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Large Volume Perspective on Branes at Singularities

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Abstract

In this paper we consider a somewhat unconventional approach for deriving worldvolume theories for D3 branes probing Calabi-Yau singularities. The strategy consists of extrapolating the calculation of F-terms to the large volume limit. This method circumvents the inherent limitations of more traditional approaches used for orbifold and toric singularities. We illustrate its usefulness by deriving quiver theories for D3 branes probing singularities where a Del Pezzo surface containing four, five or six exceptional curves collapses to zero size. In the latter two cases the superpotential depends explicitly on complex structure parameters. These are examples of probe theories for singularities which can currently not be computed by other means.

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1 Overview

The description of certain extended objects in string theory as Dirichlet branes [13] has given us a new window into the physics of highly curved geometries. The worldvolume theory of a D-brane probing an orbifold singularity at low energies is described by a gauge theory of quiver type [14]. The correspondence goes both ways; on the one hand, we can access distance scales that are shorter than the scales we can probe with closed strings. On the other hand, one may view this as a way of engineering interesting gauge theories in string theory, and then using string dualities we can sometimes learn new things about gauge theories. Let us focus on D3 branes probing

Calabi-Yau three-fold singularities, which give rise to $\mathcal{N}=1$ gauge theories in 3+1 dimensions.

Orbifolds of flat space constitute an interesting and computationally convenient set of singular space-times, and have given us a nice picture of the probe brane splitting up into fractional branes near the singularity[1]. But they are certainly not the most general type of singularity and do not lead to the most general allowed gauge theory. Another class is that of toric singularities, which contains the class of abelian orbifold singularities. These backgrounds are described by toric geometry (or linear sigma models). Adding some probe branes filling the remaining transverse dimensions leads to SUSY quiver theories where the ranks of the gauge groups are all equal and the superpotential can be written in such a way that no matter field appears more than twice. The prime example of this type of singularity is the conifold.

The way one usually analyses toric singularities is by embedding them in an orbifold singularity, which we know how to deal with [15]. Then one may partially resolve the singularity in a toric way to get the desired space. In the gauge theory this corresponds to giving large VEVs to certain fields and integrating out the very massive modes. At sufficiently low energies the gauge theory only describes the local neighbourhood of the toric singularity.

This however does not yet exhaust the class of allowed Calabi-Yau singularities or $\mathcal{N}=1$ gauge theories. We refer to the remaining cases as non-toric singularities.

In this article we would like to attack these cases by using an approach different from the ones we have mentioned above. The strategy consists of extrapolating to large volume where the correct description of the fractional branes is in terms of certain collections of sheaves[2], called exceptional collections. The matter content and superpotential can be completely determined by geometric calculations involving these collections, and the results are not affected by the extrapolation. In principle this yields a general approach to all Calabi-Yau singularities, both toric and non-toric.

In the next section we will review many of the ingredients of this approach[3]. In the following section we will apply the techniques to compute the quiver theories for branes probing certain Calabi-Yau geometries where a four-cycle collapses to zero size. If the singularity can be resolved by a single blow-up, such a four-cycle is either a $\mathbf{P}^1$ fibration over a Riemann

[1]These are the analogues of wrapped branes when we resolve the singularity and go to large volume.

[2]For our purposes these will mostly be ordinary vector bundles, even line bundles.

[3]Some references: [16],[19],[18],[1],[7],[12]

surface or a Del Pezzo surface. We will treat some toric and non-toric Del Pezzo cases in detail, computing their moduli spaces and showing how they are related to known quiver theories through the Higgs mechanism. The non-toric cases considered here are new and there is no other method we know of for calculating them. The case of Del Pezzo 6 in particular can be viewed as a four parameter non-toric family of deformations of the familiar $\mathbf{C}^3/Z_3 \times Z_3$ orbifold singularity.

In order to do the computations we introduce the notion of a 'three-block' quiver diagram in section three. Quiver diagrams of this type have some important simplifying features. Most importantly for our purposes is that they they lead to superpotentials with only cubic terms for all the Del Pezzo cases. Other quiver theories may then be deduced through Seiberg duality.

2 Large volume perspective

2.1 Relation to exceptional collections

We will start with a non-compact Calabi-Yau threefold which has a 4-cycle or 2-cycle shrinking to zero size at a singularity. Then we may put a set of D3 branes at the singularity and try to understand the low energy worldvolume theory on the D3 branes, which turns out to be an $\mathcal{N} = 1$ quiver gauge theory. The analysis may be done using exceptional collections living on the shrinking cycle.

We recall first the definition of an exceptional collection. An exceptional sheaf is a sheaf E such that dim $\mathrm{Hom}(E, E) = 1$ and $\mathrm{Ext^k(E, E)}$ is zero whenever $k > 0$. An exceptional collection is an ordered collection $\{E_i\}$ of exceptional sheaves such that $\mathrm{Ext}^k(E_i, E_j) = 0$ for any k whenever $i > j$. We will usually assume an exceptional collection to be complete in the sense that the Chern characters of the sheaves in the collection generate all the cohomologies of the shrinking cycle [4].

A sheaf living on a cycle inside the Calabi-Yau gives rise to a sheaf on the Calabi-Yau by "push-forward." Heuristically this just means that we take the sheaf on the cycle and extend it by zero to get a sheaf denoted by i_*E on the CY. An exceptional collection lifts to a set of sheaves with "spherical"

[4]A more precise definition would probably be that the collection generates the bounded derived category of coherent sheaves.

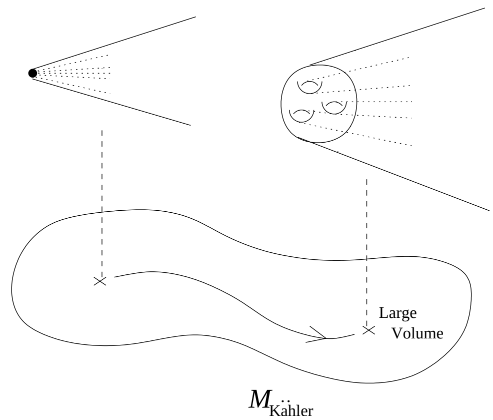

Figure 1: Extrapolation to large volume in order to perform calculations.

cohomology on the Calabi-Yau: because of the formula

$$\mathrm{Ext}^k(i_* E_i, i_* E_j) \sim \sum_{l+m=k} \mathrm{Ext}^l(E_i, E_j \otimes \Lambda^m N) \tag{2.1}$$

where N is the normal bundle to the cycle we conclude that the cohomologies of an exceptional sheaf are given by $\dim \mathrm{Ext}^k(i_* E, i_* E) = \{1, 0, 0, 1\}$ for $k = \{0, 1, 2, 3\}$. The reason for the terminology is that under mirror symmetry such sheaves get mapped to Lagrangian 3-spheres. The ordering of the sheaves is related to the ordering with respect to the imaginary coordinate on the W-plane.

The significance of exceptional collections is that the sheaves in the collection have the right properties to be the large volume descriptions of the fractional branes at the singularity. Since we are in type IIB string theory, D-branes filling the 3+1 flat directions must wrap even dimensional cycles in the CY, and therefore strings ending on such branes satisfy boundary conditions that allow for a B-type topological twist. Correlation functions in the twisted theory are independent of the choice of Kähler structure on the Calabi-Yau, and correspond to superpotential terms in the untwisted theory. The upshot is that Kähler parameters only manifest themselves in the D-terms of the gauge theory. So as long as we are asking questions only about the F-terms (superpotential terms), we can do a Kähler deformation to give the vanishing cycle a finite volume. At large volume we can describe D-branes as sheaves and the computations can be done using exceptional collections. It is important to remember that the gauge theory we will be discussing doesn't actually live at large volume, but using the justification above we will interpret the results of the geometric computations as protected quantities in a gauge theory which is a valid low energy description of the D3 branes at small volume.

2.2 Spectrum and quiver diagram

In order to describe the gauge theory on the D3 branes we need to construct the D3 brane out of the fractional branes and find the lightest modes. A D3 brane filling the 3+1 flat directions is determined by specifying a point p on the Calabi-Yau, so we will use a skyscraper sheaf $\mathcal{O}_p$ to represent it. The RR charges of a brane are combined in the Chern character of a sheaf, so if $\{E_i\}$ is an exceptional collection and n_i the multiplicities of the fractional branes then we have the condition:

$$\sum_i n_i \operatorname{ch}(i_* E_i) = \operatorname{ch}(\mathcal{O}_p). \tag{2.2}$$

Notice that some of the n_i are necessarily negative, because we have to cancel all the charges associated with wrappings of 2- and 4-cycles. So it may appear that we have both "branes" and "anti-branes" present and therefore break supersymmetry, however this is just an artefact of the large volume description. As we vary the Kähler moduli to make the Calabi-Yau singular the central charges[5] of the fractional branes all line up and thus they all break the same half of the supersymmetry at the point of interest[6].

Now the lightest modes for strings with both endpoints on the same fractional brane fit in an $\mathcal{N}=1$ vector multiplet. Since the fractional brane is rigid ($\operatorname{Ext}^1(E,E)=0$ from the sheaf point of view) we do not get any adjoint chiral multiplets describing deformations from these strings. If the fractional brane has multiplicity $|n_i|$ then the vector multiplet will transform in the adjoint of $U(|n_i|)$.

We can also have strings with endpoints on different fractional branes E_i and E_j. The lightest modes of these strings are $\mathcal{N}=1$ chiral fields transforming in the bifundamental of the gauge groups associated with the two fractional branes. To count their number we can do a computation in the B-model which we can perform at large volume. If the branes fill the whole Calabi-Yau then the ground states of these strings can be seen to arise from the cohomology of the Dolbeault operator coupled to the gauge fields of the two fractional branes, $Q = \bar{\partial}_{\bar{z}} + A^{(j)}_{\bar{z}} - A^{(i)T}_{\bar{z}}$, acting on the space of anti-holomorphic forms $\Omega^{(0,\cdot)}(E_i^* \otimes E_j)$. If the branes are wrapped

[5] The central charge is associated with a particle, not with a space-filling brane, but the two systems are closely related as far as SUSY properties is concerned so we will borrow the language.

[6] A nice picture of this for the case of $\mathbf{C}^3/Z_3$ can be found in [12].

on lower dimensional cycles then we can dimensionally reduce by changing the gauge fields with indices that do not lie along the worldvolume of the branes into normal bundle valued scalars. Schematically then the number of chiral fields is in one to one correspondence with the generators of the sheaf cohomology groups $H^{(0,m)}(E_i^* \otimes E_j \otimes \Lambda^n N)$ where N is the normal bundle of the shrinking cycle that both branes are wrapped on, or more generally the global Ext groups $\mathrm{Ext}^k(i_* E_i, i_* E_j)$ [8]. However we are double counting because given a generator we can get another one by applying Serre duality on the three-fold. As we discuss momentarily the corresponding physical mode would have opposite charges and opposite chirality, so this should be part of the vertex operator for the corresponding anti-particle. Then we should have only one chiral field for each pair of generators that are related by Serre duality. Finally the degree of the Ext group indicates ghost number k of the topological vertex operator. In the physical theory this gets related to the chirality of the multiplet through the GSO projection. If the degree is even we get say a left handed fermion so we should assign fundamental charges for the jth gauge group and anti-fundamental charges of the ith gauge group to the chiral field. Then if k is odd we get a right handed fermion and we should assign the opposite charges to the chiral field. The chirality flips if we turn a brane the string ends on into an anti-brane (or more precisely if we invert its central charge) because this shifts k by an odd integer.

We may summarise this spectrum in a quiver diagram. For each fractional brane E_i we draw a node which corresponds to a vector multiplet transforming in the adjoint of $U(|n_i|)$ (or $U(|n_i N|)$ if we started with N D3 branes). For each chiral multiplet in the fundamental of $U(|n_j|)$ and in the anti-fundamental of $U(|n_i|)$ we draw an arrow from node i to node j. Let us illustrate this in the example of $\mathbf{C}^3/Z_3$ where we can do a Kähler deformation to get a finite size $\mathbf{P}^2$ (this example was treated in [1],[12]). We'll take the exceptional collection on $\mathbf{P}^2$ to be the set $\{\mathcal{O}(-1), \Omega^1(1), \mathcal{O}(0)\}$. By $\Omega^1(1)$ we mean the cotangent bundle, tensored with the line bundle $\mathcal{O}(1)$. To reproduce the charge of a D3 brane the multiplicities will have to be $\{1, -1, 1\}$. Then we will draw three nodes with a gauge group $U(N)$ for each. The non-zero Ext's are found to be

$$\dim \mathrm{Ext}^0(E_1, E_2) = 3, \quad \dim \mathrm{Ext}^0(E_2, E_3) = 3, \quad \dim \mathrm{Ext}^0(E_1, E_3) = 3 \tag{2.3}$$

with everything else either zero or related by Serre duality. Drawing in the arrows with the proper orientations gives the following quiver diagram:

If one is only interested in the net number of arrows between two nodes, it is sufficient to find the relative Euler character of the two corresponding

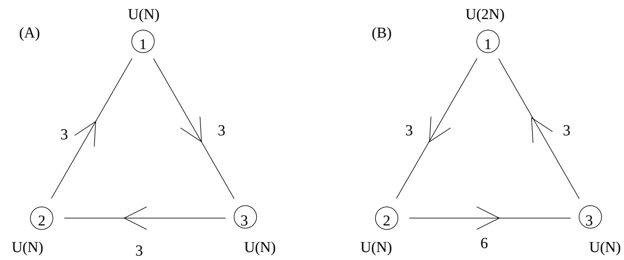

Figure 2: (A): Quiver for the orbifold $\mathbf{C}^3/Z_3$. (B): Seiberg dual quiver, after integrating out massive modes.

sheaves

$$\chi(i_*E, i_*F) = \sum_k (-1)^k \dim \operatorname{Ext}^k(i_*E, i_*F). \tag{2.4}$$

This number is frequently easier to compute than the actual number of arrows.[7] Now notice that a brane can only have non-zero intersection number with something that is localised at a point on the Calabi-Yau if it fills the whole Calabi-Yau. This does not happen for fractional branes, which are localised around the cycle that shrinks to zero. Hence we deduce that

$$\chi(\mathcal{O}_p, i_*E_j) = \sum_i n_{ij} n_i = 0 \tag{2.5}$$

where n_{ij} is the net number of arrows from E_i to E_j. In terms of field theory equation (2.5) states that the number of arrows directed towards node j exactly matches the number of arrows pointing away from it, or that the number of quarks for the jth gauge group is the same as the number of anti-quarks. We are therefore guaranteed that gauge anomalies cancel [1].[8]

2.3 Superpotential

So far we have only indicated how to obtain the matter content of the gauge theory, but we know for instance from the $\mathbf{C}^3/Z_3$ orbifold that we should also expect a superpotential. Another way to see this is by observing that the moduli space of the gauge theory should be the moduli space of the sheaf $\mathcal{O}_p$, which certainly contains the Calabi-Yau three-fold itself. We can only recover this by imposing additional constraints from a superpotential on the

[7] For instance for the case of exceptional sheaves E and F on Del Pezzo surfaces that we will discuss later on, the number of arrows is simply given by $r_E\, d_F - r_F\, d_E$ where r is the rank of the sheaf and d is the degree (i.e. the intersection with the canonical class K).

[8] At least for the non-abelian factors; for the $U(1)$'s there's mixing with closed string modes, see [22].

VEVs. As we have discussed above, the superpotential is also a piece of data we can compute at large volume.

The cubic terms in the superpotential can be computed with relative ease. Suppose we want to know if a cubic coupling for the chiral fields running between nodes i,j and k. Then after picking Ext generators which correspond to the chiral fields we may compute the Yoneda pairings

$$\begin{aligned} \mathrm{Ext}^l(i_*E_i, i_*E_j) \times \mathrm{Ext}^m(i_*E_j, i_*E_k) \times & \mathrm{Ext}^n(i_*E_k, i_*E_i) \\ & \to \mathrm{Ext}^{l+m+n}(i_*E_i, i_*E_i). \end{aligned} \tag{2.6}$$

If $l + m + n = 3$ then we may use the fact that dim $\mathrm{Ext}^3(E, E) = 1$ on a CY three-fold to get a number from the above composition. This number is the coefficient of the cubic coupling in the superpotential. If $l + m + n \neq 3$ then the cubic coupling is automatically zero. One may justify this claim by looking at a disc diagram with three vertex operators inserted at the boundary. Eg. the cubic part of a superpotential term in the action of the form

$$\mathrm{Tr}\left(\frac{\partial^2 W(\phi)}{\partial \phi^2}\psi\psi\right) \tag{2.7}$$

is computed in string theory by a disc diagram with two fermion vertex operators and one scalar vertex operator on the boundary. This is just the type of diagram that can be computed in the topologically twisted theory

$$\mathrm{Tr}\ \langle 0 \rangle V_\phi V_\psi V_{\psi\,\mathrm{untwisted}} \sim \langle 0 \rangle V^{(1)}_{NS} V^{(2)}_{NS} V^{(3)}_{NS\,\mathrm{twisted}} \tag{2.8}$$

where the $V^{(i)}$ are the internal part of the full physical vertex operators, with some spectral flow applied if the internal part was in the RR sector. At large volume the $V^{(i)}_{NS}$ can be represented as generators of the sheaf cohomology from which the physical fields descended. In the B-model the amplitude (2.8) just gives the overlap of the vertex operators and vanishes unless the total ghost number from the three vertex operators adds up to three.

For $\mathbf{P}^2$ one finds the following generators [1],[12]:

$$\begin{aligned} (C_3 z_2 - C_2 z_3) dz_1 + (C_1 z_3 - C_3 z_1) dz_2 + & (C_2 z_1 - C_1 z_2) dz_3 \\ & \in \mathrm{Hom}(\mathcal{O}(-1), \Omega^1(1)), \\ A_1 \frac{\partial}{\partial z_1} + A_2 \frac{\partial}{\partial z_2} + A_3 \frac{\partial}{\partial z_3} & \in \mathrm{Hom}(\Omega^1(1), \mathcal{O}), \\ B_1 z_1^* + B_2 z_2^* + B_3 z_3^* & \in \mathrm{Hom}^*(\mathcal{O}(-1), \mathcal{O}). \end{aligned} \tag{2.9}$$

The star indicates we have dualised the Hom, making it an Ext^3 on the CY by Serre duality. Computing the Yoneda pairings gives the usual orbifold superpotential

$$W = \mathrm{Tr}\left(A_1(B_2C_3 - B_3C_2) + A_2(B_3C_1 - B_1C_3) + A_3(B_1C_2 - B_2C_1)\right) \tag{2.10}$$

It turns out that for a clever choice of exceptional collections for Del Pezzo surfaces, namely the ones with "three-block" structure [2], the superpotential only has cubic couplings. So for our purposes we do not have to worry about higher order terms in the superpotential. However let us briefly discuss how one might go about computing the higher order terms [1].[9] Suppose we are interested in the coefficient of a possible quartic term, say involving nodes i, j, k and l. Then we would like to compute an amplitude of the form

$$\langle 0 \rangle V_{NS}^{(1)} V_{NS}^{(2)} V_{NS}^{(3)} \int V_{NS}^{(4)} \tag{2.11}$$

Here we have assumed that the operator $V_{NS}^{(4)}$ comes from an Ext^1, so its ghost number is equal to 1 and can be integrated over the boundary between the points where $V_{NS}^{(3)}$ and $V_{NS}^{(1)}$ are inserted. Suppose that instead we consider the following amplitude:

$$\langle 0 \rangle V_{NS}^{(1)} V_{NS}^{(2)} V_{NS}^{(3)} \exp\left[t \int V_{NS}^{(4)}\right] = \langle 0 \rangle \tilde{V}_{NS}^{(1)} V_{NS}^{(2)} \tilde{V}_{NSt}^{(3)} \tag{2.12}$$

In other words we use the operator $V_{NS}^{(4)}$ to deform the sigma model action. Then we may recover (2.11) by differentiating with respect to t. If $V_{NS}^{(4)}$ comes from $\mathrm{Ext}^1(E_i, E_j)$ then this deformation creates a new boundary condition corresponding to the sheaf F, where F is the deformation of $E_i \oplus E_j$ defined by the extension class. In gauge theory terms we are Higgsing down the quartic term to get a cubic term. Now we may proceed as before and compute a cubic coupling using F, E_k and E_l.[10]

Finally to completely specify the gauge theory we must also supply the Kähler terms. We will not have anything to say about this, but one should keep in mind that we are considering the far IR physics which is expected to give rise to an interacting superconformal theory based on ADS/CFT arguments [15]. In this case the Kähler terms are quite possibly completely fixed in terms of F-term data by superconformal invariance, as they are thought to be for two-dimensional superconformal theories.

[9] Apart from a direct computation in holomorphic Chern-Simons theory [16], which is usually quite hard.

[10] In general when we have Ext^n with $n \neq 1$, there still exists a deformation but it does not give a sheaf but a complex of sheaves; it should still be possible to do some calculation in this case. In the disc diagram we should have SL(2,R) invariance so it seems somewhat peculiar that the Ext^1's are singled out.

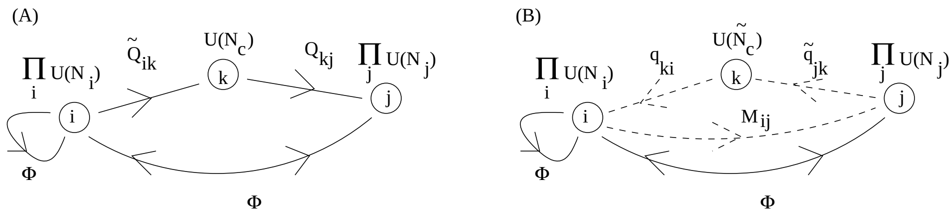

Figure 3: (A): Organising the nodes before applying Seiberg duality to node k. (B): Seiberg dual theory.

2.4 Mutations and Seiberg duality

In our discussion up till now we have suppressed the fact that there actually are many choices of exceptional collections. Indeed there is a general procedure for making new exceptional collections from old ones, called mutation. At the level of Chern characters this transformation looks very familiar. Given an exceptional collection $\{\ldots, E_{i-1}, E_i, E_{i+1}, E_{i+2}, \ldots\}$ we can make a new "left mutated" collection[11]

$$\{\ldots, E_{i-1}, L_{E_i}E_{i+1}, E_i, E_{i+2}, \ldots\} \tag{2.13}$$

where the Chern character of the new sheaf $L_{E_i}E_{i+1}$ is given by[12]

$$\mathrm{ch}(L_{E_i}E_{i+1}) = \pm\left[\mathrm{ch}(E_{i+1}) - \chi(E_i, E_{i+1})\mathrm{ch}(E_i)\right]. \tag{2.14}$$

This looks very similar to (and is in fact mirror to) Picard-Lefschetz monodromy. One may construct the sheaf $L_{E_i}E_{i+1}$ out of E_i and E_{i+1} using certain exact sequences.

For a certain class of mutations the corresponding transformation on the quiver gauge theory was interpreted as Seiberg duality [1]. Before we delve into this let us first comment on the abelian factors in the gauge groups. There is an overall factor of $U(1)$ which completely decouples and then there are relative $U(1)$'s between the nodes which become weakly coupled in the infrared. Seiberg duality is a statement about the infrared behaviour of the non-abelian part of the gauge theory, which can get strongly coupled. So when we discuss a quiver diagram with $U(N)$ gauge groups we would like to think of the $SU(N)$ parts as dynamical gauge groups and the $U(1)$'s as global symmetries, or alternatively reinstate the $U(1)$'s as gauge symmetries only at "intermediate" energy scales.

[11] There is also a right shifted version $\{\ldots, E_i, R_{E_i}E_{i-1}, E_{i+1}, \ldots\}$.

[12] When we apply this formula in the next section we will deduce the sign using charge conservation.

To see which class of mutations we need to look at let us suppose we want to do a Seiberg duality on node k and organise the quiver so that all the incoming arrows (labelled by i) are to the left of node k and all the outgoing arrows (labelled by j) to the right of node k as in Figure 2.4 A. We think $SU(N_k)$ as the colour group which gets strongly coupled and treat all the other nodes as flavour groups. The total number of flavours is

$$N_f = \sum_i n_{ik} N_i = \sum_j n_{kj} N_j \tag{2.15}$$

where the second equality comes from anomaly cancellation as we discussed above. In the Seiberg dual theory the quarks and anti-quarks $\{\tilde{Q}_{ik}, Q_{kj}\}$ are replaced by dual quarks $\{q_{ki}, \tilde{q}_{jk}\}$, which means we have to reverse the arrows going into and coming out of node k in the quiver diagram. We also have to add mesons M_{ij} between the nodes on the left of k and the nodes on the right of k. In the original theory these mesons are bound states of the quarks, $M_{ij} \sim \tilde{Q}_{ik} Q_{kj}$, but they become fundamental in the dual theory. Finally we have to replace the original gauge group $SU(N_c) = SU(N_k)$ on node k by the dual gauge group $SU(\tilde{N}_c) = SU(N_f - N_c)$. If the superpotential for the original theory is $W(\tilde{Q} \cdot Q, \Phi)$ then the superpotential for the dual theory is

$$W_{\text{dual}}(\tilde{q} \cdot q, M, \Phi) = W_{\text{orig}}(M, \Phi) + \lambda \ \text{Tr}(M \tilde{q} q). \tag{2.16}$$

Since we are only interested in F-terms, the coupling constant λ is not relevant for our purposes and can be scaled away.

This transformation can be realised in geometry by doing a mutation by E_k on either (1) all the sheaves E_i to the left of k or (2) all the sheaves E_j to the right of k [1].[13] Let us illustrate this for the case of $\mathbf{P}^2$. Starting with the collection $\{\mathcal{O}(-1), \Omega^1(1), \mathcal{O}(0)\}$ we may do a Seiberg duality on node 1 by going to the mutated collection

$$\{L_{\mathcal{O}(-1)}\Omega^1(1), \mathcal{O}(-1), \mathcal{O}(0)\} = \{\mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}(0)\}. \tag{2.17}$$

Note that in Figure 2.2 B we label the nodes to be consistent with the original quiver Figure 2.2 A, not according to the order of the exceptional collection. Now for this new collection the matter fields descend from monomials on $\mathbf{P}^2$:

$$\begin{aligned} A_i z^i &\in \text{Hom}(\mathcal{O}(-2), \mathcal{O}(-1)) \\ B_j z^j &\in \text{Hom}(\mathcal{O}(-1), \mathcal{O}(0)) \\ C^*_{ij} z^i z^j &\in \text{Hom}(\mathcal{O}(-2), \mathcal{O}(0)). \end{aligned} \tag{2.18}$$

[13] Some progress has been made towards the understanding of more general mutations [20], though as we will note in section 3 one doesn't necessarily expect an arbitrary mutation to give rise to a good field theory.

Here C_{ij} is symmetric in i and j, so there are only six fields, not nine. From the Yoneda pairings we deduce the superpotential

$$W = \mathrm{Tr} \sum_{i,j} C_{ij} A_i B_j \tag{2.19}$$

An easy calculation shows that this agrees with the Seiberg dual of Figure 2.2 A after integrating out the massive fields, which are irrelevant in the infrared. We create nine new mesons between nodes 2 and 3 but three of these mesons pair up with fields from the original theory, leaving six massless fields.

A crucial property of Seiberg duality is that it preserves the moduli space and the ring of chiral operators of the theory. The moduli space is parametrised by the gauge invariant operators modulo algebraic relations and relations coming from the superpotential. We may distinguish between the mesonic operators which are products of "quarks" and "anti-quarks" (i.e. they are gauge invariant because fundamentals are tensored with anti-fundamentals) and the baryonic operators which are gauge invariant because they contain an epsilon-tensor. In this article we will investigate the part of the moduli space parametrised by the mesonic operators. These moduli describe the motion of the branes in the background geometry. In particular for a single probe brane we expect to recover the background geometry itself, perhaps with some additional branches if the singularity is not isolated. The baryonic operators describe partial resolutions of the background (coming from twisted sector closed strings in the case of orbifolds), and we will make use of them for comparing the new quiver theories we write down to known quivers in the partially resolved background.

3 Superpotentials for toric and non-toric Calabi-Yau singularities

In light of the approach sketched in the previous section, the first step in writing down the quiver theory for branes at a singularity is identifying a suitable exceptional collection of sheaves at large volume. The Calabi-Yau three-fold singularities we would like to consider here are complex cones over Del Pezzo surfaces (though we will also make some comments on the conifold). This just means that we take the equations defining a Del Pezzo surfaces living in projective space, and regard the same equations as defining a variety over affine space, which gives a cone over the surface with a singularity at the origin where the 4-cycle shrinks to zero size.

Since the Del Pezzo's can be obtained from $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$ by blowing up generic points, it would be nice to have a simple way to construct exceptional sheaves on the blow-up from exceptional sheaves on the original variety. This is indeed possible; we can start for instance with the exceptional collection $\{\mathcal{O}(-1), \mathcal{O}(0), \mathcal{O}(1)\}$ on $\mathbf{P^2}$, and let σ denote the map from the nth Del Pezzo to $\mathbf{P^2}$ that blows down the exceptional curves E_1 through E_n. Then an exceptional collection on the nth Del Pezzo is given by $\{\sigma^*\mathcal{O}(-1), \sigma^*\mathcal{O}(0), \sigma^*\mathcal{O}(1), \mathcal{O}_{E_1}, \ldots, \mathcal{O}_{E_n}\}$ [9].[14] However this collection is probably not very useful for constructing quiver theories. The reason is that the $\mathcal{O}_{E_i}$ are needed to recover the full moduli space, but since only $\mathcal{O}_{E_i}$ carries charge for the E_i cycle and such charges must cancel, it follows that it must appear both as a brane and an anti-brane in the quiver, thereby at the least breaking supersymmetry.

Other exceptional collections are related by mutation, so we can take the exceptional collection mentioned above as a starting point and try to get better behaved collections. A list of some exceptional collections obtained in this way was given in [2]. If we group all the sheaves that have no relative cohomologies between them together in a single "block", then the exceptional collections written down in [2] only consist of three blocks. Such collections have many useful properties and provide the easiest way to access the Del Pezzo cases; eg. the matter fields are always given by Ext^0's (corollary 3.4 in [2]) and the expected superconformal invariance of the gauge theory restricts the superpotential to be purely cubic. These properties simplify the computations significantly. The allowed three-block collections were completely determined in [2] by studying a Markov-type equation

$$\alpha x^2 + \beta y^2 + \gamma z^2 = \sqrt{\alpha\beta\gamma K^2}\, xyz. \tag{3.1}$$

Here K^2 is the degree of the Del Pezzo (which is 9 minus the number of points blown up), α, β and γ are the numbers of exceptional sheaves in each of the three blocks, and x, y and z are the ranks of the sheaves in each block (which one can prove to be equal within each block). This equation is invariant under "block" Seiberg dualities and is a special case for three-block collections of a Markov equation for more general exceptional collections which is invariant under arbitrary Seiberg dualities [25]. Equation (3.1) is also satisfied if one takes x, y and z to be the ranks of the gauge groups within each block, provided the left hand side of the equation is multiplied by the number N of D3 branes we are describing. These two possibilities are related by three "block" Seiberg dualities [2]. We would like to mention that (3.1) has been interpreted as a consistency condition for the vanishing of NSVZ beta-functions in [26].

[14] We thank R. Thomas for pointing out this reference.

The nth Del Pezzo has a discrete group of global diffeomorphisms isomorphic to the exceptional group E_n. It is known that the induced action of E_n on the quiver diagram yields a symmetry of the diagram [2]. It might be interesting to understand what kind of constraints it puts on the superpotential.

In the following sections we will display our quiver diagrams in "block" notation in order to avoid cluttering the pictures with arrows. When nodes are grouped together into a single block, there are no arrows between them. Also, an arrow from one group to another group signifies an arrow from each node in the first group to each node in the second group. In the remainder we will also leave out the overall trace of the superpotentials, it being understood that this has to be added back to get a gauge invariant expression.

3.1 Del Pezzo 3

In this section we use the approach based on exceptional collections to construct a quiver gauge theory for a local DP3. We check the answer we get by Seiberg dualising to a known quiver gauge theory that was constructed by toric methods.

We choose to use the exceptional collection that was found in [2]. It is a particularly nice collection because it consists only of line bundles and because it gives rise to a superpotential with only cubic couplings. To introduce it let us first fix some notation. DP3 can be obtained as a blow-up of $\mathbf{P}^2$ at three generic points. The divisor classes are then linear combinations of the three exceptional curves E_1, E_2 and E_3, and the hyperplane class H, which satisfy the following relations:

$$E_i^2 = -1, \quad H^2 = 1, \quad E_i \cdot E_j = 0, \quad H \cdot E_i = 0. \tag{3.2}$$

Then the exceptional collection of interest is given by the following set of line bundles, presented in "three-block" form:

$$\begin{array}{lll} 1.\ \mathcal{O} & 2.\ \mathcal{O}(H) & 4.\ \mathcal{O}(2H-E_1-E_2) \\ & 3.\ \mathcal{O}(2H-E_1-E_2-E_3) & 5.\ \mathcal{O}(2H-E_1-E_3) \\ & & 6.\ \mathcal{O}(2H-E_2-E_3) \end{array} \tag{3.3}$$

The point of this grouping of the sheaves is that there are no arrows between any two members of the same block, and the number of arrows from any member of a block to another block is the same. Now to find the quiver diagram we proceed as outlined in section 2. For each pair of nodes i and j we need to find the dimensions of $\mathrm{Ext}^k(E_i, E_j)$. In the case at hand the E_i

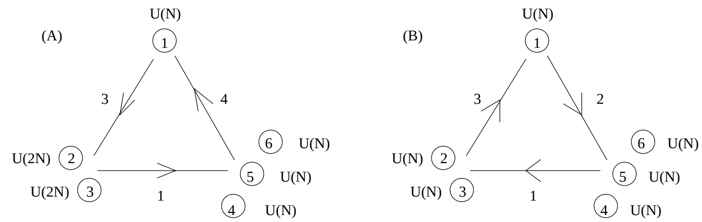

Figure 4: (A) Quiver for the exceptional collection in (3.3). (B) Seiberg dual theory, also known as "model IV".

are line bundles and therefore the Ext's just reduce to the sheaf cohomology groups $H^k(E_j \otimes E_i^*)$. The dimensions of these groups are easily computed to be zero unless $k = 0$. Let us denote an arbitrary sheaf in the ith block by $E_{(i)}$. Then the cohomologies are given by:

$$\begin{aligned} &\dim \operatorname{Ext}^0(E_{(1)}, E_{(2)}) = 3, \quad \dim \operatorname{Ext}^0(E_{(2)}, E_{(3)}) = 1, \\ &\dim \operatorname{Ext}^0(E_{(1)}, E_{(3)}) = 4. \end{aligned} \tag{3.4}$$

To find the ranks of the gauge groups, n_i, we need to satisfy the equation

$$\sum_i n_i \operatorname{ch}(E_i) = \operatorname{ch}(\mathcal{O}_p) \tag{3.5}$$

where $\mathcal{O}_p$ is the skyscraper sheaf over a point p on the Del Pezzo. This is just the statement that the brane configuration has the correct charges of a set of D3 branes filling the dimensions transverse to the Calabi-Yau, but it will also guarantee that we end up with an anomaly free theory [1]. In the present case we can take the n_i to be

$$\{n_1, n_2, n_3, n_4, n_5, n_6\} = \{1, -2, -2, 1, 1, 1\}. \tag{3.6}$$

The effect of the "-2' on the quiver diagram is that the arrows involving nodes 2 and 3 need to be reversed (because we turned a brane into an anti-brane) and the gauge groups at these nodes will be $U(2N)$ instead of $U(N)$. The resulting quiver diagram is displayed below (see the introduction to this section for an explanation of the block notation):

In order to compute the superpotential it is not sufficient to know the dimensions of $H^0(E_j \otimes E_i^*)$; we also need to know the generators. These can be represented as polynomials on the underlying $\mathbf{P}^2$.

Let us write the coordinates on $\mathbf{P}^2$ as $[x_0, x_1, x_2]$ and fix the points on $\mathbf{P}^2$ obtained by blowing down the exceptional curves as

$$E_1 \sim [1, 0, 0], \quad E_2 \sim [0, 1, 0], \quad E_3 \sim [0, 0, 1]. \tag{3.7}$$

The generators we use are listed below:

$$
\begin{aligned}
X_{12} &= A_{12}\,x_0 + B_{12}\,x_1 + C_{12}\,x_2 \\
X_{13} &= A_{13}\,x_1x_2 + B_{13}\,x_0x_2 + C_{13}\,x_0x_1 \\
X_{24} &= x_2 \\
X_{25} &= x_1 \\
X_{26} &= x_0 \\
X_{34} &= 1 \\
X_{35} &= 1 \\
X_{36} &= 1 \\
X_{14} &= A_{14}\,x_1x_2 + B_{14}\,x_0x_2 + C_{14}\,x_0x_1 + D_{14}\,x_2^2 \\
X_{15} &= A_{15}\,x_1x_2 + B_{15}\,x_0x_2 + C_{15}\,x_0x_1 + D_{15}\,x_1^2 \\
X_{16} &= A_{16}\,x_1x_2 + B_{16}\,x_0x_2 + C_{16}\,x_0x_1 + D_{16}\,x_0^2
\end{aligned}
\tag{3.8}
$$

Note that for $X_{34}\ldots X_{36}$ we have chosen generators that do not seem to vanish in the right places. However these will pull back to the right sections on Del Pezzo 3, and it is only the x_i dependence that matters for our calculations. Now we can compute any three point coupling we want by composing the generators around a loop. This gives rise to the following superpotential:

$$
\begin{aligned}
W = {} & A_{21}X_{52}C_{15} + B_{12}X_{26}C_{61} + B_{12}X_{24}A_{41} + C_{12}X_{25}A_{51} \\
& + A_{12}X_{24}B_{41} + C_{12}X_{26}B_{61} + A_{12}X_{26}D_{61} + B_{12}X_{25}D_{51} \\
& + C_{12}X_{24}D_{41} + A_{13}X_{34}A_{41} + A_{13}X_{35}A51 + A_{13}X_{36}A_{61} \\
& + B_{13}X_{34}B_{41} + B_{13}X_{35}B_{51} + B_{13}X_{36}B_{61} \\
& + C_{13}X_{34}C_{41} + C_{13}X_{35}C_{51} + C_{13}X_{36}C_{61}.
\end{aligned}
\tag{3.9}
$$

This superpotential can be compared with existing results in the literature. After applying Seiberg dualities on nodes 2 and 3 we get the quiver diagram depicted in Figure 3.1 B. It is known as model IV in [3],[4]. The superpotential obtained through Seiberg duality is

$$
\begin{aligned}
W_{\text{dual}} = {} & - A_{12}A_{41}X_{24} - A_{12}A_{51}X_{25} - B_{12}B_{41}X_{24} - A_{61}B_{12}X_{26} \\
& - B_{51}C_{12}X_{25} - B_{61}C_{12}X_{26} + A_{13}B_{41}X_{34} + A_{13}B_{51}X_{35} \\
& + A_{41}B_{13}X_{34} + B_{13}B_{61}X_{36} + A_{51}C_{13}X_{35} + A_{61}C_{13}X_{36}.
\end{aligned}
\tag{3.10}
$$

After a relabelling of the nodes and some simple field redefinitions this agrees exactly with [3],[4].

3.2 Del Pezzo 4

In this subsection we analyse the first non-toric example. Its superpotential will again be completely cubic. The notation will be similar to the previous subsection except that we add and extra exceptional curve E_4 which blows down to

$$E_4 \sim [1, 1, 1]. \tag{3.11}$$

Now we proceed with the construction of the quiver. The exceptional collection from [2] is given by:

$$\begin{array}{lll} 1.\ \mathcal{O} & 2.\ F & 3.\ \mathcal{O}(H) \\ & & 4.\ \mathcal{O}(2H - E_2 - E_3 - E_4) \\ & & 5.\ \mathcal{O}(2H - E_1 - E_3 - E_4) \\ & & 6.\ \mathcal{O}(2H - E_1 - E_2 - E_4) \\ & & 7.\ \mathcal{O}(2H - E_1 - E_2 - E_3) \end{array} \tag{3.12}$$

Here F is a rank two bundle defined as the unique extension of the line bundles $\mathcal{O}(H)$ and $\mathcal{O}(2H - \sum_i E_i)$:

$$0 \to \mathcal{O}(2H - \sum_i E_i) \to F \to \mathcal{O}(H) \to 0. \tag{3.13}$$

The cohomologies for this collection are:

$$\begin{aligned} &\dim \mathrm{Ext}^0(E_{(1)}, E_{(2)}) = 5, \dim \mathrm{Ext}^0(E_{(2)}, E_{(3)}) = 1, \\ &\dim \mathrm{Ext}^0(E_{(1)}, E_{(3)}) = 3. \end{aligned} \tag{3.14}$$

Because of the presence of a rank two bundle our computations differ from those in the previous subsection. All computations for this case were performed in by embedding the Del Pezzo in projective space using its anticanonical embedding and constructing the sheaves on it in Macaulay2 [5]. Let us make some comments about how this can be done, since it may be useful for other computations. To get the equations of the Del Pezzo one picks a linearly independent set of cubic polynomials on $\mathbf{P}^2$ which vanish on the points we want to blow up. There are six of these and we can denote them by $c_1, \ldots, c_6$. Then through the map

$$[x_0, x_1, x_2] \to [c_1(x_0, x_1, x_2), \ldots, c_6(x_0, x_1, x_2)] \tag{3.15}$$

our $\mathbf{P}^2$ parametrises a submanifold of $\mathbf{P}^5$. This map seems not so well behaved near the marked points on $\mathbf{P}^2$ which we wanted to blow up, since all the cubics are zero there but the origin is excised from $\mathbf{C}^6$ when we form $\mathbf{P}^5$. The idea is that if we find the smallest variety which contains the submanifold parametrised by $\mathbf{P}^2$ then this variety will have the marked

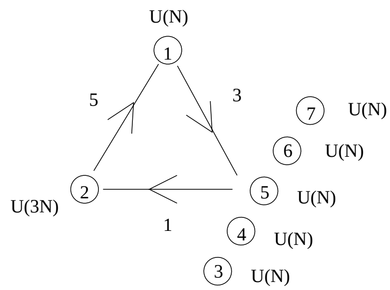

Figure 5: Quiver diagram for the exceptional collection in (3.12).

points blown up into 2-spheres, and so will be our Del Pezzo surface. Given a set of cubics, we can find this variety explicitly by elimination theory.

Once we have the equations for the variety, we need to construct the sheaves. These can be represented as modules over the coordinate ring of the variety. For example if we want the sheaf $\mathcal{O}(-E)$ where E is an exceptional curve obtained from blowing up the point p, we can use elimination theory to find the "image" of p under the map (3.15). This yields a set of linear equations forming an ideal, and the associated module is the module for the sheaf $\mathcal{O}(-E)$.

Finally we need to build the sheaf F by extension, find all the Ext generators and compute the Yoneda pairings. In order to do this we used ideas from [6]. The point of that paper is that sheaf Ext can be computed as module Ext if we first truncate the modules. A bound for the truncation was given in [6]. For our computations we found that frequently no truncation was needed at all. One may now find the sheaf F by computing the extension of the truncated modules using the "push-out" construction, and calculate the Yoneda pairings by computing compositions of Ext's (which for us were just Ext^0's) using maps between the modules.

The quiver diagram for the collection (3.12) is

The superpotential is found to be:

$$\begin{aligned} W = {} & A_{13}X_{32}C_{21} + B_{13}X_{32}D_{21} + C_{31}X_{32}E_{21} + C_{14}X_{42}A_{21} + B_{14}X_{42}B_{21} \\ & + B_{41}X_{42}C_{21} - A_{14}X_{42}C_{21} + C_{41}X_{42}E_{21} + B_{14}X_{42}E_{21} - A_{14}X_{42}E_{21} \\ & + C_{15}X_{52}A_{21} + A_{15}X_{52}B_{21} - C_{15}X_{52}C_{21} - A_{15}X_{52}D_{21} - C_{15}X_{52}E_{21} \\ & - B_{15}X_{52}E_{21} + C_{16}X_{62}A_{21} + B_{16}X_{62}B_{21} + A_{16}X_{62}B_{21} - A_{16}X_{62}C_{21} \\ & - A_{16}X_{62}D_{21} - A_{16}X_{62}E_{21} + C_{17}X_{72}A_{21} + B_{17}X_{72}B_{21} + A_{17}X_{72}B_{21} \\ & - A_{17}X_{72}D_{21} + B_{17}X_{72}E_{21}. \end{aligned} \tag{3.16}$$

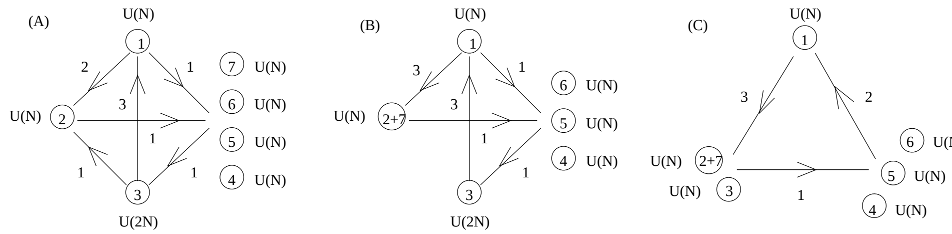

Figure 6: (A): DP4 quiver, dualised version of Figure 3.2. (B): condensing X_{27} leads to a DP3 quiver. (C): by performing an additional Seiberg duality we arrive at "model IV" for DP3.

We would like to perform two checks on this result. First we would like to show that the moduli space of this quiver is indeed Del Pezzo 4, after that we will see how one can Higgs this theory down to Del Pezzo 3.

To find the moduli space we should write down the generators of the ring of gauge invariant operators and then impose the algebraic relations between them as well as the constraints from the superpotential. We will actually only do this for the case of a single D3-brane; the more general case can be handled by thinking of the generators below as $N \times N$ matrices. Our quiver diagram for Del Pezzo 4 counts 75 generators, and from the superpotential we get 69 linear relations between them. We will take the six linearly independent ones to be

$$\begin{aligned} m_{67} &= A_{17}\,X_{72}\,C_{21}, \quad m_{68} = B_{17}\,X_{72}\,C_{21}, \quad m_{69} = C_{17}\,X_{72}\,C_{21}, \\ m_{73} &= A_{17}\,X_{72}\,E_{21}, \quad m_{74} = B_{17}\,X_{72}\,E_{21}, \quad m_{75} = C_{17}\,X_{72}\,E_{21}\,. \end{aligned} \tag{3.17}$$

Now there are 3 'obvious' quadratic relations between them and 2 non-obvious ones that make use of the F-term equations:

$$\begin{aligned} 0 &= m_{69}m_{74} - m_{68}m_{75}, \\ 0 &= m_{69}m_{73} - m_{67}m_{75}, \\ 0 &= m_{68}m_{73} - m_{67}m_{74}, \\ 0 &= m_{73}m_{74} + m_{74}^2 - m_{67}m_{75} - m_{73}m_{75} - m_{74}m_{75}, \\ 0 &= m_{67}m_{69} - m_{67}m_{74} - m_{68}m_{74} + m_{67}m_{75} + m_{68}m_{75}. \end{aligned} \tag{3.18}$$

These give rise to a degree five surface in $\mathbf{P^5}$ which is our Del Pezzo 4.

The other check we would like to perform is Higgsing down the theory in an appropriate way. The idea is make a partial resolution of the Calabi-Yau singularity such that the new singularity is the affine cone over a Del Pezzo with fewer exceptional curves, and to compare with existing result for the simpler case. In order to do this we have to write down a baryonic operator whose VEV higgses the theory to a new quiver where the E_4 charges in the Chern characters have all disappeared. It should be possible to do that directly in the quiver we have considered, but this is likely to be somewhat complicated because the E_4 charge is spread over several nodes. Recall that equation (2.14) tells us what happens with the Chern characters after a Seiberg duality. So we will apply some Seiberg dualities until the E_4 charge in the Chern characters appears in only two nodes linked by a single arrow (so as to avoid creating adjoints fields in the process of Higgsing down).

The Chern characters of the original collection (3.12) are given by

$$\begin{array}{lll} 1.\ (1,0,0) & 2.\ -(2,3H-\sum_i E_i,1/2) & 3.\ (1,H,1/2) \\ & & 4.\ (1,2H-E_2-E_3-E_4,1/2) \\ & & 5.\ (1,2H-E_1-E_3-E_4,1/2) \\ & & 6.\ (1,2H-E_1-E_2-E_4,1/2) \\ & & 7.\ (1,2H-E_1-E_2-E_3,1/2) \end{array} \tag{3.19}$$

The entries in each vector in the above indicate $(\mathrm{rank}, c_1, \mathrm{ch}_2)$ of the sheaves. We also use an extra minus sign at node 2 so that the sum of the Chern characters (with proper multiplicities) is $(0,0,1)$, the Chern character for the D3-brane. Now we can achieve our objective by performing Seiberg dualities on nodes 3 and 2, after which the quiver diagram takes the form in figure Figure 3.2 A with superpotential

$$\begin{aligned} W = {} & A_{31}\,X_{14}\,X_{43} + C_{31}\,X_{14}\,X_{43} + A_{12}\,A_{31}\,X_{24}\,X_{43} + A_{12}\,C_{31}\,X_{24}\,X_{43} \\ & + B_{12}\,C_{31}\,X_{24}\,X_{43} + X_{24}\,X_{32}\,X_{43} + C_{31}\,X_{15}\,X_{53} - A_{31}\,B_{12}\,X_{25}\,X_{53} \\ & - A_{12}\,B_{31}\,X_{25}\,X_{53} - B_{12}\,C_{31}\,X_{25}\,X_{53} + X_{25}\,X_{32}\,X_{53} - \frac{1}{2}A_{31}\,X_{16}\,X_{63} \\ & - \frac{1}{2}B_{31}\,X_{16}\,X_{63} - \frac{1}{2}C_{31}\,X_{16}\,X_{63} - \frac{1}{2}A_{12}\,A_{31}\,X_{26}\,X_{63} - \frac{1}{2}A_{12}\,B_{31}\,X_{26}\,X_{63} \\ & - \frac{1}{2}A_{12}\,C_{31}\,X_{26}\,X_{63} + X_{26}\,X_{32}\,X_{63} - \frac{1}{2}B_{31}\,X_{17}\,X_{73} - \frac{1}{2}C_{31}\,X_{17}\,X_{73} \\ & - \frac{1}{2}A_{12}\,B_{31}\,X_{27}\,X_{73} + \frac{1}{2}A_{12}\,C_{31}\,X_{27}\,X_{73} + X_{27}\,X_{32}\,X_{73} \end{aligned} \tag{3.20}$$

If we do the Seiberg dualities on nodes 3 and 2 by applying mutations to nodes $\{2\}$ and $\{4,5,6,7\}$ respectively, then the Chern characters for the new quiver diagram take the following form:

$$\begin{array}{llll} 1.\ (1,0,0) & 2.\ (1,2H-\sum_i E_i,0) & 3.\ -(1,H,1/2) & 4.\ (0,E_1,1/2) \\ & & & 5.\ (0,E_2,1/2) \\ & & & 6.\ (0,E_3,1/2) \\ & & & 7.\ (0,E_4,1/2) \end{array} \tag{3.21}$$

The E_4 charge now only appears in nodes 2 and 7, and checking the relative Euler characteristic shows that the link between them is given by an Ext^1. Now we may Higgs down to Del Pezzo 3 by turning on an expectation value for X_{27}, which gives rise to a new exceptional collection of sheaves:[15]

$$\begin{array}{lll} 1.\ (1,0,0) & 2+7.\ (1,2H-E_1-E_2-E_3,1/2) & 4.\ (0,E_1,1/2) \\ & 3.\ -(1,H,1/2) & 5.\ (0,E_2,1/2) \\ & & 6.\ (0,E_3,1/2) \end{array} \tag{3.22}$$

The resulting quiver is drawn in Figure 3.2 B.

We can do a final Seiberg duality on node 3 after which we arrive at the quiver diagram of model IV. If we perform a mutation on nodes $\{4,5,6\}$ then we get the Chern characters for the line bundles

$$\begin{array}{lll} 1.\ \mathcal{O} & 2+7.\ \mathcal{O}(2H-E_1-E_2-E_3,) & 4.\ \mathcal{O}(H-E_1) \\ & 3.\ \mathcal{O}(H) & 5.\ \mathcal{O}(H-E_2) \\ & & 6.\ \mathcal{O}(H-E_3) \end{array} \tag{3.23}$$

Upon some field redefinitions the superpotential reduces exactly to the expected known superpotential for model IV, given eg. in equation (3.10) above. Moreover a further Seiberg duality on nodes $2+7$ and 3 gives rise to the exceptional collection (3.3) we used in the previous subsection.

3.3 Del Pezzo 5

For Del Pezzo surfaces of with more than four exceptional curves we have to deal with a new phenomenon: these surfaces have a complex structure moduli space. While our computations are insensitive to the Kähler structure, they definitely do depend on the complex structure so we expect these moduli to make an appearance in the superpotential.

The appearance of complex structure moduli is easy to understand in the description of Del Pezzos we have used, as blow-ups of $\mathbf{P}^2$. The set of coordinate transformations preserving the complex structure of $\mathbf{P}^2$ is the group $PGl(3,C)$, which has 8 complex parameters. The complex structure of the Del Pezzos is completely determined by specifying which points on $\mathbf{P}^2$ get blown up. To specify a point on $\mathbf{P}^2$ one needs two complex parameters. Therefore, the first four marked points can always be fixed at some chosen reference points, but then we have used up the coordinate transformations

[15] The unique extension can be found by tensoring the fundamental sequence $0 \to \mathcal{O}(-E_4) \to \mathcal{O} \to \mathcal{O}_{E_4} \to 0$ with $\mathcal{O}(2H-E_1-E_2-E_3)$, which is consistent with the Chern characters.

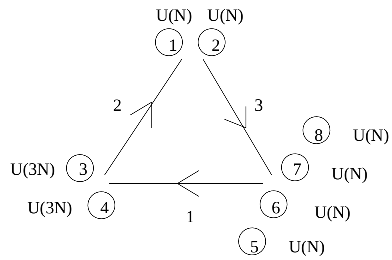

Figure 7: Quiver diagram corresponding to the exceptional collection in (3.24).

and for each additional point we have a choice of two complex parameters each of which gives rise to a different complex structure.

With this in mind, for DP5 we add a fifth exceptional curve E_5 which blows down to a point with floating coordinates. For convenience we use projective coordinates to parametrise this marked point and therefore the complex structure moduli space, $E_5 \sim [z_0, z_1, z_2]$.

The exceptional collection of [2] is given by the following collection of line bundles:

$$\begin{array}{lcl} 1.\ \mathcal{O}(E_5) & 3.\ \mathcal{O}(H) & 5.\ \mathcal{O}(2H - E_1 - E_2) \\ 2.\ \mathcal{O}(E_4) & 4.\ \mathcal{O}(2H - E_1 - E_2 - E_3) & 6.\ \mathcal{O}(2H - E_2 - E_3) \\ & & 7.\ \mathcal{O}(2H - E_1 - E_3) \\ & & 8.\ \mathcal{O}(3H - \sum_{i=1}^{5} E_i) \end{array} \tag{3.24}$$

The cohomologies for this collection are:

$$\begin{aligned} &\dim \text{Ext}^0(E_{(1)}, E_{(2)}) = 2, \dim \text{Ext}^0(E_{(2)}, E_{(3)}) = 1, \\ &\dim \text{Ext}^0(E_{(1)}, E_{(3)}) = 3. \end{aligned} \tag{3.25}$$

The quiver diagram for this collection is displayed below:

This quiver diagram allows for cubic terms and sextic terms in the action. The calculation for the cubic terms is very similar to the computation for

DP3, so we will not write down the details here. The answer we get is

$$
\begin{aligned}
W_{\text{cubic}} =& X_{53}A_{31}(-\frac{z_0}{z_2}A_{15}+\frac{z_1}{z_2}B_{15})+X_{53}B_{31}B_{15}+X_{63}A_{31}(-\frac{z_0}{z2}B_{16}+\frac{z_0}{z_2}C_{16}) \\
&+X_{63}B_{31}(-\frac{z_0}{z_1}A_{16}+\frac{z_0}{z_1}C_{16})+X_{73}A_{31}(-\frac{z_0}{z_2}B_{17}+\frac{z_1}{z_2}C_{17}) \\
&+X_{73}B_{31}C_{17}+X_{83}A_{31}(\frac{z_0}{z_2}A_{18}+\frac{z_0 z_1}{z_2(z_1-z_2)}B_{18}+\frac{z_1(z_0-z_2)}{z_0(z_1-z_2)}C_{18}) \\
&+X_{83}B_{31}(-\frac{z_0}{z_2-z_1}B_{18}+\frac{z_2(z_0-z_2)}{z_0(z_1-z_2)}C_{18})+X_{53}A_{32}(-A_{25}+B_{25}) \\
&+X_{53}B_{32}B_{25}+X_{63}A_{32}(-B_{26}+C_{26})+X_{63}B_{32}(-A_{26}+C_{26}) \\
&+X_{73}A_{32}(-B_{27}+C_{27})+X_{73}B_{32}C_{27} \\
&+X_{83}A_{32}(\frac{z_1(z_2-z_0)}{z_0(z_1-z_2)}A_{28}+B_{28}+\frac{z_0}{z_2-z_1}C_{28}) \\
&+X_{83}B_{32}(\frac{z_1(z_2-z_0)}{z_0(z_1-z_2)}A_{28}+\frac{z_0}{z_2-z_1}C_{28}) \\
&+X_{54}A_{41}(-\frac{z_0}{z_2}A_{15}+\frac{z_1}{z_2}B_{15})+X_{54}B_{41}(-\frac{z_0}{z_2}A_{15}+\frac{z_0}{z_2}C_{15})+X_{64}A_{41}A_{16} \\
&+X_{64}B_{41}B_{16}+X_{74}A_{41}A_{17}+X_{74}B_{14}C_{17}-X_{84}A_{41}C_{18} \\
&+X_{54}A_{42}(-A_{25}+B_{25})+X_{54}B_{24}(-A_{25}+C_{25})+X_{64}A_{42}A_{26} \\
&+X_{64}B_{42}B_{26}+X_{74}A_{42}A_{27}+X_{74}B_{42}C_{27}+X_{84}A_{42}A_{28} \\
&+X_{84}B_{41}(\frac{z_0(z_2-z_0)}{z_2(z_1-z_0)}A_{18}+\frac{z_0^2}{z_2(z_0-z_1)}B_{18}) \\
&+X_{84}B_{42}(\frac{z_0(z_2-z_0)}{z_2(z_1-z_0)}B_{28}+\frac{z_0^2}{z_2(z_1-z_0)}C_{28}).
\end{aligned}
\tag{3.26}
$$

The sextic terms a priori may present a problem. Since there are no Ext^1's among the cohomologies and we do not have a prescription for computing higher order couplings without Ext^1's, we cannot find the coefficients of the allowed sextic terms from first principles. On the other hand, since all cubic terms appear in the superpotential and therefore have R-charge equal to 2 and dimension 3, any sextic term would have dimension 6 and so its coefficient in the superpotential would be dimensionful. Such a term simply can't be present since we're considering the far IR physics which is given by an (interacting) scale-invariant theory. One may also take a limit in which the generic Del Pezzo becomes a toric surface and compare with toric computations which can be done independently. In this limit one finds only cubic terms. So we conclude that the cubic terms provide the full answer.

Let us do some checks on the answer we have obtained. The first test of the superpotential is a computation of the moduli space. To reduce the number of gauge invariant operators we perform Seiberg dualities on nodes 3,4,5 and 6 and make some field redefinitions after which we arrive at the quiver in Figure 3.3 A with purely quartic couplings:

$$\begin{aligned}W = \ & X_{27}X_{73}X_{35}X_{52} + X_{27}X_{73}X_{36}X_{62} + X_{27}X_{74}X_{45}X_{52} + X_{27}X_{74}X_{46}X_{62} \\ & + X_{17}X_{73}X_{36}X_{61} + z_1 X_{17}X_{74}X_{45}X_{51} + z_1 X_{17}X_{74}X_{46}X_{61} \\ & + z_2 X_{17}X_{73}X_{35}X_{51} - (z_1 - 1)X_{18}X_{83}X_{36}X_{61} + z_1(z_1-1)X_{18}X_{84}X_{46}X_{61} \\ & \quad + z_1(z_1 - z_2)X_{18}X_{84}X_{45}X_{51} + (z_2 - z_1)X_{18}X_{83}X_{35}X_{51} \\ & + (z_1 - 1)X_{28}X_{83}X_{36}X_{62} + (z_1 - z_2)X_{28}X_{83}X_{35}X_{52} \\ & \quad - z_2(z_1-1)X_{28}X_{84}X_{46}X_{62} + (z_2 - z_1)X_{28}X_{84}X_{45}X_{52}.\end{aligned} \tag{3.27}$$

The coordinate z_0 has been set to one in this expression, but may of course always be restored by replacing $z_1 \to z_1/z_0$ and $z_2 \to z_2/z_0$. When we compute the moduli space of this theory we seem to find two components. That is, the ideal I of relations we find can be expressed as the intersection $I = I_1 \cap I_2$ where I_1 contains the equations for the affine cone over the Del Pezzo and I_2 is another ideal which geometrically contains only one point, the singular point of the Calabi-Yau.[16] Geometrically therefore the moduli space is exactly as expected. For the I_1 component all the gauge invariant operators can be expressed in terms of

$$\begin{aligned} & m_{11} = X_{18}X_{83}X_{36}X_{61}, \quad m_{14} = X_{28}X_{84}X_{45}X_{52}, \quad m_{16} = X_{28}X_{84}X_{46}X_{62}, \\ & m_{12} = X_{28}X_{83}X_{36}X_{62}, \quad m_{15} = X_{18}X_{84}X_{46}X_{61}.\end{aligned} \tag{3.28}$$

There is one obvious relation between these and one non-obvious relation:

$$\begin{aligned} 0 &= m_{12}m_{15} - m_{11}m_{16}, \\ 0 &= z_1(z_1 - z_2)m_{11}m_{14} - (z_1 - z_2)m_{12}m_{14} - (z_1 - z_2)^2 m_{14}^2 \\ & \quad - z_1(z_1 - z_2)m_{14}m_{15} + z_1(z_1 - 1)m_{11}m_{16} - z_2(z_1 - 1)m_{12}m_{16} \\ & \quad - 2z_2(z_1 - 1)(z_1 - z_2)m_{14}m_{16} - z_1 z_2(z_1 - 1)m_{15}m_{16} + z_2^2(z_1 - 1)m_{16}^2.\end{aligned} \tag{3.29}$$

It would be interesting to check if this Del Pezzo has the expected complex structure.

Another possible way of testing the superpotentials is by Higgsing down to known Del Pezzo's as we did for Del Pezzo 4. The simplest way to do

[16]If the Calabi-Yau develops non-isolated singularities one finds additional branches. Their interpretation is that the D3 branes can split into fractional branes which can move along the singularities.

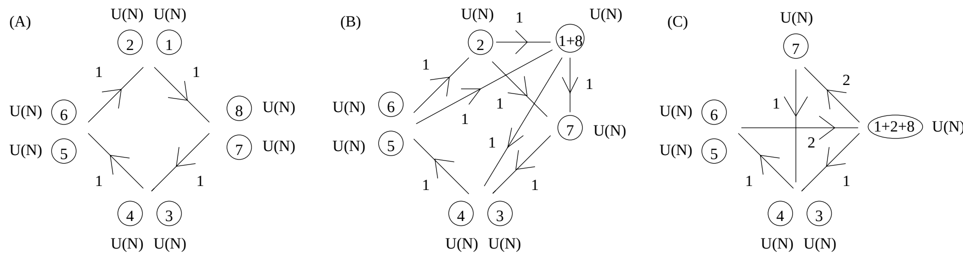

Figure 8: (A): Quiver obtained by dualising Figure 3.3. (B): Giving a VEV to X_{18} in (A), which results in a quiver for Del Pezzo 4. (C): Further condensing X_{28} gives rise to "model III" for Del Pezzo 3.

that in the present case is by giving an expectation value to X_{18} in the quiver of Figure 3.3 A, after which we get the quiver theory in Figure 3.3 B without having to do any integrating out. One can straightforwardly check that the moduli space of this quiver is exactly Del Pezzo 4. One may Higgs it down further to Del Pezzo 3 by condensing X_{28}. This quiver is known as "model III" [3],[4], and the superpotential we get coincides with the known superpotential after field redefinitions.

The quiver theory in Figure 3.3 B is related to the Del Pezzo 4 quiver we found in the previous subsection. Namely we can start with Figure 3.2 and apply Seiberg dualities on nodes 2,3,4,2,5.

Finally one can take a limit in which some of the marked points approach each other. In such a limit the shrinking 4-cycle may cease to have a positive curvature anti-canonical bundle and so would therefore no longer be a Del Pezzo surface, but one may obtain cones over toric surfaces in such limits or sometimes even orbifolds. For Del Pezzo 5 there is a limit in which the Calabi-Yau turns into the $Z_2 \times Z_2$ orbifold of the conifold, and another in which it becomes a partial toric resolution of the orbifold singularity $\mathbf{C}^3/Z_3 \times Z_3$ and hence we may compare with known answers in the literature. We have not carried out the details because the field rescalings are rather involved, but it should be possible to check this.

3.4 Del Pezzo 6

For Del Pezzo 6 two three-block models were given in [2]. We can compute the cubic couplings in either without too much difficulty, but we will stick to the collection of line bundles because it is easier to get the dependence on complex structure moduli that way. We blow up a sixth point on $\mathbf{P^2}$ with

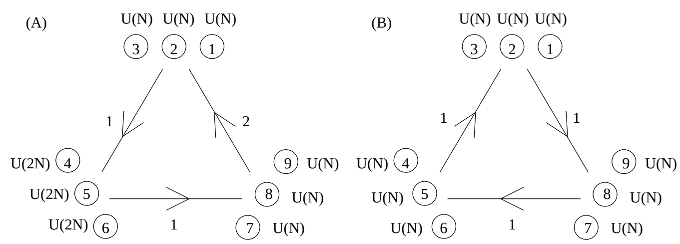

Figure 9: (A): Quiver diagram corresponding to the exceptional collection in (3.31). (B): Seiberg dual quiver; also the quiver diagram for the orbifold $\mathbf{C^3}/Z_3 \times Z_3$. Notice the similarity between these two quivers and the $\mathbf{P}^2$ quivers in Figure 2.2.

floating coordinates

$$E_6 \sim [w_0, w_1, w_2]. \tag{3.30}$$

The exceptional sheaves of the first collection in [2], which we might call " model 6.1", are:

$$\begin{array}{lll} 1.\ \mathcal{O}(E_4) & 4.\ \mathcal{O}(H-E_1) & 7.\ \mathcal{O}(2H-E_1-E_2-E_3) \\ 2.\ \mathcal{O}(E_5) & 5.\ \mathcal{O}(H-E_2) & 8.\ \mathcal{O}(H) \\ 3.\ \mathcal{O}(E_6) & 6.\ \mathcal{O}(H-E_3) & 9.\ \mathcal{O}(3H-\sum_{i=1}^6 E_i) \end{array} \tag{3.31}$$

with cohomologies given by

$$\begin{array}{l} \dim \mathrm{Ext}^0(E_{(1)}, E_{(2)}) = 1, \quad \dim \mathrm{Ext}^0(E_{(2)}, E_{(3)}) = 1, \\ \dim \mathrm{Ext}^0(E_{(1)}, E_{(3)}) = 2. \end{array} \tag{3.32}$$

The quiver for this collection is displayed in Figure 3.4 A.

Before writing down the superpotential let us introduce the following shorthand notation:

$$\begin{aligned} f_1 &= (z_1-z_0)z_2w_0^2 + (z_0^2 - z_1z_2)w_0w_2 + z_0(z_2-z_0)w_1w_2 \\ f_2 &= z_0(z_1-z_2)w_1w_2 + (z_0-z_1)z_2w_0w_1 + (z_2-z_0)z_1w_0w_2 \\ f_3 &= -z_1(z_2-z_1)w_0w_2 - (z_0-z_1)z_2w_1^2 - (z_1^2 - z_0z_2)w_1w_2. \end{aligned} \tag{3.33}$$

The cubic terms in the superpotential for model 6.1 are

$$
\begin{aligned}
W &= (-a_{71}+b_{71})\, x_{14}\, x_{47} + x_{24}\, x_{47}\left(b_{72} - \frac{a_{72}\, z_1}{z_2}\right) \\
&+ \left(b_{73} - \frac{a_{73}\, w_1}{w_2}\right) x_{34}\, x_{47} + a_{81}\, x_{14}\, x_{48} + a_{82}\, x_{24}\, x_{48} + a_{83}\, x_{34}\, x_{48} \\
&+ \left(b_{91} + \frac{a_{91}\, f_1}{f_2}\right) x_{14}\, x_{49} + \left(b_{92} + \frac{a_{92}\, f_1}{f_2}\right) x_{24}\, x_{49} \\
&+ \left(b_{93} + \frac{a_{93}\, f_1}{f_2}\right) x_{34}\, x_{49} + b_{71}\, x_{15}\, x_{57} + b_{72}\, x_{25}\, x_{57} + b_{73}\, x_{35}\, x_{57} \\
&+ b_{81}\, x_{15}\, x_{58} + b_{82}\, x_{25}\, x_{58} + b_{83}\, x_{35}\, x_{58} \\
&+ \left(b_{91} + \frac{a_{91}\, f_2}{f_3}\right) x_{15}\, x_{59} + \left(b_{92} + \frac{a_{92}\, f_2}{f_3}\right) x_{25}\, x_{59} \\
&+ \left(b_{93} + \frac{a_{93}\, f_2}{f_3}\right) x_{35}\, x_{59} + a_{71}\, x_{16}\, x_{67} + a_{72}\, x_{26}\, x_{67} + a_{73}\, x_{36}\, x_{67} \\
&+ (-a_{81}+b_{81})\, x_{16}\, x_{68} + x_{26}\, x_{68}\left(b_{82} - \frac{a_{82}\, z_0}{z_1}\right) + \left(b_{83} - \frac{a_{83}\, w_0}{w_1}\right) x_{36}\, x_{68} \\
&+ (-a_{91}+b_{91})\, x_{16}\, x_{69} + \left(b_{93} - \frac{a_{93}\, w_0}{w_1}\right) x_{36}\, x_{69} + x_{26}\, x_{69}\left(b_{92} - \frac{a_{92}\, z_0}{z_1}\right)
\end{aligned}
\tag{3.34}
$$

As for Del Pezzo 5 we expect this to be the full answer.

In order to do the moduli space computation we go to a new quiver by Seiberg dualities on nodes 4,5,6, after which we get Figure 3.4 B with superpotential

$$
\begin{aligned}
W &= X_{17}(X_{41}X_{74} - X_{51}X_{75} + X_{61}X_{76}) \\
&+ X_{18}(X_{41}X_{84} - X_{51}X_{85} + X_{61}X_{86}) \\
&+ X_{19}((-f_2^2 - f_2 f_3)X_{41}X_{94} + (f_1 f_3 + f_2 f_3)X_{51}X_{95} + (f_2^2 - f_1 f_3)X_{61}X_{96}) \\
&+ X_{27}(X_{62}X_{76}z_1 + X_{42}X_{74}z_2 - X_{52}X_{75}z_2) \\
&+ X_{28}(X_{42}X_{84}z_0 - X_{52}X_{85}z_1 + X_{62}X_{86}z_1) \\
&+ X_{29}((-f_2 f_3 z_0 - f_2^2 z_1)X_{42}X_{94} + (f_2 f_3 z_0 + f_1 f_3 z_1)X_{52}X_{95}) \\
&+ X_{29}(z_1(f_2^2 - f_1 f_3)X_{62}X_{96}) + X_{37}(w_2(X_{43}X_{74} - X_{53}X_{75}) + w_1 X_{63}X_{76}) \\
&+ X_{38}(w_0 X_{43}X_{84} - w_1 X_{53}X_{85} + w_1 X_{63}X_{86}) + X_{39}((-f_2 f_3 w_0 - f_2^2 w_1)X_{43}X_{94}) \\
&+ X_{39}((f_2 f_3 w_0 + f_1 f_3 w_1)X_{53}X_{95} + w_1(f_2^2 - f_1 f_3)X_{63}X_{96})
\end{aligned}
\tag{3.35}
$$

When we calculate the ideal of relations we find, just as for Del Pezzo 5, that it can be expressed as $I = I_1 \cap I_2$ where I_1 gives a cubic relation in four

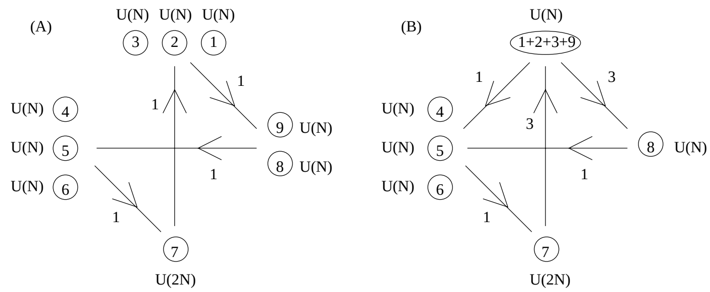

Figure 10: (A): Dual to Figure 3.4 B by Seiberg duality on node 7, superpotential given in (3.38). Upon Higgsing down we get quiver theories for DP5,DP4,DP3. (B): DP3 quiver, related to model IV by duality on node 7.

variables and I_2 only contains the origin of moduli space geometrically. As before to get the locus of the moduli space one should take the radical of I, leaving us only with a cubic equation. We were not able to get a general expression involving the complex structure parameters as in (3.29), but we can find the cubic equation for any given complex structure. The 207 gauge invariant operators can all be solved for in terms of

$$\begin{aligned} p_{23} = X_{29}X_{95}X_{52}, \quad p_{24} = X_{39}X_{95}X_{53}, \\ p_{26} = X_{29}X_{96}X_{62}, \quad p_{27} = X_{39}X_{96}X_{63}. \end{aligned} \tag{3.36}$$

If for example the complex structure is given by $[z_0, z_1, z_2] = [1, 3, 5]$ and $[w_0, w_1, w_2] = [1, 2, -2]$, we get

$$\begin{aligned} 0 = {} & p_{23}\, p_{24}\, p_{26} + \frac{4}{43}\, p_{24}^2\, p_{26} + \frac{105}{731}\, p_{24}\, p_{26}^2 + \frac{5}{43}\, p_{23}^2\, p_{27} + \frac{20}{43}\, p_{23}\, p_{24}\, p_{27} \\ & - \frac{825}{731}\, p_{23}\, p_{26}\, p_{27} - \frac{10}{731}\, p_{24}\, p_{26}\, p_{27} - \frac{350}{731}\, p_{23}\, p_{27}^2. \end{aligned} \tag{3.37}$$

We recognise the well-known realisation of Del Pezzo 6 as a degree 3 surface in $\mathbf{P}^3$.

As usual we can also try to Higgs down to other Del Pezzo's. One seems to get the simplest result by first Seiberg dualising on node 7 in Figure 3.4

B, after which one arrives at Figure 3.4 A with superpotential

$$\begin{aligned} W =& -X_{18}\,X_{47}\,X_{71}\,X_{84} - \frac{w_0}{w_2}\,X_{38}\,X_{47}\,X_{73}\,X_{84} - X_{18}\,X_{57}\,X_{71}\,X_{85} \\ & - \frac{w_1}{w_2}\,X_{38}\,X_{57}\,X_{73}\,X_{85} - X_{18}\,X_{67}\,X_{71}\,X_{86} - X_{28}\,X_{67}\,X_{72}\,X_{86} - X_{38}\,X_{67}\,X_{73}\,X_{86} \\ & + f_2^2\,X_{19}\,X_{47}\,X_{71}\,X_{94} + f_2\,f_3\,X_{19}\,X_{47}\,X_{71}\,X_{94} + \frac{f_2\,f_3\,w_0}{w_2}\,X_{39}\,X_{47}\,X_{73}\,X_{94} \\ & + \frac{f_2^2\,w_1}{w_2}\,X_{39}\,X_{47}\,X_{73}\,X_{94} + f_1\,f_3\,X_{19}\,X_{57}\,X_{71}\,X_{95} + f_2\,f_3\,X_{19}\,X_{57}\,X_{71}\,X_{95} \\ & + \frac{f_2\,f_3\,w_0}{w_2}\,X_{39}\,X_{57}\,X_{73}\,X_{95} + \frac{f_1\,f_3\,w_1}{w_2}\,X_{39}\,X_{57}\,X_{73}\,X_{95} - f_2^2\,X_{19}\,X_{67}\,X_{71}\,X_{96} \\ & + f_1\,f_3\,X_{19}\,X_{67}\,X_{71}\,X_{96} - f_2^2\,X_{29}\,X_{67}\,X_{72}\,X_{96} + f_1\,f_3\,X_{29}\,X_{67}\,X_{72}\,X_{96} \\ & - f_2^2\,X_{39}\,X_{67}\,X_{73}\,X_{96} + f_1\,f_3\,X_{39}\,X_{67}\,X_{73}\,X_{96} - \frac{z_0}{z_2}X_{28}X_{47}X_{72}X_{84} \\ & + \frac{f_2\,f_3\,z_0}{z_2}\,X_{29}\,X_{47}\,X_{72}\,X_{94} + \frac{f_2\,f_3\,z_0}{z_2}\,X_{29}\,X_{57}\,X_{72}\,X_{95} - \frac{z_1}{z_2}X_{28}\,X_{57}\,X_{72}\,X_{85} \\ & + \frac{f_2^2\,z_1}{z_2}\,X_{29}\,X_{47}\,X_{72}\,X_{94} + \frac{f_1\,f_3\,z_1}{z_2}\,X_{29}\,X_{57}\,X_{72}\,X_{95} \end{aligned} \tag{3.38}$$

One can arrange the links X_{39}, X_{29} and X_{19} to correspond to Ext^1's, and by successively condensing them one should get exceptional collections, quiver diagrams and superpotentials for Del Pezzo 5,4, and 3 respectively. There is no integrating out involved so this is very simple. By doing an additional Seiberg duality on node 7 in Figure 3.4 B, we get the quiver diagram for model IV yet again and we have checked that one recovers the known superpotential for the quiver after going through this sequence all the way.

The $\mathbf{C}^3/Z_3 \times Z_3$ orbifold is a limit of Del Pezzo 6 and its quiver diagram is given by Figure 3.4 B. The orbifold superpotential is completely cubic and our superpotential should reproduce this in the appropriate limit. This looks promising but we haven't been able to check it precisely due to the large number of allowed field rescalings.

3.5 Simple quiver diagrams for the remaining Del Pezzo's

Let us list some quiver diagrams that can be deduced from the collections in [2] for Del Pezzo surfaces of degree 1 and 2.[17] We haven't yet computed the

[17] Quiver diagrams for low degree Del Pezzo's have been proposed before (see the fourth reference in [4]) but they were found to have some problems; for degree 3 this was found by F. Cachazo and the author, and for degrees 1 and 2 in [27]

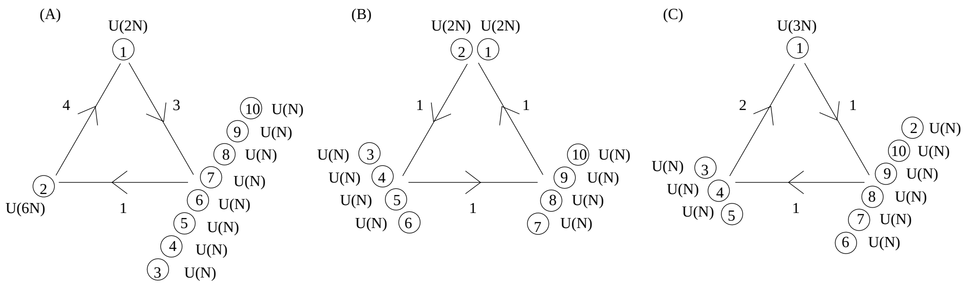

Figure 11: Three quivers diagrams for the degree 2 Del Pezzo. All these should have a cubic superpotential. (A): Type (7.1). (B): Type (7.2). (C): Type (7.3).

superpotentials for these quivers but they should give rise to cubic couplings only.

Let us start with collection (7.1) in [2]. The Chern characters are computed to be:

$$\begin{array}{lll} 1.\ (2, -2H + \sum_i E_i, -3/2) & 2.\ -(2, H, -1/2) & 3.\ (1, 3H - \sum_i E_i, 1) \\ & & 4.\ (1, H - E_1, 0) \\ & & 5.\ (1, H - E_2, 0) \\ & & 6.\ (1, H - E_3, 0) \\ & & 7.\ (1, H - E_4, 0) \\ & & 8.\ (1, H - E_5, 0) \\ & & 9.\ (1, H - E_6, 0) \\ & & 10.\ (1, H - E_7, 0) \end{array} \tag{3.39}$$

The corresponding quiver diagram is drawn in figure Figure 3.5 A. The quiver can be simplified by dualising node 2, which reduces the gauge group to $U(2N)$.

It is instructive to see how one may get the other allowed types of three-block quivers through Seiberg duality. In principle we could get them from the collections in [2], however computation of the cohomologies for the remaining collections reveals that some of them fail to be exceptional. In these cases one may write down a valid set by following the Chern characters through Seiberg dualities, as we discussed in detail for the degree 5 Del Pezzo.

To get a new type of three-block quiver we can start with (3.39) and apply Seiberg dualities on nodes $\{2, 3, 4, 5, 6, 2\}$ after which one arrives at the quiver depicted in Figure 3.5 B. This quiver belongs to the class of three-block collections called (7.2) in [2]. Starting with (3.39) and applying

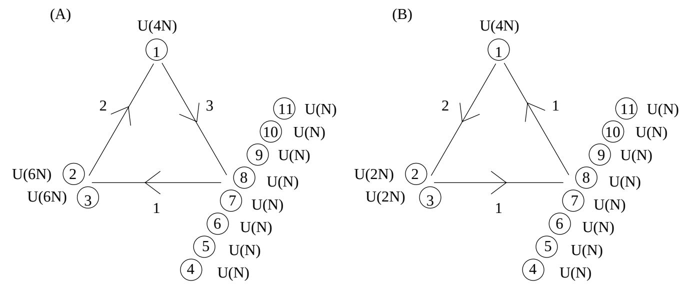

Figure 12: Quivers diagrams for the degree 1 Del Pezzo. All these should have a cubic superpotential. (A): Collection (3.40), type (8.2). (B): Seiberg dual.

Figure 13: Other quivers diagrams for the degree 1 Del Pezzo with cubic superpotential. (A): Type (8.1). (B): Type (8.3). (C): Type (8.4).

dualities on nodes $\{2, 3, 4, 5, 1, 2\}$ results in the quiver in Figure 3.5 C, which is of type (7.3).

Similarly we can get simple quiver diagrams for the degree 1 Del Pezzo. Collection (8.1) in [2] appears to be invalid so we skip to collection (8.2). The Chern characters are found to be

$$\begin{array}{lll}
1.\ (4, \sum_{i=1}^{4} E_i, -5/2) & 2.\ -(2, H, -1/2) & 4.\ (1, H-E_1, 0) \\
 & 3.\ -(2, 2H-\sum_{i=1}^{3}, -1/2) & 5.\ (1, H-E_2, 0) \\
 & & 6.\ (1, H-E_3, 0) \\
 & & 7.\ (1, 3H-\sum_i E_i+E_8, 1) \\
 & & 8.\ (1, 3H-\sum_i E_i+E_7, 1) \\
 & & 9.\ (1, 3H-\sum_i E_i+E_6, 1) \\
 & & 10.\ (1, 3H-\sum_i E_i+E_5, 1) \\
 & & 11.\ (1, 3H-\sum_i E_i+E_4, 1)
\end{array} \quad (3.40)$$

The corresponding quiver diagram is shown in Figure 3.5 A. A simpler quiver may be obtained through Seiberg duality on nodes 2 and 3, drawn in Figure 3.5 B. From here we may obtain the other types of three-block quivers through Seiberg duality. Type (8.1) can be recovered by dualising $\{4, 1, 2, 3\}$, type (8.3) by dualising $\{4, 5, 1\}$, and type (8.4) by dualising $\{4, 5, 6, 1, 4, 5, 6\}$. The resulting diagrams are given in Figure 3.5 A, Figure 3.5 B and Figure 3.5 C respectively.

3.6 Preliminary remarks on the conifold

In the case of the conifold the shrinking cycle is simply a $\mathbf{P}^1$ with normal bundle $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. An exceptional collection is given by $\{\mathcal{O}(-1), \mathcal{O}(0)\}$. The non-zero cohomology groups are

$$\dim \mathrm{Ext}^0(\mathcal{O}(-1), \mathcal{O}(0)) = 2, \qquad \dim \mathrm{Ext}^0(\mathcal{O}(-1), \mathcal{O}(0) \otimes N) = 2 \quad (3.41)$$

with the rest related by Serre duality. This gives rise to two bifundamentals stretching one way between the two nodes and two more bifundamentals stretching the other way. The first two fields describe the position of the D3-branes on the $\mathbf{P}^1$ and the last two describe deformations in the normal direction. It is hard to see how to compute the known quartic superpotential, the current known rules do not suffice. Presumably one wants to replace $\mathcal{O}_{\mathbf{P}^1}$ with some exact sequence of branes/anti-branes that fill the Calabi-Yau.

3.7 Z_k orbifolds

While for orbifolds of the form $\mathbf{C}^3/Z_k$ the quiver diagram and superpotential are easily derived using representation theory, we would like to show here that the computations are not necessarily any more difficult from a large volume perspective.

The (partial) resolution we would like to use for $\mathbf{C}^3/Z_k$ is provided by the usual linear sigma model. Suppose that the coordinates of $\mathbf{C}^3$ are labelled by (x_0, x_1, x_2) and the weights of the action of Z_k are (q_0, q_1, q_2) with $q_0 + q_1 + q_2 = k$ and all q's positive. Then the linear sigma model with four fields given by (x_0, x_1, x_2, p) and charges $(q_0, q_1, q_2, -k)$ has a moduli space given by the solutions of

$$q_0|x_0|^2 + q_1|x_1|^2 + q_2|x_2|^2 - k|p|^2 = t \quad (3.42)$$

modulo the action of the $U(1)$. For large negative t we get the orbifold $\mathbf{C}^3/Z_k$ and for large positive t we get a partial resolution with a finite size 4-cycle, namely the weighted projective space $\mathbf{WP}(q_0, q_1, q_2)$ with the sheaf $\mathcal{O}(-k)$ on top of it. In order to derive the quiver theory we need a collection of sheaves on the weighted projective space that correspond to the fractional branes for the orbifold. It was argued in [17] that these should take the form of exterior powers of cotangent sheaves tensored with invertible sheaves, and that the sections mapping between them should behave as fermionic variables. One reason we have to talk about sheaves and not bundles is because weighted projective spaces typically have orbifold singularities.

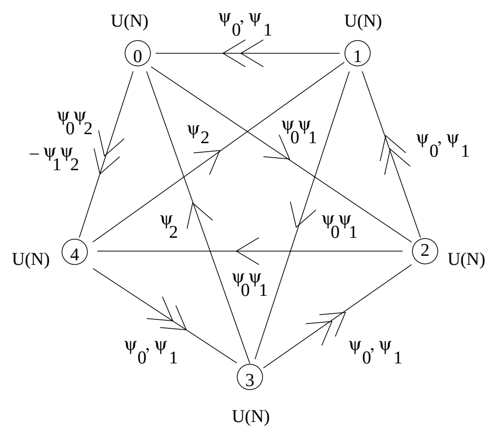

Figure 14: Quiver theory for the $\mathbf{C}^3/Z_5$ orbifold.

We have not understood how to exactly identify the sheaves, but we would like to point out that if sections behave as in [17] (see also [24],[23]) then not only do we get the correct Z_k symmetric quiver diagram, we also get the correct superpotential. Rather than setting up notation for the general case let us simply treat an illustrative example. We would like to find the quiver for the orbifold $\mathbf{C}^3/Z_5$ with weights $(1,1,3)$. We will label the sections by ψ_0, ψ_1 and ψ_2, where ψ_i carries the same $U(1)$ charge as x_i. We denote the sheaves by $S_0, \ldots, S_4$. Then the claim is that $\mathrm{Ext}^0(S_i, S_j)$ is generated by the fermionic variables of total charge $j-i$. So we obtain for example

$$\begin{aligned}
&\dim \mathrm{Ext}^0(S_0, S_1) = 2 \to \psi_0, \psi_1 \\
&\dim \mathrm{Ext}^0(S_0, S_2) = 1 \to \psi_0\psi_1 \\
&\dim \mathrm{Ext}^0(S_0, S_3) = 1 \to \psi_2 \\
&\dim \mathrm{Ext}^0(S_0, S_4) = 2 \to \psi_0\psi_2, \psi_1\psi_2
\end{aligned} \tag{3.43}$$

Moreover if the fermion number of the map from S_0 to S_i is odd, then we have to invert the Chern character for S_i in order to get the correct quiver theory for D3 branes. The quiver diagram is drawn in Figure 3.7.

Now let us try to understand the superpotential. Because of the fermionic nature of the variables we cannot multiply too many of them, in fact we can only get cubic couplings. Eg. composing the map ψ_0 from node 0 to node 1 with the map ψ_1 from node 1 to node 2 yields a map $\psi_1\psi_0$ from node 0 to node 2. Comparing with the standard generator which we chose to be $\psi_0\psi_1$, we see that we get a -1 as the coefficient of the corresponding cubic term in the superpotential. Continuing in this way we find that

$$W = (Y_{01}X_{12} - X_{01}Y_{12})Z_{20} + (Y_{12}X_{23} - X_{12}Y_{23})Z_{31} + (Y_{23}X_{34} - X_{23}Y_{34})Z_{42} + (Y_{34}X_{40} - X_{34}Y_{40})Z_{03} + (Y_{40}X_{01} - X_{40}Y_{01})Z_{14} \tag{3.44}$$

which is just the usual answer obtained by projecting the superpotential of $\mathcal{N} = 4$ Yang-Mills theory. The linear sigma model with charges $(2, 2, 1)$ provides a different partial resolution of the same orbifold, but leads to the same quiver diagram and superpotential up to a permutation of the nodes.

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