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The quantisation of Poisson structures arising in Chern-Simons theory with gauge group $G \ltimes \mathfrak{g}^*$

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Abstract

We quantise a Poisson structure on H^{n+2g}, where H is a semidirect product group of the form $G \ltimes \mathfrak{g}^*$. This Poisson structure arises in the combinatorial description of the phase space of Chern-Simons theory with gauge group $G \ltimes \mathfrak{g}^*$ on $\mathbb{R} \times S_{g,n}$, where $S_{g,n}$ is a surface of genus g with n punctures. The quantisation of this Poisson structure is a key step in the quantisation of Chern-Simons theory with gauge group $G \ltimes \mathfrak{g}^*$. We construct the quantum algebra and its irreducible representations and show that the quantum double $D(G)$ of the group G arises naturally as a symmetry of the quantum algebra.

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1 Introduction

The aim of this paper is the quantisation of a Poisson structure which arises in the study of Chern-Simons gauge theory with semidirect product gauge group $H = G \ltimes \mathfrak{g}^*$, where G is a Lie group, $\mathfrak{g}^*$ the dual of its Lie algebra $\mathfrak{g}$ viewed as a vector space and G acts on $\mathfrak{g}^*$ in the co-adjoint representation. Such gauge groups occur in the Chern-Simons formulation of (2+1)-dimensional gravity [1], where the gauge group is the three-dimensional Poincaré group or Euclidean group, depending on the signature of spacetime. Besides their mathematical interest, Chern-Simons gauge theories with gauge groups of this type are therefore of physical relevance.

The Poisson algebra studied in this article, in the following referred to as flower algebra, was first defined by Alekseev, Grosse and Schomerus [2, 3, 4] in the context of an earlier work by Fock and Rosly [5]. Fock and Rosly showed that it is possible to describe the Poisson structure of the phase space of Chern-Simons theory with gauge group H, the moduli space of flat H-connections, on an oriented, punctured surface in terms of a graph embedded into this surface. By assigning a classical r-matrix for the gauge group H to each vertex of the graph, they define a Poisson structure on the space of graph connections. After Poisson reduction with respect to graph gauge transformations, this Poisson structure agrees with the canonical Poisson structure on the moduli space [6, 7]. Alekseev, Grosse and Schomerus specialised this description to a set of curves representing the generators of the surface's fundamental group as a particularly simple graph. They then obtain a Poisson structure on the space of holonomies associated to these generating curves, which - due to the resemblance of this set of curves to a flower - we will call the flower algebra. Via Poisson reduction with respect to simultaneous conjugation of the holonomies with elements of the gauge group H, it induces the canonical Poisson structure on the moduli space. Although the case of surfaces with a boundary is more involved due to additional degrees of freedom arising at the boundary, the flower algebra still remains an important ingredient.

The relevance of the flower algebra for the phase space of Chern-Simons gauge theory on a punctured surface with a boundary makes its quantisation an important task. For the case of compact, semisimple Lie groups this has been achieved by Alekseev, Grosse and Schomerus [2, 3, 4] with their formalism of combinatorial quantisation of Chern-Simons gauge theories. However, the case of (non-compact and non-semisimple) Lie groups of type $H = G \ltimes \mathfrak{g}^*$ such as the three-dimensional Poincaré group arising in (2+1)-dimensional gravity is less well investigated. In addition to the need to establish a quantisation procedure, the physical relevance of this case also calls for a more explicit description in terms of coordinates with a direct physical meaning.

In this article we show that this can be achieved for groups of type $H = G \ltimes \mathfrak{g}^*$, where G is a finite-dimensional, simply connected and unimodular Lie group. The assumptions of simply-connectedness and unimodularity are made for convenience; dropping them would lead to technical modifications without affecting the essence of our results. By extending and adapting the work of Alekseev and Malkin [8] to this case, we construct a bijective decoupling transformation which breaks up the Poisson structure of the flower algebra into a set of Poisson-commuting building blocks, a copy of the dual Poisson-Lie group H^* for each puncture and a copy of its Heisenberg double $D_+(H)$ for each handle. We quantise these building blocks and then define a quantum counterpart of the decoupling transformation to construct the quantum algebra for the original Poisson structure and its irreducible Hilbert space representations. After investigating the action of the quantum symmetries on the representation spaces, we relate them to the quantum double $D(G)$ of the group G.

The article is structured as follows: in Sect. 2 we establish the relevant definitions and notations and discuss various Poisson structures associated to groups $G \ltimes \mathfrak{g}^*$ that are relevant for our description of the flower algebra. Extending our treatment [9] of the universal cover $\widetilde{SO(2,1)} \ltimes \mathbb{R}^3$ of the Poincaré group in three dimensions, we introduce the flower algebra on a genus g surface with n punctures as defined in [2, 3, 4] and give an explicit description of its Poisson structure for groups of type $H = G \ltimes \mathfrak{g}^*$. We define a bijective decoupling transformation that maps this Poisson structure onto the direct sum of n copies of the dual Poisson-Lie group H^* and g copies of the Heisenberg double $D_+(H)$. Finally, we show that elements of the semidirect product group $G \ltimes \mathcal{C}^\infty(G)$ act as Poisson isomorphisms on the flower algebra and relate this group action to the action of the group H by global conjugation.

Sect. 3 describes the quantisation of the flower algebra. Starting from the decoupled Poisson structure, we construct the quantum algebra and its irreducible representations for each of the building blocks. We then define a quantum counterpart of the classical decoupling transformation to obtain a quantisation of the original brackets of the flower algebra.

In Sect. 4 we discuss symmetries acting on the quantised flower algebra. We determine how the group $G \ltimes \mathcal{C}^\infty(G)$ acts on the quantum algebra and how this action can be implemented as an action on the representation spaces. We establish the relation between this quantum symmetry and the quantum double $D(G)$ of the Lie group G.

Sect. 5 contains our outlook and conclusions.

2 The classical Poisson structure

2.1 Poisson structures associated to $G \ltimes \mathfrak{g}^*$ as a Poisson Lie group

We consider groups $H = G \ltimes \mathfrak{g}^*$ which are the semidirect product of a simply connected, unimodular Lie group G and the dual $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$. All Lie algebras are considered over $\mathbb{R}$ unless stated otherwise.

We parametrise group elements $h \in H$ according to

$$(u, \boldsymbol{a}) = (u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}) \qquad \text{with } u \in G,\ \boldsymbol{a}, \boldsymbol{j} \in \mathfrak{g}^*, \tag{2.1}$$

such that the group multiplication is given by

$$(u_1, \boldsymbol{a}_1) \cdot (u_2, \boldsymbol{a}_2) = (u_1 \cdot u_2, \boldsymbol{a}_1 + \mathrm{Ad}^*(u_1^{-1})\boldsymbol{a}_2). \tag{2.2}$$

Let J_a, P^a, $a = 1, \ldots, \dim G$, denote the generators of the Lie algebra $\mathfrak{h} = \mathrm{Lie}\, H = \mathfrak{g} \ltimes \mathfrak{g}^*$, such that the generators J_a, $a = 1, \ldots, \dim G$, generate $\mathfrak{g} = \mathrm{Lie}\, G$ and P^a, $a = 1, \ldots, \dim G$, generate $\mathfrak{g}^*$. The commutator is then given by

$$[J_a, J_b] = f_{ab}{}^{c}\, J_c \qquad [J_a, P^b] = -f_{ac}{}^{b}\, P^c \qquad [P^a, P^b] = 0, \tag{2.3}$$

where $f_{ab}{}^{c}$ are the structure constants of $\mathfrak{g}$. The Lie algebra $\mathfrak{h}$ admits the non-degenerate bilinear form

$$\langle J_a,\, J_b \rangle = 0 \qquad \langle P^a, P^b \rangle = 0 \qquad \langle J_a,\, P^b \rangle = \delta_a^b. \tag{2.4}$$

This allows us to view $\mathfrak{h}$ as the classical double of the Lie bialgebra $\mathfrak{g}$ with standard commutator and trivial cocommutator, where the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ is given by (2.4). It has a coboundary Lie bialgebra structure with commutator (2.3) and cocommutator $\delta : \mathfrak{h} \to \mathfrak{h} \otimes \mathfrak{h}$

$$\delta(J_a) = 0 \qquad \delta(P^a) = f_{bc}{}^{a}\, P^b \otimes P^c, \tag{2.5}$$

which arises from the classical r-matrix

$$r = P^a \otimes J_a \in \mathfrak{h} \otimes \mathfrak{h}. \tag{2.6}$$

This Lie bialgebra structure on the space $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{g}^*$ is the infinitesimal version, the tangent Lie bialgebra, of an associated Poisson-Lie structure on the group H. If we denote by $\tilde{P}^a_L, \tilde{P}^a_R, \tilde{J}^L_a, \tilde{J}^R_a$, $a = 1, \ldots, \dim G$, the right- and left-invariant vector fields on H

$$\begin{aligned}
\tilde{P}^a_R f(u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}) &= \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f\left((u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}) \cdot tP^a\right) \\
&= -\frac{\partial f}{\partial j_a}(u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}) \\
\tilde{P}^a_L f(u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}) &= \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f\left(-tP^a \cdot (u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j})\right) \\
&= \mathrm{Ad}^*(u)_b{}^a \frac{\partial f}{\partial j_b}(u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}) \\
\tilde{J}^R_a f(u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}) &= \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f\left((u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}) \cdot e^{tJ_a}\right) \\
&= J^R_a f(u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}) + f_{ab}{}^c \frac{\partial f}{\partial j_b}(u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}) j_c \\
\tilde{J}^L_a f(u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}) &= J^L_a f(u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}),
\end{aligned} \tag{2.7}$$

with $\boldsymbol{j} = j_a P^a$, $f \in \mathcal{C}^\infty(H)$ and the left-and right-invariant vector fields J^R_a, J^L_a on G

$$J^R_a f := \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f(ue^{tJ_a}) \qquad J^L_a f := \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f(e^{-tJ_a}u) \quad \text{for } f \in \mathcal{C}^\infty(G), \tag{2.8}$$

this Poisson-Lie structure is given by the Poisson bivector

$$B_H = \tilde{P}^a_L \wedge \tilde{J}^L_a - \tilde{P}^a_R \wedge \tilde{J}^R_a. \tag{2.9}$$

Similarly, there is a Poisson-Lie group structure associated to the dual $\mathfrak{h}^*$ of the Lie bialgebra $\mathfrak{h} = \mathfrak{g} \ltimes \mathfrak{g}^*$, the dual H^* of the Poisson-Lie group H. As a group, it is the direct product $G \times \mathfrak{g}^*$ with group multiplication $(u_1, \boldsymbol{j}_1) \cdot (u_2, \boldsymbol{j}_2) = (u_1 u_2, \boldsymbol{j}_1 + \boldsymbol{j}_2)$. The global diffeomorphism $H^* \to H$, $(u, \boldsymbol{j}) \mapsto (u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j})$ [10] allows us to describe its Poisson structure in terms of the following Poisson bivector on H

$$B_{H^*} = \frac{1}{2}\left(\tilde{P}^a_L \wedge \tilde{J}^L_a + \tilde{P}^a_R \wedge \tilde{J}^R_a\right) + \tilde{P}^a_R \wedge \tilde{J}^L_a. \tag{2.10}$$

A more explicit formula in terms of the parametrisation (2.1) is given in Sect. 2.3, (2.26). The symplectic leaves of this Poisson structure are the conjugacy classes in H [11, 12]. As they play an important role in the quantisation of the flower algebra, we need to introduce some additional notation that will allow us to take a more geometric viewpoint and relate them to the conjugacy classes in the group G. For an element $u \in G$ let

$N_u = \{n \in G | nun^{-1} = u\}$ denote its stabiliser group with Lie algebra $\mathfrak{n}_u$ and dimension ν_u. Pick a basis $\{J_\alpha\}$, $\alpha = 1, \ldots, \nu_u$ of $\mathfrak{n}_u$ and complete it to a basis $\{J_a\}$, $a = 1, \ldots, \dim G$, of $\mathfrak{g}$. If we denote the dual basis of $\mathfrak{g}^*$ by $\{P_a\}$, $a = 1, \ldots, \dim G$, as above and define $\mathfrak{n}_u^* = \mathrm{Span}(P_1, \ldots, P_{\nu_u})$, we have the decomposition

$$\mathfrak{g}^* = \mathrm{Im}(1 - \mathrm{Ad}^*(u)) \oplus \mathfrak{n}_u^* \quad \text{for each } u \in G. \tag{2.11}$$

This decomposition gives rise to a convenient parametrisation of the conjugacy classes in the group H that clarifies their relation to the conjugacy classes in G. From a fixed element $(g, -\mathrm{Ad}^*(g^{-1})\boldsymbol{s}) \in H$ other elements $(u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j})$ in the same conjugacy class in H are obtained by conjugation with $(v, \boldsymbol{x}) \in H$ and explicitly given by

$$\begin{aligned} u &= vgv^{-1} \\ \boldsymbol{j} &= \mathrm{Ad}^*(v^{-1})\boldsymbol{s} + (1 - \mathrm{Ad}^*(u))\boldsymbol{x}. \end{aligned} \tag{2.12}$$

Equations (2.11), (2.12) allow us to characterise a conjugacy class in H by picking a group element $g_\mu \in G$ and an element $\boldsymbol{s} \in \mathfrak{n}_{g_\mu}^*$ in the dual Lie algebra of its stabiliser. After choosing $g_\mu \in G$, the remaining arbitrariness in the choice of $\boldsymbol{s}$ is parametrised by elements $n \in N_{g_\mu}$. Under their action $\boldsymbol{s}$ sweeps out a co-adjoint orbit $\mathcal{O}_s$ in $\mathfrak{n}_{g_\mu}^*$. Conjugacy classes $\mathcal{C}_{\mu s}$ in H are therefore uniquely characterised by G-conjugacy classes $\mathcal{C}_\mu = \{vg_\mu v^{-1} \,|\, v \in G\}$ and co-adjoint orbits $\mathcal{O}_s$ in $\mathfrak{n}_{g_\mu}^*$. With respect to fixed $g_\mu \in G$ and $\boldsymbol{s} \in \mathfrak{n}_{g_\mu}^*$, they are given as the image of the map

$$\begin{aligned} \mathrm{conj}_{\mu s} : H &\to \mathcal{C}_{\mu s} \\ (v, \boldsymbol{x}) &\mapsto (u, \boldsymbol{j}) = (v, \boldsymbol{x})(g_\mu, -\boldsymbol{s})(v, \boldsymbol{x})^{-1}. \end{aligned} \tag{2.13}$$

In geometric terms, this amounts to the following. The identification $T_u^*\mathcal{C}_\mu = \mathrm{Im}(1 - \mathrm{Ad}^*(u))$ allows us to write the H-conjugacy classes $C_{\mu s}$ locally as the product of the cotangent bundle $T^*\mathcal{C}_\mu$ and $\mathcal{O}_s$. With the projection $\pi_u : \mathfrak{g}^* \mapsto \mathfrak{g}^*/\mathfrak{n}_u^* \simeq \mathrm{Im}(1 - \mathrm{Ad}^*(u))$ we then have the bundle

$$\begin{aligned} \mathcal{C}_{\mu s} &\to T^*\mathcal{C}_\mu \\ (u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}) &\mapsto (u, (-\mathrm{Ad}^*(u^{-1}))\pi_u(\boldsymbol{j})) \end{aligned} \tag{2.14}$$

with typical fibre $\mathcal{O}_s$.

The third Poisson structure associated to the Lie bialgebra $\mathfrak{h}$ that we will be relevant in this article is the so-called Heisenberg double $D_+(H)$ of the Poisson-Lie group H. It is the Poisson structure on the direct product $H \times H$ defined by the following Poisson bivector

$$\begin{aligned} B_{D_+(H)} =& \frac{1}{2}\left(\tilde{P}^a_{R_1} \wedge \tilde{J}^{R_1}_a + \tilde{P}^a_{L_1} \wedge \tilde{J}^{L_1}_a + \tilde{P}^a_{R_2} \wedge \tilde{J}^{R_2}_a + \tilde{P}^a_{L_2} \wedge \tilde{J}^{L_2}_a\right) \\ &+ \tilde{P}^a_{R_1} \wedge \tilde{J}^{R_2}_a + \tilde{P}^a_{L_1} \wedge \tilde{J}^{L_2}_a, \end{aligned} \tag{2.15}$$

where $\tilde{P}^a_{R_i}$, $\tilde{P}^a_{L_i}$, $\tilde{J}^{R_i}_a$, $\tilde{J}^{L_i}_a$ denote the left-and right-invariant vector fields on the two copies of H. It has been shown by [12, 13] that this Poisson structure is symplectic; however, it is *not* a Poisson-Lie structure. We derive an explicit formula for this Poisson structure in Sect. 2.3,(2.27) and show that in suitable coordinates it is the canonical Poisson structure of the cotangent bundle $T^*(G \times G)$.

2.2 The flower algebra

After introducing the relevant concepts and definitions, we are now ready to discuss the flower algebra for semidirect product gauge groups $H = G \ltimes \mathfrak{g}^*$ on a genus g surface $S_{g,n}$ with n punctures and, possibly, a connected boundary.

The phase space of Chern-Simons theory on the surface $S_{g,n}$ is the moduli space of flat connections. It can be described in terms of a graph embedded into the surface [5]. The moduli space *as a space* is then obtained as the quotient of the space of flat graph connections modulo graph gauge transformations. However, as the canonical Poisson structure of the underlying Chern-Simons gauge theory does in general *not* induce a Poisson structure on the space of graph connections, this description can a priori not provide a description of the canonical Poisson structure on the moduli space. However, Fock and Rosly [5] succeeded in defining a (non-canonical) a Poisson structure on the space of graph connections that induces the canonical Poisson structure on the moduli space. They assign a classical r-matrix for the group H to each vertex of the graph, whose components with respect to a given basis of $\mathfrak{h}$ act as the structure constants of the resulting Poisson structure, and then show that reduction of this structure with respect to graph gauge transformations agrees with the canonical Poisson structure on the moduli space. Alekseev, Grosse and Schomerus [2, 3, 4] specialised this description to the simplest graph that can be used to describe the underlying surface: a set of curves representing the generators of its fundamental group. The space of graph connections is then simply the set of holonomies along these curves, and graph gauge transformations act on the holonomies via simultaneous conjugation. The resulting Poisson structure on the space of holonomies is the flower algebra.

The case of surfaces with boundaries is more involved, as gauge transformations that are nontrivial at the boundary acquire a physical meaning and are no longer divided out of the phase space. Depending on the boundary conditions imposed, there are additional degrees of freedom associated to the boundary which enter into the phase space. The Poisson structure then contains a contribution of these boundary degrees of freedom as well as a bulk term representing the internal degrees of freedom, subject to constraints relating the two contributions.

We can now define the flower algebra, summarising the results and definitions of [2, 3, 4]. The first ingredient is a set of generators for the fundamental group of the underlying Riemann surface. The fundamental group $\pi_1(S_{g,n})$ of a genus g surface $S_{g,n}$ with n punctures is generated by the equivalence classes of a loop m_i, $i = 1, \ldots, n$, around each puncture and two curves a_j, b_j, $j = 1, \ldots, g$ for each handle, see Fig. 1.

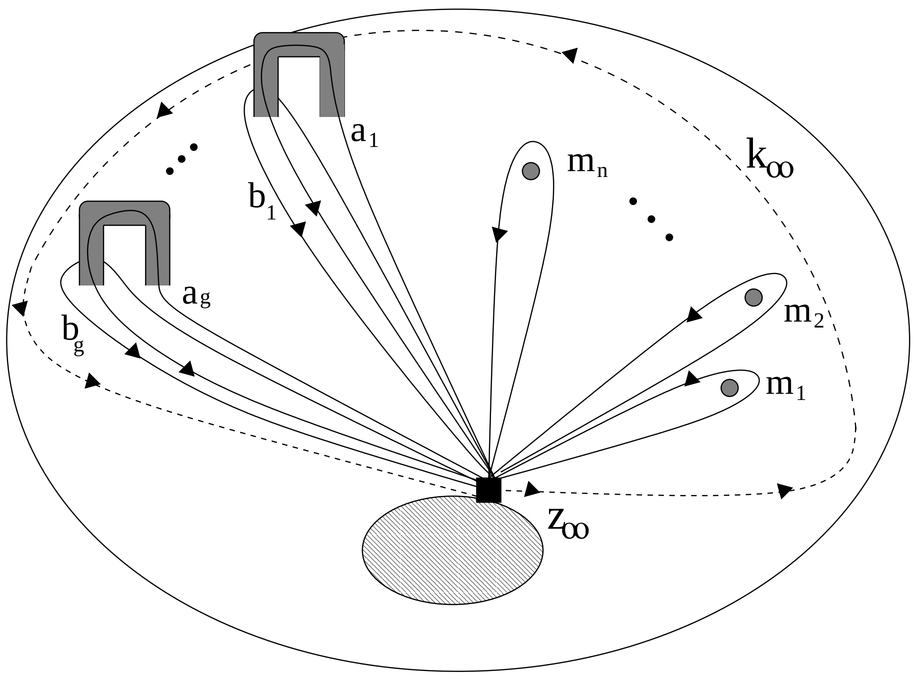

Fig. 1

The generators of the fundamental group of the surface $S^{\infty}_{g,n}$ with a boundary (shaded)

For a closed surface, these generators are subject to a single, defining relation

$$k_\infty = [b_g, a_g^{-1}] \cdot \ldots \cdot [b_1, a_1^{-1}] \cdot m_n \cdot \ldots \cdot m_1 = 1, \tag{2.16}$$

with group commutator $[b_i, a_i^{-1}] = b_i a_i^{-1} b_i^{-1} a_i$. In the case of a surface with connected boundary as shown in Fig. 1, they generate the fundamental group freely. Whereas the holonomies of the curves for each handle are general elements of the gauge group H, the holonomies corresponding to the punctures are restricted to fixed H-conjugacy classes $\mathcal{C}_{\mu_i s_i} \subset H$. Therefore, the space $\mathcal{A}_{g,n}$ of graph connections, or holonomies, is given by

$$\mathcal{A}_{g,n} = \big\{(M_1, \ldots, M_n, A_1, B_1, \ldots, A_g, B_g) \in \mathcal{C}_{\mu_1 s_1} \times \ldots \times \mathcal{C}_{\mu_n s_n} \times H^{2g} \mid \\ [B_g, A_g^{-1}] \cdot \ldots \cdot [B_1, A_1^{-1}] \cdot M_n \cdot \ldots \cdot M_1 = 1 \,\big\}. \tag{2.17}$$

The moduli space $\mathcal{M}_{g,n}$ of flat H-connections on a closed surface $S_{g,n}$ is obtained from this space by dividing out simultaneous conjugation of all holonomies by the group H

$$\mathcal{M}_{g,n} = \mathcal{A}_{g,n}/\sim\ , \tag{2.18}$$

where $\sim$ denotes simultaneous conjugation by an element of the group H. Following the work of Alekseev, Grosse and Schomerus [2, 3, 4], we then have the following definition of the flower algebra (see Theorem 2 in [4]).

Definition 2.1. *(Flower algebra)*

The flower algebra for gauge group H on a genus g surface $S_{g,n}$ with n punctures is the Poisson algebra $\mathcal{C}^\infty(H^{n+2g})$ defined by the following Poisson bivector

$$\begin{aligned} B_{FR} =& \sum_{i=1}^{n} r^{\alpha\beta}\left(\frac{1}{2}R^{M_i}_\alpha \wedge R^{M_i}_\beta + \frac{1}{2}L^{M_i}_\alpha \wedge L^{M_i}_\beta + R^{M_i}_\alpha \wedge L^{M_i}_\beta\right) \\ &+ \sum_{i=1}^{g} r^{\alpha\beta}\Big(\frac{1}{2}\left(R^{A_i}_\alpha \wedge R^{A_i}_\beta + L^{A_i}_\alpha \wedge L^{A_i}_\beta + R^{B_i}_\alpha \wedge R^{B_i}_\beta + L^{B_i}_\alpha \wedge L^{B_i}_\beta\right) \\ &\qquad + R^{A_i}_\alpha \wedge (R^{B_i}_\beta + L^{A_i}_\beta + L^{B_i}_\beta) + R^{B_i}_\alpha \wedge (L^{A_i}_\beta + L^{B_i}_\beta) + L^{A_i}_\alpha \wedge L^{B_i}_\beta\Big) \\ &+ \sum_{1\leq i<j\leq n} r^{\alpha\beta}(R^{M_i}_\alpha + L^{M_i}_\alpha) \wedge (R^{M_j}_\beta + L^{M_j}_\beta) \\ &+ \sum_{1\leq i<j\leq g} r^{\alpha\beta}(R^{A_i}_\alpha + L^{A_i}_\alpha + R^{B_i}_\alpha + L^{B_i}_\alpha) \wedge (R^{A_j}_\beta + L^{A_j}_\beta + R^{B_j}_\beta + L^{B_j}_\beta) \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{g} r^{\alpha\beta}(R^{M_i}_\alpha + L^{M_i}_\alpha) \wedge (R^{A_j}_\beta + L^{A_j}_\beta + R^{B_j}_\beta + L^{B_j}_\beta)\,, \end{aligned} \tag{2.19}$$

where elements of H^{n+2g} are denoted by $(M_1, \ldots, M_n, A_1, B_1, \ldots, A_g, B_g)$. The coefficients $r^{\alpha\beta}$ are the components of a classical r-matrix $r \in \mathfrak{h} \otimes \mathfrak{h}$ for H with respect to a given basis Z_α, $\alpha = 1, \ldots, \dim H$ of $\mathfrak{h}$ and L^α_X, R^α_X, $X = M_1, \ldots, M_n, A_1, B_1, \ldots, A_g, B_g$ the right-and left invariant vector fields corresponding to this basis.

With an expression for the classical r-matrix and the right-and left invariant vector fields on the different copies of H, formula (2.19) determines the Poisson brackets of two functions in $\mathcal{C}^\infty(H^{n+2g})$. However, in the case of groups of type $H = G \ltimes \mathfrak{g}^*$, there is an advantage in working with a slightly different definition of the flower algebra. We expand the vector $\boldsymbol{j}$ in (2.1) as $\boldsymbol{j} = j_b P^b$, and denote by the same symbol the maps $j_a \in \mathcal{C}^\infty(H) : (u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}) \mapsto j_a$. Instead of $\mathcal{C}^\infty(H)$ we then consider the algebra generated by the functions in $\mathcal{C}^\infty(G)$ together with these maps j_a. By inserting the r-matrix (2.6) into (2.19) together with the expressions (2.7) for the left- and right invariant vector fields, we obtain the Poisson brackets of these generating functions j_a and functions $F \in \mathcal{C}^\infty(G^{n+2g})$, resulting in the following alternative definition of the flower algebra.

Definition 2.2. *(Flower algebra for groups $G \ltimes \mathfrak{g}^*$)*

The flower algebra $\mathcal{F}$ for gauge group $H = G \ltimes \mathfrak{g}^*$ on a genus g surface $S_{g,n}$ with n punctures is the commutative Poisson algebra

$$\mathcal{F} = S\left(\bigoplus_{k=1}^{n+2g} \mathfrak{g}\right) \otimes \mathcal{C}^\infty(G^{n+2g}), \tag{2.20}$$

where $S\left(\bigoplus_{k=1}^{n+2g} \mathfrak{g}\right)$ is the symmetric envelope of the real Lie algebra $\bigoplus_{k=1}^{n+2g} \mathfrak{g}$, i.e. the polynomials with real coefficients on the vector space $\bigoplus_{l=1}^{n+2g} \mathfrak{g}$. In terms of a fixed basis $\mathcal{B} = \{j_a^{M_i}, j_a^{A_k}, j_a^{B_k},\ i = 1, \ldots, n,\ k = 1, \ldots, g,\ a = 1, \ldots, dim\, G\}$, its Poisson structure is given by

$$\begin{aligned}
&\{j_a^X \otimes 1, j_b^X \otimes 1\} = -f_{ab}{}^c\, j_c^X \otimes 1\\
&\{j_a^X \otimes 1, j_b^Y \otimes 1\} = -f_{db}{}^c\, j_c^Y \otimes (\delta_a{}^d - Ad^*(u_X)_a{}^d)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall X, Y \in \{M_1, \ldots, B_g\},\ X < Y\\
&\{j_a^{A_i} \otimes 1, j_b^{B_i} \otimes 1\} = -f_{ab}{}^c\, j_c^{B_i} \otimes 1\\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall i = 1, \ldots, g
\end{aligned}$$

$$\begin{aligned}
\{j_a^{M_i} \otimes 1, 1 \otimes F\} &= -1 \otimes (J_a^{R_{M_i}} + J_a^{L_{M_i}})F\\
&\quad - 1 \otimes (\delta_a{}^b - Ad^*(u_{M_i})_a{}^b)\left(\sum_{Y > M_i} (J_b^{R_Y} + J_b^{L_Y})F\right)
\end{aligned}$$

$$
\begin{aligned}
\{j_a^{A_i} \otimes 1, 1 \otimes F\} = & -1 \otimes (J_a^{R_{A_i}} + J_a^{L_{A_i}})F - 1 \otimes (J_a^{R_{B_i}} + J_a^{L_{B_i}})F \\
& - 1 \otimes Ad^*(u_{B_i}^{-1} u_{A_i})_a{}^b J_b^{R_{B_i}} \\
& - 1 \otimes (\delta_a{}^b - Ad^*(u_{A_i})_a{}^b) \left(\sum_{Y > A_i} (J_b^{R_Y} + J_b^{L_Y})F \right) \\
\{j_a^{B_i} \otimes 1, 1 \otimes F\} = & -1 \otimes J_a^{L_{A_i}} F - 1 \otimes (J_a^{R_{B_i}} + J_a^{L_{B_i}})F \\
& - 1 \otimes (\delta_a{}^b - Ad^*(u_{B_i})_a{}^b) \left(\sum_{Y > B_i} (J_b^{R_Y} + J_b^{L_Y})F \right),
\end{aligned}
\tag{2.21}
$$

where $F \in \mathcal{C}^\infty(G^{n+2g})$, $M_1 < \ldots < M_n < A_1, B_1 < \ldots < A_g, B_g$ and $J_a^{L_X}$, $J_a^{R_X}$ denote the right- and left invariant vector fields (2.8) *on the different copies of G.*

2.3 The decoupling transformation

We will now show how the flower algebra for groups of type $H = G \ltimes \mathfrak{g}^*$ can be broken down into a set of Poisson commuting building blocks. In doing this, we follow closely the work of Alekseev and Malkin [8] who treated the case of compact, semisimple Lie groups H. They give a bijective transformation that maps the Poisson structure on the moduli space $\mathcal{M}_{n,g}$ to the direct sum of n symplectic forms on H-conjugacy classes and g copies of the Heisenberg double $D_+(H)$. While, in general, this transformation is quite complicated, which makes it difficult to obtain an explicit expression in terms of coordinates, the picture is a lot simpler in the case $H = G \ltimes \mathfrak{g}^*$. Not only can the transformation of Alekseev and Malkin be generalised to this setting, but it is then possible to obtain an explicit expression in terms of the generators defined in Def. 2.2. This allows us to verify the asserted properties of this transformation by direct calculation.

Definition 2.3. *(Decoupling transformation)*

The decoupling transformation is the bijective transformation $K : \mathcal{F} \to \mathcal{F}$ of the flower algebra

$$
K: \qquad 1 \otimes F \mapsto 1 \otimes F, \quad j_a^{M_i} \otimes 1 \mapsto j_a^{M_i'}, \quad j_a^{A_j} \otimes 1 \mapsto j_a^{A_j'}, \quad j_a^{B_j} \otimes 1 \mapsto j_a^{B_j'}
\tag{2.22}
$$

for $F \in \mathcal{C}^\infty(G^{n+2g})$, $i = 1, \ldots, n$ and $j = 1, \ldots, g$. The transformed genera-

tors are given by

$$j_a^{M_i'} = j_a^{M_i} - ({\delta_a}^b - Ad^*(u_{M_i}){_a}^b)\Big(\sum_{k=i+1}^{n} Ad^*\big(u_{M_{k-1}}\cdots u_{M_{i+1}}\big){_b}^c \cdot j_c^{M_k}$$

$$+\sum_{k=1}^{g} Ad^*\big(u_{K_{k-1}}\cdots u_{K_1}\cdot u_{M_n}\cdots u_{M_{i+1}}\big){_b}^c\cdot j_c^{H_k}\Big) \tag{2.23}$$

$$j_a^{A_i'} = j_a^{A_i} + Ad^*(u_{A_i}^{-1}){_a}^b(j_b^{A_i} - j_b^{B_i}) + (Ad^*(u_{A_i}) - Ad^*(u_{B_i}u_{A_i}^{-1})){_a}^b\Big(\sum_{k=i+1}^{g} Ad^*\big(u_{K_{k-1}}\cdots u_{K_{i+1}}\big){_b}^c j_c^{H_k}\Big)$$

$$j_a^{B_i'} = j_a^{B_i} + Ad^*(u_{A_i}^{-1}){_a}^b(j_b^{A_i} - j_b^{B_i}) + (Ad^*(u_{B_i}) - Ad^*(u_{B_i}u_{A_i}^{-1})){_a}^b\Big(\sum_{k=i+1}^{g} Ad^*\big(u_{K_{k-1}}\cdots u_{K_{i+1}}\big){_b}^c j_c^{H_k}\Big),$$

where we write j_a^X *for* $j_a^X\otimes 1$ *and* F *for* $1\otimes F$. *The expressions* $Ad^*(u_X){_a}^b : (u_{M_1},\ldots,u_{B_g})\mapsto Ad^*(u_X){_a}^b$ *for* $X\in\{M_1,\ldots,B_g\}$ *are to be interpreted as functions in* $\mathcal{C}^\infty(G^{n+2g})$, *and we set*

$$u_{K_i} := u_{B_i}u_{A_i}^{-1}u_{B_i}^{-1}u_{A_i} \tag{2.24}$$

$$j_a^{H_i} := \big({\delta_a}^b - Ad^*(u_{A_i}^{-1}u_{B_i}^{-1}u_{A_i}){_a}^b\big)j_b^{A_i} + \big(Ad^*(u_{A_i}^{-1}u_{B_i}^{-1}u_{A_i}) - Ad^*(u_{B_i}^{-1}u_{A_i})\big){_a}^b j_b^{B_i}.$$

Its inverse is given by

$$j_a^{M_i} = j_a^{M_i'} + ({\delta_a}^b - Ad^*(u_{M_i}){_a}^b)\left(\sum_{k=i+1}^{n} j_b^{M_k'}\right) + ({\delta_a}^b - Ad^*(u_{M_i}){_a}^b)\cdot \tag{2.25}$$

$$\left(\sum_{k=1}^{g}\Big({\delta_b}^c - Ad^*(u_{A_k}^{-1}){_b}^c\Big) j_c^{A_k'} + \Big(Ad^*(u_{A_k}^{-1}) - Ad^*(u_{B_k}^{-1}u_{A_k})\Big){_b}^c j_c^{B_k'}\right)$$

$$j_a^{A_i} = j_a^{A_i'} - Ad^*(u_{A_i}^{-1}){_a}^b(j_b^{A_i'} - j_b^{B_i'}) + ({\delta_a}^b - Ad^*(u_{A_i}){_a}^b)\cdot \left(\sum_{k=i+1}^{g}\Big({\delta_b}^c - Ad^*(u_{A_k}^{-1}){_b}^c\Big) j_c^{A_k'} + \Big(Ad^*(u_{A_k}^{-1}) - Ad^*(u_{B_k}^{-1}u_{A_k})\Big){_b}^c j_c^{B_k'}\right)$$

$$j_a^{B_i} = j_a^{B_i'} - Ad^*(u_{A_i}^{-1}){_a}^b(j_b^{A_i'} - j_b^{B_i'}) + ({\delta_a}^b - Ad^*(u_{B_i}){_a}^b)\cdot \left(\sum_{k=i+1}^{g}\Big({\delta_b}^c - Ad^*(u_{A_k}^{-1}){_b}^c\Big) j_c^{A_k'} + \Big(Ad^*(u_{A_k}^{-1}) - Ad^*(u_{B_k}^{-1}u_{A_k})\Big){_b}^c j_c^{B_k'}\right).$$

With this definition, we can calculate the transformed bracket and verify that the transformation does indeed decouple the mixed contributions in (2.21) into Poisson-commuting building blocks.

Theorem 2.4. *The decoupling transformation K maps the Poisson structure* (2.21) *to the direct sum of n Poisson structures on the dual H^* and g copies of the symplectic structure of the Heisenberg double $D_+(H)$:*

$$
\begin{aligned}
&\{j_a^{M_i'}, j_b^{M_j'}\} = -\delta_{ij} f_{ab}{}^c\, j_c^{M_i'} \\
&\{j_a^{M_i'}, 1\otimes F\} = -1\otimes (J_a^{R_{M_i}} + J_a^{L_{M_i}})F \qquad \forall i,j=1,\ldots,n
\end{aligned}
\tag{2.26}
$$

$$
\begin{aligned}
&\{j_a^{A_i'}, j_b^{A_j'}\} = -\delta_{ij} f_{ab}{}^c\, j_c^{A_i'} \\
&\{j_a^{B_i'}, j_b^{B_j'}\} = -\delta_{ij} f_{ab}{}^c\, j_c^{B_i'} \\
&\{j_a^{A_i'}, j_b^{B_j'}\} = -\delta_{ij} f_{ab}{}^c\, j_c^{B_i'} \\
&\{j_a^{A_i'}, 1\otimes F\} = -1\otimes J_a^{R_{A_i}} F - 1\otimes (1 + Ad^*(u_{B_i}^{-1} u_{A_i}))_a{}^b J_b^{R_{B_i}} F \\
&\{j_a^{B_i'}, 1\otimes F\} = -1\otimes J_a^{R_{B_i}} F \qquad \forall i,j=1,\ldots,g
\end{aligned}
\tag{2.27}
$$

$$
\{j_a^{M_i'}, j_b^{A_j'}\} = \{j_a^{M_i'}, j_b^{B_j'}\} = 0 \qquad \forall i=1,\ldots,n,\ j=1,\ldots,g. \tag{2.28}
$$

Proof: The Poisson brackets (2.26), (2.27) of the transformed generators $j_a^{M_i'}, j_a^{A_i'}, j_a^{B_i'}$ can be calculated directly from the Poisson brackets (2.21) of the flower algebra. We then insert the expressions (2.7) for the vector fields on H in the Poisson bivectors (2.10) and (2.15) of the dual H^* and the Heisenberg double $D_+(H)$ and apply them to the functions $j_a \in \mathcal{C}^\infty(H)$: $(u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}) \mapsto j_a$ to verify that the result does agree with the decoupled brackets (2.26) and (2.27). □

Note that K may also be viewed as the pullback of a map $H^{n+2g} \to H^{n+2g}$ which is the identity on G^{n+2g} and leaves each of the conjugacy classes $\mathcal{C}_{\mu_i s_i}$ invariant. From (2.23) we see that these maps add to $\boldsymbol{j}_{M_i}$ respectively $\boldsymbol{j}_{M_i'}$ an element of $\bigoplus_{k=1}^{n+2g} \mathfrak{g}\otimes \mathcal{C}^\infty(G^{n+2g})$ preceded by a factor $(1-\mathrm{Ad}^*(u_{M_i}))$. It follows from (2.12) that such transformations map a given conjugacy class into itself.

The transformation K simplifies the Poisson structure of the flower algebra considerably by decoupling the contributions of different punctures and handles. However, it is still possible to simplify the resulting Poisson structure further by breaking up the Heisenberg double Poisson structure (2.27) associated to each handle. Defining a map $L : H^{2g+n} \to H^{2g+n}$ via

$$\begin{array}{rcll} (u_{A_i}, u_{B_i}) & \mapsto & (w_{1,i}, w_{2,i}) = (u_{A_i}, u_{B_i}^{-1} u_{A_i}) & \\ (\boldsymbol{j}_{A_i'}, \boldsymbol{j}_{B_i'}) & \mapsto & (\boldsymbol{k}_{1,i}, \boldsymbol{k}_{2,i}) = (\boldsymbol{j}_{A_i'} - \boldsymbol{j}_{B_i'}, \boldsymbol{j}_{B_i'}) & i = 1, \dots, g \\ u_{M_i} & \mapsto & u_{M_i} & \\ \boldsymbol{j}_{M_i'} & \mapsto & \boldsymbol{j}_{M_i'} & i = 1, \dots, n, \end{array} \tag{2.29}$$

we can transform each Heisenberg double into the cotangent bundle symplectic structure $T^*(G \times G) \cong T^*(G) \times T^*(G)$

$$\begin{array}{l} \{k_a^{1,i}, k_b^{1,i}\} = -f_{ab}{}^c \, k_c^{1,i} \qquad \{k_a^{2,i}, k_b^{2,i}\} = -f_{ab}{}^c \, k_c^{2,i} \qquad \{k^a_{1,i}, k^b_{2,i}\} = 0 \\ \{k_a^{1,i}, F\}(w_{1,i}, w_{2,i}) = -\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} F(w_{1,i}\, e^{tJ_a}\,,\, w_{2,i}) \\ \{k_a^{2,i}, F\}(w_{1,i}, w_{2,i}) = -\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} F(w_{1,i}\,,\, e^{-tJ_a} w_{2,i}). \end{array} \tag{2.30}$$

Combining the decoupling transformation K with the pull-back L^* and using the notation (2.13) then yields the following theorem

Theorem 2.5. *The bijective map $L^* \circ K : \mathcal{F} \to \mathcal{F}$ maps the Poisson structure* (2.21) *of the flower algebra to the direct sum of n copies of the dual Poisson-Lie group H^* and $2g$ copies of the cotangent bundle Poisson structure $T^*(G)$. The symplectic leaves of the Poisson manifold $\big(H^*\big)^n \times \big(T^*(G)\big)^{2g}$ are of the form $\mathcal{C}_{\mu_1 s_1} \times \ldots \times \mathcal{C}_{\mu_n s_n} \times H^{2g}$ and the pull-back of the symplectic structure on each leaf via the map*

$$conj_{\mu_1 s_1} \times \ldots \times conj_{\mu_n s_n} \times id^{2g} : H^{n+2g} \to \mathcal{C}_{\mu_1 s_1} \times \ldots \times \mathcal{C}_{\mu_n s_n} \times H^{2g} \tag{2.31}$$

is the exterior derivative of the symplectic potential

$$\begin{aligned} \Theta \;=\; & -\sum_{i=1}^{n} \langle dv_{M_i} v_{M_i}^{-1}\,,\, j_a^{M_i'} P^a\rangle + \sum_{i=1}^{g} \langle w_{1,i}^{-1} dw_{1,i}\,,\, k_a^{1,i} P^a\rangle \\ & - \langle dw_{2,i} w_{2,i}^{-1}\,,\, k_a^{2,i} P^a\rangle. \end{aligned} \tag{2.32}$$

Proof: The expression for the pull-back of the symplectic potential on the symplectic leaves of H^* was derived for the case $G = \widetilde{SO}(2,1)$ in [9], but the derivation is valid for any Lie group G. The expression for the symplectic potential on $T^*(G \times G)$ is standard and uses the identification of T^*G with $G \times \mathfrak{g}^*$. The identification can be made using either the left- or the right-multiplication of G; our definition of $k_a^{1,i}$ and $k_a^{2,i}$ is such that we use right-multiplication for one copy and left-multiplication for the other copy of G in $T^*(G \times G)$. The reasons for this choice are related to the natural action of the quantum double of G in the quantum theory, and will become clear in Sect. 4.2. $\square$

This theorem provides us with an interpretation of the map $L^* \circ K$. In the decoupled coordinates $\boldsymbol{j}_{M_i'}, u_{M_i'}, \boldsymbol{k}_{\alpha,i}$ and $w_{\alpha,i}$ the symplectic structure on symplectic leaves has the canonical form (2.32). We shall see in Sect. 3 that the Poisson structure expressed in terms of the decoupled coordinates is amenable to a rather straightforward quantisation procedure. The map $L^* \circ K$ thus establishes a link between two important sets of coordinates, the holonomy coordinates with a direct gauge-theoretical interpretation and the decoupled coordinates which are convenient for quantisation.

2.4 Symmetries

As the flower algebra for the group $H = G \ltimes \mathfrak{g}^*$ is closely related to the phase space of Chern-Simons gauge theory with gauge group H, the moduli space of flat H-connections, it is to be expected that at least some of the invariance transformations of the underlying Chern-Simons theory give rise to symmetries of the flower algebra. We identify two such groups of symmetries of the flower algebra and show how they are related to the invariance transformations of the underlying Chern-Simons gauge theory.

The first such group of symmetries of the flower algebra is the mapping class group $\mathrm{Map}(S_{g,n})$ of the underlying surface $S_{g,n}$. As topological field theories, Chern-Simons theories on a surface $S_{g,n}$ are invariant under diffeomorphisms of this surface, in particular, under large diffeomorphisms, which form its mapping class group $\mathrm{Map}(S_{g,n})$. In [9] it was shown for the case $H = \widetilde{SO}(2,1) \ltimes \mathbb{R}^3$ that elements of the mapping class group act on the flower algebra as Poisson isomorphisms. The proof can be extended to general H, but we will not give it here. Instead we defer a full discussion of the mapping class group in classical and in particular quantised Chern-Simons theories with gauge groups $G \ltimes \mathfrak{g}^*$ to a future paper.

The second type of symmetry acting on the flower algebra is related to the other class of invariance transformations in Chern-Simons theory, Chern-Simons gauge transformations. In the description of Chern-Simons gauge theory by means of a set of curves representing the generators of the fundamental group, Chern-Simons gauge transformations that are nontrivial at the basepoint act on the associated holonomies by global conjugation with an element of H. Via the identification of the holonomies with the different copies of H in definitions Def. 2.1, Def. 2.2, this action on the holonomies induces transformations of the flower algebra. However, these transformations are in general *not* Poisson isomorphisms unless we take into account the nontrivial Poisson structure (2.9) of the group H. For semisimple H, it was shown in [5] that, interpreted as maps $H \times H^{n+2g} \to H^{n+2g}$, they are Poisson isomorphisms with respect to the flower algebra Poisson structure on H^{n+2g} and the Poisson structure on $H \times H^{n+2g}$ that is the direct product of the Poisson structure (2.9) on H and the flower algebra Poisson structure on H^{n+2g}. However, we will see that there is a much larger group of Poisson symmetries of the flower algebra that generalises the conjugation with elements of the group H. In particular, for the case where the exponential map $\exp : \mathfrak{g} \to G$ is bijective, these Poisson symmetries give rise to a Poisson action of H on the flower algebra that can be interpreted as a deformed conjugation.

Theorem 2.6. *(Action of $G \ltimes \mathcal{C}^\infty(G)$)*

1. *Consider the group $G \ltimes \mathcal{C}^\infty(G)$ with multiplication law*

$$(h_1, f_1) \cdot (h_2, f_2) = (h_1 h_2, f_1 + f_2 \circ Ad_{h_1^{-1}}) \qquad \forall h_1, h_2 \in G,\ f_1, f_2 \in \mathcal{C}^\infty(G). \tag{2.33}$$

It acts on the flower algebra via

$$\begin{aligned}(h, f) :\ \ & j_a^X \otimes 1 \mapsto j_b^X \otimes Ad^*(h)_a{}^b + 1 \otimes (Ad^*(h)_a{}^b \{ j_b^X \otimes 1, 1 \otimes f \circ \Phi_\infty \}) \\ & 1 \otimes F \mapsto 1 \otimes F \circ Ad^{n+2g}_{h^{-1}},\end{aligned} \tag{2.34}$$

where $Ad_h^{n+2g} : (u_{M_1}, \ldots, u_{B_g}) \mapsto (h u_{M_1} h^{-1}, \ldots, h u_{B_g} h^{-1})$ denotes the global conjugation by $h \in G$, $\{\ ,\ \}$ the Poisson bracket (2.21) on the flower algebra and $\Phi_\infty : G^{n+2g} \to G$ is the map

$$\Phi_\infty(u_{M_1}, \ldots, u_{B_g}) = u_{tot} = u_{K_g} \cdots u_{K_1} \cdot u_{M_n} \cdots u_{M_1} \tag{2.35}$$

with u_{K_i} given by (2.24).

2. *This action is a Poisson action. The infinitesimal generators of the action of $G \subset G \ltimes \mathcal{C}^\infty(G)$ are the elements*

$$\begin{aligned} j_a^{tot} &= \sum_{i=1}^{n} j_b^{M_i} \otimes Ad^*(u_{M_{i-1}} \cdots u_{M_1})_a{}^b + \sum_{i=1}^{g} 1 \otimes Ad^*(u_{K_{i-1}} \cdots u_{M_1})_a{}^b \cdot j_b^{H_i} \\ &= \sum_{i=1}^{n} j_a^{M_i'} + \sum_{k=1}^{g} 1 \otimes (\delta_a{}^b - Ad^*(u_{A_k}^{-1})_a{}^b) \cdot j_b^{A_k'} \\ &\quad + 1 \otimes (Ad^*(u_{A_k}^{-1}) - Ad^*(u_{B_k}^{-1} u_{A_k}))_a{}^b \cdot j_b^{B_k'}, \end{aligned} \tag{2.36}$$

with $j_a^{H_i}$ given by (2.24), and the action of $\mathcal{C}^\infty(G) \subset G \ltimes \mathcal{C}^\infty(G)$ is generated by functions $1 \otimes (f \circ \Phi_\infty)$, $f \in \mathcal{C}^\infty(G)$. We have for all elements $\varphi \in \mathcal{F}$

$$\frac{d}{dt}|_{t=0}(e^{-tJ_a}\,,\,0)(\varphi) = \{j_a^{tot}, \varphi\} \tag{2.37}$$
$$\frac{d}{dt}|_{t=0}(1\,,\,-t\cdot f)(\varphi) = \{1 \otimes (f \circ \Phi_\infty)\,,\,\varphi\}.$$

Proof: That (2.34) defines a group action of $G \ltimes \mathcal{C}^\infty(G)$ on the flower algebra with infinitesimal generators (2.37) can be shown by direct computation using the Poisson brackets (2.21) or, alternatively, (2.26),(2.27) of the flower algebra. That it is a Poisson action follows from the fact that it is infinitesimally generated via the Poisson brackets [14], but can also be verified directly from (2.34) and the Poisson bracket of the flower algebra. □

Note that the map $\Phi_\infty : (u_{M_1}, \ldots, u_{B_g}) \mapsto u_{tot}$ as well as the algebra elements j_a^{tot} occurring in Theorem 2.6 have a geometric meaning [9]. If we parametrise the holonomies $M_1, \ldots, B_g$ associated to the generators $m_1, \ldots, b_g$ of $\pi_1(S_{g,n})$ according to (2.1)

$$(u_X, -\mathrm{Ad}^*(u_X^{-1})\boldsymbol{j}_X) = (u_X, -\mathrm{Ad}^*(u_X^{-1}) j_a^X P^a) \quad \forall X \in \{M_1, \ldots, B_g\} \tag{2.38}$$

and identify the parameters j_a^X with generators $j_a^X \otimes 1$, then the holonomy associated to the curve k_∞ defined in (2.16) is $\mathrm{Hol}(k_\infty) = (u_{tot}\,,\,-\mathrm{Ad}^*(u_{tot}^{-1})\boldsymbol{j}_{tot})$ with u_{tot} and $\boldsymbol{j}_{tot}$ given by (2.35) and (2.36).

We will now relate this action of the group $G \ltimes \mathcal{C}^\infty(G)$ on the flower algebra to the transformations induced by simultaneous conjugation of the holonomies with a fixed element of $G \ltimes \mathfrak{g}^*$. From the group multiplication law (2.2), it follows that $(h, \boldsymbol{x}) \in G \ltimes \mathfrak{g}^*$ acts on elements of the flower algebra by conjugation with the inverse $(h^{-1}, -\mathrm{Ad}^*(h)\boldsymbol{x})$ according to

$$1 \otimes F \mapsto 1 \otimes (F \circ \mathrm{Ad}_{h^{-1}}^{n+2g}) \tag{2.39}$$
$$j_a^X \otimes 1 \mapsto j_b^X \otimes \mathrm{Ad}^*(h)_a{}^b - 1 \otimes (\mathrm{Ad}^*(h) - \mathrm{Ad}^*(u_X h))_a{}^b x_b.$$

This action is not a Poisson action. On the other hand, we can use (2.21) together with the definition of the map Φ_∞ to evaluate the bracket $\{j_a^X\,,\,f \circ \Phi_\infty\}$ in the definition (2.34) of the $G \ltimes \mathcal{C}^\infty(G)$-action on the flower algebra and obtain the Poisson action

$$(h, f): \quad 1 \otimes F \mapsto 1 \otimes (F \circ \mathrm{Ad}_{h^{-1}}^{n+2g}) \tag{2.40}$$
$$j_a^X \otimes 1 \mapsto j_b^X \otimes \mathrm{Ad}^*(h)_a{}^b - 1 \otimes ((\mathrm{Ad}^*(h) - \mathrm{Ad}^*(u_X h))_a{}^b (J_b^L f) \circ \Phi_\infty).$$

Comparing expressions (2.39) and (2.40), we see that the action of $G \ltimes \mathcal{C}^\infty(G)$ on the flower algebra can be interpreted as a generalised conjugation: the action of $G \subset G \ltimes \mathcal{C}^\infty(G)$ agrees with the action of G via global conjugation, whereas the action of $\mathcal{C}^\infty(G) \subset G \ltimes \mathcal{C}^\infty(G)$ mimics global conjugation with $\mathfrak{g}^*$, only that now the transformation vector $\boldsymbol{x} \in \mathfrak{g}^*$ is replaced by a function of the group element u_{tot}.

If the exponential map $\exp : \mathfrak{g} \to G$ is bijective, group elements $u \in G$ can be parametrised as $u = \exp(p^a J_a)$ and the relation between the action of $G \ltimes \mathcal{C}^\infty(G)$ and the action of the group $G \ltimes \mathfrak{g}^*$ by global conjugation becomes more explicit. Denoting the inverse of $\exp : \mathfrak{g} \to G$ by $\log : G \to \mathfrak{g}$, we can define an embedding $\iota : \mathfrak{g}^* \to \mathcal{C}^\infty(G)$ via

$$\iota(\boldsymbol{x})(u) = \langle \boldsymbol{x}, \log(u) \rangle \qquad \forall \boldsymbol{x} \in \mathfrak{g}^*, u \in G, \tag{2.41}$$

where $\langle\, , \rangle$ denotes the pairing (2.4). In particular, we have the coordinate functions $\iota(P^a) : u = \exp(p^b J_b) \mapsto p^a$. Then, the group multiplication (2.33) restricted to $G \ltimes \iota(\mathfrak{g}^*)$ becomes simply the group multiplication of H, and we obtain the following lemma

Lemma 2.7. *If the exponential map* $\exp : \mathfrak{g} \to G$ *is bijective, there is a Poisson action of the group* $G \ltimes \mathfrak{g}^*$ *on the flower algebra given by*

$$\begin{array}{rcl} (h, \boldsymbol{x}) :\ 1 \otimes F & \mapsto & 1 \otimes (F \circ Ad^{n+2g}_{h^{-1}}) \\ j^X_a \otimes 1 & \mapsto & j^X_b \otimes Ad^*(h)_a{}^b - 1 \otimes (Ad^*(h) - Ad^*(u_X h))_a{}^b q_b. \end{array} \tag{2.42}$$

where $q_b = T_b{}^c(u_{tot})\, x_c$ *and*

$$T(u_{tot}) = \frac{1 - Ad^*(u_{tot})}{ad^*(p^a_{tot} J_a)} \tag{2.43}$$

is a linear map depending on the total holonomy u_{tot}.

Proof: Apply the formula (2.40) for the Poisson action of $G \ltimes \mathcal{C}^\infty(G)$ to the function $f = \iota(\boldsymbol{x})$. The lemma then follows from the formula $(J^L_b \iota(\boldsymbol{x})) \circ \Phi_\infty = T_b{}^c(u_{tot})\, x_c$, which can be found, for example, in [15], p. 179. □

3 Quantisation

By Theorem 2.5, we have reduced the task of quantising the flower algebra to the quantisation of the symplectic structure on the cotangent bundle $T^*(G)$ of the group G and the dual Poisson structure H^*. Both of these Poisson structures are relatively simple and special cases of the following general situation. We have a commutative Poisson algebra given as a tensor product

$$\mathcal{Q} = S(\mathfrak{e}) \otimes \mathcal{C}^\infty(E), \tag{3.1}$$

where E is a finite dimensional, simply connected Lie group with Lie algebra $\mathfrak{e} = \mathrm{Lie}\, E$ and $S(\mathfrak{e})$ denotes the symmetric envelope of $\mathfrak{e}$. The Poisson structure is that of a semidirect product: Poisson brackets of two elements of $S(\mathfrak{e})$ are given by the derivative extension of the Lie bracket on $\mathfrak{e}$, Poisson brackets of two functions in $\mathcal{C}^\infty(E)$ vanish and Poisson brackets of elements of $S(\mathfrak{e})$ with functions in $\mathcal{C}^\infty(E)$ are derived from a group action $.$ of E on itself

$$\begin{aligned}
&\{\xi \otimes 1, \eta \otimes 1\} = [\xi, \eta]_{\mathfrak{e}} \otimes 1 \qquad\qquad \{1 \otimes F, 1 \otimes K\} = 0 \\
&\{\xi \otimes 1, 1 \otimes F\}(d) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} F(e^{-t\xi} .u) \qquad \forall \xi, \eta \in \mathfrak{e},\ u \in E,\ F, K \in \mathcal{C}^\infty(E).
\end{aligned} \tag{3.2}$$

In the case of the cotangent bundle $T^*(G)$, we have $E = G$ and the group action is the inverse left multiplication or, alternatively, right multiplication by G. For the dual Poisson structure H^*, it is conjugation with G. We proceed now to discuss the quantisation of a general Poisson algebra $\mathcal{Q} = S(\mathfrak{e}) \otimes \mathcal{C}^\infty(E)$ of this type and then apply the results to the cotangent bundle $T^*(G)$, the dual H^* and the decoupled flower algebra in Sect. 3.2.

3.1 Quantisation of Poisson algebras $\mathcal{Q} = S(\mathfrak{e}) \otimes \mathcal{C}^\infty(E)$

Let $\mathcal{Q} = S(\mathfrak{e}) \otimes \mathcal{C}^\infty(E)$ be a Poisson algebra with a semidirect product Poisson structure arising from an action $.$ of the simply connected Lie group E on itself as in (3.2). The Poisson algebra $\mathcal{Q}$ inherits a $\mathbb{N}$-grading from the canonical $\mathbb{N}$-grading of the symmetric envelope

$$\mathcal{Q} = \bigoplus_{k=0}^{\infty} \mathcal{Q}^{(k)} \qquad \mathcal{Q}^{(k)} = S^{(k)}(\mathfrak{e}) \otimes \mathcal{C}^\infty(E), \tag{3.3}$$

where $S^{(k)}(\mathfrak{e})$ is the space of monomials of degree k in a basis $\mathcal{B}_{\mathfrak{e}} = \{\xi_1, \ldots, \xi_{\dim E}\}$ with real coefficients. The multiplication of homogeneous elements adds their degrees, whereas the Poisson bracket (3.2) adds their degrees and subtracts one

$$\mathcal{Q}^{(k)} \cdot \mathcal{Q}^{(l)} \subset \mathcal{Q}^{(k+l)} \qquad\qquad \{\mathcal{Q}^{(k)}, \mathcal{Q}^{(l)}\} \subset \mathcal{Q}^{(k+l-1)}. \tag{3.4}$$

In quantisation, the commutative Poisson algebra $\mathcal{Q}$ is to be replaced by an associative *-algebra $\widehat{\mathcal{Q}}$, which depends on a deformation parameter

$\hbar$ and has to exhibit certain structural properties relating it to the classical algebra $\mathcal{Q}$. Every element of the quantum algebra $\widehat{\mathcal{Q}}$ must correspond to a unique element in the complexified classical algebra $\mathcal{Q}_{\mathbb{C}}$ and conversely, it must be possible to assign to every element of the complexified classical algebra a quantum counterpart in $\widehat{\mathcal{Q}}$. This is equivalent to the existence of a vector space isomorphism $Q : \mathcal{Q}_{\mathbb{C}} \to \widehat{\mathcal{Q}}$. In order to obtain an algebra $\widehat{\mathcal{Q}}$ that merits the name quantisation, we must furthermore request that the product of two of its elements is given as the quantum counterpart of the product of the corresponding classical elements plus a quantum correction of order $O(\hbar)$, and that their commutator is equal to $i\hbar$ times the quantum counterpart of the Poisson bracket of the corresponding classical elements plus a quantum correction of order $O(\hbar^2)$:

$$Q(W) \cdot Q(Z) = Q(W \cdot Z) + O(\hbar) \tag{3.5}$$

$$[Q(W), Q(Z)] = i\hbar Q(\{W, Z\}) + O(\hbar^2) \qquad \forall W, Z \in \mathcal{Q}. \tag{3.6}$$

In general, the quantum corrections in (3.6) cannot be eliminated for all elements $W, Z \in \mathcal{Q}$. (For the case of the Heisenberg algebra, this is a consequence of the no-go theorem by Groenewold and van Hove [16, 17, 18].) But there should be a Poisson subalgebra of the classical algebra $\mathcal{Q}$ containing the generating elements $\xi_1 \otimes 1, \ldots, \xi_{\dim E} \otimes 1$ and $1 \otimes F$, for which this is possible.

As our Poisson algebra $\mathcal{Q}$ is of a particularly simple type and related to the symmetric envelope of a Lie algebra $\mathfrak{e}$, a framework for the construction of the quantum algebra is provided by the theory of universal enveloping algebras and the theorem of Poincaré-Birkhoff-Witt [19]. We obtain the following theorem defining a quantum algebra with the requested properties

Theorem 3.1. *(Construction of the quantum algebra $\widehat{\mathcal{Q}}$)*

Let $\widehat{\mathcal{Q}}$ be the associative algebra $\widehat{\mathcal{Q}} = U(\mathfrak{e}) \otimes \mathcal{C}^\infty(E, \mathbb{C})$ with a semidirect multiplication defined by

$$(\xi \otimes F) \cdot (\eta \otimes K) = \xi \cdot_U \eta \otimes FK + i\hbar\, \eta \otimes F\, \{\xi \otimes 1, 1 \otimes K\} \qquad \forall \xi, \eta \in \mathfrak{e},\ F, K \in \mathcal{C}^\infty(E, \mathbb{C}), \tag{3.7}$$

where $U(\mathfrak{e})$ denotes the universal enveloping algebra of the Lie algebra $\mathfrak{e}$ with Lie bracket multiplied by a factor $i\hbar$, $\cdot_U$ the multiplication in $U(\mathfrak{e})$ and $\{\,,\,\}$ the Poisson bracket (3.2). *Then*

1. *$\widehat{\mathcal{Q}}$ has a *-structure given by $(\xi \otimes 1)^* = \xi \otimes 1$, $(1 \otimes F)^* = 1 \otimes \bar{F}$ for $\xi \in \mathfrak{e}$, $F \in \mathcal{C}^\infty(E, \mathbb{C})$.*

2. *The algebra $\widehat{\mathcal{Q}}$ inherits a filtration from the canonical filtration of the universal enveloping algebra $U(\mathfrak{e})$*

$$\widehat{\mathcal{Q}} = \bigcup_{k=0}^{\infty} \widehat{\mathcal{Q}}^{(k)} \qquad \widehat{\mathcal{Q}}^{(k)} = U^{(k)}(\mathfrak{e}) \otimes \mathcal{C}^\infty(E,\mathbb{C}) \subset \widehat{\mathcal{Q}}^{(k+1)} \qquad \forall k \in \mathbb{N} \tag{3.8}$$

with $\widehat{\mathcal{Q}}^{(0)} \cong \mathcal{C}^\infty(E,\mathbb{C}) \cong \mathcal{Q}^{(0)}_{\mathbb{C}}$, $\widehat{\mathcal{Q}}^{(k)}/\widehat{\mathcal{Q}}^{(k-1)} \cong \mathcal{Q}^{(k)}_{\mathbb{C}}$ $\forall k \geq 1$ *and*

$$\widehat{\mathcal{Q}}^{(k)} \cdot \widehat{\mathcal{Q}}^{(l)} \subset \widehat{\mathcal{Q}}^{(k+l)} \qquad [\widehat{\mathcal{Q}}^{(k)}, \widehat{\mathcal{Q}}^{(l)}] \subset \widehat{\mathcal{Q}}^{(k+l-1)} \qquad \forall k,l \in \mathbb{N}. \tag{3.9}$$

In particular, the commutator $[\,,\,]$ of $\widehat{\mathcal{Q}}$ defines a Lie bracket on the space $\widehat{\mathcal{Q}}^{(1)}$.

3. *In terms of elements $\xi_1 < \ldots < \xi_{dim\,E}$ of an ordered basis of $\mathfrak{e}$, a vector space isomorphism $Q: \mathcal{Q}_{\mathbb{C}} \to \widehat{\mathcal{Q}}$ is defined by*

$$\begin{aligned} Q(\xi_{a_1} \ldots \xi_{a_m} \otimes F) :=& \xi_{a_1} \cdot_U \ldots \cdot_U \xi_{a_m} \otimes F \\ =& Q(1 \otimes F) \cdot Q(\xi_{a_1} \otimes 1) \cdots Q(\xi_{a_m} \otimes 1) \\ & \forall a_1 \leq \ldots \leq a_m, F \in \mathcal{C}^\infty(E,\mathbb{C}). \end{aligned} \tag{3.10}$$

It satisfies

$$\Pi^{(k+l)}\big(\, Q(W) \cdot Q(Z) - Q(WZ)\,\big) = 0 \tag{3.11}$$
$$\Pi^{(k+l-1)}\big(\, [Q(W), Q(Z)] - i\hbar\, Q(\{W,Z\})\,\big) = 0 \tag{3.12}$$
$$\forall W \in \mathcal{Q}^{(k)}_{\mathbb{C}}, Z \in \mathcal{Q}^{(l)}_{\mathbb{C}},$$

where $\Pi^{(k)}$ is the canonical projection $\widehat{\mathcal{Q}}^{(k)} \to \widehat{\mathcal{Q}}^{(k)}/\widehat{\mathcal{Q}}^{(k-1)}$. In particular, $Q|_{(\mathcal{Q}^{(0)}\oplus\mathcal{Q}^{(1)})_{\mathbb{C}}} : \mathcal{Q}^{(0)}_{\mathbb{C}} \oplus \mathcal{Q}^{(1)}_{\mathbb{C}} \to \widehat{\mathcal{Q}}^{(1)}$ is a Lie algebra isomorphism with respect to the brackets $\{\,,\,\}|_{(\mathcal{Q}^{(0)}\oplus\mathcal{Q}^{(1)})_{\mathbb{C}}}$ and $[\,,\,]|_{\widehat{\mathcal{Q}}^{(1)}}$.

Note that for general elements $\theta, \chi \in U(\mathfrak{e})$ of the universal enveloping algebra $U(\mathfrak{e})$ the multiplication law defined by (3.7) can be written using Sweedler notation

$$(\theta \otimes F) \cdot (\chi \otimes K) = \sum (\theta_{(1)} \cdot_U \chi) \otimes (F \cdot \theta^L_{(2)} K), \tag{3.13}$$

where $\Delta_U : U(\mathfrak{e}) \to U(\mathfrak{e}) \otimes U(\mathfrak{e})$, $\Delta_U(\theta) = \sum \theta_{(1)} \otimes \theta_{(2)}$ is the co-multiplication of the universal enveloping algebra defined inductively by $\Delta_U(\xi) = 1 \otimes \xi + \xi \otimes 1$ for $\xi \in \mathfrak{e}$ and $\Delta_U(\theta \cdot_U \chi) = \Delta_U(\theta) \cdot_U \Delta_U(\chi)$ for $\theta, \chi \in U(\mathfrak{e})$.

Proof:

1. The canonical filtration of universal enveloping algebra $U(\mathfrak{e}) = \bigcup_{k=0}^{\infty} U^{(k)}(\mathfrak{e})$, where $U^{(k)}(\mathfrak{e})$ is the space of non-commutative polynomials of degree $\leq k$ in the generators of the Lie algebra $\mathfrak{e}$, satisfies

$$U^{(k)}(\mathfrak{e}) \cdot_U U^{(l)}(\mathfrak{e}) \subset U^{(k+l)}(\mathfrak{e}), \quad [U^{(k)}(\mathfrak{e}), U^{(l)}(\mathfrak{e})]_U \subset U^{(k+l-1)}(\mathfrak{e}) \; \forall k,l \in \mathbb{N} \tag{3.14}$$

and $U^{(k)}(\mathfrak{e})/U^{(k-1)}(\mathfrak{e}) \cong S^{(k)}(\mathfrak{e})$ for $k \geq 1$. This implies $\widehat{\mathcal{Q}}^{(k)}/\widehat{\mathcal{Q}}^{(k-1)} = (U^{(k)}(\mathfrak{e}) \otimes \mathcal{C}^\infty(E,\mathbb{C}))/(U^{(k-1)}(\mathfrak{e}) \otimes \mathcal{C}^\infty(E,\mathbb{C})) \cong (U^{(k)}(\mathfrak{e})/U^{(k-1)}(\mathfrak{e})) \otimes \mathcal{C}^\infty(E,\mathbb{C}) \cong S^{(k)}(\mathfrak{e}) \otimes \mathcal{C}^\infty(E,\mathbb{C}) = \mathcal{Q}^{(k)}_{\mathbb{C}}$ for $k \geq 1$. The identities (3.9) then follow directly from (3.14) and the multiplication law (3.7). In particular, $[\widehat{\mathcal{Q}}^{(1)}, \widehat{\mathcal{Q}}^{(1)}] \subset \widehat{\mathcal{Q}}^{(1)}$, which makes the subspace $\widehat{\mathcal{Q}}^{(1)}$ with the commutator a Lie algebra.

2. According to the theorem of Poincaré-Birkhoff-Witt, see for example [19], an ordered basis of the vector space $U(\mathfrak{e})$ is given by the ordered monomials $\xi_{a_1} \cdot \ldots \cdot \xi_{a_m}$, $a_1 \leq \ldots \leq a_m$, $m \in \mathbb{N}$, in the elements of an ordered basis $\xi_1 < \ldots < \xi_{\dim E}$ of the Lie algebra $\mathfrak{e}$. Therefore, the map Q in (3.10) defines a vector space isomorphism from $\mathcal{Q}_{\mathbb{C}}$ to $\widehat{\mathcal{Q}}$. From the multiplication law (3.7) and the definition (3.2) of the Poisson bracket we have for the commutator of two Lie algebra elements $\xi, \eta \in \mathfrak{e}$ and the commutator of a Lie algebra element with a function $F \in \mathcal{C}^\infty(E,\mathbb{C})$

$$\begin{aligned} &[\xi \otimes 1, \eta \otimes 1] = (\xi \cdot_U \eta - \eta \cdot_U \xi) \otimes 1 = i\hbar \{\xi \otimes 1, \eta \otimes 1\} \\ &[\xi \otimes 1, 1 \otimes F] = i\hbar\, 1 \otimes \left(-\frac{d}{dt}|_{t=0} F(e^{-t\xi}.\cdot) \right) = i\hbar \{\xi \otimes 1, 1 \otimes F\}. \end{aligned} \tag{3.15}$$

For elements $W \in \mathcal{Q}^{(k)}_{\mathbb{C}}$, $Z \in \mathcal{Q}^{(l)}_{\mathbb{C}}$, the product $Q(W) \cdot Q(Z)$ differs from $Q(WZ)$ only by factor ordering, which can be seen from the multiplication law (3.7) and commutators (3.15) to give rise to a quantum correction in $\widehat{\mathcal{Q}}^{(k+l-1)}$ preceded by a factor $\hbar$. The same applies to the commutator $[Q(W), Q(Z)]$ and $i\hbar Q(\{W,Z\})$, only that now the quantum correction is an element of $\widehat{\mathcal{Q}}^{(k+l-2)}$ with a factor $\hbar^2$. This proves identities (3.11) and (3.12), in particular, that $Q|_{(\mathcal{Q}^{(0)} \oplus \mathcal{Q}^{(1)})_{\mathbb{C}}} : \mathcal{Q}^{(0)}_{\mathbb{C}} \oplus \mathcal{Q}^{(1)}_{\mathbb{C}} \to \widehat{\mathcal{Q}}^{(1)}$ is a Lie algebra isomorphism. □

Theorem 3.1 provides us with a way of constructing the quantum algebra for Poisson algebras $\widehat{\mathcal{Q}} = U(\mathfrak{e}) \otimes \mathcal{C}^\infty(E,\mathbb{C})$ with a semidirect product Poisson structure (3.2). The next step is the study of their representation theory, i.e. the classification of all irreducible Hilbert space representations. For this, we must decide which representations we want to consider and which meaning we want to give to the requirement of irreducibility. These questions arise in a similar way in the standard examples treated in textbooks on quantum mechanics, for the case of $T^*\mathbb{R}^N$ see [20] and also the papers [21] for a more detailed treatment. Physical requirements usually result in restrictions on the admissible representations, which are reflected in the concept of integrability, and a specific interpretation of irreducibility.

The simplest case is that of a quantum algebra of observables that is the universal enveloping algebra $U(\mathfrak{e})$ of a Lie algebra $\mathfrak{e}$. Via differentiation, representations of the corresponding Lie group E on a Hilbert space give rise to representations of the universal enveloping algebra $U(\mathfrak{e})$ on a dense, invariant subspace, the Garding space or space of $\mathcal{C}^\infty$-vectors [22], [23]. However, in general not all representations of the universal enveloping algebra arise that way. Following [22] we call a representation Π of $U(\mathfrak{e})$ *integrable* if it is derived from a unitary Hilbert space representation π of E according to the rule

$$\Pi(\xi)\psi = \frac{d}{dt}|_{t=0}\pi(\exp(-t\xi))\psi, \qquad \xi \in \mathfrak{g}, \tag{3.16}$$

for all $\mathcal{C}^\infty$-vectors ψ. Because the elements of the Lie group E often correspond to physically meaningful transformations on the phase space, it is usually requested that the Lie group E be represented on the Hilbert space and only integrable representations are considered. Similarly, irreducibility is usually understood with respect to the representation of the Lie group E, i.e. one classifies integrable representations of the universal enveloping algebra $U(\mathfrak{e})$ for which the corresponding representation of the Lie group E is irreducible.

In our case, we want to impose analogous requirements for the representations of the subalgebra $U(\mathfrak{e}) \subset \widehat{\mathcal{Q}}$, but we need to combine the associated representations of the group E with representations of the function algebra $\mathcal{C}^\infty(E,\mathbb{C})$. To do this, we recall that the product (3.7) in $\widehat{\mathcal{Q}}$ involves the action (3.2) of $\mathfrak{e}$ on $\mathcal{C}^\infty(E,\mathbb{C})$ that is derived from the group action $.$ of E on $\mathcal{C}^\infty(E,\mathbb{C})$. Thus we want to combine the group E with the functions $\mathcal{C}^\infty(E,\mathbb{C})$ in such a way that the product of group elements and functions involves the action $g \in E$ which sends $F \in \mathcal{C}^\infty(E,\mathbb{C})$ to $(u \mapsto F(g^{-1}.u))$. Tensoring group elements with the function algebra $\mathcal{C}^\infty(E,\mathbb{C})$ only makes sense at the level of the group algebra of E. If we realise the group algebra of E as (a certain class of) functions on E with multiplication defined by convolution, the combination with the function algebra $\mathcal{C}^\infty(E,\mathbb{C})$ can be achieved in the framework of transformation group algebras, initiated by Dixmier [24] and continued by Glimm [25]. In the most general definition of transformation group algebras one starts with a topological group E which is Hausdorff, locally compact and second countable and acts on a space X.

Here we only need to consider the case where $X = E$, and E is a finite-dimensional Lie group. We summarise the key results following the paper [26], which gives a treatment that is closely related to our situation.

Definition 3.2. *Let E be a unimodular Lie group acting continuously on itself via $. : E \times E \to E$. Then the space $C_0(E \times E, \mathbb{C})$ of continuous functions on $E \times E$ with compact support is a transformation group algebra if it is equipped with the multiplication and $*$-operation given by*

$$F_1 \bullet F_2(v,u) = \int_E F_1(z,u) F_2(z^{-1}v, z^{-1}.u)\, dz \tag{3.17}$$

$$F^*(v,u) = \overline{F(v^{-1}, v^{-1}.u)}. \tag{3.18}$$

With the norm $||F||_1 = \int_E ||F(z,\cdot)||_\infty\, dz$ we have the inequality $||F_1 \bullet F_2||_1 \leq ||F_1||_1 ||F_2||_1$.

We now define irreducible integrable representations of $\widehat{\mathcal{Q}}$ to be those which, in a sense to be specified below, are derived from irreducible unitary and bounded representation of $C_0(E \times E, \mathbb{C})$. The task of classifying integrable and irreducible representations of $\widehat{\mathcal{Q}}$ then reduces, by definition, to the task of classifying irreducible unitary and bounded representations of the transformation algebra $C_0(E \times E, \mathbb{C})$.

Following the study in [26], we make the technical but important assumption that the orbit space of the E-action on itself is T_0 in the quotient topology (A topological space is T_0 if for any two distinct points at least one of the points has a neighbourhood to which the other does not belong). Also, we assume for simplicity the existence of an invariant measure dm on the orbits of the group action of E on itself. Then, the bounded irreducible representations of a transformation group algebra are characterised by the following theorem [26].

Theorem 3.3. *Let E be as above and assume that the orbit space of the E-action on itself is T_0 in the quotient topology and that there exists invariant measures dm on each orbit. Then the irreducible $||\cdot||_1$-bounded unitary representations of the transformation group algebra $C_0(E \times E, \mathbb{C})$ are labelled by orbits $O_\mu = \{v.g_\mu \,|\, v \in E\}$, $g_\mu \in E$, of the action of E on itself and irreducible unitary representations Π_s of the associated stabilisers $N_\mu = \{n \in E \,|\, n.g_\mu = g_\mu\}$ on Hilbert spaces V_s. The representation spaces $V_{\mu s}$ are, up to equivalence, given by the following construction. Let $L^2_{\mu s}$ be the space*

$$\begin{aligned} L^2_{\mu s} := \{\psi : E \to V_s \mid \psi(vn) = \Pi_s(n^{-1})\psi(v), \ \ \forall n \in N_\mu, \ \ \forall v \in E, \\ \textit{and} \quad \|\psi\|^2 := \int_{E/N_\mu} \|\psi(z)\|^2_{V_s}\, dm(zN_\mu) < \infty\} \end{aligned} \tag{3.19}$$

with the positive semi-definite inner product

$$\langle \psi_1, \psi_2 \rangle := \int_{E/M_\mu} \langle \psi_1(z), \psi_2(z) \rangle_{V_s} \, dm(zN_\mu). \tag{3.20}$$

Then one obtains a Hilbert space by taking the quotient space with respect to the subspace of functions with norm zero:

$$V_{\mu s} = L^2_{\mu s} \,/\, \{\psi \in L^2_{\mu s} \mid \|\psi\| = 0\}, \tag{3.21}$$

on which elements $F \in C_0(E \times E, \mathbb{C})$ act via

$$(\pi_{\mu s}(F)\psi)\ (v) = \int_E F(z, v.g_\mu)\psi(z^{-1}v)\ dz. \tag{3.22}$$

It remains to show that there is a dense invariant subspace of the Hilbert space $V_{\mu s}$ that carries a representation of the quantum algebra $\widehat{\mathcal{Q}}$ and to specify how this representation is derived from that of $C_0(E \times E, \mathbb{C})$. For this purpose, note that the group E acts unitarily (and reducibly) on $V_{\mu s}$ via

$$(\pi(g)\psi)\,(v) = \psi(g^{-1}v). \tag{3.23}$$

Following [23], we define $\mathcal{C}^\infty$-vectors in a representation of E to be those for which the map $\psi \mapsto \pi(g)\psi$ is infinitely often differentiable. The $\mathcal{C}^\infty$-vectors in $V_{\mu s}$ viewed as a representation of E are precisely those $\psi \in V_{\mu s}$ which are smooth functions $E \to V_s$. On these vectors the derived action of the Lie algebra $\mathfrak{e}$ is then obtained by differentiation as in (3.16). In order to obtain a subspace on which $\mathcal{C}^\infty(E, \mathbb{C})$ acts we need to impose the additional restriction that the map $\|\psi\|^2_{V_s} : E/N_\mu \mapsto \mathbb{C}$ has compact support. We define

$$V^\infty_{\mu s} = \{\psi \in \mathcal{C}^\infty(E, V_s) |\ \psi(vn) = \Pi_s(n^{-1})\psi(v) \quad \forall n \in N_\mu, v \in E \\ \text{and} \quad ||\psi||^2_{V_s} \in \mathcal{C}^\infty_0(E/N_\mu)\} \tag{3.24}$$

and have

Theorem 3.4. *The space $V^\infty_{\mu s}$ is a dense subspace of the Hilbert space $V_{\mu s}$ and carries the derived representation of the quantum algebra $\widehat{\mathcal{Q}}$ defined by*

$$\Pi_{\mu s}(\xi \otimes 1)\psi(v) = -i\hbar \frac{d}{dt}|_{t=0}\psi(e^{-t\xi}\, v) \quad \Pi_{\mu s}(1 \otimes F)\psi(v) = F(v.g_\mu) \cdot \psi(v) \tag{3.25}$$

for $\xi \in \mathfrak{e}$, $F \in \mathcal{C}^\infty(E, \mathbb{C})$.

Proof: The density of $V^\infty_{\mu s}$ in $V_{\mu s}$ follows from the density of $\mathcal{C}^\infty_0(M, \mathbb{C})$ in $L^2(M, \mathbb{C})$ for any domain M [23]. To see that the action (3.25) of $\widehat{\mathcal{Q}}$ on $V^\infty_{\mu s}$ is well-defined and leaves $V^\infty_{\mu s}$ invariant note that if $||\psi||^2_{V_s}$ has compact support so does $|F(\cdot g_\mu)|^2||\psi||^2_{V_s}$. Checking that (3.25) defines a representation is an easy algebraic exercise. □

3.2 Quantisation of the decoupled flower algebra

After constructing the quantum algebra and its representations for Poisson algebras of type (3.1) with brackets (3.2) we can now apply these results to the cotangent bundle symplectic structure $T^*(G)$, the dual Poisson structure H^* and, finally, the decoupled flower algebra $\mathcal{F}$. For the case of the cotangent bundle $T^*(G)$ with the Poisson brackets given by (2.30), the corresponding group action is simply the inverse left multiplication or, alternatively, the right multiplication with G. There is only a single orbit, the group G, and its stabiliser group is trivial. We obtain the following quantum algebra

Theorem 3.5. *(Quantum algebra for $T^*(G)$)*

1. *The quantum algebra for the cotangent bundle Poisson structure $T^*(G)$ is the associative algebra $U(\mathfrak{g})\otimes\mathcal{C}^\infty(G,\mathbb{C})$ with the multiplication defined by*

$$(\xi \otimes F) \cdot (\eta \otimes K) = (\xi \cdot_U \eta) \otimes (FK) - i\hbar\, \eta \otimes (\xi^L K), \tag{3.26}$$

   where $\xi, \eta \in \mathfrak{g}$, $F, K \in \mathcal{C}^\infty(G,\mathbb{C})$, ξ^L is the right-invariant vector field associated to ξ and $\cdot_U$ denotes the multiplication in the universal enveloping algebra $U(\mathfrak{g})$.

2. *The corresponding transformation group algebra has a single irreducible representation on the Hilbert space $V = L^2(G,\mathbb{C})$. Elements of $U(\mathfrak{g})\otimes\mathcal{C}^\infty(G,\mathbb{C})$ act on the dense invariant subspace $\mathcal{C}_0^\infty(G,\mathbb{C})$ according to*

$$\Pi(\xi \otimes 1)\psi(u) = -i\hbar\, \xi^L \psi(u) \qquad \Pi(1 \otimes F)\psi(u) = F(u) \cdot \psi(u) \tag{3.27}$$

   for $u \in G, \xi \in \mathfrak{g}$, $F \in \mathcal{C}^\infty(G,\mathbb{C})$.

Of course, we could just as well have G let act on itself via the right multiplication and simply exchanged left and right in Theorem 3.5. Combining one copy of $T^*(G)$, where G acts by right multiplication, and one where it acts by inverse left multiplication, we obtain the quantum algebra associated to the Poisson structure (2.30) of each handle:

Definition 3.6. *(Handle algebra $\widehat{\mathcal{H}}$)*

1. *The handle algebra is the associative algebra $\widehat{\mathcal{H}} = U(\mathfrak{g}\oplus\mathfrak{g})\otimes\mathcal{C}^\infty(G\times G,\mathbb{C})$ with generators $k_a^1, k_a^2 \in \mathfrak{g}$ for the two copies of $\mathfrak{g}$ and multiplication defined by*

$$(\xi \otimes F) \cdot (\eta \otimes K) = (\xi \cdot_U \eta) \otimes (FK) + i\hbar\, \eta \otimes \{\xi \otimes 1, 1 \otimes K\}, \tag{3.28}$$

where $\xi, \eta \in \mathfrak{g} \oplus \mathfrak{g}, F, K \in \mathcal{C}^\infty(G \times G, \mathbb{C})$, $\cdot_U$ *is the multiplication in* $U(\mathfrak{g} \oplus \mathfrak{g})$ *and the bracket* $\{\,,\,\}$ *is given by* (2.30).

2. *The corresponding transformation group algebra has a single irreducible representation on the space* $L^2(G\times G, \mathbb{C})$ *and elements of* $\widehat{\mathcal{H}}$ *act on the dense subspace* $\mathcal{C}_0^\infty(G \times G, \mathbb{C})$ *via*

$$\Pi_{\widehat{\mathcal{H}}}((k_a^1, k_b^2) \otimes 1)\psi(w_1, w_2) = -i\hbar \frac{d}{dt}|_{t=0}\psi(w_1 e^{tJ_a}, e^{-tJ_b} w_2), \qquad (3.29)$$
$$\Pi_{\widehat{\mathcal{H}}}(1 \otimes F)\psi(w_1, w_2) = F(w_1, w_2) \cdot \psi(w_1, w_2).$$

The case of the dual Poisson structure H^* is slightly more complicated. Here, the group action associated to the Poisson bracket is conjugation with G. Consequently, its orbits are G-conjugacy classes, and its irreducible representations are labelled by G-conjugacy classes and unitary irreducible representations of the associated stabilisers.

Theorem 3.7. *(Puncture algebra* $\widehat{\mathcal{P}}$*)*

1. *The quantum algebra* $\widehat{\mathcal{P}}$ *associated to the Poisson structure* (2.26) *on the dual Poisson-Lie group* H^*, *in the following referred to as puncture algebra, is the associative algebra* $\widehat{\mathcal{P}} = U(\mathfrak{g})\otimes\mathcal{C}^\infty(G, \mathbb{C})$ *with multiplication defined by*

$$(\xi \otimes F) \cdot (\eta \otimes K) = (\xi \cdot_U \eta) \otimes (FK) - i\hbar\, \eta \otimes ((\xi^L + \xi^R)K), \qquad (3.30)$$

where $\xi, \eta \in \mathfrak{g}$, $F, K \in \mathcal{C}^\infty(G, \mathbb{C})$, ξ^L, ξ^R *are the right- and left-invariant vector fields associated to* ξ *and* $\cdot_U$ *denotes the multiplication in the universal enveloping algebra* $U(\mathfrak{g})$.

2. *The corresponding transformation group algebra is called the quantum double of* G *and denoted* $D(G)$. *Under the technical assumptions of Theorem 3.3 its irreducible representations are labelled by the* G-*conjugacy classes* $\mathcal{C}_\mu = \{v \cdot g_\mu \cdot v^{-1} \,|\, v \in G\}$ *and irreducible unitary representations* Π_s *of associated stabilisers* $N_\mu = \{n \in G | n \cdot g_\mu \cdot n^{-1} = g_\mu\}$ *on Hilbert spaces* V_s. *The representation spaces are*

$$V_{\mu s} = \{\psi : G \to V_s \mid \psi(vn) = \Pi_s(n^{-1})\psi(v), \;\; \forall n \in N_\mu, \;\; \forall v \in G,$$
$$\text{and} \quad \|\psi\|^2 := \int_{G/N_\mu} \|\psi(z)\|^2_{V_s}\, dm(zN_\mu) < \infty\}/\sim \qquad (3.31)$$

where $\sim$ *denotes division by zero-norm states and* dm *is an invariant measure on* G/N_μ. *The algebra* $\widehat{\mathcal{P}}$ *acts on the dense subspace* $V_{\mu s}^\infty$ *via*

$$\Pi_{\mu s}(\xi \otimes 1)\psi(v) = -i\hbar\xi^L\psi(v), \;\; \Pi_{\mu s}(1 \otimes F)\psi(v) = F(vg_\mu v^{-1}) \cdot \psi(v) \qquad (3.32)$$

for $\xi \in \mathfrak{g}$, $F \in \mathcal{C}^\infty(G, \mathbb{C})$.

After quantising the Poisson algebras (2.26) and (2.30) associated to each puncture and handle, we can now combine these building blocks to construct the quantum algebra for the decoupled flower algebra and its irreducible representations. By inverting transformation (2.29), we obtain the quantum algebra $\hat{\mathcal{F}}$ associated to the Poisson algebra (2.26),(2.27). Note that this Poisson algebra is again of type (3.1) with Poisson brackets (3.2), where now $E = G^{n+2g}$ and the action of G^{n+2g} on itself given by combining n copies of the group action associated to the puncture algebra $\widehat{\mathcal{P}}$ and g copies of the group action for the handle algebra $\widehat{\mathcal{H}}$:

$$\begin{aligned}&(h_{M_1},\ldots,h_{M_n},h_{A_1},h_{B_1},\ldots,h_{A_g},h_{B_g}).(u_{M_1},\ldots,u_{M_n},u_{A_1},u_{B_1},\ldots,u_{A_g},u_{B_g})\\&=(h_{M_1}u_{M_1}h_{M_1}^{-1},\ldots,h_{M_n}u_{M_n}h_{M_n}^{-1},u_{A_1}h_{A_1},u_{A_1}h_{A_1}u_{A_1}^{-1}u_{B_1}h_{B_1},\ldots,u_{A_g}h_{A_g},\\&\qquad u_{A_g}h_{A_g}u_{A_g}^{-1}u_{B_g}h_{B_g}).\end{aligned}$$

Theorem 3.8. *(Quantisation of the decoupled flower algebra)*

1. *The quantum algebra for the decoupled flower algebra in Theorem 2.4 is the associative algebra*

$$\hat{\mathcal{F}} = U\left(\bigoplus_{k=1}^{n+2g}\mathfrak{g}\right)\otimes\mathcal{C}^\infty(G^{n+2g},\mathbb{C}), \tag{3.33}$$

with the multiplication defined by

$$(\xi\otimes F)\cdot(\eta\otimes K) = \xi\cdot_U\eta\otimes FK + i\hbar\,\eta\otimes F\{\xi\otimes 1, 1\otimes K\}, \tag{3.34}$$

where $\xi,\eta\in\bigoplus_{k=1}^{n+2g}\mathfrak{g}$, $F,K\in\mathcal{C}^\infty(G^{n+2g},\mathbb{C})$ and $\cdot_U$ denotes the multiplication in $U\left(\bigoplus_{k=1}^{n+2g}\mathfrak{g}\right)$. The bracket $\{\,,\}$ is given by (2.26),(2.27) via the identification of the generators of the different copies of $\mathfrak{g}$ with the elements $j_a^{M_1'},\ldots,j_a^{M_n'}$, $j_a^{A_1'}-j_a^{B_1'},j_a^{B_1'},\ldots,j_a^{A_g'}-j_a^{B_g'},j_a^{B_g'}$ in Theorem 2.4. The algebra $\hat{\mathcal{F}}$ has a $*$-structure given by $(j_a^{X'})^* = j_a^{X'}$, $(1\otimes F)^* = 1\otimes\bar{F}$.

2. *If the technical assumptions of Theorem 3.3 hold the irreducible representations of the transformation group algebra corresponding to* (3.33) *are labelled by n G-conjugacy classes $\mathcal{C}_{\mu_1},\ldots,\mathcal{C}_{\mu_n}$ and irreducible unitary representations Π_{s_i} of stabilisers $N_{\mu_1},\ldots,N_{\mu_n}$ of chosen elements $g_{\mu_1},\ldots g_{\mu_n}$ in those conjugacy classes on Hilbert spaces V_{s_i}. Let*

$$\begin{aligned}L^2_{\mu_1 s_1\ldots\mu_n s_n} = \big\{\psi : G^{n+2g}\to V_{s_1}\otimes\ldots\otimes V_{s_n})\mid\\ \psi(v_1h_1,\ldots,v_{M_n}h_n,u_{A_1},\ldots,u_{B_g})\\ = (\Pi_{s_1}(h_1^{-1})\otimes\ldots\otimes\Pi_{s_n}(h_n^{-1}))\;\psi(v_{M_1},\ldots,v_{M_n},u_{A_1},\ldots,u_{B_g})\\ \forall h_i\in N_{\mu_i},\ ||\psi||^2<\infty\big\},\end{aligned}$$

with norm $||\cdot||$ derived from the inner product

$$\langle\psi,\phi\rangle = \int_{G/N_{\mu_1}\times\ldots\times G/N_{\mu_n}\times G^{2g}} (\,\psi\,,\,\phi\,)\,(v_{M_1},\ldots,v_{M_n},u_{A_1},\ldots u_{B_g}) \qquad (3.35)$$
$$dm_1(v_{M_1}\cdot N_1)\cdots dm_n(v_{M_n}\cdot N_n)\cdot du_{A_1}\cdots du_{B_g},$$

where $(\,,\,)$ is the canonical inner product on the tensor product of Hilbert spaces $V_{s_1}\otimes\ldots\otimes V_{s_n}$. The representation spaces are

$$V_{\mu_1 s_1\ldots\mu_n s_n} = L^2_{\mu_1 s_1\ldots\mu_n s_n}/\sim \qquad (3.36)$$

where $\sim$ denotes division by zero-norm states. Elements of $\hat{\mathcal{F}}$ act on the dense subspace $V^\infty_{\mu_1 s_1\ldots\mu_n s_n}$ according to

$$\begin{aligned}
&\Pi_{\mu_1 s_1\ldots\mu_n s_n}(1\otimes F)\psi(v_{M_1},\ldots,v_{M_n},u_{A_1},\ldots,u_{B_g}) \qquad (3.37)\\
&\quad= F(v_{M_1}g_{\mu_1}v_{M_1}^{-1},\ldots,v_{M_n}g_{\mu_n}v_{M_n}^{-1},u_{A_1},\ldots,u_{B_g})\\
&\qquad\quad \Psi(v_{M_1},\ldots,v_{M_n},u_{A_1},\ldots,u_{B_g})\\
&\quad= ((F\circ\beta)\cdot\psi)(v_{M_1},\ldots,v_{M_n},u_{A_1},\ldots,u_{B_g})\\
&\Pi_{\mu_1 s_1\ldots\mu_n s_n}(j_a^{M_i'}\otimes 1)\psi(v_{M_1},\ldots,v_{M_n},u_{A_1},\ldots,u_{B_g})\\
&\quad= -i\hbar\,\frac{d}{dt}|_{t=0}\psi(v_{M_1},\ldots,e^{-tJ^a}v_{M_i},\ldots,v_{M_n},u_{A_1},\ldots,u_{B_g})\\
&\quad= -i\hbar\,J_a^{L_{M_i}}\,\psi(v_{M_1},\ldots,v_{M_n},u_{A_1},\ldots,u_{B_g})\\
&\Pi_{\mu_1 s_1\ldots\mu_n s_n}((j_a^{A_i'}-j_a^{B_i})\otimes 1)\psi(v_{M_1},\ldots,v_{M_n},u_{A_1},\ldots,u_{B_g})\\
&\quad= -i\hbar\,\frac{d}{dt}|_{t=0}\psi(v_{M_1},\ldots,u_{A_i}e^{tJ^a},u_{A_i}e^{tJ^a}u_{A_i}^{-1}u_{B_i},\ldots,u_{B_g})\\
&\quad= -i\hbar\,(J_a^{R_{A_i}}+Ad^*(u_{A_i}^{-1}u_{B_i})_a{}^b J_b^{R_{B_i}})\\
&\qquad\quad \psi(v_{M_1},\ldots,v_{M_n},u_{A_1},\ldots,u_{B_g})\\
&\Pi_{\mu_1 s_1\ldots\mu_n s_n}(j_a^{B_i'}\otimes 1)\psi(v_{M_1},\ldots,v_{M_n},u_{A_1},\ldots,u_{B_g})\\
&\quad= -i\hbar\,\frac{d}{dt}|_{t=0}\psi(v_{M_1},\ldots,u_{A_i},u_{B_i}e^{tJ^a},\ldots,u_{B_g})\\
&\quad= -i\hbar\,J_a^{R_{B_i}}\,\psi(v_{M_1},\ldots,v_{M_n},u_{A_1},\ldots,u_{B_g}),
\end{aligned}$$

with $\beta : G^{n+2g}\to G^{n+2g}$ given by

$$\begin{aligned}
&\beta(v_{M_1},\ldots,v_{M_n},u_{A_1},\ldots,u_{B_g}) \qquad (3.38)\\
&\qquad := (v_{M_1}g_{\mu_1}v_{M_1}^{-1},\ldots,v_{M_n}g_{\mu_n}v_{M_n}^{-1},u_{A_1},\ldots,u_{B_g}).
\end{aligned}$$

3. *With the induced inner product on the subspace $V^\infty_{\mu_1 s_1\ldots\mu_n s_n}$, the representations (3.37) are $*$-representations with respect to the $*$-structure given above and the operators $\Pi_{\mu_1 s_1\ldots\mu_n s_n}(j_a^{X'})$, $\Pi_{\mu_1 s_1\ldots\mu_n s_n}(1\otimes F)$ are Hermitian.*

3.3 The quantum decoupling transformation

We can now use the quantisation of the decoupled Poisson structure in Theorem 3.8 to obtain the quantum version of the original, undecoupled Poisson brackets (2.21) of the flower algebra. The idea is to define a quantum counterpart of the inverse decoupling transformation K^{-1} given by (2.25). In trying to implement this idea one encounters ordering ambiguities in the part of K^{-1} that involves the generators $j_a^{A_i}, j_a^{B_i}$ associated to the handles, such that the quantum versions of the brackets (2.21) associated to different choices of ordering would differ by quantum corrections. However, a closer look at the structure of the quantum algebra $\hat{\mathcal{F}}$ provides us with a canonical definition of the inverse quantum decoupling transformation. Note that the classical decoupling transformation and its inverse are *linear* in the generators j_a^X and $j_a^{X'}$. We can therefore interpret them as bijective maps on the vector space $\mathcal{F}^{(0)} \oplus \mathcal{F}^{(1)}$ and use the vector space isomorphism Q in Theorem 3.1 to define its quantum counterpart.

Theorem 3.9. *(Quantum decoupling transformation)*

Define the quantum decoupling transformation as

$$\hat{K} := Q|_{01} \circ K \circ Q|_{01}^{-1} : \ \hat{\mathcal{F}}^{(1)} \to \hat{\mathcal{F}}^{(1)}, \tag{3.39}$$

where $Q|_{01}$ denotes the map $Q|_{(\mathcal{F}^{(0)} \oplus \mathcal{F}^{(1)})_{\mathbb{C}}} : \mathcal{F}_{\mathbb{C}}^{(0)} \bigoplus \mathcal{F}_{\mathbb{C}}^{(1)} \to \hat{\mathcal{F}}^{(1)}$ to simplify notation, and transform the generators of $\hat{\mathcal{F}}^{(1)}$ with its inverse $\hat{K}^{-1} = Q|_{01} \circ K^{-1} \circ Q|_{01}^{-1}$. Then the commutators of the transformed generators are given by applying the map $i\hbar\, Q|_{01}$ to the right-hand side of equations (2.21)*.*

Note that, although this construction looks quite formal, it amounts to the choice of an ordering in (2.23),(2.25), namely ordering all the generators j_a^X, $j_a^{X'}$ that occur in these expressions to the right. The quantisation of the Poisson brackets (2.21) obtained this way is canonical in the following sense. Although the right-hand sides of (2.21) contain products of generators $j_a^Y \in \bigoplus_{k=1}^{n+2g} \mathfrak{g}$ with functions $\mathrm{Ad}^*(u_X)_b{}^c \in \mathcal{C}^\infty(G^{n+2g})$, these products do not give rise to ordering ambiguities. This is due to the fact that $X < Y$ in all of them, for which the last three brackets in (2.21) imply that the factors commute.

4 Quantum symmetries and the quantum double $D(G)$

4.1 Quantum action of $G \ltimes \mathcal{C}^\infty(G)$

With the definition of the quantum flower algebra $\hat{\mathcal{F}}$ and its irreducible representations in Theorems 3.8 and 3.9, we can now determine how the group $G \ltimes \mathcal{C}^\infty(G)$ acts on this algebra. The crucial observation for constructing the quantum action of this symmetry group is the fact that it induces *linear* transformations of the classical generators j_a^X, as explained in Theorem 2.6. This allows us to define their action on the generators of the quantum algebra by means of the map Q in Theorem 3.1:

Theorem 4.1. *(Action of $G \ltimes \mathcal{C}^\infty(G)$ on $\hat{\mathcal{F}}$)*

For an element $(h, f) \in G \ltimes \mathcal{C}^\infty(G)$, define its action $\widehat{(h, f)}$ on the subalgebra $\hat{\mathcal{F}}^{(1)}$ by

$$\widehat{(h, f)} = Q|_{01} \circ (h, f) \circ Q|_{01}^{-1}. \tag{4.1}$$

This map is a Lie algebra automorphism of $\hat{\mathcal{F}}^{(1)}$ and can be extended canonically to an algebra isomorphism $\widehat{(h, f)} : \hat{\mathcal{F}} \to \hat{\mathcal{F}}$ of the quantised flower algebra.

Proof: Let τ be a Poisson isomorphism of the classical flower algebra that restricts to a Lie algebra automorphism of the linear subalgebra $\mathcal{F}^{(0)}_{\mathbb{C}} \oplus \mathcal{F}^{(1)}_{\mathbb{C}}$ and therefore defines a Lie algebra automorphism $\hat{\tau} : \hat{\mathcal{F}}^{(1)} \to \hat{\mathcal{F}}^{(1)}$ by $\hat{\tau} = Q|_{01} \circ \tau \circ Q|_{01}^{-1}$. As $Q|_{01} : \mathcal{F}^{(0)}_{\mathbb{C}} \oplus \mathcal{F}^{(1)}_{\mathbb{C}} \to \hat{\mathcal{F}}^{(1)}$ is a Lie algebra isomorphism and τ a Poisson automorphism of $\mathcal{F}$, we have

$$\begin{aligned}
[\hat{\tau}(\theta), \hat{\tau}(\chi)] &= i\hbar\, Q|_{01}(\{\tau(Q|_{01}^{-1}(\theta)), \tau(Q|_{01}^{-1}(\chi))\}) \\
&= i\hbar\, Q|_{01} \circ \tau(\{Q|_{01}^{-1}(\theta), Q|_{01}^{-1}(\chi)\}) \\
&= Q|_{01} \circ \tau \circ Q|_{01}^{-1}([\theta, \chi]) = \hat{\tau}([\theta, \chi]) \qquad \forall \theta, \chi \in \hat{\mathcal{F}}^{(1)}.
\end{aligned} \tag{4.2}$$

Via the choice of an ordered basis of $U\left(\bigoplus_{k=1}^{n+2g} \mathfrak{g}\right)$, for example the ordered polynomials in the elements of the ordered basis $j_1^{M'_1} < \ldots < j_{\dim G}^{M'_1} < j_1^{M'_2} < \ldots < j_{\dim G}^{M'_n} < j_1^{A'_1} < \ldots < j_{\dim G}^{A'_1} < j_1^{B'_1} < \ldots < j_{\dim G}^{B'_g}$ of the Lie algebra $\bigoplus_{k=1}^{n+2g} \mathfrak{g}$, and setting

$$\begin{aligned}
\hat{\tau}((j_{a_1}^{X'_1} \cdots j_{a_m}^{X'_m}) \otimes 1) &:= \hat{\tau}(j_{a_1}^{X'_1} \otimes 1) \cdots \hat{\tau}(j_{a_m}^{X'_m} \otimes 1) \\
\hat{\tau}((1 \otimes F) \cdot (\theta \otimes 1)) &:= \hat{\tau}(1 \otimes F) \cdot \hat{\tau}(\theta \otimes 1)
\end{aligned} \tag{4.3}$$

for elements θ of this ordered basis, $\hat{\tau}$ can be extended to a vector space isomorphism on $\hat{\mathcal{F}}$. Because $\hat{\tau}$ has been extended multiplicatively to the ordered basis and $[\hat{\tau}(\xi\otimes F),\hat{\tau}(\eta\otimes K)]=\hat{\tau}([\xi\otimes F,\eta\otimes K])$ for $\xi,\eta\in\bigoplus_{k=1}^{n+2g}\mathfrak{g}$ and $F,K\in\mathcal{C}^\infty(G^{n+2g},\mathbb{C})$, $\hat{\tau}:\hat{\mathcal{F}}\to\hat{\mathcal{F}}$ is also an algebra isomorphism. □

The action of the group $G\ltimes\mathcal{C}^\infty(G)$ as an algebra isomorphism of the quantised flower algebra raises the question if it can be implemented unitarily in the representation spaces (3.36). The following theorems show that this is indeed possible and give explicit formulae for the action of $G\ltimes\mathcal{C}^\infty(G)$ in the representation spaces (3.36).

Theorem 4.2. *(Representations of $G\ltimes\mathcal{C}^\infty(G)$)*

Let $\Pi_{\mu_1 s_1\ldots\mu_n s_n}$ be a representation of the flower algebra as defined in Theorem 3.8. Let the maps $\beta,\widetilde{Ad}_h^{n+2g}:G^{n+2g}\to G^{n+2g}$ be given by (3.38) and

$$\widetilde{Ad}_h^{n+2g}(v_{M_1},\ldots,v_{M_n},u_{A_1},\ldots,u_{B_g})=(hv_{M_1},\ldots,hv_{M_n},hu_{A_1}h^{-1},\ldots,hu_{B_g}h^{-1}). \tag{4.4}$$

Then $\Gamma:G\ltimes\mathcal{C}^\infty(G)\to End(V_{\mu_1 s_1\ldots\mu_n s_n})$

$$\begin{aligned}\Gamma(h,f)\Psi(v_{M_1},\ldots,v_{M_n},u_{A_1},\ldots,u_{B_g}) \qquad\qquad (4.5)\\ =\left(e^{\frac{i}{\hbar}f\circ\Phi_\infty\circ\beta}\cdot(\Psi\circ\widetilde{Ad}_{h^{-1}}^{n+2g})\right)(v_{M_1},\ldots,v_{M_n},u_{A_1},\ldots,u_{B_g})\\ =e^{\frac{i}{\hbar}f(u_{tot})}\,\Psi(h^{-1}v_{M_1},\ldots,h^{-1}v_{M_n},h^{-1}u_{A_1}h,\ldots,h^{-1}u_{B_g}h)\end{aligned}$$

with u_{tot} given by by (2.35), defines a representation of the group $G\ltimes\mathcal{C}^\infty(G)$ on the representation space $V_{\mu_1 s_1\ldots\mu_n s_n}$ that satisfies

$$\Pi_{\mu_1 s_1\ldots\mu_n s_n}(\widehat{(h,f)}Z)=\Gamma(h,f)\circ\Pi_{\mu_1 s_1\ldots\mu_n s_n}(Z)\circ\Gamma^{-1}(h,f)\qquad\forall Z\in\hat{\mathcal{F}} \tag{4.6}$$

on the dense invariant subspace $V^\infty_{\mu_1 s_1\ldots\mu_n s_n}$. If the conditions in theorem 3.8 are fulfilled, this representation is unitary.

Proof: To simplify notation, we write V for $V_{\mu_1 s_1\ldots\mu_n s_n}$ and Π for $\Pi_{\mu_1 s_1\ldots\mu_n s_n}$. The identity $f\circ\Phi_\infty\circ\beta\circ\widetilde{Ad}_{h^{-1}}^{n+2g}=f\circ\mathrm{Ad}_{h^{-1}}\circ\Phi_\infty\circ\beta$ implies that (4.5) defines a representation of $G\ltimes\mathcal{C}^\infty(G)$ on V. Since the group elements corresponding to the punctures are left-multiplied by elements of G and the elements corresponding to the handles conjugated in 4.4, the conditions on the measures in Theorem 3.8 guarantee unitarity for the representations of elements $(h,0)\in G\ltimes\mathcal{C}^\infty(G)$. For elements $(1,f)\in G\ltimes\mathcal{C}^\infty(G)$, which act via multiplication by a phase, this is trivial.

To prove identity 4.6, we calculate

$$\begin{aligned}
&\Pi\big(\widehat{(h,f)}(1\otimes K)\big)\circ\Gamma(h,f)\Psi\\
&\qquad = \Pi\big(1\otimes(K\circ \mathrm{Ad}_{h^{-1}})\big)\,\big(\,e^{\frac{i}{\hbar}f\circ\Phi_\infty\circ\beta}\cdot(\Psi\circ\widetilde{\mathrm{Ad}}{}^{n+2g}_{h^{-1}})\,\big) \qquad (4.7)\\
&\qquad = (K\circ\mathrm{Ad}_{h^{-1}}\circ\beta)\cdot e^{\frac{i}{\hbar}f\circ\Phi_\infty\circ\beta}\cdot(\Psi\circ\widetilde{\mathrm{Ad}}{}^{n+2g}_{h^{-1}})\\
&\qquad = \Gamma(h,f)\circ\Pi(1\otimes K)\Psi \qquad \forall\Psi\in V
\end{aligned}$$

and, with the Poisson bracket (2.26),(2.27) for the generators $j_a^{X'}$ of the decoupled flower algebra

$$\begin{aligned}
&\Pi(\widehat{(h,f)}j_a^{X'})\circ\Gamma(h,f)\Psi\\
&\quad = \Pi\big((1\otimes \mathrm{Ad}^*(h)_a{}^b)\cdot j_b^{X'} + 1\otimes\mathrm{Ad}^*(h)_a{}^b\{j_b^{X'}, f\circ\Phi_\infty\}\big)(\Gamma(h,f)\Psi)\\
&\quad = \mathrm{Ad}^*(h)_a{}^b\,(\,\{j_b^{X'}, f\circ\Phi_\infty\}\circ\beta\,)\cdot e^{\frac{i}{\hbar}f\circ\Phi_\infty\circ\beta}\cdot(\Psi\circ\widetilde{\mathrm{Ad}}{}^{n+2g}_{h^{-1}}) \qquad (4.8)\\
&\qquad + \mathrm{Ad}^*(h)_a{}^b\,e^{\frac{i}{\hbar}f\circ\Phi_\infty\circ\beta}\cdot\big(\Pi(j_b^{X'})(\Psi\circ\widetilde{\mathrm{Ad}}{}^{n+2g}_{h^{-1}})\big)\\
&\qquad + \tfrac{i}{\hbar}\mathrm{Ad}^*(h)_a{}^b\,e^{\frac{i}{\hbar}f\circ\Phi_\infty\circ\beta}\cdot(\Psi\circ\widetilde{\mathrm{Ad}}{}^{n+2g}_{h^{-1}})\cdot\big(\Pi(j_b^{X'})(f\circ\Phi_\infty\circ\beta)\big)\\
&\quad = e^{\frac{i}{\hbar}f\circ\Phi_\infty\circ\beta}\cdot(\Pi(j_a^{X'})\Psi)\circ\widetilde{\mathrm{Ad}}{}^{n+2g}_{h^{-1}} = \Gamma(h,f)\circ\Pi(j_a^{X})\Psi \qquad \forall\Psi\in V,
\end{aligned}$$

where we used the identities

$$\begin{aligned}
&\mathrm{Ad}^*(h)_a{}^b\,\Pi(j_b^{X'})(\Psi\circ\widetilde{\mathrm{Ad}}{}^{n+2g}_{h^{-1}}) = \big(\Pi(j_b^{X'})\Psi\big)\circ\widetilde{\mathrm{Ad}}{}^{n+2g}_{h^{-1}} \qquad (4.9)\\
&\Pi(j_b^{X'})(f\circ\Phi_\infty\circ\beta) = i\hbar\,\{j_b^{X'}\otimes 1, f\circ\Phi_\infty\}\circ\beta.
\end{aligned}$$

Therefore, we have $\Pi(\widehat{(h,f)})\theta) = \Gamma(h,f)\circ\Pi(\theta)\circ\Gamma^{-1}(h,f)$ for all $\theta\in\mathcal{F}$. $\Box$

4.2 The relation to the quantum double $D(G)$

In the combinatorial approach to quantising Chern-Simons theory [2, 3, 27, 28] quantum groups or, more precisely, ribbon-Hopf-*-algebras and associated structures play a central role. Since our approach to quantising Chern-Simons theory with gauge group $G\ltimes\mathfrak{g}^*$ begins with a description of the classical phase space which is analogous to that used in [2, 3] it is perhaps surprising the we arrived at a quantisation without making explicit use of quantum group theory so far.

In this section we discuss which role quantum groups, more precisely, the quantum double $D(G)$ of the Lie group G play in our formalism. We already encountered $D(G)$ as the transformation group algebra associated with the puncture algebra in Theorem 3.7. Here we will exhibit its ribbon-Hopf properties. We show that the quantum double $D(G)$ also acts on the representation spaces of the handle algebra $\widehat{\mathcal{H}}$ given in Def. 3.6. Using this action we obtain a representation of the quantum double on the representation spaces of the quantised flower algebra defined in Theorem 3.8. By comparison with the quantum symmetries discussed in Sect. 4.1, it then becomes apparent that the quantum double can be viewed as a generalisation of the symmetry group $G \ltimes \mathcal{C}^\infty(G)$.

The quantum double of a Lie group G has been studied in various publications. We adopt the conventions used in [26] where the quantum double is identified with continuous functions on $G \times G$, except that we exchange the roles played by the two copies of G in order to match our conventions for the semidirect product group H. Thus we identify $D(G) = C_0(G \times G, \mathbb{C})$ as a vector space. In order to exhibit the structure of $D(G)$ as a ribbon-Hopf-*-algebra, we need to include Dirac delta functions which are not strictly in $C_0(G \times G, \mathbb{C})$. One can avoid this problem by modifying the definition of $D(G)$ as explained in [29] or by simply adjoining the singular elements. In practice the latter approach is more convenient. Later we shall see that it is precisely the singular elements $\delta_g \otimes f\ (v,u) = \delta_g(v)f(u)$ which have a conceptually simple interpretation.

Thus we define multiplication $\bullet$, identity 1, co-multiplication Δ, co-unit ϵ, antipode S and involution * via

$$
\begin{aligned}
(F_1 \bullet F_2)(v,u) &:= \int_G F_1(z,u)\, F_2(z^{-1}v, z^{-1}uz)\, dz, && (4.10)\\
1(v,u) &:= \delta_e(v), && (4.11)\\
(\Delta F)(v_1,u_1;v_2,u_2) &:= F(v_1,u_1u_2)\,\delta_{v_1}(v_2). && (4.12)\\
\epsilon(F) &:= \int_G F(v,e)\, dv, && (4.13)\\
(SF)(v,u) &:= F(v^{-1}, v^{-1}u^{-1}v), && (4.14)\\
F^*(v,u) &:= \overline{F(v^{-1}, v^{-1}uv)}, && (4.15)
\end{aligned}
$$

so that we have for the singular elements

$$
\begin{aligned}
&(\delta_{g_1} \otimes f_1) \bullet (\delta_{g_2} \otimes f_2) = \delta_{g_1g_2} \otimes (f_1 \cdot f_2 \circ \mathrm{Ad}_{g_1^{-1}}) && (4.16)\\
&\Delta(\delta_g \otimes f)(v_1,u_1;v_2,u_2) = \delta_g(v_1)\delta_g(v_2)f(u_1u_2) && (4.17)\\
&\epsilon(\delta_g \otimes f) = f(e) && (4.18)\\
&S(\delta_g \otimes f) = \delta_{g^{-1}} \otimes (f \circ \mathrm{Ad}_g \circ (\,)^{-1}) && (4.19)\\
&(\delta_g \otimes f)^* = \delta_{g^{-1}} \otimes (\overline{f} \circ \mathrm{Ad}_g). && (4.20)
\end{aligned}
$$

The universal R-matrix of $D(G)$ is

$$
R(v_1,u_1;v_2,u_2) = \delta_e(v_1)\delta_e(u_1v_2^{-1}) \qquad (4.21)
$$

and the central ribbon element c

$$c(v,u) = \delta_v(u). \tag{4.22}$$

It satisfies the ribbon relation

$$\Delta c = (R_{21} \bullet R) \bullet \big(c \otimes c\big) \tag{4.23}$$

with $R_{21}(v_1,u_1;v_2,u_2) := R(v_2,u_2;v_1,u_1)$.

The irreducible representations of $D(G)$ are given in Theorem 3.7. With the notation introduced there, the singular elements have the simple action

$$\Pi_{\mu s}(\delta_g \otimes f)\psi(v) = f(vg_\mu v^{-1})\psi(g^{-1}v). \tag{4.24}$$

There is a further representation of $D(G)$ which will be relevant in the following, but which is not irreducible. It is obtained by letting $D(G)$ act on itself via the adjoint action. Using Sweedler notation as defined in (3.13) the adjoint action of $F \in D(G)$ on $\phi \in D(G)$ is

$$\begin{aligned} \mathrm{ad}(F)\phi(w_1,w_2) &= \sum F_{(1)} \bullet \phi \bullet SF_{(2)} \\ &= \int_G F(v, w_1 w_2^{-1} w_1^{-1} w_2)\phi(v^{-1}w_1 v, v^{-1}w_2 v)\, dv \end{aligned} \tag{4.25}$$

implying

$$\mathrm{ad}(\delta_g \otimes k)\phi(w_1,w_2) = k(w_1 w_2^{-1} w_1^{-1} w_2)\phi(g^{-1}w_1 g, g^{-1}w_2 g). \tag{4.26}$$

Now note that the same action can be used to let $D(G)$ act on the irreducible representation of the handle algebra $\widehat{\mathcal{H}}$. Combining the representations (3.31) and (4.25) and using the co-multiplication of $D(G)$ repeatedly we obtain an action of $D(G)$ on the representation spaces $V_{\mu_1 s_1 \ldots \mu_n s_n}$ of the flower algebra:

$$\begin{aligned} \Pi(F)\psi = (\Pi_{\mu_1 s_1} \otimes \cdots \otimes \Pi_{\mu_n s_n} \otimes \mathrm{ad} \otimes \cdots \otimes \mathrm{ad}) \circ (\Delta \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(n+g-2)\times}) \circ \ldots \\ \ldots \circ (\Delta \otimes 1 \otimes 1) \circ (\Delta \otimes 1)(F)\psi, \end{aligned} \tag{4.27}$$

and for the singular elements

$$\begin{aligned} &\Pi(\delta_h \otimes k)\psi(v_{M_1},\ldots,v_{M_n},w_{1,1},w_{2,1},\ldots,w_{1,g},w_{2,g}) \\ &\quad = (k \circ \Phi_\infty \circ \beta)\cdot(\psi \circ \widetilde{\mathrm{Ad}}^{n+2g}_{h^{-1}})(v_{M_1},\ldots,v_{M_n},w_{1,1},w_{2,1},\ldots,w_{1,g},w_{2,g}) \\ &\quad = k(u_{tot})\psi(h^{-1}v_{M_1},\ldots,h^{-1}v_{M_n},h^{-1}w_{1,1}h,h^{-1}w_{2,1}h,\ldots \\ &\qquad\qquad ,h^{-1}w_{1,g}h,h^{-1}w_{2,g}h), \end{aligned} \tag{4.28}$$

where β, $\widetilde{\mathrm{Ad}}_{h^{-1}}^{n+2g}$ are given by (3.38), (4.4) and Φ_∞ and u_{tot} by (2.35) with the identification $w_{1,i} = u_{A_i}$, $w_{2,i} = u_{B_i}^{-1} u_{A_i}$ as in (2.29).

Comparing this representation of the quantum double $D(G)$ on the representation space $V_{\mu_1 s_1 \ldots \mu_n s_n}$ with the quantum action (4.5) of the group $G \ltimes \mathcal{C}^\infty(G)$ in Theorem 4.2, we see that they are identical if we identify $h \leftrightarrow \delta_h$ and $k \leftrightarrow e^{\frac{i}{\hbar} f}$. Furthermore, with this identification, the multiplication law (4.16) of $D(G)$ agrees with the multiplication (2.33) of the group $G \ltimes \mathcal{C}^\infty(G)$ and the $*$-operation (4.20) maps each element of $G \ltimes \mathcal{C}^\infty(G)$ to its inverse. In other words, the map

$$(h, f) \mapsto \delta_h \otimes e^{\frac{i}{\hbar} f} \tag{4.29}$$

is a group morphism from the symmetry group $G \ltimes \mathcal{C}^\infty(G)$ into the semidirect product $G \ltimes \mathcal{C}^\infty(G, U(1))$ of G with smooth $U(1)$-valued functions on G, realised as a group of (singular) elements in the quantum double $D(G)$. In this sense the quantum group $D(G)$ generalises the symmetry group $G \ltimes \mathcal{C}^\infty(G)$. Moreover, the formula (4.27) shows that the action of this generalised symmetry on the Hilbert space $V_{\mu_1 s_1 \ldots \mu_n s_n}$ is naturally expressed in terms of the co-multiplication of $D(G)$.

This relation between the group $G \ltimes \mathcal{C}^\infty(G)$ and the quantum double $D(G)$ fits in nicely with the role quantum groups play in the formalism of combinatorial quantisation of Chern-Simons gauge theories. In Sect. 2.2, we explained how the phase space of Chern-Simons gauge theory with gauge group H, the moduli space of flat H-connections, is related to the classical flower algebra. It is obtained from the space of holonomies by dividing out the residual gauge transformations that act on the holonomies by global conjugation with H, i.e. by imposing the constraint arising from (2.16). In the combinatorial quantisation scheme [2, 3, 4], the representation spaces of the quantised moduli space are then constructed by imposing invariance under the action of a corresponding quantum group on the representation spaces of the quantum flower algebra.

In our formalism, the constraint $(u_{tot}, -\mathrm{Ad}(u_{tot})\boldsymbol{j}) \approx 1$ arising from (2.16) appears as the infinitesimal generator of the classical action of the symmetry group $G \ltimes \mathcal{C}^\infty(G)$ in Theorem 2.6. Its action on the flower algebra can be interpreted as a generalised or deformed conjugation. Implementing this constraint via the Dirac formalism [2, 3, 4] would then amount to selecting the states on the representation spaces $V_{\mu_1 s_1 \ldots \mu_n s_n}$ of the quantum flower algebra that are invariant under the action of the group $G \ltimes \mathcal{C}^\infty(G)$.

5 Outlook and conclusions

In this paper we quantised the flower algebra associated to Chern-Simons gauge theories with semidirect product gauge groups $H = G \ltimes \mathfrak{g}^*$ on a punctured surface. We showed how its Poisson structure can be broken up into a set of Poisson commuting building blocks and discussed the Poisson action of the group $G \ltimes \mathcal{C}^\infty(G)$. This allowed us to construct the corresponding quantum algebra and its irreducible Hilbert space representations by means of a rather straightforward quantisation procedure. After determining the action of the group $G \ltimes \mathcal{C}^\infty(G)$ on the quantum algebra, we were then able to relate this group action to the quantum double $D(G)$ of the Lie group G. This clarified how this quantum group arises as a quantum symmetry.

It is interesting to compare our approach to the formalism of combinatorial quantisation of Chern-Simons gauge theories developed for compact, semisimple gauge groups in [2, 3, 4] and its extension to the case of the semisimple but non-compact group $SL(2,\mathbb{C})$ in [27, 28]. Both start from the classical flower algebra and in both cases quantum groups play an important role. However, the generalisation of combinatorial quantisation scheme to groups of the form $H = G \ltimes \mathfrak{g}^*$ is beset by technical difficulties. For this reason, we did not use the combinatorial quantisation scheme as a guideline for quantisation but based our approach on a detailed investigation of the structure of the classical algebra.

We see explicitness and simplicity as an advantage of our approach. It describes the classical flower algebra in terms of quantities that can easily be related to the physical content of the underlying Chern-Simons gauge theory. For instance, in the case of (2+1)-dimensional gravity in its formulation as a Chern-Simons theory with the three-dimensional Poincaré group as gauge group, our parameters represent momenta and angular momenta of handles and massive particles with spin [9]. All of the structural properties of the flower algebra, the action of the symmetry group $G \ltimes \mathcal{C}^\infty(G)$ as well as the transformation that decouples the contributions of different punctures and handles, can be expressed in terms of these quantities. This allows us to perform concrete calculations and to gain insight into their physical interpretation. Similarly, the corresponding quantum algebra is given explicitly as a semidirect product of an universal enveloping algebra and an Abelian algebra of functions, rather than implicitly by a set of generating matrix elements and relations as in the combinatorial quantisation formalism. This facilitates the investigation of its structure and the study of its representation theory.

We did not impose the constraint that the holonomies around punctures lie in fixed H-conjugacy classes either in the classical or in the quantised flower algebra, mainly for technical reasons. Instead, we found that the irreducible representations of the quantised flower algebra correspond to the symplectic leaves of the classical flower algebra. Recall that the latter are of the form $\mathcal{C}_{\mu_1 s_1} \times \ldots \times \mathcal{C}_{\mu_n s_n} \times H^{2g}$, where μ_i label G-conjugacy classes and s_i co-adjoint orbits of associated stabiliser groups. The irreducible representations (3.36) of the quantised flower algebra are labelled by G-conjugacy classes and irreducible representations of associated stabiliser groups. The correspondence between symplectic leaves and irreducible representations is thus the familiar correspondence between co-adjoint orbits and irreducible representations [30] which typically holds for quantised values of the parameters labelling the co-adjoint orbits. Details depend on the group G and seem worth investigating further. In particular, one should be able to obtain the representation spaces $V_{\mu s}$ of the puncture algebra directly by geometric quantisation of the conjugacy classes $\mathcal{C}_{\mu s}$. For an approach to the quantisation of closely related spaces $(T^*G)/N$ via C^*-algebras see [31].

The quantisation of the flower algebra for Chern-Simons gauge theories with gauge groups $H = G \ltimes \mathfrak{g}^*$ constitutes an important step towards the quantisation of the moduli space of flat H-connections. To obtain a quantisation of the moduli space from this quantisation of the flower algebra, one would have to implement the constraint arising from (2.16) which acts as the infinitesimal generator of the action of the group $G \ltimes \mathcal{C}^\infty(G)$. Doing this via the Dirac quantisation procedure would amount to determining the subspaces of the representation spaces of the flower algebra that are invariant under the action of $G \ltimes \mathcal{C}^\infty(G)$ or the quantum double $D(G)$. This requires a Clebsch-Gordon analysis of tensor product representations of the quantum double $D(G)$. For compact groups G, a general framework for doing this was developed in [32], but explicit calculations of Clebsch-Gordon coefficients depend on the particular choice of G. The case $G = SU(2)$ was studied in [33]. For other groups such as the group $G = \widetilde{SO}(2,1)$ occurring in (2+1)-dimensional gravity, this remains an open question and possible subject of further investigations.

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