice of dual cell decomposition or subordination map. That is justified by Claim 2.3.

Claim 2.3. *The expression $hol_Y(\omega)$ varies by an element in $2\pi\mathbb{Z}$ under modifications of the dual cell decomposition Δ_i or of the subordination map ρ.*

Proof. We follow two steps. First, we show that varying the subordination map for the same cell decomposition changes expression (2.9) by an element in $2\pi\mathbb{Z}$. Secondly, we will see that refining a cell decomposition under the same subordination map does not change (2.9). These will be enough to prove Claim 2.3.

To start with the first step, let us assume that the initial subordination map of $(\Delta_i)_{i\in I}$:
$$\rho\colon I \to \mathcal{A}$$
is modified over a unique cell Δ_{i_o} such that
$$\widetilde{\rho}(i) = \begin{cases} \rho(i) & \text{if } i \neq i_o; \\ \widetilde{\rho}(i_o) \neq \rho(i_o) & \text{if } i = i_o. \end{cases}$$

Let $\mathrm{hol}_Y(\omega)$ be the holonomy defined using the subordination map ρ and $\widetilde{\mathrm{hol}}_Y(\omega)$ be the holonomy defined using $\widetilde{\rho}$. We claim the following:
$$\mathrm{hol}_Y(\omega) - \widetilde{\mathrm{hol}}_Y(\omega) = \sum_{(i)=(i_o=i_1>i_2>\cdots i_{n+1})} \int_{\Delta_{(i)}} \omega^{-1}_{\rho(i_1)\widetilde{\rho}(i_1)\rho(i_2)\cdots\rho(i_{n+1})}$$
The quantity on the right is then a sum of numbers in $2\pi\mathbb{Z}$ (integrals are just functions evaluated in points) and the difference of the two holonomies is therefore 2π times an integer. This completes the first step. Let us prove the earlier relation. Assume that the order relation for indices $i \in I$ is chosen such that i_o is the largest index. We make the following notations:
$$P_k = \sum_{(i)=(i_o=i_1>i_2>\cdots>i_k)} \int_{\Delta_{(i)}} \left(\omega^{n+1-k}_{\rho(i_1)\rho(i_2)\cdots\rho(i_k)} - \omega^{n+1-k}_{\widetilde{\rho}(i_1)\rho(i_2)\cdots\rho(i_k)}\right)$$
and
$$S_k = \sum_{(i)=(i_o=i_1>i_2>\cdots>i_k)} \int_{\Delta_{(i)}} d\omega^{n-k}_{\rho(i_1)\widetilde{\rho}(i_1)\rho(i_2)\cdots\rho(i_k)}.$$
Therefore,
$$\mathrm{hol}_Y(\omega) - \widetilde{\mathrm{hol}}_Y(\omega) = \sum_{k=1}^{n+1} (-1)^{k+1} P_k$$
But $P_1 = S_1$ and in general for $2 < k$,
$$S_{k-1} + (-1)^{k+1} P_k = S_k.$$
This results by just applying Stokes' theorem. Hence,
$$\mathrm{hol}_Y(\omega) - \widetilde{\mathrm{hol}}_Y(\omega) = \sum_{k=1}^{n+1} (-1)^{k+1} P_k = S_1 + \sum_{k=2}^{n+1} (S_k - S_{k-1}) = S_{n+1}.$$
But since
$$S_{n+1} = \sum_{(i)=(i_o=i_1>i_2>\cdots>i_{n+1})} \int_{\Delta_{(i)}} \omega^{-1}_{\rho(i_1)\widetilde{\rho}(i_1)\rho(i_2)\cdots\rho(i_{n+1})},$$
the needed identity follows.

We now go over the second step. Let us assume that we have two different dual cell decompositions $(\Delta_i)_{i\in I}$ and $(\Delta_j)_{j\in J}$ with the latter one being a refinement of the former. Say, there is a map

$$\varphi\colon J \to I \quad \text{such that } \Delta_j \subset \Delta_{\varphi(j)}.$$

Moreover, both decompositions are subordinated to the covering U_a through maps

$$\rho\colon I \to \mathcal{A},$$
$$\rho\circ\varphi\colon J \to \mathcal{A}.$$

Assume the indices in both I and J are ordered such that the refinement map φ is increasing. We have two expressions for holonomy, depending on which decomposition we are using

$$\sum_{k=1}^{n+1}(-1)^{k+1}\sum_{(i)=(i_1>i_2>\cdots>i_k)}\int_{\Delta_{(i)}}\omega^{n+1-k}_{\rho(i_1)\rho(i_2)\cdots\rho(i_k)}$$

and

$$\sum_{k=1}^{n+1}(-1)^{k+1}\sum_{(j)=(j_1>j_2>\cdots>j_k)}\int_{\Delta_{(j)}}\omega^{n+1-k}_{\rho(j_1)\rho(j_2)\cdots\rho(j_k)}.$$

They are the same though. This is (roughly speaking) because $\omega^{n+1-k}_{\rho(j_1)\rho(j_2)\cdots\rho(j_k)}$ vanishes as soon as two subindices are the same. This completes the second step. □

2.3 Definition of the B-field

We are in position to give a definition for the B-field [5]. Recall the notation in (2.6).

Definition 2.4. Let space-time X be a smooth manifold. A B-field on X is a non-flat differential 2-cochain $B \in \mathcal{B}^2(X)$ defined over an open covering $\mathcal{U}$.

Let us recall that for non-flat differential n-cochains in

$$\mathcal{B}^n(X) = \left\{\omega \in N\check{C}^n(\mathcal{U})|\ \mathcal{U} \text{ covering}, \check{d}\omega \in \Omega^{n+1}(X)\right\},$$

one has the equivalence relation "$\sim$" defined in section 2.1 (Definition 2.2), which extends the standard cocycle equivalence.

Definition 2.5. Two B-fields $B_1, B_2 \in \mathcal{B}^2(X)$ are said to be (gauge) equivalent if $B_1 \sim B_2$.

Two B-fields cannot be normally added up unless they are defined on the same open covering. However, there is a well-defined summation rule on the set of equivalence classes

$$\mathcal{B}^2(X)/\sim. \tag{2.10}$$

For $B_1, B_2 \in \mathcal{B}^2(X)$, one defines $[B_1] + [B_2] = [B_1' + B_2']$, where B_1', B_2' are restrictions of B_1, B_2 on an open covering refining both coverings underlying B_1 and B_2. This definition does not depend on the choice of refinement or subordination maps. In this respect, (2.10) becomes a group.

Let $\omega \in \Omega^3(X)$ be a fixed 3-form and

$$\mathcal{T}_\omega = \{[B] | B \in \mathcal{B}^2(X), \check{d}B = \omega\}.$$

The equivalence relation does not modify the field strength H_B of a B-field $B \in \mathcal{B}^2(X)$. There is then an exact sequence

$$0 \to \check{H}^2(X) \hookrightarrow \mathcal{B}^2(X)/\sim \xrightarrow{\Phi} \Omega^3(X) \times \Omega^3(X)$$

with $\Phi(B) = (H_B, \check{d}B)$. Clearly, $\mathcal{T}_\omega = \Phi^{-1}(\Omega^3(X) \times \{\omega\})$. One can therefore conclude that the field strength projection map

$$\mathcal{T}_\omega \to \Omega^3(X), \quad [B] \to H_B$$

realizes a fibration over the image, the fibers being principal homogeneous spaces for the second smooth Deligne cohomology, $\check{H}^2(X)$.

Let us review the geometrical meaning attached to these objects. Any B-field contains a 3-form $\omega = \check{d}B$, which (as a differential 3-cocycle in the sense of (2.5)) can be seen as a connection $\mathcal{A}_\omega$ on the trivial 2-gerbe over X. $\mathcal{A}_\omega$ has exact field strength (given by $dH_B = d\omega$), and it may very well carry holonomy along closed 3-manifolds embedded in X. The B-field can be seen as a (geometrical) section trivializing the 2-gerbe. Since B is not required to be flat, the section is not necessarily parallel with respect to $\mathcal{A}_\omega$. In fact, its covariant derivative with respect to $\mathcal{A}_\omega$ is H_B.

In general, according to earlier definition, B-fields are just non-flat, differential 2-cochains. They do not carry holonomy in the standard sense. (Non-flat 2-cocycles can be viewed as connections on gerbes and do carry holonomy along closed surfaces.) However, as mentioned earlier, B can be regarded then as a section trivializing a 2-gerbe. But a non-flat 2-cocycle can be seen (integrated down, see Appendix A for details) to give a circle bundle connection over the space of embedded closed surfaces in X. Its underlying circle bundle is trivial, but has no natural trivialization. A B-field which is a section of a 2-gerbe integrates down to produce a section of this circle bundle. Thus, a B-field assigns to any closed surface Σ mapping to X, a point in the circle fiber over the point of the space of mappings given by $\Sigma \to X$. We denote this by

$$\exp\left(i\int_{\Sigma} B\right)$$

and interpret it as the holonomy of the B-field along Σ. If B is a cocycle ($\check{d}B = 0$), the circle bundle has a canonical trivialization and with respect to that the quantity can be seen as a unitary complex number, well defined up to an overall phase factor independent of the mapping. It realizes the standard gerbe connection holonomy.

2.4 Chern-Simons cocycle

Let G be a compact, simply connected, simple Lie group and $P \to X$ a principal G-bundle. To any choice of connection A on $P \to X$, one can associate a non-flat differential Chern–Simons 3-cocycle $\mathcal{CS}_A \in N\check{Z}^3(X)$. As shown by Freed [5], a general definition for $\mathcal{CS}_A$ can be achieved by pulling back a universal choice on BG through classifying maps. However, in this way, $\mathcal{CS}_A$ is well defined just up to a flat differential 3-coboundary. For computational reasons, we would like to introduce $\mathcal{CS}_A$ directly as a 3-cocycle. Although not canonically, this can be achieved by fixing a local trivialization of the bundle. Our task is further eased by the fact that all G-bundles we shall be dealing with throughout this paper are trivializable.

Let g be the Lie algebra of G and $\mathrm{ad}\colon G \to \mathrm{End}(g)$ the adjoint representation. Chern–Weil theory provides an isomorphism

$$H^4(BG, \mathbb{R}) \simeq I^2(G),$$

where $I^2(G)$ represents the family of ad-invariant quadratic forms on g. Such a quadratic form is called integral if it represents an integral cohomology class in $H^4(BG, \mathbb{R})$. For compact, simply connected, simple, Lie groups G, $H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}$. The integral ad-invariant forms make a free group a rank one. A generator is given by

$$q(a) = \frac{1}{16\pi^2 c_G} < a, a >_k . \tag{2.11}$$

Here, the right-hand side bracket denotes the Killing form on g. The number c_G is the Dynkin index of G, which is always integer or half-integer . $c_G = 1/2$ for $G = \mathrm{SU}(n)$. For $G = E_8$, the number c_G equals the dual Coxeter number 30.

The Lie bracket on g can be naturally extended to a graded Lie algebra bracket on $\Omega^*(X,\mathbf{g})$. Precisely,

$$[\omega^p,\omega^q](v_1,v_2,\ldots,v_{p+q}) = \frac{1}{(p+q)!}\sum_{\sigma\in S_{p+q}}(-1)^{\mathrm{sgn}(\sigma)} \times [\omega^p(v_{\sigma(1)},v_{\sigma(2)}\cdots v_{\sigma(p)}),\omega^q(v_{\sigma(p+1)},\cdots v_{\sigma(p+q)})].$$

It satisfies

$$[\omega^p,\omega^q] = (-1)^{pq+1}\,[\omega^q,\omega^p]$$

as well as the Jacobi identity

$$[\omega^p,[\omega^q,\omega^r]] = [[\omega^q,\omega^r]\,,\omega^p] + (-1)^{pq}\,[\omega^q,[\omega^p,\omega^r]]\,.$$

The inner product (2.11) extends as well to a pairing

$$\langle\cdot,\cdot\rangle : \Omega^p(X,g)\otimes\Omega^q(X,g)\to\Omega^{p+q}(X)$$

$$\langle\omega^p,\omega^q\rangle\,(v_1,v_2\cdots v_{p+q}) = \frac{1}{(p+q)!}\sum_{\sigma\in S_{p+q}}(-1)^{\mathrm{sgn}(\sigma)} \times \left\langle\omega^p(v_{\sigma(1)},v_{\sigma(2)}\cdots v_{\sigma(p)}),\omega^q(v_{\sigma(p+1)},\cdots v_{\sigma(p+q)})\right\rangle_k.$$

One can check that

$$\langle\omega^p,\omega^q\rangle = (-1)^{pq}\,\langle\omega^q,\omega^p\rangle \quad \text{and} \quad \langle\omega^p,[\omega^q,\omega^r]\rangle = \langle[\omega^p,\omega^q]\,,\omega^r\rangle\,.$$

On G, one has the g-valued Maurer–Cartan 1-form θ_G, which assigns to each vector its left-invariant extension. This satisfies

$$L_g^*\theta_G = \theta_G, \quad \text{and} \quad R_g^*\theta_G = ad_{g^{-1}}\theta_G,$$

where $L_g, R_g\colon G\to G$ represent left and right multiplication with $g\in G$. Its differential verifies the Maurer–Cartan equation:

$$d\theta_G + \frac{1}{2}\,[\theta_G,\theta_G] = 0.$$

Combining the two pairing operators, we obtain a new 3-form on G:

$$\mathcal{W}_G = -\frac{1}{6}\,\langle\theta_G,[\theta_G,\theta_G]\rangle\in\Omega^3(G,\mathbb{R}).$$

It is a closed bi-invariant 3-form with integral periods.

We introduce then the connection data. A connection on $P \to X$ can be seen as a 1-form $A \in \Omega^1(P, g)$ satisfying

$$L_p^* A = \theta_G \quad \text{and} \quad R_g^* A = \mathrm{ad}_{g^{-1}} A.$$

The curvature

$$F = dA + \frac{1}{2}[A, A]$$

verifies

$$L_p^* F = 0 \quad \text{and} \quad R_g^* F = \mathrm{ad}_{g^{-1}} F$$

as well as the Bianchi identity

$$dF + \frac{1}{2}[A, F] = 0.$$

The Chern–Simons 3-form is by definition a global form on the total space:

$$CS_A = \left\langle A, dA + \frac{1}{3}\,[A, A] \right\rangle = \left\langle A,\ F - \frac{1}{6}\,[A, A] \right\rangle \in \Omega^3(P, \mathbb{R}).$$

Its basic properties are $L_p^* CS_A = \mathcal{W}_G$, $R_g^* CS_A = CS_A$ and $dCS_A = \langle F, F\rangle$; the last relation being the standard computation for the Chern–Simons action.

So far, all objects were defined on the total space of the bundle. However, by employing a choice of local trivializations, all data can be transfered down to X. Here, we simplify our discussion. We assume that the G-bundle $P \to X$ is trivial. We fix, once for all, a global section $s\colon X \to P$ trivializing the bundle. In this setting, everything can be pulled back on X. We shall keep the same notation for the curvature 1-form and its curvature and Chern–Simons 3-form, although what we really mean is their pull-back through the section s. The Chern-Simons non-flat differential 3-cocycle can then be defined as the 3-cocycle induced by the global 3-form:

$$2\pi \left\langle A, dA + \frac{1}{3}\,[A, A] \right\rangle = 2\pi \left\langle A, F - \frac{1}{6}\,[A, A] \right\rangle \in \Omega^3(X, \mathbb{R}).$$

In differential cocycle language, the Chern–Simons 3-cocycle can be represented in a random open covering as

$$\mathcal{CS}_A = 2\pi \left(\langle F, F\rangle\,, \left\langle A, dA + \frac{1}{3}\,[A, A] \right\rangle |_{U_a}, 0, 0, 0, 0 \right). \tag{2.12}$$

This makes a well-defined cocycle since the global 4-form $2\pi H = 2\pi\,\langle F, F\rangle \in \Omega^4(X)$ has periods in $2\pi\mathbb{Z}$. If $G = \mathrm{SU}(n)$, $H = \mathrm{ch}_2(F)$ and its associated 4-cohomology class recover the second Chern class $c_2(P) \in H^4(X, \mathbb{Z})$. Since the bundle is trivial, this cohomology class vanishes.

The formulation (2.12) of $\mathcal{CS}_A$ depends on the choice of trivialization s. However, a variation of s changes $\mathcal{CS}_A$ by a flat 3-coboundary. The gauge class of the 3-cocycle does not change. The above definition of the Chern–Simons 3-cocycle can be extended to non-trivial G-bundles. The construction process involves fixing a family of local trivializations. However, we shall not deal with the non-trivial bundle case here.

We continue with:

Lemma 2.6. *The holonomy of $\mathcal{CS}_A$ along a closed 3-manifold W embedded in X recovers the Chern–Simons invariant* $\mathrm{cs}(W, A)$.

Proof. This follows from standard arguments in Chern–Simons theory. Let $W \hookrightarrow X$ be an embedded closed 3-manifold. There always exists a 4-manifold M such that $\partial M = W$. The bounding 4-manifold M is not necessarily embedded in X. Due to its triviality, the bundle P extends to a G-bundle $\widetilde{P}$ over M. The global section S and connection A extend also to $\widetilde{s}$ and $\widetilde{A}$ on $\widetilde{P}$. Therefore, the restriction on W of the Chern–Simons cocycle $\mathcal{CS}$ extends to a non-flat 3-cocycle $\mathcal{CS}_{\widetilde{A}}$ on M. In this setting,

$$\mathrm{hol}_W(\mathcal{CS}_A) = \exp\left(2\pi \cdot \int_M \left\langle \widetilde{F}, \widetilde{F} \right\rangle\right). \tag{2.13}$$

But (2.13) is exactly the Chern–Simons invariant $\mathrm{cs}(W, A)$. $\square$

In what follows we analyze the way Chern–Simons cocycles (seen here as global 3-forms due to triviality of the bundle) change under the action of symmetry group $\mathcal{G}$ of bundle automorphisms φ for P covering orientation preserving diffeomorphisms $\bar{\varphi}$ on X.

Theorem 2.7.

1) For each $\varphi \in \mathcal{G}$ and connection A, there exist a differential flat 2-chain $\theta_{(A,\varphi)} \in \mathcal{B}^2(X)$ such that

$$\mathcal{CS}_{\varphi^* A} = \bar{\varphi}^* \mathcal{CS}_A + \check{d}\theta_{(A,\varphi)}. \tag{2.14}$$

2) For any two $\varphi_1, \varphi_2 \in \mathcal{G}$, the quantity

$$\left[\theta_{(A,\varphi_1\circ\varphi_2)}\right] - \left[\theta_{(\varphi_1^* A,\varphi_2)}\right] - \left[\bar{\varphi}_2^* \theta_{(A,\varphi_1)}\right] \tag{2.15}$$

vanishes in $\mathcal{B}^2(X)/\sim$.

Proof. We look at the first part of the theorem. The symmetry group $\mathcal{G}$ consist of all automorphisms φ of the bundle P covering orientation preserving diffeomorphisms $\bar{\varphi}$ on X. It includes as a normal subgroup the group of standard gauge transformations $\mathcal{G}(P)$. Technically, we have the following short exact sequence of groups

$$\{1\} \to \mathcal{G}(P) \to \mathcal{G} \to \mathrm{Diff}^+(X) \to \{1\}. \tag{2.16}$$

Due to triviality of the bundle the above sequence splits. Indeed, there is a map $\mathrm{Diff}^+(X) \to \mathcal{G}$, $\bar{\varphi} \to \bar{\varphi}_o$ sending a diffeomorphism $\bar{\varphi}$ of X to the unique automorphism of P leaving the section s invariant. This map builds a section for the second projection in the exact sequence (2.16). Therefore, any automorphism $\varphi \in \mathcal{G}$ can be decomposed uniquely as

$$\varphi = \psi \circ \bar{\varphi}_{\mathrm{o}} \tag{2.17}$$

with $\psi \in \mathcal{G}(P)$. The symmetry group can then be understood as a semi-direct product

$$\mathcal{G}(P) \rtimes \mathrm{Diff}^+(X).$$

Let ψ be a standard gauge transformation. In the chosen trivialization s, ψ can be seen through a smooth function $t\colon X \to G$. For any connection A, we get

$$\psi^* A = \mathrm{ad}_{t^{-1}} A + t^* \theta_G.$$

The Chern–Simons three-form gets modified as follows:

$$\begin{aligned} &\left\langle \psi^* A, d\psi^* A + \frac{1}{3}\left[\psi^* A, \psi^* A\right] \right\rangle - \left\langle A, dA + \frac{1}{3}\left[A, A\right] \right\rangle \\ &\quad = d\left\langle \mathrm{ad}_{t^{-1}} A, t^* \theta_G \right\rangle + t^* \mathcal{W}_G. \end{aligned} \tag{2.18}$$

The 3-form $\mathcal{W}_G$ is closed and has integral periods. Therefore, one could interpret $2\pi \cdot \mathcal{W}_G$ as a flat 3-cocycle with trivial holonomy. Hence, we can pick a universal choice of a flat 2-cochain $\eta \in \check{C}^2(G)$ satisfying $\check{d}\eta = 2\pi \cdot \mathcal{W}_G$. On the basis of this assumption, we define

$$\theta_{(A,\psi)} = 2\pi \cdot \left\langle \mathrm{ad}_{t^{-1}} A, t^* \theta_G \right\rangle + t^* \eta \tag{2.19}$$

The first term on the right-hand side of (2.19) is to be interpreted as a non-flat differential 2-cochain representable in a random covering U_a as

$$2\pi \cdot \left(0, \left\langle \mathrm{ad}_{t^{-1}} A, t^* \theta_G \right\rangle |_{U_a}, 0, 0, 0\right).$$

In this setting, $\theta_{(A,\psi)}$ as defined in (2.19) becomes a differential flat 2-cochain. A straightforward computation based on (2.18) shows that

$$\mathcal{CS}_{\psi^* A} = \mathcal{CS}_A + \check{d}\theta_{(A,\psi)}. \tag{2.20}$$

Here, we make use of the fact that $\mathcal{CS}_A$ and $\mathcal{CS}_{\psi^* A}$, as 3-cocycles associated to global 3-forms, can be represented in any choice of open covering; in particular, on the covering underlying $\theta_{(A,\psi)}$. The earlier expression is then (2.14) in the case of a standard gauge transformation.

Let us consider $\varphi \in \mathcal{G}$. According to (2.17), we can uniquely decompose φ as

$$\varphi = \psi \circ \bar{\varphi}_{\mathrm{o}},$$

where $\bar{\varphi}_o$ is the unique automorphism covering $\bar{\varphi}$ on base space, preserving the trivializing section s and $\psi \in \mathcal{G}(P)$. Clearly,

$$\mathcal{CS}_{\varphi^* A} = \mathcal{CS}_{(\psi\circ\bar{\varphi}_o)^* A} = \mathcal{CS}_{\bar{\varphi}_o^*(\psi^* A)} = \bar{\varphi}^* \mathcal{CS}_{\psi^* A} = \bar{\varphi}^* \left(\mathcal{CS}_A + \check{d}\theta_{(A,\psi)}\right). \tag{2.21}$$

Defining

$$\theta_{(A,\varphi)} = \bar{\varphi}^* \theta_{(A,\psi)}$$

and using this in (2.21), we obtain

$$\mathcal{CS}_{\psi^* A} = \bar{\varphi}^* \mathcal{CS}_A + \check{d}\theta_{(A,\psi)},$$

which is exactly Eq. (2.14).

We prove now the second part of the theorem. As before, we start by analyzing the case when the two automorphisms involved are just gauge transformations. Let $\psi_1, \psi_2 \in \mathcal{G}(P)$. We have

$$\begin{array}{ccccc} P & \stackrel{\psi_1}{\to} & P & \stackrel{\psi_2}{\to} & P \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ X & = & X & = & X \end{array}$$

The three flat 2-cochains involved, $\theta_{(A,\psi_1\circ\psi_2)}$, $\theta_{(\psi_1^* A,\psi_2)}$ and $\theta_{(A,\psi_1)}$, live on different open coverings. However, our goal is to prove an equality involving their equivalence classes. We take a common refinement and make choices of subordination maps restricting, therefore, the three flat 2-cochains on a unique common open covering where they can be added up. The equivalence classes will not be affected by the choice of refinement. In this setting,

$$\theta_{(A,\psi_1\circ\psi_2)} - \theta_{(\psi_1^* A,\psi_2)} - \theta_{(A,\psi_1)} \tag{2.22}$$

$$\begin{aligned} &= 2\pi \cdot \left\langle \mathrm{ad}_{(t_2 t_1)t^{-1}} A, (t_2 t_1)^* \theta_G \right\rangle - 2\pi \cdot \left\langle \mathrm{ad}_{t_2^{-1}} (\mathrm{ad}_{t_1^{-1}} A + t_1^* \theta_G), t_2^* \theta_G \right\rangle \\ &\quad - 2\pi \cdot \left\langle \mathrm{ad}_{t_1^{-1}} A, t_1^* \theta_G \right\rangle + (t_2 t_1)^* \eta - t_2^* \eta - t_1^* \eta \\ &= 2\pi \cdot \left\langle \mathrm{ad}_{t_2^{-1}} t_1^* \theta_G, t_2^* \theta_G \right\rangle + (t_2 t_1)^* \eta - t_2^* \eta - t_1^* \eta. \end{aligned} \tag{2.23}$$

Therefore, expression (2.22) does not depend on connection A. Now, clearly, (2.23) makes a differential 2-cocycle, because

$$\begin{aligned} &\check{d}\left(2\pi \cdot \left\langle \mathrm{ad}_{t_2^{-1}} t_1^* \theta_G, t_2^* \theta_G \right\rangle + (t_2 t_1)^* \eta - t_2^* \eta - t_1^* \eta\right) \\ &\quad = 2\pi \cdot d \left\langle \mathrm{ad}_{t_2^{-1}} t_1^* \theta_G, t_2^* \theta_G \right\rangle + (t_2 t_1)^* \mathcal{W}_G - t_2^* \mathcal{W}_G - t_1^* \mathcal{W}_G = 0. \end{aligned}$$

It is also a flat 2-cocycle since all 2-cochains $\theta_{(A,\psi)}$ are flat. We show (2.23) is actually a coboundary of a flat differential 1-cochain. That follows from the fact that its holonomy along any embedded compact oriented 2-manifold vanishes. Let $\Sigma \subset X$ be an embedded smooth surface. There always exist a compact, oriented 3-manifold W such that $\partial W = \Sigma$. It is not necessarily that W is embedded in X. The bundle P extends to $\widetilde{P}$ over W and so does the trivialization s. Obstruction theory (based on $\pi_1(G) = 0$) shows that the two gauge transformations $\psi_i, i \in \{1,2\}$, extend to gauge transformations $\widetilde{\psi}_i, i \in \{1,2\}$, in $\widetilde{P}$. They correspond to functions $\widetilde{t}_i \colon W \to G$. Accordingly, the flat 2-cocycle (2.23) extends to a flat 2-cocycle

$$2\pi \cdot \left\langle \mathrm{ad}_{\widetilde{t}_2^{-1}} \widetilde{t}_1^* \theta_G, \widetilde{t}_2^* \theta_G \right\rangle + (\widetilde{t}_2 \widetilde{t}_1)^* \eta - \widetilde{t}_2^* \eta - \widetilde{t}_1^* \eta. \tag{2.24}$$

But, in such situations, the holonomy of (2.23) along Σ is just the exponential of the strength field of (2.24) integrated over W. The latter vanishes since the strength field of (2.24) is zero.

We look now at the general case. Assume that $\varphi_1, \varphi_2 \in \mathcal{G}$ covering $\bar{\varphi}_1$, $\bar{\varphi}_1 \in \mathrm{Diff}^+(X)$

$$\begin{array}{ccccc} P & \stackrel{\varphi_1}{\to} & P & \stackrel{\varphi_2}{\to} & P \\ \downarrow & & \downarrow & & \downarrow \\ X & \stackrel{\bar{\varphi}_1}{\to} & X & \stackrel{\bar{\varphi}_1}{\to} & X \end{array}$$

and they decompose as

$$\varphi_i = \psi_i \circ \bar{\varphi}_{io}, \quad i \in \{1,2\}.$$

By definition,

$$\theta_{(A,\varphi_i)} = \bar{\varphi}_i^* \theta_{(A,\psi_i)}.$$

Let $\psi_1' = \bar{\varphi}_{2o} \circ \psi_1 \circ \bar{\varphi}_{2o}^{-1} \in \mathcal{G}(P)$. For any connection A, we then have

$$\theta_{(\bar{\varphi}_{2o}^* A, \psi_1)} = \bar{\varphi}_2^* \theta_{(A,\psi_1')}. \tag{2.25}$$

Therefore, expression (2.15) becomes

$$\begin{aligned} &\left[\theta_{(A,\varphi_2\circ\varphi_1)}\right] - \left[\theta_{(\varphi_2^* A,\varphi_1)}\right] - \left[\bar{\varphi}_1^* \theta_{(A,\varphi_2)}\right] \\ &\quad = (\bar{\varphi}_{2o} \circ \bar{\varphi}_{1o})^* \left[\theta_{(A,\psi_2\circ\psi_1')}\right] - \bar{\varphi}_1^* \left[\theta_{(\varphi_2^* A,\psi_1)}\right] - \bar{\varphi}_1^* \bar{\varphi}_2^* \left[\theta_{(A,\psi_2)}\right]. \end{aligned} \tag{2.26}$$

Now, from the case of pure gauge transformations, we know that on some common refining subcovering,

$$\theta_{(A,\psi_2\circ\psi_1')} - \theta_{(\psi_2^* A,\psi_1')} - \theta_{(A,\psi_2)} \tag{2.27}$$

is a coboundary of a differential flat 1-cochain. Putting together relations (2.26) and (2.27) we obtain that

$$\begin{aligned}&\left[\theta_{(A,\varphi_2\circ\varphi_1)}\right] - \left[\theta_{(\varphi_2^*A,\varphi_1)}\right] - \bar{\varphi}_1^*\left[\theta_{(A,\varphi_2)}\right]\\ &\quad = \bar{\varphi}_1^*\bar{\varphi}_2^*\left[\theta_{(\psi_2^*A,\psi_1')}\right] - -\bar{\varphi}_1^*\left[\theta_{(\varphi_2^*A,\psi_1)}\right] + \left[\check{d}\{\text{flat 1-cochain}\}\right].\end{aligned}$$

Using equality (2.25) with connection ψ_2^*A, we notice that the first term in the right-hand side of above expression vanishes. We can therefore conclude that

$$\left[\theta_{(A,\varphi_2\circ\varphi_1)}\right] - \left[\theta_{(\varphi_2^*A,\varphi_1)}\right] - \bar{\varphi}_1^*\left[\theta_{(A,\varphi_2)}\right] = 0.$$

□

These features are the basic facts about Chern–Simons cocycles when G is a simply connected, simple Lie group. However, the two compact groups we shall use throughout the paper, $(E_8 \times E_8) \rtimes \mathbb{Z}_2$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$, do not quite fall in this category. In what follows, we adapt the previous discussion to those cases.

$G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$. A trivializable G-bundle $P \to X$ can be understood as a sum $P_1 \times P_2$ with $P_i \to X, i \in \{1, 2\}$ trivializable E_8-bundles. A section s is P can be fixed in the form of a pair (s_1, s_2) with s_i section in P_i. Following the pattern, one can view a G-connection on P as a pair $A = (A_1, A_2)$ with A_i, $i = 1, 2$, as E_8-connections on P_i's. We define then the Chern–Simons cocycle as

$$\mathcal{CS}_A^G = \mathcal{CS}_{A_1}^{E_8} + \mathcal{CS}_{A_2}^{E_8}. \tag{2.28}$$

The differential 2-cochains $\theta_{(A,\varphi)}$ can be reconstructed as well. The group of automorphisms φ of P has as index-two subgroup the group of automorphisms $\varphi = (\varphi_1, \varphi_2)$ with φ_i automorphisms of P_i. The $\mathbb{Z}_2$ quotient is generated by the automorphism that exchanges the two E_8 factors. We go ahead and define for $\varphi = (\varphi_1, \varphi_2)$:

$$\theta_{(A,\varphi)}^G = \theta_{(A_1,\varphi_1)}^{E_8} + \theta_{(A_2,\varphi_2)}^{E_8}. \tag{2.29}$$

A straightforward computation shows that Theorem 2.7 is still true.

$G = Spin(32)/\mathbb{Z}_2$. Let $P \to X$ be a trivializable $G = \mathrm{Spin}(32)/\mathbb{Z}_2$-connection. We set a section s. The Lie group projection $\mathrm{Spin}(32) \to \mathrm{Spin}(32)/\mathbb{Z}_2$ induces an isomorphism at Lie algebra level. We define the Chern–Simons cocycle corresponding to a connection A by using the trace pairing of the standard 32-dimensional representation of Spin(32). To view it in a different way, say $\widetilde{P}$ is the trivial lifting of P to a Spin(32)-bundle

$$\begin{array}{ccc} \widetilde{P} & \xrightarrow{p} & P \\ \Big\downarrow & & \Big\downarrow \\ X & = & X \end{array} \tag{2.30}$$

together with a section $\widetilde{s} = p^* s$. There is then a one-to-one correspondence (given by pull-back through p) between G-connections on P and Spin(32)-connections on $\widetilde{P}$. The Chern–Simons cocycle can then be defined as

$$\mathcal{CS}^G_A = \frac{1}{2} \cdot \mathcal{CS}^{\mathrm{Spin}(32)}_{p^*A}. \tag{2.31}$$

The $\theta_{(A,\varphi)}$ 2-cochains can be extended similarly. Here, we make use of the fact that, as we mentioned earlier, the symmetry group $\mathcal{G}$ in the $\mathrm{Spin}(32)/\mathbb{Z}_2$ case consists of only liftable automorphisms. So, any $\varphi \in \mathcal{G}$ covering $\bar{\varphi} \in \mathrm{Diff}^+(X)$ can be seen as coming from an automorphism $\widetilde{\varphi}$ of $\widetilde{P}$. We define then

$$\theta^G_{(A,\varphi)} = \frac{1}{2} \cdot \theta_{(p^*A, \widetilde{\varphi})}. \tag{2.32}$$

It can be seen that up to a coboundary of a differential flat 1-cochain, the right side of (2.32) does not depend on the choice of automorphism lifting $\widetilde{\varphi}$. Indeed, say $\widetilde{\widetilde{\varphi}}$ is a different lifting. Then, $\widetilde{\widetilde{\varphi}} = \xi \circ \widetilde{\varphi}$, where ξ is the gauge transformation in $\widetilde{P}$, which consist in each fiber of right multiplication with q, the exotic element in $\mathbb{Z}_2$. Let $\widetilde{\varphi} = \psi \circ \bar{\varphi}_o$ with ψ gauge transformation. Then, for any connection $\widetilde{A}$ on $\widetilde{P}$,

$$\theta_{(\widetilde{A},\widetilde{\varphi})} = \bar{\varphi}^* \theta_{(\bar{\varphi}^*\widetilde{A},\psi)} \quad \text{and} \quad \theta_{(\widetilde{A},\widetilde{\widetilde{\varphi}})} = \bar{\varphi}^* \theta_{(\bar{\varphi}^*\widetilde{A},\xi\circ\psi)}.$$

But, up to a coboundary of a flat 1-chain,

$$\theta_{(\bar{\varphi}^*\widetilde{A},\xi\circ\psi)} = \theta_{(\bar{\varphi}^*\widetilde{A},\psi)} + \theta_{(\psi^*\bar{\varphi}^*\widetilde{A},\xi)}.$$

The last term on the right side above vanishes. That happens because denoting $B = \psi^* \bar{\varphi}^* \widetilde{A}$ and applying definition (2.19), one can write

$$\theta_{(B,\xi)} = 2\pi \cdot \langle \mathrm{ad}_{t^{-1}} B, t^*\theta_G \rangle + t^*\eta \tag{2.33}$$

where $t \colon X \to \mathrm{Spin}(32)$ is the map associated to the gauge transformation in the given trivialization. In the case of ξ, the map t is constant. Therefore, both terms in the right side of (2.33) vanish. Going back to the two equations earlier, that is enough to conclude

$$\theta_{(\widetilde{A},\widetilde{\varphi})} = \theta_{(\widetilde{A},\widetilde{\widetilde{\varphi}})}$$

up to a coboundary of a flat 1-cochain.

Expression (2.32) gives then a well-defined formulation for $\theta^G_{(A,\varphi)}$. Theorem 2.7 is verified.

We finish this section with a short remark about the gravitational $SO(10)$ Chern–Simons.

$G = SO(10)$. The gravitational Chern–Simons term $\mathcal{CS}_g$ appears in the anomaly cancellation condition (3.1) described in next section. It represents the Chern–Simons cocycle associated to the Levi-Civita connection induced by Riemannian metric g on TX, which can be seen as a connection on the SO(10)-principal bundle of orthonormal oriented frames associated to TX. Therefore, $\mathcal{CS}_g$ is basically the Chern–Simons cocycle of a SO(10)-connection. However, the space-time X is endowed from the beginning with a spin structure. The Levi-Civita connection lifts uniquely to a Spin(10) connection on the fixed spin bundle. We define $\mathcal{CS}_g$ to be half of the Chern–Simons associated to the Spin(10)-lifted Levi-Civita. The Lie group Spin(10) is simply connected (and in our particular case of space-time, the spin structure is trivializable), so the entire previous discussion applies.

3 String data

In this section, we analyze the parameter data for heterotic string theory. In general, a 10-dimensional spin manifold X is chosen as a space-time and a gauge bundle is fixed to be a G-principal bundle $P \to X$. For anomaly cancellation reasons [10], the Lie group G must be one of the two choices $(E_8 \times E_8) \rtimes \mathbb{Z}_2$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$. The string data specifying a theory consist of a triplet (g, A, B) involving a metric on X, a G-connection on P and a B-field in the form of a differential non-flat 2-cochain on X. Equations of Motion [10] impose the following conditions:

1) The metric g on X is Ricci flat.
2) The G-connection A on P has vanishing curvature.
3) The B-field B has vanishing field strength.

These classical solutions to the equations of motion are to be considered up to gauge equivalence. However, before factoring to equivalence classes, there is one more constraint to be taken into account. This is the anomaly cancellation condition interpreted by Freed [5] as

$$\check{d}B = \mathcal{CS}_A - \mathcal{CS}_g. \tag{3.1}$$

$\mathcal{CS}_A$ and $\mathcal{CS}_g$ are the Chern–Simons 3-cocycles corresponding to the connection A, respectively, the lifted Levi-Civita connection associated to the metric g on the fixed spin bundle of X. Their construction was explained in Section 2.4. The possibility of imposing such a constraint requires an a priori topological condition on space-time X. Namely,

$$\lambda(P) = p_1(TX) \quad \text{in} H^4(X, \mathbb{Z}). \tag{3.2}$$

Here, $\lambda \in H^4(BG, \mathbb{Z})$ represents the level of the theory.

In case $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$, one obtains $H^4(BG, \mathbb{Z}) = H^4(BE_8, \mathbb{Z}) \times H^4(BE_8, \mathbb{Z})$. It is a known fact [14] that for a simply connected, simple Lie group H, $H^4(BH, \mathbb{Z}) \simeq \mathbb{Z}$. A generator ξ for $H^4(BE_8, \mathbb{Z})$ is obtained via Chern–Weil theory from the ad-invariant quadratic form

$$I_o \colon e_8 \to \mathbb{R}, \quad I_o(a) = \frac{1}{16\pi^2 c_{E_8}} \langle a, a \rangle_k,$$

where e_8 is the Lie algebra of E_8, $\langle \cdot, \cdot \rangle$ represents the Killing form on e_8 and $c_{E_8} = 30$ is the dual Coxeter number. The level of the $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$ theory is then chosen as $\lambda = (\xi, \xi) \in H^4(BG, \mathbb{Z})$.

The case $G = \mathrm{Spin}(32)/\mathbb{Z}_2$ can be treated similarly. One has

$$H^4(BG, \mathbb{Z}) \simeq \mathbb{Z},$$

and the level of the theory is chosen to be the generator $\lambda \in H^4(BG, \mathbb{Z})$ representing in Chern–Weil theory the quadratic form

$$\frac{1}{2} I_o \colon \mathrm{Spin}(32) \to \mathbb{R}, \quad \frac{1}{2} I_o(a) = \frac{1}{2} \cdot \frac{1}{16\pi^2 c_{\mathrm{Spin}(32)}} \langle a, a \rangle_k .$$

The particular case of eight-dimensional heterotic string is simpler from this point of view. One compactifies from 10 to 8 dimensions. The space-time is $X = E \times \mathbb{R}^8$, with E a 2-dimensional torus. Condition (3.2) is satisfied by default, as there is no 4-cohomology. In this particular framework, the classical solutions to the equations of motion are invariant under orientation-preserving isometries of $\mathbb{R}^8$ and we consider only $\mathbb{R}^8$-invariant solutions. Therefore, we can equivalently regard the three objects, involved as

1) g, flat metric on E;
2) A, G-connection on a fixed principal bundle $P \to E$;
3) B, flat differential cochain on E with $\check{d}B = \mathcal{CS}_A - \mathcal{CS}_g$.

If $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$, all G-bundles over a 2-torus are topologically trivial. However, for $G = \mathrm{Spin}(32)/\mathbb{Z}_2$, there exist a non-trivial topological type. The topological type of the bundle is characterized by the generalized Stiefel-Whitney class $\widetilde{w}_2 \in H^2(X, \mathbb{Z}_2)$, which measures the obstruction to lifting the structure group of the bundle to Spin(32). Nevertheless, the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string assumes that the gauge bundle allows such a lifting (P carries a vector structure in physics terminology). Therefore, we consider from now on that the G-bundle $P \to E$ is topologically trivial.

Let us denote by $\widetilde{\mathcal{M}}_{\mathrm{het}}$ the set of triplets $(g, A, [B])$ such that g is a flat metric, $F_A = 0$, $H_B = 0$ and $[B] \in \mathcal{T}_\omega$, where

$$\omega = \mathcal{CS}_A - \mathcal{CS}_g.$$

In addition, set

$$\mathcal{P}_G = \{(g, A) | g \text{ flat metric}, F_A = 0\}.$$

One cannot consider the equivalence class of a general differential 2-cochain. However, all bundles involved here are trivializable. Therefore, as explained in Section 2.4, the two Chern–Simons cocycles $\mathcal{CS}_A$ and $\mathcal{CS}_g$ are global 3-forms. The B-fields appearing in the definition of $\widetilde{\mathcal{M}}_{\mathrm{het}}$ are then included in $\mathcal{B}^2(X)$, and their equivalence class is well defined.

The $\widetilde{\mathcal{M}}_{\mathrm{het}}$ and $\mathcal{P}_G$ spaces carry natural smooth structures. There is a projection map

$$\pi \colon \widetilde{\mathcal{M}}_{\mathrm{het}} \to \mathcal{P}_G, \tag{3.3}$$

realizing a smooth fibration. The space of gauge classes of flat 2-cocycles, $\check{H}^2(E) \simeq S^1$ acts freely and transitively on each fiber. In this sense, (3.3) becomes a principal circle bundle. This raw picture must be divided out by symmetries. The symmetry group $\mathcal{G}$ consists of bundle automorphisms covering any orientation-preserving diffeomorphism of the base:

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & P \\ \downarrow & & \downarrow \\ E & \xrightarrow{\bar{\varphi}} & E \end{array}$$

$$\mathcal{G} \overset{\text{def}}{=} \left\{\varphi \in \mathrm{Aut}_G(P) | \bar{\varphi} \in \mathrm{Diff}^+(E)\right\}. \tag{3.4}$$

(Strictly speaking, this is the symmetry group for $(E_8 \times E_8) \rtimes \mathbb{Z}_2$ case. If $G = \mathrm{Spin}(32)/\mathbb{Z}_2$, $\mathcal{G}$ is only made out of those automorphisms that can be lifted to Spin(32) automorphisms.) The group $\mathcal{G}$ is a non-trivial extension of the diffeomorphism group of E by the group of changes of gauge of the bundle P:

$$\{1\} \to \mathcal{G}(P) \to \mathcal{G} \to \mathrm{Diff}^+(E) \to \{1\}.$$

$\mathcal{G}$ acts naturally on both the total and base spaces of (3.3). The action on $\mathcal{P}_G$ is the standard one:

$$\mathcal{P}_G \times \mathcal{G} \to \mathcal{P}_G, \quad (g, A).\varphi = (\bar{\varphi}^* g, \varphi^* A).$$

In order to establish the action on the total space, we use the following facts:

$$\mathcal{CS}_{\bar{\varphi}^* g} = \bar{\varphi}^* \mathcal{CS}_g$$

and

$$\mathcal{CS}_{\varphi^* A} = \bar{\varphi}^* \mathcal{CS}_A + \check{d}\theta_{(A,\varphi)}.$$

Here, $\theta_{(A,\varphi)}$ is a flat 2-cochain in $\mathcal{B}^2(X)$ as described in Section 2.4. The action $\widetilde{\mathcal{M}}_{\mathrm{het}} \times \mathcal{G} \to \widetilde{\mathcal{M}}_{\mathrm{het}}$ can then be described as

$$(g, A, [B]).\varphi = (\bar{\varphi}^* g, \varphi^* A, [\bar{\varphi}^* B] + [\theta_{(A,\varphi)}]).$$

It is a well defined action since, as is established in Section 2.4,

$$[\theta_{(A,\varphi_1 \circ \varphi_2)}] - [\theta_{(\varphi_1^* A, \varphi_2)}] - [\bar{\varphi}_2^* \theta_{(A,\varphi_1)}] = 0.$$

The symmetry group action commutes with the projection π, factoring the circle bundle (3.3). However, as we shall see, there are points in the base for which the stabilizer group does not act trivially on the earlier fiber. Nevertheless, all stabilizer groups are finite. The circle bundle (3.3) descends to a circle fibration:

$$\widetilde{\mathcal{M}}_{\mathrm{het}}/\mathcal{G} \to \mathcal{P}_G/\mathcal{G}. \tag{3.5}$$

The total space represents exactly the moduli space of heterotic string parameters $\mathcal{M}_{\mathrm{het}}$. Our goal is to describe this space in detail and to characterize the circle fibration (3.5).

4 Anomaly cancellation: Chern–Simons versus Pfaffians

The discussion in Section 3 was based on a smooth family of metrics and G-connections on a 2-torus. The associated space of B-field equivalence classes forms the total space of a principal circle bundle over the parameter space. Moreover, this bundle carries a natural connection. Now, let ρ be a complex unitary representation of the Lie group G. Using ρ, one can construct for each pair (A, g) a complex elliptic operator, the coupled Dirac operator. A determinant line bundle can be associated to this family of operators. It carries canonical Quillen connection and metric. Under certain conditions, a tensor combination of such determinant bundles recovers completely the B-field circle bundle with its connection. The determinant tensor combination forms the so-called fermionic anomaly of the family. The identification existing between the two bundles is known as the Green–Schwartz anomaly cancellation. This section provides the details of the construction.

4.1 Chern–Simons bundle

To begin with, we establish the following general setting. Let $p\colon Z \to Y$ be a smooth fibration of manifolds with all fibers isomorphic to a 2-torus E. We assume the following geometrical data:

- a spin structure and a metric $g^{(Z/Y)}$ on the tangent bundle along the fibers $T(Z/Y) \to Z$;
- a projection $TZ \to T(Z/Y)$;
- a (trivializable) $\mathrm{SO}(n)$-bundle $U \to Z$ endowed with a connection A^{grav};
- a (trivializable) G-principal bundle $Q \to Z$ endowed with a connection A^{gauge};
- a complex unitary representation $\rho\colon G \to U(r)$ carrying a real structure (meaning that representation ρ can be obtained by complexification from a real representation).

The tangent bundle over the fibers of p can be endowed with a canonical connection. Indeed, let g^Y be an arbitrary Riemannian metric on Y. Since $TZ = p^*TY \oplus T(Z/Y)$, the pull-back p^*g^Y together with $g^{(Z/Y)}$ generates a Riemannian metric on Z. Let ∇^Z be its Levi-Civita connection. Projecting ∇^Z on the tangent bundle along fibers, we obtain a connection $\nabla^{(Z/Y)}$. It is independent of the choice of metric g^Y on Y [2].

The Chern–Simons cocycle associated to the family

$$\mathrm{CS} = \mathcal{CS}_{A^{\mathrm{gauge}}} - \mathcal{CS}_{A^{\mathrm{grav}}}. \tag{4.1}$$

is a non-flat 3-cocycle[2] in $N\check{Z}^3(Z)$. It can be seen as a connection on a 2-gerbe over Z. Fixing an $y \in Y$, we obtain a 2-torus $E_y = p^{-1}(y)$ together with a G-connection $A_y^{\mathrm{gauge}} = A^{\mathrm{gauge}}|_{E_y}$ and a metric g_y on TE_y. The restriction $A_y^{\mathrm{grav}} = A^{\mathrm{grav}}|_{E_y}$ recovers the Levi-Civita connection associated to g_y. Therefore,

$$\mathrm{CS}|_{E_y} = \mathcal{CS}_{A_y^{\mathrm{gauge}}} - \mathcal{CS}_{A_y^{\mathrm{grav}}} \in N\check{Z}^3(E_y).$$

[2]Owing to the fact that the two bundles involved are trivializable, the Chern–Simons cocycle CS actually represents, as explained in Section 2.4, a global 3-form.

The Chern–Simons 3-cocycle (4.1) can be pushed-forward (Appendix A) to Y, defining a 1-cocycle:

$$\omega = \int_{Z/Y} \mathrm{CS} \in N\check{Z}^1(Y), \tag{4.2}$$

which can be geometrically interpreted as a circle connection on a certain principal circle bundle

$$\mathcal{R}_Y \to Y. \tag{4.3}$$

We call this the Chern–Simons bundle associated to the family.

Now, we incorporate the B-fields. Let

$$\widetilde{M}_Y = \left\{ (y, [B]) | [B] \in \mathcal{T}_\omega, \text{ where } \omega = \mathcal{CS}_{A_y^{\text{gauge}}} - \mathcal{CS}_{A_y^{\text{grav}}} \right\}. \tag{4.4}$$

$\widetilde{M}_Y$ can be endowed with a canonical topology and smooth structure. The projection $\pi\colon \widetilde{M}_Y \to Y$ makes then a circle bundle under the action on the total space of $\check{H}^2(E) \simeq S^1$.

Theorem 4.1. *There exist a canonical circle bundle isomorphism Φ_Y:*

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_Y & \overset{\Phi_Y}{\to} & \mathcal{R}_Y \\ \downarrow & & \downarrow \\ Y & = & Y \end{array}$$

identifying the B-field parameter space $\widetilde{\mathcal{M}}_Y$ to the total space of the circle bundle (4.3).

Proof. The correspondence Φ_Y goes as follows. Let $(y, [B])$ be an element in $\widetilde{\mathcal{M}}_Y$. B is a non-flat 2-cochain on E_y, satisfying:

$$\check{d}B = \mathcal{CS}_{A_y^{\text{gauge}}} - \mathcal{CS}_{A_y^{\text{grav}}}. \tag{4.5}$$

We take

$$\Phi_Y([B]) = \left[\int_{E_y} B \right]. \tag{4.6}$$

The integral inside (4.6) is explained in Appendix A. Although the construction of the integral itself on 2-cochains in not canonical, its (gauge) equivalence class is well defined. $\Phi_Y([B])$ is simply an equivalence class of a 0-cochain over the point $y \in Y$. (Roughly speaking, such a 0-cochain over a point is just a real number, two 0-cochains being equivalent if the difference of the two numbers is in $2\pi\mathbb{Z}$.) But $\mathcal{CS}_{A_y^{\text{gauge}}} - \mathcal{CS}_{A_y^{\text{grav}}} = CS|_{E_y}$ and Eq. (4.5) show that B can be interpreted as a section in the 2-gerbe underlying $\mathcal{CS}_{A_y^{\mathrm{gauge}}} - \mathcal{CS}_{A_y^{\mathrm{grav}}}$. The integration mechanism commutes with differentiation (Appendix A). Hence, formally,

$$\check{d}\left(\int_{E_y} B\right) = \int_{E_y} \left(\mathcal{CS}_{A_y^{\mathrm{gauge}}} - \mathcal{CS}_{A_y^{\mathrm{grav}}}\right).$$

In this light, $\Phi_Y([B])$ can be regarded as a geometrical section in the circle bundle underlying the 1-cocycle

$$\int_{E_y} \left(\mathcal{CS}_{A_y^{\mathrm{gauge}}} - \mathcal{CS}_{A_y^{\mathrm{grav}}}\right) = \left(\int_{Z/Y} \mathrm{CS}\right)\Big|_y = \omega|_y. \tag{4.7}$$

Yet, this circle bundle is just the circle $\mathcal{R}_y$ over y. Therefore, a section represents just a point in $\mathcal{R}_y$. Hence, $\Phi_Y([B])$ can be seen as being an element inside the total space of the Chern–Simons bundle (4.3).

The map Φ_Y is well defined since two gauge-equivalent B-fields integrate to gauge-equivalent 0-cochains generating the same section-point in $\mathcal{R}_Y$. One checks immediately that it realizes an equivariant isomorphism. $\square$

Remark 4.2. As seen earlier, the total space of the Chern–Simons bundle (4.3) describes the space of equivalence classes of B-fields associated to a family of metrics and connections Y. The connection ω naturally induced on this circle bundle has curvature

$$\Omega_\omega = \int_{Z/Y} \left(H_{A^{\mathrm{gauge}}} - H_{A^{\mathrm{gauge}}}\right),$$

where $H_{A^{\mathrm{gauge}}}$ and $H_{A^{\mathrm{grav}}}$ are the strength fields of the Chern–Simons cocycles $\mathcal{CS}_{A^{\mathrm{gauge}}}$ and $\mathcal{CS}_{A^{\mathrm{grav}}}$. The holonomy along a smooth loop $\gamma\colon S^1 \to Y$ can be obtained as

$$\mathrm{hol}_\omega(\gamma) = \mathrm{cs}(W, A_{\mathrm{gauge}}) \cdot \mathrm{cs}(W, A_{\mathrm{grav}})^{-1},$$

the two factors on the right side representing Chern–Simons invariants along the closed 3-manifold W swept by γ inside Z.

4.2 Chern–Simons versus Pfaffians

Next, we establish a relation between the Chern–Simons circle bundle and determinant bundles for analytic Dirac operators. We follow arguments described in [7]. There are fixed metrics and spin structures on each 2-torus E_y in the family. Therefore, one can define Dirac operators

$$\mathcal{D}_y^+\colon S_y^+ \to S_y^-. \tag{4.8}$$

There is a holomorphic interpretation for these operators. A metric on a surface induces a complex structure through its conformal class. The spin structure is then a holomorphic root $K^{1/2}$ for the canonical line K. Dirac operators become just $\overline{\partial}$-operators:

$$\overline{\partial}_{K^{1/2}} \colon \mathcal{E}^0(K^{1/2}) \to \mathcal{E}^{0,1}(K^{1/2}). \tag{4.9}$$

They build a determinant line bundle $\mathrm{Det}(\mathcal{D}_y^+) \to Y$, which comes endowed with a Quillen metric and connection [16]. Moreover, the natural pairing

$$\mathcal{E}^0(K^{1/2}) \otimes \mathcal{E}^{0,1}(K^{1/2}) \to \mathbb{C}, \quad a \otimes b \rightsquigarrow \int_E a \otimes b \tag{4.10}$$

gives an isomorphism

$$\mathcal{E}^{0,1}(K^{1/2}) \simeq \mathcal{E}^0(K^{1/2})^*.$$

The Dirac operators (4.9) can then be regarded as skew-adjoint operators

$$\overline{\partial}_{K^{1/2}} \colon \mathcal{E}^0(K^{1/2}) \to \mathcal{E}^0(K^{1/2})^*. \tag{4.11}$$

In these conditions, the corresponding determinant line bundle admits a square root. This square root is the Pfaffian complex line bundle (see [7] for details):

$$\mathrm{Pfaff}(\mathcal{D}_y^+) \to Y. \tag{4.12}$$

It comes equipped with a metric and connection, which are half the ones on the determinant line.

The framework can be further developed. One couples the Dirac operators (4.8) to the following families of connections:

- connection ∇_y^{grav} induced by A_y^{grav} in the rank n complex bundle associated to U_y through the (complexified) standard representation of $\mathrm{SO}(n)$;
- connection $\nabla_y^{\mathrm{gauge}}$ induced by A_y^{gauge} in the rank r complex hermitian bundle associated to Q_y through the representation ρ;
- connection $\nabla_y^{Z/Y}$ on $\mathrm{TE}_y \otimes_{\mathbb{Z}} \mathbb{C}$.

We obtain three families of elliptic operators: $\mathcal{D}_y^{\mathrm{grav}}$, $\mathcal{D}_y^{\mathrm{gauge}}$ and $\mathcal{D}_y^{(Z/Y)}$. Each family builds a Pfaffian line bundle. The Pfaffians can be constructed as all three complex bundles supporting the coupling connections have natural real structures.

We consider the following fermionic combination depending on two integers α and β:

$$\mathcal{L}_{\text{ferm}} = \text{Pfaff}(\mathcal{D}_y^{\text{gauge}}) \otimes (\text{Pfaff}(\mathcal{D}_y))^{\otimes\beta} \otimes \left(\text{Pfaff}(\mathcal{D}_y^{(Z/Y)})\right)^{\otimes\alpha} \otimes \left(\text{Pfaff}(\mathcal{D}_y^{\text{grav}})\right)^{\otimes(-\alpha)}. \tag{4.13}$$

The tensor product (4.13) produces a complex line bundle $\mathcal{L}_{\text{ferm}} \to Y$ and is endowed with a canonical Quillen metric g_q and compatible unitary connection ∇_q [7, 2]. Moreover, in the case of a holomorphic family Y, (4.13) inherits a holomorphic structure compatible with the connection. Restricting (4.13) to unit circles in each fiber, we obtain a circle bundle $\widetilde{\mathcal{L}}_{\text{ferm}} \to Y$, the phase Pfaffian. The Quillen connection descends to $\widetilde{\mathcal{L}}_{\text{ferm}}$.

Under certain conditions, the phase Pfaffian can be identified (up to an overall phase) to the Chern–Simons circle bundle (4.3). Explicitly, let us assume that G is a simply connected, compact Lie group and $\lambda \in H^4(BG, \mathbb{Z})$ is the integral class giving the level of the Chern–Simons cocycle. The strength field of the Chern–Simons cocycle is then a de Rham representative for the image of λ in real cohomology. The unitary representation $\rho\colon G \to U(r)$ induces a cohomology map at the level of classifying spaces:

$$B\rho^4\colon H^4(\text{BU}(r), \mathbb{Z}) \to H^4(BG, \mathbb{Z}).$$

Theorem 4.3. *If $B\rho^4(c_2) = 2\alpha \cdot \lambda$ and $\alpha(n + 22) = r + \beta$, then the Quillen connection ∇_q on the phase Pfaffian $\widetilde{\mathcal{L}}_{\text{ferm}}$ and the α-multiple of the Chern–Simons connection ω existing on the αth tensor power of* (4.3) *have the same curvature and holonomy.*

Proof. To begin with, we compute the curvature of the fermionic anomaly (4.13). The basic ingredient here is Bismut–Freed formula [7], which computes the curvature of the Quillen connection on a general Pfaffian line bundle. In our particular case, we obtain

$$\Omega^{\mathcal{L}_{\text{ferm}}} = \pi \left(\int_{Z/Y} \hat{A}(\Omega^{(Z/Y)})(\text{ch}(\Omega^{\text{gauge}}) + \beta + \alpha\text{ch}(\Omega^{(Z/Y)}) - \alpha\text{ch}(\Omega^{\text{grav}})) \right)_{(2)}. \tag{4.14}$$

Let

$$p = p_1\left(\nabla^{(Z/Y)}\right)$$

be the Chern–Weil representative for the first Pontriagin class written in terms of connection $\nabla^{(Z/Y)}$ on the vertical tangent space $T(Z/Y) \to Y$.

Then,

$$\begin{aligned}\hat{A}(\Omega^{(Z/Y)}) &= 1 - \frac{p}{24} + \cdots \\ \mathrm{ch}(\Omega^{\mathrm{gauge}}) &= r + \mathrm{ch}_1(\Omega^{\mathrm{gauge}}) + \mathrm{ch}_2(\Omega^{\mathrm{gauge}}) + \cdots \\ \mathrm{ch}(\Omega^{\mathrm{grav}}) &= n + \mathrm{ch}_1(\Omega^{\mathrm{grav}}) + \mathrm{ch}_2(\Omega^{\mathrm{grav}}) + \cdots \\ \mathrm{ch}(\Omega^{(Z/Y)}) &= 2 + p + \cdots\end{aligned}$$

Rewriting the relevant terms inside the integral (4.14), one obtains

$$\Omega^{\mathcal{L}_{\mathrm{ferm}}} = \pi \int_{Z/Y} \left(\frac{\alpha(22+n) - \beta - r}{24} p + \mathrm{ch}_2(\Omega^{\mathrm{gauge}}) - \alpha \mathrm{ch}_2(\Omega^{\mathrm{grav}}) \right).$$

Assuming $r + \beta = \alpha(22+n)$, the expression becomes

$$\Omega^{\mathcal{L}_{\mathrm{ferm}}} = \pi \int_{Z/Y} \left(\mathrm{ch}_2(\Omega^{\mathrm{gauge}}) - \alpha \mathrm{ch}_2(\Omega^{\mathrm{grav}}) \right). \tag{4.15}$$

The strength field of the gauge Chern–Simons is

$$H_{\mathrm{CS}_{A^{\mathrm{gauge}}}} = 2\pi \cdot \frac{1}{2\alpha} \cdot \mathrm{ch}_2(\Omega^{\mathrm{grav}}),$$

whereas for the gravitational Chern–Simons,

$$H_{\mathrm{CS}_{A^{\mathrm{grav}}}} = 2\pi \cdot \frac{1}{2} \cdot \mathrm{ch}_2(\Omega^{\mathrm{grav}}).$$

Combining this with (4.15), we obtain

$$\Omega^{\mathcal{L}_{\mathrm{ferm}}} = \alpha \int_{Z/Y} \left(H_{\mathrm{CS}_{A^{\mathrm{gauge}}}} - H_{\mathrm{CS}_{A^{\mathrm{grav}}}} \right).$$

□

The equality of holonomies is more delicate. It relies on a key relation between ξ-invariants and Chern–Simons invariants [1]. The facts we need here are best summarized in the following lemma ([6], Proposition 3.20).

Lemma 4.5. *Let M be a spin three-manifold, g a Riemannian metric on M and $E \to M$ a complex hermitian vector bundle with compatible connection A. We denote by $\mathcal{D}^+_{(g,A)}$ the standard Dirac operator coupled to connection A. Its ξ-invariant is defined as*

$$\xi(A) = \frac{\eta(A) + h(A)}{2},$$

where $\eta(A)$ is the spectral eta-invariant [3] *of $\mathcal{D}^+_{(g,A)}$ and $h(A)$ is the dimension of the kernel of this operator.*

Assume that for a certain positive integer N, the formal combination

$$\left[N \cdot \hat{A}(\Omega^g)\mathrm{ch}(\Omega^A)\right]_{(4)}$$

represents the Chern–Weil representative of an integral characteristic class $c \in H^4(\mathrm{BU}, \mathbb{Z})$. We denote by $\mathrm{cs}(A) \in U(1)$ *the level* c *Chern–Simons invariant associated to the connection A. Then, the unitary complex number*

$$\mathrm{e}^{2N\pi i \cdot \xi(A)} \cdot \mathrm{cs}(A)^{-1} \tag{4.16}$$

is a spin bordism invariant depending only on the class of $E \to M$ in Ω_3^{spin} (BU). In particular, if E extends over a spin 4-manifold bounding M, then (4.16) *vanishes.*

We use this technical lemma for our purposes. Let $\gamma\colon S^1 \to Y$ be a smooth loop. We denote by W the spin 3-manifold swept by γ inside Z. We pick a metric on S^1. It induces a metric on g_o on W. For each ϵ, we construct the Dirac operator associated to the scaled metric g_o/ϵ^2. Coupling this Dirac operator with the restrictions of each of the three connections existing on Z, A^{gauge}, A^{grav} and $\nabla^{(Z/Y)}$, we obtain three elliptic operators. The three ξ-invariants associated to them are obtained on the following pattern [3]:

$$\xi_\epsilon = \frac{\eta_\epsilon + h_\epsilon}{2},$$

where η_ϵ represents the eta-invariant associated to the corresponding Dirac operator and h_ϵ is the dimension of the kernel. In fact, as shown in [6], in dimension 3, the ξ-invariant is independent of the metric and therefore the index ϵ can be ignored. The fermionic holonomy along γ can be obtained through Witten's adiabatic limit formula [3], which for this particular case stands as

$$\begin{aligned} \mathrm{hol}_{\mathcal{L}_{\mathrm{ferm}}}(\gamma) &= e^{-\pi i \xi}, \\ & \quad \text{where } \xi = \xi^{\mathrm{gauge}} + \alpha\xi^{(Z/Y)} - \alpha\xi^{\mathrm{grav}}. \end{aligned} \tag{4.17}$$

Now, the curvature considerations explained earlier together with Lemma 4.5 led us to conclude that

$$\mathrm{e}^{-\pi i \xi} \cdot \mathrm{cs}(W, A_{\mathrm{gauge}})^{-\alpha} \cdot \mathrm{cs}(W, A_{\mathrm{grav}})^{\alpha} \tag{4.18}$$

is a spin bordism invariant. As any given closed spin 3-manifold can be realized as the boundary of a spin 4-manifold and all connections considered here extend, (4.18) must vanish.

Theorem 4.3 allows one to identify (up to an overall phase) the two circle bundles:

$$\begin{array}{ccc} \mathcal{R}_Y^{\otimes\alpha} & \simeq & \widetilde{\mathcal{L}}_{\text{ferm}} \\ \downarrow & & \downarrow \\ Y & = & Y \end{array} \tag{4.19}$$

Next, we are going to apply this pattern in two particular cases relevant to our discussion.

4.3 Adjoint Pfaffian

Let us return to heterotic variables on a 2-torus. For simplicity, we analyze the case $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$. Let us consider $Y = \mathcal{P}_G$ be the family of pairs of flat metrics on E and flat G-connections on the trivial G-bundle $P \to E$. The group of symmetries $\mathcal{G}$ consists of bundle automorphisms covering orientation-preserving diffeomorphisms on the base space X. As explained earlier, the moduli space of string variables $\mathcal{M}_{\text{het}}$ makes the total space of a fibration

$$\widetilde{\mathcal{M}}_{\text{het}}/\mathcal{G} \to \mathcal{P}_G/\mathcal{G}, \tag{4.20}$$

which is obtained by factoring out the action of the symmetry group $\mathcal{G}$ from a circle line bundle

$$\pi \colon \widetilde{\mathcal{M}}_{\text{het}} \to \mathcal{P}_G. \tag{4.21}$$

Moreover, the action of $\mathcal{G}$ on the base space has finite stabilizer groups. (4.20) makes a circle fibration.

We apply the earlier discussion to a particular framework, involving a family Y given by the following geometrical data:

- $Y = \mathcal{P}_G$, $Z = E \times \mathcal{P}_G$, fibration $p\colon Z \to Y$ is just projections on the first factor.
- the tangent bundle along the fibers, $T(Z/Y)$ is just $TE \times \mathcal{P}_G \to E \times \mathcal{P}_G$. It is endowed with a metric $g^{(Z/Y)}$ and a connection $\nabla^{(Z/Y)}$.
- $U \to E \times \mathcal{P}_G$ is the rank 10 real vector bundle obtained by direct summing $T(Z/Y)$ with the rank 8 trivial vector bundle over $E \times \mathcal{P}_G$. It is endowed with metric $g^{\text{grav}} = g^{(Z/Y)} \oplus g_{\text{prod}}$ and connection $A^{\text{grav}} = \nabla^{(Z/Y)} \oplus \nabla_{\text{prod}}$ (g_{prod} and ∇_{prod} are the product metric and connection, respectively, on the rank 8 trivial real bundle over $E \times \mathcal{P}_G$).
- $Q = P \times \mathcal{P}_G$. We endow it with a gauge connection A^{gauge} such that $A^{\text{gauge}}|_{P\times(A,g)} = A$.
- ρ is the adjoint representation of G. It has rank 496.

By Theorem 4.1, the line bundle (4.21) can be canonically identified to the Chern–Simons circle bundle associated to the family $Y = \mathcal{P}_G$:

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_{\text{het}} & \stackrel{\simeq}{\to} & \mathcal{R}_{\mathcal{P}_G} \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{P}_G & = & \mathcal{P}_G \end{array} \tag{4.22}$$

The right-hand side bundle comes with a Chern–Simons circle connection. We consider the fermionic Pfaffian phase combination $\widetilde{\mathcal{L}}_{\text{ad}}$ associated to this particular geometric data with $\alpha = c_{E_8} = 30$ and $\beta = 464$. Interpreting Theorem 4.3 accordingly, we obtain:

Corollary 4.6. *There exist a circle bundle isomorphism (unique up to multiplication by a unitary complex number) between the order c_{E_8} tensor power of circle bundle* (4.21) *and the adjoint phase Pfaffian $\widetilde{\mathcal{L}}_{\text{ad}}$,*

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_{\text{het}}^{\otimes c_{E_8}} & \simeq & \widetilde{\mathcal{L}}_{\text{ad}} \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{P}_G & = & \mathcal{P}_G, \end{array} \tag{4.23}$$

identifying the Chern–Simons connection to a c_G order fraction of the Quillen connection.

Proof. For a simply connected, simple, compact Lie group G, $H^4(BG,\mathbb{Z}) \simeq \mathbb{Z}$. An explicit generator θ corresponds to the integral ad-invariant quadratic form

$$q\colon g \to \mathbb{R}, \quad q(a) = \frac{1}{16\pi^2 c}\langle a, a\rangle_k,$$

where the above right-hand side pairing $< \cdot, \cdot >_k$ represents the Killing form. The number c is always integer or half-integer. For $G = SU(n)$, $c = 1/2$. For $G = E_8$, c equals the dual Coxeter number $c_{E_8} = 30$.

Here, we analyze the case $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$. The Chern–Simons level $\lambda \in H^4(BG,\mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}$ corresponds to (θ, θ), where θ is the generator in $H^4(BE_8,\mathbb{Z})$. The adjoint representation $\text{ad}\colon E_8 \to \text{SU}(248)$ induces a cohomology map:

$$\begin{aligned} B\,\text{ad}^4\colon H^4(B\,\text{SU}(248),\mathbb{Z}) \to H^4(BE_8,\mathbb{Z}), \quad &\text{with } B\,\text{ad}^4(c_2) \\ &= 2c_{E_8}\cdot\theta = 60\cdot\theta. \end{aligned}$$

The adjoint representation of G is then just $\rho = (\text{ad}, \text{ad})\colon E_8 \times E_8 \to \text{SU}(248) \times \text{SU}(248) \hookrightarrow \text{SU}(496)$ and induces

$$\begin{aligned} B\rho^4\colon H^4(B\,\text{SU}(496),\mathbb{Z}) &\longrightarrow H^4(B\ \text{SU}(248),\mathbb{Z}) \times H^4(B\,\text{SU}(248),\mathbb{Z}) \\ &\xrightarrow{B\,\text{ad}^4 \times B\,\text{ad}^4} H^4(BG,\mathbb{Z}) \end{aligned}$$

with $B\rho^4(c_2) = (2c_{E_8} \cdot \theta, 2c_{E_8} \cdot \theta) = 2c_{E_8} \cdot \lambda$. Taking then $\alpha = c_{E_8} = 30$ and $\beta = 464$, both conditions in Theorem 4.3 are satisfied. $\square$

In the light of this statement, the heterotic fibration $\widetilde{\mathcal{M}}_{\text{het}} \to \mathcal{P}_G$ can be regarded naturally as a root of order c_{E_8} for the adjoint Pfaffian phase $\widetilde{\mathcal{L}}_{\text{ad}}$. The latter can be further simplified. One can write

$$\mathcal{D}^{\text{grav}}_{(A,g)} = \mathcal{D}^{(Z/Y)}_{(A,g)} \oplus \mathcal{D}_g(\mathbb{C}^8).$$

The second operator on the right-hand side represents just the Dirac operator on E associated to the metric g coupled with the product connection on the trivial rank 8 vector bundle over E. This does not depend on connection A and is just eight times the standard Dirac operator $\mathcal{D}_g$ associated to metric g. Therefore,

$$\det(\mathcal{D}^{\text{grav}}_{(A,g)}) = \det(\mathcal{D}^{(Z/Y)}_{(A,g)}) \otimes \det(\mathcal{D}_g)^8.$$

Rewriting the fermionic Pfaffian combination (4.13), one obtains

$$\begin{aligned} \mathcal{L}_{\text{ad}} &= \operatorname{Pfaff}(\mathcal{D}^{\text{gauge}}_{(A,g)}) \otimes \operatorname{Pfaff}(\mathcal{D}_g)^{\otimes(\beta - 8\alpha)} \\ &= \operatorname{Pfaff}(\mathcal{D}^{\text{gauge}}_{(A,g)}) \otimes \operatorname{Pfaff}(\mathcal{D}_g)^{\otimes 224}. \end{aligned} \tag{4.24}$$

The $\widetilde{\mathcal{L}}_{\text{ad}}$ represents just the phase of (4.24).

The action of symmetry group $\mathcal{G}$ on $\mathcal{P}_G$ preserves the geometric data. Therefore, the group $\mathcal{G}$ acts [7, 6] on each of the Pfaffians involved in (4.24), preserving the Quillen metrics and connections. Hence, there is an action of $\mathcal{G}$ on the phase Pfaffian $\widetilde{\mathcal{L}}_{\text{ad}}$. It makes (4.23) an equivariant isomorphism. The identification can be pushed down to quotients. One obtains an isomorphism of fibrations:

$$\begin{array}{ccc} \widetilde{\mathcal{M}}^{\otimes c_{E_8}}_{\text{het}}/\mathcal{G} & \overset{\simeq}{\to} & \widetilde{\mathcal{L}}_{\text{ad}}/\mathcal{G} \\ \downarrow & & \downarrow \\ \mathcal{P}_G/\mathcal{G} & = & \mathcal{P}_G/\mathcal{G} \end{array} \tag{4.25}$$

This correspondence provides a first link relating the heterotic parameters to fermionic Pfaffian phases. However, it does not offer quite enough input about the string parameter moduli space, as one obtains information about, roughly speaking, the order c_{E_8} tensor power of the moduli space and not the moduli space itself. One way to use the same construction and to decrease the order of the power would be to choose a lower rank representation. E_8 does not have irreducible representations in rank lower than 248. However, reducing the group structure, one can get lower rank representations.

4.4 Spin Pfaffian

Let $H = \mathrm{Spin}(16) \times \mathrm{Spin}(16)$. There is a copy of $\mathrm{Spin}(16)/\mathbb{Z}_2$ sitting as a subgroup inside E_8. Taking its double cover, one obtains a Lie group morphism $j\colon \mathrm{Spin}(16) \to E_8$. It is known [14] that the induced cohomology map:

$$Bj^4\colon H^4(BE_8,\mathbb{Z}) \to H^4(B\mathrm{Spin}(16),\mathbb{Z}) \tag{4.26}$$

is an isomorphism.

We denote by $\mathcal{P}_H$ the family of pairs (A, g) of flat connections on the trivial H-bundle over E and flat metrics on E. There is a projection map

$$\sigma\colon \mathcal{P}_H \to \mathcal{P}_G, \tag{4.27}$$

assigning to each H-connection its associated G-connection through

$$j \times j\colon \mathrm{Spin}(16) \times \mathrm{Spin}(16) \to (E_8 \times E_8) \rtimes \mathbb{Z}_2.$$

We use the earlier geometric data on $\mathcal{P}_H$, taking as representation ρ the standard 32-dimensional representation available for H. This will provide a Pfaffian, as it is actually a real representation. The space of gauge classes of B-fields associated to the $\mathcal{P}_H$ family makes the top space of the Chern–Simons circle bundle

$$\mathcal{R}_H \to \mathcal{P}_H. \tag{4.28}$$

Moreover, due to (4.26), the circle bundle (4.28) is exactly the pull-back of the Chern–Simons bundle from $\mathcal{P}_G$ through projection σ:

$$\begin{array}{ccccccc}
\mathcal{R}_H & \simeq & \sigma^*\mathcal{R}_G & \to & \mathcal{R}_G & \simeq & \widetilde{\mathcal{M}}_{\mathrm{het}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{P}_H & = & \mathcal{P}_H & \xrightarrow{\sigma} & \mathcal{P}_G & = & \mathcal{P}_G
\end{array} \tag{4.29}$$

The Chern–Simons connection on $\mathcal{R}_H$ gets identified with the pull-back through σ of the Chern–Simons connection on $\mathcal{R}_G$.

Next, we build the fermionic combination of Pfaffians corresponding to the 32-dimensional representation ρ and $\alpha = 1, \beta = 0$:

$$\mathcal{L}_\rho = \mathrm{Pfaff}(\mathcal{D}^{\mathrm{gauge}}_{(A,g)}) \otimes \mathrm{Pfaff}(\mathcal{D}^{(Z/Y)}_g) \otimes \mathrm{Pfaff}(\mathcal{D}^{\mathrm{grav}}_g)^{\otimes -1}.$$

It can be simplified to

$$\mathcal{L}_\rho = \mathrm{Pfaff}(\mathcal{D}^{\mathrm{gauge}}_{(A,g)}) \otimes \mathrm{Pfaff}(\mathcal{D}_g)^{\otimes -8}. \tag{4.30}$$

Employing Theorem 4.3, one obtains:

Corollary 4.7. *There is a circle bundle isomorphism (unique up to multiplication by a unitary complex number) between the phase Pfaffian combination* (4.30) *and the pull-back to $\mathcal{P}_H$ of the raw $(E_8 \times E_8) \rtimes \mathbb{Z}_2$ heterotic bundle* (3.3)*:*

$$\begin{array}{ccccc} \widetilde{\mathcal{L}}_\rho & \simeq & \sigma^* \widetilde{\mathcal{M}}_{\text{het}} & \to & \widetilde{\mathcal{M}}_{\text{het}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{P}_H & = & \mathcal{P}_H & \stackrel{\sigma}{\to} & \mathcal{P}_G \end{array} \tag{4.31}$$

transporting the Chern–Simons connection on $\widetilde{\mathcal{M}}_{\text{het}}$ back to the Quillen connection existing on the Pfaffian phase bundle.

Proof. We verify the conditions in Theorem 4.3. Let ξ be the generator in $H^4(B\ \mathrm{Spin}(16), \mathbb{Z}) \simeq \mathbb{Z}$. If θ is the generator in $H^4(BE_8, \mathbb{Z})$, then, as mentioned earlier, under the map $j\colon \mathrm{Spin}(16) \to E_8$, we get an isomorphism

$$Bj^4 H^4(BE_8, \mathbb{Z}) \to H^4(B\,\mathrm{Spin}(16), \mathbb{Z}),$$

with $Bj^4(\theta) = \xi$. The Chern–Simons cocycle of level $\lambda = (\theta, \theta) \in H^4(BG, \mathbb{Z})$ pulls then back to the Chern–Simons theory, corresponding to level $\widetilde{\lambda} = (\xi, \xi) \in H^4(BH, \mathbb{Z})$.

We just have to check then the conditions needed in Theorem 4.3. Let $s\colon \mathrm{Spin}(16) \to \mathrm{SU}(16)$ be the complexification of the standard 16-dimensional real representation of Spin(16). It induces a cohomology map

$$Bs^4\colon H^4(B\,\mathrm{SU}(16), \mathbb{Z}) \to H^4(B\,\mathrm{Spin}(16), \mathbb{Z}),$$

with $Bs^4(c_2) = 2 \cdot \xi$. The 32-dimensional representation

$$\rho = (s, s)\colon H \to \mathrm{SU}(16) \times \mathrm{SU}(16) \hookrightarrow \mathrm{SU}(32)$$

induces then

$$\begin{aligned} B\rho^4\colon H^4(B\,\mathrm{SU}(32), \mathbb{Z}) & \to H^4(B\,\mathrm{SU}(16), \mathbb{Z}) \times H^4(B\,\mathrm{SU}(16), \mathbb{Z}) \\ & \xrightarrow{Bs^4 \times Bs^4} H^4(BH, \mathbb{Z}), \end{aligned}$$

with $B\rho^4(c_2) = (2 \cdot \xi, 2 \cdot \xi) = 2\widetilde{\lambda}$. $\square$

The symmetry group $\mathcal{G}_H$ acts on $\mathcal{P}_H$. It preserves the geometric data necessary for building the Pfaffians and determines [7, 6] an unitary action on

$$\operatorname{Pfaff}(\mathcal{D}^{\mathrm{gauge}}_{(A,g)}) \otimes \operatorname{Pfaff}(\mathcal{D}_g)^{\otimes -8}.$$

Moreover, any H-gauge transformation induces a G-gauge transformation, and therefore, we have a group morphism

$$i\colon \mathcal{G}_H \to \mathcal{G}.$$

The two actions commute, making the isomorphisms in diagram (4.31) equivariant. Then, there exist a circle fibration identification at quotient level:

$$\begin{array}{ccccc} \widetilde{\mathcal{L}}_\rho/\mathcal{G}_H & \simeq & \rho^*\left(\widetilde{\mathcal{M}}_{\mathrm{het}}/\mathcal{G}\right) & \to & \widetilde{\mathcal{M}}_{\mathrm{het}}/\mathcal{G} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{P}_H/\mathcal{G}_H & = & \mathcal{P}_H/\mathcal{G}_H & \xrightarrow{\sigma} & \mathcal{P}_G/\mathcal{G} \end{array} \tag{4.32}$$

Therefore, the heterotic fibration pulls back to recover the spin Pfaffian. Summarizing the information obtained throughout this section, we can say:

Proposition 4.8. *1) The moduli space of $G = E_8 \times E_8$ heterotic parameters, $\mathcal{M}_{\mathrm{het}}$, makes the total space of a circle fibration*

$$\mathcal{M}_{\mathrm{het}} \to \mathcal{P}_G/\mathcal{G}. \tag{4.33}$$

This is obtained by factoring out the action of the symmetry group $\mathcal{G}$ from circle bundle $\widetilde{\mathcal{M}}_{\mathrm{het}} \to \mathcal{P}_G$.

2) As an equivariant model, (4.33) *represents a root of order $c_{E_8} = 30$ for the adjoint Pfaffian phase fibration:*

$$\widetilde{\mathcal{L}}_{\mathrm{ad}}/\mathcal{G} \to \mathcal{P}_G/\mathcal{G}.$$

3) The pull-back of (4.33) *through the σ projection of* (4.27) *recovers the Pfaffian phase corresponding to the* 32*-dimensional representation of* $\mathrm{Spin}(16) \times \mathrm{Spin}(16)$.

We finish this section with a short note about the $G = \mathrm{Spin}(32)/\mathbb{Z}_2$ case. This can be handled similarly to the $E_8 \times E_8$ case. However, there are a couple of differences which we enumerate here. First of all, as mentioned earlier, the bundle P is chosen such that it carries a vector structure. As we are working over the 2-torus, P must be topologically trivializable. We choose the trivial $\mathrm{Spin}(32)$ bundle $\widetilde{P}$ as a lift for P.

The Lie group projection $\mathrm{Spin}(32) \to \mathrm{Spin}(32)/\mathbb{Z}_2$ induces an isomorphism at Lie algebra level. The Chern–Simons cocycles for $\mathrm{Spin}(32)/\mathbb{Z}_2$-connections on P are then defined (Section 2.4) using the Killing form of $\mathrm{Spin}(32)$. The symmetry group $\mathcal{G}$ is made out of liftable symmetries, namely, those automorphisms of P which can be lifted to automorphisms on $\widetilde{P}$.

On the basis of this framework, the entire $E_8 \times E_8$ discussion in this section can be adapted to work for $G = \mathrm{Spin}(32)/\mathbb{Z}_2$. The spin Pfaffian arrives via the standard 32-dimensional spin representation ρ of $H = \mathrm{Spin}(32)$. The analog of Proposition 4.8 can be formulated:

Proposition 4.9. *1) The moduli space of $G = \mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic parameters, $\mathcal{M}_{\mathrm{het}}$, makes the total space of a circle fibration*

$$\mathcal{M}_{\mathrm{het}} \to \mathcal{P}_G/\mathcal{G}. \tag{4.34}$$

This can be regarded as an equivariant model. (4.34) *is obtained by factoring out the action of the symmetry group $\mathcal{G}$ from a circle bundle $\widetilde{\mathcal{M}}_{\mathrm{het}} \to \mathcal{P}_G$.*

2) As an equivariant model, (4.34) *represents a root of order* 30 *for the adjoint Pfaffian phase fibration:*

$$\widetilde{\mathcal{L}}_{\mathrm{ad}}/\mathcal{G} \to \mathcal{P}_G/\mathcal{G}.$$

3) The pull-back of (4.34) *through $\mathcal{P}_H \to \mathcal{P}_G$ recovers the Pfaffian phase corresponding to the* 32*-dimensional representation of* Spin(32)*. Moreover, the pull-back commutes with the actions of the symmetry group $\mathcal{G}$, creating an identification of equivariant models.*

One can interpret this description of $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic parameters along the lines set by Witten [19] in his analysis of world-sheet anomaly cancellation. Our definition of B-fields as differential 2-cochains recovers the properties Witten enlists in his discussion on the nature of these fields. The $\mathrm{Spin}(32)/\mathbb{Z}_2$ analog of Pfaffian combination (4.30) represents the supergravity fermionic anomaly circle bundle, and its cancellation mechanism can be seen through ideas similar to world-sheet anomaly cancellation.

Evaluating the supergravity path integral for a fixed Riemannian surface (namely, the choice $E \subset E \times \mathbb{R}^8$), one obtains a product combination

$$\mathcal{L}_\rho^{-1}(A, g)\exp\left(i\int_E B\right). \tag{4.35}$$

Here, $\mathcal{L}_\rho^{-1}(A, g)$ represents the inverse of the Pfaffian combination (4.30), whereas the second factor is the "holonomy" of the B-field B along the 2-torus E. Both factors represent sections of non-trivial line bundles. The first one makes a section in the inverse of the line bundle $\mathcal{L}_\rho$. The second quantity, as discussed in Section 1, takes values in the total space of a circle bundle, the Chern–Simons bundle $\mathcal{R}$. Both bundles carry unitary connections. In order for the path integral combination (4.35) to be a complex number, not just a point in the total space of a tensor bundle, one needs to covariantly trivialize the tensor combination

$$\mathcal{L}_\rho^{-1} \otimes \mathcal{R}.$$

A trivialization can be achieved only if the 2-phase bundles can be made to cancel each other. In other words, if a smooth isomorphism $\widetilde{\mathcal{L}_\rho} \simeq \mathcal{R}$ identifying the two circle holonomies does exist. The third statement in Proposition 4.9 provides exactly such an isomorphism.

5 Moduli of heterotic data and the character fibration

Let G be one of the two Lie group choices $E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$. We analyze the family $\mathcal{P}_G$ up to gauge equivalence. The family is made out of pairs consisting of flat G-connections A and flat metrics g over the 2-torus E. By convention, if $G = \mathrm{Spin}(32)/\mathbb{Z}_2$, we consider only those connections that can be lifted to Spin(32). The gauge group $\mathcal{G}$ is represented by automorphisms φ of the bundle covering orientation-preserving diffeomorphisms $\bar{\varphi}$ on E (as mentioned earlier, if $G = \mathrm{Spin}(32)/\mathbb{Z}_2$, one considers only liftable automorphisms). $\mathcal{G}$ enters a short exact sequence,

$$\{1\} \to \mathcal{G}(P) \hookrightarrow \mathcal{G} \to \mathrm{Diff}^+(E) \to \{1\}$$

and acts on $\mathcal{P}_G$ as $\varphi \cdot (A, g) = (\varphi^* A, \bar{\varphi}^* g)$.

The space of flat metrics splits as

$$\mathrm{Met}_o(E) = \mathrm{Conf}(E) \times \mathbb{R}_+^*$$

for any given metric producing a conformal class and a volume. It is known that, on a 2-torus, a conformal class produces a complex structure. The group of orientation-preserving diffeomorphisms $\mathrm{Diff}^+(E)$ acts on $\mathrm{Met}_o(E)$, leaving the volume component invariant. Therefore,

$$\mathrm{Met}_o(E)/\mathrm{Diff}^+(E) = \mathrm{Conf}(E)/\mathrm{Diff}^+(E) \times \mathbb{R}_+^*.$$

Two conformal classes determine isomorphic complex structures on E if and only if they can be transformed one into the other through an orientation-preserving diffeomorphism. The orbit space

$$\mathcal{M}_E = \mathrm{Conf}(E)/\mathrm{Diff}^+(E)$$

represents then the family of isomorphism classes of elliptic curves. For practical reasons, we shall identify a complex structure on a 2-torus by a complex number τ inside the upper half-plane $\mathcal{H}$. The corresponding elliptic curve is obtained by factoring the lattice $\mathbb{Z} \oplus \tau\mathbb{Z}$ out of the complex plane $\mathbb{C}$.

Two points in the upper half-plane τ_1 and τ_2 determine isomorphic complex structures if and only if

$$\tau_2 = \frac{a\tau_1 + b}{c\tau_1 + d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

One can therefore describe the moduli space of isomorphic classes of elliptic curves as a complex orbifold:

$$\mathcal{M}_E = \mathrm{PSL}(2, \mathbb{Z}) \backslash \mathcal{H}. \tag{5.1}$$

Turning to connections, the family of gauge equivalence classes of G-connections over a 2-torus is identified to $\mathrm{Hom}(\pi_1(E), G)/G$, the group G acting on the space of homomorphisms by conjugation. A gauge class can therefore be seen as a pair of commuting elements in G, up to simultaneous conjugation. For simply connected groups, this picture can be refined [9]. Any two commuting elements in G can be simultaneously conjugated inside a maximal torus. This also holds for liftable G-connections if G is connected, but not necessarily simply connected. We make a choice of maximal torus $T \hookrightarrow G$. Let $W = W(T, G)$ be the associated Weyl group and $t_{\mathbb{R}} \subset g$ be the Cartan subalgebra. There is then a 1-to-1 correspondence:

$$\mathrm{Hom}(\pi_1(E), G)/G \simeq \mathrm{Hom}(\pi_1(E), T)/W. \tag{5.2}$$

We consider the coroot lattice $\Lambda_G \subset \mathbf{t}_{\mathbb{R}}$. The Weyl group acts on Λ_G preserving the set of coroots. There is an identification $T \simeq U(1) \otimes_{\mathbb{Z}} \Lambda_G$ commuting with the Weyl action. Under this identification,

$$\mathrm{Hom}(\pi_1(E), T) \simeq \mathrm{Hom}(\pi_1(E), U(1) \otimes_{\mathbb{Z}} \Lambda_G) \simeq \mathrm{Hom}(\pi_1(E), U(1)) \otimes_{\mathbb{Z}} \Lambda_G. \tag{5.3}$$

The set $\mathrm{Hom}(\pi_1(E), U(1))$ represents the family of gauge equivalence classes of flat hermitian line bundles. In the presence of a complex structure E_τ, this can be regarded as $\mathrm{Pic}^o(E_\tau)$. Hence, using (5.2) and (5.3), on a complex torus E_τ, the space of gauge classes of G-connections can be identified to the complex orbifold:

$$\mathrm{Pic}^o(E) \otimes_{\mathbb{Z}} \Lambda_G / W. \tag{5.4}$$

For $G = E_8 \times E_8$, this is a product of two identical 8-dimensional complex weighted projective spaces [13]. If $G = \mathrm{Spin}(32)/\mathbb{Z}_2$, then (5.4) makes a quotient of a 16-dimensional complex weighted projective space by $\mathbb{Z}_2 \times \mathbb{Z}_2$. Bringing things together, one concludes:

Theorem 5.1 ([9, 13]). *The moduli space $\mathcal{M}_{E,G}$ of equivalence classes of flat G-connections over elliptic curves can be given the structure of a*

17-dimensional complex variety. It fibers over the moduli space of elliptic curves

$$\mathcal{M}_{E,G} \to \mathcal{M}_E; \tag{5.5}$$

all fibers being 16-dimensional compact complex orbifolds. Moreover, there is a diffeomorphism of stratified spaces:

$$\mathcal{P}_G/\mathcal{G} \simeq \mathcal{M}_{E,G} \times \mathbb{R}^*_+; \tag{5.6}$$

the second factor on the right-hand side representing the volume of the flat metric on E.

Let us recapitulate now the facts of previous sections. Following Section 3, in both relevant cases $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$ and $G = \mathrm{Spin}(32)/\mathbb{Z}_2$, the moduli space of heterotic string data $\mathcal{M}^G_{\text{het}}$ is obtained by factoring out the action of the symmetry group $\mathcal{G}$ in the principal circle bundle of (3.3):

$$\widetilde{\mathcal{M}}^G_{\text{het}} \to \mathcal{P}_G. \tag{5.7}$$

As discussed in Section 4, this bundle is endowed with a Chern–Simons S^1-connection. Furthermore, the quotient space $\mathcal{P}_G/\mathcal{G}$ splits as in (5.6). Since the gravitational Chern–Simons invariant does not vary under flat conformal transformations, the bundle (5.7) can be trivialized along the volume direction, and therefore, the space $\mathcal{M}^G_{\text{het}}$ can be seen as the total space of a C^∞ fibration with fibers isomorphic to $\mathbb{R}^*_+ \times S^1 = \mathbb{C}^*$:

$$\mathcal{M}^G_{\text{het}} \to \mathcal{M}_{E,G}. \tag{5.8}$$

One can see at this point that $\mathcal{M}^G_{\text{het}}$ inherits a natural structure of analytic space. Indeed, looking at the $\mathbb{C}^*$-fibration that covers (5.8), one sees that this fibration is associated to a C^∞ fibration with complex lines over a smooth complex manifold, which is endowed with a connection and compatible hermitian metric. These features induce a natural complex structure on the total space of the line bundle, which descends to an analytic structure on $\mathcal{M}^G_{\text{het}}$. Moreover, in this setting, fibration (5.8) becomes a holomorphic (Seifert) $\mathbb{C}^*$-fibration. The analysis of this holomorphic structure of (5.8) is the central goal of this paper.

In future considerations, we shall use an abstract, coordinate oriented, working model for $\mathcal{M}_{E,G}$. Namely, let

$$V_G \stackrel{\text{def}}{=} \mathcal{H} \times (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_G)\,.$$

Every pair $(\tau, z) \in V_G$ can be seen to determine an elliptic curve E_τ, together with a flat G-connection connection on E_τ. Indeed, one can take

$$E_\tau = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$$

and then the factor z can be seen to determine an element in

$$\mathrm{Pic}^o(E_\tau) \otimes_{\mathbb{Z}} \Lambda_G/W \simeq E_\tau \otimes_{\mathbb{Z}} \Lambda_G/W, \tag{5.9}$$

which according to the previous arguments define a corresponding G-flat connection on E_τ. Let $\mathrm{pr}_1\colon V_G \to \mathbb{H}$ be the projection on the first factor. For each $\tau \in \mathcal{H}$, take

$$\mathcal{L}_\tau\colon\ = \{\tau\} \times ((\mathbb{Z} \oplus \tau\mathbb{Z}) \otimes \Lambda_G) \subset \mathrm{pr}_1^{-1}(\tau).$$

Then, $\mathcal{L}_\tau$ represents a family of 32-dimensional lattices, sitting fiber-wise inside the fibration pr_1.

Definition 5.2. Let Π_G be the group of holomorphic automorphisms of the fibration pr_1, which preserve the lattice family $\mathcal{L}$ and cover $\mathrm{PSL}(2,\mathbb{Z})$ transformations on $\mathcal{H}$.

It turns out that two elements (τ, z) and (τ', z') of $\mathcal{H} \times \Lambda_{\mathbb{C}}$ determine isomorphic pairs of elliptic curves and flat G-connections if and only if they can be transformed one into another through an isomorphism in Π_G. In this sense, the analytic space

$$\Pi_G \backslash \left(\mathcal{H} \times \Lambda_{\mathbb{C}}\right) \tag{5.10}$$

can be seen as the moduli space of pairs of elliptic curves and flat G-bundles[3] $\mathcal{M}_{E,G}$. Also, there is a canonical projection $p\colon \Pi_G \to \mathrm{PSL}(2,\mathbb{Z})$, which commutes with pr_1 inducing a fibration

$$\Pi_G \backslash V_G \to \mathrm{PSL}(2,\mathbb{Z}) \backslash \mathcal{H}.$$

This is the model for projection (5.5) in Theorem 5.1.

Let us single out the following three particular subgroups of Π_G:

- $\mathcal{S}_G = \left\{ \psi \in \Pi_G \middle| \psi(\tau,z) = \left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \right\}$
- $\mathcal{T}_G = \left\{ \psi \in \Pi_G \middle| \psi(\tau,z) = (\tau, z + q_1 + \tau q_2),\ (q_1,q_2) \in \Lambda_G \oplus \Lambda_G \right\}$
- $\mathcal{W}_G = \left\{ \psi = \mathrm{id} \oplus f \in \Pi_G \ \middle| f \in O(\Lambda) \right\}.$

[3] Again, in the case $G = \mathrm{Spin}(32)/\mathbb{Z}_2$, one considers only $\mathrm{Spin}(32)$-liftable connections.

The three subgroups $\mathcal{S}_G$, $\mathcal{T}_G$ and $\mathcal{W}_G$ generate the entire Π_G. In addition, note that $\mathcal{S}_G \cap \mathcal{W}_G = \{\pm\mathrm{id}\}$ and that $\mathrm{Ker}(p)$ is generated by $\mathcal{W}_G$ and $\mathcal{T}_G$. One concludes from these facts that Π_G is a semi-direct product:

$$\Pi_G = \mathcal{T}_G \rtimes \left(\mathcal{W}_G \times_{\{\pm\mathrm{id}\}} \mathcal{S}_G\right). \tag{5.11}$$

Now, the model (5.10) allows us to describe easily holomorphic line bundles over the moduli space of of elliptic curves and flat G-bundles, $\mathcal{M}_{E,G}$. Such bundles over a complex orbifold are best described in terms of equivariant line bundles over the cover. These equivariant bundles are line bundles $L \to V_G$, where the action of the group Π_G on the base is given a lift to the fibers. All holomorphic line bundles over V_G are trivial and a lift of the action to fibers can be obtained through a set of automorphy factors $(\varphi_a)_{a\in\Pi_G}$ with $\varphi_a \in H^0(V_G, \mathcal{O}^*_{V_G})$ satisfying

$$\varphi_{ab}(x) = \varphi_a(b\cdot x)\varphi_b(x).$$

Such a set makes a 1-cocycle φ in $Z^1(\Pi_G, H^0(V_G, \mathcal{O}^*_{V_G}))$. Two automorphy factors provide isomorphic line bundles on $\mathcal{M}_{E,G}$ if and only if they determine the same cohomology class in $H^1(\Pi_G, H^0(V_G, \mathcal{O}^*_{V_G}))$. To state this rigorously, there is a canonical map ϕ entering the following exact sequence:

$$\begin{aligned}\{1\} \to H^1(\Pi_G, H^0(V_G, \mathcal{O}^*_{V_G})) &\xrightarrow{\phi} H^1(\mathcal{M}_{E,G}, \mathcal{O}^*_{\mathcal{M}_{E,\ G}}) \\ &\to H^1(V_G, \mathcal{O}^*_{V_G}) \simeq \{1\}.\end{aligned} \tag{5.12}$$

There is an important holomorphic line bundle living on $\mathcal{M}_{E,G}$. Leaving aside more advanced interpretation, one can minimally define this bundle as the fibration with lines supporting the character function of Λ_G. Indeed, in both cases $G = E_8 \times E_8$ and $G = \mathrm{Spin}(32)/\mathbb{Z}_2$, the coroot lattice Λ_G is positive definite, unimodular and even, and therefore, one can define then an associated holomorphic theta-function (see [12, 15] for details):

$$\Theta_{\Lambda_G}\colon V_G \to \mathbb{C}, \quad \Theta_{\Lambda_G}(\tau, z) = \sum_{\gamma\in\Lambda} \mathrm{e}^{\pi i(2(z,\gamma)+\tau(\gamma,\gamma))}. \tag{5.13}$$

The pairing appearing earlier represents the bilinear complexification of the integral pairing on Λ_G. The **Λ_G-character** function can be written then as a quotient of Θ_{Λ_G}:

$$B_{\Lambda_G}\colon V_G \to \mathbb{C}, \quad B_\Lambda(\tau, z) = \frac{\Theta_{\Lambda_G}(\tau, z)}{\eta(\tau)^{16}}. \tag{5.14}$$

Here, $\eta(\tau)$ is Dedekind's eta-function:

$$\eta(\tau) = \mathrm{e}^{\pi i\tau/12} \prod_{m=1}^{\infty} \left(1 - \mathrm{e}^{2\pi i m\tau}\right),$$

which is an automorphic form of weight 1/2 and multiplier system given by a group homomorphism $\chi\colon \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Z}/24\mathbb{Z}$, in the sense that (see also [18]):

$$\eta(\gamma \cdot \tau) = \chi(\gamma)\sqrt{c\tau + d}\,\eta(\tau) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

The character terminology for (5.14) is justified by its role in the representation theory of infinite-dimensional Lie algebras. The function B_{Λ_G} represents the zero-character of the level $l = 1$ basic highest weight representation of the Kac–Moody algebra associated to G (see [12] for details).

The function B_{Λ_G} obeys the following transformation properties.

Proposition 5.3 ([12]). *Under the action of the group Π_G, the Λ_G-character function* (5.14) *transforms as*

$$B_{\Lambda_G}\left(g \cdot (\tau, z)\right) = \varphi_g^{\mathrm{ch}}(\tau, z) \cdot B_{\Lambda_G}(\tau, z), \quad g \in \Pi_G.$$

The factors φ_a^{ch} can be described on the generators of the group Π_G as:

- $\varphi_g^{\mathrm{ch}}(\tau, z) = \mathrm{e}^{\pi i(-2<q_2,z>-\tau<q_2,q_2>)}$ *for $g \in \mathcal{T}_G$ induced by $(q_1, q_2) \in \Lambda_G \oplus \Lambda_G$;*
- $\varphi_g^{\mathrm{ch}} = \mathrm{e}^{\pi ic<z,z>/c\tau+d}$ *for $g \in \mathcal{S}_G$ induced by* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$*;*
- $\varphi_g^{\mathrm{ch}} = 1$ *for $g \in \mathcal{W}_G$.*

The function B_{Λ_G} descends therefore to a holomorphic section of a holomorphic $\mathbb{C}$-fibration:

$$\mathcal{Z} \to \Pi_G \backslash \left(\mathcal{H} \times \Lambda_{\mathbb{C}}\right) = \mathcal{M}_{E,G}, \tag{5.15}$$

for which a set of automorphy factors are given by the φ_g^{ch} in Proposition 5.3.

The main result of this paper asserts

Theorem 5.4. *There exist a holomorphic isomorphism of $\mathbb{C}^*$-fibrations between* (5.8) *and the $\mathbb{C}^*$, the Λ_G-character fibration* (5.15). *The isomorphism is unique up to twisting with a non-zero complex number.*

The rest of the paper is dedicated to proving this theorem.

6 Proof of Theorem 5.4

Our strategy is as follows. We shall exploit the link described in Section 4 between the heterotic fibrations:

$$\mathcal{M}^G_{\rm het} \to \mathcal{P}_G/\mathcal{G} \tag{6.1}$$

and the Pfaffian combinations (4.24) and (4.30) in order to explicitly compute a set of automorphy factors $\varphi^{\rm het}_g, g \in \Pi_G$ for (5.8). Then, we shall compare such computed $\varphi^{\rm het}_g$ with the automorphy factors $\varphi^{\rm ch}_g$ of the Λ_G-character fibration (5.15) provided by Proposition 5.3.

6.1 Determinants of flat line bundles over elliptic curves

One can regard a pair consisting of an elliptic curve and a flat hermitian line bundle as an element $(\tau, u) \in \mathcal{H} \times \mathbb{C}$. The imaginary complex number τ determines a complex structure E_τ on the 2-torus, whereas a complex number $u = \alpha\tau + \beta \in \mathbb{C}$ determines a flat bundle of holonomy map

$$h\colon \pi_1(E) \simeq \mathbb{Z} \oplus \mathbb{Z} \to U(1), \quad h(m,n) = \mathrm{e}^{2\pi i(\alpha m + \beta n)}.$$

The projection $\mathrm{pr}_1\colon \mathbb{H} \times \mathbb{C} \to \mathbb{H}$ carries fiber-wise a continuous family of lattices $U_\tau = \mathbb{Z} \oplus \tau\mathbb{Z} \subset \mathcal{H} \times \mathbb{C}$. We denote by $\Pi_{U(1)}$ the group of automorphisms of projection pr_1, preserving the lattice family U_τ and covering $\mathrm{PSL}(2,\mathbb{Z})$ transformations on the base. Two pairs (τ_1, u_1) and (τ_2, u_2) determine isomorphic pairs of elliptic curves and flat line bundles if and only if they belong to the same orbit under the action of $\Pi_{U(1)}$. The moduli space of elliptic curves and flat hermitian line bundles can then be seen as:

$$\mathcal{M}_{E,U(1)} = \Pi_{U(1)} \backslash (\mathcal{H} \times \mathbb{C}) \,. \tag{6.2}$$

The induced projection

$$\mathcal{M}_{E,U(1)} \to \mathcal{M}_E = \mathrm{PSL}(2,\mathbb{Z}) \backslash \mathcal{H}$$

makes a holomorphic torus fibration, the fiber over $[\tau] \in \mathcal{M}_E$ being $\mathrm{Pic}^o(E_\tau)$. The symmetry group $\Pi_{U(1)}$ is generated by two particular subgroups:

- $\mathcal{S}_{U(1)} = \left\{ \psi \in \Pi_{U(1)} \middle| \psi(\tau, u) = \left(\frac{a\tau+b}{c\tau+d}, \frac{u}{c\tau+d} \right) \right\}$
  where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$. One has that $\mathcal{S}_{U(1)} \simeq \mathrm{SL}(2,\mathbb{Z})$.
- $\mathcal{T}_{U(1)} = \left\{ \psi \in \Gamma_{U(1)} \middle| \psi(\tau, u) = (\tau, u + a_1 + a_2\tau) \right\}$
  where $(a_1, a_2) \in \mathbb{Z} \oplus \mathbb{Z}$. In this case, $\mathcal{T}_{U(1)} \simeq \mathbb{Z} \oplus \mathbb{Z}$.

The group $\Pi_{U(1)}$ can then be generated as a semi-direct product $\mathcal{T}_{U(1)} \rtimes \mathcal{S}_{U(1)}$.

Consider then the odd spin structure (trivial double cover of the frame bundle) on E. In the presence of a metric on E, there is then an induced Dirac operator. Moreover, this operator can be coupled to the flat connections creating therefore a family of elliptic differential operators. Interpreting the flat line bundles as holomorphic line bundles, one can see these operators as $\overline{\partial}_L$ operators. The family determines then a determinant $\mathbb{C}$-fibration (see [7] for more details):

$$\mathrm{Det}(\mathcal{D}_L) \to \mathcal{M}_{E,U(1)} \tag{6.3}$$

carrying a natural determinant section $\det\colon \mathcal{M}_{E,U(1)} \to \mathrm{Det}(\mathcal{D}_L)$. One can use then the model (6.2) to describe a set of automorphy factors for this fibration.

Lemma 6.1. *The determinant fibration* (6.3) *can be obtained by factoring the trivial holomorphic line bundle over $\mathbb{H} \times \mathbb{C}$ by a set of automorphy factors φ_g, $g \in \Pi_{U(1)}$ with:*

1) $\varphi_g(\tau, u) = (-1)^{q_1+q_2}\, \mathrm{e}^{\pi i(-2uq_2 - \tau q_2^2)}$
for $g \in \mathcal{T}_{U(1)}$ *determined by* $(q_1, q_2) \in \mathbb{Z} \oplus \mathbb{Z}$

2) $\varphi_g(\tau, u) = \chi(m)^2\, \mathrm{e}^{\pi i(cu^2/c\tau + d)}$
for $g \in \mathcal{S}_{U(1)}$ *associated to* $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$.

Moreover, the determinant section det *is obtained in this setting as the factorization of the holomorphic function:*

$$f\colon \mathcal{H} \times \mathbb{C} \to \mathbb{C}, \quad f(\tau, u) = \frac{\vartheta_1(\tau, u)}{\eta(\tau)}.$$

Here, $\vartheta_1(\tau, u)$ *is Siegel's twisted theta-function:*

$$\vartheta_1(\tau, u) = \sum_{n \in \mathbb{Z}} \mathrm{e}^{2\pi i(u-1/2)(n+1/2) + \pi i \tau (n+1/2)^2}.$$

Proof. We follow arguments from [7]. The determinant line (6.3) can be endowed with Quillen metric and connection compatible with the holomorphic structure. The Quillen norm of the determinant section det is then computed as regularized determinant. According to [17, 7], one has

$$||\det(\tau, u)|| = \left| \mathrm{e}^{\pi i u u_2} \frac{\vartheta_1(\tau, u)}{\eta(\tau)} \right| \tag{6.4}$$

with u_2 coming from the decomposition $u = u_1 + \tau u_2$.

Let us consider the pull-back $\pi^* \operatorname{Det}(\mathcal{D}_L)$ through the covering map $\pi\colon \mathcal{H} \times \mathbb{C} \to \mathcal{M}_{E,U(1)}$. The determinant section det pulls back to a new section $\widetilde{\det}$ in $\pi^* \operatorname{Det}(\mathcal{D}_L)$. The Quillen metric and connection pull back to $\pi^* \operatorname{Det}(\mathcal{D}_L)$ as well. One computes then the curvature of the pull-back connection as

$$\Omega = \overline{\partial}\partial \log \|\widetilde{\det}\|^2 \tag{6.5}$$

Combining with (6.4), we get

$$\Omega = \overline{\partial}\partial(2\pi i u u_2) = 2\pi i \overline{\partial}\partial \left(\frac{u^2 - u\overline{u}}{\tau - \overline{\tau}} \right) \tag{6.6}$$

We choose now a holomorphic trivialization for $\pi^* \operatorname{Det}(\mathcal{D}_L)$. It is not possible to find a covariant trivialization (with respect to the pull-back Quillen connection ∇_Q), since the curvature does not vanish. However, one can deform the connection to

$$\nabla' = \nabla_Q - \mu, \quad \mu \in \Omega^1(\mathcal{H} \times \mathbb{C})$$

and remove the curvature. In performing this deformation, we keep in mind that we need a holomorphic trivialization, so we wish to keep the connection compatible with the holomorphic structure. In order to achieve this, the deformation 1-form must satisfy

$$d\mu = \Omega \quad \text{and} \quad \mu^{0,1} = 0. \tag{6.7}$$

We make the choice:

$$\mu = 2\pi i \partial \left(\frac{u^2 - u\overline{u}}{\tau - \overline{\tau}} \right), \tag{6.8}$$

which satisfies condition (6.7). The new connection ∇' is flat, and therefore, it is possible now to choose a holomorphic covariant section denoted σ. Clearly, $\widetilde{\det} = f\sigma$ with f holomorphic function. In connection with (6.4) one gets then

$$|f| = \left| \frac{\vartheta_1(\tau, z)}{\eta(\tau)} \right|.$$

As f is holomorphic, we see that, up to multiplication by an unitary complex number,

$$f = \frac{\vartheta_1(\tau, z)}{\eta(\tau)}. \tag{6.9}$$

The automorphy factors result then from the well-known transformation rules for (6.9). They are (as described in [18]):

$$f(\tau, u + q_1 + \tau q_2) = (-1)^{q_1+q_2}\, \mathrm{e}^{\pi i(-2uq_2 - \tau q_2^2)} f(\tau, u) \tag{6.10}$$

for $(q_1, q_2) \in \mathbb{Z} \oplus \mathbb{Z}$ and

$$f\left(\frac{a\tau + b}{c\tau + d}, \frac{u}{c\tau + d} \right) = \chi(m)^2\, \mathrm{e}^{\frac{\pi i c z^2}{c\tau + d}} f(\tau, z) \tag{6.11}$$

for $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$. □

There is a particular case of this construction that we need to also analyze. Suppose that we give up the coupling with flat connections and just consider the Dirac operators induced by the flat metrics of the elliptic curves. As showed in (4.11), these operators can be interpreted as skew-symmetric operators. In this setting, there exists a square root for the determinant, the Pfaffian. This $\mathbb{C}$-fibration:

$$\mathrm{Pfaff}(\overline{\partial}_\tau) \to \mathcal{M}_E = \mathrm{PSL}(2,\mathbb{Z})\backslash\mathcal{H}. \tag{6.12}$$

appeared earlier in the fermionic combinations (4.30) and (4.24). Removing the flat connections from Lemma 6.1, one obtains:

Corollary 6.2. *The holomorphic Pfaffian fibration* (6.12) *can be obtained by factoring out the trivial line over $\mathcal{H}$ by a set of automorphy factors:*

$$\varphi^o_m(\tau) = \chi(m) \quad \text{for } m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

6.2 Automorphy factors for fermionic pfaffians

We employ now the data of previous section to compute automorphy factors for the line bundles (4.30) and (4.24). Each of the two is a tensor product involving Pfaffian bundles. Automorphy factors for the simplest one are given by Corollary 6.2. In this section, we compute the automorphy factors for the other two.

The first one to look at is the adjoint Pfaffian:

$$\mathrm{Pfaff}(\mathcal{D}_{\mathrm{ad}}) \to \mathcal{M}_{E,G}. \tag{6.13}$$

This holomorphic line bundle is obtained as the Pfaffian of the family of Dirac operators coupled to flat connections on the complex vector bundle associated to the adjoint representation of G. This complex vector bundle has a natural real structure. The Pfaffian carries a canonical holomorphic section $\mathrm{Pfaff}_{\mathrm{ad}} \colon \mathcal{M}_{E,G} \to \mathrm{Pfaff}(\mathcal{D}_{\mathrm{ad}})$. Working under coordinate model

$$\mathcal{M}_{E,G} = \Pi_G \backslash V_G$$

we can describe the bundle as follows.

Lemma 6.3. *The Pfaffian line bundle* (6.13) *can be obtained by factoring out the trivial line over the universal cover V_G by means of automorphy factors φ^{ad}_g, $g \in \Pi_G$ defined on generators as*:

1) $\varphi_g^{\rm ad}(\tau,z) = {\rm e}^{c_G\pi i(-2<z,q_2>-\tau<q_2,q_2>)}$
for $g\in\mathcal{T}_G$ *associated to* $(q_1,q_2)\in\Lambda_G\oplus\Lambda_G$*;*

2) $\varphi_g^{\rm ad}(\tau,z) = \chi(m)^N{\rm e}^{c_G\pi i(c<z,z>/c\tau+d)}$
for $g\in\mathcal{S}_G$ *associated to* $m=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in{\rm SL}(2,\mathbb{Z})$*;*

3) $\varphi_g^{\rm ad}(\tau,z)=1$ *for* $g\in\mathcal{W}_G$*.*

Here, $c_G=30$ *is the Coxeter number of the group* G *(it takes the same value for both choices* $E_8\times E_8$ *and* ${\rm Spin}(32)/\mathbb{Z}_2$*) and* $N=\dim(G)=496$*.*

Proof. Let r be a root of G. One can also look at r as a particular weight $r\colon T\to U(1)$, where T is a choice of maximal torus $T\subset G$. For each flat T-connection A, one can use the weight to associate a flat hermitian line bundle or a holomorphic line bundle. This association leads to an analytic projection:

$$\sigma_r\colon \mathcal{M}_{E,T}\to\mathcal{M}_{E,U(1)}.$$

Employing coordinate models, one can see this map being factored from

$$\widetilde{\sigma}_r\colon V_G\to\mathcal{H}\oplus\mathbb{C},\quad \widetilde{\sigma}_r(\tau,z)=(\tau,<r^\vee,z>).$$

In this setting, the adjoint determinant appears as a tensor product over all roots of G (including the zero ones):

$$\widetilde{\rm Det}(\mathcal{D}_{\rm ad}) = \bigotimes_r \sigma_r^*\,{\rm Det}(\mathcal{D}_L). \tag{6.14}$$

After factoring out the action of the Weyl group, the line bundle (6.14) descends to the determinant fibration ${\rm Det}(\mathcal{D}_{\rm ad})\to\mathcal{M}_{E,G}$. Lemma 6.1 provides a set of automorphy factors φ_g^r, $g\in\Pi_G$ for each $\sigma_r^*\,{\rm Det}(\mathcal{D}_L)$. These are defined on generators of Π_G as:

1) $\varphi_g^r(\tau,z) = (-1)^{(<r^\vee,q_1>+<r^\vee,q_2>)}\,{\rm e}^{\pi i(-2<r^\vee,z><r^\vee,q_2>-\tau<r^\vee,q_2>^2)}$
for $g\in\mathcal{T}_G$ associated to $(q_1,q_2)\in\Lambda_G\oplus\Lambda_G$;

2) $\varphi_g^r(\tau,z)=\chi(m)^2\,{\rm e}^{\pi i\frac{c<r^\vee,z>^2}{c\tau+d}}$
for $g\in\mathcal{S}_G$ associated to $m=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in{\rm SL}(2,\mathbb{Z})$.

One can then construct a set of automorphy factors $\widetilde{\varphi}_g^{\rm ad}$, $g\in\Pi_G$, for the adjoint determinant by just taking the product of the above quantities over all roots of G. An useful shortcut here is provided by the formula:

$$\sum_r <r^\vee,z_1><r^\vee,z_2> = 2c_G<z_1,z_2>. \tag{6.15}$$

One obtains therefore that $\widetilde{\varphi}_g^{\rm ad}$ can be given on generators of Π_G as:

1) $\widetilde{\varphi}_g^{\mathrm{ad}}(\tau, z) = \mathrm{e}^{2c_G\pi i(-2<z,q_2>-\tau<q_2,q_2>)}$
for $g \in \mathcal{T}_G$ associated to $(q_1, q_2) \in \Lambda_G \oplus \Lambda_G$;

2) $\widetilde{\varphi}_g^{\mathrm{ad}}(\tau, z) = \chi(m)^{2N}\, \mathrm{e}^{2c_G\pi i \frac{c<z,z>^2}{c\tau+d}}$
for $g \in \mathcal{S}_G$ associated to $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$.

It is straightforward then that a set of automorphy factors φ_g^{ad} for the adjoint Pfaffian can be obtained by factoring the exponents of $\widetilde{\varphi}_g^{\mathrm{ad}}$ by 2. The Weyl action factors are constant 1 due to the Weyl invariance of the Pfaffian section. □

As opposed to the adjoint Pfaffian, the spin Pfaffian line bundle appearing in fermionic combination (4.30) corresponds to a lower rank representation. However, such representations do not exist on G, and therefore, in order to lower the rank, one has to reduce the bundle group. For the sake of clarity, let us assume that we analyze the case $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$. Set then $H = \mathrm{Spin}(16) \times \mathrm{Spin}(16)$. There exists then a Lie group homomorphism: $\alpha\colon \mathrm{Spin}(16) \to E_8$, which doubles to $\alpha \times \alpha\colon H \to G$. Any flat H-connection, induces then an associated flat G-connection and this correspondence produces an analytic morphism between the respective moduli spaces

$$\sigma\colon \mathcal{M}_{E,H} \to \mathcal{M}_{E,G}. \tag{6.16}$$

This map can be explicitly realized in coordinate models. Let $x_1, x_2, \ldots, x_8$ be the fundamental (co)weights of SO(16). The (co)roots of Spin(16) can then be seen as

$$\frac{1}{2}\left(\pm x_i \pm x_j\right), \quad 1 \le i < j \le 8,$$

and they generate the (co)root lattice $\Lambda_{\mathrm{Spin}(16)}$ of Spin(16). The (co)root lattice of E_8 makes a sublattice of $\Lambda_{\mathrm{Spin}(16)}$. The (co)roots are obtained in two series:

$\pm x_i \pm x_j$, these are from $\mathrm{Spin}(16)/\mathbb{Z}_2$ and there are 112 of them;
$\frac{1}{2}\left(\pm x_1 \pm x_2 \cdots \pm x_8\right)$, where there are an even number of minus signs.

The Lie group homomorphism $\alpha\colon \mathrm{Spin}(16) \to E_8$ induces then a morphism of coroot lattices:

$$\alpha_\Lambda\colon \Lambda_{\mathrm{Spin}(16)} \to \Lambda_{E_8}, \quad \alpha_\Lambda\left(\frac{1}{2}\left(\pm x_i \pm x_j\right)\right) = \pm x_i \pm x_j.$$

This homomorphism, in turn, extends canonically to a homomorphism

$$\alpha\colon \Lambda_H \to \Lambda_G,$$

commuting with the Weyl projection $p\colon W_H \to W_G$. One gets therefore the map

$$\mathrm{id} \oplus (\mathrm{id} \otimes \alpha) : \Pi_H \backslash \mathcal{H} \times (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_H) \to \Pi_G \backslash \mathcal{H} \times (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_G)\,, \tag{6.17}$$

which models exactly the complex variety morphism (6.16).

As mentioned earlier, unlike E_8, Spin(16) admits a representation in rank 16, the spin representation. Taking the direct sum of two such representations, one obtains a representation ρ for H in dimension 32. Coupling the Dirac operators on elliptic curves with flat connections induced by representation ρ, one obtains a ρ-pfaffian $\mathbb{C}$-fibration:

$$\mathrm{Pfaff}(\mathcal{D}_\rho) \to \mathcal{M}_{E,H}. \tag{6.18}$$

It turns out that in the coordinate model

$$\mathcal{M}_{E,H} = \Pi_H \backslash \mathcal{H} \times (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_H),$$

we have Lemma 6.4.

Lemma 6.4. *The holomorphic Pfaffian fibration* (6.18) *can be by factoring out the trivial line bundle over the universal cover* $\mathcal{H} \oplus (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_H)$ *by means of a set of automorphy factors* φ_g^ρ, $g \in \Pi_H$, *which can be given on generators of* Π_H *as*:

1) $\varphi_g^\rho(\tau, z) = \mathrm{e}^{\pi i(-2<z,q_2>-\tau<q_2,q_2>)}$
for $g \in \mathcal{T}_H$ *associated to* $(q_1, q_2) \in \Lambda_H \oplus \Lambda_H$
2) $\varphi_g^\rho(\tau, z) = \chi(m)^{16}\,\mathrm{e}^{\pi i(c<z,z>/c\tau+d)}$
for $g \in \mathcal{S}_H$ *associated to* $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$
3) $\varphi_g^\rho(\tau, z) = 1$ *for* $g \in \mathcal{W}_H$.

Proof. The representation ρ is determined by 32 weights on the maximal torus of H represented by two copies of $x_1, x_2, \ldots, x_8,\ -x_1, -x_2, \ldots, -x_8$. Each weight x determines an analytic map

$$\sigma_x\colon \mathcal{M}_{E,T} \to \mathcal{M}_{E,U(1)},$$

which can be described at the level of universal covers as

$$\mathcal{H} \oplus (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_H) \to \mathcal{H} \oplus \mathbb{C}, \quad (\tau, u \otimes \lambda) \rightsquigarrow (\tau, u \cdot x(\lambda)).$$

The pull-back of the determinant bundle on $\mathcal{M}_{E,U(1)}$ through the morphism σ_x determines a holomorphic line bundle $\sigma_x^* \,\mathrm{Det}(\mathcal{D}_L) \to \mathcal{M}_{E,T}$. Tensoring over all weights, we obtain

$$\widetilde{\mathrm{Det}}(\mathcal{D}_\rho) = \bigotimes_w \sigma_x^* \operatorname{Det}(\mathcal{D}_L).$$

This bundle and its holomorphic section are invariant under the action of the Weyl group W_H. Factoring out the Weyl action, one obtains therefore the ρ-determinant fibration $\operatorname{Det}(\mathcal{D}_\rho) \to \mathcal{M}_{E,H}$, which is the square of (6.18). Once again, Lemma 6.1 can be used to determine a set of automorphy factors φ_g^x, $g \in \Pi_H$, for each $\sigma_x^* \operatorname{Det}(\mathcal{D}_L)$. These factors can be read on generators of Π_H as

1) $\varphi_g^x(\tau, z) = (-1)^{x(q_1)+x(q_2)}\, \mathrm{e}^{\pi i(-2x(z)x(q_2)-\tau x(q_2)^2)}$
   for $g \in \mathcal{T}_H$ associated to $(q_1, q_2) \in \Lambda_H \oplus \Lambda_H$
2) $\varphi_g^x(\tau, z) = \chi(m)^2\, \mathrm{e}^{\pi i(c \cdot x(z)^2/c\tau+d)}$
   for $g \in \mathcal{S}$ associated to for $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Z})$.

Taking the product over all weights and using the fact that

$$2 \cdot \sum_{i=1}^{8} x_i(a) \cdot x_i(b) =< a, b >$$

for all $a, b \in \Lambda_H$, we obtain a set of automorphy factors $\widetilde{\varphi}_g^\rho$, $g \in \Pi_H$, for the ρ-determinant fibration. They are:

1) $\widetilde{\varphi}_g^\rho(\tau, z) = \mathrm{e}^{2\pi i(-2<z,q_2>-\tau<q_2,q_2>)}$
   for $g \in \mathcal{T}_H$ associated to $(q_1, q_2) \in \Gamma_H \oplus \Gamma_H$;
2) $\widetilde{\varphi}_g^\rho(\tau, z) = \chi(m)^{32} \mathrm{e}^{2\pi i(cu^2/c\tau+d)}$
   for $g \in \mathcal{S}_H$ associated to $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$.

The factor corresponding to the Weyl action is identically 1. The Pfaffian fibration is obtained by grouping together the weights of opposite sign. Therefore, one can obtain a set φ_g^ρ of automorphy factors for the ρ-Pfaffian fibration (6.18) by just dividing the exponents in $\widetilde{\varphi}_g^\rho$ by 2. □

There is a similar $\mathrm{Spin}(32)/\mathbb{Z}_2$ analog for Lemma 6.4. In this case, one takes H to be Spin(32) and ρ to be the spin representation of Spin(32).

6.3 Heterotic fibration = character fibration

We are now in position to finish the proof of Theorem 5.4. Recall the basic facts. The moduli space of G-heterotic data makes an 18-dimensional complex variety $\mathcal{M}_{\text{het}}^G$, which is the total space of the heterotic Chern–Simons $\mathbb{C}^*$-fibration:

$$\mathcal{M}_{\text{het}} \to \mathcal{M}_{E,G}. \tag{6.19}$$

Under the coordinate description

$$\mathcal{M}_{E,G} = \Pi_G \backslash \mathcal{H} \times (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_G),$$

the holomorphic type of fibration (6.19) is given, equivariantly, by the class of a set of automorphy factors φ_g^{het}, $g \in \Pi_G$, which describe a way to factor (6.19) from the trivial line bundle over the universal cover $\mathcal{H} \times (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_G)$ of the base.

The fermionic tensor products constructed as Pfaffian phase combinations in Section 4 were holomorphic $\mathbb{C}$-fibrations:

$$\mathcal{L}_{\text{ad}} \to \mathcal{M}_{E,G} \quad \text{and} \quad \mathcal{L}_\rho \to \mathcal{M}_{E,H},$$

obtained as in (4.24) and (4.30):

$$\mathcal{L}_{\text{ad}} = \text{Pfaff}(\mathcal{D}_{\text{ad}}) \otimes (\text{Pfaff}(\mathcal{D}_g))^{\otimes 224} \tag{6.20}$$

and

$$\mathcal{L}_\rho = \text{Pfaff}(\mathcal{D}_\rho) \otimes (\text{Pfaff}(\mathcal{D}_g))^{\otimes -8}. \tag{6.21}$$

Proposition 6.5.

- *The adjoint anomaly fibration*

  $$\mathcal{L}_{\text{ad}} \to \mathcal{M}_{E,G} = \Pi_G \backslash \mathcal{H} \times (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_G)$$

  can be described by a set of automorphy factors ϕ_g^{ad}, $g \in \Pi_G$, given on generators of Π_G as:
  1) $\phi_g^{\text{ad}}(\tau, z) = \mathrm{e}^{c_G \pi i(-2<z,q_2>-\tau<q_2,q_2>)}$
  for $g \in \mathcal{T}_G$ associated to $(q_1, q_2) \in \Lambda_G \oplus \Lambda_G$;
  2) $\phi_g^{\text{ad}}(\tau, z) = \mathrm{e}^{c_G \pi i(c<z,z>/c\tau+d)}$
  for $g \in \mathcal{S}_G$ associated to $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$;
  3) $\phi_g^{\text{ad}}(\tau, z) = 1$ for $g \in \mathcal{W}_G$.
  In both cases $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$ and $G = \mathrm{Spin}(32)/\mathbb{Z}_2$, the Coxeter number $c_G = 30$.
- *The ρ-anomaly fibration*

  $$\mathcal{L}_\rho \to \mathcal{M}_{E,H} = \Pi_H \backslash \mathcal{H} \times (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_H)$$

  can be described by a set of automorphy factors ϕ_g^ρ, $g \in \Pi_H$, given on generators of Π_H as:
  1) $\phi_g^\rho(\tau, z) = \mathrm{e}^{\pi i(-2<z,q_2>-\tau<q_2,q_2>)}$
  for $g \in \mathcal{T}_H$ associated to $(q_1, q_2) \in \Lambda_H \oplus \Lambda_H$;

2) $\phi^\rho_g(\tau, z) = \mathrm{e}^{\pi i (c<z,z>/c\tau+d)}$

for $g \in \mathcal{S}_H$ *associated to* $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$*;*

3) $\phi^\rho_g(\tau, z) = 1$ *for* $g \in \mathcal{W}_H$*.*

Proof. This is a straightforward computation based on Lemmas (6.3) and (6.4) and Corollary (6.2). □

In this setting, after checking the earlier formulas upon the character bundle automorphy factors of Proposition 5.3 one concludes:

Remark 6.6. Under the analytic map $\sigma \colon \mathcal{M}_{E,H} \to \mathcal{M}_{E,G}$, the two fermionic fibrations (6.20) and (6.21) are related to the Λ_G-character fibration $\mathcal{Z} \to \mathcal{M}_{E,G}$ of (5.15) as follows:

$$\mathcal{L}_\rho \simeq \sigma^* \mathcal{Z}, \quad \mathcal{L}_{\mathrm{ad}} \simeq \mathcal{Z}^{\otimes c_G}. \tag{6.22}$$

We exploit this remark in order to extract information about the automorphy factors φ^{het}_g of (6.19). The considerations in Section 4 allow us to affirm the following two holomorphic fibration isomorphisms: $\mathcal{L}_\rho \simeq \sigma^* \mathcal{L}_{\mathrm{het}}$ and $\mathcal{L}_{\mathrm{ad}} \simeq \mathcal{L}^{\otimes c_G}_{\mathrm{het}}$.

These statements can be subsequently translated into the following equivalences of automorphy factors:

1) $\left[\left(\varphi^{\mathrm{het}}_a \right)_{a \in \Pi_G} \right]^{c_G} \sim \left(\phi^{\mathrm{ad}}_a \right)_{a \in \Pi_G}$,

2) $\left(\varphi^{\mathrm{het}}_{p(b)} \right)_{b \in \Pi_H} \sim \left(\phi^\rho_b \right)_{b \in \Pi_H}$.

The equivalence relation "$\sim$" asserts the factors on the two sides realize the same group cohomology class in

$$H^1(\Pi_G, H^0(\mathcal{O}^*_{\mathcal{H} \times (\mathbb{C} \otimes \Lambda_G)}))$$

in the first assertion or

$$H^1(\Pi_H, H^0(\mathcal{O}^*_{\mathcal{H} \times (\mathbb{C} \otimes \Lambda_H)})$$

in the second assertion or, in geometrical terms, the two sets of automorphy factors determine isomorphic equivariant line bundles on the universal cover of the base space. As defined earlier, c_G represents the Coxeter number of G. For both $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$ and $G = \mathrm{Spin}(32)/\mathbb{Z}_2$, we have $c_G = 30$.

The two equivalence relations mentioned earlier and Remark 6.6 allow us to conclude that the two sets of automorphy factors φ_a^{het} and φ_a^{ch} are related:

$$\varphi_a^{\mathrm{het}} \sim f_a \cdot \varphi_a^{\mathrm{ch}}, \tag{6.23}$$

where $f_a \in H^1(\Pi_G, H^0(\mathcal{O}^*_{\mathcal{H}\times(\mathbb{C}\otimes\Lambda_G)}))$ and satisfies

1) $(f_a)^{30} = 1$ for any $a \in \Pi_G$.
2) $f_{p(b)} = 1$ for any $b \in \Pi_H$.

Since all f_a are holomorphic functions, the first condition above implies that they are all constant functions. Therefore, the set f_a can be reduced to a group homomorphism $f\colon \Pi_G \to \mathbb{Z}_{30}$ with $p(\Pi_H) \subset \mathrm{Ker}(f)$.

We note immediately that $f_a = 0$ for $a \in \mathcal{S}_G$ as $\mathcal{S}_G = p(\mathcal{S}_H)$. Let us next analyze the restriction of f on $\mathcal{W}_G$. In the case $G = \mathrm{Spin}(32)/\mathbb{Z}_2$ and $H = \mathrm{Spin}(32)$ one has $p(\mathcal{W}_H) = \mathcal{W}_G$ and therefore, f vanishes on $\mathcal{W}_G$. For the other case, $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$, $H = \mathrm{Spin}(16) \times \mathrm{Spin}(16)$, and the image $p(\mathcal{W}_H)$ is a subgroup of index $135 = 3^3 \cdot 5$ in $\mathcal{W}_G$. This can be deduced from the orders of the Weyl groups of Spin(16) and E_8, whose explicit computations one can find, for example, in [22]. Therefore, the image of the character f restricted to $\mathcal{W}_G$ is either $\{0\}$ or a subgroup of $\mathbb{Z}_{30}$ of odd order. But $\mathcal{W}_G$ is generated by elements of order 2. Hence $\mathrm{Im}(f) = \{0\}$. We conclude then that, in both cases, the character f vanishes on the Weyl transformations in $\mathcal{W}_G$. Let us then continue with an analysis of the behavior of f on $\mathcal{T}_G$. This group is acted upon by $\mathcal{W}_G$ and, under the identification, $\mathcal{T}_G \simeq \Lambda_G \times \Lambda_G$, this action is the Weyl action. Since the Weyl action is transitive on the roots, we conclude that the homomorphism f takes the same value on all roots in $\Lambda_G \times \Lambda_G$. However, some of the roots are coming from the roots of H and therefore all roots must belong to $\mathrm{Ker}(f)$. This shows that $f_a = 0$ for any $a \in \mathcal{T}_G$ and since we have already proved that f vanishes on $\mathcal{W}_G$ and $\mathcal{S}_G$, conclude that f is identically zero on Π_G.

By (6.23), the automorphy factors φ_a^{het} and φ_a^{ch} are equivalent and hence the heterotic $\mathbb{C}^*$-fibration (6.19) is holomorphically isomorphic to the Λ_G-character fibration $\mathcal{Z} \to \mathcal{M}_{E,G}$ of (5.15). This completes the proof of Theorem 5.4.

Appendix A. The push-forward map

In this appendix we clarify the push-forward mechanism for differential cocycles. This feature has been used in Section 4 in order to construct the Chern–Simons bundle (4.3) associated to a family of heterotic parameters.

As mentioned in [5], differential cohomology classes can be integrated. In other words, there exists a push-forward homomorphism of the following type:

$$\int_E : \check{H}^n(X \times E) \to \check{H}^{n-d}(X) \tag{A.1}$$

where X and E are smooth closed manifolds and $\dim_R E = d$. Moreover, there are (non-canonical) extensions of this cohomology map to push-forward morphisms for non-flat differential cochains:

$$\int_E : N\check{C}^n(X \times E) \to N\check{C}^{n-d}(X). \tag{A.2}$$

Such kind of map integrates non-flat n-cochains from $X \times E$ down to non-flat $(n-d)$-cochains on X. And carries certain properties as diffeomorphism invariance, sensitivity to orientation, Stokes' theorem and gluing law. The goal of this appendix is to describe in detail the construction of (A.2).

Recall that there is an integration map for differential forms:

$$\int_E : \Lambda^n(X \times E) \to \Lambda^{n-d}(X) \tag{A.3}$$

which gives rise to a morphism of the de Rham complexes:

$$\int_E : \Lambda^\bullet(X \times E) \to \Lambda^{\bullet - d}(X). \tag{A.4}$$

The induced homomorphism

$$\int_E : H^n_{dR}(X \times E) \to H^{n-d}_{dR}(X) \tag{A.5}$$

represents the cap product with the fundamental homology class $[E] \in H^d(E)$. We would like to reproduce the same pattern for differential cohomology.

The push-forward mechanism, in the framework we are going to use, is in some sense a generalization of the higher gerbe connection holonomy, which a non-flat n-cocycle defines along a closed embedded n-manifold. This has been discussed briefly in Section 2.2. namely, if one assumes that X reduces to just a point and $d = n$ then the particular map :

$$\int_E : N\check{Z}^n(E) \to N\check{Z}^0(\{\text{point}\}) \simeq \mathbf{R} \tag{A.6}$$

returns a real number whose exponential gives precisely the holonomy of the n-cocycle along E. Accordingly, we construct the push-forward mechanism along the lines holonomy was defined in Section 2.2.

Let X and E be closed smooth manifolds and the real dimension of E is d. Assume that X and E are endowed with (contractible) open coverings:

$$X = \bigcup_{a\in\mathcal{A}} U_a, \quad E = \bigcup_{b\in\mathcal{B}} U_b$$

and that the covering of E is admissible, in the sense that it admits a subordinated dual cell decomposition. This creates a product covering of $X \times E$:

$$X \times E = \bigcup_{(ab)\in\mathcal{A}\times\mathcal{B}} U_{(ab)} \quad \text{with} U_{(ab)} = U_a \times U_b.$$

As with holonomy, we make a choice of dual cell decomposition for E, $(\Delta_i)_{i\in I}$, Δ_i representing the top d-cells. We keep the same orientation convention. $\Delta_{(i_1>i_2\cdots>i_k)}$ is the $(d+1-k)$-face obtained by intersecting $\Delta_{i_1}, \Delta_{i_2}, \ldots, \Delta_{i_k}$ if such a thing does exist. The orientation we consider on $\Delta_{(i_1>i_2>\cdots>i_k)}$ is the one obtained when regarding $\Delta_{(i_1>i_2>\cdots>i_k)}$ as a boundary component in $\Delta_{(i_1>i_2>\cdots>i_{k-1})}$. As before, suppose the decomposition is subordinated to the covering $(U_b)_{b\in\mathcal{B}}$ through a subordination map

$$\rho\colon I \to \mathcal{B}, \Delta_i \subset U_{\rho(i)}. \tag{A.7}$$

We now describe the push-forward map. Let ω be a non-flat n-cochain on $X \times E$. We assume that the product covering $U_{(ab)} = U_a \times U_b$ is chosen to be sufficiently small such that ω can be represented in this covering as a multiplet:

$$\begin{aligned}\omega = (H, \omega^n_{(a_1b_1)}, \omega^{n-1}_{(a_1b_1)(a_2b_2)}, \omega^{n-2}_{(a_1b_1)(a_2b_2)(a_3b_3)}, \ldots,\\ \omega^{-1}_{(a_1b_1)(a_2b_2)\cdots(a_{n+2}b_{n+2})}).\end{aligned} \tag{A.8}$$

The integration map associates then to each ω a non-flat $(n-d)$-cochain:

$$A = \int_E \omega \in N\check{C}^{n-d}(X).$$

and such an object can be represented in the covering $(U_a)_{a\in\mathcal{A}}$ as a multiplet:

$$A = (T, A^{n-d}_{a_1}, A^{n-d-1}_{a_1a_2}, \ldots, A^{-1}_{a_1a_2\cdots a_{n-d+2}}). \tag{A.9}$$

Definition A.1. The components T and $A_{(a)}$ are defined as:

$$T = \int_E H \tag{A.10}$$

and for higher indices $(a) = (a_1 a_2 \cdots a_r)$,

$$A_{(a)}^{n-d+1-r} = \sum_{k=1}^{d+1} (-1)^{(n-d+1)(k+1)} \times \sum_{(i)=(i_1>i_2>\cdots>i_k)} \int_{\Delta_{(i)}} T_{\rho((i))}^{(a)}\omega \quad \text{if } 1 \leq r \leq n+1-d \tag{A.11}$$

and

$$A_{(a)}^{-1} = (-1)^{(n-d+1)d} \sum_{(i)=(i_1>i_2>\cdots>i_{d+1})} \int_{\Delta_{(i)}} T_{\rho((i))}^{(a)}\omega \tag{A.12}$$

for $r = n+2-d$. For a multi-index $(i) = (i_1 > i_2 > \cdots > i_k)$ in the second sum, $\rho((i))$ is defined as $(\rho(i_1)\rho(i_2)\cdots\rho(i_k))$. The symbols:

$$T_{(b_1 b_2 \cdots b_k)}^{(a_1 a_2 \cdots a_r)}\omega \tag{A.13}$$

represent local $(n+2-r-k)$-forms (local constant functions with values in $2\pi\mathbb{Z}$ if $n+2-r-k=-1$) living on

$$U_{a_1} \cap U_{a_2} \cap \cdots \cap U_{a_r}) \times (U_{b_1} \cap U_{b_2} \cap \cdots \cap U_{b_k}).$$

The exact formulation of symbols (A.13) is given by the next definition. By convention, in expressions (A.11) and (A.12), integration over points ($\Delta_{(i)}$ with $|(i)| = d+1$) means just evaluating the respective forms on $\Delta_{(i)}$.

Definition A.2. Let the two multi-indices be:

$$(a) = (a_1, a_2, \cdots, a_r) \quad \text{and} \quad (b) = (b_1, b_2, \ldots, b_k).$$

We set

$$T_{(b)}^{(a)}\omega = \sum_{\gamma\in\mathcal{D}} (-1)^{A(d)} \ \omega_d^{n+2-k-r} \tag{A.14}$$

where $\mathcal{D}$ stands for the set of paths in the rectangular network (a_p, b_q) generated by $(a) = (a_1, a_2, \cdots, a_r)$ and $(b) = (b_1, b_2, \cdots, b_k)$ joining (a_1, b_1) to (a_r, b_k) and moving only to the right or upward. $A(\gamma)$ stands for the area of the domain bounded by the path γ and the B-axis. For

$$\gamma = \big((a_{p_1} b_{q_1}), (a_{p_2} b_{q_2}), \ldots, (a_{p_{r+k-1}} b_{q_{t+r-1}})\big),$$

we define

$$\omega_\gamma^{n+2-k-r} = \omega^{n+2-r-k}_{(a_{p_1} b_{q_1})(a_{p_2} b_{q_2})\cdots(a_{p_{r+k-1}} b_{q_{k+r-1}})}. \tag{A.15}$$

We make the remark here that in (A.15), if the number of nodes in the path γ is $n+2$ then $\omega_{\gamma}^{-1} \in 2\pi\mathbb{Z}$. Relation (A.12) let us then conclude that $A_{(a)}^{-1} \in 2\pi\mathbb{Z}$ for $|(a)| = n - d + 2$, and therefore, the multiplet A described in Definition A.1 is indeed a differential cochain. One obtains therefore a push-forward map

$$\int_E : N\check{C}^n(X \times E) \to N\check{C}^{n-d}(X). \tag{A.16}$$

It satisfies the following propriety:

Theorem A.3. *For every non-flat differential cocycle $\omega \in N\check{Z}^N(X \times E)$, the push-forward*:

$$\int_E \omega$$

is a $(n-d)$-dimensional non-flat differential cocycle on X.

Moreover, Theorem A.3 is just a particular case of a Stokes-like integration argument.

Theorem A.4. *The push-forward map defined above for differential cochains commutes with the differentiation operator $\check{d}$. Namely,*

$$\int_E \check{d}\omega = \check{d}\left(\int_E \omega\right) \tag{A.17}$$

Based on above statement, one concludes that the integration mechanism constructed so far sends differential cocycles to differential cocycles and coboundaries of flat differential cochains to coboundaries of flat cochains. This induces therefore a push-forward cohomology map:

$$\int_E : \check{H}^n(X \times E) \to \check{H}^{n-d}(X). \tag{A.18}$$

Proof of Theorem A.4. Theorem A.3 will follow as a corollary. Assume that the input ω is a non-flat differential n-cochain on a product open covering of $X \times E$ and $\eta = \check{d}\omega \in N\check{C}^{n+1}(X \times E)$. We are going to show that the push-forward cochains:

$$A = \int_E \omega = (T, A^{n-d}_{a_1}, A^{n-d-1}_{a_1 a_2}, \ldots, A^{-1}_{a_1 a_2 \cdots a_{n-d+2}}) \tag{A.19}$$

and

$$B = \int_E \eta = (Q, B^{n-d}_{a_1}, B^{n-d-1}_{a_1 a_2}, \ldots, B^{-1}_{a_1 a_2 \cdots a_{n-d+2}}) \tag{A.20}$$

satisfy $\check{d}A = B$. That means:

$$dT = Q, \;\; T_{|U_{a_1}} - dA_{a_1} = B_{a_1} \quad \text{and} \quad (\delta A)_{(a)} + (-1)^{|(a)|}\; dA_{(a)} = B_{(a)}.$$

The first identity can be quickly proved:

$$dT = d\left(\int_E H\right) = \int_E d_x H = \int_E dH \; - \int_E d_y H = \int_E dH = Q.$$

Here d_x and d_y are derivatives along X, respectively, Y directions.

Let us prove that $T_{|U_{a_1}} - dA_{a_1} = B_{a_1}$. We have

$$\begin{aligned}
T - dA_{a_1} &= \int_E H - dA_{a_1} \\
&= \int_E H - \left(\sum_{k=1}^{d+1} (-1)^{(n-d+1)(k+1)} \sum_{(i)=(i_1>i_2>\cdots>i_k)} \int_{\Delta_{(i)}} d_x T^{a_1}_{(\rho(i_1)\rho(i_2)\cdots\rho(i_k))} \omega \right) \\
&= \sum_{i_1} \int_{\Delta_{i_1}} H - \sum_{k=1}^{d+1} (-1)^{(n-d+1)(k+1)} \\
&\quad \times \sum_{(i)=(i_1>i_2>\cdots>i_k)} \int_{\Delta_{(i)}} (dT^{a_1}_{(\rho(i_1)\rho(i_2)\cdots\rho(i_k))}\omega - d_y T^{a_1}_{(\rho(i_1)\rho(i_2)\cdots\rho(i_k))}\omega).
\end{aligned}$$

But Stockes' theorem provides:

$$\begin{aligned}
&\sum_{(i)=(i_1>i_2>\cdots>i_{k-1})} \int_{\Delta_{(i)}} d_y T^{a_1}_{(\rho(i_1)\rho(i_2)\cdots\rho(i_{k-1}))}\omega \\
&\quad = \sum_{(i)=(i_1>i_2>\cdots>i_k)} (-1)^{n-d+1+k+1} \int_{\Delta_{(i)}} [\delta_b T^{a_1}_{(.)}\omega]_{(\rho(i_1)\rho(i_2))\cdots\rho(i_k))}.
\end{aligned}$$

Here δ_b means Cech differentiation with respect to the second indices. Continuing along these lines one obtains that $T - dA_{a_1}$ equals:

$$\begin{aligned}
&\sum_{i_1} \int_{\Delta_{i_1}} (H - dT^{a_1}_{\rho(i_1)}\omega) \; - \sum_{k=2}^{d+1} (-1)^{(n-d+1)(k+1)} \\
&\sum_{(i)=(i_1>i_2>\cdots>i_k)} \left(\int_{\Delta_{(i)}} (dT^{a_1}_{(\rho(i_1)\rho(i_2)\cdots\rho(i_k))} + (-1)^k [\delta_b T^{a_1}_{(.)}\omega]_{(\rho(i_1)\rho(i_2))\cdots\rho(i_k))}) \right).
\end{aligned}$$

However, $\check{d}\omega = \eta$ and therefore:

$$H - dT^{a_1}_{\rho(i_1)}\omega = H - d\omega^n_{(a_1\rho(i_1))} = \eta^{n+1}_{(a_1\rho(i_1))} = T^{a_1}_{\rho(i_1)}\eta$$

and

$$\begin{aligned} dT^{a_1}_{(\rho(i_1)\rho(i_2)\cdots\rho(i_k))}\omega &+ (-1)^k[\delta_b T^{a_1}_{(.)}\omega]_{(\rho(i_1)\rho(i_2))\cdots\rho(i_k))} \\ &= d\omega^{n+1-k}_{(a\rho(i_1))(a\rho(i_2))\cdots(a\rho(i_k))} + (-1)^k[\delta\omega^{n+1-k}]_{(a\rho(i_1))(a\rho(i_2))\cdots(a\rho(i_k))} \\ &= (-1)^k\eta^{n+2-k}_{(a\rho(i_1))(a\rho(i_2))\cdots(a\rho(i_k))} = (-1)^k T^{a_1}_{(\rho(i_1)\rho(i_2)\cdots\rho(i_k))}\eta. \end{aligned}$$

Hence:

$$\begin{aligned} T - dA_{a_1} &= \sum_{i_1}\int_{\Delta_{i_1}} T^{a_1}_{\rho(i_1)}\eta + \sum_{k=2}^{d+1}(-1)^{(n-d+1)(k+1)} \\ &\quad\times \sum_{(i)=(i_1>i_2>\cdots>i_k)}\int_{\Delta_{(i)}}(-1)^k T^{a_1}_{(\rho(i_1)\rho(i_2)\cdots\rho(i_k))}\eta \\ &= \sum_{k=1}^{d+1}(-1)^{(n-d+2)(k+1)} \sum_{(i)=(i_1>i_2>\cdots>i_k)} \\ &\quad\times \int_{\Delta_{(i)}} T^{a_1}_{(\rho(i_1)\rho(i_2)\cdots\rho(i_k))}\eta \ = B_{a_1}. \end{aligned}$$

We are now in position to prove the cocycle condition for higher indices. Namely:

$$[\delta A]_{(a)} + (-1)^{|(a)|}dA_{(a)} = B_{(a)} \quad \text{for}(a) = (a_1a_2\cdots a_r), |(a)| = r \le n-d+2.$$

We have:

$$dA_{(a)} = \sum_{k=1}^{d+1}(-1)^{(n-d+1)(k+1)} \sum_{(i)=(i_1>i_2>\cdots>i_k)}\int_{\Delta_{(i)}} d_x T^{(a)}_{\rho((i))}\omega$$

and

$$(\delta A)_{(a)} = \sum_{k=1}^{d+1}(-1)^{(n-d+1)(k+1)} \sum_{(i)=(i_1>i_2>\cdots>i_k)}\int_{\Delta_{(i)}} [\delta_a T^{(.)}_{\rho((i))}\omega]^{(a)}.$$

Taking into account

$$d_x T^{(a)}_{\rho((i))}\omega = dT^{(a)}_{\rho((i))}\omega - \ d_y T^{(a)}_{\rho((i))}\omega$$

we write:

$$
\begin{aligned}
&[\delta A]_{(a)} + (-1)^r dA_{(a)} \\
&\quad = \sum_{k=1}^{d+1} (-1)^{(n-d+1)(k+1)} \sum_{(i)=(i_1>i_2>\cdots>i_k)} \\
&\qquad \times \int_{\Delta_{(i)}} \left([\delta_a T^{(.)}_{\rho((i))}\omega]^{(a)} + (-1)^r d_x T^{(a)}_{\rho((i))}\omega \right) \\
&\quad = \sum_{k=1}^{d+1} (-1)^{(n-d+1)(k+1)} \sum_{(i)=(i_1>i_2>\cdots>i_k)} \\
&\qquad \times \int_{\Delta_{(i)}} \left([\delta_a T^{(.)}_{\rho((i))}\omega]^{(a)} + (-1)^r dT^{(a)}_{\rho((i))}\omega - (-1)^r d_y T^{(a)}_{\rho((i))}\omega \right).
\end{aligned}
$$

But

$$
\begin{aligned}
&\sum_{(i)=(i_1>i_2>\cdots>i_{k-1})} \int_{\Delta_{(i)}} d_y T^{(a)}_{\rho((i))}\omega \\
&\quad = (-1)^{n-d+r+k+1} \sum_{(i)=(i_1>i_2>\cdots>i_k)} \int_{\Delta_{(i)}} [\delta_b T^{(a)}_{(.)}\omega]_{\rho((i))}.
\end{aligned}
$$

Therefore we continue:

$$
\begin{aligned}
&[\delta A]_{(a)} + (-1)^r dA_{(a)} \\
&\quad = \sum_{i_1} \int_{\Delta_{i_1}} \left([\delta_a T^{(.)}_{\rho(i_1)}\omega]^{(a)} + (-1)^r dT^{(a)}_{\rho(i_1)}\omega \right) \\
&\qquad + \sum_{k=2}^{d+1} (-1)^{(n-d+1)(k+1)} \sum_{(i)=(i_1>i_2>\cdots>i_k)} \\
&\qquad \times \int_{\Delta_{(i)}} \left([\delta_a T^{(.)}_{\rho((i))}\omega]^{(a)} + (-1)^r dT^{(a)}_{\rho((i))}\omega + (-1)^{k+1} [\delta_b T^{(a)}_{(.)}\omega]_{\rho((i))} \right).
\end{aligned} \tag{A.21}
$$

We claim now that the right part of identity (A.21) is exactly $B_{(a)}$. To clarify this assertion we need the following lemma:

Lemma A.5. *Let $\omega \in N\check{C}^n(X \times E)$ be a non-flat differential n-cocycle and $\check{d}\omega = \eta \in N\check{C}^{n+1}(X \times E)$. We consider the symbols $T^{(a)}_{(b)}\omega$ and $T^{(a)}_{(b)}\eta$ introduced by Definition* A.2 *. They satisfy the following relation:*

$$
T^{(a)}_{(b)}\eta = (-1)^{|(a)|+|(b)|+1} dT^{(a)}_{(b)}\omega + (-1)^{|(b)|+1}\ [\delta_a T^{(.)}_{(b)}\omega]^{(a)} + [\delta_b T^{(a)}_{(.)}\omega]_{(b)}. \tag{A.22}
$$

If $|(a)| = 1$ or $|(b)| = 1$ the second or the third term in (A.22) *disappears.*

Before giving a proof for Lemma A.5 , let us notice that this this ends the proof of Theorem A.4 . Indeed, one can see that the terms appearing in summation (A.21) can be immediately rewritten as:

$$[\delta_a T^{(.)}_{\rho(i_1)}\omega]^{(a)} + (-1)^r dT^{(a)}_{\rho(i_1)}\omega = T^{(a)}_{\rho(i_1)}\eta \tag{A.23}$$

and

$$\begin{aligned}&[\delta_a T^{(.)}_{\rho((i))}\omega]^{(a)} + (-1)^r dT^{(a)}_{\rho((i))}\omega \\ &\quad + (-1)^{k+1}[\delta_b T^{(a)}_{(.)}\omega]_{\rho((i))} = (-1)^{k+1} T^{(a)}_{\rho((i))}\eta.\end{aligned} \tag{A.24}$$

Therefore

$$\begin{aligned}&[\delta A]_{(a)} + (-1)^r dA_{(a)} \\ &\quad = \sum_{i_1}\int_{\Delta_{i_1}} T^{(a)}_{\rho(i_1)}\eta \\ &\quad\quad + \sum_{k=2}^{d+1}(-1)^{(n-d+1)(k+1)}\sum_{(i)=(i_1>i_2>\cdots>i_k)}\int_{\Delta_{(i)}}(-1)^{k+1}T^{(a)}_{\rho((i))}\eta \\ &\quad = \sum_{k=1}^{d+1}(-1)^{(n-d+2)(k+1)}\sum_{(i)=(i_1>i_2>\cdots>i_k)}\int_{\Delta_{(i)}} T^{(a)}_{\rho((i))}\eta = B_{(a)}.\end{aligned}$$

This finishes the proof. □

We still have to provide a proof for Lemma A.5.

Proof of Lemma A.5. Let us recall the definition of the symbols $T^{(a)}_{(b)}$. For $(a) = (a_1, a_2, \cdots, a_r)$ and $(b) = (b_1, b_2, \cdots, b_k)$, these objects are expressed as.

$$T^{(a)}_{(b)}\omega = \sum_{\gamma\in\mathcal{D}}(-1)^{A(\gamma)}\omega_\gamma^{n+2-k-r} \tag{A.25}$$

where $\mathcal{D}$ is the set of paths

$$\gamma = ((a_{p_1}b_{q_1}), (a_{p_2}b_{q_2}), \cdots, (a_{p_{r+k-1}}b_{q_{r+k-1}}))$$

in the rectangular network

$$\{(a_ib_j) | i \in \{1\cdots r\}, j \in \{1\cdots k\}\}$$

starting at (a_1b_1), ending up at (a_rb_k) and moving only upward or to the right. $A(\gamma)$ represents the area of the domain bounded by the path γ and the b-axis. With this in mind:

$$\begin{aligned}
T^{(a)}_{(b)}\eta &= \sum_{\gamma\in\mathcal{D}}(-1)^{A(d)}\eta^{n+2-k-r}_{\gamma} \\
&= \sum_{\gamma\in\mathcal{D}}(-1)^{A(\gamma)}\left((\delta\omega^{n+2-k-r})_{\gamma}+(-1)^{r+k-1}d\omega^{n+1-k-r}_{\gamma}\right) \\
&= \sum_{\gamma\in\mathcal{D}}(-1)^{A(\gamma)}(\delta\omega^{n+2-k-r})_{\gamma} \\
&\quad + (-1)^{r+k-1}\left(\sum_{\gamma\in\mathcal{D}}(-1)^{A(\gamma)}\omega^{n+1-k-r}_{\gamma}\right).
\end{aligned}$$

The second term on the left side of above is exactly

$$(-1)^{|(a)|+|(b)|+1}dT^{(a)}_{(b)}\omega,$$

and therefore, in order to complete the proof of the lemma we just have to show

$$\begin{aligned}
&\sum_{\gamma\in\mathcal{D}}(-1)^{A(\gamma)}(\delta\omega^{n+2-k-r})_{\gamma} \\
&\qquad = (-1)^{|(b)|+1}[\delta_a T^{(.)}_{(b)}\omega]^{(a)} + [\delta_b T^{(a)}_{(.)}\omega]_{(b)}.
\end{aligned} \tag{A.26}$$

One can prove this by observing that both sides of (A.26) are made of terms of type:

$$\pm\omega^{n+3-r-k}_{(a_{p_1}b_{q_1})(a_{p_2}b_{q_2})\cdots(a_{p_{k+r-2}}q_{i_{k+r-2}})}.$$

We have to show that, after cancellations, the same terms appear in both sides, with the same signatures. Let

$$\gamma = [(a_{p_1}b_{q_1})(a_{p_2}b_{q_2})\cdots(a_{p_{k+r-1}}b_{q_{k+r-1}})]$$

be a path in $\mathcal{D}$. We have: $(a_{p_1}b_{p_1}) = (a_1b_1)$ and $(a_{p_{k+r-1}}b_{q_{k+r-1}}) = (a_rb_k)$. Let $t \in 1,\cdots,k+r-1$. We denote by γ_t the broken path obtained after removing the point $(a_{p_t}b_{q_t})$ from γ. In other words:

$$\gamma_t = [(a_{p_1}b_{q_1})(a_{p_2}b_{q_2})\cdots(a_{p_{t-1}}b_{q_{t-1}})(a_{p_{t+1}}b_{q_{t+1}})\cdots(a_{p_{k+r-1}}b_{q_{k+r-1}})].$$

Let

$$\omega^{n+3-k-r}_{\gamma_t} = \omega^{n+3-k-r}_{(a_{p_1}b_{q_1})(a_{p_2}q_{i_2})\cdots(a_{p_{t-1}}b_{q_{t-1}})(a_{p_{t+1}}b_{q_{t+1}})\cdots(a_{p_{k+r-1}}b_{q_{k+r-1}})}.$$

$\omega^{n+3-k-r}_{\gamma_t}$ appears in the right-hand side of Eq. (A.26) with signature

$$(-1)^{A(\gamma)+t+1}.$$

We analyze what happens on the other side. There are four cases to be considered.

1) $b_{q_{t-1}} = b_{q_t} = b_{q_{t+1}}$. This happens if the path γ keeps going horizontally in the vicinity of the node $(a_{p_t} b_{q_t})$.
2) $a_{p_{t-1}} = a_{p_t} = a_{p_{t+1}}$. This happens when the path keeps climbing vertically in the vicinity of the node $(a_{p_t} b_{q_t})$.
3) $b_{q_{t-1}} = b_{q_t} = b_{q_{t+1}} - 1$. The path γ makes a lower L with center in $(a_{p_t} b_{q_t})$.
4) $a_{p_{t-1}} = a_{p_t} = a_{p_{t+1}} - 1$. The path makes a upper L with center in $(a_{p_t} b_{q_t})$.

In each of the above cases, one obtains a term of type:

$$\begin{aligned}&(-1)^{A(\gamma)+t+1} \omega^{n+3-k-r}_{\gamma_t}\\ &\quad = (-1)^{A(\gamma)+t+1} \omega^{n+3-k-r}_{(a_{p_1} b_{q_1})(a_{p_2} b_{q_2}) \cdots (a_{p_{t-1}} b_{q_{t-1}})(a_{p_{t+1}} b_{q_{t+1}}) \cdots (a_{p_{k+r-1}} b_{q_{k+r-1}})}\end{aligned}$$

on the right side of the equation. Let us notice that the last two cases, 3 and 4, cancel each other in some way. For example, if case 3 happens then the corresponding term

$$(-1)^{A(\gamma)+t+1} \omega^{n+3-k-r}_{\gamma_t}$$

gets canceled by the term

$$(-1)^{A(\widetilde{\gamma})+t+1} \omega^{n+3-k-r}_{\widetilde{\gamma}_t}$$

where $\widetilde{\gamma}$ is the path :

$$[(a_{p_1} b_{q_1})(a_{p_2} b_{q_2}) \cdots (a_{p_{t-1}}, b_{q_{t-1}})(a_{p_t - 1}, b_{q_t + 1})(a_{p_{t+1}}, b_{q_{t+1}}) \cdots (a_{p_{k+r-1}} b_{q_{k+r-1}})].$$

($\widetilde{\gamma}$ is obtained from γ by downgrading the upper L at step t to a lower L.) Clearly:

$$\omega^{n+3-k-r}_{\widetilde{\gamma}_t} = \omega^{n+3-k-r}_{\gamma_t}$$

(as we are removing exactly the step where the two paths differ) and since $A(\widetilde{\gamma}) = A(\gamma) - 1$ we get:

$$(-1)^{A(\gamma)+t+1} \omega^{n+3-k-r}_{\gamma_t} + (-1)^{A(\widetilde{\gamma})+t+1} \omega^{n+3-k-r}_{\widetilde{\gamma}_t} = 0.$$

It is also clear that such terms do not appear on the left side of the Eq. (A.26).

It remains to look at the first two cases. Let us analyze the first case. In this situation, the term $\omega^{n+3-k-r}_{\gamma_t}$ appears in

$$(\delta_b T^{(a_1 a_2 \cdots a_r)}_{(.)} \omega)_{(b_1 b_2 \cdots b_k)}$$

but not in

$$(\delta_a T^{(.)}_{(b_1 b_2 \cdots b_k)} \omega)_{(a_1 a_2 \cdots a_r)}.$$

We check the sign which $\omega^{n+3-k-r}_{\gamma_t}$ carries on the left side of Eq. (A.26). Assume $q_t = f$ and $p_t = g$. Then γ_t can be interpreted as a path in the rectangular network $(a_p b_q)$ generated by $(a_1, a_2, \cdots, a_r)$ and $(b_1, b_2, \cdots b_{f-1}, b_{f+1}, \cdots b_{f_k})$. The area bounded by γ_t and b-axis is:

$$A(\gamma_t) = A(\gamma) - (\gamma - 1).$$

Therefore $\omega_{\gamma_t}^{n+3-k-r}$ appears in

$$T^{(a_1 a_2 \cdots a_r)}_{(b_1 b_2 \cdots b_{f-1} b_{f+1} \cdots b_k)} \omega$$

with sign

$$(-1)^{A(\gamma_t)} = (-1)^{A(\gamma) - g + 1}.$$

But $T^{(a_1 a_2 \cdots a_r)}_{(b_1 b_2 \cdots b_{f-1} b_{f+1} \cdots b_k)} \omega$ is a term in $(\delta_b T^{(a_1 a_2 \cdots a_r)}_{(.)} \omega)_{(b_1 b_2 \cdots b_k)}$ with sign $(-1)^{f+1}$ and $(\delta_b T^{(a_1 a_2 \cdots a_r)}_{(.)} \omega)_{(b_1 b_2 \cdots b_k)}$ appears on the left-hand side of expression (A.26) with positive sign. Therefore, the following term appears once in the left-hand side of (A.26):

$$(-1)^{A(\gamma) - g + 1 + f + 1} \ \omega_{\gamma_t}^{n+3-k-r}.$$

Since $f + g = t + 1$, $A(\gamma) - g + 1 + f + 1 \equiv A(\gamma) + t + 1 (\mathrm{mod}\ 2)$, and hence, the sign for $\omega_{\gamma_t}^{n+3-k-r}$ in the left-hand side of (A.26) coincides with the sign in the right-hand side.

One clears the second case in a similar manner. Then, drawing a conclusion, the entire right-hand side of expression (A.26) appears exactly the same in the left-hand side. It can immediately be checked that the reverse holds too. Each term in the left-hand side of Eq. (A.26) appears in the mirror on the right-hand side. Therefore, the equality holds and the proof of the lemma is complete. □

The integration mechanism:

$$\int_E \omega = A = (T, A^{n-d}_{a_1}, A^{n-d-1}_{a_1 a_2}, \cdots, A^{-1}_{a_1 a_2 \cdots a_{n-d+2}}) \tag{A.27}$$

with:

$$T = \int_E H, \tag{A.28}$$

$$A_{(a)} = \sum_{k=1}^{d+1} (-1)^{(n-d+1)(k+1)} \sum_{(i)=(i_1 > i_2 > \cdots > i_k)} \int_{\Delta_{(i)}} T^{(a)}_{\rho((i))} \omega, \ |(a)| \leq n - d + 1 \tag{A.29}$$

and

$$A^{-1}_{a_1 a_2 \cdots a_{n-d+2}} = (-1)^{(n-d+1)d} \sum_{(i)=(i_1 > i_2 > \cdots > i_k)} \int_{\Delta_{(i)}} T^{a_1 a_2 \cdots a_{n-d+2}}_{\rho((i))} \omega, \tag{A.30}$$

is now complete. However, there is one more issue to be discussed. The push-forward map obtained:

$$\int_E : N\check{C}^n(X \times E) \to N\check{C}^{n-d}(X) \tag{A.31}$$

commutes with differentiation operator $\check{d}$, realizing therefore a morphism of cochain complexes.

$$\begin{array}{ccccccc}
\xrightarrow{\check{d}} & N\check{C}^{n-1}(X \times E) & \xrightarrow{\check{d}} & N\check{C}^{n}(X \times E) & \xrightarrow{\check{d}} & N\check{C}^{n+1}(X \times E) & \xrightarrow{\check{d}} \\
& \Big\downarrow \int_E & & \Big\downarrow \int_E & & \Big\downarrow \int_E & \\
\xrightarrow{\check{d}} & N\check{C}^{n-d-1}(X) & \xrightarrow{\check{d}} & N\check{C}^{n-d}(X) & \xrightarrow{\check{d}} & N\check{C}^{n-d+1}(X) & \xrightarrow{\check{d}}
\end{array}$$

This morphism is far from being canonical. It depends on choices of dual cell decomposition Δ_i of the compact manifold E and subordination map ρ. However, one can measure its variation when using different sets of choices.

Proposition A.6. *Assume that (Δ_i, ρ) and (Δ_i', ρ') are two distinct pairs of dual cell decompositions and corresponding subordination maps. Let:*

$$\int_E' \quad \textit{and} \quad \int_E$$

be the associated push-forward morphisms. There exists a homotopy operator

$$k^n : N\check{C}^n(X \times E) \to N\check{C}^{n-d-1}(X) \tag{A.32}$$

such that for any differential non-flat n-cocycle $\omega \in N\check{Z}^n(X \times E)$,

$$\int_E \omega - \int_E' \omega = \check{d}k^n(\omega) + k^{n+1}(\check{d}\omega). \tag{A.33}$$

Proof. The proof goes in two steps. Initially, we show that the above statement holds if the two cell decompositions are the same and only the subordination maps are different. Secondly, we prove that refining a decomposition in the same subordination map does not change the integration process.

Let us proceed with the first step and construct the homotopy operator.

$$
\begin{array}{ccccccccc}
\xrightarrow{\check{d}} & N\check{C}^{n-1}(X\times E) & \xrightarrow{\check{d}} & N\check{C}^{n}(X\times E) & \xrightarrow{\check{d}} & N\check{C}^{n+1}(X\times E) & \xrightarrow{\check{d}} \\
{\scriptstyle k^{n-1}}\swarrow & \int'_E\Big\downarrow\Big\downarrow\int_E & {\scriptstyle k^{n}}\swarrow & \int'_E\Big\downarrow\Big\downarrow\int_E & {\scriptstyle k^{n+1}}\swarrow & \int'_E\Big\downarrow\Big\downarrow\int_E & \\
\xrightarrow{\check{d}} & N\check{C}^{n-d-1}(X) & \xrightarrow{\check{d}} & N\check{C}^{n-d}(X) & \xrightarrow{\check{d}} & N\check{C}^{n-d+1}(X) & \xrightarrow{\check{d}}
\end{array}
$$

We keep the notations used earlier. The subordination maps are defined as:

$$\rho, \rho' \colon I \to \mathcal{B}.$$

Let ω be a n-cocycle in $N\check{Z}^n(X\times E)$ described in the product open covering as

$$\omega = (H, \omega^n_a, \omega^{n-1}_{a_1a_2}, \omega^{n-2}_{a_1a_2a_3}, \dots.\omega^0_{a_1a_2\dots.a_{n+1}},\ \omega^{-1}_{a_1a_2\dots.a_{n+2}}). \tag{A.34}$$

We define $\Theta = k^n(\omega) \in N\check{C}^{n-d-1}(X)$ represented in the open covering $(U_a)_{a\in\mathcal{A}}$ of X as

$$\Theta = \left(0, \Theta^{n-d-1}_{a_1}, \Theta^{n-d-2}_{a_1a_2}, \cdots, \Theta^{-1}_{a_1a_2\cdots a_{n-d+1}}\right). \tag{A.35}$$

The components of (A.35) are obtained as follows. Let $(a) = (a_1a_2\cdots a_r)$, $1 \le k \le n-d+1$. Then:

$$\begin{aligned}\Theta^{n-d-r}_{(a)} = &\sum_{k=1}^{d+1}(-1)^{(n-d)(k+1)} \sum_{(i)=(i_1>i_2>\cdots>i_k)} \\ &\times \int_{\Delta_{(i)}}\left(\sum_{t=1}^{k}(-1)^{t+1}T^{(a)}_{(\rho(i_1)\rho(i_2)\cdots\rho(i_t)\rho'(i_t)\cdots\rho'(i_k))}\omega\right).\end{aligned} \tag{A.36}$$

The integrals over points are considered to be restrictions over the corresponding X-slices. If $r = n-d+1$, the above summation is made for $k = d+1$ only.

Denote by $A_{(a)}$, respectively, $A'_{(a)}$, the components of the $(n-d)$-cochains on X obtained from integrating ω using the subordination ρ, respectively,

ρ'. We compute:

$$
\begin{aligned}
(\delta\Theta)_{(a)} + (-1)^{|(a)|} d\Theta_{(a)} = &\sum_{k=1}^{d+1} (-1)^{(n-d)(k+1)} \\
&\sum_{t=1}^{k} (-1)^{t+1} \sum_{(i)=(i_1>\cdots>i_k)} \int_{\Delta_{(i)}} \\
\Big([\delta_a T^{(.)}_{(\rho(i_1)\cdots\rho(i_t)\rho'(i_t)\cdots\rho'(i_k))}\omega]^{(a)} &+ (-1)^{|(a)|} d_x T^{(a)}_{(\rho(i_1)\cdots\rho(i_t)\rho'(i_t)\cdots\rho'(i_k))}\omega \Big) \\
= &\sum_{k=1}^{d+1}\sum_{t=1}^{k} (-1)^{(n-d)(k+1)+t+1} \\
&\times \sum_{(i)=(i_1>\cdots>i_k)} ([\delta_a T^{(.)}_{(\rho(i_1)\cdots\rho(i_t)\rho'(i_t)\cdots\rho'(i_k))}\omega]^{(a)} \\
&+(-1)^{|(a)|} dT^{(a)}_{(\rho(i_1)\cdots\rho(i_t)\rho'(i_t)\cdots\rho'(i_k))}\omega \\
&+(-1)^{|(a)|+1} d_y T^{(a)}_{(\rho(i_1)\cdots\rho(i_t)\rho'(i_t)\cdots\rho'(i_k))}\omega).
\end{aligned}
$$

But, for a fixed k:

$$
\begin{aligned}
&\sum_{t=1}^{k} \sum_{(j)=(j_1>\cdots>j_{k-1})} (-1)^{t+1} \int_{\Delta_{(j)}} d_y T^{(a)}_{(\rho(j_1)\cdots\rho(j_t)\rho'(i_t)\cdots\rho'(i_{k-1}))}\omega \\
&\quad = (-1)^{(n-d+|(a)|+k+1)} \sum_{t=1}^{k+1} \sum_{(i)=(i_1>\cdots>i_k)} (-1)^{t+1} \\
&\quad\quad \times \int_{\Delta_{(i)}} ([\delta_b T^{(a)}_{(.)}\omega]_{(\rho(i_1)\cdots\rho(i_t)\rho'(i_t)\cdots\rho'(i_k))} \\
&\quad\quad + (T^{(a)}_{(\rho(i_1)\rho(i_2)\cdots\rho(i_k))}\omega - T^{(a)}_{(\rho'(i_1)\rho'(i_2)\cdots\rho'(i_k))}\omega)).
\end{aligned}
$$

On can check the above equality by just applying Stokes' theorem and then removing the terms that cancel. Using this in the earlier expression provides us with:

$$
\begin{aligned}
&(\delta\Theta)_{(a)} + (-1)^{|(a)|} d\Theta_{(a)} \\
&\quad = \sum_{k=1}^{d+1} \sum_{t=1}^{k} (-1)^{(n-d)(k+1)+t+1} \\
&\quad\quad \times \sum_{(i)=(i_1>\cdots>i_k)} ([\delta_a T^{(.)}_{(\rho(i_1)\cdots\rho(i_t)\rho'(i_t)\cdots\rho'(i_k))}\omega]^{(a)} \\
&\quad\quad + (-1)^{|(a)|} dT^{(a)}_{(\rho(i_1)\cdots\rho(i_t)\rho'(i_t)\cdots\rho'(i_k))}\omega \\
&\quad\quad + \ (-1)^k [\delta_b T^{(a)}_{(.)}\omega]_{(\rho(i_1)\cdots\rho(i_t)\rho'(i_t)\cdots\rho'(i_k))} \\
&\quad\quad + \ (-1)^{k+1} (T^{(a)}_{(\rho(i_1)\rho(i_2)\cdots\rho(i_k))}\omega - T^{(a)}_{(\rho'(i_1)\rho'(i_2)\cdots\rho'(i_k))}\omega)).
\end{aligned}
$$

According to Lemma A.5, the first three terms inside the above summation make

$$(-1)^k T^{(a)}_{(\rho(i_1)\cdots\rho(i_t)\rho'(i_t)\cdots\rho(i_k))} \check{d}\omega.$$

Therefore:

$$
\begin{aligned}
&(\delta\Theta)_{(a)} + (-1)^{|(a)|} d\Theta_{(a)} \\
&\quad = -\sum_{k=1}^{d+1} \sum_{t=1}^{k} (-1)^{(n-d+1)(k+1)+t+1} \\
&\quad\quad \times \sum_{(i)=(i_1>i_2>\cdots>i_k)} \int_{\Delta_{(i)}} \left(T^{(a)}_{(\rho(i_1)\rho(i_2)\cdots\rho(i_t)\rho'(i_t)\cdots\rho'(i_k))} \check{d}\omega \right) \\
&\quad\quad + \sum_{k=1}^{d+1} (-1)^{(n-d+1)(k+1)} \\
&\quad\quad \times \sum_{(i)=(i_1>\cdots>i_k)} \int_{\Delta_{(i)}} \left(T^{(a)}_{(\rho(i_1)\rho(i_2)\cdots\rho(i_k))}\omega - T^{(a)}_{(\rho'(i_1)\rho'(i_2)\cdots\rho'(i_k))}\omega \right).
\end{aligned}
$$

The first sum on the right-hand side of earlier equality represents exactly the (a)-component of $k^{n+1}(\check{d}\omega)$. Therefore we obtain:

$$(\delta\Theta)_{(a)} + (-1)^{|(a)|} d\Theta_{(a)} = \ - \left[k^{n+1}(\check{d}\omega)\right]_{(a)} + A_{(a)} - A'_{(a)},$$

In other words:

$$\left[\check{d}k^n(\omega)\right]_{(a)} + \left[k^{n+1}(\check{d}\omega)\right]_{(a)} = A_{(a)} - A'_{(a)}.$$

Equality (A.33) follows. The first part of the proof is complete.

Let us turn our attention to the second step. Assume that we are dealing with two different cell decompositions $(\Delta_i)_{i\in I}$ and $(\Delta_j)_{j\in J}$ and that the latter one is a refinement of the former. Say, there is a map:

$$\varphi\colon J \to I \quad \text{such that} \quad \Delta_j \subset \Delta_{\varphi(j)}.$$

In addition, assume that both decompositions are subordinated to covering $(U_b)_{b\in\mathcal{B}}$ and the subordination map for Δ_i is

$$\rho\colon I \to \mathcal{B}.$$

The subordination on Δ_j can be taken as $\rho\circ\varphi$.

Say, the integration of ω using the first pair (Δ_i, ρ) is :

$$\int_E \omega = A = (T, A^{n-d}_{a_1}, A^{n-d-1}_{a_1 a_2}, \ldots, A^{-1}_{a_1 a_2\cdots a_{n-d+2}}) \tag{A.37}$$

whereas the integration of ω using $(\Delta_j, \rho\circ\varphi)$ is:

$$\widetilde{\int_E} \omega = \widetilde{A} = (T, \widetilde{A}^{n-d}_{a_1}, \widetilde{A}^{n-d-1}_{a_1 a_2}, \ldots, \widetilde{A}^{-1}_{a_1 a_2\cdots a_{n-d+2}}). \tag{A.38}$$

Our goal is to prove that these two multiplets are actually the same. We have:

$$A_{(a)} = \sum_{k=1}^{d} (-1)^{(n-d+1)(k+1)} \sum_{(i)=(i_1>i_2>\cdots>i_k)} \int_{\Delta_{(i)}} T^{(a)}_{\rho((i))}\omega. \tag{A.39}$$

and

$$\widetilde{A}_{(a)} = \sum_{k=1}^{d} (-1)^{(n-d+1)(k+1)} \sum_{(j)=(j_1>j_2>\cdots>j_k)} \int_{\Delta_{(j)}} T^{(a)}_{\rho(\varphi(j))}\omega. \tag{A.40}$$

(the order relations on J and I are such chosen to make φ an increasing map).

But, $\omega_{(c)}$ vanishes as soon as the multi-index (c) includes two identical subindices (by convention). Therefore, in sum (A.40), all terms:

$$\int_{\Delta_{(j)}} T^{(a)}_{\rho(\varphi(j))}\omega$$

for which (j) contains two subindices j_u and j_v with $\varphi(j_u) = \varphi(j_v)$ vanish. After removing those, the terms in expression (A.40) can be summed up to make the ones in (A.39). The second step is complete. □

Let us finish up the appendix by making the following observations related to Proposition A.6. First, although the integration mechanism we devised for differential cochains depends on ingredient choices (dual cell decomposition and subordination assignment), the induced cohomology push-forward map does not depend on those. One can extend the arguments in the proof and take into account a variation of the product open covering too. Under a change of admissible open covering, the integration mechanism stays again in the same homotopy class. One can therefore conclude that the induced cohomology push-forward map:

$$\int_E : \check{H}^n(X \times E) \to \check{H}^{n-d}(X)$$

does not depend at all on the choice of product open covering, dual cell decomposition and subordination relation.

Secondly, the construction can be extended to work for differential cochains having as derivatives global forms. Namely, let $\mathcal{B}^n(X)$ be the set of non-flat differential n-cochains ω such that $\check{d}\omega \in \Omega^n(X)$. We know from Section 1 that the flat coboundary equivalence relation extends to $\mathcal{B}^n(X)$. One can show then that, for $\omega_1, \omega_2 \in \mathcal{B}^n(X \times E)$, defined on admissible product coverings, with $\omega_1 \sim \omega_2$

$$\int_E \omega_1 \sim \int_E \omega_2 \quad \text{in } \mathcal{B}^{n-d}(X)$$

regardless of the choice of dual cell decomposition subordinated to the two distinct open coverings. One obtains, therefore, a canonical push-forward homomorphism:

$$\int_E : \frac{\mathcal{B}^n(X \times E)}{\sim} \to \frac{\mathcal{B}^{n-d}(X)}{\sim}.$$

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