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Virtual class of zero loci and Mirror Theorems

Department of Mathematics and Statistics
American University
4400 Massachusetts Ave, NW
Washington, DC 20016

aelezi@american.edu

Artur Elezi

Abstract

Let Y be the zero loci of a regular section of a convex vector bundle E over X. We provide a proof of a conjecture of Cox, Katz and Lee for the virtual class of the genus zero moduli of stable maps to Y. This in turn yields the expected relationship between Gromov-Witten theories of Y and X which together with Mirror Theorems allows for the calculation of enumerative invariants of Y inside of X.

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0 Introduction

Let X be a smooth, projective variety over $\mathbb{C}$. A vector bundle

$$E \to X$$

is called *convex* if $H^1(f^*(E)) = 0$ for any nonconstant morphism $f : \mathbb{P}^1 \to X$. Let $Y = Z(s) \subset X$ be the zero locus of a regular section s of a convex vector bundle E and let i denote the embedding of Y in X. It is the relationship between the Gromov-Witten theories of Y and X that we study here.

0.1 Virtual class of the zero loci

Let $\overline{M}_{0,n}(X,d)$ be the $\mathbb{Q}$-scheme that represents coarsely genus zero, n-pointed stable maps

$$(C, x_1, x_2, ..., x_n, f : C \to X)$$

of class $d \in H_2(X, \mathbb{Z})$. Since E is convex the vector spaces $H^0(f^*(E))$ fit into a $\mathbb{Q}$-vector bundle E_d on $\overline{M}_{0,n}(X,d)$. The section s of E induces a section $\tilde{s}$ of E_d over $\overline{M}_{0,0}(X,d)$ via $\tilde{s}((C,f)) = s \circ f$. If $i_*(\beta) = d$ the map $i : Y \hookrightarrow X$ yields an inclusion $i_\beta : \overline{M}_{0,0}(Y,\beta) \to \overline{M}_{0,0}(X,d)$. Clearly

$$Z(\tilde{s}) = \coprod_{i_*(\beta)=d} \overline{M}_{0,0}(Y,\beta).$$

The map $i_* : H_2Y \to H_2X$ is not injective in general, hence the zero locus $Z(\tilde{s})$ may have more then one connected component. An example is the quadric surface in $\mathbb{P}^3$.

The $\mathbb{Q}$-normal bundle of $Z(\tilde{s})$ in $\overline{M}_{0,0}(X,d)$ is $E_d|_{Z(\tilde{s})}$. In Section 2.1 of this paper we prove the following theorem [5]

Theorem 0.1.1. *For any $d \in H_2(X,\mathbb{Z})$,*

$$\sum_{i_*\beta=d} (i_\beta)_*[\overline{M}_{0,0}(Y,\beta)]^{vir} = c_{top}(E_d) \cap [\overline{M}_{0,0}(X,d)]^{vir}.$$

where c_{top} denotes the top chern operator. The first proof appeared in [10] and uses Behrend-Fantechi construction of the virtual fundamental class. Our approach is based on the alternative Tian-Lian construction.

Remark 0.1.1. *Theorem* 0.1.1 *was conjectured in [5] in the language of stacks. It is easily seen that our formulation is equivalent to the stack one, at least from a point of view of calculations. The original conjecture is stated for any n-pointed moduli stack. But one of the functorial properties of the virtual class is that*

$$\pi_n^*([\overline{M}_{0,n-1}(Y,\beta)]^{virt}) = [\overline{M}_{0,n}(Y,\beta)]^{virt}.$$

It is also true that $\pi_{n+1}^*(E_d) = E_d$ [6]. Suffices then to prove the theorem for 0*-pointed stable maps.*

0.2 Mirror Theorems

Theorem 0.1.1 together with Mirror Theorem provide a complete answer to the relationship between the enumerative invariants of Y and those of X.

Let $\{T_0 = 1, T_1, ..., T_m\}$ a basis of $H^*(Y,\mathbb{Q})$ such that $\mathrm{Span}\{T_1, ..., T_r\} = H^2(Y,\mathbb{Q})$. Let $t_1, ..., t_r$ be variables and $tT = \sum_{i=1}^r t_i T_i$. We denote by c the cotangent line class on $\overline{M}_{0,1}(Y,\beta)$, i.e. the first chern class of the line bundle whose fiber over moduli point (C, x_1, f) is $T^{\vee}_{C,x_1}$. Let e be the evaluation map on $\overline{M}_{0,1}(Y,\beta)$. Let $\hbar$ be a formal parameter. Then the quantum $\mathcal{D}$-module structure of the pure quantum cohomology of Y is determined by the following formal function [8]:

$$J_Y := e^{\frac{tT}{\hbar}} \cap \left(1 + \sum_{0\neq\beta\in H_2(Y,\mathbb{Z})} q^\beta e_* \left(\frac{[\overline{M}_{0,1}(Y,\beta)]^{\mathrm{vir}}}{\hbar(\hbar - c)}\right)\right). \tag{1}$$

Note that $e^{\frac{tT}{\hbar}}$ acts via its power series expansion (which is finite). For our purposes J_Y will be viewed as an element of the Novikov completion $H_*Y[[t_1, ..., t_r, \hbar^{-1}]][[q^\beta]]$ of the ring $H_*Y[[t_1, ..., t_r, \hbar^{-1}]]$ along the semigroup of rational curves β in Y. This generator encodes Gromov-Witten invariants and gravitational descendants of Y.

For each stable map $(C, x_1, f) \in \overline{M}_{0,1}(X, d)$ the sections of $H^0(C, f^*(E))$ that vanish at x_1 form a bundle E'_d that fits into an exact sequence:

$$0 \to E'_d \to E_d \to e^*(E) \to 0 \tag{2}$$

The quantum $\mathcal{D}$-module structure of the E-twisted quantum cohomology of X [6][8] is determined by the following formal function:

$$J_E := e^{\frac{tT'}{\hbar}} c_{\mathrm{top}}(E) \cap \left(1 + \sum_{0\neq d\in H_2(X,\mathbb{Z})} q^d e_* \left(\frac{c_{\mathrm{top}}(E'_d) \cap [\overline{M}_{0,1}(X,d)]^{\mathrm{vir}}}{\hbar(\hbar - c)}\right)\right)$$

where tT' and c denote similar expressions to those in J_Y. Mirror theorem states that for a large class of smooth varieties X, the generator J_E is computable via hypergeometric series [2],[6],[8],[11]. While important in itself, this fact is relevant with respect to the Gromov-Witten theory of Y only if one can show that J_E is intrinsically related to Y. The basic example is when X is a projective bundle and E is a direct sum positive line bundles. It has been shown in [4] that in this case $i_*(J_Y) = J_E$. Coupled with the fact that J_E is computable this allows for the calculation of (at least some of) gravitational descendants of Y. This is for example how one computes the enumerative invariants of the quintic threefold. In section 2.2 we use Theorem 0.1.1 to prove the following generalization: Assume that the map $i_* : H_2(Y) \to H_2(X)$ is surjective. Complete a basis $\{T'_1, T'_2, ..., T'_r\}$ of H^2X into a basis $\{T_1 = i^*(T'_1), ..., T_r = i^*(T'_r), ..., T_m\}$ of H^2Y. We extend the map $i_* : H_*Y \to H_*X$ to a homomorphism of completions $i_* : H_*Y[[t_1, ..., t_m, \hbar^{-1}]][[q^\beta]] \to H_*X[[t_1, ..., t_r, \hbar^{-1}]][[q^d]]$ via $i_*(t_k) = 0$ for $k > r$ and $i_*(q^\beta) = q^{i_*(\beta)}$.

Theorem 0.2.1. *Assume that $i_* : H_2(Y) \to H_2(X)$ is surjective. Then $i_*(J_Y) = J_E$.*

1 Background

1.1 Stable maps and Gromov-Witten invariants

Let g, n nonnegative numbers and $d \in H_2(X, \mathbb{Z})$. *A stable map* of genus g with n-markings consists of a nodal curve C, an n-tuple $(x_1, x_2, ..., x_n)$ of smooth points of C and a map $f : C \to X$ that has a finite group of automorphisms. A stable map $(C, x_1, ..., x_n, f)$ is said to *represent the curve class d* if $f_*[C] = d$. The moduli functor that parameterizes such stable maps is a proper Deligne-Mumford stack $\overline{\mathcal{M}}_{g,n}(X, d)$ and is coarsely represented by a $\mathbb{Q}$-scheme denoted by $\overline{M}_{g,n}(X, d)$ (see section 1.3 for a discussion on the category of $\mathbb{Q}$-schemes). The expected dimension of this moduli stack is $(\dim X - 3)(1 - g) + n - K_X \cdot \beta$. Let x_k be one of the marked points. The evaluation morphism $e_k : \overline{M}_{g,n}(X, d) \to X$ sends a closed point $(C, x_1, ..., x_n, f)$ to $f(x_k)$. The contangent bundle at x_k is denoted by $\mathcal{L}_k$. Its fiber over the closed point $(C, x_1, ..., x_n, f)$ is $T^\vee_{C,x_k}$. The forgetful morphism $\pi_k : \overline{M}_{g,n+1}(X, d) \to \overline{M}_{g,n}(X, d)$ forgets the k-th marking and stabilizes the source curve. The universal stable map over $\overline{M}_{g,n}(X,d)$ is

$$\begin{array}{ccc} \overline{M}_{g,n+1}(X,d) & \xrightarrow{e_{n+1}} & X \\ \big\downarrow{\scriptstyle \pi_{n+1}} & & \\ \overline{M}_{g,n}(X,d). & & \end{array}$$

The bundle E_d from the introduction may be precisely defined as $E_d := \pi_{n+1_*} e^*_{n+1}(E)$.

1.2 The virtual fundamental class and the associativity of the refined Gysin maps

The moduli spaces of stable maps may behave badly in families and they may have components whose dimension is bigger than the expected dimension. However there is a cycle of the expected dimension either in the Chow ring of the stack [1] or in the Chow ring of the coarse moduli space [13] which is deformation invariant. This class is used instead of the true fundamental class for intersection theory purposes. In this section we review the Li-Tian construction of the virtual fundamental class and a key lemma about the associativity of the refined Gysin maps.

The virtual fundamental class of a moduli functor is constructed in general using solely a choice of a tangent-obstruction complex. Our interest here is the moduli functor $\mathcal{F}^d_X$ of 0-pointed, genus zero, degree d stable maps to X. We describe the natural tangent-obstruction complex of $\mathcal{F}^d_X$. Let $\eta \in \mathcal{F}^d_X(S)$ be represented by the following diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & X \\ \big\downarrow{\scriptstyle \pi} & & \\ S & & \end{array}$$

The deformations and obstructions of η are described respectively by the global sections of the sheaves $\mathcal{T}^1\mathcal{F}^d_X(\eta) := \mathcal{E}xt^1_{\mathcal{X}/S}([f^*(\Omega_X) \to \Omega_{\mathcal{X}/S}], \mathcal{O}_{\mathcal{X}})$ and $\mathcal{T}^2\mathcal{F}^d_X(\eta) := \mathcal{E}xt^2_{\mathcal{X}/S}([f^*(\Omega_X) \to \Omega_{\mathcal{X}/S}], \mathcal{O}_{\mathcal{X}})$. The natural tangent obstruction-complex for this moduli problem is $\mathcal{T}^\bullet_\eta := [\mathcal{T}^1\mathcal{F}_X(\eta) \to \mathcal{T}^2\mathcal{F}_X(\eta)]$ with the zero arrow. This complex is *perfect* in the sense that locally, there is a 2-term complex of locally free sheaves $\mathcal{E}^\bullet_\eta := [\mathcal{E}_{\eta,1} \to \mathcal{E}_{\eta,2}]$ whose sheaf cohomology yields the tangent-obstruction complex $\mathcal{H}^\bullet\mathcal{E}^\bullet_\eta = \mathcal{T}^\bullet_\eta$.

The *virtual fundamental class* of $\mathcal{T}^\bullet_\eta$ is a Chow class in the Chow ring of the $\mathbb{Q}$-scheme $\overline{M}_{g,n}(X,d)$. We denote it by $[\overline{M}_{g,n}(X,d)]^{\mathrm{vir}}$. This class corresponds under Poincaré duality to $1 \in H^*(\overline{M}_{g,n}(X,d))$.

The key to the proof of Theorem 0.1.1 is a lemma about the associativity of the refined Gysin maps. Let us first formulate it for representable functors. Consider a fibre diagram

$$\begin{array}{ccc} W_0 & \xrightarrow{\delta_0} & W \\ \Big\downarrow{\scriptstyle \alpha_0} & & \Big\downarrow{\scriptstyle \alpha} \\ T_0 & \xrightarrow{\delta} & T \end{array} \tag{3}$$

where δ is a regular embedding. Let $\mathcal{N}$ be the normal bundle of T_0 in T. Assume that W and W_0 admit perfect tangent obstruction-complexes $\mathcal{T}^\bullet_W$ and $\mathcal{T}^\bullet_{W_0}$. They are said to be *compatible* relative to the fibre diagram (3) if for each affine scheme S and for any morphism $\eta : S \to W_0 \subset W$ there is an exact sequence

$$0 \to \mathcal{T}^1_{W_0}(\eta) \to \mathcal{T}^1_W(\eta) \to (\alpha_0 \circ \eta)^*\mathcal{N} \to \mathcal{T}^2_{W_0}(\eta) \to \mathcal{T}^2_W(\eta) \to 0. \tag{4}$$

Assume that this compatibility satisfies a *technical condition*. Namely, there exists a short exact sequence of 2-term complexes

$$0 \to [0 \to \alpha_0^*\mathcal{N}] \to \tilde{\mathcal{E}}^\bullet_\eta \to \mathcal{E}^\bullet_\eta \to 0 \tag{5}$$

such that

- Its long exact sequence of cohomologies is precisely the exact sequence of the compatibility.
- The cohomologies of $\tilde{\mathcal{E}}^\bullet_\eta$ and $\mathcal{E}^\bullet_\eta$ yield the tangent-obstruction complex of W_0.

Proposition 1.2.1. *[13] Assume that $\mathcal{T}^\bullet_W$ and $\mathcal{T}^\bullet_{W_0}$ are compatible and the technical condition (5) is satisfied. Then*

$$\delta^![W]^{vir} = [W_0]^{vir}$$

where the virtual cycles are with respect to $\mathcal{T}^\bullet_W$ and $\mathcal{T}^\bullet_{W_0}$.

The case of interest for us is the localized top chern class [7] of (E, s), where $E \to Z$ is a vector bundle and s is a section of E. We use diagram (3) with $T_0 = W = Z$ and T the total space of E. Let $\delta = s_E$ be the zero section of E and $\alpha = s$. It follows that $W_0 = Z(s)$ is the zero locus of s.

Corollary 1.2.1. *With the assumptions of the previous proposition*

$$\alpha_{0*}[Z(s)]^{vir} = c_{top}(E) \cap [Z]^{vir} \tag{6}$$

1.3 The category of $\mathbb{Q}$-schemes

Stable maps have nontrivial automorphisms hence the moduli functor of stable maps is only locally representable. It follows that in dealing with virtual fundamental class and Gromov-Witten invariants one has to work either in the category of $\mathbb{Q}$-schemes or in the category of stacks. We chose to work in the category of $\mathbb{Q}$-schemes after a failed attempt to prove a stack version of Proposition 1.2.1. Such a version has since appeared in [10]. The category of $\mathbb{Q}$-schemes is a natural generalization of Mumford's notion of $\mathbb{Q}$-varieties. In the étale topology, $\mathbb{Q}$-schemes look like finite quotients of quasiprojective schemes. The precise definition is due to Lian-Tian.

Definition 1.3.1. *A $\mathbb{Q}$-scheme is a scheme U together with the following data:*

- *A finite collection $(U_\alpha, G_\alpha, q_\alpha)$ where U_α is a quasi-projective scheme, G_α is a finite group acting on U_α and $q_\alpha : U_\alpha/G_\alpha \to U$ is an étale map such that $U = \cup Im(q_\alpha)$.*
- *For each pair of indices (α, β) there is*
$$(U_{(\alpha,\beta)}, G_{(\alpha,\beta)} = G_\alpha \times G_\beta, q_{(\alpha,\beta)})$$
together with with equivariant finite étale maps
$$U_{(\alpha,\beta)} \to U_\alpha, U_{(\alpha,\beta)} \to U_\beta$$
such that $Imq_{(\alpha,\beta)}) = Imq_\alpha) \cap Imq_\beta$ and the map $q_{(\alpha,\beta)}$ factors through both q_α and q_β via the above maps.
- *For any triple (α, β, γ) there is $(U_{(\alpha,\beta,\gamma)}, G_{(\alpha,\beta,\gamma)}, q_{(\alpha,\beta,\gamma)})$ with $G_{(\alpha,\beta,\gamma)} = G_\alpha \times G_\beta \times G_\gamma$ together with equivariant, finite, étale maps from $U_{(\alpha,\beta,\gamma)}$ to $U_{(\alpha,\beta)}, U_{(\alpha,\gamma)}, U_{(\beta,\gamma)}$ which commute with the maps introduced in the second condition and such that*
$$Imq_{(\alpha,\beta,\gamma)} = Imq_\alpha \cap Imq_\beta \cap Imq_\gamma.$$

Features such as morphisms, sheaves, cones in the category of $\mathbb{Q}$-schemes are defined as equivariant local objects that descend to the level of $\mathbb{Q}$-schemes. Using this recipe we define $\mathbb{Q}$-sheaves and $\mathbb{Q}$-complexes on a $\mathbb{Q}$-scheme. The chern classes of a $\mathbb{Q}$-bundle E on a $\mathbb{Q}$-scheme U may be defined as cohomology classes with rational coefficients by imitating the definition of the chern classes of vector bundles on Deligne-Mumford stacks. They are the chern classes of the $\mathbb{Q}$-cone on U obtained by the patching of the

G_α-quotients of E on U_α. Let s be a section of E given locally by s_α. The zero locus $Z(s)$ is a closed $\mathbb{Q}$-subscheme of U with chart $Z(s_\alpha) \cap U_\alpha$. The localized top chern class of (E, s) may be defined by imitating the construction of Fulton [7] as follows. Let F be a pure n-dimensional subscheme of U. The group G_α acts on the normal cone to $Z(s_\alpha) \cap q_\alpha^{-1}(F)$ in $q_\alpha^{-1}(F)$. The quotients can be patched together to a $\mathbb{Q}$-cone $C_{Z(s)/F}$ inside of the restriction to $Z(s)$ of the $\mathbb{Q}$-bundle E . Let i be the zero section of this cone. Then the action of the localized top chern class of (E, s) on F is $i^![C_{Z(s)/F}]$. One can easily check that, with this definition, Corollary 1.2.1 holds. The technical assumption is the same; the only difference is that the complexes become $\mathbb{Q}$-complexes.

2 The Proofs

2.1 The associativity of the refined Gysin maps in the category of $\mathbb{Q}$-schemes.

It is known that $\overline{M}_{0,0}(X, d)$ has a $\mathbb{Q}$-scheme structure so that the functors $\mathrm{Hom}(-, \overline{M}_{0,0}(X, d))$ and $\mathcal{F}_X^d$ are equivalent. Here is a brief sketch. Let $(C, f) \in \overline{M}_{0,0}(X, d)$ be a stable map. Choose divisors $H_1, H_2, ..., H_r$ so that f intersects each H_i transversally at $y_{i1}, ..., y_{id_i}$ and $(\tilde{C}, y_{ij})$ has no automorphisms. Now $\mathrm{Aut}(f)$ acts on $(\tilde{C}, f) \in \overline{M}_{0,dr}(X, d) \cap e_{ij}^*(H_i)$ by permuting the markings. Choose a quasiprojective $\mathrm{Aut}(f)$-equivariant neighborhood $U_f \subset \overline{M}_{0,dr}(X, d) \cap e_{ij}^*(H_i)$ of $(\tilde{C}, f)$ such that all the stable maps in U_f have no automorphisms and the map that forgets the markings does not change the source curve. There is an action of $\mathrm{Aut}(f)$ on the universal stable map $(\mathcal{C}_f, F_f)$ over U_f. The classifying map $U_f/\mathrm{Aut}(f) \to \overline{M}_{0,0}(X, d)$ is étale. The neighborhoods U_f satisfy the conditions of a $\mathbb{Q}$-scheme.

It has been pointed out in [13] that the virtual fundamental class can be constructed alternatively as follows: First, one constructs a local virtual class $[U_f]^{\mathrm{vir}}$ using the tangent-obstruction complex of the universal family over U_f. Then the $\mathrm{Aut}f$-quotients of these local virtual classes patch to the virtual class $[\overline{M}_{0,0}(X, d)]^{\mathrm{vir}}$. Clearly, the group $\mathrm{Aut}f$ acts on the restriction of the universal family $(\mathcal{C}_f, F_f)$ over $U_f^s := Z(\tilde{s}) \cap U_f$ and the classifying map $U_f^s/\mathrm{Aut}(f) \to Z(\tilde{s})$ is again étale. The local Gysin diagrams (3) patch to a global Gysin diagram in the category of $\mathbb{Q}$-schemes. Of crucial importance for us is that *Corollary 1.2.1 holds in the category of* $\mathbb{Q}$*-schemes.*

2.2 Virtual class of the zero loci

The proof of theorem 0.1.1 uses corollary 1.2.1. We need to check that the *technical condition* is satisfied. Let η be a 0-pointed, genus zero stable map of class d over an affine scheme S represented by the following diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & Y \subset X \\ \Big\downarrow{\scriptstyle \pi} & & \\ S & & \end{array}$$

The deformations and obstructions of η are described respectively by the global sections of the sheaves $\mathcal{T}^1\mathcal{F}_X^d(\eta) := \mathcal{E}xt^1_{\mathcal{X}/S}([f^*(\Omega_X) \to \Omega_{\mathcal{X}/S}], \mathcal{O}_{\mathcal{X}})$ and $\mathcal{T}^2\mathcal{F}_X^d(\eta) := \mathcal{E}xt^2_{\mathcal{X}/S}([f^*(\Omega_X) \to \Omega_{\mathcal{X}/S}], \mathcal{O}_{\mathcal{X}})$. Since the normal bundle of Y in X is $E|_Y$ the conormal exact sequence writes

$$0 \to f^*(E^*|_Y) \to f^*(\Omega_X|_Y) \to f^*(\Omega_Y) \to 0 \tag{7}$$

Recall from [13] that there is a short exact sequence of $\mathcal{O}_{\mathcal{X}}$-sheaves

$$0 \to W_2 \to W_1 \to f^*\Omega_X \to 0 \tag{8}$$

such that:

- $\mathcal{E}xt^i_{\mathcal{X}/S}([W_1 \to \Omega_{\mathcal{X}/S}], \mathcal{O}_{\mathcal{X}})$ and $\mathcal{E}xt^i([W_2 \to 0], \mathcal{O}_{\mathcal{X}})$ vanish for $i \neq 1$.
- Both $\mathcal{E}_{\eta,1} = \mathcal{E}xt^1_{\mathcal{X}/S}([W_1 \to \Omega_{\mathcal{X}/S}], \mathcal{O}_{\mathcal{X}})$ and $\mathcal{E}_{\eta,2} = \mathcal{E}xt^1([W_2 \to 0], \mathcal{O}_{\mathcal{X}})$ are locally free.
- The sheaf cohomology of the complex $\mathcal{E}_\eta^\bullet = [\mathcal{E}_{\eta,1} \to \mathcal{E}_{\eta,2}]$ is the tangent-obstruction complex of the stable map η, i.e. there is an exact sequence

$$0 \to \mathcal{T}^1\mathcal{F}_X^d(\eta) \to \mathcal{E}_{\eta,1} \to \mathcal{E}_{\eta,2} \to T^2\mathcal{F}_X^d(\eta) \to 0.$$

We pull back (8) exact sequence via (7) and obtain the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & W_2 \arrow[r] \arrow[d] & \mathcal{A} \arrow[r] \arrow[d] & f^*(E^*) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W_2 \arrow[r] \arrow[d] & W_1 \arrow[r] \arrow[d] & f^*(\Omega_X|_Y) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & f^*(\Omega_Y) \arrow[r] \arrow[d] & f^*(\Omega_Y) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Let $\tilde{\mathcal{E}}_{\eta,2} := \mathcal{E}xt^1_{\mathcal{X}/S}([\mathcal{A} \to 0], \mathcal{O}_{\mathcal{X}})$. We apply the long exact sequence for $\mathcal{E}xt$ to various exact sequences obtained from the above diagram. The middle vertical sequence yields a short exact sequence

$$0 \to [\mathcal{A} \to 0] \to [W_1 \to \Omega_{\mathcal{X}/S}] \to [f^*(\Omega_Y) \to \Omega_{\mathcal{X}/S}] \to 0.$$

Its long exact sequence for $\mathcal{E}xt$ yields

$$0 \to \mathcal{T}^1\mathcal{F}^\beta_Y(\eta) \to \mathcal{E}_{\eta,1} \to \tilde{\mathcal{E}}_{\eta,2} \to \mathcal{T}^2\mathcal{F}^\beta_Y(\eta) \to 0.$$

Let $\tilde{\mathcal{E}}^\bullet_\eta := [\mathcal{E}_{\eta,1} \to \tilde{\mathcal{E}}_{\eta,2}]$. The last exact sequence says that the cohomology of $\tilde{\mathcal{E}}^\bullet_\eta$ is the tangent obstruction complex of η. Next, we apply the long exact sequence for $\mathcal{E}xt$ to the top horizontal row of the diagram and use the conditions for W_i's. We obtain

$$0 \to \mathcal{E}xt^1_{\mathcal{X}/S}([f^*(E^*) \to 0], \mathcal{O}_{\mathcal{X}}) \to \tilde{\mathcal{E}}_{\eta,2} \to \mathcal{E}_{\eta,2} \to \mathcal{E}xt^2_{\mathcal{X}/S}([f^*(E^*) \to 0], \mathcal{O}_{\mathcal{X}}).$$

But one easily sees that

$$\mathcal{E}xt^2_{\mathcal{X}/S}([f^*(E^*) \to 0], \mathcal{O}_{\mathcal{X}}) \simeq \mathcal{E}xt^1(f^*(E^*), \mathcal{O}_{\mathcal{X}}) = 0$$

and

$$\mathcal{E}xt^1_{\mathcal{X}/S}([f^*(E^*) \to 0], \mathcal{O}_{\mathcal{X}}) \simeq \pi_* f^*(E)$$

as $\mathcal{O}_S$-sheaves. It follows that there is an exact sequence

$$0 \to [0 \to \pi_*(f^*(E))] \to \tilde{\mathcal{E}}^\bullet_\eta \to \mathcal{E}^\bullet_\eta \to 0. \tag{9}$$

Its long exact sequence of sheaf cohomologies is easily seen to be

$$0 \to \mathcal{T}^1\mathcal{F}^\beta_Y(\eta) \to \mathcal{T}^1\mathcal{F}^d_X(\eta) \to \pi_* f^* E \to \mathcal{T}^2\mathcal{F}^\beta_Y(\eta) \to \mathcal{T}^2\mathcal{F}^d_X(\eta) \to 0. \tag{10}$$

The technical condition is satisfied.

2.3 Mirror Theorems

We recall the setup. Let i denote the embedding of Y in X. Assume that the map $i_* : H_2(Y) \to H_2(X)$ is surjective. Complete a basis $\{T'_1, T'_2, ..., T'_r\}$ of H^2X to a basis $\{T_1 := i^*(T'_1), ..., T_r := i^*(T'_r), T_{r+1}, ..., T_m\}$ of H^2Y. Let $tT = \sum_{i=1}^m t_iT_i$ and $tT' = \sum_{i=1}^r t_iT'_i$ where $t_1, ..., t_n$ are variables. The map $i_* : H_*Y \to H_*X$ extends to a homomorphism of completions

$$i_* : H_*Y[[t_1, ..., t_m, \hbar^{-1}]][[q^\beta]] \to H_*X[[t_1, ..., t_r, \hbar^{-1}]][[q^d]]$$

via $i_*(t_k) = 0$ for $k > r$ and $i_*(q^\beta) = q^{i_*(\beta)}$.

Theorem 2.3.1. *Assume that $i_* : H_2(Y) \to H_2(X)$ is surjective. Then $i_*(J_Y) = J_E$.*

Proof. Recall that

$$J_Y := e^{\frac{tT}{\hbar}} \cap \left(1 + \sum_{\beta \neq 0} q^\beta e_* \left(\frac{[\overline{M}_{0,1}(Y,\beta)]^{\mathrm{vir}}}{\hbar(\hbar - c)}\right)\right)$$

By the definition of i_* and the projection formula we obtain

$$i_*(J_Y) = e^{\frac{tT'}{\hbar}} \cap \left(e_*(1) + \sum_{\beta \neq 0} q^\beta i_* \left(e_* \left(\frac{[\overline{M}_{0,1}(Y,\beta)]^{\mathrm{vir}}}{\hbar(\hbar - c)}\right)\right)\right)$$

As we have said before any $\beta \in H_2(Y, \mathbb{Z})$ such that $i_*(\beta) = d$ induces a morphism $i_\beta : \overline{M}_{0,0}(Y, \beta) \to \overline{M}_{0,0}(X, d)$. Consider the following commutative diagram:

$$\begin{array}{ccc} \overline{M}_{0,1}(Y,\beta) & \xrightarrow{i_\beta} & \overline{M}_{0,1}(X,d) \\ \downarrow e & & \downarrow e \\ Y & \xrightarrow{i} & X \end{array}$$

The line bundle $\mathcal{L}_1$ on $\overline{M}_{0,1}(Y, \beta)$ is the pullback via i_β of the bundle on $\overline{M}_{0,1}(X, d)$. By the projection formula and Theorem 2.0.1:

$$i_* \left(\sum_{i_*(\beta)=d} e_* \left(\frac{[\overline{M}_{0,1}(Y,\beta)]^{\mathrm{vir}}}{\hbar(\hbar - c)}\right)\right) = e_* \left(\frac{\sum_{i_*\beta=d} (i_\beta)_*[\overline{M}_{0,0}(Y,\beta)]^{\mathrm{vir}}}{\hbar(\hbar - c)}\right)$$

$$= e_* \left(\frac{c_{\mathrm{top}}(E_d) \cap [\overline{M}_{0,0}(X,d)]^{\mathrm{vir}}}{\hbar(\hbar - c)}\right) \qquad (11)$$

The exact sequence (2) implies

$$c_{\text{top}}(E_d) = c_{\text{top}}(E'_d)e^*(c_{\text{top}}(E)) \tag{12}$$

hence by the projection formula

$$i_* \left(\sum_{i_*(\beta)=d} e_* \left(\frac{[\overline{M}_{0,1}(Y,\beta)]^{\text{vir}}}{\hbar(\hbar - c)} \right) \right)$$

$$= c_{\text{top}}(E) \cap e_* \left(\frac{c_{\text{top}}(E'_d) \cap [\overline{M}_{0,0}(X,d)]^{\text{vir}}}{\hbar(\hbar - c)} \right)$$

Note also that $i_*(1) = i_*([Y]) = c_{\text{top}}(E) \cap [X]$. Now the theorem follows readily.† □

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