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Addendum: Generalized Spencer Cohomology and filtered Deformations of $\mathbb{Z}$-graded Lie superalgebras

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In [K] all the possibilities for the non-positive part $\mathfrak{g}_{\leq 0} = \bigoplus_{j=-h}^{0} \mathfrak{g}_j$ of the associated graded Lie superalgebra $\mathfrak{g} = \bigoplus_{j \geq -h} \mathfrak{g}_j$ of a simple linearly compact Lie superalgebra L, for a "good" choice of its filtration, were obtained. In [CK2] the transitive $\mathbb{Z}$-graded $\mathfrak{g}$ with those $\mathfrak{g}_{\leq 0}$ were classified, and in [CK1] and [K], in order to reconstruct L from $\mathfrak{g}$, their filtered deformations were classified. However, some cases had been inadvertently omitted. In this note we take the opportunity to amend this.

The list for $\bigoplus_{j=-h}^{0} \mathfrak{g}_j$ (written below as the $h+1$-tuple $(\mathfrak{g}_{-h}, \mathfrak{g}_{-h+1}, \dots, \mathfrak{g}_{-1}, \mathfrak{g}_0)$) is as follows [K]:

Inconsistent gradations of depth 1:

(I1) $(\mathbb{C}^{m|n}, gl(m,n))$.

(I2) $(\mathbb{C}^{m|n}, sl(m,n))$.

(I3) $(\mathbb{C}^{m|n}, spo(m,n))$.

(I4) $(\mathbb{C}^{m|n}, cspo(m,n))$.

(I5) $(\mathbb{C}^{m|0} \otimes \Lambda(1), sl(m) \otimes \Lambda(1) + \mathfrak{a})$.

(I6) $(\mathbb{C}^{2n|0} \otimes \Lambda(1), sp(2n) \otimes \Lambda(1) + \mathfrak{a})$.

(I7) $(\mathbb{C}^{m|m}, \tilde{P}(m))$.

(I8) $(\mathbb{C}^{m|m}, c\tilde{P}(m))$.

(I9) $(\mathbb{C}^{m|m}, P(m))$.

(I10) $(\mathbb{C}^{m|m}, cP(m))$.

(I11) $(\mathbb{C}^{4|4}, \hat{P}(4))$

(I12) $(\mathbb{C}^{2|2}, spin_4^0 + \mathfrak{a})$.

(I13) $(\mathbb{C}^{m|m}, Q(m))$.

(I14) $(\mathbb{C}^{m|m}, cQ(m))$.

(I15) $(\Pi(\Lambda(2)^\lambda), W(0,2))$, $\lambda \neq 0, 1$.

(I16) $(\Pi(\Lambda(2)^\lambda), cW(0,2))$, $\lambda \neq 0, 1$.

(I17) $(\Pi(\Lambda(2)), W(0,2) + \Lambda(2))$.

(I18) $(\Pi(\Lambda(2)), S(0,2) + \Lambda(2))$.

(I19) $(\Pi(\Lambda(2)), S(0,2) + \mathbb{C}1 + \mathbb{C}\xi_1 + \mathbb{C}\xi_2)$.

(I20) $(\Pi(\Lambda(3)^\lambda)/\mathbb{C}\xi_1\xi_2\xi_3, W(0,3))$, $\lambda = 1$.

Inconsistent gradations of depth 2:

(J1) $(\mathbb{C}^{1|0}, \mathbb{C}^{m|n}, spo(m,n))$.

(J2) $(\mathbb{C}^{1|0}, \mathbb{C}^{m|n}, cspo(m,n))$.

(J3) $(\mathbb{C}^{0|1}, \mathbb{C}^{m|m}, \tilde{P}(m))$.

(J4) $(\mathbb{C}^{0|1}, \mathbb{C}^{m|m}, c\tilde{P}(m))$.

(J5) $(\mathbb{C}^{0|1}, \mathbb{C}^{m|m}, P(m))$.

(J6) $(\mathbb{C}^{0|1}, \mathbb{C}^{m|m}, cP(m))$.

(J7) $(\mathbb{C}^{0|1}, \mathbb{C}^{m|m}, P(m) + \mathbb{C}(I + \beta\Phi))$.

(J8) $(\mathbb{C}^{1|0} \otimes \xi, \mathbb{C}^{2n|0} \otimes \Lambda(1), sp(2n) \otimes \Lambda(1) + \mathbb{C}\xi\frac{\partial}{\partial\xi} + \mathbb{C}\frac{\partial}{\partial\xi})$.

(J9) $(\mathbb{C}^{1|0} \otimes \Lambda(1), \mathbb{C}^{2n|0} \otimes \Lambda(1), csp(2n) \otimes \Lambda(1) + \mathfrak{a})$.

Consistent gradations:

(C1) $(\mathbb{C}, \mathbb{C}^n, cso(n))$, $n \geq 1$ and $n \neq 2$.

(C2) $(\mathbb{C}^{5*}, \Lambda^2(\mathbb{C}^5), sl(5))$.

(C3) $(\mathbb{C}^{5*}, \Lambda^2(\mathbb{C}^5), gl(5))$.

(C4) $(\mathbb{C}^{3*}, \mathbb{C}^3 \boxtimes \mathbb{C}^2, gl(3) \oplus sl(2))$.

(C5) $(\mathbb{C}^{3*}, \mathbb{C}^3 \boxtimes \mathbb{C}^2, sl(3) \oplus sl(2))$.

(C6) $(\mathbb{C}^2, \mathbb{C}^{3*}, \mathbb{C}^3 \boxtimes \mathbb{C}^2, gl(3) \oplus sl(2))$.

(C7) $(\mathbb{C}^2, \mathbb{C}^{3*}, \mathbb{C}^3 \boxtimes \mathbb{C}^2, sl(3) \oplus sl(2))$.

Remark 0.1. Above $\Lambda(n)$ is the Grassmann superalgebra in the indeterminates $\xi_1, \ldots, \xi_n$ and $\mathfrak{a}$ is a subalgebra of $\mathbb{C}1 + \mathbb{C}\xi + \mathbb{C}\xi\frac{\partial}{\partial\xi} + \mathbb{C}\frac{\partial}{\partial\xi}$ which projects non-trivially onto $\mathbb{C}\frac{\partial}{\partial\xi}$. For further explanation of the above notations see page 220–221 of [CK2]. We want to point out that (I18) and (I19) on page 220 of [CK2] contain typos and that (I20) was inadvertently omitted in [K]. Namely, to the (empty) list of Lemma 3.5 and to the list of Theorem 3.1 of [K] one should add the representation (I20).

In the case when $\mathfrak{g}_0$ contains a grading operator, it is well-known that the $\mathbb{Z}$-graded transitive Lie superalgebra $\mathfrak{g}$ allows no non-trivial filtered deformations (c.f. Corollary 2.2 [CK1]). This takes care of all cases except for (I2), (I3), (I5), (I6), (I7), (I9), (I13), (I15), (I20), (J1), (J3), (J5), (J7), (J8), (C2), (C5) and (C7). By [CK2] we have the following possibilities for $\mathfrak{g}$ with prescribed $\mathfrak{g}_{\leq 0}$ for these remaining cases (see [K] or [CK2] for notations and definitions):

(I2) $S(m,n)$ and $S'(m,n)$ in principal gradation.

(I3) $H(m,n)$ in principal gradation.

(I5) $S(n,0)\otimes\Lambda(1)+\mathfrak{a}$, $S(n,1)$ and $S(n,1)+\mathbb{C}E$ in subprincipal gradation.

(I6) $H(n,0)\otimes\Lambda(1)+\mathfrak{a}$ in subprincipal gradation.

(I7) $SHO(n,n)+\mathbb{C}\Phi$, $SHO'(n,n)+\mathbb{C}\Phi$ and $HO(n,n)$ in principal gradation.

(I9) $SHO(n,n)$ and $SHO'(n,n)$ in principal gradation.

(I13) No infinite-dimensional prolongation by Section 2.7 of [CK2].

(I15) $SKO(2,3;1-\frac{1}{\lambda})$, $\lambda\neq 0,1$, in subprincipal gradation.

(I20) $SKO(3,4;\frac{1}{3})$ in subprincipal gradation.

(J1) $\widehat{H}(m,n)$ in principal gradation.

(J3) $\widehat{HO}(n,n)$, $\widehat{SHO}(n,n)+\mathbb{C}\Phi$ and $\widehat{SHO}'(n,n)+\mathbb{C}\Phi$ in principal gradation.

(J5) $\widehat{SHO}(n,n)$ and $\widehat{SHO}'(n,n)$ in principal gradation.

(J7) $SKO(n,n+1;\beta)$, $SKO'(n,n+1;\beta)$, $\widehat{SHO}(n,n)+\mathbb{C}(\tau+\beta\Phi)$ and $\widehat{SHO}'(n,n)+\mathbb{C}(\tau+\beta\Phi)$ in principal gradation.

(J8a) $H(2n,2)$ in subprincipal gradation.

(J8b) $\widehat{H}(2n,0)\otimes\Lambda(1)+\mathbb{C}\xi\frac{\partial}{\partial\xi}+\mathbb{C}\frac{\partial}{\partial\xi}/\mathbb{C}1$.

(C2) $E(5,10)$.

(C5) $SHO(3,3)+sl_2$.

(C7) $\mathbb{C}^2+SHO(3,3)+sl_2$.

Remark 0.2. One can show, arguing as in Section 2.6 of [CK2], that the full prolongation of $(\Pi(\Lambda(3)^1)/\mathbb{C}\xi_1\xi_2\xi_3, W(0,3))$ is $SKO(3,4;\frac{1}{3})$ in the subprincipal gradation.

We will now discuss the above cases one by one.

(I2) No non-trivial filtered deformations by Lemma 6.4 of [K]. As the proof of Lemma 6.4 [K] contains a gap (which can be easily fixed) we will provide below an independent proof using Spencer cohomology.

(I3) No non-trivial filtered deformations by Theorem 4.4 of [CK1].

(I7) No non-trivial filtered deformations by Remark 4.3 and Theorem 4.2 of [CK1].

(I9) $SHO(n,n)$ has no non-trivial filtered deformations by Propositions 4.1 and 4.2 of [CK1], while $SHO'(n,n)$ has a unique non-trivial filtered deformation by Theorem 5.1 (i) of [CK1].

(J1) No non-trivial filtered deformations by Proposition 2.7 of [CK1].

(J3) No non-trivial filtered deformations by Remarks 4.1 and 4.2 of [CK1].

(J5) $\widehat{SHO}(n,n)$ has a unique non-trivial filtered deformation by Theorem 5.1 (ii) of [CK1], while $\widehat{SHO}'(n,n)$ has no non-trivial filtered deformations by Theorem 5.1 (iii) of [CK1].

(J7) Only $SKO(n,n+1;\frac{n+2}{n})$, for n odd, has a unique non-trivial filtered deformation by Theorem 5.2 of [CK1]. The remaining cases are taken care of by Proposition 2.7, Remarks 4.1, 4.3 and Theorem 4.3 of [CK1].

(J8b) $\widehat{H}(2n,0)\otimes\Lambda(1)+\mathbb{C}\xi\frac{\partial}{\partial\xi}+\mathbb{C}\frac{\partial}{\partial\xi}/\mathbb{C}1$ in (J8) has no filtered deformations, for which L_0 is a maximal subalgebra, by Proposition 2.7 of [CK1].

(C2) No non-trivial filtered deformations by Lemma 6.3 of [K].

Therefore we are left to consider the following cases: (I2), (I5), (I6), (I15), (I20), (J8a), (C5) and (C7).

We follow the strategy of [CK1] for determining filtered deformations of a $\mathbb{Z}$-graded transitive Lie superalgebra $\mathfrak{g}$. We briefly summarize the idea here. A filtered deformation gives rise to a *defining sequence* μ_i, $i\geq 1$, as defined in (2.2) of [CK1]. The first non-zero term is a 2-cocycle of $\mathfrak{g}$ with coefficients in $\mathfrak{g}$, which, by Proposition 2.2 of [CK1], when restricted to $\mathfrak{g}_-=\oplus_{j<0}\mathfrak{g}_j$ is a $\mathfrak{g}_0$-invariant (necessarily even) Spencer 2-cocycle, i.e. an even $\mathfrak{g}_0$-invariant element in the second cohomology group $H^{*,2}(\mathfrak{g}_-, \mathfrak{g})$ of $\mathfrak{g}_-$ with coefficients in $\mathfrak{g}$. Recall that the $\mathbb{Z}$-grading of $\mathfrak{g}$ induces a $\mathbb{Z}_+$-grading (referred to by the first superscript) of its Spencer cohomology.

If the subspace of $\mathfrak{g}_0$-invariants in $H^{*,2}(\mathfrak{g}_-, \mathfrak{g})_{\bar{0}}$ is zero, then $\mathfrak{g}$ has no filtered deformations, provided that $\mathfrak{g}$ is a full or an almost full prolongation (Corollary 2.3 of [CK1]). Let $\mathfrak{s}$ denote a reductive Lie subalgebra of $(\mathfrak{g}_0)_{\bar{0}}$. By complete reducibility it follows, in particular, that if $(\Lambda^2(\mathfrak{g}_-^*) \otimes \mathfrak{g})_{\bar{0}}^{\mathfrak{s}} = 0$, then the $\mathfrak{g}$ as above has no filtered deformations. Moreover, provided that again $\mathfrak{g}$ is a full or an almost full prolongation, Corollary 2.5 of [CK1] implies that if the even $\mathfrak{g}_0$-invariant part of the second Spencer cohomology group is one-dimensional, then a non-trivial filtered deformation is necessarily unique.

1 (I2) has no non-trivial filtered deformations

We recall that $W(m,n)$ is the Lie superalgebra of derivations of $\Lambda(m,n)$, which is generated by the even elements x_i, $i = 1, \ldots, m$, and odd elements ξ_j, $j = 1, \ldots, n$. We shall prove that for $m, n > 0$ any graded subalgebra $\mathfrak{g}$ of $W(m,n)$, in its principal gradation, containing $S(m,n)_{-1}$ and $S(m,n)_0$, and thus in particular the element $D = n \sum_{i=1}^m x_i \frac{\partial}{\partial x_i} + m \sum_{i=j}^n \xi_j \frac{\partial}{\partial \xi_j}$, has no non-trivial $\mathfrak{g}_0$-invariants in $H^2(\mathfrak{g}_-, \mathfrak{g})$. This in particular will show that the two algebras of (I2) have no non-trivial $\mathfrak{g}_0$-invariants in $H^2(\mathfrak{g}_-, \mathfrak{g})$.

An elementary calculation, using the fact that $m, n > 0$, shows that the only $sl(m) \oplus sl(n) \oplus \mathbb{C}D = (S(m,n)_0)_{\bar{0}}$-submodules of $\Lambda^2(\mathfrak{g}_{-1}^*) \otimes W(m,n)$ on which D acts trivially are the following:

(a) $S^2(\mathbb{C}^n) \otimes \left(\Lambda^l(\mathbb{C}^n) \boxtimes \mathbb{C}^{m*}\right)$, if there exists $l > 0$ such that $m(l+2) = n$,

(b) $\Lambda^2(\mathbb{C}^m) \otimes \left(S^k(\mathbb{C}^m) \boxtimes \mathbb{C}^{n*}\right)$, if there exists $k > 0$ such that $m = (k+2)n$.

But it is evident that these two modules contain no $sl(m) \oplus sl(n)$-invariants. Now in the case when $S(m,n) = S'(m,n)$, it is the full prolongation of (I2). If $S(m,n) \subsetneq S'(m,n)$, then $S'(m,n)$ is the full prolongation and $S(m,n)$ is an almost full prolongation of (I2). Hence Corollary 2.3 of [CK1] is applicable and thus neither $S(m,n)$ nor $S'(m,n)$ can have non-trivial filtered deformations.

2 (I5) has no non-trivial filtered deformations

The following lemma can be verified directly (c.f. [K] Example 3.4).

Lemma 2.1. *The following are all possibilities for a subalgebra* $\mathfrak{a} \subseteq gl(1|1) = \mathbb{C}1 + \mathbb{C}\xi + \mathbb{C}\frac{\partial}{\partial\xi} + \mathbb{C}\xi\frac{\partial}{\partial\xi}$ *with a non-trivial projection onto* $\mathbb{C}\frac{\partial}{\partial\xi}$*:*

(a) $gl(1|1)$,

(b) $\mathbb{C}1 + \mathbb{C}\frac{\partial}{\partial\xi} + \mathbb{C}\xi\frac{\partial}{\partial\xi}$,

(c) $\mathbb{C}(\alpha 1 + \beta\xi\frac{\partial}{\partial\xi}) + \mathbb{C}\frac{\partial}{\partial\xi}$, $\alpha, \beta \in \mathbb{C}$, *and one of them is non-zero,*

(d) $\mathbb{C}\frac{\partial}{\partial\xi}$,

(e) $\mathbb{C}1 + \mathbb{C}(\frac{\partial}{\partial\xi} + \xi)$,

(f) $\mathbb{C}\frac{\partial}{\partial\xi} + \mathbb{C}\xi + \mathbb{C}1$.

Lemma 2.2. *Suppose the associated graded* $\mathfrak{g}$ *of* L *is* $S(n,0) \otimes \Lambda(1) + \mathfrak{a}$*, where* $\mathfrak{a}$ *is listed in Lemma* 2.1*. Then* $[L,L] \neq L$ *and hence* L *is not simple.*

Proof. Let $\mathfrak{g} = \bigoplus_{j=-1}^{\infty} \mathfrak{g}_j$. We have $sl_n \subseteq (\mathfrak{g}_0)_{\bar{0}}$ and it is easy to write down explicitly the structure of $\mathfrak{g}_j$ as an sl_n-module for each j. Now $\mathfrak{a}_{\bar{1}} \neq 0$ and sl_n acts trivially on it. Consider the sl_n-module decomposition of

$$L = \prod_{j \geq -1} \mathfrak{m}_j$$

of (6.1) in [K] so that $\mathfrak{m}_j \cong \mathfrak{g}_j$ as sl_n-modules. One verifies that the component isomorphic to $\mathfrak{a}_{\bar{1}}$ inside $\mathfrak{m}_0$ cannot be obtained from $[L,L]$. Indeed, it can only be obtained from $[(\mathfrak{m}_{-1})_{\bar{0}}, (\mathfrak{m}_{-1})_{\bar{1}}]$, $[(\mathfrak{m}_{-1})_{\bar{0}}, (\mathfrak{m}_1)_{\bar{1}}]$ or $[(\mathfrak{m}_1)_{\bar{0}}, (\mathfrak{m}_{-1})_{\bar{1}}]$. Now we have $(\mathfrak{m}_{-1})_\epsilon \cong R(\pi_{n-1})$ and $(\mathfrak{m}_1)_\epsilon \cong R(2\pi_1 + \pi_{n-1})$, $\epsilon = \bar{0}, \bar{1}$. Here as usual π_i denotes the i-th fundamental weight and $R(\pi_i)$ is the corresponding irreducible $sl(n)$-module, etc. Since $\mathfrak{a}_{\bar{1}}$ is a trivial sl_n-module and neither of the three pairs of modules above are contragredient to each other, it follows that $\mathfrak{a}_{\bar{1}}$ cannot lie in $[L,L]$. □

The remaining cases of (I5) are taken care of as follows. $S(n,1) + \mathbb{C}E$ contains a grading operator E, while $S(n,1)$ has no non-trivial filtered deformations by Lemma 6.5 of [K].

3 (I6) has a unique non-trivial filtered deformation (isomorphic to $H(2n, 1)$)

As in the proof of Lemma 2.2 one verifies that only when $\mathfrak{a} = \mathbb{C}\frac{\partial}{\partial\xi}$ can one possibly have a simple non-trivial filtered deformation for (I6).

Consider $H(2n, 1)$, which we identify with $\mathbb{C}[p_i, q_i, \xi]/\mathbb{C}1$, $i = 1, \ldots, n$, equipped with the induced Poisson bracket. Let $C^{\geq j}$ be the span of homogeneous polynomials of degree $\geq j$ in p_i and q_i. Let $L_0 = C^{\geq 2} + C^{\geq 2}\xi + \mathbb{C}\xi$ and $L_j = C^{\geq j+2} + C^{\geq j+2}\xi$, for $j > 0$. This gives rise to a filtration on $H(2n, 1)$ such that the associated graded is isomorphic to $H(2n, 0) \otimes \Lambda(1) + \mathbb{C}\frac{\partial}{\partial\xi}$, with the $\mathbb{Z}$-gradation induced from the standard $\mathbb{Z}$-gradation of $H(2n, 0)$ by letting $\deg\xi = 0$ [CaK]. This is the $\mathbb{Z}$-graded Lie superalgebra of type (I6).

We will show that the above filtered deformation $H(2n, 1)$ of this $\mathbb{Z}$-graded Lie superalgebra is the unique non-trivial one.

Proposition 3.1. *Let $\mathfrak{g} = H(2n, 0) \otimes \Lambda(1) + \mathbb{C}\frac{\partial}{\partial\xi}$. Then we have*

$$H^{l,2}(\mathfrak{g}_{-1}, \mathfrak{g})_{\bar{0}}^{sp(2n)} = 0, \quad \text{if } l \neq 2,$$
$$H^{2,2}(\mathfrak{g}_{-1}, \mathfrak{g})_{\bar{0}}^{sp(2n)} = \mathbb{C}.$$

Proof. We have $\mathfrak{g}_0 = sp(2n) \otimes \Lambda(1) + \mathbb{C}\frac{\partial}{\partial\xi}$. Write $\mathfrak{g} = \bigoplus_{j\geq -1} \mathfrak{g}_j$, then the $sp(2n)$-module structure of each $\mathfrak{g}_j$ is easily computed:

$$\begin{aligned}
(\mathfrak{g}_{-1})_{\bar{0}} &= R(\pi_1), & (\mathfrak{g}_{-1})_{\bar{1}} &= R(\pi_1), \\
(\mathfrak{g}_0)_{\bar{0}} &= R(2\pi_1), & (\mathfrak{g}_0)_{\bar{1}} &= R(2\pi_1) \oplus R(0), \\
(\mathfrak{g}_j)_{\bar{0}} &= R((j+2)\pi_1), & (\mathfrak{g}_j)_{\bar{1}} &= R((j+2)\pi_1), \quad j \geq 1.
\end{aligned}$$

As an $sp(2n)$-module $\Lambda^2(\mathfrak{g}_{-1}^*)$ is as follows:

$$\Lambda^2(\mathfrak{g}_{-1}^*)_{\bar{0}} \cong \Lambda^2(\mathfrak{g}_{-1}^*)_{\bar{1}} \cong R(0) \oplus R(\pi_2) \oplus R(2\pi_1).$$

Now we consider the $sp(2n)$-module $(\Lambda^2(\mathfrak{g}_{-1}^*) \otimes \mathfrak{g})_{\bar{0}}$. It is easy to see that the trivial $sp(2n)$-module can only appear in $(\Lambda^2(\mathfrak{g}_{-1}^*) \otimes \mathfrak{g}_0)_{\bar{0}}$, from which it follows immediately that $H^{l,2}(\mathfrak{g}_{-1}; \mathfrak{g})_{\bar{0}}^{sp(2n)} = 0$, if $l \neq 2$.

The space of $sp(2n)$-invariants in $(\Lambda^2(\mathfrak{g}_{-1}^*) \otimes \mathfrak{g}_0)_{\bar{0}}$ is three-dimensional. In order to write down a basis for it, we need some more notation. Let p_i, q_i, $i = 1, \ldots, n$, be the standard basis of $\mathbb{C}^{2n}$, on which $sp(2n)$ acts naturally. For $f \in \mathbb{C}[p_i, q_i]$ we let $\tilde{f}_i = f \otimes \xi$. Choose the standard basis $\{p_i, q_i\}$ for

$(\mathfrak{g}_{-1})_{\bar{0}}$ and the standard basis $\{\tilde{p}_i, \tilde{q}_i\}$ for $(\mathfrak{g}_{-1})_{\bar{1}}$. Let p_i^* etc. denote the corresponding dual. Then $(\Lambda^2(\mathfrak{g}_{-1}^*) \otimes \mathfrak{g}_0)_{\bar{0}}^{sp(2n)}$ is spanned by:

$$\begin{aligned}
c_1 &= \left(\sum_{i=1}^n p_i^* \otimes \tilde{q}_i^* - q_i^* \otimes \tilde{p}_i^*\right) \otimes \frac{\partial}{\partial \xi},\\
c_2 &= \frac{1}{2}\sum_{i,j}\left(\tilde{p}_i^* \otimes \tilde{q}_j{}^* + \tilde{q}_j^* \otimes \tilde{p}_i{}^*\right) \otimes p_i q_j\\
&\quad + \frac{1}{2}\sum_{i\leq j}\left(\tilde{p}_i^* \otimes \tilde{p}_j{}^* + \tilde{p}_j^* \otimes \tilde{p}_i{}^*\right) \otimes p_i p_j + \left(\tilde{q}_i^* \otimes \tilde{q}_j{}^* + \tilde{q}_j^* \otimes \tilde{q}_i{}^*\right) \otimes q_i q_j,\\
c_3 &= \sum_{i,j}\left(p_i^* \otimes \tilde{q}_j^* + q_j^* \otimes \tilde{p}_i^*\right) \otimes \widetilde{p_i q_j} + \sum_{i\leq j}\left(p_i^* \otimes \tilde{p}_j^* + p_j^* \otimes \tilde{p}_i^*\right) \otimes \widetilde{p_i p_j}\\
&\quad + \left(q_i^* \otimes \tilde{q}_j^* + q_j^* \otimes \tilde{q}_i^*\right) \otimes \widetilde{q_i q_j}.
\end{aligned}$$

It is straightforward to check that $c_1 + c_2$ is a 2-cocycle. On the other hand one computes

$$\begin{aligned}
&(dc_1)(q_1, \tilde{p}_1, \tilde{p}_1) = -2p_1, \quad &&(dc_1)(p_1, q_1, \tilde{p}_1) = 0,\\
&(dc_2)(q_1, \tilde{p}_1, \tilde{p}_1) = 2p_1, &&(dc_2)(p_1, q_1, \tilde{p}_1) = 0,\\
&(dc_3)(q_1, \tilde{p}_1, \tilde{p}_1) = 0, &&(dc_3)(p_1, q_1, \tilde{p}_1) = -5\tilde{p}_1,
\end{aligned}$$

from which it follows that if a linear combination of the form $\lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3$ is a cocyle, then $\lambda_3 = 0$. Thus a cocycle is of the form $\lambda_1 c_1 + \lambda_2 c_2$, and the above calculation also shows that we must have $\lambda_1 = \lambda_2$. This shows that $H^{2,2}(\mathfrak{g}_{-1}, \mathfrak{g})_{\bar{0}}^{sp(2n)} = \mathbb{C}$. □

The filtered deformation corresponding to the non-trivial Spencer cocycle can be realized as follows. Consider the Lie superalgebra $H(2m, n+s)$, for $s \geq 1$, which we identify with $\Lambda(2m, n+s)/\mathbb{C}$, in the variables p_i, q_i, ξ_j, $i = 1, \ldots, m$, $j = 1, \ldots, n+s$, equipped with the Poisson bracket.

For $f, g \in \Lambda(2m, n)$ and $a, b \in \Lambda(s)$ the Lie bracket in $H(2m, n+s)$ can be written as

$$[f \otimes a, g \otimes b] = (-1)^{p(a)p(g)}\Big([f, g] \otimes ab + fg \otimes [a, b]\Big). \tag{3.1}$$

Let ϵ be a new (even) variable and f and g be homogeneous polynomials in the variables p_i, q_i, ξ_j, $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Consider the following degeneration of (3.1):

$$[f \otimes a, g \otimes b]_{\deg} = (-1)^{p(a)p(g)}\Big([f, g] \otimes ab + \lim_{\epsilon\to 0}\frac{fg(\epsilon p_i, \epsilon q_i, \epsilon \xi_j)}{\epsilon^{\max\{\deg f, \deg g\}}} \otimes [a, b]\Big).$$

One can verify directly that $[\cdot,\cdot]_{\deg}$ is precisely the Lie bracket on $H(2m,n)\otimes\Lambda(s)+H(0,s)$. Hence for $s>0$ $H(2m,n+s)$ is a filtered deformation of $H(2m,n)\otimes\Lambda(s)+H(0,s)$.

Finally by Proposition 2.4.4 of [CK2] $H(2n,0)\otimes\Lambda(1)+\mathbb{C}\frac{\partial}{\partial\xi}$ is the full prolongation of (I6) with $\mathfrak{a}=\mathbb{C}\frac{\partial}{\partial\xi}$ and hence combining this fact with Proposition 3.1 we have by Corollary 2.5 of [CK1] that $H(2n,1)$ in the above filtration is the unique non-trivial filtered deformation of $H(2n,0)\otimes\Lambda(1)+\mathbb{C}\frac{\partial}{\partial\xi}$.

4 (I15) has no non-trivial filtered deformations

Consider the Lie superalgebra $\mathfrak{g}=SKO(2,3;\beta)=\bigoplus_{j\geq -1}\mathfrak{g}_j$ in its subprincipal gradation. We regard $\mathfrak{g}$ as a subalgebra of $KO(2,3)$, which is identified with $\mathbb{C}[x_1,x_2,\xi_1,\xi_2,\tau]$ with reversed parity, equipped with the odd contact bracket. In $\mathfrak{g}_0$ we have a copy of $gl(2)$ spanned by $x_1\xi_2$, $x_2\xi_1$, $x_1\xi_1-x_2\xi_2$ and $\tau+\beta\Phi$. The $sl(2)$-module structure of $\mathfrak{g}_j$ are as follows:

$$
\begin{aligned}
(\mathfrak{g}_{-1})_{\bar 0} &= R(1), & (\mathfrak{g}_{-1})_{\bar 1} &= 2R(0),\\
(\mathfrak{g}_{0})_{\bar 0} &= R(2)\oplus R(0), & (\mathfrak{g}_{0})_{\bar 1} &= 2R(1),\\
(\mathfrak{g}_{j})_{\bar 0} &= R(j+2)\oplus R(j), & (\mathfrak{g}_{j})_{\bar 1} &= 2R(j+1).
\end{aligned}
$$

Furthermore we have $\Lambda^2(\mathfrak{g}_{-1}^*)_{\bar 0}=4R(0)$, and $\Lambda^2(\mathfrak{g}_{-1}^*)_{\bar 1}=2R(1)$. From this it follows that $sl(2)$-invariants of $\Lambda^2(\mathfrak{g}_{-1}^*)\otimes\mathfrak{g}$ can only occur in $\Lambda^2(\mathfrak{g}_{-1}^*)\otimes\mathfrak{g}_j$, $j=-1,0,1$. But the $sl(2)$-invariants in $\Lambda^2(\mathfrak{g}_{-1}^*)\otimes\mathfrak{g}_j$, $j=-1,1$ are all odd, hence they cannot give rise to filtered deformations. Thus we are left to consider the even $sl(2)$-invariants in $\Lambda^2(\mathfrak{g}_{-1}^*)\otimes\mathfrak{g}_0$. This is an 8-dimensional space, and it is easy to write down a linear basis for this space. We now use the action of $\tau+\beta\Phi$ on this space to determine the $gl(2)$-invariants. A simple calculation using the fact that $\beta\neq 0$ shows that only in the case when $\beta=-1$ can we have $gl(2)$-invariants. In this case the space of $gl(2)$-invariants is two-dimensional and it is spanned by

$$c_1=\xi_1^*\wedge\xi_2^*\otimes(\tau-\Phi),\quad c_2=(1^*\otimes(\xi_1\xi_2)^*)\otimes(\tau-\Phi).$$

Now we compute

$$
\begin{aligned}
(dc_1)(\xi_1,\xi_2,1)&=2, & (dc_1)(1,1,\xi_1\xi_2)&=0,\\
(dc_2)(\xi_1,\xi_2,1)&=0, & (dc_2)(1,1,\xi_1\xi_2)&=4.
\end{aligned}
$$

Hence no non-zero linear combination of c_1 and c_2 can give rise to a 2-cocycle. Thus we have the following.

Proposition 4.1. *In the subprincipal gradation for $\beta \neq 0$ we have*

$$H^{l,2}(SKO(2,3;\beta)_{-1}, SKO(2,3;\beta))_{\bar{0}}^{gl(2)} = 0, \quad l \geq 0.$$

Since by Theorem 2.6.1 of [CK2] $SKO(2,3;1-\frac{1}{\lambda})$, $\lambda \neq 0,1$, is the full prolongation of (I15) we conclude from Proposition 4.1, using Corollary 2.3 of [CK1], that $SKO(2,3;1-\frac{1}{\lambda})$ in the subprincipal gradation has no non-trivial filtered deformations.

5 (I20) has no non-trivial filtered deformations

As in (I15) one computes the $gl(3)$-module structure of $SKO(3,4;\frac{1}{3}) = \mathfrak{g} = \bigoplus_{j\geq -1} \mathfrak{g}_j$ in the subprincipal gradation. It is then straightforward to verify directly that the center of $gl(3)$, given by the element $\tau + \frac{1}{3}\Phi$, acts non-trivially on $(\Lambda^2(\mathfrak{g}_{-1}^*) \otimes \mathfrak{g})_{\bar{0}}$. Since the calculation is quite similar to that of (I15), we omit the details. From this we obtain the following.

Proposition 5.1. *In the subprincipal gradation we have*

$$H^{l,2}\left(SKO\left(3,4;\frac{1}{3}\right)_{-1}, SKO\left(3,4;\frac{1}{3}\right)\right)_{\bar{0}}^{gl(3)} = 0, \quad l \geq 0.$$

Since by Remark 0.2 $SKO(3,4;\frac{1}{3})$ is the full prolongation of (I20), it follows that $SKO(3,4;\frac{1}{3})$ has no non-trivial filtered deformations.

6 (J8a) has no non-trivial filtered deformations

Proposition 6.1. *In the subprincipal gradation we have*

$$H^{l,2}(H(2n,2)_-, H(2n,2))_{\bar{0}}^{\mathfrak{g}_0} = 0, \quad l \geq 0,$$

where $H(2n,2)_- = H(2n,2)_{-2} \oplus H(2n,2)_{-1}$.

Proof. We identify $H(2n,2)$ with $\mathbb{C}[p_i, q_i, \xi_1, \xi_2]/\mathbb{C}1$, $i = 1, \ldots, n$. We have $\mathfrak{g} = \bigoplus_{j\geq -2} \mathfrak{g}_j$. $\mathfrak{g}_0$ is spanned by vectors of the form $p_i q_j$, $p_i q_j \xi_1$, $\xi_2 \xi_1$ and ξ_2 and hence $\mathfrak{g}_0 \cong sp(2n) \otimes \Lambda(1) + W(0,1)$. Thus $\mathfrak{g}_0$ contains $csp(2n)$. As a $\mathfrak{g}_0$-module the other graded components are as follows:

$$\mathfrak{g}_{-2} = R(0), \quad \mathfrak{g}_{-1} = R(\pi_1) \otimes \Lambda(1),$$
$$\mathfrak{g}_j = (R((j+2)\pi_1) \otimes \Lambda(1)) + (R(j\pi_1) \otimes W(1)),$$

Here as usual $R(\pi_1)$ denotes the standard module of $sp(2n)$ etc. Let $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ and consider $\Lambda^2(\mathfrak{g}_-^*)$. The $sp(2n)$-module structure of $\Lambda^2(\mathfrak{g}_-^*)$ is not hard to find, and from this one obtains that the even $csp(2n)$-invariants of $\Lambda^2(\mathfrak{g}_{-2}^*) \otimes \mathfrak{g}$ form a 3-dimensional vector space spanned by the following basis vectors:

$$
\begin{aligned}
c_1 &= \left(\sum_{i=1}^n p_i^* \wedge q_i^* \right) \otimes \xi_1 \xi_2, \\
c_2 &= \sum_{i=1}^n (p_i^* \otimes \xi_1^*) \otimes \tilde{p}_i + (q_i^* \otimes \xi_1^*) \otimes \tilde{q}_i, \\
c_3 &= \sum_{i \leq j} (p_i^* \otimes \tilde{p}_j^* + p_j^* \otimes \tilde{p}_i^*) \otimes \widetilde{p_i p_j} + (q_i^* \otimes \tilde{q}_j^* + q_j^* \otimes \tilde{q}_i^*) \otimes \widetilde{q_i q_j} \\
&\quad + \sum_{i,j} (p_i^* \otimes \tilde{q}_j^* + q_j^* \otimes \tilde{p}_i^*) \otimes \widetilde{p_i q_j}.
\end{aligned}
$$

Here $\tilde{p}_i = p_i \otimes \xi_1$ etc. and $*$ denotes taking the dual as usual. Now we compute

$$
\begin{aligned}
&(dc_1)(p_1, q_1, \xi_1) = \xi_1, \quad (dc_1)(p_1, q_1, \tilde{p}_1) = \tilde{p}_1, \quad (dc_1)(p_1, p_2, \tilde{q}_1) = 0, \\
&(dc_2)(p_1, q_1, \xi_1) = -2\xi_1, \quad (dc_2)(p_1, q_1, \tilde{p}_1) = 0, \quad (dc_2)(p_1, p_2, \tilde{q}_1) = 0, \\
&(dc_3)(p_1, q_1, \xi_1) = 0, \quad (dc_3)(p_1, q_1, \tilde{p}_1) = -5\tilde{p}_1, \quad (dc_3)(p_1, p_2, \tilde{q}_1) = -\tilde{p}_2.
\end{aligned}
$$

It follows that no non-zero linear combination of c_1, c_2 and c_3 can be a 2-cocycle. □

Now by Lemma 3.3.2 of [CK2] $H(2n, 2)$ in its subprincipal gradation is the full prolongation of (J8a). Hence it follows from Corollary 2.3 of [CK1] that $H(2n, 2)$ in the subprincipal gradation has no non-trivial filtered deformations.

7 (C5) and (C7) have no non-trivial filtered deformations

Consider the case of (C5) so that the associated graded $\mathrm{Gr}L = SHO(3,3) + sl_2$. Consider the decomposition $L = \prod_{j=-2} \mathfrak{m}_j$ as an $sl_3 \oplus sl_2$-module as in (6.1) of [K]. The module structure of each component $\mathfrak{m}_j$ is easily written down explicitly. In particular we have $\mathfrak{m}_{-1} = \mathbb{C}^3 \boxtimes \mathbb{C}^2$. It can be verified that $[\mathfrak{m}_{-1}, \mathfrak{m}_{-1}] = \mathfrak{m}_{-2}$ and $[\mathfrak{m}_1, \mathfrak{m}_{-1}] \subseteq \mathfrak{m}_0$. We can now apply Lemma 6.2 of [K] to conclude that $L \cong SHO(3,3) + sl_2$.

Now if $\mathrm{Gr}L = \mathbb{C}^2 + SHO(3,3) + sl_2$, and $L = \prod_{j=-3} \mathfrak{m}_j$ is the decomposition of $sl_3 \oplus sl_2$-modules, then one finds that in addition to $[\mathfrak{m}_{-1}, \mathfrak{m}_{-1}] = \mathfrak{m}_{-2}$ and $[\mathfrak{m}_1, \mathfrak{m}_{-1}] \subseteq \mathfrak{m}_0$, one has also $[\mathfrak{m}_{-1}, \mathfrak{m}_{-2}] = \mathfrak{m}_{-3}$. So Lemma 6.2 of [K] is applicable and we conclude that $\mathbb{C}^2 + SHO(3,3) + sl_2$ has no non-trivial filtered deformations.

Thus the only additional non-trivial filtered deformation of $\mathbb{Z}$-graded Lie superalgebras that occur in [CK2] and [K] (including the missing case (I20)) is $H(2n, 1)$. This produces no new simple linearly compact Lie superalgebras.

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