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(0,2) Duality

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Abstract

We construct dual descriptions of $(0,2)$ gauged linear sigma models. In some cases, the dual is a $(0,2)$ Landau-Ginzburg theory, while in other cases, it is a non-linear sigma model. The duality map defines an analogue of mirror symmetry for $(0,2)$ theories. Using the dual description, we determine the instanton corrected chiral ring for some illustrative examples. This ring defines a $(0,2)$ generalization of the quantum cohomology ring of $(2,2)$ theories.

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1 Introduction

Mirror symmetry is one of the more spectacular predictions of string theory [1]. Strings propagating on topologically distinct spaces can give rise to the same effective space-time physics. This duality is best understood for theories that can be constructed as $(2,2)$ gauged linear sigma models (GLSM) [2].

In the space of perturbative heterotic string compactifications, $(2,2)$ world-sheet theories are quite special. The more general supersymmetric string compactification only requires $(0,2)$ supersymmetry. To describe a geometric heterotic string compactification (without fluxes), we need to specify a Kähler space, $\mathcal{M}$, with tangent bundle $T\mathcal{M}$ and a holomorphic bundle, $\mathcal{V}$, satisfying the conditions

$$\mathrm{c}_1(T\mathcal{M}) = 0, \quad \mathrm{c}_1(\mathcal{V}) = 0 \ (\mathrm{mod}\ 2) \tag{1.1}$$

$$\mathrm{ch}_2(\mathcal{V}) = \mathrm{ch}_2(T\mathcal{M}). \tag{1.2}$$

The assumption of world-sheet $(2,2)$ supersymmetry corresponds to the choice,

$$\mathcal{V} = T\mathcal{M}. \tag{1.3}$$

The moduli space of $\mathcal{M}$ locally decomposes into Kähler and complex structure deformations which are exchanged under the mirror map.

It is natural to ask whether a generalization of mirror symmetry exists for the larger class of $(0,2)$ theories. At the outset, there is a potential problem; namely, specifying an $(\mathcal{M},\mathcal{V})$ obeying (1.1) and (1.2) does not guarantee the existence of a corresponding superconformal field theory. Except under special conditions [3, 4], we expect world-sheet instantons to destabilize most $(0,2)$ non-linear sigma models. Fortunately, this potential problem vanishes for $(0,2)$ theories that can be realized as linear sigma models [5, 6, 7]; this vanishing can be quite non-trivial, as shown in [8], because individual instantons can give non-zero contributions. However, the net contribution to the space-time superpotential is zero.

The next basic issue is defining a non-perturbative duality. The moduli space for a geometric $(0,2)$ superconformal field theory consists of Kähler and complex structure deformations together with deformations of the gauge bundle. We could imagine many different dualities permuting these three kinds of moduli. A natural extension of $(2,2)$ mirror symmetry was studied in a special class of solvable $(0,2)$ models by Blumenhagen et. al. [9]. Some mirror pairs related by quotient actions (as in the original $(2,2)$ construction [1]) were described in [10, 11]. This notion of mirror symmetry intimately involves a superpotential in both the original and dual descriptions. In related work, a description of equivariant sheaves and their relevence to $(0,2)$ mirror symmetry appears in [12], while an extension of the monomial divisor mirror map [13] to a class of $(0,2)$ theories appears in [14]. Note that unlike the $(2,2)$ case, we believe that $(0,2)$ mirror symmetry should map certain instanton sums on $\mathcal{M}$ to instanton sums on the mirror. Generically, both sides of the duality receive non-perturbative corrections.

Our aim in this effort is to define a non-perturbative $(0,2)$ duality for theories that can be constructed from gauged linear sigma models. We generalize an approach used recently by Hori and Vafa to construct $(2,2)$ mirror pairs [15]. Their approach suggests an equivalence between certain $(2,2)$ gauged linear sigma models and $(2,2)$ Landau-Ginzburg theories. This equivalence is derived using a generalization of world-sheet abelian duality, and is closely related to an earlier attempt at deriving mirror symmetry [16]. The manifold associated with the gauged linear sigma model is a toric variety with non-negative first Chern class. The basic approach used in [15] is to dualize the torus action which is implemented via an abelian gauge symmetry in the GLSM. This dualization exchanges charged fields for uncharged fields. However, because the circle action is not free, a superpotential is generated by instantons. The dual description is therefore a Landau-Ginzburg theory.

We construct an analogue of abelian duality for $(0,2)$ theories. Applied to a GLSM, this duality generates a non-perturbative dual. There are some important points to note: in this analysis, we dualize models without a superpotential. In a sense, the parameter space that is natural for us is orthogonal to the one studied in [9, 10, 11]. To connect the two notions of duality will require understanding the dualization process in the presence of a superpotential. It seems to us that this problem can be addressed (at least for special models).

We consider both conformal and non-conformal models. For non-conformal models, we can relax condition (1.1) and permit the weaker constraint

$$c_1(T\mathcal{M}) > 0. \tag{1.4}$$

Using the dual description, we can determine the exact chiral ring of the original theory, including instanton corrections. We use this ring to define a generalization of the quantum cohomology ring of a $(2,2)$ model [17, 18]. In some particularly nice illustrative examples based on $\mathbb{P}^1 \times \mathbb{P}^1$, we determine this instanton corrected ring precisely.

The structure of our dual theory depends sensitively on whether $\mathrm{rk}(\mathcal{V}) \geq \mathrm{rk}(T\mathcal{M})$ or whether $\mathrm{rk}(\mathcal{V}) < \mathrm{rk}(T\mathcal{M})$. In the former case, the low-energy dual theory is generically a $(0,2)$ Landau-Ginzburg model with isolated supersymmetric vacua. In the latter case, the dual theory is typically a $(0,2)$ non-linear sigma model. The sigma model metric is singular on certain loci where the accompanying dilaton also diverges. It is worth noting that the duality maps the canonical Kähler moduli of a GLSM to superpotential terms in the dual description.

In section 2, we establish our $(0,2)$ superspace, superfield, and component field conventions. Section 3 contains a derivation of the perturbative duals to both ungauged and gauged $(0,2)$ models. In section 4, we determine the exact form of non-perturbative corrections to the dual superpotential. The vacuum structure, and the nature of instanton corrections to $(0,2)$ theories are described in section 5. Finally, in section 6 we present an analysis of some illustrative examples, together with an explanation of how the different dual descriptions emerge depending on the relation between $\mathrm{rk}(\mathcal{V})$ and $\mathrm{rk}(T\mathcal{M})$.

We should mention a few of the future directions that seem worth exploring to us. Restricting our dualization results to the case of $(0,4)$ theories should help classify heterotic compactifications on a $K3$ surface, extending the classification given in [19] for tori. The analogue of the quantum cohomology ring that we have described should be computable in a large class of examples (conformal and non-conformal), perhaps with the help of localization techniques [20]. In section 6.2, we found a nice example of a bundle degeneration which is easily generalizable. The two-dimensional field theory should provide a resolution of the singularity which seems worth comparing with the cases studied in [21]. There should be some space-time duality argument, generalizing the SYZ construction [22] for $(2,2)$ theories. In a related vein, the heterotic instanton corrections we consider are related, via S-duality, to type I D-string corrections. The relation between the open and closed string instanton moduli spaces is likely to be fascinating (see, for example, [23]). Lastly, we would like to know how much we can learn about the Yukawa couplings of generic $(0,2)$ heterotic theories, and perhaps about superpotentials for vector bundle moduli (studied recently, for example, in [24]).

2 The Structure of $(0,2)$ Theories

In this section we review some basic facts, and fix our notation for $(0,2)$ supersymmetric field theories in $1+1$ dimensions.

2.1 $(0,2)$ Supersymmetry

Chiral $(0,2)$ supersymmetry is generated by two supercharges, Q_+ and $\bar{Q}_+ = Q_+^\dagger$, the bosonic generators H, P and M of translations and rotations, and the generator F_+ of a $U(1)$ R-symmetry. The algebra itself is

$$\begin{aligned} Q_+^2 = \bar{Q}_+^2 &= 0 & \{Q_+, \bar{Q}_+\} &= 2(H-P) \\ [M, Q_+] &= -Q_+ & \left[M, \bar{Q}_+\right] &= -\bar{Q}_+ \\ [F_+, Q_+] &= -Q_+ & \left[F_+, \bar{Q}_+\right] &= +\bar{Q}_+. \end{aligned}$$

Much of what follows is simplified by the use of superspace. Let the $(0,2)$ superspace coordinates be $(y^+, y^-, \theta^+, \bar{\theta}^+)$, where $y^\pm = (y^0 \pm y^1)$. Spinor conventions are as in Wess & Bagger [25]. The superderivatives are

$$\mathrm{D}_+ = \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+\partial_+ \qquad \bar{\mathrm{D}}_+ = -\frac{\partial}{\partial\bar{\theta}^+} + i\theta^+\partial_+ \tag{2.1}$$

$$\{\mathrm{D}_+, \mathrm{D}_+\} = \{\bar{\mathrm{D}}_+, \bar{\mathrm{D}}_+\} = 0 \qquad \{\bar{\mathrm{D}}_+, \mathrm{D}_+\} = 2i\partial_+. \tag{2.2}$$

Unconstrained superfields are arbitrary functions of $(y^+, y^-, \theta^+, \bar{\theta}^+)$. In general, we will work with superfields constrained in different ways. For this reason, it is worth noting that $\bar{\mathrm{D}}_+$ annihilates the combinations $z^+ = y^+ - i\theta^+\bar{\theta}^+$, $z^- = y^-$, and θ^+.

2.1.1 The $(0,2)$ Gauge Multiplet

To construct gauge theories, we need to extend our superspace derivatives, D_+ and $\bar{\mathrm{D}}_+$, to gauge covariant superderivatives. The gauge covariant superderivatives $\mathcal{D}_+, \bar{\mathcal{D}}_+$ acting on charge 1 fields, and $\mathcal{D}_\alpha$ $(\alpha = 1,2)$ satisfy the algebra

$$\mathcal{D}_+^2 = \bar{\mathcal{D}}_+^2 = 0, \qquad \{\ \mathcal{D}_+, \bar{\mathcal{D}}_+\}\ = 2i(\mathcal{D}_0 + \mathcal{D}_1). \tag{2.3}$$

The first two equations imply that $\mathcal{D}_+ = e^{-\Psi} D_+ e^{\Psi}$ and $\bar{\mathcal{D}}_+ = e^{\bar{\Psi}} \bar{D}_+ e^{-\bar{\Psi}}$ where Ψ is a superfield taking values in the Lie algebra of the gauge group. We will restrict to abelian theories in our discussion. In Wess-Zumino gauge, the component expansion of Ψ gives

$$\Psi = \theta^+\bar{\theta}^+(A_0 + A_1)(y^\alpha),$$

while

$$\begin{aligned} \mathcal{D}_0 + \mathcal{D}_1 &= \partial_0 + \partial_1 + i(A_0 + A_1), & (2.4)\\ \mathcal{D}_+ &= \frac{\partial}{\partial\theta^+} - i\bar\theta^+(\mathcal{D}_0 + \mathcal{D}_1), & (2.5)\\ \bar{\mathcal{D}}_+ &= -\frac{\partial}{\partial\bar\theta^+} + i\theta^+(\mathcal{D}_0 + \mathcal{D}_1), & (2.6)\\ \mathcal{D}_0 - \mathcal{D}_1 &= \partial_0 - \partial_1 + iV. & (2.7) \end{aligned}$$

The vector superfield V is given by,

$$V = A_0 - A_1 - 2i\theta^+\bar\lambda_- - 2i\bar\theta^+\lambda_- + 2\theta^+\bar\theta^+ D. \tag{2.8}$$

We see that the A_- component of the gauge-field has two real gaugino partners, while A_+ does not. Under a gauge transformation with gauge parameter Λ satisfying a chiral constraint $\bar{\mathrm{D}}_+\Lambda = 0$, the two gauge-fields V and Ψ transform as follows

$$\begin{aligned} \delta_\Lambda V &= \partial_-(\Lambda + \bar\Lambda),\\ \delta_\Lambda \Psi &= i(\Lambda - \bar\Lambda). \end{aligned}$$

Finally, the natural field strength is an uncharged fermionic chiral superfield,

$$\Upsilon = [\bar{\mathcal{D}}_+, \mathcal{D}_0 - \mathcal{D}_1] = \bar{\mathrm{D}}_+(\partial_-\Psi + iV) = -2\{\lambda_-(z) - i\theta^+(D - iF_{01})\}, \tag{2.9}$$

for which the natural action is

$$S_\Upsilon = \frac{1}{8e^2}\int \mathrm{d}^2y\, d^2\theta\ \bar\Upsilon\Upsilon = \frac{1}{e^2}\int \mathrm{d}^2y\ \left\{\frac{1}{2}F_{01}^2 + i\bar\lambda_-(\partial_0 + \partial_1)\lambda_- + \frac{1}{2}D^2\right\}. \tag{2.10}$$

Since Υ is a chiral fermion, we can also add an FI term of the form

$$S_{\mathrm{FI}} = \frac{t}{4}\int \mathrm{d}^2y\, d\theta^+\ \Upsilon|_{\bar\theta^+=0} + \mathrm{h.c.} \tag{2.11}$$

where $t = ir + \frac{\theta}{2\pi}$ is the complexified FI parameter.

2.1.2 Chiral Multiplets

An uncharged chiral superfield is one which satisfies $\bar{\mathrm{D}}_+\Phi^0 = 0$. Chiral superfields are therefore naturally expanded in the z coordinates $z^+ = y^+ - i\theta^+\bar\theta^+$, $z^- = y^-$, and θ^+. Bosonic chiral superfields contain the components fields,

$$\begin{aligned} \Phi^0 &= \phi(z) + \sqrt{2}\theta^+\psi_+(z) & (2.12)\\ &= \phi(y) + \sqrt{2}\theta^+\psi_+(y) - i\theta^+\bar\theta^+\partial_+\phi(y). \end{aligned}$$

The action for a chiral boson is

$$S_{\Phi^0} = -\frac{i}{2}\int \mathrm{d}^2y\,\mathrm{d}^2\theta\;\bar{\Phi}^0\partial_-\Phi^0. \tag{2.13}$$

With the definition $\Phi^0 = e^{-Q\Psi}\Phi$, we note that Φ satisfies the covariant chirality constraint $\bar{\mathcal{D}}_+\Phi = 0$ for a field with $U(1)$ charge Q. In components,

$$\Phi = \phi(y) + \sqrt{2}\theta^+\psi_+(y) - i\theta^+\bar{\theta}^+(D_0 + D_1)\phi(y), \tag{2.14}$$

where $D_\alpha = \partial_\alpha + iQA_\alpha$. The corresponding gauge invariant Lagrangian is given by,

$$\begin{aligned} S_\Phi &= -\frac{i}{2}\int \mathrm{d}^2y\,\mathrm{d}^2\theta\;\bar{\Phi}(\mathcal{D}_0 - \mathcal{D}_1)\Phi, \qquad (2.15)\\ &= \int \mathrm{d}^2y\Big\{-|D_\alpha\phi|^2 + i\bar{\psi}_+(D_0 - D_1)\psi_+ - iQ\sqrt{2}\bar{\phi}\lambda_-\psi_+ \\ &\quad +iQ\sqrt{2}\phi\bar{\psi}_+\bar{\lambda}_- + QD|\phi|^2\Big\}. \end{aligned}$$

2.1.3 Fermi Multiplets

In addition to bosonic chiral multiplets, there are also fermionic multiplets which, for uncharged fields, satisfy the condition

$$\bar{\mathrm{D}}_+\Gamma^0 = \sqrt{2}E^0 \tag{2.16}$$

where E^0 satisfies

$$\bar{\mathrm{D}}_+E^0 = 0.$$

A component expansion gives the terms

$$\Gamma^0 = \chi_- - \sqrt{2}\theta^+G - i\theta^+\bar{\theta}^+\partial_+\chi_- - \sqrt{2}\bar{\theta}^+E^0. \tag{2.17}$$

Note that fermi multiplets have negative chirality.

To satisfy the covariant chirality condition, we again define $\Gamma^0 = e^{-Q\Psi}\Gamma$, and $E^0 = e^{-Q\Psi}E$ so that

$$\bar{\mathcal{D}}_+\Gamma = \sqrt{2}E. \tag{2.18}$$

The choice of E plays an important role in our discussion for reasons that we will describe later. We follow [2] and assume that E is a holomorphic function of chiral superfields Φ_i. The action for Γ is given by

$$\begin{aligned} S_\Gamma &= -\frac{1}{2}\int \mathrm{d}^2y\,d^2\theta\;\bar{\Gamma}\Gamma \qquad (2.19)\\ &= \int \mathrm{d}^2y\,\left\{i\bar{\chi}_-(D_0 + D_1)\chi_- + |G|^2 - |E|^2 - \left(\bar{\chi}_-\frac{\partial E}{\partial\phi_i}\psi_{+i} + \bar{\psi}_{+i}\frac{\partial\bar{E}}{\partial\bar{\phi}_i}\chi_-\right)\right\}. \end{aligned}$$

A special case of (2.19) of particular importance to us; namely, where $E = \Sigma\mathcal{E}(\Phi_i)$ and Σ is an uncharged chiral superfield with component expansion

$$\Sigma = \sigma + \sqrt{2}\theta^+\bar{\lambda}_+ - i\theta^+\bar{\theta}^+\partial_+\sigma. \tag{2.20}$$

Then the action for Γ is given by

$$\begin{aligned} S_\Gamma &= \int \mathrm{d}^2y \Bigg\{ i\bar{\chi}_-(D_0 + D_1)\chi_- + |G|^2 - |\sigma\mathcal{E}|^2 \\ &\quad - \left(\sigma\bar{\chi}_- \frac{\partial\mathcal{E}}{\partial\phi_i}\psi_{+i} + \bar{\sigma}\bar{\psi}_{+i}\frac{\partial\bar{\mathcal{E}}}{\partial\bar{\phi}_i}\chi_-\right) - \left(\mathcal{E}\bar{\chi}_-\bar{\lambda}_+ + \bar{\mathcal{E}}\lambda_+\chi_-\right) \Bigg\}. \end{aligned} \tag{2.21}$$

2.1.4 $(0,2)$ Superpotentials

In general, we can also add superpotential terms. These terms depend on Fermi superfields, Γ_a, and holomorphic functions, J^a, of the chiral superfields

$$\begin{aligned} S_{\mathcal{W}} &= -\frac{1}{\sqrt{2}}\int \mathrm{d}^2y\, d\theta^+\ \Gamma_a J^a|_{\bar{\theta}^+=0} - \text{h.c.}, \\ &= -\int \mathrm{d}^2y\ \left\{ G_a J^a(\phi_i) + \chi_{-a}\psi_{+i}\frac{\partial J^a}{\partial\phi_i} \right\} - \text{h.c..} \end{aligned} \tag{2.22}$$

Since Γ_a is not an honest chiral superfield but satisfies (2.18), we need to impose the condition

$$E \cdot J = 0 \tag{2.23}$$

to ensure that the superpotential is chiral. Lastly, note that gauge invariance requires

$$Q_{\Gamma_a} = -Q_{J^a}.$$

2.2 $(2,2)$ Supersymmetry

A special class of $(0,2)$ theories have enhanced $(2,2)$ supersymmetry. To describe these theories, we enlarge our superspace by adding two fermionic coordinates, $(y^+, y^-, \theta^+, \bar{\theta}^+, \theta^-, \bar{\theta}^-)$, and we introduce additional supercovariant derivatives

$$D_- = \frac{\partial}{\partial\theta^-} - i\bar{\theta}^-\partial_-, \tag{2.24}$$

$$\bar{D}_- = -\frac{\partial}{\partial\bar{\theta}^-} + i\theta^-\partial_-. \tag{2.25}$$

We normalize integrals over all the fermionic coordinates of superspace with the convention that

$$\int d^4\theta\ \theta^+\bar{\theta}^+\theta^-\bar{\theta}^- = 1. \tag{2.26}$$

Unlike the $(0,2)$ case, there are two kinds of chiral multiplet. Conventional chiral multiplets, Φ, satisfy the conditions

$$\bar{\mathrm{D}}_+\Phi = \bar{\mathrm{D}}_-\Phi = 0, \tag{2.27}$$

while twisted chiral multiplets, Y, satisfy the conditions

$$\bar{\mathrm{D}}_+Y = \mathrm{D}_-Y = 0. \tag{2.28}$$

Both kinds of multiplet can be reduced to $(0,2)$ multiplets. An uncharged $(2,2)$ chiral multiplet gives a $(0,2)$ chiral and Fermi multiplet,

$$\Phi^{(0,2)} = \Phi|_{\theta^-=\bar{\theta}^-=0}, \qquad \Gamma^{(0,2)} = \frac{1}{\sqrt{2}}\mathrm{D}_-\Phi|_{\theta^-=\bar{\theta}^-=0}. \tag{2.29}$$

Similarly, a twisted chiral multiplet (which is always uncharged) also gives a chiral and Fermi multiplet,

$$Y^{(0,2)} = Y|_{\theta^-=\bar{\theta}^-=0}, \qquad F^{(0,2)} = -\frac{1}{\sqrt{2}}\bar{\mathrm{D}}_-Y|_{\theta^-=\bar{\theta}^-=0}. \tag{2.30}$$

There is also a $(2,2)$ vector superfield, V, whose field strength is a twisted chiral multiplet (often denoted Σ). On reduction to $(0,2)$, we obtain a chiral multiplet, $\Sigma^{(0,2)}$, and a vector multiplet, $V^{(0,2)}$, as follows:

$$\bar{\theta}^+\Sigma^{(0,2)} = -\frac{1}{\sqrt{2}}\mathrm{D}_-V|_{\theta^-=\bar{\theta}^-=0}, \qquad V^{(0,2)} - i\partial_-\Psi^{(0,2)} = -\bar{\mathrm{D}}_-\mathrm{D}_-V|_{\theta^-=\bar{\theta}^-=0}. \tag{2.31}$$

Lastly, we note that a $(2,2)$ chiral multiplet with $U(1)$ charge Q reduces to a charged $(0,2)$ chiral multiplet, $\Phi^{(0,2)}$, and a charged Fermi multiplet, $\Gamma^{(0,2)}$, with a particular non-vanishing E so that

$$\bar{\mathcal{D}}_+\Gamma^{(0,2)} = \sqrt{2}E$$

where E is given by [2]

$$E = \sqrt{2}Q\Sigma^{(0,2)}\Phi^{(0,2)}. \tag{2.32}$$

3 Duality in $(0,2)$ Models

3.1 Duality in Free $(0,2)$ Theories

The essential magic of mirror symmetry is that a priori distinct target spaces may lead to identical string spectra. A simple example of a mirror symmetry is T-duality, which identifies the spectrum of strings on tori of radii R and $1/R$. Since the world-sheet theory on a torus is exactly solvable, T-duality of tori, unlike general mirror symmetry, is easily derived directly in the world-sheet theory. In this section, we recall the standard prescription for deriving such dualities, following Roček-Verlinde (RV) [26]. We then apply this prescription to $(0,2)$ models; this will play an essential role in our dualization of $(0,2)$ GLSMs in the following sections. We begin by reviewing the dualization procedure for free $(0,2)$ theories before addressing the more interesting case of $(0,2)$ GLSMs.

3.1.1 T-duality as Abelian Duality

T-duality identifies the momentum (winding) modes on a circle of radius R with the winding (momentum) modes on a circle of radius $1/R$. As such, it may be implemented via a Legendre transformation in a way we now recall. The theory must admit a $U(1)$ isometry, and the simplest example is a free scalar on a circle of radius R with action

$$S = \frac{R^2}{4\pi}\int d^2y\ (\partial\phi)^2. \tag{3.1}$$

To dualize the shift symmetry of ϕ, we introduce a Lagrange multiplier 1-form, B, with modified action

$$S = \frac{1}{4\pi R^2}\int B\wedge *B - \frac{i}{2\pi}\int \phi\, dB. \tag{3.2}$$

Path-integrating out B in Euclidean space amounts to solving the B equation of motion giving

$$B = -iR^2 * d\phi.$$

When plugged into the action, we recover our original theory (3.1).

To obtain the dual description, we instead integrate out ϕ. This enforces the condition that B be closed,

$$dB = 0.$$

Locally, we can express B in the form

$$B = d\theta$$

where θ is not necessarily single-valued. The only caveat to this argument is that ϕ is periodic so for the action (3.2) to be well-defined, we require that B be an integral class.

The dual action is therefore

$$S = \frac{1}{4\pi R^2} \int d^2y \, (\partial\theta)^2. \tag{3.3}$$

We note that θ must be periodic with radius $1/R$. This is most easily seen by comparing the spectra of the original and dual descriptions. A momentum mode for ϕ can only correspond to a solitonic excitation for θ implying that θ is compact.

3.1.2 Dualization in $(0, 2)$ Superspace

The above reasoning can be extended to $(0, 2)$ superspace. To dualize a $(0, 2)$ chiral multiplet, Y, we consider the action

$$\mathcal{S}_{ch} = -\frac{1}{4} \int d^2y \, d^2\theta \, (R^2 AB + A(Y + \bar{Y}) - iB\partial_-(Y - \bar{Y})), \tag{3.4}$$

where A and B are unconstrained real superfields without kinetic terms. Integrating out these non-dynamical real superfields gives the relations

$$A = \frac{i}{R^2}\partial_-(Y - \bar{Y}), \tag{3.5}$$

$$B = -\frac{1}{R^2}(Y + \bar{Y}). \tag{3.6}$$

Inserted back into the action, these relations give, up to total derivatives, the standard action for Y

$$\mathcal{S}_{ch} = -\frac{i}{2R^2} \int d^2y \, d^2\theta \, \bar{Y}\partial_- Y. \tag{3.7}$$

To obtain the dual description, we instead integrate out the chiral superfield, Y, which gives the relation,

$$\bar{\mathrm{D}}_+(A + i\partial_- B) = 0, \tag{3.8}$$

allowing us to write $A = i\partial_-(\Phi - \bar{\Phi})$ and $B = (\Phi + \bar{\Phi})$, where Φ is a bosonic chiral superfield.[1] The resulting dual action is

$$\mathcal{S}_{ch} = -\frac{iR^2}{2}\int d^2y\ d^2\theta\ \bar{\Phi}\partial_-\Phi. \tag{3.9}$$

The duality map is therefore

$$(Y + \bar{Y}) = -R^2(\Phi + \bar{\Phi}) \qquad \partial_-(Y - \bar{Y}) = R^2\partial_-(\Phi - \bar{\Phi}). \tag{3.10}$$

We can also dualize a chiral Fermi multiplet in a similar way. Let F be a chiral Fermi multiplet satisfying $\bar{\mathrm{D}}_+F = 0$, and let $\mathcal{N}$ be an unconstrained Fermi superfield. To induce dual descriptions, we consider the following first-order action

$$\mathcal{S}_f = \int d^2y\ d^2\theta \left\{ -\frac{R^2}{2}\bar{\mathcal{N}}\mathcal{N} \ - \frac{1}{2}(F\bar{\mathcal{N}} + \mathcal{N}\bar{F}) \right\}. \tag{3.11}$$

Integrating out $\bar{\mathcal{N}}$ gives the relation

$$\mathcal{N} = \frac{1}{R^2}F, \tag{3.12}$$

which when substituted into the action gives

$$\mathcal{S}_f = \frac{1}{2R^2}\int d^2y\ d^2\theta\ \bar{F}F. \tag{3.13}$$

Integrating out F instead gives the relation

$$\bar{\mathrm{D}}_+\bar{\mathcal{N}} = 0 \tag{3.14}$$

which gives a dual action

$$\mathcal{S}_f = \frac{R^2}{2}\int d^2y\ d^2\theta\ \bar{\Gamma}\Gamma \tag{3.15}$$

where the chiral superfield $\Gamma = \bar{\mathcal{N}}$ and $\bar{\mathcal{N}}$ satisfies the chirality constraint (3.14). The corresponding duality map is

$$\bar{\Gamma} = \frac{1}{R^2}F. \tag{3.16}$$

[1] The general solution for B includes an arbitrary superfield anihilated by ∂_-, i.e., $B = \Phi + \bar{\Phi} + \mathcal{S}$, where $\partial_-\mathcal{S} = 0$. Plugging this solution into the action and integrating by parts reveals that all terms involving $\mathcal{S}$ vanish. We can therefore safely neglect any such $\mathcal{S}$.

From this map, we see that the action on fermions is no more than a rescaling at the level of free-fields. We should point out that this map becomes more complicated when we include interactions with chiral bosons (via the $\bar{\mathrm{D}}_+\Gamma = \sqrt{2}E(\Phi)$ coupling), as we shall see in detail in section 3.2.3.

The Duality Map in Components

It is useful to restate these dualities in component form. Start with a supersymmetric sigma model on a cylinder of radius R with action,

$$S \ = R^2 \int d^2y \ \left(-\partial_\alpha\bar{\phi}\ \partial^\alpha\phi + i\ \bar{\psi}_+\partial_-\psi_+\right), \tag{3.17}$$

where

$$\phi = \rho + i\varphi \tag{3.18}$$

is the lowest component of a $(0,2)$ bosonic chiral multiplet, $\Phi = \phi(z) + \sqrt{2}\theta^+\psi_+$, whose imaginary part is periodic

$$\varphi \sim \varphi + 2\pi.$$

Dualizing this isometry amounts to dualizing φ and ψ_+. Starting with the bosonic fields, we see that the resulting dual metric is

$$ds^2 = R^2 d\rho^2 + \frac{1}{R^2}d\vartheta^2 = \frac{1}{R^2}(R^4 d\rho^2 + d\vartheta^2), \tag{3.19}$$

where

$$\vartheta \sim \vartheta + 2\pi.$$

This suggests that the natural dual coordinate is $\eta = R^2\rho - i\vartheta$. Written in terms of η, the full dual lagrangian takes the simple form

$$S = \frac{1}{R^2}\int d^2y\ \left(-\partial_\alpha\bar{\eta}\partial^\alpha\eta\ + i\bar{\widetilde{\psi}}_+\partial_-\widetilde{\psi}_+\right). \tag{3.20}$$

If we have correctly identified our dual fields, their supervariations should again close with the correct normalizations. By dualizing the original susy variations, we find

$$\delta\psi_+ = \sqrt{2}i\partial_+\phi\bar{\epsilon}_- \quad \Rightarrow \quad \delta\widetilde{\psi}_+ = \sqrt{2}i\partial_+\eta\bar{\epsilon}_-, \tag{3.21}$$

where $\widetilde{\psi}_+ = R^2\psi_+$, so our dualization is consistent with supersymmetry and the dual fields also fill out a $(0,2)$ chiral multiplet

$$Y = \eta(z) + \sqrt{2}\theta^+\widetilde{\psi}_+.$$

The case of a free fermionic $(0,2)$ chiral supermultiplet, $\Gamma = \chi_-(z) + \sqrt{2}\theta^+ g$, can also be expressed in components. The initial action is

$$S = iR^2 \int d^2y \; \left(\bar{\chi}_- \partial_+ \chi_- \; + \; |g|^2\right) \tag{3.22}$$

and the dual action is simply

$$S = \frac{i}{R^2} \int d^2y \; \left(\bar{\psi}_- \partial_+ \psi_- \; + \; |f|^2\right), \tag{3.23}$$

where $\psi_- = R^2 \bar{\chi}_-$, and we can assemble ψ_- and f into a chiral Fermi superfield $F = \psi_-(z) + \sqrt{2}\theta^+ f$.

3.2 Duality in $(0,2)$ Gauge Theories

We next consider the dualization of $(0,2)$ gauged linear sigma models. The dualization for $(2,2)$ gauged linear sigma models has been carried out in [15]. Since a $(2,2)$ model is a special case of a $(0,2)$ model, we can reduce the duality map of [15] to a map on $(0,2)$ fields. This gives us a particular case of a $(0,2)$ duality. Next we generalize this duality to arbitrary $(0,2)$ theories.

It is important to keep in mind that the $U(1)$ action we wish to dualize is no longer free. This is an issue that we will ignore for the moment. The way this issue emerges in the dual description is via the generation of a non-perturbative superpotential to which we turn in section 4. This is, perhaps, the most critical aspect of the dualization procedure.

Warm-up: $(2,2)$ Duality in $(0,2)$ Superspace

We begin by expressing the results of Hori and Vafa [15] in $(0,2)$ language. This gives a special case of a more general $(0,2)$ duality map. The simplest such theory is that of a single chiral $(2,2)$ multiplet with charge Q coupled to a $(2,2)$ vector multiplet. The slightly involved rewriting of the original $(2,2)$ theory in $(0,2)$ language is performed in Appendix A.

When reduced to $(0,2)$ language, the end result is a $(0,2)$ gauge theory with a chiral multiplet, Φ, a chiral Fermi multiplet, Γ, both with $U(1)$ charge Q. In addition, the $(2,2)$ vector multiplet reduces to a $(0,2)$ vector multiplet with field strength Υ, and an uncharged chiral multiplet, Σ. The Lagrangian for these fields is given by

$$
\begin{aligned}
L &= -\frac{i}{2}\int d^2\theta\ \bar{\Phi}(\mathcal{D}_0-\mathcal{D}_1)\Phi - \frac{1}{2}\int d^2\theta\ \bar{\Gamma}\Gamma + \frac{i}{2e^2}\int d^2\theta\ \bar{\Sigma}\partial_-\Sigma \\
&\quad +\frac{1}{8e^2}\int d^2\theta\ \bar{\Upsilon}\Upsilon + \left\{\frac{t}{4}\int d\theta^+\ \Upsilon|_{\bar{\theta}^+=0} + \text{h.c.}\right\}
\end{aligned} \tag{3.24}
$$

where e is the gauge coupling, and $t = ir + \frac{\theta}{2\pi}$ is the complexified FI parameter. The Fermi superfield satisfies $\bar{\mathcal{D}}_+\Gamma = \sqrt{2}E$ with E given by [2]

$$
E = \sqrt{2}Q\Sigma\Phi. \tag{3.25}
$$

Dualizing an isometry means exchanging the roles of the generator of the isometry and its canonical conjugate. This means that under this generalized world-sheet T-duality, a charged field maps to an uncharged field. The dual variables, a chiral superfield Y and chiral Fermi superfield F, are therefore neutral.

The dual action is again obtained by reducing the $(2,2)$ result in Appendix A. The result is,

$$
\begin{aligned}
\widetilde{L} &= \frac{i}{8}\int d^2\theta\ \left[\frac{Y-\bar{Y}}{Y+\bar{Y}}\partial_-(Y+\bar{Y}) + 2i\frac{\bar{F}F}{Y+\bar{Y}}\right] \\
&\quad -\left(\frac{Q}{2}\int d\theta^+[\Sigma F + \frac{i}{2}Y\Upsilon] + \text{h.c.}\right) + \frac{i}{2e^2}\int d^2\theta\ \bar{\Sigma}\partial_-\Sigma \\
&\quad +\frac{1}{8e^2}\int d^2\theta\ \bar{\Upsilon}\Upsilon + \left\{\frac{t}{4}\int d\theta^+\ \Upsilon|_{\bar{\theta}^+=0} + \text{h.c.}\right\},
\end{aligned}
$$

where $\bar{\mathrm{D}}_+Y = \bar{\mathrm{D}}_+F = 0$. The $(2,2)$ duality map can also be expressed in $(0,2)$ language as described in Appendix A. The map becomes,

$$
\bar{\Phi}\Phi = \frac{1}{2}(Y+\bar{Y}), \tag{3.26}
$$

$$
-i\bar{\Phi}(\overleftrightarrow{\partial}_- + iQV)\Phi + \bar{\Gamma}\Gamma = \frac{i}{4}\partial_-(Y-\bar{Y}), \tag{3.27}
$$

$$
\frac{1}{2}\bar{F} = \bar{\Phi}\Gamma, \tag{3.28}
$$

where

$$
\bar{\Phi}\overleftrightarrow{\partial}_-\Phi = \frac{1}{2}\left(\bar{\Phi}\partial_-\Phi - \Phi\partial_-\bar{\Phi}\right).
$$

Since it is rather important, we must emphasize that Y_i is not a conventional $\mathbb{C}$-valued field. Rather,

$$
\mathrm{Im}(Y_i) \sim \mathrm{Im}(Y_i) + 2\pi, \qquad \mathrm{Re}(Y_i) \geq 0. \tag{3.29}
$$

One must interpret this duality map (and the $(2,2)$ map) with great care. As an equivalence between superfields, the map does not make sense. The component expansions on both sides of the equivalence do not agree. However, we will only use the relations between the lowest components when we need explicit relations. Those relations and the dualization procedure itself (as an equivalence between theories) do make sense.

3.2.1 Dualizing $(0,2)$ Chiral Multiplets

In string compactifications, chiral multiplets describe the geometry of our target space, while chiral Fermi multiplets define a vector bundle over this space. Our current task is to dualize charged chiral and Fermi multiplets. We begin by considering just chiral multiplets with no coupled Fermi multiplets.

We need a starting action along the lines described earlier: let us start with the candidate action

$$\mathcal{S}_{ch} = \int d^2y\, d^2\theta\, \left\{ -\frac{i}{2} e^{2(\Psi+B)}(iV + iA) \; - i\mathcal{F}\, \bar{\mathrm{D}}_+(\partial_- B + iA) + \text{h.c.} \right\} \tag{3.30}$$

where $\mathcal{F}$ is a neutral unconstrained fermionic superfield, while A and B are unconstrained real superfields.

Integrating out the unconstrained Lagrange multiplier field $\mathcal{F}$ yields the constraint

$$\bar{\mathrm{D}}_+(\partial_- B + iA) = 0, \tag{3.31}$$

the general solution of which is[2]

$$2B = \Pi + \bar{\Pi} \qquad 2iA = \partial_-(\Pi - \bar{\Pi}) \tag{3.32}$$

where Π is a chiral superfield. Plugging this back into the action gives, after some reordering,

$$\mathcal{S}_{ch} = -\frac{i}{2} \int d^2y\, d^2\theta\, e^{\Psi+\bar{\Pi}}\, (\partial_- + iV)\, e^{\Psi+\Pi}. \tag{3.33}$$

We can make the kinetic term canonical by changing variables to the covariantly chiral field $\Phi = e^{\Psi+\Pi}$, in terms of which the action reads

$$\mathcal{S}_{ch} = -\frac{i}{2} \int d^2y\, d^2\theta\, \bar{\Phi}\, (\mathcal{D}_0 - \mathcal{D}_1)\, \Phi. \tag{3.34}$$

[2] A note of caution is in order. In general we should write $2B = \Pi + \bar{\Pi} + 2\mathcal{S}_R$, where $\mathcal{S}_R$ is a real bosonic superfield annihilated by ∂_-. However, a real bosonic $(0,2)$ superfield can always be written as the real part of a complex chiral superfield, $2\mathcal{S}_R = (\mathcal{S} + \bar{\mathcal{S}})$; both have four independent real components. Absorbing $\mathcal{S}$ into Π gives the Lagrangian written above up to a shift $V \to V + c$, where c is a constant c-number; since this may be absorbed by a gauge transformation, (3.32) is indeed the most general solution of the constraint.

Integrating out instead the auxiliary gauge fields A and B requires first integrating the constraint terms by parts. Defining $\frac{1}{4}Y = \bar{\mathrm{D}}_+\mathcal{F}$, the auxiliary field variations give

$$\delta_A \quad \Rightarrow \quad \frac{1}{2}e^{2(\Psi+B)} - \frac{1}{4}(Y+\bar{Y}) = 0, \tag{3.35}$$

$$\delta_B \quad \Rightarrow \quad -ie^{2(\Psi+B)}(iV+iA) - \frac{i}{4}\partial_-(Y-\bar{Y}) = 0. \tag{3.36}$$

Solving these gives

$$2B = -2\Psi + \ln(\frac{Y+\bar{Y}}{2}) \qquad iA = -iV - \frac{\partial_-(Y-\bar{Y})}{2(Y+\bar{Y})}. \tag{3.37}$$

Plugging back into the action and simplifying gives

$$\mathcal{S}_{ch} = \frac{i}{8}\int d^2y\; d^2\theta\; \frac{(Y-\bar{Y})\,\partial_-(Y+\bar{Y})}{(Y+\bar{Y})} - \frac{i}{4}\int d^2y\; d\theta^+ Y\Upsilon + \text{h.c.} \tag{3.38}$$

Comparing (3.32) and (3.37), we see that the duality map is

$$\bar{\Phi}\Phi \;=\; \frac{1}{2}(Y+\bar{Y}), \qquad \bar{\Phi}(\overleftrightarrow{\partial}_- + iV)\Phi = -\frac{1}{4}\partial_-(Y-\bar{Y}). \tag{3.39}$$

On comparing with (3.27), we see that the fermion bilinear has dropped out as we intuitively expect for this special case with no coupling to the left-moving fermions.

3.2.2 Dualizing $(0,2)$ Fermi Multiplets

We can similarly dualize Fermi supermultiplets. The first order Lagrangian is

$$\mathcal{S}_f = \int d^2y\; d^2\theta\; \left\{-\frac{1}{2}\,\bar{\mathcal{N}}\mathcal{N} + \mathcal{S}\left(\bar{\mathcal{D}}_+\mathcal{N} - \sqrt{2}E\right) \;-\bar{\mathcal{S}}\left(\mathcal{D}_+\bar{\mathcal{N}} + \sqrt{2}\bar{E}\right)\right\}, \tag{3.40}$$

where $\mathcal{N}$ is an unconstrained Fermi superfield, $\mathcal{S}$ is an unconstrained bosonic superfield, and E is a bosonic (covariantly) chiral multiplet. Both $\mathcal{N}$ and E have charge Q while S has charge $-Q$. Integrating out $\mathcal{S}$ gives the equation of motion

$$\bar{\mathcal{D}}_+\mathcal{N} = \sqrt{2}E,$$

which is solved by $\mathcal{N} = \Gamma$, where Γ is a chiral Fermi superfield in the general sense of (2.18). The corresponding Lagrangian is just

$$\mathcal{S}_f = \;-\frac{1}{2}\int d^2y\; d^2\theta\; \bar{\Gamma}\Gamma. \tag{3.41}$$

Solving the $\mathcal{N}$ equation of motion instead gives the relation

$$\bar{\mathcal{D}}_+\mathcal{S} = -\frac{1}{2}\bar{\mathcal{N}} \tag{3.42}$$

Let us set $\bar{\mathcal{N}} = \mathcal{G}$ so (3.42) implies that

$$\bar{\mathcal{D}}_+\mathcal{G} = 0.$$

Substituting gives the action,

$$\mathcal{S}_f = \int d^2y \; d^2\theta \; \left\{ -\frac{1}{2}\bar{\mathcal{G}}\mathcal{G} - \sqrt{2}\mathcal{S}E - \sqrt{2}\bar{\mathcal{S}}\bar{E} \right\}. \tag{3.43}$$

We now write,

$$\int d^2y \; d^2\theta \; \sqrt{2}\mathcal{S}E = -\int d^2y \; d\theta^+ \sqrt{2}\left(\bar{\mathcal{D}}_+\mathcal{S}\right) E = \int d^2y \; d\theta^+ \frac{1}{\sqrt{2}}\mathcal{G}E$$

since $\bar{\mathcal{D}}_+ E = 0$. Note that $\mathcal{G}$ has charge $-Q$ while E has charge Q. Let us define a neutral superfield $F = \mathcal{G}E$. The reason to do this is so that (in nice cases) we can express the action in terms of the dual chiral fields, Y. In terms of F, the action takes the form

$$\mathcal{S}_f = -\frac{1}{2}\int d^2y \; d^2\theta \; \frac{\bar{F}F}{\bar{E}E} - \left\{ \int d^2y \; d\theta^+ \frac{1}{\sqrt{2}}F + \text{h.c.} \right\}. \tag{3.44}$$

Let us define

$$|E|^2 = \frac{Y_E + \bar{Y}_E}{2} \tag{3.45}$$

so

$$\mathcal{S}_f = -\int d^2y \; d^2\theta \; \frac{\bar{F}F}{Y_E + \bar{Y}_E} - \left\{ \int d^2y \; d\theta^+ \frac{1}{\sqrt{2}}F + \text{h.c.} \right\}. \tag{3.46}$$

The duality map for Fermi superfields is then given by

$$F = E\,\bar{\Gamma}. \tag{3.47}$$

In nice cases, we can find explicit expressions for $|E|^2$ using the duality map (3.39); for example, if E is a monomial.

An important special case, related to the discussion around eq. (2.20), is when $E = \Sigma\mathcal{E}$, where Σ is an uncharged chiral boson. In this case we can rescale $\mathcal{G}$ by $\mathcal{E}$ rather than E to get

$$\mathcal{S}_f = -\int d^2y \; d^2\theta \; \frac{\bar{F}F}{\bar{\mathcal{E}}\mathcal{E}} - \left\{ \int d^2y \; d\theta^+ \frac{1}{\sqrt{2}}\Sigma F + \text{h.c.} \right\}. \tag{3.48}$$

3.2.3 Dualizing General $(0, 2)$ Models

Things get more interesting when we dualize chiral multiplets, Φ_i, coupled to Fermi multiplets, Γ_a, via constraints of the form

$$\bar{\mathcal{D}}_+\Gamma_a = \sqrt{2}\Sigma\mathcal{E}_a(\Phi_i).$$

In a situation like this, we can perform our previous dualization procedure but we can only explicitly solve for the dual action when $\mathcal{E}$ is a monomial.

Start with the sum of first order actions

$$S = S_{ch} + S_f$$

where S_{ch} is given in (3.30) and S_f is given in (3.40). We permit $\mathcal{E}$ to be an arbitrary (generally non-local) function of A, B and Ψ. As before, integrating out $\mathcal{S}$ and $\mathcal{F}$ gives an action,

$$S = -\frac{i}{2}\int d^2y\; d^2\theta\; \bar{\Phi}\; (\mathcal{D}_0 - \mathcal{D}_1)\, \Phi\; - \frac{1}{2}\int d^2y\; d^2\theta\; \bar{\Gamma}\Gamma,$$

where $\bar{\mathcal{D}}_+\Gamma = \sqrt{2}\Sigma\mathcal{E}(\Phi)$.

To get the dual description, we integrate out A, B and $\mathcal{N}$. From integrating out $\mathcal{N}$, we get $|\mathcal{E}|^2$ in the kinetic term for the fermions as in (3.48). In general, the A and B equations of motion are complicated (non-local) functions of A and B. For the particular case,

$$|\mathcal{E}|^2 = e^{-2N(\Psi+B)}, \tag{3.49}$$

the A equation of motion is unchanged from (3.35), but the B equation of motion gives

$$A = -V + \frac{i}{2}\frac{\partial_-(Y-\bar{Y})}{Y+\bar{Y}} - N\left(\frac{2}{Y+\bar{Y}}\right)^{N+1}\frac{\bar{F}F}{|\Sigma|^2}. \tag{3.50}$$

In the original theory, this corresponds to the case $\mathcal{E} = \Phi^N$.

The corresponding dual action is given by,

$$\begin{aligned} S \;\; = \;\; & \int d^2y\; d^2\theta\; \left[\frac{i}{8}\frac{(Y-\bar{Y})\,\partial_-(Y+\bar{Y})}{(Y+\bar{Y})} - \frac{2^{N-1}\bar{F}F}{(Y+\bar{Y})^N}\right] \\ & - \int d^2y\; d\theta^+\left[\frac{1}{\sqrt{2}}\Sigma F - \frac{i}{4}Y\Upsilon\right] + \text{h.c.}, \end{aligned} \tag{3.51}$$

so the action takes the same form we found before. What has changed is the duality map, which now reads

$$\bar{\Phi}\Phi \;=\; \frac{1}{2}(Y+\bar{Y}), \qquad \bar{\Phi}(\overleftrightarrow{\partial}_- + iV)\Phi - iN\bar{\Gamma}\Gamma = -\frac{1}{4}\partial_-(Y-\bar{Y}). \tag{3.52}$$

On comparing with (3.39), we note the appearance of a fermion bilinear; for the special case $N = 1$, this reproduces the $(2, 2)$ result (3.27), as expected.

Unfortunately, things rapidly become difficult once we consider general functions $\mathcal{E}(\Phi)$, because the A, B equations of motion involve complicated functions of A and B. So the action (and duality map) cannot, in general, be written in closed form. There are really two issues: the first is that we cannot express $|\mathcal{E}|^2$ in terms of Y and $\bar{Y}$. However, this only affects the kinetic terms for the dual Fermi multiplets, but not any holomorphic quantities. The second issue is finding the exact duality map. Fortunately, the correction to the naive duality map always involves terms with two or more fermions. This kind of correction will play no role in our subsequent computations, so we can safely ignore it.

4 The Exact Dual Superpotential

4.1 Lagrangians and Conventions

We have derived the perturbative superpotential of the dual theory. It is easy to extend the analysis of the previous sections to theories with several superfields carrying arbitrary charges. Let us consider a theory containing chiral superfields, Φ_i, with charges Q_i and Fermi superfields, Γ_a, with charges Q_a. We shall always assume that the charges satisfy the gauge anomaly cancellation condition required for a consistent quantum field theory

$$\sum_i Q_i^2 = \sum_a Q_a^2. \tag{4.1}$$

This condition is equivalent in the infra-red to the geometric constraint given in (1.2).

When dualizing $(0, 2)$ models, we are faced with the natural question: which fields should we dualize? To answer this question, we need to consider different choices for E. The first choice we might consider is $E = 0$, but this is problematic because (in general) there is no natural way to construct a neutral dual Fermi superfield. In section 6.1.4, we will describe a particular model in which there is a natural choice.

One possible way to proceed for $E = 0$ is to dualize the chiral superfields leaving the Fermi fields untouched. This seems reasonable because chiral and the Fermi superfields interact only indirectly via their coupling to gauge-fields. In this situation, the fields map as follows from the original to the dual description

$$(\Phi_i, \Gamma_a) \to (Y_i, \Gamma_a).$$

The chiral superfields Y_i are uncharged, while the Fermi superfields Γ_a are charged. The difficulty we seem to encounter is with the superpotential. Under a partial dualization where the theory is described in terms of (Y_i, Γ_a), it is hard to even define what is meant by a superpotential. There is clearly no perturbative superpotential of the form appearing in (3.46) because of gauge invariance. It is also unclear how to take into account instanton effects in the original theory; it seems likely that these non-perturbative effects result in a non-local dual theory. For these reasons, for the most part we restrict to $E \neq 0$.

When the charged chiral and Fermi superfields interact with each other via $E_a \neq 0$, we must dualize both the chiral and Fermi superfields

$$(\Phi_i, \Gamma_a) \to (Y_i, F_a)$$

where both Y_i and F_a are neutral.

We give the Lagrangians for the dual theory for two classes of E_a. Omitted is the kinetic term for the vector multiplet with field strength Υ given, for example, in (3.24). For $E_a = f_a(\Phi_i)$,

$$\widetilde{L} = \frac{i}{8} \sum_i \int d^2\theta \, \frac{Y_i - \bar{Y}_i}{Y_i + \bar{Y}_i} \partial_-(Y_i + \bar{Y}_i) - \sum_a \int d^2\theta \, \frac{\bar{F}_a F_a}{Y_{f_a} + \bar{Y}_{f_a}} + (\int d\theta^+ \, \widetilde{W} + \text{h.c.}), \tag{4.2}$$

where,

$$\widetilde{W} = -\frac{i\Upsilon}{4}(\sum_i Q_i Y_i + it) - \frac{1}{\sqrt{2}} \sum_a F_a \tag{4.3}$$

and

$$\frac{1}{2}(Y_{f_a} + \bar{Y}_{f_a}) = |f_a(\phi_i)|^2. \tag{4.4}$$

For the second case, $E_a = \Sigma g_a(\Phi_i)$, and Σ is a neutral chiral superfield with canonical kinetic terms. Rescaling as in (3.48) gives the Lagrangian

$$\widetilde{L} = \frac{i}{8} \sum_i \int d^2\theta \, \frac{Y_i - \bar{Y}_i}{Y_i + \bar{Y}_i} \partial_-(Y_i + \bar{Y}_i) - \sum_a \int d^2\theta \, \frac{\bar{F}_a F_a}{Y_{g_a} + \bar{Y}_{g_a}} + (\int d\theta^+ \, \widetilde{W} + \text{h.c.}), \tag{4.5}$$

where,

$$\widetilde{W} = -\frac{i\Upsilon}{4}(\sum_i Q_i Y_i + it) - \frac{\Sigma}{\sqrt{2}} \sum_a F_a, \tag{4.6}$$

and,

$$\frac{1}{2}(Y_{g_a} + \bar{Y}_{g_a}) = |g_a(\phi_i)|^2. \tag{4.7}$$

The dual superpotential is exact in perturbation theory because of perturbative non-renormalization theorems [27, 28]. However, there can be non-perturbative corrections. Our aim is to determine the exact form of the dual superpotential taking into account the non-perturbative effects generated by vortex instantons in the original theory [29, 30]. We should note, however, that the superpotential of the original theory does not receive non-perturbative corrections as recently shown in [6].

Before proceeding further, we state our field expansion conventions and some relevant formulae that we need both here and in later discussion. In the original theory, the charged chiral superfields, Φ_i, satisfy $\bar{\mathcal{D}}_+\Phi_i = 0$, and have the component field expansion

$$\Phi_i = \phi_i + \sqrt{2}\theta^+\psi_{+i} - i\theta^+\bar{\theta}^+ D_+\phi_i. \tag{4.8}$$

The charged Fermi superfields, Γ_a, satisfy $\bar{\mathcal{D}}_+\Gamma_a = \sqrt{2}E_a$, and have the component field expansion

$$\Gamma_a = \chi_{-a} - \sqrt{2}\theta^+ G_a - i\theta^+\bar{\theta}^+ D_+\chi_{-a} - \sqrt{2}\bar{\theta}^+ E_a. \tag{4.9}$$

In the dual theory, the neutral chiral superfields, Y_i, satisfy $\bar{D}_+ Y_i = 0$, and have the component field expansion

$$Y_i = y_i + \sqrt{2}\theta^+\bar{\xi}_{+i} - i\theta^+\bar{\theta}^+\partial_+ y_i, \tag{4.10}$$

while the neutral Fermi superfields, F_a, satisfy $\bar{D}_+ F_a = 0$, and have the component field expansion

$$F_a = \eta_{-a} - \sqrt{2}\theta^+ H_a - i\theta^+\bar{\theta}^+\partial_+\eta_{-a}. \tag{4.11}$$

Finally let us state some general results obtained from the duality maps that we derived in section 3. We will need these formulae for studying non-perturbative corrections to the dual superpotential, and later for verifying various dual descriptions. We define

$$\begin{aligned} \phi_i &= \rho_i e^{i\varphi_i}, \\ y_i &= \varrho_i - i\vartheta_i. \end{aligned} \tag{4.12}$$

From (3.39), we find from the first relation that

$$\begin{aligned} \varrho_i &= \rho_i^2, \\ \bar{\xi}_{+i} &= 2\bar{\phi}_i\psi_{+i}, \\ \xi_{+i} &= 2\phi_i\bar{\psi}_{+i}, \\ \partial_+\vartheta_i &= 2\left[-\rho_i^2(\partial_+\varphi_i + Q_i A_+) + \bar{\psi}_{+i}\psi_{+i}\right]. \end{aligned} \tag{4.13}$$

From the second relation, we see that

$$\partial_-\vartheta_i = 2\rho_i^2(\partial_-\varphi_i + Q_i A_-). \tag{4.14}$$

Note the difference in the expressions for $\partial_+\vartheta_i$ and $\partial_-\vartheta_i$. Since vortices play a crucial role in the construction of the superpotential, we begin by briefly reviewing vortex instantons.

4.2 A Review of Vortex Instantons

We briefly review the vortex instanton solution of the two dimensional Abelian Higgs model. In order to construct the one instanton solution, we wick rotate to Euclidean space sending

$$y^0 \to -iy^2, \quad F_{01} \to -iF_{12}.$$

The Euclideanized action for the Abelian Higgs model is

$$S = \int \mathrm{d}^2 y \left[\sum_i |\mathrm{D}_i\phi|^2 + \frac{1}{2e^2}F_{12}^2 + \frac{i\theta}{2\pi}F_{12} + \frac{D^2}{2e^2} \right], \tag{4.15}$$

where $i = 1, 2$ and D is given by

$$D = -e^2(Q|\phi|^2 - r). \tag{4.16}$$

In polar coordinates (ρ, θ), the one-instanton configuration is given by

$$A_\rho = 0, \quad A_\theta = A(\rho), \quad \phi = f(\rho)e^{i\theta} \tag{4.17}$$

where for large ρ,

$$A(\rho) \sim \frac{1}{\rho} + \text{constant} \times \frac{e^{-\sqrt{r}\rho}}{\sqrt{\rho}}, \tag{4.18}$$

$$f(\rho) \sim \sqrt{r} + \text{constant} \times e^{-\sqrt{2r}\rho}, \tag{4.19}$$

and $A(0) = f(0) = 0$. In writing the expression for $A(\rho)$ and $f(\rho)$, we have set $Q = e = 1$. The fields go to zero at the location of the instanton and also fall off exponentially at spatial infinity. The Bogomol'nyi equations which determine BPS instanton configurations are

$$(D_1 + iD_2)\ \phi = 0 \tag{4.20}$$

and

$$D + F_{12} = 0. \tag{4.21}$$

On evaluating the instanton action (4.15) in this background, we obtain $S = -2\pi it$, where $t = ir + \frac{\theta}{2\pi}$. In the supersymmetric theories that we consider, there are fermion zero modes in the instanton background which are crucial in our analysis of non-perturbative corrections to the dual superpotential. We now turn to the construction of the dual superpotential.

4.3 R-charge Assignments

We will restrict to the case where both chiral and Fermi superfields are dualized, and where $E \neq 0$. So we proceed by constructing the dual theory in terms of the neutral chiral superfields, Y_i, and the neutral Fermi superfields, F_a. We recall from our previous analysis that the relation between the original and dual Fermi superfields is a local one where

$$F_a = \bar{\Gamma}_a \mathcal{E}(\Phi_i). \tag{4.22}$$

Clearly, this definition is not unique and can be subject to field redefinitions by gauge-invariant combinations of the original superfields. This possibility will play a role when we construct explicit examples. That the relation between the original and dual Fermi superfields is a local one will make our life easier in determining instanton corrections.

Recall that the component expansion for Σ takes the form

$$\Sigma = \sigma + \sqrt{2}\theta^+ \bar{\lambda}_+ - i\theta^+ \bar{\theta}^+ \partial_+ \sigma. \tag{4.23}$$

We need only consider the case of $E_a = \Sigma g_a(\Phi_i)$ since the case $E_a = f_a(\Phi_i)$ follows by giving Σ an expectation value,

$$< \Sigma > \neq 0.$$

The Lagrangian of the original $(0,2)$ theory given in (3.24) admits a classical $U(1)_R$ symmetry under which

$$\begin{aligned} \theta^+ \rightarrow &\quad e^{-i\alpha}\theta^+, \\ \Upsilon \rightarrow &\quad e^{-i\alpha}\Upsilon, \end{aligned}$$

while Φ_i and Γ_a are left invariant. In terms of component fields, the non-trivial transformations are given by

$$\psi_{+i} \rightarrow e^{i\alpha}\psi_{+i}, \qquad \lambda_- \rightarrow e^{-i\alpha}\lambda_-, \qquad E_a \rightarrow e^{-i\alpha}E_a, \tag{4.24}$$

which means that $\sigma \rightarrow e^{-i\alpha}\sigma$. To avoid confusion, we should note that E_a has mass dimension 1. The dimensionful parameter in E_a can either be absorbed in the definition of Σ, or inserted by hand. Either way, we call this mass parameter σ, and it carries all the R-charge of E_a. This classical R-symmetry is generally anomalous, and leads to a shift of the theta angle given by

$$\theta \rightarrow \theta - \sum_i Q_i \, \alpha. \tag{4.25}$$

How do the dual superfields transform under $U(1)_R$? In cases where $\mathcal{E}_a$ is not zero, we see from (4.22) that the corresponding F_a is uncharged since the mass parameter σ does not appear in the relation. When $E_a = 0$, the relation is even simpler

$$\mathcal{F}_a = \bar{\Gamma}_a$$

and again the dual Fermi superfield is uncharged. In this case, however, the dual Fermi field is charged under the gauge symmetry.

We also require the transformation properties of Y_i under the classical R-symmetry. In order to find the transformation properties of Y_i, we follow the procedure in [15]. The classical $U(1)_R$ symmetry has a conserved current given by

$$J_+^R = \sum_i \bar{\psi}_{i+}\psi_{i+} + \frac{i}{e^2}\sigma\partial_+\bar{\sigma} \tag{4.26}$$

and

$$J_-^R = -\frac{1}{e^2}\bar{\lambda}_-\lambda_- - \frac{i}{e^2}\bar{\sigma}\partial_-\sigma. \tag{4.27}$$

Using these currents and the expressions for $\partial_+\vartheta_i$ and $\partial_-\vartheta_i$ from (4.13) and (4.14), we get that

$$J_+^R(x)\partial_+\vartheta_i(y) \sim \frac{2}{(x^+ - y^+)^2}, \qquad J_\pm^R(x)\partial_-\vartheta_i(y) \sim 0, \qquad J_-^R(x)\partial_+\vartheta_i(y) \sim 0, \tag{4.28}$$

where we have dropped the regular terms in the operator product expansion. This leads to the singularity structure

$$J_+^R(x)\vartheta_i(y) \sim \frac{2}{(x^+ - y^+)}. \tag{4.29}$$

Constructing the classically conserved charge Q^R given by

$$Q^R = \frac{1}{2\pi}\int dx^1 (J_+^R + J_-^R), \tag{4.30}$$

we obtain the relation

$$[Q^R, \vartheta_i(y)] = -i \quad \Rightarrow \quad [Q^R, Y_i(y)] = -1. \tag{4.31}$$

In evaluating the integral we have used the OPE (4.29) and also wick rotated to Euclidean space. So we obtain the result

$$e^{i\alpha Q^R} Y_i(\theta^+, \bar{\theta}^+) e^{-i\alpha Q^R} = Y_i(e^{-i\alpha}\theta^+, e^{i\alpha}\bar{\theta}^+) - i\alpha. \tag{4.32}$$

Therefore the perturbative dual superpotential

$$\widetilde{W} = -\frac{i\Upsilon}{4}\Big(\sum_i Q_i Y_i + it\Big) + \frac{\Sigma}{\sqrt{2}}\sum_a F_a \tag{4.33}$$

yields the correct $U(1)_R$ anomaly under the shift of the Y_i fields. From this we learn that the possible non-perturbative corrections to $\widetilde{W}$, which we denote $\widetilde{W}_{non-pert}$, must have $U(1)_R$ charge one.

4.4 The Structure of Instanton Corrections

The fermionic nature of the superpotential forces non-perturbative corrections to be of the form

$$\Upsilon A + \sum_a B^a F_a \tag{4.34}$$

where A carries no R-charge and B^a has R-charge one.

First let us determine A. A cannot be just a parameter since such a term is ruled out by the perturbative non-renormalization theorem (note that $\widetilde{W}$ already contains the term $\frac{t\Upsilon}{4}$). Also, A cannot depend solely on Σ which has R-charge one. Suppose A is only a function of Y_i. Demanding that the function be analytic in Y_i allows us to expand

$$A = a_0 + \sum_i a_1^i Y_i + \sum_{ij} a_2^{ij} Y_i Y_j + \sum_{ijk} a_3^{ijk} Y_i Y_j Y_k + \dots . \tag{4.35}$$

Then $[Q^R, A] = 0$ evaluated using (4.31) implies that A only depends on the Y_i in the combination

$$\sum_i \alpha_i Y_i$$

where

$$\sum_i \alpha_i = 0. \tag{4.36}$$

Perturbative contributions to the superpotential are ruled out, so we must look for single-valued terms of the form

$$e^{\left(\sum_i \alpha_i Y_i\right)}.$$

However, because of condition (4.36), this kind of term always grows as we make one or more of the Y_i large. These non-perturbative contributions are therefore ruled out, and we conclude that A cannot depend solely on the Y_i.

Suppose A depends on both Σ and Y_i. Demanding regular behaviour in Σ allows us to expand A in the form

$$A = \Sigma f_1(Y_j) + \Sigma^2 f_2(Y_j) + \Sigma^3 f_3(Y_j) + \dots = \sum_{k>0} \Sigma^k f_k(Y_j), \tag{4.37}$$

where $f_k(Y_i)$ has R-charge $-k$. We construct a solution in a way similar to the prior case. Insisting that f_k has R-charge $-k$ tells us that

$$\sum_i \frac{\partial f_k}{\partial Y_i} = k f_k.$$

A single-valued solution of this equation contains terms of the form

$$e^{\left(k \sum_i \alpha_i^k Y_i\right)} \tag{4.38}$$

where, unlike the prior case,

$$\sum_i \alpha_i^k = 1.$$

Again, as some combination of Y_i become large, terms of the form (4.38) must diverge and are therefore ruled out. We conclude that $A = 0$.

Next we proceed to constrain B^a which must have R-charge 1. Clearly B^a cannot depend only on Σ since this would be a perturbative term modifying the already present $-\frac{\Sigma}{\sqrt{2}} \sum_a F_a$ coupling. So we must consider the possibility that B^a depends on both Σ and Y_i. Demanding regular behaviour in Σ allows us to put B^a in the form

$$B^a = f_0^a(Y_j) + \Sigma f_1^a(Y_j) + \Sigma^2 f_2^a(Y_j) + \ldots = \sum_k \left\{ \Sigma^k f_k^a(Y_j) \right\}, \tag{4.39}$$

where $f_k^a(Y_i)$ has R-charge $1-k$. From our prior discussion, we know that each f_k^a (for $k \neq 1$) contains terms of the form

$$e^{\left(\{k-1\} \sum_i \alpha_i^k Y_i\right)}, \tag{4.40}$$

where,

$$\sum_i \alpha_i^k = 1.$$

The case $k = 1$ involves terms of the form $e^{\left(\sum_i \alpha_i^1 Y_i\right)}$, where

$$\sum_i \alpha_i^1 = 0.$$

The only case that admits terms that decay as $\sum_i Y_i \to \infty$ in all possible ways is $k = 0$. Every other case is ruled out. This leads to a possible non-perturbative superpotential

$$\widetilde{W}_{non-pert} = \sum_{\mu a} \beta_{\mu a} F_a e^{-\sum_i \alpha_{\mu i} Y_i}. \tag{4.41}$$

where $\sum_i \alpha_{\mu i} = 1$ for each μ.

4.5 Constraining the Superpotential

Let us now constrain $\widetilde{W}_{non-pert}$ further. On integrating over the superspace variables, we see that $\widetilde{W}_{non-pert}$ leads to a term in the Lagrangian

$$L = \ldots + \sqrt{2}\sum_{\mu a i} \beta_{\mu a}\alpha_{\mu i} e^{-\sum_j \alpha_{\mu j} y_j} \eta_{-a}\bar{\xi}_{+i}. \tag{4.42}$$

If such a term exists in the Lagrangian, then

$$\langle \bar{\eta}_{-a}\xi_{+i}\rangle \neq 0$$

for all i. It is instructive for us to calculate this 2-point function in the original theory. It can only be non-vanishing in an instanton background. Let use the duality map of (4.13),

$$\xi_{+i} = 2\phi_i\bar{\psi}_i,$$

from which we see that the 2-point function in the original theory involves a factor of ϕ_i. If the instanton is embedded in ϕ_m, then $\phi_i = 0$ for $i \neq m$. Hence, only

$$\langle \bar{\eta}_{-a}\xi_{+m}\rangle$$

can possibly be non-zero while all the other terms $\langle \bar{\eta}_{-a}\xi_{+i}\rangle$ for $i \neq m$ vanish trivially. For any instanton configuration, only one term of this kind can possibly be non-zero (this term may still vanish because of additional fermion zero modes, as we shall see in later examples).

The structure of BPS instanton contributions tells us that B^a must be of the form,

$$B^a = \sum_i \beta_{ia} e^{-Y_i}, \tag{4.43}$$

giving the non-perturbative superpotential

$$\widetilde{W}_{non-pert} = \sum_{ia} \beta_{ia} F_a e^{-Y_i}. \tag{4.44}$$

This can also be seen in a different way. Periodicity of Y_i implies that

$$\alpha_{\mu i} \in \mathbb{Z}.$$

When combined with the constraint $\sum_i \alpha_{\mu i} = 1$ and the decay condition on $\widetilde{W}_{non-pert}$, we are lead to the same conclusion: namely, that the exact dual superpotential is given by

$$\widetilde{W}_{exact} = -\frac{i\Upsilon}{4}(\sum_i Q_i Y_i + it) + \frac{\Sigma}{\sqrt{2}}\sum_a F_a + \mu\sum_{ia}\beta_{ia}F_a e^{-Y_i}, \tag{4.45}$$

where we have explicitly exhibited the mass scale μ in the superpotential. What remains is the determination of the β_{ia} parameters of the dual theory. Unlike the case of $(2,2)$ theories, these parameters depend on the particular theory under consideration.

5 The Vacuum Structure and Observables

We want to begin by studying the vacuum structure of these $(0,2)$ theories. In the absence of a superpotential, minimizing the bosonic potential imposes the constraints

$$E_a(\phi_i, \Sigma) = 0, \qquad \sum_i Q_i |\phi_i|^2 = r. \tag{5.1}$$

where $i = 1, \dots, N$, and each E_a is associated to a left-moving fermion, χ_{-a}. With a superpotential, there are additional holomorphic constraints

$$J^a(\phi_i, \Sigma) = 0. \tag{5.2}$$

Note that there need not be a Σ field in the theory. There are typically multiple phases for these models, with $r >> 0$ corresponding to a geometric phase, while $r << 0$ corresponds to a Landau-Ginzburg phase. With multiple $U(1)$ factors, hybrid phases are also possible. There are a myriad of models that we could examine, but in this effort, we will restrict to a few classes that we find particularly interesting.

5.1 Without a Σ field

There are really two distinct cases that we will consider: let us first suppose that there is no Σ field. Each E_a depends only on Φ_i, and is a section of the line-bundle

$$\mathcal{O}(Q_a). \tag{5.3}$$

Similarly, each J^a is a section of $\mathcal{O}(-Q_a)$. Minimizing the bosonic potential restricts us to the surface $E_a = J^a = 0$. Usually, we consider non-singular surfaces where $\frac{\partial E_a}{\partial \phi_i} \neq 0$ and $\frac{\partial J^a}{\partial \phi_i} \neq 0$ on the locus $E_a = J^a = 0$. This is not really a necessary condition for the physical theory but it does simplify our analysis.

Suppose we have a single field χ_a. The chirality condition $E \cdot J = 0$ tells us that either E or J must be zero. If we have more than a single left-moving field, there can be non-trivial solutions to the chirality condition. However, if (E_a, J^a) are both non-zero for any a, the resulting surface is singular since

$$dE_1 \wedge \cdots dE_{a_{max}} \wedge dJ^1 \wedge \cdots dJ^{a_{max}} = 0.$$

The linear sigma model is likely to be perfectly regular in this case but again, for simplicity, we will restrict to non-singular surfaces. For the moment, let us also take each $Q_i \geq 0$ so the ambient space $\mathcal{A}$, defined by $\sum_i Q_i|\phi_i|^2 = r$, is compact. We will consider models with some negatively charged fields later. Lastly, we note that $a_{max} \leq N$ for a non-singular surface.

The last element of the low-energy description is the fermions. The right-handed fermions are fixed by supersymmetry to be sections of the tangent bundle to the hypersurface (5.1) and (5.2) regardless of whether there is or is not a Σ field. It is worth seeing how this emerges directly from the Yukawa couplings in this case since we will use the same techniques for the left-moving fermions. The Yukawa couplings are,

$$-\left\{ iQ_i\sqrt{2}\bar{\phi}^i\lambda_-\psi_{+i} + \bar{\chi}_{-a}\frac{\partial E_a}{\partial \phi_i}\psi_{+i} + \chi_{-a}\psi_{+i}\frac{\partial J^a}{\partial \phi_i}\right\} - \text{h.c.} \tag{5.4}$$

We want to determine which of the ψ_{+i} fermions is massless. Massless fermions satisfy the conditions

$$\sum_i Q_i\bar{\phi}^i\psi_{+i} = 0, \qquad \sum_i \frac{\partial E_a}{\partial \phi_i}\psi_{+i} = 0, \qquad \sum_i \frac{\partial J^a}{\partial \phi_i}\psi_{+i} = 0, \tag{5.5}$$

for each a. Following [2], we interpret the first condition as a gauge-fixing condition on the holomorphic equivalence

$$\psi_{+i} \sim \psi_{+i} + \phi_i\psi. \tag{5.6}$$

We encode this condition in a short sequence,

$$0 \to \mathcal{O} \xrightarrow{\alpha} \oplus_i \ \mathcal{O}(Q_i) \to 0, \tag{5.7}$$

where α is the map $\psi \to \phi_i\psi$. This defines the tangent bundle to the ambient space, $\mathcal{A}$, defined by $\sum_i Q_i|\phi^i|^2 = r$ in terms of a quotient of line bundles $\oplus_i \ \mathcal{O}(Q_i)/\text{Im}(\alpha)$.

We can now impose the remaining conditions in turn. For example, for a particular E_a, we consider the sequence

$$0 \to T\mathcal{A} \overset{\alpha_E}{\to} \mathcal{O}(Q_a) \to 0, \tag{5.8}$$

where

$$\alpha_E : \ s_i \mapsto \sum_i \frac{\partial E_a}{\partial \phi_i}s_i$$

and $\{s_i\}$ is a section of $T\mathcal{A}$. This sequence simply defines the restriction of $T\mathcal{A}$ to the hypersurface $E_a = 0$. In a similar way, we impose all the remaining

Yukawa conditions (5.5). What we learn (as expected from supersymmetry) is that the surviving light ψ_{+i} transform as sections of the tangent bundle to the surface $E_a = J^a = 0$.

More interesting are the left-moving fermions, χ_{-a}, with charge Q_a. These fermions satisfy the conditions

$$\sum_a \frac{\partial \bar{E}_a}{\partial \bar{\phi}_i} \chi_{-a} = 0, \qquad \sum_a \frac{\partial J^a}{\partial \phi_i} \chi_{-a} = 0 \tag{5.9}$$

for each i. The condition that the surface be non-singular guarantees that for a given a, either E_a or J^a is non-zero but not both. The first condition of (5.9) is a gauge-fixing condition for a holomorphic identification akin to (5.6)

$$\chi_{-a} \sim \chi_{-a} + \sum_i \frac{\partial E_a}{\partial \phi_i} \chi^i \tag{5.10}$$

where χ^i has charge Q_i.

We must first dispense with fermions, χ_{-a}, for which both E_a and J^a are zero. These fermions come along for the ride as we flow into the IR where they transform as sections of $\mathcal{O}(Q_a)$ restricted to the surface. They also contribute to the low-energy anomaly in a straightforward way since,

$$\mathrm{ch}(\oplus_a \mathcal{O}(Q_a)) = \sum_a \mathrm{ch}(\mathcal{O}(Q_a)).$$

Any fermion for which E_a or J^a is non-trivial must satisfy (5.10) for each i. However, this imposes N equations on $a_{max} \leq N$ variables so there are no surviving left-moving fermions.

The low-energy theory is then a non-linear sigma model on the surface $\mathcal{M}$ obtained by setting

$$E_a(\Phi) = J^a(\Phi) = 0, \qquad \sum_i Q_i |\phi_i|^2 = r. \tag{5.11}$$

The Chern classes of the surface can be computed using the adjunction formula which tells us that

$$c(T\mathcal{M}) = \frac{\prod_i (1 + Q_i J)}{\prod_{E_a \neq 0} (1 + Q_a J) \prod_{J^a \neq 0} (1 - Q_a J)} \tag{5.12}$$

from which we see that

$$c_1(T\mathcal{M}) = \sum_i Q_i - \sum_{E_a \neq 0} Q_a + \sum_{J^a \neq 0} Q_a, \qquad \mathrm{ch}_2(T\mathcal{M}) = \frac{1}{2} c_1^2 - c_2 = 0. \tag{5.13}$$

This low-energy theory is free of anomalies as we expect with no left-moving fermions at all. It is worth pointing out that we can even construct simple conformal models ($c_1 = 0$) of this kind.

5.2 With a Σ field

So far, our examples have given theories with no low-energy left-moving fermions at all. To obtain interesting models with left-movers, we need to include an uncharged field, Σ. We consider cases where

$$E_a = \Sigma \mathcal{E}_a(\Phi_i), \qquad J^a = J^a(\Phi_i).$$

Minimizing the bosonic potential gives two branches. If $< \Sigma > \neq 0$ then we must set $\mathcal{E}_a = J^a = 0$, and the corresponding low-energy analysis is exactly as before except there is an extra uncharged decoupled chiral multiplet in the IR.

More interesting is the case where $\Sigma = 0$. This allows us to have non-trivial $\mathcal{E}$ without the constraint $\mathcal{E} = 0$. The analysis for the right-moving fermions, ψ_{+i}, is as before. Again, we conclude that they are tangent to the surface

$$J^a(\phi_i) = 0, \qquad \sum_i Q_i |\phi_i|^2 = r.$$

The only non-vanishing Yukawa couplings for the left-moving fermions teach us that

$$\sum_a \bar{\mathcal{E}}_a \chi_{-a} = 0, \qquad \sum_a \frac{\partial J^a}{\partial \phi_i} \chi_{-a} = 0. \tag{5.14}$$

Suppose there are no J^a in the UV theory. The single remaining constraint from (5.14) is a gauge-fixing condition on the equivalence,

$$\chi_{-a} \sim \chi_{-a} + \mathcal{E}_a \chi,$$

which tells us that the left-movers are sections of the quotient bundle $\oplus_a \mathcal{O}(Q_a)/\mathrm{Im}(\alpha_{\mathcal{E}})$ where

$$0 \to \mathcal{O} \stackrel{\alpha_{\mathcal{E}}}{\to} \oplus_a \mathcal{O}(Q_a) \to 0, \tag{5.15}$$

where

$$\alpha_{\mathcal{E}} : \chi \mapsto \mathcal{E}_a \chi.$$

This construction includes the special class of theories where for each Φ_i, we include one χ_i ($a = i$) with charge Q_i and

$$\mathcal{E}_i = \sqrt{2} Q_i \Phi_i.$$

For this particular choice, the left-movers are also sections of the tangent bundle, and theory has enhanced $(2,2)$ supersymmetry. The target space is the ambient space, $\mathcal{A}$.

Now suppose that some J^a are non-trivial in the UV. We are then confined to the surface $J^a = 0$ in $\mathcal{A}$. The second condition from (5.14) has no solutions for the partner χ_{-a} except the pure gauge solution,

$$\chi_{-a} = \mathcal{E}_a \chi,$$

which one can check is a solution on the surface using $\mathcal{E} \cdot J = 0$. Those χ_{-a} whose corresponding J^a do vanish in the UV survive. The bundle that appears in the IR can, however, now be more interesting than a direct sum of line bundles. The holomorphic bundle, $\mathcal{V}$, is defined by the cohomology of the sequence

$$0 \to \mathcal{O} \stackrel{\alpha_{\mathcal{E}}}{\to} \oplus_a \mathcal{O}(Q_a) \stackrel{\beta_J}{\to} \oplus_i \mathcal{O}(-Q_i) \to 0, \tag{5.16}$$

where

$$\alpha_{\mathcal{E}} : \chi \mapsto \mathcal{E}_a \chi, \qquad \beta_J : \chi_{-a} \mapsto \sum_a \frac{\partial J^a}{\partial \phi_i} \chi_{-a}. \tag{5.17}$$

The left-movers are therefore sections of $\mathcal{V}$ given by the $\mathrm{Ker}(\beta_J)/\mathrm{Im}(\alpha_{\mathcal{E}})$. The rank of $\mathcal{V}$ is $a_{max} - \{\#(J^a \neq 0) + 1\}$. It is easy to generalize this construction to cases where some E_a, J^a depend on Σ while some do not.

5.3 Vacua for Non-Linear Sigma Models

In the geometric phase, the low-energy physics is captured by a non-linear sigma model on the surface $\mathcal{M}$, with the left-moving fermions taking values in the holomorphic bundle, $\mathcal{V}$, of rank r. For corresponding $(2,2)$ models, the semi-classical ground states of the sigma model are in one-to-one correspondence with elements of de Rham cohomology, $H^*(\mathcal{M}, \mathbb{R})$.

For $(0,2)$ theories, the situation is different. In a sector of the Hilbert space with m left-moving fermions excited, the supercharge acts as the Dolbeault operator, $\bar{\partial}_E$, twisted in the holomorphic bundle $E = \wedge^m \mathcal{V}^*$. The semi-classical ground states of the sigma model are therefore in correspondence with the cohomology groups,

$$H^*(\mathcal{M}, \wedge^m \mathcal{V}^*), \quad m = 0, \ldots, r-1 \tag{5.18}$$

with dimension $h^*(\mathcal{M}, \wedge^m \mathcal{V}^*)$. Some of these ground states might pair up and become massive but the Witten index,

$$\mathrm{Tr}(-1)^F = \sum_{p,m} (-1)^{p+m}\, h^p(\mathcal{M}, \wedge^m \mathcal{V}^*), \tag{5.19}$$

should remain invariant. Lastly, we should mention the existence of BPS solitons interpolating between these vacua with mass gap. These are quite fascinating excitations that merit further exploration, perhaps with the aim of generalizing the structure of helices of coherent sheaves [31], and the attempt to classify massive N=2 theories [32].

5.4 Moduli for Conformal Models

In the case of conformal models where $\sum_i Q_i = 0$, there are particularly interesting operators that control the moduli of the non-linear sigma model. The simplest to describe are the moduli for the Kähler metric. Deformations of the Kähler and complex structure correspond, respectively, to elements of

$$H^1(\mathcal{M}, T^*\mathcal{M}), \quad H^1(\mathcal{M}, T\mathcal{M}).$$

For models with a space-time interpretation, each cohomology element gives rise to a space-time scalar field. Ignoring effects that are non-perturbative in the string coupling, the potential for these scalar fields has flat directions.

The last class of moduli parametrize continuous deformations of the holomorphic bundle, $\mathcal{V}$, and correspond to elements of

$$H^1(\mathcal{M}, \mathrm{End}\mathcal{V}).$$

Each of these deformations also gives rise to a space-time scalar. Even in non-conformal models, these deformations are interesting because they are relevant deformations. For example, starting with the tangent bundle, $\mathcal{V} = T\mathcal{M}$, where the theory is $(2,2)$, we can find families of $(0,2)$ theories by deforming the bundle.

5.5 Instanton Corrections

The most natural set of observables to study both in massive and conformal models are chiral operators. Both the vacua (via the state-operator correspondence) and the moduli described above correspond to particular chiral operators. A chiral operator, $\mathcal{O}$, satisfies

$$\{\bar{Q}_+, \mathcal{O}\} = 0.$$

Consider a correlator of chiral operators,

$$< \mathcal{O}_1(y_1) \cdots \mathcal{O}_n(y_n) > . \tag{5.20}$$

Chirality ensures that the correlator is independent of the insertion points, y_i, on the world-sheet Σ.[3] The correlator must also depend on the parameters of the theory in a holomorphic way, and so is protected from perturbative corrections.

While there are no perturbative contributions to the correlation function, there can be non-perturbative contributions arising from instantons. In the linear sigma model, an instanton corresponds to a BPS solution of the abelian Higgs model reviewed in section 4.2. In the IR non-linear sigma model, these BPS instantons correspond to holomorphic maps

$$\phi : \quad \Sigma \to \mathcal{M}.$$

Each map is characterized by winding number n, which is given by

$$n = \frac{1}{2\pi} \int_\Sigma \phi^*(\omega)$$

where ω is the Kähler form of the target space $\mathcal{M}$. Both in the linear and non-linear sigma model, an n instanton contribution to a correlator function is suppresed by the instanton action (taking $n > 0$),

$$S_{inst} \sim e^{2\pi i n t}, \qquad t = ir + \frac{\theta}{2\pi}.$$

However, the linear sigma model contains point-like instanton contributions in addition to the usual smooth instantons [2]. The effect of these point-like instantons is to renormalize t as we flow from the UV to the IR. The relation between t in the linear and non-linear sigma models has been computed for $(2,2)$ theories in [33], where in some cases, the parameters were found to agree.

We can use symmetries to further constrain the correlation functions. The main symmetry that we will consider is the right-moving $U(1)_R$ symmetry under which the right-moving ψ_+ fermions have charge one. To obtain a selection rule, we need to determine the number of right-moving fermion zero-modes in a sector with instanton number n. On a genus g world-sheet Σ, the count of fermion zero-modes follows from an index theorem. In instanton sector n, there are

$$\dim(\mathcal{M}) * (1-g) + n\, c_1(\mathcal{M})$$

right-moving zero modes. We will primarily consider the plane (or equivalently a genus 0 world-sheet). For the perturbative sector where $n = 0$ where we consider constant maps (the only holomorphic maps) from $\Sigma \to \mathcal{M}$, we

[3] We apologize for the multiple uses of Σ, but this notation for the world-sheet is conventional.

learn that the correlator (5.20) is non-vanishing only when the product of chiral operators, each associated to an element of twisted Dolbeault cohomology, has anti-holomorphic degree $\dim(\mathcal{M})$, i.e., only when it is a top form. The semi-classical value of the correlator (5.20) then defines a map

$$H^*(\mathcal{M}, E_1) \times \ldots \times H^*(\mathcal{M}, E_m) \to \mathbb{C} \tag{5.21}$$

where each E_i is a bundle of the form $\wedge^* \mathcal{V}^*$, and the total anti-holomorphic form degree is $\dim(\mathcal{M})$ or the correlator vanishes. This is a kind of intersection form on $\mathcal{M}$ [4].

Let us consider the left-movers. To constrain the left-moving fermions, we want to restrict to $(0,2)$ non-linear sigma models which are the IR limits of GLSMs. In the UV GLSM, there is a classical $U(1)$ charge Q_L where

$$Q_L \sim \int dx^1 \sum_a \bar{\chi}_{-a} \chi_{-a}. \tag{5.22}$$

In general, this is not a conserved charge like the $U(1)_R$ charge. However, the charge violation is proportional to the instanton number. As we flow to the IR, some of the χ_- fermions become massive. There is an index theorem that counts the net number of χ_- zero modes,

$$\dim(\mathcal{V}) * (1-g) + n\, c_1(\mathcal{V}).$$

Absorbing these zero modes for $n = 0, g = 0$ gives a selection rule: the correlator (5.20) must contain $\dim(\mathcal{V})$ left-moving fermions. Note that $\dim(\mathcal{V}) = \mathrm{rk}(\mathcal{V})$ for these holomorphic bundles so this constraint is again a statement that the correlator be a top form.

In non-conformal models, a combination of the $U(1)_L$ and $U(1)_R$ charges is conserved exactly in the UV. Both charges are individually violated by instantons. This permits a quantum deformation of the classical geometric rings which satisfy the $n = 0$ selection rules. In the $(2,2)$ case, the instanton corrected ring is known as the quantum cohomology ring [17, 18]. In the following section, we will find analogous structures for $(0,2)$ theories.

The last issue we need to address is the coefficient of the instanton corrections to a chiral correlator in a low-energy conformal non-linear sigma model. Since the model is conformal, $U(1)_R$ is conserved. In a conformal model, $c_1(\mathcal{M}) = 0$ so there are no additional right-moving zero modes for $n > 0$. This, combined with the conservation of $U(1)_R$, implies that the only way that the chiral ring is modified quantum mechanically is via instanton corrections to the classical ring coefficients.

In the $(2,2)$ case, this coefficient 'counted' the number of holomorphic curves in some suitable sense. In the $(0,2)$ case, the basic picture is similar. Consider the moduli space of instantons with charge n, which we denote $\mathcal{M}_n$. There are subtle issues surrounding the compactification of this space. We will take the physical compactification provided by the linear sigma model. The zero-modes for the left-moving χ_- fermions (which transform as a section of $\mathcal{V}$) in the sector with instanton charge n define a holomorphic bundle $\mathcal{V}_n$ on $\mathcal{M}_n$. The effective theory of the instanton moduli is a sigma model with target $\mathcal{M}_n$ and with a supercharge acting as the $\bar{\partial}$ operator twisted in the bundle $\oplus_m \wedge^m \mathcal{V}_n^*$. The leading contribution of the path-integral over the moduli gives instanton contributions

$$< \cdots > \quad = \quad \sum_{n>0} \left(\sum_m (-1)^m \mathrm{Ind}(\bar{\partial}_{\wedge^m \mathcal{V}_n^*}) \right) e^{-2\pi i n t}. \tag{5.23}$$

More precisely, the path-integral computation gives the integral over the index density over $\mathcal{M}_n$ which need not necessarily agree with the index. When non-vanishing, these instanton contributions modify the ring coefficients. In the $(2,2)$ case, the coefficient of the instanton correction reduces to $\chi(\mathcal{M}_n)$. In the $(0,2)$ case, we find a natural generalization that depends on the choice of holomorphic bundle, $\mathcal{V}$.

6 Examples of Dual Pairs

We now turn to the construction of specific $(0,2)$ dual pairs. There are three broad classes of models. These classes are characterized by whether the rank of the left-moving bundle, $\mathcal{V}$, is less than, equal to, or greater than the rank of the tangent bundle $T\mathcal{M}$. As we will see, the dual theory in the first case is quite different from the latter two cases. Unlike the latter two cases, the dual theory for $\mathrm{rk}(\mathcal{V}) < \mathrm{rk}(T\mathcal{M})$ is typically a non-linear sigma model so the duality relates two geometric theories. In the remaining cases, the dual theory is typically a $(0,2)$ Landau-Ginzburg theory with no flat directions in the superpotential.

For brevity, in our subsequent discussion, we will not explicitly write the gauge kinetic terms, the FI-terms, and the θ terms in either the original or the dual theories. We will always assume they are present. The first examples that we will consider fall in the category $\mathrm{rk}(\mathcal{V}) = \mathrm{rk}(T\mathcal{M})$.

6.1 One Chiral & One Fermi Field

We start with the simplest possible model containing one chiral superfield, Φ, and one Fermi superfield, Γ, both with charge Q. The Lagrangian of the theory is given by

$$L = -\frac{i}{2}\int d^2\theta\ \bar{\Phi}(\mathcal{D}_0 - \mathcal{D}_1)\Phi - \frac{1}{2}\int d^2\theta\ \bar{\Gamma}\Gamma, \tag{6.1}$$

In the definition of Γ, we have some freedom in our choice of E. We consider two choices for E below, and construct the dual theories. In the first case, we find no non-perturbative corrections to the dual superpotential, while in the second case there is a correction.

6.1.1 $E = i\alpha\Phi$

For the first case, take $E = i\alpha\Phi$ so that E is itself a chiral superfield of charge Q which satisfies $\bar{\mathcal{D}}_+ E = 0$ for some parameter, α. This theory is free of anomalies. Note that for this choice of E, this theory is a $(0,2)$ theory that never has enhanced $(2,2)$ supersymmetry for any choice of α. This is the case because there is no Σ superfield, and so no right-moving gauginos. Hence, the left-moving fermions in the Fermi multiplet do not couple to the gauginos at all. We could also equivalently start with a Σ field and the choice $E = \Sigma\Phi$, and set

$$< \Sigma >= i\alpha$$

while setting the right-moving gauginos in Σ to zero.

Using the component field expansions for Φ and Γ, we get that

$$\begin{aligned} L \ &= \ (\partial_+\rho)(\partial_-\rho) + \rho^2(\partial_+\varphi + QA_+)(\partial_-\varphi + QA_-) \\ &\quad +i\bar{\psi}_+ D_-\psi_+ - \sqrt{2}iQ\bar{\phi}\lambda_-\psi_+ + \sqrt{2}iQ\phi\bar{\psi}_+\bar{\lambda}_- + QD\rho^2 \\ &\quad +i\bar{\chi}_- D_+\chi_- - |\alpha\phi|^2 - i\alpha\bar{\chi}_-\psi_+ + i\bar{\alpha}\bar{\psi}_+\chi_-, \end{aligned} \tag{6.2}$$

where we have set $G = 0$ by its classical equation of motion. In the dual theory, we have a single neutral chiral superfield Y, and a neutral Fermi superfield F. The relation between the original and the dual Fermi fields follows from the component expansion of the duality map (3.47)

$$\bar{\eta}_- = -\bar{\phi}\chi_-,\ \eta_- = -\phi\bar{\chi}_-. \tag{6.3}$$

These relations will be useful in determining the non-perturbative corrections to the dual superpotential. The perturbative dual theory is given by the

Lagrangian

$$\begin{aligned}\widetilde{L} = \frac{1}{8}\int d^2\theta\ &\left[\frac{i(Y-\bar{Y})}{Y+\bar{Y}}\partial_-(Y+\bar{Y}) - 8\frac{\bar{F}F}{Y+\bar{Y}}\right] \\ &-\left[\frac{iQ}{4}\int d\theta^+ Y\Upsilon - \frac{i\alpha}{\sqrt{2}}\int d\theta^+ F + \text{h.c.}\right].\end{aligned} \tag{6.4}$$

This dual description can be checked using the various duality maps together with the identity (true up to total derivatives),

$$\frac{(\partial_+\vartheta)(\partial_-\vartheta)}{2\varrho} - Q\vartheta F_{01} = \frac{(\partial_-\vartheta)}{8\varrho^2}\xi_+\bar{\xi}_+. \tag{6.5}$$

We also have to integrate out the auxiliary field H in the superfield F using its classical equation of motion to explicitly check the duality.

So in the dual theory, we find the perturbative superpotential

$$\widetilde{W} = -\frac{i\Upsilon}{4}(QY+it) + \frac{i\alpha}{\sqrt{2}}F. \tag{6.6}$$

Now we must consider the possibility of non-perturbative corrections to the dual superpotential: namely, is there an Fe^{-Y} addition to the superpotential? We will argue that such a term does not arise. The non-perturbative correction to the dual superpotential is generated by instantons in the original theory. Because of the $|\alpha\phi|^2$ term in the original action, there is no BPS instanton because ϕ must be set to zero. For any non-zero α, there is no non-perturbative correction. The perturbative dual superpotential is exact

$$\widetilde{W}_{exact} = -\frac{i\Upsilon}{4}(QY+it) + \frac{i\alpha}{\sqrt{2}}F. \tag{6.7}$$

On integrating out Υ, we find an effective potential

$$\widetilde{W}_{eff} = \frac{i\alpha}{\sqrt{2}}F, \tag{6.8}$$

with the constraint $QY = -it$. Note that supersymmetry is spontaneously broken in both the original and dual theories.

6.1.2 A Vanishing Result for More General Cases

We can extend the prior result to a more general setting. Non-perturbative terms in the dual superpotential of the form $\beta_{ia}F_a e^{-Y_i}$ lead to terms in the Lagrangian given by

$$L = \ldots + \sqrt{2}\sum_{ai}\beta_{ia}e^{-y_i}\eta_{-a}\bar{\xi}_{+i}. \tag{6.9}$$

The existence of these terms implies that the correlator $\langle \bar{\eta}_{-a} \xi_{+i} \rangle$ must be non-vanishing. Consider the case $E_a = f_a(\Phi_i)$ which is a generalization of the case just considered. For this choice of E_a, we see that the Lagrangian of the original theory contains the term

$$L = \ldots - \sum_i |f_a(\phi_i)|^2. \tag{6.10}$$

So the condition for a BPS instanton solution is $f_a(\phi_i) = 0$ for all a. From the duality map $\bar{F}_a = \Gamma_a \bar{\mathcal{E}}_a$, we see that $\bar{\eta}_{-a} = -\chi_{-a} \bar{f}_a(\bar{\phi}_i)$, which is zero for all a using the BPS condition. Hence the two point function always vanishes, and so do the non-perturbative corrections to the dual superpotential. There is an apparent caveat to this argument; namely, the kinetic terms for the F_a superfields diverge like $1/|f_a|^2$ since for an instanton configuration $f_a = 0$. However, in the dual theory, in terms of Y variables, $1/|f_a|^2$ is not holomorphic and so this divergence should not affect the determination of the superpotential.

6.1.3 $E = c\Sigma\Phi$

Next we consider a case where, as we shall show, there is a non-perturbative correction to the dual superpotential. We consider the case where $E = c\Sigma\Phi$, where c is a non-zero parameter. The key difference is the appearance of Σ in E. This case can easily be generalized to a theory with N chiral and Fermi superfields with charge Q_i where

$$E_i = c_i \Sigma \Phi_i.$$

These models are deformations of theories with $(2,2)$ supersymmetry which is restored at the point $c_i = \sqrt{2} Q_i$. However, this particular deformation is not a relevant deformation although it does break supersymmetry. We can see this from the low-energy perspective by considering the target space, $W\mathbb{P}^N$. The left-moving bundle $\mathcal{V}$ is a deformation of the tangent bundle specified by the sequence (5.15); however, the bundles obtained from this deformation are all equivalent. We will see this reflected in the low-energy physics of the dual description. Note, however, that the bundle can degenerate by taking some $c_i \to 0$.

Determining the β_{ia} Coefficients

While it is difficult to determine the β_{ia} coefficients in the superpotential for most models, in this case, we can explicitly determine these parameters.

The dual superpotential takes the form,

$$\widetilde{W}_{exact} = -\frac{i\Upsilon}{4}(\sum_i Q_i Y_i + it) + \frac{\Sigma}{\sqrt{2}} \sum_i c_i F_i + \mu \sum_{ij} \beta_{ij} F_i e^{-Y_j}. \qquad (6.11)$$

We have replaced β_{ia} by β_{ij} since we have an equal number of chiral and Fermi superfields. We have also rescaled F_i and β_{ij} by a factor of c_i in (4.5) and (4.45) to get this form.

We shall see that we can determine β_{ij} exactly. In the original theory, we take σ, the lowest component field of Σ, to be very large and slowly varying, and we give it a specific expectation value. Then from the terms in the Lagrangian given by

$$L = \ldots - |\sigma|^2 \sum_i |c_i \phi_i|^2 - \sigma \sum_i c_i \bar{\chi}_{-i} \psi_{+i} - \bar{\sigma} \sum_i \bar{c}_i \bar{\psi}_{+i} \chi_{-i}, \qquad (6.12)$$

we see that Φ_i and Γ_i both get a large mass of order $c_i \sigma$. We can therefore consider integrating out the massive superfields, Φ_i and Γ_i, for a fixed value of σ, together with the high frequency modes of Σ (in the sense of Wilsonian R.G.). This will give us an effective superpotential, $\widetilde{W}_{eff}(\Upsilon, \Sigma)$, for the remaining low energy degrees of freedom. We can also integrate out the neutral superfields Y_i and F_i in the dual theory to get another expression for $\widetilde{W}_{eff}(\Upsilon, \Sigma)$. Equating the two expressions gives a constraint on the β_{ij} coefficients.

First we focus on integrating out the massive superfields in the original theory. The superpotential $\widetilde{W}_{eff}(\Upsilon, \Sigma)$, on demanding analyticity in Υ, is of the form

$$\widetilde{W}_{eff}(\Upsilon, \Sigma) = W^0_{eff}(\Sigma) + \Upsilon W_{eff}(\Sigma). \qquad (6.13)$$

The Grassmann odd nature of the superpotential forces $W^0_{eff}(\Sigma) = 0$, leading to

$$\widetilde{W}_{eff}(\Upsilon, \Sigma) = \Upsilon W_{eff}(\Sigma). \qquad (6.14)$$

This gives terms in the Lagrangian

$$\frac{1}{4} \int d\theta^+ \ \widetilde{W}_{eff}(\Upsilon, \Sigma) + \text{h.c.} = -D \ \text{Im}\{W_{eff}(\sigma)\} + F_{01} \ \text{Re}\{W_{eff}(\sigma)\} + \ldots \qquad (6.15)$$

where Im and Re are the imaginary and real parts of the complex quantity. Therefore, in order to determine $\widetilde{W}_{eff}$, it is enough to consider only the terms in the effective action that are linear in D and F_{01}. We need to evaluate

$$e^{iS_{eff}(\Upsilon,\Sigma)} = \int \mathcal{D}\Phi_i \mathcal{D}\bar{\Phi}_i \mathcal{D}\Gamma_i \mathcal{D}\bar{\Gamma}_i \ e^{iS(\Upsilon,\Sigma,\Phi_i,\bar{\Phi}_i,\Gamma_i,\bar{\Gamma}_i)}. \qquad (6.16)$$

Because each E_i is linear in Φ_i, we can exactly evaluate the path integral and hence compute S_{eff}. In the limit of large σ, the wick rotated Lagrangian in Euclidean space reduces to

$$\begin{aligned} L^E \quad &= \sum_i \quad \Big[|D_\alpha \phi_i|^2 + i\bar{\psi}_{+i} D_-^E \psi_{+i} - i\bar{\chi}_{-i} D_+^E \chi_{-i} - Q_i D |\phi_i|^2 + |c_i \sigma \phi_i|^2 \\ &\quad + \sigma c_i \bar{\chi}_{-i} \psi_{+i} + \bar{\sigma} \bar{c}_i \bar{\psi}_{+i} \chi_{-i} \Big], \end{aligned} \tag{6.17}$$

where $D_\pm^E = D_1 \pm iD_2$. Let us now extract the dependence of L^E on the phase of σ and the c_i. We define $\sigma = |\sigma| e^{i\omega}$ and $c_i = |c_i| e^{i\tau_i}$. Classically, these phases can be absorbed by a phase rotation of the fermions given by

$$\psi_{+i} \to e^{-\frac{i}{2}(\omega + \tau_i)} \psi_{+i}, \qquad \chi_{-i} \to e^{\frac{i}{2}(\omega + \tau_i)} \chi_{-i}. \tag{6.18}$$

However, this chiral rotation of the fermions is anomalous and shifts the effective Lagrangian by

$$-i \sum_i Q_i (\omega + \tau_i) F_{12}. \tag{6.19}$$

Hence,

$$L_{eff}^E(\sigma, c_i) = L_{eff}^E(|\sigma|, |c_i|) - i \sum_i Q_i (\omega + \tau_i) F_{12}. \tag{6.20}$$

We calculate $L_{eff}^E(|\sigma|, |c_i|)$ finding

$$e^{-\int d^2x L_{eff}^E(|\sigma|, |c_i|)} = \prod_i \frac{\det \begin{pmatrix} -|\sigma c_i| & iD_+^E \\ iD_-^E & |\sigma c_i| \end{pmatrix}}{\det(-D_\mu^2 - Q_i D + |\sigma c_i|^2)}. \tag{6.21}$$

The square of the Dirac operator in the numerator is

$$\begin{pmatrix} -|\sigma c_i| & iD_+^E \\ iD_-^E & |\sigma c_i| \end{pmatrix}^2 = \begin{pmatrix} -D_\mu^2 + Q_i F_{12} + |\sigma c_i|^2 & 0 \\ 0 & -D_\mu^2 - Q_i F_{12} + |\sigma c_i|^2 \end{pmatrix} \tag{6.22}$$

which gives an effective action

$$\begin{aligned} \textstyle\int d^2x \; L_{eff}^E(|\sigma|, |c_i|) &= \textstyle\sum_i \Big\{ \log \det(-D_\mu^2 - Q_i D + |\sigma c_i|^2) \qquad (6.23) \\ -\tfrac{1}{2} \log \det(-D_\mu^2 + Q_i F_{12} + |\sigma c_i|^2) &- \tfrac{1}{2} \log \det(-D_\mu^2 - Q_i F_{12} + |\sigma c_i|^2) \Big\}. \end{aligned}$$

It is easy to see that this gives no linear term in F_{12}. However it has a term linear in D given by

$$\int d^2x \; L_{eff}^E(|\sigma|, |c_i|) = -D \sum_i Q_i \mathrm{tr}\Big(\frac{1}{-\partial_\mu^2 + |\sigma c_i|^2}\Big) + \dots. \tag{6.24}$$

So we obtain an effective action

$$L^E_{eff}(|\sigma|,|c_i|) = -\frac{D}{2}\sum_i Q_i \ln(\frac{\Lambda^2_{UV}+|\sigma c_i|^2}{|\sigma c_i|^2}) + \ldots \tag{6.25}$$

which in the continuum limit $\Lambda_{UV} \to \infty$ reduces to

$$L^E_{eff}(|\sigma|,|c_i|) = -D\sum_i Q_i \ln(\frac{\Lambda_{UV}}{|\sigma c_i|}) + \ldots. \tag{6.26}$$

Putting together these results, we find that

$$L^E_{eff}(\sigma, c_i) = -D\sum_i Q_i \ln(\frac{\Lambda_{UV}}{|\sigma c_i|}) - iF_{12}\sum_i Q_i(\omega+\tau_i) + \ldots. \tag{6.27}$$

Using (6.15), we read off the effective superpotential

$$\widetilde{W}_{eff}(\Upsilon,\sigma,c_i) = -\frac{i\Upsilon}{4}(\sum_i Q_i \ln(\frac{\Lambda_{UV}}{c_i\sigma}) + it_0). \tag{6.28}$$

Now we use the one-loop renormalization of t given by

$$t(\mu) = i\sum_i Q_i \ln(\frac{\mu}{\Lambda}), \tag{6.29}$$

where Λ is the RG invariant dynamical scale of the theory given by $\Lambda = \mu e^{it(\mu)/\sum_i Q_i}$, to obtain

$$\widetilde{W}_{eff}(\Upsilon,\sigma,c_i) = -\frac{i\Upsilon}{4}\sum_i Q_i \ln\left(\frac{\Lambda}{c_i\sigma}\right). \tag{6.30}$$

We will now argue that this is an exact result which receives no corrections from integrating out the high frequency modes of Σ. Previously, we described a classical R-symmetry under which σ transforms as

$$\sigma \to e^{-i\alpha}\sigma.$$

The RG invariant scale $\Lambda \to e^{-i\alpha}\Lambda$ under this classical symmetry, so that $\widetilde{W}_{eff}$ remains invariant. Now for $\frac{\sigma}{\Lambda} \to \infty$, $\widetilde{W}_{eff}$ must reduce to (6.30). This constrains the form of the effective superpotential

$$\widetilde{W}_{eff}(\Upsilon,\sigma,c_i) = -\frac{i\Upsilon}{4}\sum_i Q_i \ln(\frac{\Lambda}{c_i\sigma}) + \sum_{n>0} a_n(\frac{\Lambda}{\sigma})^n. \tag{6.31}$$

However, these corrections are non-perturbative in nature because of the positive powers of Λ. We have obtained the result simply by perturbatively integrating out the high frequency modes so there should not be any non-perturbative corrections to $\widetilde{W}_{eff}$. Hence all the a_n vanish. For Σ large and slowly varying, we obtain the low-energy effective superpotential of the original theory

$$\widetilde{W}_{eff}(\Upsilon,\Sigma) = \frac{i\Upsilon}{4}\left\{\sum_i Q_i \ln\left(\frac{c_i\Sigma}{\mu}\right) - it(\mu)\right\}. \tag{6.32}$$

Now consider the dual theory with the exact superpotential taking the form (6.11). On taking σ to be large and slowly varying, we see that the neutral superfields, Y_i and F_i, get masses of order $\frac{c_i\sigma}{\sqrt{r}}$. We can therefore integrate out the Y_i and F_i to get a low-energy effective superpotential $\widetilde{W}_{eff}(\Upsilon,\Sigma)$. Integrating out F_i teaches us that

$$\frac{\Sigma c_i}{\sqrt{2}} = -\mu\sum_j \beta_{ij}e^{-Y_j}. \tag{6.33}$$

On substituting the value of Y_i obtained from (6.33) in the dual superpotential, we get $\widetilde{W}_{eff}(\Upsilon,\Sigma)$. However, in general, we cannot solve (6.33) exactly for Y_i. Consider the case where the matrix $\mathcal{B}$ (with entries β_{ij}) is invertible ($\mathcal{B}^{-1}$ has entries β^{ij}). This is actually the case of interest in our example, but to show this requires an instanton analysis that we will temporarily postpone. Using the invertibility of $\mathcal{B}$, we find that

$$Y_i = -\ln\left\{\frac{-\Sigma}{\sqrt{2}\mu}\sum_j c_j\beta^{ij}\right\}. \tag{6.34}$$

This leads to the effective superpotential

$$\widetilde{W}_{eff}(\Upsilon,\Sigma) = \frac{i\Upsilon}{4}\left\{\sum_i Q_i \ln\left(\frac{-\Sigma}{\sqrt{2}\mu}\sum_j c_j\beta^{ij}\right) - it(\mu)\right\}. \tag{6.35}$$

On equating this with (6.32), we find a general constraint on the β_{ij} given by

$$\prod_i\left(\frac{-\sqrt{2}c_i}{\sum_j c_j\beta^{ij}}\right)^{Q_i} = 1. \tag{6.36}$$

Let us now show that the matrix $\mathcal{B}$ is actually diagonal. Consider a term $\beta_{ij}F_i e^{-Y_j}$ in the dual superpotential. If this term is non-zero, then the two point function

$$\langle\bar{\eta}_{-i}\xi_{+j}\rangle$$

must be non-zero. Using the duality maps, we obtain the relations, $\bar{\eta}_{-i} = -\bar{\phi}_i \chi_{-i}$ and $\xi_{+j} = 2\phi_j \bar{\psi}_{+i}$. We evaluate this two point function in the instanton background in the original theory. If $i \neq j$ then the correlator vanishes trivially since either ϕ_i or ϕ_j is zero. We therefore define $\beta_{ij} = \delta_{ij} \frac{\beta_i}{\sqrt{2}}$. From our assumption that the matrix $\mathcal{B}$ is invertible, we see that all the β_i are non-vanishing. The constraint (6.36) becomes

$$\prod_i (-\beta_i)^{Q_i} = 1. \tag{6.37}$$

Actually, it is possible to obtain the β_i from this constraint. To do this, we use the result obtained in [15] for $(2,2)$ theories. For $(2,2)$ theories, the dual theory has a non-perturbative superpotential

$$\widetilde{W}^{(2,2)}_{non-pert} = \mu \sum_i e^{-\widetilde{Y}_i}, \tag{6.38}$$

where $\widetilde{Y}_i$ is a neutral twisted chiral superfield satisfying $\bar{D}_+ \widetilde{Y}_i = D_- \widetilde{Y}_i = 0$. We simply reduce this to $(0,2)$ form

$$\widetilde{W}_{non-pert} = -\frac{\mu}{\sqrt{2}} \sum_i F_i e^{-Y_i}, \tag{6.39}$$

where $Y_i = \widetilde{Y}_i|_{\theta^- = \bar{\theta}^- = 0}$ and $-\sqrt{2}F_i = \bar{D}_- \widetilde{Y}_i|_{\theta^- = \bar{\theta}^- = 0}$. We have scaled μ suitably for notational convenience.

Consider a specific i, say $i = m$, and take $E_m = c_m \Sigma \Phi_m$ where c_m is arbitrary and non-zero. For all $i \neq m$, we take $E_i = \sqrt{2} Q_i \Sigma \Phi_i$, i.e., $c_i = \sqrt{2} Q_i$. Hence for all $i \neq m$ we have $\beta_i = -1$. So the constraint (6.37) gives us that

$$(-\beta_m)^{Q_m} = 1, \tag{6.40}$$

leading to

$$\beta_m = -e^{\frac{2\pi i k}{Q_m}} \tag{6.41}$$

where $k = 0, 1, \ldots, Q_m - 1$. Note that β_m is independent of c_m, so we can determine it by considering any non-zero value of c_m. For $c_m = \sqrt{2} Q_m$, we know from the $(2,2)$ result that $\beta_m = -1$ so this must be true for all values of c_m; hence $k = 0$. We can repeat this analysis for each β_i leading to the result

$$\beta_i = -1. \tag{6.42}$$

We therefore have the exact dual superpotential given by

$$\widetilde{W}_{exact} = -\frac{i\Upsilon}{4} (\sum_i Q_i Y_i + it) + \frac{\Sigma}{\sqrt{2}} \sum_i c_i F_i - \frac{\mu}{\sqrt{2}} \sum_i F_i e^{-Y_i}. \tag{6.43}$$

A few comments about the superpotential are in order. First, as mentioned above, the non-perturbative corrections to the superpotential are independent of c_i for any non-zero c_i. We might ask what happens as we take a particular $c_m \to 0$. In the original theory, the bundle degenerates. In the dual theory, this limit is singular because our procedure for arriving at the effective superpotential involved integrating out massive fields with masses of $O(c_m)$ in the original theory. In the dual theory, we integrated out Y_m and F_m with masses of $O(\frac{c_m \sigma}{\sqrt{r}})$. These fields become massless as $c_m \to 0$ so the integration procedure leads to singularities in the effective superpotential.

The effective superpotential (6.32) gives us information about the vacuum structure of the theory for large Σ. For large Σ, the charged heavy fields Φ_i and Γ_i are frozen at zero vacuum expectation value. As is standard, the potential energy of the theory is then given by

$$U = \frac{e^2 r^2}{2} + \frac{e^2}{2}(\frac{\tilde{\theta}}{2\pi})^2 = \frac{e^2}{2}|\tilde{t}|^2, \tag{6.44}$$

where $(\frac{\tilde{\theta}}{2\pi})^2$ is the minimum value of $(\frac{\theta}{2\pi} - n)^2$ for $n \in \mathbb{Z}$ [34]. In the expression for U, the first term comes from the D-term while the second term comes from the energy density generated by the θ-term. Here, $\tilde{t}$ (defined with appropriate shifts in $\frac{\theta}{2\pi}$ by integer amounts) is basically due to the FI-term in the Lagrangian

$$\frac{t}{4}\int d\theta^+ \ \Upsilon|_{\bar{\theta}^+=0} + \text{h.c.}$$

In the calculation above for the effective superpotential, we allowed Φ_i and Γ_i to fluctuate about their classical zero expectation values to take quantum effects into account. From (6.32), we see that this leads to a renormalization of t

$$U = \frac{e^2_{eff}}{2}|t_{eff}(\sigma)|^2, \tag{6.45}$$

where

$$t_{eff}(\sigma) = t(\mu) + i\sum_i Q_i \ln(\frac{c_i \sigma}{\mu}). \tag{6.46}$$

This can also be determined from the one-loop renormalization of t. The supersymmetric ground states of the theory for large Σ are then given by $t_{eff}(\sigma) = 0$ which has solutions,

$$\sigma^{\sum_i Q_i} = \frac{\mu^{\sum_i Q_i} e^{it(\mu)}}{\prod_i c_i^{Q_i}} = \frac{\Lambda^{\sum_i Q_i}}{\prod_i c_i^{Q_i}}. \tag{6.47}$$

Hence for large Σ, there are $|\sum_i Q_i|$ vacuum states labelled by

$$\sigma = \frac{\mu e^{\frac{it}{\sum_i Q_i}}}{(\prod_i c_i^{Q_i})^{\frac{1}{\sum_i Q_i}}} \times e^{\frac{2\pi i k}{\sum_i Q_i}}, \tag{6.48}$$

for $k = 0, 1, \ldots, \sum_i Q_i - 1$. For $(2, 2)$ theories where $c_i = \sqrt{2}Q_i$ for all i, we recover the relation

$$\sigma^{\sum_i Q_i} = \frac{\Lambda^{\sum_i Q_i}}{\prod_i(\sqrt{2}Q_i)^{Q_i}} \tag{6.49}$$

which is indeed true.

Let us turn to the $(0, 2)$ $\mathbb{P}^{N-1}$ model. For generic choices of c_i where we only have $(0, 2)$ supersymmetry in the UV, we find the relation

$$\sigma^N = \frac{\mu^N e^{it}}{\prod_i c_i} = \frac{\Lambda^N}{\prod_i c_i}, \tag{6.50}$$

which shows us that quantum cohomology ring is unchanged by the deformation modulo a numerical scaling. This is in accord with our expectation that this deformation is not a relevant one. The number of vacua is also unchanged with N vacua given by

$$\sigma = \frac{\mu e^{it/N}}{(\prod_i c_i)^{1/N}} \times e^{2\pi ik/N} \tag{6.51}$$

for $k = 0, \ldots, N - 1$.

<u>*A Direct Computation via Instantons*</u>

So far, we determined the superpotential by using symmetries, the effective superpotential, and the known $(2, 2)$ result. For the case $\sqrt{2}Q|c| = 1$ but a non-trivial phase, we can do better. In this case, the fermion zero modes can be explicitly constructed in a one instanton background, and a non-perturbative correction to the dual superpotential can be directly exhibited.

Let us return to the original case of one chiral and one Fermi field. Consider the Lagrangian

$$L = -\frac{i}{2}\int d^2\theta\ \bar{\Phi}(\mathcal{D}_0 - \mathcal{D}_1)\Phi - \frac{1}{2}\int d^2\theta\ \bar{\Gamma}\Gamma. \tag{6.52}$$

Using the duality map for the Fermi superfield, we see that

$$\bar{\eta}_- = -\frac{1}{\sqrt{2}}\bar{\phi}\chi_-,\ \eta_- = -\frac{1}{\sqrt{2}}\phi\bar{\chi}_-. \tag{6.53}$$

The dual theory has a perturbative superpotential given by

$$\widetilde{W} = -\frac{i\Upsilon}{4}(QY + it) + cQ\Sigma F. \tag{6.54}$$

Is there an Fe^{-Y} correction to the superpotential? As before, this implies that $\langle\bar{\eta}_-\xi_+\rangle$ should be non-zero, which we now argue directly is the case.

The Euclidean action of the original theory has vortex instantons for $\sigma = 0$. There are two fermion zero modes in this instanton background. The first is given by,

$$\mu^0 = \begin{pmatrix} \bar{\psi}_+^0 \\ \lambda_-^0 \end{pmatrix} \begin{pmatrix} -\sqrt{2}(\bar{D}_1 + i\bar{D}_2)\bar{\phi} \\ D - F_{12} \end{pmatrix} \tag{6.55}$$

and the second is,

$$\nu^0 = \begin{pmatrix} \chi_-^0 \\ \bar{\lambda}_+^0 \end{pmatrix} = \begin{pmatrix} -2Qc(D_1 - iD_2)\phi \\ D - F_{12} \end{pmatrix}. \tag{6.56}$$

The fact that $|c| = 1$ is necessary to show that the ν^0 zero mode is annihilated by the Dirac-Higgs operator. So,

$$\langle\bar{\eta}_-\xi_+\rangle \sim c \int d^2x_0 e^{-2\pi it}|\phi(D_1 - iD_2)\phi|^2 \tag{6.57}$$

which is clearly non-zero. Hence the exact superpotential is

$$\widetilde{W}_{exact} = -\frac{i\Upsilon}{4}(QY + it) + cQ\Sigma F + \beta\mu Fe^{-Y}, \tag{6.58}$$

where μ is the energy scale of the theory and β is a non-zero constant. Using our prior discussion, we see that β is independent of c and is given by $\beta = -\frac{1}{\sqrt{2}}$, which leads to the exact result

$$\widetilde{W}_{exact} = -\frac{i\Upsilon}{4}(QY + it) + cQ\Sigma F - \frac{\mu}{\sqrt{2}}Fe^{-Y}. \tag{6.59}$$

The Vacuum Structure

We can now directly analyze the vacuum structure of the $(0,2)$ $\mathbb{P}^{N-1}$ model with $E_i = c_i\Sigma\Phi_i$. Earlier from the large Σ analysis, we obtained N vacuua and the chiral ring relation (6.50). Using the dual theory, we show that these conclusions are indeed correct.

For the $\mathbb{P}^{N-1}$ model, $Q_i = 1$ for all i. The exact superpotential is given by (6.43). We will determine the vacua of this superpotential. Integrating out Υ gives the constraint

$$\sum_i Y_i = -it$$

which is solved by setting $Y_i = -\Theta_i$ (for $i = 1, \ldots, N-1$) and $Y_N = -it + \sum_{i=1}^{N-1} \Theta_i$. Each Θ_i is a periodic variable with period 2π. Integrating out Σ gives the constraint

$$\sum_i c_i F_i = 0$$

which is solved by $F_i = \frac{1}{c_i}\mathcal{G}_i$ (for $i = 1, \ldots, N-1$) and $F_N = -\frac{1}{c_N}\sum_{i=1}^{N-1} \mathcal{G}_i$. Finally, defining $X_i = e^{\Theta_i}$, we exhibit an effective superpotential

$$\widetilde{W}_{eff} = -\frac{\mu}{\sqrt{2}} \sum_{i=1}^{N-1} \mathcal{G}_i \left(\frac{X_i}{c_i} - \frac{e^{it}}{c_N X_1 \cdots X_{N-1}} \right). \tag{6.60}$$

We obtain the supersymmetric ground states by solving $\frac{\partial \widetilde{W}_{eff}}{\partial \mathcal{G}_i} = 0$ for all i. This gives us

$$\frac{X_1}{c_1} = \frac{X_2}{c_2} = \ldots = \frac{X_{N-1}}{c_{N-1}} = \frac{e^{it}}{c_N X_1 \cdots X_{N-1}}. \tag{6.61}$$

Also the linearity of $\widetilde{W}_{eff}$ in $\mathcal{G}_i$ sets $\widetilde{W}_{eff} = 0$. Setting $\frac{X_i}{c_i} = \frac{x}{\mu}$, we see that

$$x^N = \frac{\mu^N e^{it}}{\prod_i c_i} = \frac{\Lambda^N}{\prod_i c_i}, \tag{6.62}$$

which is the quantum cohomology ring (or chiral ring) relation for this theory. The vacuum states are given by

$$x = \frac{\mu e^{it/N}}{(\prod_i c_i)^{1/N}} \times e^{2\pi i k/N}, \tag{6.63}$$

for $k = 0, 1, \ldots, N-1$. There are indeed N supersymmetric vacua, which confirms that the large Σ analysis did capture all the vacuum states.

6.1.4 The Case of Equal and Opposite Charges

Next we consider a theory with one chiral superfield Φ of charge Q, and one Fermi superfield Γ of charge $-Q$. With these charge assignments, this theory is never $(2,2)$, but it is a consistent $(0,2)$ theory. Because Γ carries charge $-Q$, we see that E has to be zero in the theory. This is because the only possibility consistent with chirality and the charge assignments is $E \sim \frac{1}{\Phi}$ which is singular. So the theory described has the Lagrangian

$$L = -\frac{i}{2} \int d^2\theta \; \bar{\Phi}(\mathcal{D}_0 - \mathcal{D}_1)\Phi - \frac{1}{2} \int d^2\theta \; \bar{\Gamma}\Gamma. \tag{6.64}$$

The case of $E = 0$ is problematic for us since it corresponds to a singular choice of section. This model is simple enough that we can postulate a reasonable dual description as follows. We dualize only the chiral superfield, initially leaving the Fermi superfield untouched. In the dual theory, we find a neutral chiral superfield, Y, and a charged Fermi superfield Γ.

However, as we discussed earlier, it is difficult to study (and perhaps even define) the dual theory in terms of Y and Γ. So we proceed by constructing the dual in terms of Y, and a neutral Fermi superfield F. We will define F by

$$F = \Phi\Gamma$$

so that

$$\eta_- = \phi\chi_-. \tag{6.65}$$

Now the dual Lagrangian is

$$\widetilde{L} = \frac{1}{8}\int d^2\theta \left[\frac{i(Y-\bar{Y})}{Y+\bar{Y}}\partial_-(Y+\bar{Y}) - 8\frac{\bar{F}F}{Y+\bar{Y}}\right] - \left[\frac{iQ}{4}\int d\theta^+ Y\Upsilon + \text{h.c.}\right] \tag{6.66}$$

The perturbative dual superpotential is given by

$$\widetilde{W} = -\frac{i\Upsilon}{4}(QY + it). \tag{6.67}$$

We now consider the possibility of non-perturbative corrections to the dual superpotential of the usual form Fe^{-Y}. We can check if there is such a term by computing,

$$\langle \bar{\eta}_- \xi_+ \rangle \sim \int d^2x_0 |\phi|^2 \phi(\bar{D}_1 + i\bar{D}_2)\bar{\phi}. \tag{6.68}$$

To obtain this expression, we have used the $\bar{\psi}_+$ zero mode given by (6.55), and the $\bar{\chi}_-$ zero mode given by

$$\bar{\chi}_-^0 = \phi.$$

However, the integral vanishes using the identity (in Euclidean space)

$$2i\phi(\bar{D}_1 + i\bar{D}_2)\bar{\phi} + (\partial_1 + i\partial_2)(D - F_{12}) = 0. \tag{6.69}$$

So this non-perturbative correction is absent. Our conjectured dual superpotential is therefore

$$\widetilde{W}_{exact} = -\frac{i\Upsilon}{4}(QY + it), \tag{6.70}$$

leading to $\widetilde{W}_{eff} = 0$ with the constraint $QY = -it$. This is consistent with the original theory where there is a single supersymmetric vacuum with mass gap.

6.2 Relevant Deformations of $\mathbb{P}^1 \times \mathbb{P}^1$

We now want to construct the dual of a theory that admits non-trivial bundle deformations. As a particularly simple example, we take $\mathcal{M} = \mathbb{P}^1 \times \mathbb{P}^1$.[4] Deformations of the tangent bundle are parametrized by $H^1(\mathcal{M}, \mathrm{End}(T\mathcal{M}))$. In this case, the tangent bundle is a sum of line-bundles over each $\mathbb{P}^1$ which we denote

$$T\mathcal{M} = \mathcal{O}(2,0) \oplus \mathcal{O}(0,2).$$

The cohomology of $\mathrm{End}(T\mathcal{M}) = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-2,2) \oplus \mathcal{O}(2,-2)$ can be computed easily by using the Kunneth formula and the relations

$$H^1(\mathbb{P}^1, \mathcal{O}(-2)) = \mathbb{C}, \qquad H^0(\mathbb{P}^1, \mathcal{O}(2)) = \mathbb{C}^3. \tag{6.71}$$

Therefore $H^1(\mathcal{M}, \mathrm{End}(T\mathcal{M})) = \mathbb{C}^6$. We want to both realize these 6 deformations in a GLSM, and explicitly construct the dual description. This will allow us to solve for the instanton corrected chiral ring of the IR sigma model.

The Original Theory

To realize $\mathbb{P}^1 \times \mathbb{P}^1$, we need a GLSM with a $U(1)_1 \times U(1)_2$ gauge symmetry. The fields are

$$\Phi_1, \Phi_2, \widetilde{\Phi}_1, \widetilde{\Phi}_2, \Gamma_1, \Gamma_2, \widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \Sigma, \widetilde{\Sigma}.$$

The fields with charge 1 under $U(1)_1$ are Φ_1, Φ_2, Γ_1 and Γ_2, while the fields with charge 1 under $U(1)_2$ are $\widetilde{\Phi}_1, \widetilde{\Phi}_2, \widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$. Both Σ and $\widetilde{\Sigma}$ are neutral under both $U(1)$ factors. We take the following choices for E and $\widetilde{E}$

$$\begin{aligned}
E_1 &= \sqrt{2}\{\Phi_1\Sigma + \widetilde{\Sigma}(\alpha_1\Phi_1 + \alpha_2\Phi_2)\}, \\
E_2 &= \sqrt{2}\{\Phi_2\Sigma + \widetilde{\Sigma}(\alpha_1'\Phi_1 + \alpha_2'\Phi_2)\}, \\
\widetilde{E_1} &= \sqrt{2}\{\widetilde{\Phi}_1\widetilde{\Sigma} + \Sigma(\beta_1\widetilde{\Phi}_1 + \beta_2\widetilde{\Phi}_2)\}, \\
\widetilde{E_2} &= \sqrt{2}\{\widetilde{\Phi}_2\widetilde{\Sigma} + \Sigma(\beta_1'\widetilde{\Phi}_1 + \beta_2'\widetilde{\Phi}_2)\}.
\end{aligned} \tag{6.72}$$

Here $\alpha_i, \alpha_i', \beta_i, \beta_i'$ are complex parameters. Not all of these parameters correspond to independent deformations. Rescaling α_i, α_i' and β_i, β_i' independently by any non-zero complex number correspond to trivial deformations. These projective identifications leave us with the six degrees of freedom parametrizing deformations of $T\mathcal{M}$. Intuitively, these deformations couple the tangent bundles of each $\mathbb{P}^1$. Note that when the deformation parameters are taken to zero, we recover a $(2,2)$ GLSM.

[4] It is our pleasure to thank Sheldon Katz for suggesting this example, and describing the following computation of $H^1(\mathcal{M}, \mathrm{End}(T\mathcal{M}))$.

The vacuum solution of the GLSM is given by

$$\sum_i |\phi_i|^2 = r_1, \quad \sum_i |\widetilde{\phi}_i|^2 = r_2, \tag{6.73}$$

i.e., the product of $\mathbb{P}^1 \times \mathbb{P}^1$ with Kähler classes r_1 and r_2 respectively, and

$$E_i = \widetilde{E}_i = 0. \tag{6.74}$$

Generically, $E_i = \widetilde{E}_i = 0$ has a solution $\sigma = \widetilde{\sigma} = 0$. However there do exist vacuum solutions with $\sigma \neq 0$ and $\widetilde{\sigma} \neq 0$. These correspond to new branches in the moduli space of solutions. Typically, where these branches meet is extremely interesting since there is usually a singularity at the intersection locus which should be resolved in the full two-dimensional field theory. In this case, such a singularity must be a kind of bundle degeneration.

For example, let us construct a vacuum solution with

$$(\phi_1 = \sqrt{r}_1, \phi_2 = 0) \qquad (\widetilde{\phi}_1 = \sqrt{r}_2, \widetilde{\phi}_2 = 0).$$

Now we can have a solution with $\sigma \neq 0$ and $\widetilde{\sigma} \neq 0$ given by

$$\alpha_1' = \beta_1' = 0, \qquad \sigma = -\alpha_1 \widetilde{\sigma}, \qquad \alpha_1 \beta_1 = 1.$$

In this case, we see that Σ is proportional to $\widetilde{\Sigma}$, and from the analysis of the left-moving Yukawa couplings (which we described in section 5), we see that the rank of the bundle decreases by 1 instead of decreasing by 2 when Σ and $\widetilde{\Sigma}$ are linearly independent. This is in accord with our general expectations. Although these degeneration locii are fascinating, we will continue by considering the generic vacuum solution where $\Sigma = \widetilde{\Sigma} = 0$.

We now consider the massless fermionic degrees of freedom of the low-energy theory. Let the $U(1)_1 \times U(1)_2$ gauginos be λ_{-1} and λ_{-2}, respectively. From the Yukawa couplings for λ_{-1}, we see that the massless right-moving fermions satisfy

$$\sum_i \bar{\phi}_i \psi_{+i} = 0, \tag{6.75}$$

which we can interpret as a gauge fixing constraint as before. From the Yukawa couplings for λ_{-2}, we see that the massless right-moving fermions satisfy

$$\sum_i \bar{\widetilde{\phi}}_i \widetilde{\psi}_{+i} = 0, \tag{6.76}$$

which we again interpret as a gauge fixing constraint.

Let us denote the fermionic component field of Σ and $\widetilde{\Sigma}$ by $\bar{\lambda}_+$ and $\bar{\widetilde{\lambda}}_+$, respectively. From their Yukawa couplings, we see that the left-moving massless fermions satisfy

$$\sum_i \bar{\phi}_i \chi_{-i} + \widetilde{\chi}_{-1} \sum_i \bar{\beta}_i \bar{\widetilde{\phi}}_i + \widetilde{\chi}_{-2} \sum_i \bar{\beta}'_i \bar{\widetilde{\phi}}_i = 0 \tag{6.77}$$

and,

$$\sum_i \bar{\widetilde{\phi}}_i \widetilde{\chi}_{-i} + \chi_{-1} \sum_i \bar{\alpha}_i \bar{\phi}_i + \chi_{-2} \sum_i \bar{\alpha}'_i \bar{\phi}_i = 0. \tag{6.78}$$

These are again interpretable as gauge fixing constraints.

The Dual Description

Let us analyse the dual theory. The dual classical Lagrangian is given by

$$\begin{aligned}\widetilde{L} = \frac{i}{8} \sum_i \int d^2\theta \, \frac{Y_i - \bar{Y}_i}{Y_i + \bar{Y}_i} \partial_-(Y_i + \bar{Y}_i) + \frac{i}{8} \sum_i \int d^2\theta \, \frac{\widetilde{Y}_i - \bar{\widetilde{Y}}_i}{\widetilde{Y}_i + \bar{\widetilde{Y}}_i} \partial_-(\widetilde{Y}_i + \bar{\widetilde{Y}}_i) \\ -\frac{1}{2} \sum_i \int d^2\theta \, \bar{\mathcal{F}}_i \mathcal{F}_i - \frac{1}{2} \sum_i \int d^2\theta \, \bar{\widetilde{\mathcal{F}}}_i \widetilde{\mathcal{F}}_i + \int d\theta^+ \, \widetilde{W} + \text{h.c.}\end{aligned} \tag{6.79}$$

where

$$\widetilde{W} = -\frac{i\Upsilon_1}{4}(\sum_i Y_i + it_1) - \frac{i\Upsilon_2}{4}(\sum_i \widetilde{Y}_i + it_2) + \frac{1}{\sqrt{2}} \sum_i E_i \mathcal{F}_i + \frac{1}{\sqrt{2}} \sum_i \widetilde{E}_i \widetilde{\mathcal{F}}_i. \tag{6.80}$$

Here the $\mathcal{F}_i, \widetilde{\mathcal{F}}_i$ are charged Fermi superfields. The duality maps (modulo fermion bilinears) for the bosonic superfields are

$$\bar{\Phi}_i \Phi_i = \frac{1}{2}(Y_i + \bar{Y}_i), \qquad \bar{\Phi}_i (\overleftrightarrow{\partial}_- + iV_1) \Phi_i = -\frac{1}{4} \partial_-(Y_i - \bar{Y}_i), \tag{6.81}$$

$$\bar{\widetilde{\Phi}}_i \widetilde{\Phi}_i = \frac{1}{2}(\widetilde{Y}_i + \bar{\widetilde{Y}}_i), \qquad \bar{\widetilde{\Phi}}_i (\overleftrightarrow{\partial}_- + iV_2) \widetilde{\Phi}_i = -\frac{1}{4} \partial_-(\widetilde{Y}_i - \bar{\widetilde{Y}}_i), \tag{6.82}$$

while the fermionic superfields map according to,

$$\bar{\Gamma}_i = \mathcal{F}_i, \quad \bar{\widetilde{\Gamma}}_i = \widetilde{\mathcal{F}}_i.$$

The dual Fermi superpotential term in the action can be written as

$$\int d\theta^+ \, \Sigma(F_1 + F_2) + \int d\theta^+ \, \widetilde{\Sigma}(\widetilde{F}_1 + \widetilde{F}_2) + \text{h.c.} \tag{6.83}$$

where

$$
\begin{aligned}
F_1 &= \Phi_1\mathcal{F}_1 + (\beta_1\widetilde{\Phi}_1 + \beta_2\widetilde{\Phi}_2)\widetilde{\mathcal{F}}_1, \\
F_2 &= \Phi_2\mathcal{F}_2 + (\beta_1'\widetilde{\Phi}_1 + \beta_2'\widetilde{\Phi}_2)\widetilde{\mathcal{F}}_2, \\
\widetilde{F_1} &= \widetilde{\Phi}_1\widetilde{\mathcal{F}}_1 + (\alpha_1\Phi_1 + \alpha_2\Phi_2)\mathcal{F}_1, \\
\widetilde{F_2} &= \widetilde{\Phi}_2\widetilde{\mathcal{F}}_2 + (\alpha_1'\Phi_1 + \alpha_2'\Phi_2)\mathcal{F}_2,
\end{aligned}
\tag{6.84}
$$

where $F_i, \widetilde{F_i}$ are neutral Fermi superfields. Note that there is no unique way of defining F in terms of $\mathcal{F}$, but there is a natural choice given in (6.84). With this choice, Σ only couples to F while $\widetilde{\Sigma}$ only couples to $\widetilde{F}$ in the superpotential (6.83).

It is worth noting that the kinetic terms for the dual neutral Fermi superfields are not singular, even for field configurations that correspond to instantons in the original theory. To see this, we consider generic deformations of the left-moving bundle given in (6.72). We can solve for $\mathcal{F}, \widetilde{\mathcal{F}}$ in terms of $F, \widetilde{F}$ and $\Phi, \widetilde{\Phi}$

$$
\begin{aligned}
\mathcal{F}_1 &= \tfrac{\widetilde{\Phi}_1 F_1 - A\widetilde{F}_1}{\widetilde{\Phi}_1\Phi_1 - AC}, &\qquad \mathcal{F}_2 &= \frac{\widetilde{\Phi}_2 F_2 - B\widetilde{F}_2}{\widetilde{\Phi}_2\Phi_2 - BD}, \\
\widetilde{\mathcal{F}}_1 &= \tfrac{F_1 - \Phi_1\mathcal{F}_1}{A}, &\qquad \widetilde{\mathcal{F}}_2 &= \frac{F_2 - \Phi_2\mathcal{F}_2}{B},
\end{aligned}
$$

where $A = \beta_1\widetilde{\Phi}_1 + \beta_2\widetilde{\Phi}_2,\ B = \beta_1'\widetilde{\Phi}_1 + \beta_2'\widetilde{\Phi}_2,\ C = \alpha_1\Phi_1 + \alpha_2\Phi_2$ and $D = \alpha_1'\Phi_1 + \alpha_2'\Phi_2$. So for generic choices of the parameters, all the denominators are non-vanishing, even in instanton backgrounds. Consider embedding an instanton in ϕ_1 (or ϕ_2) and $\widetilde{\phi}_1$ (or $\widetilde{\phi}_2$), then it is easy to see that A, B, C, D are each non-vanishing.

Note that on the degeneration locus described before where Σ is proportional to $\widetilde{\Sigma}$, we find that

$$
\widetilde{\Phi}_1\Phi_1 - AC = 0, \qquad B = \widetilde{\Phi}_2\Phi_2 = 0. \tag{6.85}
$$

Only A is non-zero and equal to $\beta_1\widetilde{\Phi}_1$. This leads to singular kinetic energy terms which is natural for a singular locus.

We therefore obtain the exact dual superpotential

$$
\begin{aligned}
\widetilde{W} \;=\; & -\frac{i\Upsilon_1}{4}(\sum_i Y_i + it_1) - \frac{i\Upsilon_2}{4}(\sum_i \widetilde{Y}_i + it_2) + \Sigma\sum_i F_i + \widetilde{\Sigma}\sum_i \widetilde{F}_i \\
& +\mu\sum_{ij}\left(\beta_{ij}F_i e^{-Y_j} + \beta_{\tilde{i}\tilde{j}}\widetilde{F}_i e^{-\widetilde{Y}_j} + \beta_{i\tilde{j}}F_i e^{-\widetilde{Y}_j} + \beta_{\tilde{i}j}\widetilde{F}_i e^{-Y_j}\right).
\end{aligned}
\tag{6.86}
$$

The duality map for the Fermi superfields is given by

$$
\begin{aligned}
F_1 &= \Phi_1\bar{\Gamma}_1 + (\beta_1\widetilde{\Phi}_1 + \beta_2\widetilde{\Phi}_2)\bar{\widetilde{\Gamma}}_1,\\
F_2 &= \Phi_2\bar{\Gamma}_2 + (\beta_1'\widetilde{\Phi}_1 + \beta_2'\widetilde{\Phi}_2)\bar{\widetilde{\Gamma}}_2,\\
\widetilde{F_1} &= \widetilde{\Phi}_1\bar{\widetilde{\Gamma}}_1 + (\alpha_1\Phi_1 + \alpha_2\Phi_2)\bar{\Gamma}_1,\\
\widetilde{F_2} &= \widetilde{\Phi}_2\bar{\widetilde{\Gamma}}_2 + (\alpha_1'\Phi_1 + \alpha_2'\Phi_2)\bar{\Gamma}_2.
\end{aligned}
\tag{6.87}
$$

Our task is to relate the β parameters to the original bundle deformation parameters given in (6.72). The difficulty in determining this map is easy to explain. The β parameters are determined by instanton computations in the original theory. In an instanton background, the right-moving fermion zero modes can be determined exactly. However, the left-moving zero modes depend sensitively on the choice of $E, \widetilde{E}$ given in (6.72). To determine the β parameters, we need to be able to evaluate exactly instanton corrections to various two point functions in the original theory. This is a hard task so we will need to be more clever.

<u>*The Vacuum Structure*</u>

Before determining the parameter map, let us examine the general vacuum structure for the dual theory. Integrating out the massive field strength multiplets, $\Upsilon, \widetilde{\Upsilon}$, we obtain the constraint

$$
Y_1 + Y_2 = -it_1, \qquad \widetilde{Y}_1 + \widetilde{Y}_2 = -it_2. \tag{6.88}
$$

On integrating out the massive Σ and $\widetilde{\Sigma}$ fields we find

$$
F_1 + F_2 = 0, \qquad \widetilde{F}_1 + \widetilde{F}_2 = 0. \tag{6.89}
$$

We solve these constraints by setting

$$
Y_1 = Y, \qquad Y_2 = -Y - it_1, \qquad F_1 = -F_2 = F, \tag{6.90}
$$

and

$$
\widetilde{Y}_1 = \widetilde{Y}, \qquad \widetilde{Y}_2 = -\widetilde{Y} - it_2, \qquad \widetilde{F}_1 = -\widetilde{F}_2 = \widetilde{F}. \tag{6.91}
$$

Recall that the imaginary parts of the $Y, \widetilde{Y}$ variables are periodic. Let us define the low-energy theory in terms of single-valued degrees of freedom X and $\widetilde{X}$ where

$$
X = e^{-Y}, \qquad \widetilde{X} = e^{-\widetilde{Y}}.
$$

In terms of these variables,

$$\begin{aligned}\mu^{-1}\widetilde{W}_{eff} &= F\left[X(\beta_{11}-\beta_{21})+\frac{e^{it_1}}{X}(\beta_{12}-\beta_{22})\right.\\ &\qquad\left.+\widetilde{X}(\beta_{1\tilde{1}}-\beta_{2\tilde{1}})+\frac{e^{it_2}}{\widetilde{X}}(\beta_{1\tilde{2}}-\beta_{2\tilde{2}})\right]\\ &\quad+\widetilde{F}\left[\widetilde{X}(\beta_{\tilde{1}\tilde{1}}-\beta_{\tilde{2}\tilde{1}})+\frac{e^{it_2}}{\widetilde{X}}(\beta_{\tilde{1}\tilde{2}}-\beta_{\tilde{2}\tilde{2}})\right.\\ &\qquad\left.+X(\beta_{\tilde{1}1}-\beta_{\tilde{2}1})+\frac{e^{it_1}}{X}(\beta_{\tilde{1}2}-\beta_{\tilde{2}2})\right].\end{aligned} \tag{6.92}$$

Because we deformed the bundle for the left-movers, the chiral ring of the IR (or low-energy) theory is deformed. This will define our analogue of the usual quantum cohomology ring of $(2,2)$ theories.

In order to construct the chiral ring, we set

$$\frac{\partial\widetilde{W}_{eff}}{\partial F}=\frac{\partial\widetilde{W}_{eff}}{\partial\widetilde{F}}=0$$

from which we obtain the deformed chiral ring relations

$$X+p\frac{e^{it_1}}{X}+q\widetilde{X}+s\frac{e^{it_2}}{\widetilde{X}}=0, \tag{6.93}$$

and

$$\widetilde{X}+\widetilde{p}\frac{e^{it_2}}{\widetilde{X}}+\widetilde{q}X+\widetilde{s}\frac{e^{it_1}}{X}=0. \tag{6.94}$$

In these equations,

$$p=\frac{\beta_{12}-\beta_{22}}{\beta_{11}-\beta_{21}},\quad q=\frac{\beta_{1\tilde{1}}-\beta_{2\tilde{1}}}{\beta_{11}-\beta_{21}},\quad s=\frac{\beta_{1\tilde{2}}-\beta_{2\tilde{2}}}{\beta_{11}-\beta_{21}}, \tag{6.95}$$

and

$$\widetilde{p}=\frac{\beta_{\tilde{1}\tilde{2}}-\beta_{\tilde{2}\tilde{2}}}{\beta_{\tilde{1}\tilde{1}}-\beta_{\tilde{2}\tilde{1}}},\quad \widetilde{q}=\frac{\beta_{\tilde{1}1}-\beta_{\tilde{2}1}}{\beta_{\tilde{1}\tilde{1}}-\beta_{\tilde{2}\tilde{1}}},\quad \widetilde{s}=\frac{\beta_{\tilde{1}2}-\beta_{\tilde{2}2}}{\beta_{\tilde{1}\tilde{1}}-\beta_{\tilde{2}\tilde{1}}}. \tag{6.96}$$

So the $(0,2)$ chiral ring relations mix the generators of the chiral ring for each $\mathbb{P}^1$; these generators correspond to the Kähler classes of each $\mathbb{P}^1$. The mixing occurs because we have deformed the bundle for the left-movers away from the tangent bundle (in a holomorphic way).

In the limit in which the bundle deformations vanish, we should recover two decoupled chiral rings; one for each $\mathbb{P}^1$. It is easy to see that this is true. As the bundle deformations vanish, we recover $(2,2)$ supersymmetry and only the diagonal β parameters survive giving

$$p=-1,\quad q=s=0,\qquad \widetilde{p}=-1,\quad \widetilde{q}=\widetilde{s}=0. \tag{6.97}$$

Therefore, we find a decoupled ring

$$X^2 = e^{it_1}, \qquad \widetilde{X}^2 = e^{it_2}, \tag{6.98}$$

for each $\mathbb{P}^1$ as we expect.

<u>*Determining the Exact Parameter Map*</u>

We now want to solve this theory completely by determining the exact parameter map. We want to know how the β parameters depend on $\alpha_i, \alpha_i', \beta_i, \beta_i'$. Our tools for this task will be global $U(1)$ symmetries and a large $\Sigma, \widetilde{\Sigma}$ analysis of the kind described in section 6.1.3. The strategy in constructing a $U(1)$ global symmetry is to assign suitable $U(1)$ charges to the various superfields as well as to the deformation parameters. This $U(1)$ is, in general, anomalous. In the dual theory, the $U(1)$ acts by shifting the $Y_i, \widetilde{Y}_i$ fields, and the anomaly is realized by a non-invariant term in the perturbative dual superpotential. This is exactly analogous to the case of the R-symmetry. If the β parameters are charged under the global $U(1)$, we can use the symmetry to constrain their dependence on the deformation parameters.

However, we now show that unless some of the deformation parameters are set to zero, no choice of $U(1)$ symmetry will help us fix the β parameters. To see this, let us go back to the definitions given in (6.72) to make charge assignments. Assign the superfields the following charges

$$(\Phi_1, p_1), \quad (\Phi_2, p_2), \quad (\widetilde{\Phi}_1, \widetilde{p}_1), \quad (\widetilde{\Phi}_2, \widetilde{p}_2), \quad (\Sigma, k), \quad (\widetilde{\Sigma}, \widetilde{k})$$

where, for example, Φ_1 has charge p_1. We then see that the deformation parameters have the following charges:

$$(\alpha_1, \alpha_2', k - \widetilde{k}), \quad (\alpha_2, k - \widetilde{k} + p_1 - p_2), \quad (\alpha_1', k - \widetilde{k} - p_1 + p_2),$$

$$(\beta_1, \beta_2', \widetilde{k} - k), \quad (\beta_2, \widetilde{k} - k + \widetilde{p}_1 - \widetilde{p}_2), \quad (\beta_1', \widetilde{k} - k - \widetilde{p}_1 + \widetilde{p}_2).$$

So in particular, we see that arbitrary powers of

$$\alpha_1\beta_1, \quad \alpha_1\beta_2', \quad \alpha_2'\beta_1, \quad \alpha_2'\beta_2'$$

carry zero charge. The β parameters could depend on these combinations in arbitrary ways.

Let us therefore set some deformation parameters to zero in order to usefully employ global $U(1)$ symmetries. In (6.72), we take $\alpha_1 = \epsilon_1$ and

$\alpha'_2 = \epsilon_2$ and set all other deformation parameters to zero. Thus we start with

$$\begin{aligned} E_1 &= \sqrt{2}(\Phi_1\Sigma + \epsilon_1\widetilde{\Sigma}\Phi_1), \qquad & \widetilde{E_1} &= \sqrt{2}\widetilde{\Phi}_1\widetilde{\Sigma}, \\ E_2 &= \sqrt{2}(\Phi_2\Sigma + \epsilon_2\widetilde{\Sigma}\Phi_2), \qquad & \widetilde{E_2} &= \sqrt{2}\widetilde{\Phi}_2\widetilde{\Sigma}. \end{aligned} \tag{6.99}$$

This choice gives the following expressions for the dual fermions

$$\begin{aligned} F_1 &= \Phi_1\bar{\Gamma}_1, \qquad & \widetilde{F_1} &= \widetilde{\Phi}_1\bar{\widetilde{\Gamma}}_1 + \epsilon_1\Phi_1\bar{\Gamma}_1, \\ F_2 &= \Phi_2\bar{\Gamma}_2, \qquad & \widetilde{F_2} &= \widetilde{\Phi}_2\bar{\widetilde{\Gamma}}_2 + \epsilon_2\Phi_2\bar{\Gamma}_2. \end{aligned} \tag{6.100}$$

The exact dual superpotential is given by (6.86). We assign global $U(1)$ charges as discussed above. (Note that ϵ_1 and ϵ_2 have the same charge $k - \widetilde{k}$.) So the terms ΣF_i and $\widetilde{\Sigma}\widetilde{F_i}$ in the dual perturbative superpotential are charge zero. However, this $U(1)$ symmetry is anomalous: the $U(1)_1$ gauge symmetry shifts $\sum_i Y_i$ by $-2k$, while the $U(1)_2$ gauge symmetry shifts $\sum_i \widetilde{Y}_i$ by $-2\widetilde{k}$. However, this does not tell us the amount by which each individual Y_i or $\widetilde{Y}_i$ shifts under the anomaly. The individual shifts can be determined from the duality maps if we know the complete maps including the fermion bilinear terms in (6.81) and (6.82).

In the limit in which the deformations vanish, we have a $(2,2)$ theory with

$$\beta_{11} = \beta_{22} = \beta_{\tilde{1}\tilde{1}} = \beta_{\tilde{2}\tilde{2}} = -\frac{1}{\sqrt{2}} \tag{6.101}$$

with all other β parameters vanishing. From the $U(1)$ invariance of

$$F_1 e^{-Y_1}, \quad F_2 e^{-Y_2}, \quad \widetilde{F}_1 e^{-\widetilde{Y}_1}, \quad \widetilde{F}_2 e^{-\widetilde{Y}_2},$$

we see that e^{-Y_i} has $U(1)$ charge k while $e^{-\widetilde{Y}_i}$ has $U(1)$ charge $\widetilde{k}$. In this way, we determine the individual shifts of the $Y_i, \widetilde{Y}_i$ fields without knowing the fermion bilinear terms in the duality map.

We can now determine the $U(1)$ charges for the remaining β parameters. The parameter $\beta_{i\tilde{j}}$ has charge $k - \widetilde{k}$ while $\beta_{\tilde{i}j}$ has charge $\widetilde{k} - k$. Because the β parameters depend smoothly on the deformation parameters, we conclude that $\beta_{i\tilde{j}}$ is proportional to ϵ_1 or ϵ_2, while $\beta_{\tilde{i}j}$ is zero. This is also fixes the diagonal β parameters at their $(2,2)$ value given in (6.101). We therefore find,

$$\begin{aligned} \widetilde{W}_{non-pert} = & -\frac{\mu}{\sqrt{2}}\sum_i (F_i e^{-Y_i} + \widetilde{F}_i e^{-\widetilde{Y}_i}) \\ & -\frac{\epsilon_1\mu}{\sqrt{2}}(c_1 F_1 e^{-\widetilde{Y}_1} + c_2 F_1 e^{-\widetilde{Y}_2} + c_3 F_2 e^{-\widetilde{Y}_1} + c_4 F_2 e^{-\widetilde{Y}_2}) \\ & -\frac{\epsilon_2\mu}{\sqrt{2}}(d_1 F_1 e^{-\widetilde{Y}_1} + d_2 F_1 e^{-\widetilde{Y}_2} + d_3 F_2 e^{-\widetilde{Y}_1} + d_4 F_2 e^{-\widetilde{Y}_2}). \end{aligned} \tag{6.102}$$

The particular deformation we are considering does not distinguish between $\widetilde{Y}_1$ and $\widetilde{Y}_2$. There is also an obvious $\mathbb{Z}_2$ symmetry exchanging ϵ_1 and ϵ_2, and all the 1 and 2 fields. Together, these symmetries imply

$$c_1 = c_2 = d_3 = d_4 \equiv \frac{a}{2}, \qquad c_3 = c_4 = d_1 = d_2 \equiv \frac{b}{2}.$$

Thus,

$$\begin{aligned}\widetilde{W}_{non-pert} = & -\frac{\mu}{\sqrt{2}} \sum_i (F_i e^{-Y_i} + \widetilde{F}_i e^{-\widetilde{Y}_i}) \\ & -\frac{\mu}{2\sqrt{2}} \left[\epsilon_1 (aF_1 + bF_2) + \epsilon_2 (bF_1 + aF_2)\right] \left(e^{-\widetilde{Y}_1} + e^{-\widetilde{Y}_2}\right)\end{aligned} \tag{6.103}$$

Here a and b are numbers which we now evaluate. These numbers can be evaluated using the large $\Sigma, \widetilde{\Sigma}$ approach along the lines of section 6.1.3, so we shall be brief. In the original theory, take $\Sigma, \widetilde{\Sigma}$ to be large and slowly varying. Integrate out the chiral and Fermi superfields exactly; since the Lagrangian is quadratic, we can do this exactly giving an effective action

$$\widetilde{W}_{eff}(\Sigma, \widetilde{\Sigma}, \Upsilon_1, \Upsilon_2) = \Upsilon_1 W_1(\Sigma, \widetilde{\Sigma}) + \Upsilon_2 W_2(\Sigma, \widetilde{\Sigma}). \tag{6.104}$$

This superpotential gives terms in the action

$$\begin{aligned}\frac{1}{4}\int d\theta^+ \widetilde{W}_{eff}(\Sigma, \widetilde{\Sigma}, \Upsilon_1, \Upsilon_2) + \text{h.c.} = & -D_1 \text{Im}\{W_1(\sigma, \widetilde{\sigma})\} - D_2 \text{Im}\{W_2(\sigma, \widetilde{\sigma})\} \\ & + F_{01} \text{Re}\{W_1(\sigma, \widetilde{\sigma})\} + \widetilde{F}_{01} \text{Re}\{W_2(\sigma, \widetilde{\sigma})\} + \ldots,\end{aligned} \tag{6.105}$$

where D_1, D_2 $(F_{01}, \widetilde{F}_{01})$ are the D terms (field strengths) for $U(1)_1$ and $U(1)_2$, respectively. We have only included terms that are linear in the D_i fields, and in the field strengths. In order to determine $W_1(\Sigma, \widetilde{\Sigma})$ and $W_2(\Sigma, \widetilde{\Sigma})$, we only need to retain terms linear in the D_i fields and the field strengths. It turns out that there are no terms linear in the field strengths so the entire contribution comes from terms linear in the D_i fields. The calculation is very similar to the one in section 6.1.3, giving the result

$$\begin{aligned}\widetilde{W}_{eff}(\Sigma, \widetilde{\Sigma}, \Upsilon_1, \Upsilon_2) = & \frac{i\Upsilon_1}{4}\Big\{ \sum_i \ln\left[\frac{\sqrt{2}(\Sigma + \epsilon_i \widetilde{\Sigma})}{\mu}\right] - it_1 \Big\} \\ & + \frac{i\Upsilon_2}{4}\Big\{ 2\ln\left[\frac{\sqrt{2}\widetilde{\Sigma}}{\mu}\right] - it_2 \Big\}.\end{aligned} \tag{6.106}$$

Now we proceed to the dual theory and integrate out F_i and $\widetilde{F}_i$. It is easy to solve for $Y_i, \widetilde{Y}_i$ from the four resulting equations

$$Y_1 = -\ln\left[\frac{\sqrt{2}(\Sigma - (a\epsilon_1 + b\epsilon_2)\widetilde{\Sigma})}{\mu}\right], \quad Y_2 = -\ln\left[\frac{\sqrt{2}(\Sigma - (b\epsilon_1 + a\epsilon_2)\widetilde{\Sigma})}{\mu}\right], \tag{6.107}$$

$$\widetilde{Y}_1 = \widetilde{Y}_2 = -\ln\left[\frac{\sqrt{2}\widetilde{\Sigma}}{\mu}\right].$$

Thus in the dual theory, we get

$$\begin{aligned}
&\widetilde{W}_{eff}(\Sigma, \widetilde{\Sigma}, \Upsilon_1, \Upsilon_2) = \\
&\frac{i\Upsilon_1}{4}\Big\{\ln\left[\frac{\sqrt{2}(\Sigma - (a\epsilon_1 + b\epsilon_2)\widetilde{\Sigma})}{\mu}\right] + \ln\left[\frac{\sqrt{2}(\Sigma - (b\epsilon_1 + a\epsilon_2)\widetilde{\Sigma})}{\mu}\right] - it_1\Big\} \\
&\quad + \frac{i\Upsilon_2}{4}\Big\{2\ln\left[\frac{\sqrt{2}\widetilde{\Sigma}}{\mu}\right] - it_2\Big\}. \qquad (6.108)
\end{aligned}$$

Equating coefficients in (6.106) and (6.108) gives the relation

$$(\Sigma + \epsilon_1\widetilde{\Sigma})(\Sigma + \epsilon_2\widetilde{\Sigma}) = (\Sigma - \{a\epsilon_1 + b\epsilon_2\}\widetilde{\Sigma})(\Sigma - \{b\epsilon_1 + a\epsilon_2\}\widetilde{\Sigma}). \qquad (6.109)$$

Equating the coefficients of $\Sigma\widetilde{\Sigma}$ and $\widetilde{\Sigma}^2$ gives two equations from which we determine a and b. There are two solutions given by (i) $a = 0, b = -1$ and (ii) $a = -1, b = 0$. In the first case,

$$\widetilde{W}_{non-pert} = -\frac{\mu}{\sqrt{2}}\sum_i(F_i e^{-Y_i} + \widetilde{F}_i e^{-\widetilde{Y}_i}) + \frac{\mu}{2\sqrt{2}}(\epsilon_1 F_2 + \epsilon_2 F_1)(e^{-\widetilde{Y}_1} + e^{-\widetilde{Y}_2}), \qquad (6.110)$$

while in the second case,

$$\widetilde{W}_{non-pert} = -\frac{\mu}{\sqrt{2}}\sum_i(F_i e^{-Y_i} + \widetilde{F}_i e^{-\widetilde{Y}_i}) + \frac{\mu}{2\sqrt{2}}(\epsilon_1 F_1 + \epsilon_2 F_2)(e^{-\widetilde{Y}_1} + e^{-\widetilde{Y}_2}). \qquad (6.111)$$

Note that the two superpotentials explicitly exhibit the symmetry of the theory under interchange of ϵ_1 and ϵ_2. Using (6.90) and (6.91), we obtain the chiral ring relations

$$\widetilde{X} = \frac{e^{it_2}}{\widetilde{X}}, \qquad X - \frac{e^{it_1}}{X} \pm (\epsilon_1 - \epsilon_2)\widetilde{X} = 0, \qquad (6.112)$$

where the $\pm$ is corresponds to (6.110) and (6.111), respectively. Note that this sign ambiguity in the ring relation has no physical meaning because (ϵ_1, ϵ_2) are projective coordinates, and can be freely rescaled by any non-zero complex number.

Since $\widetilde{E}_1$ and $\widetilde{E}_2$ are at their $(2, 2)$ values, the chiral ring relation for the $\mathbb{P}^1$ corresponding to $U(1)_2$ is undeformed. The other ring for the $\mathbb{P}^1$ corresponding to $U(1)_1$ is deformed because E_1 and E_2 involve $\widetilde{\Sigma}$ couplings. This is an example of a non-trivial bundle deformation where we have explicitly solved for the dual superpotential, and determined the chiral ring. It should be possible to directly compute this ring by studying instantons in the IR $(0,2)$ non-linear sigma model along the lines described in section 5. Lastly, note that for $\epsilon_1 = \epsilon_2$, the ring relations remain undeformed and correspond to two decoupled $\mathbb{P}^1$ spaces.

6.3 Examples of Conformal Models

Next we consider conformal cases where the total space is a non-compact Calabi-Yau manifold. The two examples that we consider are the total spaces of bundles over $\mathbb{P}^1 \times \mathbb{P}^1$, with the bundles suitably chosen so that the models are conformal. We continue to use the same notation of section 6.2 for the fields of the $\mathbb{P}^1 \times \mathbb{P}^1$ GLSM. In our first example, the dual IR theory is a $\mathbb{Z}_2$ Landau-Ginzburg (LG) orbifold, while in our second example, the dual is a $(\mathbb{Z}_2)^2$ LG orbifold.

6.3.1 A Uniquely $(0,2)$ Example

To the fields of the $\mathbb{P}^1 \times \mathbb{P}^1$ GLSM described in the last section, we add a chiral superfield P and a Fermi superfield Γ. Both P and Γ carry charge -2 under both $U(1)_1$ and $U(1)_2$. Since the sum of the charges for the right-movers is zero, the model is conformal: the IR theory is a non-linear sigma model on a non-compact Calabi-Yau space. The target space is the total space of the line-bundle $\mathcal{O}(-2,-2)$ over $\mathbb{P}^1 \times \mathbb{P}^1$.

We keep the same choice of E_i, $\widetilde{E}_i$ as in (6.72). For the E associated to Γ, we take the choice

$$E = -2\sqrt{2}(\Sigma + \widetilde{\Sigma})P. \tag{6.113}$$

Note that with this choice of E, this model never enjoys $(2,2)$ supersymmetry; hence the title of this section. The vacuum solution of the GLSM is given by

$$\sum_i |\phi_i|^2 - 2|p|^2 = r_1, \qquad \sum_i |\widetilde{\phi}_i|^2 - 2|p|^2 = r_2, \tag{6.114}$$

and

$$E_i = \widetilde{E}_i = E = 0. \tag{6.115}$$

Once again, we choose the generic vacuum solution $\Sigma = \widetilde{\Sigma} = 0$. Now because of the presence of the superfield P, the two D term equations for the vacuum solution are no longer decoupled from each other.

Let us define

$$P = p + \sqrt{2}\theta^+\psi_+ + \ldots, \qquad \Gamma = \chi_- + \ldots.$$

From the various Yukawa couplings, we see that the massless fermionic degrees of freedom of the low-energy theory satisfy

$$\begin{aligned}
&\sum_i \bar{\phi}_i \psi_{+i} - 2\bar{p}\psi_+ = 0, \qquad \sum_i \bar{\widetilde{\phi}}_i \widetilde{\psi}_{+i} - 2\bar{p}\psi_+ = 0,\\
&\sum_i \bar{\phi}_i \chi_{-i} + \widetilde{\chi}_{-1} \sum_i \bar{\beta}_i \bar{\widetilde{\phi}}_i + \widetilde{\chi}_{-2} \sum_i \bar{\beta}'_i \bar{\widetilde{\phi}}_i - 2\bar{p}\chi_- = 0\\
&\sum_i \bar{\widetilde{\phi}}_i \widetilde{\chi}_{-i} + \chi_{-1} \sum_i \bar{\alpha}_i \bar{\phi}_i + \chi_{-2} \sum_i \bar{\alpha}'_i \bar{\phi}_i - 2\bar{p}\chi_- = 0.
\end{aligned} \tag{6.116}$$

In the dual theory, the classical Lagrangian has the Kähler terms given in the $\mathbb{P}^1 \times \mathbb{P}^1$ example along with the following additional terms

$$\widetilde{L} = \ldots + \frac{i}{8}\int d^2\theta \, \frac{Y - \bar{Y}}{Y + \bar{Y}} \partial_-(Y + \bar{Y}) - \frac{1}{2}\int d^2\theta \, \bar{\mathcal{F}}\mathcal{F}, \tag{6.117}$$

where we have the duality map (again, modulo fermion bilinears)

$$\bar{P}P \;=\; \frac{1}{2}(Y + \bar{Y}), \qquad \bar{P}(\overleftrightarrow{\partial}_- - 2iV_1 - 2iV_2)P = -\frac{1}{4}\partial_-(Y - \bar{Y}), \tag{6.118}$$

and

$$\bar{\Gamma} = \mathcal{F}. \tag{6.119}$$

The classical dual superpotential is given by

$$\begin{aligned}
\widetilde{W} = &- \frac{i\Upsilon_1}{4}(\sum_i Y_i - 2Y + it_1) - \frac{i\Upsilon_2}{4}(\sum_i \widetilde{Y}_i - 2Y + it_2)\\
&- \frac{1}{\sqrt{2}}(\sum_i E_i \mathcal{F}_i + \sum_i \widetilde{E}_i \widetilde{\mathcal{F}}_i + E\mathcal{F}).
\end{aligned} \tag{6.120}$$

The last term can be written in the form

$$-\int d\theta^+ \, \Sigma(F_1 + F_2 - 2F) - \int d\theta^+ \, \widetilde{\Sigma}(\widetilde{F}_1 + \widetilde{F}_2 - 2F), \tag{6.121}$$

where $F = P\mathcal{F}$. The exact dual superpotential is therefore given by

$$\begin{aligned}
\widetilde{W} \;=\; &-\frac{i\Upsilon_1}{4}(\sum_i Y_i - 2Y + it_1) - \frac{i\Upsilon_2}{4}(\sum_i \widetilde{Y}_i - 2Y + it_2)\\
&-\Sigma(\sum_i F_i - 2F) - \widetilde{\Sigma}(\sum_i \widetilde{F}_i - 2F)\\
&+\mu(\sum_{ij} \beta_{ij} F_i e^{-Y_j} + \beta_{\tilde{i}\tilde{j}} \widetilde{F}_i e^{-\widetilde{Y}_j} + \beta_{i\tilde{j}} F_i e^{-\widetilde{Y}_j} + \beta_{\tilde{i}j} \widetilde{F}_i e^{-Y_j})\\
&+2\mu F(\omega e^{-Y} + \sum_i \omega_i e^{-Y_i} + \sum_i \widetilde{\omega}_i e^{-\widetilde{Y}_i}) + \mu \sum_i (\nu_i F_i + \widetilde{\nu}_i \widetilde{F}_i) e^{-Y}.
\end{aligned} \tag{6.122}$$

We now analyse the vacuum solutions of this theory for generic β, ω and ν parameters. The vacuum solution is determined by solving

$$Y_1 + Y_2 - 2Y = -it_1, \quad \widetilde{Y}_1 + \widetilde{Y}_2 - 2Y = -it_2, \tag{6.123}$$

and

$$F_1 + F_2 - 2F = 0, \qquad \widetilde{F}_1 + \widetilde{F}_2 - 2F = 0. \tag{6.124}$$

We construct solutions where

$$X_1 = e^{-Y_1/2}, \qquad X_2 = e^{-Y_2/2}, \qquad e^{-Y} = e^{-it_1/2} X_1 X_2, \tag{6.125}$$

and

$$X_3 = e^{-\widetilde{Y}_1}, \qquad e^{-\widetilde{Y}_2} = e^{i(t_2-t_1)} \frac{(X_1X_2)^2}{X_3}, \tag{6.126}$$

for the Bose superfields while

$$G_1 = F_1, \quad G_2 = F_2, \quad F = \frac{G_1+G_2}{2}, \quad G_3 = \widetilde{F}_1, \quad \widetilde{F}_2 = G_1 + G_2 - G_3, \tag{6.127}$$

for the Fermi superfields. Note that by definition, (X_1, X_2) are not single-valued and, as we shall soon see, the low-energy Landau-Ginzburg theory is an orbifold conformal field theory.

After some straight forward algebra, the effective superpotential of the low-energy theory turns out to be

$$\begin{aligned} \mu^{-1}\widetilde{W}_{eff} \;=\; & AG_1(X_1^2 + pX_2^2 + qX_3 + s\frac{(X_1X_2)^2}{X_3} + uX_1X_2) \\ & + BG_2(X_2^2 + p'X_1^2 + q'X_3 + s'\frac{(X_1X_2)^2}{X_3} + u'X_1X_2 \\ & + CG_3(X_3 + p''X_1^2 + q''X_2^2 + s''\frac{(X_1X_2)^2}{X_3} + u''X_1X_2), \end{aligned} \tag{6.128}$$

where

$$\begin{aligned} A &= \beta_{11} + \beta_{\tilde{2}1} + \omega_1, \\ B &= \beta_{22} + \beta_{\tilde{2}2} + \omega_2, \\ C &= \beta_{\tilde{1}\tilde{1}} - \beta_{\tilde{2}\tilde{1}}, \end{aligned} \tag{6.129}$$

and

$$\begin{aligned} &p = (\beta_{12} + \beta_{\tilde{2}2} + \omega_2)/A, && q = (\beta_{1\tilde{1}} + \beta_{\tilde{2}\tilde{1}} + \widetilde{\omega})/A, \\ &s = e^{i(t_2-t_1)}(\beta_{1\tilde{2}} + \beta_{\tilde{2}\tilde{2}} + \widetilde{\omega}_2)/A, && u = e^{-it_1/2}(\nu_1 + \widetilde{\nu}_2 + \omega)/A, \\ &p' = (\beta_{21} + \beta_{\tilde{2}1} + \omega_1)/B, && q' = (\beta_{\tilde{2}\tilde{1}} + \beta_{2\tilde{1}} + \widetilde{\omega}_1)/B, \\ &s' = e^{i(t_2-t_1)}(\beta_{2\tilde{2}} + \beta_{\tilde{2}\tilde{2}} + \widetilde{\omega}_2)/B, && u' = e^{-it_1/2}(\nu_2 + \widetilde{\nu}_2 + \omega)/B, \\ &p'' = (\beta_{\tilde{1}1} - \beta_{\tilde{2}1})/C, && q'' = (\beta_{\tilde{1}2} - \beta_{\tilde{2}2})/C, \\ &s'' = e^{i(t_2-t_1)}(\beta_{\tilde{1}\tilde{2}} - \beta_{\tilde{2}\tilde{2}})/C, && u'' = e^{-it_1/2}(\widetilde{\nu}_1 - \widetilde{\nu}_2)/C. \end{aligned} \tag{6.130}$$

We see that the effective superpotential is invariant under the diagonal $\mathbb{Z}_2$ which sends

$$X_1 \to \pm X_1, \qquad X_2 \to \pm X_2$$

while keeping X_1X_2 invariant. The low-energy theory is therefore a well-defined $\mathbb{Z}_2$ orbifold of the low-energy Landau-Ginzburg theory. Lastly, the chiral ring relations are given by

$$\begin{aligned}
&X_1^2 + pX_2^2 + qX_3 + s\frac{(X_1X_2)^2}{X_3} + uX_1X_2 = 0,\\
&X_2^2 + p'X_1^2 + q'X_3 + s'\frac{(X_1X_2)^2}{X_3} + u'X_1X_2 = 0,\\
&X_3 + p''X_1^2 + q''X_2^2 + s''\frac{(X_1X_2)^2}{X_3} + u''X_1X_2 = 0.
\end{aligned} \tag{6.131}$$

6.3.2 A $(2,2)$ Deformation

Now we start with our base $\mathbb{P}^1 \times \mathbb{P}^1$ GLSM, and add a chiral superfield P and a Fermi superfield Γ carrying charge -2 only under $U(1)_1$, and a chiral superfield $\widetilde{P}$ and a Fermi superfield $\widetilde{\Gamma}$ carrying charge -2 only under $U(1)_2$. The model is again conformal, but the bundle is quite different from the prior case. In this case, the target space for the low-energy theory is the total space of $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$ over $\mathbb{P}^1 \times \mathbb{P}^1$. We will see the difference between the two cases reflected in the dual description.

We take as our choice of E in the definition of Γ

$$E \equiv E_\Gamma = -2\sqrt{2}(\Sigma + \widetilde{\epsilon}\widetilde{\Sigma})P, \tag{6.132}$$

while in defining $\widetilde{\Gamma}$ we take

$$\widetilde{E} \equiv E_{\widetilde{\Gamma}} = -2\sqrt{2}(\widetilde{\Sigma} + \epsilon\Sigma)\widetilde{P}. \tag{6.133}$$

The vacuum solution of the GLSM is

$$\sum_i |\phi_i|^2 - 2|p|^2 = r_1, \qquad \sum_i |\widetilde{\phi}_i|^2 - 2|\widetilde{p}|^2 = r_2. \tag{6.134}$$

The generic vacuum has $\Sigma = \widetilde{\Sigma} = 0$. Now, unlike the previous example, the D term equations decouple.

As before, let us define

$$P = p + \sqrt{2}\theta^+\psi_+ + \dots, \qquad \Gamma = \chi_- + \dots,$$

and,

$$\widetilde{P} = \widetilde{p} + \sqrt{2}\theta^+\widetilde{\psi}_+ + \ldots, \qquad \widetilde{\Gamma} = \widetilde{\chi}_- + \ldots.$$

From the various Yukawa couplings, we see that the massless fermionic degrees of freedom of the low-energy theory satisfy

$$\begin{aligned} &\sum_i \bar{\phi}_i\psi_{+i} - 2\bar{p}\psi_+ = 0, \qquad \sum_i \bar{\widetilde{\phi}}_i\widetilde{\psi}_{+i} - 2\bar{\widetilde{p}}\widetilde{\psi}_+ = 0,\\ &\sum_i \bar{\phi}_i\chi_{-i} + \widetilde{\chi}_{-1}\sum_i \bar{\beta}_i\bar{\widetilde{\phi}}_i + \widetilde{\chi}_{-2}\sum_i \bar{\beta}'_i\bar{\widetilde{\phi}}_i - 2\bar{p}\chi_- - 2\bar{\epsilon}\bar{\widetilde{p}}\widetilde{\chi}_- = 0,\\ &\sum_i \bar{\widetilde{\phi}}_i\widetilde{\chi}_{-i} + \chi_{-1}\sum_i \bar{\alpha}_i\bar{\phi}_i + \chi_{-2}\sum_i \bar{\alpha}'_i\bar{\phi}_i - 2\bar{\widetilde{\epsilon}}\bar{p}\chi_- - 2\bar{\widetilde{p}}\widetilde{\chi}_- = 0. \end{aligned} \tag{6.135}$$

The dual theory has a classical Lagrangian with the Kähler terms given in the $\mathbb{P}^1 \times \mathbb{P}^1$ example along with the additional terms

$$\begin{aligned} \widetilde{L} \;=\; & \ldots + \frac{i}{8}\int d^2\theta\, \frac{Y - \bar{Y}}{Y + \bar{Y}}\partial_-(Y + \bar{Y}) + \frac{i}{8}\int d^2\theta\, \frac{\widetilde{Y} - \bar{\widetilde{Y}}}{\widetilde{Y} + \bar{\widetilde{Y}}}\partial_-(\widetilde{Y} + \bar{\widetilde{Y}})\\ & -\frac{1}{2}\int d^2\theta\, \bar{\mathcal{F}}\mathcal{F} - \frac{1}{2}\int d^2\theta\, \bar{\widetilde{\mathcal{F}}}\widetilde{\mathcal{F}}, \end{aligned} \tag{6.136}$$

where the duality map is (again, modulo fermion bilinears)

$$\begin{aligned} \bar{P}P &= \frac{1}{2}(Y + \bar{Y}), \qquad \bar{P}(\overleftrightarrow{\partial}_- - 2iV_1)P = -\frac{1}{4}\partial_-(Y - \bar{Y}),\\ \bar{\widetilde{P}}\widetilde{P} &= \frac{1}{2}(\widetilde{Y} + \bar{\widetilde{Y}}), \qquad \bar{\widetilde{P}}(\overleftrightarrow{\partial}_- - 2iV_2)\widetilde{P} = -\frac{1}{4}\partial_-(\widetilde{Y} - \bar{\widetilde{Y}}), \end{aligned}$$

and

$$\bar{\Gamma} = \mathcal{F}, \qquad \bar{\widetilde{\Gamma}} = \widetilde{\mathcal{F}}. \tag{6.137}$$

The perturbative dual superpotential is given by

$$\begin{aligned} \widetilde{W} \;=\; & -\frac{i\Upsilon_1}{4}(\sum_i Y_i - 2Y + it_1) - \frac{i\Upsilon_2}{4}(\sum_i \widetilde{Y}_i - 2\widetilde{Y} + it_2)\\ & -\frac{1}{\sqrt{2}}(\sum_i E_i\mathcal{F}_i + \sum_i \widetilde{E}_i\widetilde{\mathcal{F}}_i + E\mathcal{F} + \widetilde{E}\widetilde{\mathcal{F}}), \end{aligned} \tag{6.138}$$

where again we write the last term in the form

$$-\int d\theta^+\Sigma(F_1 + F_2 - 2F - 2\epsilon\widetilde{F}) - \int d\theta^+\widetilde{\Sigma}(\widetilde{F}_1 + \widetilde{F}_2 - 2\widetilde{F} - 2\widetilde{\epsilon}F), \tag{6.139}$$

where $F = P\mathcal{F}$ and $\widetilde{F} = \widetilde{P}\widetilde{\mathcal{F}}$.

The exact dual superpotential is then given by the lengthy expression

$$\widetilde{W} \;=\; -\frac{i\Upsilon_1}{4}(\sum_i Y_i - 2Y + it_1) - \frac{i\Upsilon_2}{4}(\sum_i \widetilde{Y}_i - 2\widetilde{Y} + it_2)$$

$$
\begin{aligned}
&-\Sigma(\sum_i F_i - 2F - 2\epsilon\widetilde{F}) - \widetilde{\Sigma}(\sum_i \widetilde{F}_i - 2\widetilde{F} - 2\widetilde{\epsilon}F) \\
&+\mu(\sum_{ij} \beta_{ij}F_i e^{-Y_j} + \beta_{\tilde{i}\tilde{j}}\widetilde{F}_i e^{-\widetilde{Y}_j} + \beta_{i\tilde{j}}F_i e^{-\widetilde{Y}_j} + \beta_{\tilde{i}j}\widetilde{F}_i e^{-Y_j}) \\
&+2\mu F(\omega e^{-Y} + \widetilde{\omega}e^{-\widetilde{Y}} + \sum_i \omega_i e^{-Y_i} + \sum_i \widetilde{\omega}_i e^{-\widetilde{Y}_i}) \\
&+\mu\sum_i(\nu_i F_i + \widetilde{\nu}_i\widetilde{F}_i)e^{-Y} \\
&+2\mu\widetilde{F}(\omega' e^{-Y} + \widetilde{\omega}' e^{-\widetilde{Y}} + \sum_i \omega_i' e^{-Y_i} + \sum_i \widetilde{\omega}_i' e^{-\widetilde{Y}_i}) \\
&+\mu\sum_i(\nu_i' F_i + \widetilde{\nu}_i'\widetilde{F}_i)e^{-\widetilde{Y}}.
\end{aligned}
\tag{6.140}
$$

The vacuum solution is given by

$$Y_1 + Y_2 - 2Y = -it_1, \qquad \widetilde{Y}_1 + \widetilde{Y}_2 - 2\widetilde{Y} = -it_2, \tag{6.141}$$

and

$$F_1 + F_2 - 2F - 2\epsilon\widetilde{F} = 0, \qquad \widetilde{F}_1 + \widetilde{F}_2 - 2\widetilde{F} - 2\widetilde{\epsilon}F = 0. \tag{6.142}$$

We solve these constraints in the following way

$$
\begin{array}{lll}
X_1 = e^{-Y_1/2}, & X_2 = e^{-Y_2/2}, & e^{-Y} = e^{-it_1/2}X_1X_2, \\
\widetilde{X}_1 = e^{-\widetilde{Y}_1/2}, & \widetilde{X}_2 = e^{-\widetilde{Y}_2/2}, & e^{-\widetilde{Y}} = e^{-it_2/2}\widetilde{X}_1\widetilde{X}_2,
\end{array}
\tag{6.143}
$$

for the bosonic superfields. For the fermionic superfields, we define

$$
\begin{array}{lll}
G_1 = F_1, & G_2 = F_2, & F = \dfrac{1}{2(1-\epsilon\widetilde{\epsilon})}(G_1 + G_2 - \epsilon(\widetilde{G}_1 + \widetilde{G}_2)), \\
\widetilde{G}_1 = \widetilde{F}_1, & \widetilde{G}_2 = \widetilde{F}_2, & \widetilde{F} = \dfrac{1}{2(1-\epsilon\widetilde{\epsilon})}(\widetilde{G}_1 + \widetilde{G}_2 - \widetilde{\epsilon}(G_1 + G_2)).
\end{array}
\tag{6.144}
$$

Again, $(X_1, X_2, \widetilde{X}_1, \widetilde{X}_2)$ are not single-valued, and the low-energy theory will be an orbifold.

So the low-energy theory has the effective superpotential

$$
\begin{aligned}
\mu^{-1}\widetilde{W}_{eff} = \ & AG_1(X_1^2 + pX_2^2 + q\widetilde{X}_1^2 + s\widetilde{X}_2^2 + uX_1X_2 + v\widetilde{X}_1\widetilde{X}_2) + \\
& BG_2(X_2^2 + p'X_1^2 + q'\widetilde{X}_1^2 + s'\widetilde{X}_2^2 + u'X_1X_2 + v'\widetilde{X}_1\widetilde{X}_2) + \\
& \widetilde{A}\widetilde{G}_1(\widetilde{X}_1^2 + \widetilde{p}X_1^2 + \widetilde{q}X_2^2 + \widetilde{s}\widetilde{X}_2^2 + \widetilde{u}X_1X_2 + \widetilde{v}\widetilde{X}_1\widetilde{X}_2) + \\
& \widetilde{B}\widetilde{G}_2(\widetilde{X}_2^2 + \widetilde{p}'X_1^2 + \widetilde{q}'X_2^2 + \widetilde{s}'\widetilde{X}_1^2 + \widetilde{u}'X_1X_2 + \widetilde{v}'\widetilde{X}_1\widetilde{X}_2)
\end{aligned}
\tag{6.145}
$$

where

$$
\begin{aligned}
A &= \beta_{11} + \kappa\omega_1 - \widetilde{\epsilon}\kappa\omega_1', \\
B &= \beta_{22} + \kappa\omega_2 - \widetilde{\epsilon}\kappa\omega_2', \\
\widetilde{A} &= \beta_{\tilde{1}\tilde{1}} + \kappa\widetilde{\omega}_1' - \epsilon\kappa\widetilde{\omega}_1, \\
\widetilde{B} &= \beta_{\tilde{2}\tilde{2}} + \kappa\widetilde{\omega}_2' - \epsilon\kappa\widetilde{\omega}_2,
\end{aligned}
\tag{6.146}
$$

and $\kappa = 1/(1 - \epsilon\widetilde{\epsilon})$. All the remaining parameters appearing in (6.145) can be expressed in terms of the parameters appearing in (6.140). For example,

$$p = (\beta_{12} + \kappa\omega_2 - \widetilde{\epsilon}\kappa\omega_2')/A.$$

We will not list the remaining lengthy expressions since they are not particularly enlightening.

The effective superpotential is invariant under a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry sending

$$X_1 \to \pm X_1, \qquad X_2 \to \pm X_2$$

holding the product $X_1 X_2$ invariant, and also sending

$$\widetilde{X}_1 \to \pm \widetilde{X}_1, \qquad \widetilde{X}_2 \to \pm \widetilde{X}_2$$

holding the product $\widetilde{X}_1 \widetilde{X}_2$ invariant. Hence the IR theory is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of the Landau-Ginzburg theory. This is quite different from the previous example.

Finally, the chiral ring relations are given by

$$\begin{aligned}
&X_1^2 + pX_2^2 + q\widetilde{X}_1^2 + s\widetilde{X}_2^2 + uX_1X_2 + v\widetilde{X}_1\widetilde{X}_2 = 0,\\
&X_2^2 + p'X_1^2 + q'\widetilde{X}_1^2 + s'\widetilde{X}_2^2 + u'X_1X_2 + v'\widetilde{X}_1\widetilde{X}_2 = 0,\\
&\widetilde{X}_1^2 + \widetilde{p}X_1^2 + \widetilde{q}X_2^2 + \widetilde{s}\widetilde{X}_2^2 + \widetilde{u}X_1X_2 + \widetilde{v}\widetilde{X}_1\widetilde{X}_2 = 0,\\
&\widetilde{X}_2^2 + \widetilde{p}'X_1^2 + \widetilde{q}'X_2^2 + \widetilde{s}'\widetilde{X}_1^2 + \widetilde{u}'X_1X_2 + \widetilde{v}'\widetilde{X}_1\widetilde{X}_2 = 0.
\end{aligned} \qquad (6.147)$$

6.4 Models With $\mathrm{rk}(\mathcal{V}) > \mathrm{rk}(T\mathcal{M})$

So far in all our examples, we have considered cases where we have an equal number of Fermi and chiral superfields. At special loci in their parameter spaces, many of these models enjoy enhanced $(2,2)$ supersymmetry. These models flow in the IR to non-linear sigma models with $\mathrm{rk}(\mathcal{V}) = \mathrm{rk}(T\mathcal{M})$. We now turn to cases where the number of Fermi superfields is greater than the number of chiral superfields; in the IR sigma model, the bundles satisfy $\mathrm{rk}(\mathcal{V}) > \mathrm{rk}(T\mathcal{M})$.

On general grounds, we expect the low-energy dual theory to be quite different from the previous examples. As in our prior discussion, to find the low-energy theory, we need to solve the constraints

$$\sum_{i=1}^{N} Q_i Y_i = -it, \qquad \sum_{a=1}^{M} Q_a F_a = 0,$$

where now $M > N$. We are left with $N-1$ Y variables, and $M-1$ Fermi superfields. A generic non-perturbative superpotential of the form $\mu\sum_{ia}\beta_{ia}F_a e^{-Y_i}$ imposes a further $M-1$ constraints on the Y fields – one for each light Fermi superfield. Since $M > N$, generically the only solution is $Y_i \to \infty$ for all i.[5] This is clearly quite different from the $\mathrm{rk}(\mathcal{V}) = \mathrm{rk}(T\mathcal{M})$ cases.

However, there can be interesting non-generic cases where we get non-trivial vacuum solutions of the theory. This happens when some of the vacuum solution equations are linearly dependent. There can then be solutions for finite values of the Y fields, even when the rank of the left-moving vector bundle is greater than the rank of the tangent bundle! We now consider two examples which illustrate two possible situations: in the first, the vacuum manifold consists of isolated points, while in the second, the vacuum manifold is a geometric surface.

6.4.1 A Model With Isolated Vacua

Let us describe an example where generically we find isolated points as the vacua of the theory. In the GLSM, we take 3 chiral superfields Φ_1, Φ_2 and Φ_3 carrying gauge charges $1, 1$ and -2, respectively under a single $U(1)$ gauge group. This model is conformal and flows in the IR to a NLSM with a target space given by the total space of $\mathcal{O}(-2)$ over $\mathbb{P}^1$.

We also take 6 Fermi superfields, $\Gamma_1, \ldots, \Gamma_6$, with gauge charges $(1, 1, 1, -1, -1, -1)$, respectively. To each Γ_i, there is a concomitant E given by

$$\begin{aligned} &E_1 = E_3 = \sqrt{2}\Sigma\Phi_1, \qquad E_2 = \sqrt{2}\Sigma\Phi_2, \\ &E_4 = E_5 = E_6 = -\sqrt{2}\Sigma\Phi_3(\Phi_1 + \Phi_2). \end{aligned} \tag{6.148}$$

However our analysis goes through for any (generic) E_4, E_5, E_6 satisfying $E_4 = E_5 = E_6$. The only constraint on the choice of E_4 comes from demanding charge conservation and non-singularity. Our choice of E_4, E_5, E_6 is just a particular one chosen to illustrate the general vacuum structure.

The vacuum solution of the GLSM requires solving

$$|\phi_1|^2 + |\phi_2|^2 - 2|\phi_3|^2 = r, \tag{6.149}$$

[5] If we were to consider a superpotential, the situation is likely to be quite different. There should then be many examples with $\mathrm{rk}(\mathcal{V}) > \mathrm{rk}(T\mathcal{M})$ where the dual theory flows to an interacting SCFT. This illustrates some of the subtleties we expect to encounter when attempting to dualize with a tree-level superpotential.

while the analysis of the massless fermions follows straightforwardly from the Yukawa couplings as done in the previous examples. The perturbative dual theory is given by

$$\widetilde{L} = \frac{i}{8}\sum_{i=1}^{3}\int d^2\theta\, \frac{Y_i - \bar{Y}_i}{Y_i + \bar{Y}_i}\partial_-(Y_i+\bar{Y}_i) - \frac{1}{2}\sum_{a=1}^{6}\int d^2\theta\, \frac{\bar{F}_a F_a}{|Y_{\mathcal{E}_a} + \bar{Y}_{\mathcal{E}_a}|^2} + \int d\theta^+ \widetilde{W} + \text{h.c.}, \tag{6.150}$$

where

$$\mathcal{E}_1 = \mathcal{E}_3 = \Phi_1, \qquad \mathcal{E}_2 = \Phi_2, \qquad \mathcal{E}_4 = \mathcal{E}_5 = \mathcal{E}_6 = \Phi_3(\Phi_1 + \Phi_2),$$

and

$$\widetilde{W} = -\frac{i\Upsilon_1}{4}\left(Y_1 + Y_2 - 2Y_3 + it\right) - \frac{\Sigma}{\sqrt{2}}\left(\sum_{i=1}^{3} F_i - \sum_{i=4}^{6} F_i\right). \tag{6.151}$$

The duality maps are the standard ones, and have not been written down for brevity. The non-perturbative dual superpotential is given by

$$\widetilde{W}_{non-pert} = \mu \sum_{ia} \beta_{ia} F_a e^{-Y_i}. \tag{6.152}$$

The β parameters are highly constrained because of our symmetric choice of E_a. These symmetries imply that

$$\begin{aligned} \beta_{11} = \beta_{22} = \beta_{13} \equiv a, &\qquad \beta_{12} = \beta_{21} = \beta_{23} \equiv b, \\ \beta_{31} = \beta_{33} \equiv c, &\qquad \beta_{14} = \beta_{15} = \beta_{16} \equiv p, \\ \beta_{24} = \beta_{25} = \beta_{26} \equiv q, &\qquad \beta_{34} = \beta_{35} = \beta_{36} \equiv s. \end{aligned} \tag{6.153}$$

We also set $d = \beta_{32}$.

Now we can determine the vacuum structure. We solve the constraint $Y_1 + Y_2 - 2Y_3 = -it$ by setting

$$X_1 = e^{-Y_1/2}, \qquad X_2 = e^{-Y_2/2}$$

so that

$$e^{-Y_3} = e^{-it/2} X_1 X_2.$$

The other constraint yields $F_1 + F_2 + F_3 = F_4 + F_5 + F_6$. This gives the effective superpotential

$$\begin{aligned} \mu^{-1}\widetilde{W}_{eff} \;=\; & (F_1 + F_3)\left[(a+p)X_1^2 + (b+q)X_2^2 + e^{-it/2}(c+s)X_1X_2\right] + \\ & F_2\left[(b+p)X_1^2 + (a+q)X_2^2 + e^{-it/2}(d+s)X_1X_2\right]. \end{aligned} \tag{6.154}$$

Note that only F_1, F_2 and F_3 are required to specify the effective superpotential, because of the symmetries of the E_a. We also clearly see from (6.154) that the vacuum equations for F_1, F_3 are dependent.

So the vacua are given by the solutions of

$$\begin{aligned}(a+p)X_1^2 + (b+q)X_2^2 + e^{-it/2}(c+s)X_1X_2 = 0,\\ (b+p)X_1^2 + (a+q)X_2^2 + e^{-it/2}(d+s)X_1X_2 = 0.\end{aligned} \tag{6.155}$$

We have two equations for the two independent variables X_1, X_2. For generic choices of E_a, we get isolated vacua. Also, the low-energy theory is invariant under

$$X_1 \to \pm X_1, \qquad X_2 \to \pm X_2$$

with X_1X_2 held invariant. The low-energy theory is again a $\mathbb{Z}_2$ orbifold SCFT. We should stress that we assumed that the parameters of (6.153) are generically non-zero (but subject to symmetry constraints). This is actually a worse case scenario; if some of the parameters vanish, we would find additional vacua.

6.4.2 A Model With a Continuum of Vacua

Now we consider an example where we get a geometric surface, and not isolated points, as the vacuuum manifold. The field content of the GLSM is exactly as in the previous example, but now we take

$$E_1 = E_2 = E_3 = \sqrt{2}\Sigma(\Phi_1 + \Phi_2), \qquad E_4 = E_5 = E_6 = -\sqrt{2}\Sigma\Phi_3(\Phi_1 + \Phi_2). \tag{6.156}$$

Again, the analysis of the vacuum structure really only relies on the relation

$$E_1 = E_2 = E_3, \qquad E_4 = E_5 = E_6,$$

and we have just made a special choice.

We go directly to the analysis of the non-perturbative superpotential

$$\widetilde{W}_{non-pert} = \mu \sum_{ia} \beta_{ia} F_a e^{-Y_i}. \tag{6.157}$$

From the symmetries of the E_a, we obtain the effective superpotential

$$\mu^{-1}\widetilde{W}_{eff} = (F_1 + F_2 + F_3)\left[(\widetilde{a} + \widetilde{p})X_1^2 + (\widetilde{b} + \widetilde{q})X_2^2 + e^{-it/2}(\widetilde{c} + \widetilde{s})X_1X_2\right]. \tag{6.158}$$

So the vacuum is given by the solution of

$$(\widetilde{a}+\widetilde{p})X_1^2+(\widetilde{b}+\widetilde{q})X_2^2+e^{-it/2}(\widetilde{c}+\widetilde{s})X_1X_2=0. \tag{6.159}$$

Thus there is only one equation constraining the two independent variables, X_1 and X_2. The vacuum is a one (complex) dimensional surface (6.159) in (X_1, X_2) space. The effective field theory is a $\mathbb{Z}_2$ orbifold SCFT as before. However, the low-energy theory is itself a non-linear sigma model. There is an issue we have not yet addressed in this model; namely, the kinetic terms become singular in this model, and all models where the effective potential has flat directions. We now turn to this issue in the context of $\mathrm{rk}(\mathcal{V}) < \mathrm{rk}(T\mathcal{M})$ models for which this situation is generic.

6.5 Models With $\mathrm{rk}(\mathcal{V}) < \mathrm{rk}(T\mathcal{M})$

The last class of examples have $\mathrm{rk}(\mathcal{V}) < \mathrm{rk}(T\mathcal{M})$. The dual descriptions are generically quite different from any of the prior cases. The reason is a matter of counting constraints. The vacuum is determined by solving the constraints

$$\sum_{i=1}^{N} Q_iY_i=-it, \qquad \sum_{a=1}^{M} Q_aF_a=0,$$

where now $N > M$. We are left with $N-1$ Y variables, and $M-1$ Fermi superfields. A generic non-perturbative superpotential of the form $\mu\sum_{ia}\beta_{ia}F_ae^{-Y_i}$ imposes a further $M-1$ constraints on the Y fields, as before. However, this potential must have flat directions corresponding to excitations of the remaining $N-M$ light Y fields. The low-energy theory is not a Landau-Ginzburg theory, but a $(0,2)$ non-linear sigma model with the vacuum manifold as a target space.

We need to examine the metric on this target space. After dualizing a single charged chiral field, we see from (3.38) that the dual theory, parametrized by Y, has a Kähler metric with Kähler potential

$$K(Y,\bar{Y})=(Y+\bar{Y})\ln(Y+\bar{Y}) \quad \Rightarrow \quad g_{y\bar{y}}=\frac{dyd\bar{y}}{(y+\bar{y})}. \tag{6.160}$$

Recall that $\mathrm{Re}(Y)\geq 0$ so the metric singularity at $Y=0$ is at finite distance. How is this singularity resolved?

The situation is actually quite similar to string theory on the two-dimensional black-hole solution found in [35]. We expect this metric to be accompanied by a non-trivial dilaton diverging at $Y=0$. To see that this is the case, we recall that under T-duality, the dilaton is usually shifted by a metric factor $g_{\varphi\varphi}$ where φ is the isometry direction [36].

In our case, the metric factor is $\ln(y+\bar{y})$ but there is a subtlety involving the gauge field. To see how this works, consider the first order action

$$S = \int d^2\xi \left[-\frac{1}{4\rho^2}\sqrt{\gamma}\gamma^{\mu\nu}B_\mu B_\nu + \epsilon^{\mu\nu}B_\mu(\partial_\nu\varphi + A_\nu) + \sqrt{\gamma}R^{(2)}\Phi \right] \quad (6.161)$$

where $B\mu$ is a 1-form, and $\gamma_{\mu\nu}$ is the world-sheet metric. Integrating out B_μ generates the dilaton shift [36]

$$\Phi \to \Phi - \frac{1}{2}\ln(-g_{\varphi\varphi}) = \Phi + \frac{1}{2}\ln(4\rho^2). \quad (6.162)$$

If we integrate out A, we expect an analogous shift of the dilaton but with respect to the dual metric $g_{\vartheta\vartheta} = 1/g_{\varphi\varphi}$,

$$\Phi \to \Phi - \frac{1}{2}\ln(4\rho^2). \quad (6.163)$$

These two shifts should cancel for this model as also argued in [15].

In the general case where we have many chiral fields with charge Q_i, it appears that the shift is given by

$$\Phi \to \Phi - \frac{1}{2}\sum_i \ln(-g_{\varphi_i\varphi_i}) - \frac{1}{2}\ln(-\sum_i \frac{Q_i}{g_{\varphi_i\varphi_i}}). \quad (6.164)$$

With many $U(1)$ factors, there are just more terms like the last one appearing in (6.164). This also makes sense from the low-energy target space perspective: We are T-dualizing one phase for each chiral superfield but each gauged $U(1)$ kills one combination of chiral superfields, reducing the overall dilaton shift.

Therefore, whenever we have a non-trivial vacuum manifold in the dual description, we expect a corresponding dilaton diverging at the location of the metric singularitites. This is a $(0, 2)$ generalization of the duality between minimal models and a sigma model dual with diverging dilaton (see, for example, [37]).

6.5.1 A Surface in $\mathbb{P}^3$

To conclude our discussion, we will examine two models based on the examples of [6]. For the first case, the target space geometry is a hypersurface in

$\mathbb{P}^3$. Our basic GLSM has 4 superfields of charge 1 under a single $U(1)$ gauge symmetry. We take 1 Fermi superfield Γ with charge 2. Associated to Γ is a choice of E, and we consider the case

$$E = \alpha^{ij}\Phi_i\Phi_j. \tag{6.165}$$

Note there is no Σ in E so the constraint $E = 0$ restricts us to a hypersurface, $\mathcal{M}$, in $\mathbb{P}^3$.

The low-energy theory is quite beautiful. There are no left-moving fermions at all, but $\mathrm{ch}_2(\mathcal{V}) = 0$ as described in section 5.1. Whether supersymmetry is broken in the low-energy theory can be tested by computing $\mathrm{Ind}(\bar\partial)$ which counts (with sign) the number of supersymmetric ground states. First we note that the hypersurface $\mathcal{M}$ has Chern classes,

$$c_1(\mathcal{M}) = 2, \qquad c_2(\mathcal{M}) = 2.$$

The index is given by

$$\begin{aligned}\mathrm{Ind}(\bar\partial) &= \int_{\mathcal{M}} \mathrm{td}(\mathcal{M}),\\ &= \int_{\mathcal{M}} \left(\frac{c_1^2 + c_2}{12}\right) = \frac{1}{2}\int_{\mathbb{P}^3} J^2 \wedge 2J = 1,\end{aligned} \tag{6.166}$$

where J is the Kähler form of $\mathbb{P}^3$. So supersymmetry is unbroken, and we generically expect a single vacuum state with mass gap.

Now we turn to the dual description. We want to determine whether there are non-perturbative corrections to the dual superpotential. Let us take a particular choice of E, say $E = \Phi_4^2$.[6] To perform an instanton zero mode analysis, we need the following relevant terms in the action,

$$i\bar\chi_- D_+ \chi_- - |\phi_4^2|^2 - 2(\phi_4\bar\chi_-\psi_{+4} + \bar\phi_4\bar\psi_{+4}\chi_-) + \dots. \tag{6.167}$$

A BPS instanton requires setting $\phi_4 = 0$. We must embed the instanton in some other ϕ, say ϕ_1. In this (and any other BPS configuration), all the potential terms in (6.167) vanish and we can exactly determine the fermion zero modes: there are 4 right-moving zero modes. For ψ_{+1}, the zero mode is given by

$$\mu^0 = \begin{pmatrix} \bar\psi_{+1}^0 \\ \lambda_-^0 \end{pmatrix} = \begin{pmatrix} -\sqrt{2}(\bar D_1 + i\bar D_2)\bar\phi_1 \\ D - F_{12} \end{pmatrix},$$

while $\psi_{+i}^0 = \bar\phi_1$ for $i = 2, 3, 4$. For the left-mover, there is a single zero mode $\chi_-^0 = \phi_1^2$. Any two-point function can only absorb two zero modes.

[6]This is actually a degenerate section of $\mathcal{O}(2)$ since $E = dE = 0$ has a solution. Fortunately, this will not affect the subsequent analysis since we can always perturb E by a small amount with no real change in the analysis.

Quantum effects could, in principle, lift zero modes but since the remaining 3 zero modes are right-moving, they must remain massless. These zero modes kill the correlation function. We conclude that there are no non-perturbative corrections to the dual superpotential. This is very similar to the argument in [6].

The exact dual Lagrangian is therefore given by

$$\begin{aligned}\widetilde{L} &= \frac{i}{8}\int d^2\theta\ \sum_i \frac{Y_i - \bar{Y}_i}{Y_i + \bar{Y}_i}\partial_-(Y_i + \bar{Y}_i) - 2\int d^2\theta\ \frac{\bar{F}F}{(Y_4 + \bar{Y}_4)^2} \\ &\quad - \left[\frac{i}{4}\sum_i \int d\theta^+ Y_i \Upsilon - \frac{1}{\sqrt{2}}\int d\theta^+ F + \text{h.c.}\right].\end{aligned} \tag{6.168}$$

The vacuum solution is obtained by setting

$$\sum_i Y_i = -it. \tag{6.169}$$

Integrating out F generates a potential for Y_4 of the form

$$V \sim |y_4 + \bar{y}_4|^2. \tag{6.170}$$

To find the vacuum manifold, we must set $Y_4 = 0$. The low-energy theory is therefore a non-linear sigma model on a two-dimensional target space with metric determined by solving these constraints. There are no left-moving fermions at all, and the space has metric singularities at loci where the dilaton diverges. From our analysis of the original model, we can predict that supersymmetry is unbroken and that the index is 1. It should be possible to verify these predictions directly in the low-energy dual theory. It may also be possible to relate the dual theory to a construction involving $(0,2)$ gauged WZW models.

Next we consider a special case where the potential term $|E|^2$ itself has flat directions. A simple specific choice is $E = \Phi_1\Phi_2$. The relevant terms in the action are,

$$i\bar{\chi}_- D_+ \chi_- - |\phi_1\phi_2|^2 - (\phi_2\bar{\chi}_-\psi_{+1} + \phi_1\bar{\chi}_-\psi_{+2} + \bar{\phi}_2\bar{\psi}_{+1}\chi_- + \bar{\phi}_1\bar{\psi}_{+2}\chi_-) + \ldots . \tag{6.171}$$

We argue that there are no non-perturbative corrections to the dual superpotential in the following way: perturb E by an infinitessimal amount so the resulting section of $\mathcal{O}(2)$ is generic. By our previous analysis, the non-perturbative superpotential must vanish. Since the dual superpotential varies holomorphically with the deformation parameter, it cannot depend on the parameter at all. Therefore, there are no corrections for this case. The only difference from the prior case is that on integrating out F, we obtain a potential

$$V \sim |(y_1 + \bar{y}_1)(y_2 + \bar{y}_2)|$$

which has a different structure from (6.170).

6.5.2 A Bundle Over $\mathbb{P}^3$

Let us take the same model just discussed but consider a different choice for E where

$$E = \Sigma\mathcal{E} = \Sigma\alpha^{ij}\Phi_i\Phi_j. \tag{6.172}$$

Because of the appearance of Σ in E, we expect the low-energy theory to contain a left-moving fermion which is a section of $\mathcal{O}(2)$ over $\mathbb{P}^3$. There is a subtlety here worth explaining: the Yukawa couplings described in section 5.2 would seem to give mass to the single χ_- fermion in the UV. How can there be a low-energy left-moving fermion at all? The resolution of this puzzle goes as follows. The Σ superfield becomes massive when $\mathcal{E} \neq 0$, and can be integrated out. However, on performing this integration, we see that χ_- does not pick up a mass but picks up a derivative coupling. It therefore survives as a light degree of freedom as required by consistency of the low-energy theory.

We count the number of supersymmetric vacua in this theory (weighted by signs) by evaluating the Witten index,

$$\mathrm{Tr}(-1)^F = \sum_{p,m}(-1)^{p+m}\, h^p(\mathcal{M}, \wedge^m\mathcal{V}), \tag{6.173}$$

where $\mathcal{V} = \mathcal{O}(2)$. This is easily done. For the sector with no excited left-moving fermion ($m = 0$), the only contribution comes from $\bar{\partial}$-cohomology which consists only of constant functions so $h^0(\mathbb{P}^3) = 1$. For the other case where $m = 1$, the only contribution comes from $h^0(\mathbb{P}^3, \mathcal{O}(2)) = 10$. In total, there are a net 9 fermionic ground states. Supersymmetry is unbroken.

Now we turn to the dual theory. It is easy to see that there are no non-perturbative corrections to the superpotential. In any instanton background, there are always 3 right-moving zero modes that cannot be paired. These zero modes kill any instanton contributions. The resulting superpotential is,

$$\widetilde{W}_{exact} = -\frac{i\Upsilon}{4}(\sum_i Y_i + it) + \frac{1}{\sqrt{2}}\Sigma F. \tag{6.174}$$

The only resulting constraint is $\sum_i Y_i = -it$. The ΣF coupling gives a mass to Σ so the low-energy theory is again a non-linear sigma model with no effective superpotential. We predict that supersymmetry is unbroken in this theory.

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A Expressing $(2,2)$ Theories in $(0,2)$ Notation

In this Appendix, we express a $(2,2)$ GLSM in terms of $(0,2)$ fields. Our starting point is the $(2,2)$ Lagrangian describing a chiral field, Φ, and the gauge field V with field strength Σ,

$$L = \int d^4\theta \ \bar{\Phi} e^{2QV} \Phi - \frac{1}{4e^2} \int d^4\theta \ \bar{\Sigma}\Sigma - \left(\frac{it}{2\sqrt{2}} \int d^2\widetilde{\theta} \ \Sigma + \text{h.c.} \right). \qquad (A.1)$$

The gauge coupling constant is given by e, while $t = ir + \frac{\theta}{2\pi}$ is the complexified Fayet-Iliopoulos parameter. We also use the short hand,

$$d^2\widetilde{\theta} = d\theta^+ d\bar{\theta}^-.$$

To obtain a $(0,2)$ Lagrangian, we just need to integrate out $\theta^-, \bar{\theta}^-$ which we can do by noting

$$\begin{aligned} L_\Phi &= \int d^4\theta \ \bar{\Phi} e^{2QV} \Phi & (A.2) \\ &= -\int d^2\theta \ \bar{\mathrm{D}}_- \mathrm{D}_- \left(\bar{\Phi} e^{2QV} \Phi \right). & (A.3) \end{aligned}$$

Next we reduce this expression to a $(0,2)$ Lagrangian by explicitly applying the supercovariant derivatives,

$$L_\Phi = -\int d^2\theta \quad \Big[\quad 2Q\ (\bar{\mathrm{D}}_-\bar{\Phi})(\mathrm{D}_-V)e^{2QV}\Phi + 2Q\ \bar{\Phi}(\bar{\mathrm{D}}_-\mathrm{D}_-V)e^{2QV}\Phi$$
$$-4Q^2\ \bar{\Phi}(\mathrm{D}_-V)(\bar{\mathrm{D}}_-V)e^{2QV}\Phi + (\bar{\mathrm{D}}_-\bar{\Phi})e^{2QV}(\mathrm{D}_-\Phi) \quad \text{(A.4)}$$
$$+2Q\ \bar{\Phi}(\bar{\mathrm{D}}_-V)e^{2QV}(\mathrm{D}_-\Phi) + \bar{\Phi}e^{2QV}(\bar{\mathrm{D}}_-\mathrm{D}_-\Phi)\ \Big]|_{\theta^-=\bar{\theta}^-=0}.$$

From now on for brevity, we will not explicitly write $\theta^- = \bar{\theta}^- = 0$. This final reduction will always be implied. Let us reduce term by term to $(0,2)$ superspace.

Term 1

The first term to consider is

$$-2Q\int d^2\theta\ (\bar{\mathrm{D}}_-\bar{\Phi})(\mathrm{D}_-V)e^{2QV}\Phi \tag{A.5}$$

We use that the results of section 2.2 to write

$$\bar{\mathrm{D}}_-\bar{\Phi} = \sqrt{2}(\bar{\psi}_- - \sqrt{2}\bar{\theta}^+\bar{F} + i\theta^+\bar{\theta}^+\partial_+\bar{\psi}_-) = \sqrt{2}e^{-Q\Psi}\bar{\Gamma} + 2\theta^+\bar{E} \tag{A.6}$$

where Γ is a charged $(0,2)$ Fermi superfield satisfying

$$\bar{\mathcal{D}}_+\Gamma = \sqrt{2}E.$$

Also $\Psi = \theta^+\bar{\theta}^+A_+$. We also recall that

$$\mathrm{D}_-V = -\sqrt{2}\bar{\theta}^+\Sigma_0, \tag{A.7}$$

where $\Sigma_0 = \Sigma|_{\theta^-=\bar{\theta}^-=0}$.[7] Finally note that the uncharged field $\Phi|_{\theta^-=\bar{\theta}^-=0}$ satisfies

$$\bar{\mathrm{D}}_+\Phi|_{\theta^-=\bar{\theta}^-=0} = 0,$$

and is given by

$$\Phi|_{\theta^-=\bar{\theta}^-=0} = \phi + \sqrt{2}\theta^+\psi_+ - i\theta^+\bar{\theta}^+\partial_+\phi = e^{-Q\Psi}\Phi_0 \tag{A.8}$$

where Φ_0 satisfies $\bar{\mathcal{D}}_+\Phi_0 = 0$, and is a $(0,2)$ charged chiral superfield. Therefore, in terms of $(0,2)$ superfields, we express term 1 as

$$-4Q\int d^2\theta\ \bar{\theta}^+\bar{\Gamma}\Sigma_0\Phi_0 + 4\sqrt{2}Q\int d^2\theta\ \theta^+\bar{\theta}^+\bar{E}\Sigma_0\Phi_0. \tag{A.9}$$

[7] In section 2.2, we used the notation $\Sigma^{(0,2)}$ for the $\theta^- = \bar{\theta}^- = 0$ component of the $(2,2)$ chiral field Σ. For notational simplicity, here we just use Σ_0.

Term 2

The second term to consider is

$$-2Q\int d^2\theta\ \bar{\Phi}(\bar{\mathrm{D}}_-\mathrm{D}_-V)e^{2QV}\Phi. \tag{A.10}$$

Recall from section 2.2 that

$$(\bar{\mathrm{D}}_-\mathrm{D}_-V) = -V_0 + i\partial_-\Psi \tag{A.11}$$

where V_0 is given by

$$V_0 = A_- - 2i\theta^+\bar{\lambda}_- - 2i\bar{\theta}^+\lambda_- + 2\theta^+\bar{\theta}^+D. \tag{A.12}$$

Term 2 is therefore

$$-2Q\int d^2\theta\ \bar{\Phi}_0(-V_0 + i\partial_-\Psi)\Phi_0. \tag{A.13}$$

Term 3

The next term is immediately reduced

$$4Q^2\int d^2\theta\ \bar{\Phi}(\mathrm{D}_-V)(\bar{\mathrm{D}}_-V)e^{2QV}\Phi = 8Q^2\int d^2\theta\ \bar{\theta}^+\theta^+|\Phi_0\Sigma_0|^2. \tag{A.14}$$

Term 4

Similarly for term 4,

$$\begin{aligned}-\int d^2\theta\ (\bar{\mathrm{D}}_-\bar{\Phi})e^{2QV}(\mathrm{D}_-\Phi) = -2\int d^2\theta\ \bar{\Gamma}\Gamma + 2\sqrt{2}\int d^2\theta\ \bar{\theta}^+\bar{\Gamma}E \\ -2\sqrt{2}\int d^2\theta\ \theta^+\Gamma\bar{E} + 4\int d^2\theta\ \bar{\theta}^+\theta^+|E|^2.\end{aligned} \tag{A.15}$$

Term 5

We see that term 5,

$$\begin{aligned}-2Q\int d^2\,\theta\ \bar{\Phi}(\bar{\mathrm{D}}_-V)e^{2QV}(\mathrm{D}_-\Phi) = \\ 4Q\int d^2\theta\ \theta^+\Gamma\bar{\Sigma}_0\bar{\Phi}_0 + 4\sqrt{2}Q\int d^2\theta\ \theta^+\bar{\theta}^+E\bar{\Sigma}_0\bar{\Phi}_0,\end{aligned} \tag{A.16}$$

is just the conjugate of term 1.

Term 6

Lastly, we come to

$$-\int d^2\theta\ \bar{\Phi}e^{2QV}(\bar{\mathrm{D}}_-\mathrm{D}_-\Phi) = -2i\int d^2\theta\ \bar{\Phi}_0(\partial_-\Phi_0 - Q\Phi_0\partial_-\Psi). \qquad \text{(A.17)}$$

Some Simplifications

Consider adding terms 2 and 6. The sum gives the gauge covariant combination

$$-2i\int d^2\theta\ \bar{\Phi}_0(\mathcal{D}_0 - \mathcal{D}_1)\Phi_0 \qquad \text{(A.18)}$$

where $(\mathcal{D}_0 - \mathcal{D}_1) = \partial_- + iQV_0$. On summing all terms, we find

$$\begin{aligned} L_\Phi \;=\; & -2i\int d^2\theta\ \bar{\Phi}_0(\mathcal{D}_0 - \mathcal{D}_1)\Phi_0 - 2\int d^2\theta\ \bar{\Gamma}\Gamma - 2\sqrt{2}\int d^2\theta\ \theta^+\Gamma\bar{E} \\ & +4Q\int d^2\theta\ \theta^+\Gamma\bar{\Sigma}_0\bar{\Phi}_0 + 2\sqrt{2}\int d^2\theta\ \bar{\theta}^+\bar{\Gamma}E - 4Q\int d^2\theta\ \bar{\theta}^+\bar{\Gamma}\Sigma_0\Phi_0 \\ & +8Q^2\int d^2\theta\ \bar{\theta}^+\theta^+|\Phi_0\Sigma_0|^2 + 4\int d^2\theta\ \bar{\theta}^+\theta^+|E|^2 \\ & +4\sqrt{2}Q\int d^2\theta\ \theta^+\bar{\theta}^+\bar{E}\Sigma_0\Phi_0 + 4\sqrt{2}Q\int d^2\theta\ \theta^+\bar{\theta}^+E\bar{\Sigma}_0\bar{\Phi}_0. \end{aligned} \qquad \text{(A.19)}$$

This is the $(2,2)$ theory reduced to $(0,2)$ variables. Finally for a $(2,2)$ theory reduced this way,

$$E = \sqrt{2}Q\Sigma_0\Phi_0.$$

Substituting this explicit expression leads to a large number of cancellations. When the dust settles, we are left with the simple Lagrangian

$$L_\Phi = -2i\int d^2\theta\ \bar{\Phi}(\mathcal{D}_0 - \mathcal{D}_1)\Phi - 2\int d^2\theta\ \bar{\Gamma}\Gamma, \qquad \text{(A.20)}$$

where $\bar{\mathcal{D}}_+\Phi = 0$ and $\bar{\mathcal{D}}_+\Gamma = \sqrt{2}E$. We have also dropped the subscript in the definition of the chiral superfield. When rescaled by a factor of $1/4$, this is the standard $(0,2)$ Lagrangian. The only remaining terms involve Σ and they reduce straightforwardly to give,

$$L_\Sigma = \frac{i}{2e^2}\int d^2\theta\ \bar{\Sigma}_0\partial_-\Sigma_0 + \frac{1}{8e^2}\int d^2\theta\ \bar{\Upsilon}\Upsilon + \left\{\frac{t}{4}\int d\theta^+\ \Upsilon|_{\bar{\theta}^+=0} + \text{h.c.}\right\}, \qquad \text{(A.21)}$$

where Υ is the field strength for the $(0,2)$ vector multiplet.

The Dual Description

The dual Lagrangian is given in terms of the $(2,2)$ field strength Σ and an uncharged chiral multiplet Y

$$\widetilde{L} = L_\Sigma - \frac{1}{8}\int d^4\theta\ (Y+\bar{Y})\ln(Y+\bar{Y}) - \left(\frac{Q}{2\sqrt{2}}\int d^2\widetilde{\theta}\ \Sigma Y + \text{h.c.}\right). \quad (A.22)$$

The first term is given in $(A.21)$ so need only consider the remaining terms. We start with the twisted superpotential. As before, we want to reduce it to $(0,2)$ superspace,

$$\widetilde{L} = \ldots + \left(\frac{Q}{2\sqrt{2}}\int d\theta^+[(\bar{\mathrm{D}}_-\Sigma)Y + \Sigma(\bar{\mathrm{D}}_-Y)]|_{\theta^-=\bar{\theta}^-=0} + \text{h.c.}\right). \quad (A.23)$$

Using the results $\bar{\mathrm{D}}_-\Sigma = -\frac{i}{\sqrt{2}}\Upsilon$, where

$$\Upsilon = -2\lambda_- + 2i\theta^+(D - iF_{01}) + 2i\theta^+\bar{\theta}^+\partial_+\lambda_-, \quad (A.24)$$

and $\bar{\mathrm{D}}_-Y = -\sqrt{2}F$, where $\bar{\mathrm{D}}_+F = 0$, we get that

$$\widetilde{L} = \ldots - (\frac{Q}{2}\int d\theta^+[\Sigma_0 F + \frac{i}{2}Y_0\Upsilon] + \text{h.c.}). \quad (A.25)$$

where $Y_0 = Y|_{\theta^-=\bar{\theta}^-=0}$. Note that $\bar{\mathrm{D}}_+Y_0 = 0$ and Y_0 is a neutral $(0,2)$ chiral superfield. Next consider the kinetic term

$$\widetilde{L} = \frac{1}{8}\int d^2\theta\, \bar{\mathrm{D}}_-\mathrm{D}_-(Y+\bar{Y})\ln(Y+\bar{Y}) + \ldots \quad (A.26)$$

Up to a total derivative, this gives us

$$\widetilde{L} = \frac{1}{8}\int d^2\theta\ \left[i\frac{Y_0-\bar{Y}_0}{Y_0+\bar{Y}_0}\partial_-(Y_0+\bar{Y}_0) - 2\frac{\bar{F}F}{Y_0+\bar{Y}_0}\right] + \ldots. \quad (A.27)$$

Excluding the terms involving only Σ given in $(A.21)$, we obtain the dual Lagrangian

$$\begin{aligned}\widetilde{L} &= \frac{i}{8}\int d^2\theta\ \left[\frac{Y-\bar{Y}}{Y+\bar{Y}}\partial_-(Y+\bar{Y}) + 2i\frac{\bar{F}F}{Y+\bar{Y}}\right] \\ &\quad - \left(\frac{Q}{2}\int d\theta^+\left[\Sigma F + \frac{i}{2}Y\Upsilon\right] + \text{h.c.}\right) + \ldots,\end{aligned} \quad (A.28)$$

where $\bar{\mathrm{D}}_+Y = \bar{\mathrm{D}}_+F = 0$ and we have dropped the subscript on the neutral chiral superfield Y.

The Duality Map

The $(2,2)$ duality map is given by

$$\bar{\Phi}\, e^{2QV}\Phi = \frac{1}{2}\left(Y + \bar{Y}\right). \tag{A.29}$$

To express this map $(0,2)$ language, we will make use of the relations

$$\Phi = e^{-Q\Psi}\Phi_0 + \theta^-(\sqrt{2}e^{-Q\Psi}\Gamma + 2\bar{\theta}^+E) - i\theta^-\bar{\theta}^-\partial_-(e^{-Q\Psi}\Phi_0), \tag{A.30}$$

$$Y = Y_0 + \sqrt{2}\bar{\theta}^-F + i\theta^-\bar{\theta}^-\partial_-Y_0, \tag{A.31}$$

and

$$V = \Psi - \sqrt{2}\theta^-\bar{\theta}^+\Sigma_0 - \sqrt{2}\theta^+\bar{\theta}^-\bar{\Sigma}_0 + \theta^-\bar{\theta}^-V_0. \tag{A.32}$$

Substituting these expressions into $(A.29)$, we obtain the $(0,2)$ duality map. Equating terms independent of the fermionic superspace coordinates, we get

$$\bar{\Phi}\Phi = \frac{1}{2}(Y + \bar{Y}). \tag{A.33}$$

Again, here we have dropped the subscript on the $(0,2)$ fields for brevity. Equating terms proportional to $\theta^-\bar{\theta}^-$, we get

$$-i\bar{\Phi}(\overleftrightarrow{\partial}_- + iQV)\Phi + \bar{\Gamma}\Gamma = \frac{i}{4}\partial_-(Y - \bar{Y}). \tag{A.34}$$

Finally on equating terms proportional to θ^-, we get the relation

$$\frac{1}{2}\bar{F} = \bar{\Phi}\Gamma. \tag{A.35}$$

and its conjugate from terms proportional to $\bar{\theta}^-$.

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