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Matrix Integrals and Feynman Diagrams in the Kontsevich Model

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Abstract

We review some relations occurring between the combinatorial intersection theory on the moduli spaces of stable curves and the asymptotic behavior of the 't Hooft-Kontsevich matrix integrals. In particular, we give an alternative proof of the Witten-Di Francesco-Itzykson-Zuber theorem —which expresses derivatives of the partition function of intersection numbers as matrix integrals— using techniques based on diagrammatic calculus and combinatorial relations among intersection numbers. These techniques extend to a more general interaction potential.

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1 Introduction

The aim of this paper is to describe some relations occurring between combinatorial intersection theory on moduli spaces of stable curves $\overline{\mathcal{M}}_{g,n}$ and the asymptotic expansion of the matrix integral

$$\int_{\mathcal{H}(N)} \exp\left\{-\sqrt{-1}\sum_{j=0}^{\infty}(-1/2)^j s_j \frac{\operatorname{tr} X^{2j+1}}{2j+1}\right\} \mathrm{d}\mu_\Lambda(X), \tag{1.1}$$

where $\mathcal{H}(N)$ is the space of $N \times N$ Hermitian matrices, $\Lambda \in \mathcal{H}(N)$ is diagonal and positive definite, and μ_Λ is the Gaussian measure defined by normalization of $\exp(-\frac{1}{2}\operatorname{tr}\Lambda X^2)\mathrm{d}X$.

The correspondence between these two apparently unrelated theories is given by ribbon graphs, which appear on the one side as the cells of an orbifold cellularization of the moduli spaces of curves, and on the other side as the Feynman diagrams occurring in the asymptotic expansion of the Gaussian integral (1.1). The idea to relate the intersection theory on $\overline{\mathcal{M}}_{g,n}$ to the theory of Feynman diagrams is roughly the following: given a cohomology class $\omega \in H^*(\overline{\mathcal{M}}_{g,n})$, one seeks a set of Feynman rules such that, for any ribbon graph Γ, the integral of ω on the cell corresponding to Γ equals the amplitude of Γ as a Feynman diagram. In particular, if one considers monomials of Miller's classes $\psi_i \in H^2(\mathcal{M}_{g,n})$, then such a set of rules was found by Kontsevich (see [Kon92]): the Feynman diagrams expansion of

$$\int_{\mathcal{H}(N)} \exp\Big\{\frac{\sqrt{-1}}{6}\operatorname{tr} X^3\Big\}\mathrm{d}\mu_\Lambda(X)$$

also computes the partition function $Z(t_*)$ of intersection numbers of the ψ classes, namely,

$$Z(t_*)\Big|_{t_*(\Lambda)} \asymp \int_{\mathcal{H}(N)} \exp\Big\{\frac{\sqrt{-1}}{6}\operatorname{tr} X^3\Big\}\mathrm{d}\mu_\Lambda(X), \tag{1.2}$$

where

$$t_k(\Lambda) := -(2k-1)!!\operatorname{tr}\Lambda^{-2k-1}$$

are the Miwa coordinates on $\mathcal{H}(N)/U(N)$. More generally, one can define combinatorial relatives $\mathcal{M}_{m_*;n}$ of $\mathcal{M}_{g,n}$ and develop an intersection theory on them; in [Kon92] it is shown that

$$Z(s_*;t_*)\Big|_{t_*(\Lambda)} \asymp \int_{\mathcal{H}(N)} \exp\left\{-\sqrt{-1}\sum_{j=0}^{\infty}(-1/2)^j s_j \frac{\operatorname{tr} X^{2j+1}}{2j+1}\right\} \mathrm{d}\mu_\Lambda(X), \tag{1.3}$$

where $Z(s_*, t_*)$ is the partition function of combinatorial intersection numbers. Equation (1.2) is recovered as a special case by setting $s_* = (0, 1, 0, \dots)$.

In a follow-up [Wit92] to Kontsevich' paper, Witten proposed a conjecture extending the above relation (1.2): derivatives (up to any order) of the partition function $Z(s_*; t_*)$ with respect to the t_* variables should admit a matrix integral interpretation, namely, they should correspond to asymptotic expansions of Gaussian integrals of the form

$$\int_{\mathcal{H}(N)} Q(\operatorname{tr} X, \operatorname{tr} X^3, \operatorname{tr} X^5, \dots) \exp\left\{\frac{\sqrt{-1}}{6} X^3\right\} \mathrm{d}\mu_\Lambda(X), \tag{1.4}$$

where Q is a polynomial. Witten also checked the first cases of his conjecture by Feynman diagrams techniques close to Kontsevich' ones. A proof of the full statement was later given by Di Francesco-Itzykson-Zuber [DFIZ93]. They solved the problem explicitly, but their argument rests upon checking some non-trivial algebraic-combinatorial identities, and does not touch the geometrical aspect of the problem.

Yet, it is clear by equation (1.3) that the asymptotic expansions of (1.4) can be seen as derivatives of $Z(s_*; t_*)$ with respect to the s_* variables, evaluated at the point $s_* = (0, 1, 0, 0, \dots)$. Therefore, the Di Francesco-Itzykson-Zuber theorem is equivalent to the existence of a linear isomorphism $D : \mathbb{C}[\partial/\partial t_*] \to \mathbb{C}[\partial/\partial s_*]$, $D : P \mapsto D_P$ such that, for any differential operator $P(\partial/\partial t_*)$,

$$P(\partial/\partial t_*) Z(s_*; t_*)\Big|_{s_*=(0,1,0,0,\dots)} = D_P(\partial/\partial s_*) Z(s_*; t_*)\Big|_{s_*=(0,1,0,0,\dots)}. \tag{1.5}$$

Up to our knowledge, this has first been remarked by Arbarello-Cornalba in [AC96]. From a geometrical point of view, it means that —in a certain sense— combinatorial classes on $\overline{\mathcal{M}}_{g,n}$ are Poincaré duals to the Miller classes. For a formalization of this remark see, in addition to the already cited papers, also the recent preprints by Igusa [Igu02, Igu03] and Mondello [Mon03].

We are going to show how, suitably recasting Witten's computations from [Wit92] in the language of graphical calculus, one can prove a statement which generalizes the above equation (1.5). Indeed, denote by $\mathbb{C}\langle \partial/\partial t_* \rangle$ the free non-commutative algebra generated by the $\partial/\partial t_*$ (acting on the formal power series in s_* and t_* via its abelianization $\mathbb{C}[\partial/\partial t_*]$); then one can prove the following.

Main Theorem. *There exist an algebra homomorphism*

$$D\colon \mathbb{C}\langle \partial/\partial t_* \rangle \to \mathbb{C}\langle\!\langle s_*, \partial/\partial s_* \rangle\!\rangle$$

with values in a suitable algebra of formal differential operators in the variables s_*, such that, for any $P \in \mathbb{C}\langle \partial/\partial t_* \rangle$,

$$P(\partial/\partial t_*) Z(s_*; t_*) = D_P(s_*; \partial/\partial s_*) Z(s_*; t_*). \tag{1.6}$$

Moreover,

$$D_{\partial/\partial t_k} = c_k {s_1}^{2k+1} \partial/\partial s_k + \textit{lower order terms}, \qquad c_k \in \mathbb{C}. \tag{1.7}$$

If $s_*^\circ = (s_0^\circ, s_1^\circ, \dots, s_\nu^\circ, 0, 0, \dots)$, where the s_i° are complex constants, then evaluating both sides of (1.6) at $s_* = s_*^\circ$ and translating the result into matrix integral terms, we obtain that there exists a linear map $Q^{s_*^\circ} : \mathbb{C}[\partial/\partial t_*] \to \mathbb{C}[s_*^\circ; \operatorname{tr} X, \operatorname{tr} X^3, \dots]$ such that, for $N \gg 0$,

$$\begin{aligned} P(\partial/\partial t_*) \int_{\mathcal{H}(N)} \exp\left\{ -\sqrt{-1} \sum_{j=0}^{\nu} (-1/2)^j s_j^\circ \frac{\operatorname{tr}(X^{2j+1})}{2j+1} \right\} d\mu_\Lambda(X) = \\ = \int_{\mathcal{H}(N)} Q_P^{s_*^\circ}(X) \exp\left\{ -\sqrt{-1} \sum_{j=0}^{\nu} (-1/2)^j s_j^\circ \frac{\operatorname{tr}(X^{2j+1})}{2j+1} \right\} d\mu_\Lambda(X), \end{aligned}$$

in the sense of asymptotic expansions. Moreover, equation (1.7) implies that, at the point $s_*^\circ = (0, 1, 0, 0, \dots)$, the map $Q^{s_*^\circ}$ is a vector space isomorphism, i.e., there exists a vector space isomorphism $Q : \mathbb{C}[\partial/\partial t_*] \to \mathbb{C}[\operatorname{tr} X, \operatorname{tr} X^3, \operatorname{tr} X^5, \dots]$ such that, for $N \gg 0$,

$$\begin{aligned} P(\partial_{t_*}) \int_{\mathcal{H}(N)} \exp\left\{ \frac{\sqrt{-1}}{6} \operatorname{tr} X^3 \right\} d\mu_\Lambda(X) \\ = \int_{\mathcal{H}(N)} Q_P(X) \exp\left\{ \frac{\sqrt{-1}}{6} \operatorname{tr} X^3 \right\} d\mu_\Lambda(X) \end{aligned}$$

in the sense of asymptotic expansions, which is precisely the statement of the Di Francesco-Itzykson-Zuber (henceforth referred to as "DFIZ").

Plan of the paper

The paper is organized as follows.

Section 2 contains a brief glossary of intersection theory on moduli spaces of stable curves and its combinatorial description; moreover the Kontsevich' Main Identity, relating the intersection numbers to the combinatorics of ribbon graphs is recalled. Finally, the partition function $Z(s_*; t_*)$ of combinatorial intersection numbers is introduced.

In section 3, ribbon graphs are introduced from a different point of view, namely, as Feynman diagrams appearing in asymptotic expansions of certain Gaussian integrals. The approach to Feynman diagrams theory is through graphical calculus functors; rules of graphical calculus are recalled and used extensively all through this paper.

In section 4, a cyclic algebra structure, depending on a positive definite Hermitian matrix Λ, is introduced on the space of $N \times N$ complex matrices; we call it the N-dimensional 't Hooft Kontsevich model. Feynman rules for this algebra reproduce the combinatorial terms in Kontsevich' Main Identity. As a corollary, it is shown how the partition function $Z(s_*; t_*)$ is an asymptotic expansion for the partition function of the 't Hooft-Kontsevich model.

In section 5, we use an observation by Witten to relate first order derivatives of the partition function $Z(s_*; t_*)$ to Laurent coefficients of amplitudes (taken in the $(N+1)$-dimensional 't Hooft-Kontsevich model) of graphs with a distinguished hole.

The long section 6 contains the proof of the main result. In rough details, it goes as follows. Laurent coefficients appearing in Witten's formula of Section 5 are polynomials in the eigenvalues of Λ. They can be expressed as amplitudes of ribbon graphs in an extended N-dimensional 't Hooft-Kontsevich model, where vertices are allowed to have polynomial amplitudes. One can then apply an recursive procedure to lower the degree of these polynomials. In the end, a canonical form for the expectation values of the graphs related to the first order derivatives of $Z(s_*; t_*)$ is found. This canonical form is seen to equal $D(s_*; \partial/\partial s_*)Z(s_*; t_*)$ for some D in a certain (non-commutative) algebra $\mathbb{C}\langle\langle s_*; \partial/\partial s_* \rangle\rangle$ of power series in s_* and $\partial/\partial s_*$. As a corollary the main result of this paper follows, i.e., the existence of an algebra homomorphism $D\colon \mathbb{C}\langle \partial/\partial t_* \rangle \to \mathbb{C}\langle\langle s_*; \partial/\partial s_* \rangle\rangle$ such that $D(\partial/\partial t_*)Z(s_*; t_*) = D_P(s_*; \partial/\partial s_*)Z(s_*; t_*)$, for any $P \in \mathbb{C}\langle \partial/\partial t_* \rangle$.

Finally, in Section 7, various corollaries and examples of the main result are given; it is shown how a geometrical interpretation identifies the combinatorial classes on the moduli spaces of curves with the Poincaré duals of the ψ classes and how a matrix integral translation implies the Di Francesco-Itzykson-Zuber theorem [DFIZ93].

2 Intersection numbers on the Moduli Space of Curves

Fix integers $g \geqslant 0$, $n \geqslant 1$ with $2-2g-n<0$. Let $\mathcal{M}_{g,n}$ be the moduli space of smooth complete curves of genus g with n marked points $\{x_1, x_2, \ldots, x_n\}$, and let $\overline{\mathcal{M}}_{g,n}$ be its Deligne-Mumford compactification [DM69]. The moduli space $\overline{\mathcal{M}}_{g,n+1}$ is naturally isomorphic (as a stack) to the universal curve over $\overline{\mathcal{M}}_{g,n}$; that is, if

$$\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$$

is the projection map which "forgets the marking on the point x_{n+1}", then the fiber $\pi^{-1}(p)$ at the generic point p of $\overline{\mathcal{M}}_{g,n}$ has a natural structure of a genus g stable curve C_p with n marked points $\{x_1(p), x_2(p), \ldots, x_n(p)\}$, lying in the isomorphism class represented by p. So we have n canonical sections

$$\begin{aligned} x_i : \overline{\mathcal{M}}_{g,n} &\to \overline{\mathcal{M}}_{g,n+1}, \\ p &\mapsto x_i(p) \in C_p. \end{aligned}$$

Define line bundles $\mathcal{L}_i$ on $\overline{\mathcal{M}}_{g,n}$ by

$$\mathcal{L}_i\big|_p := T^*_{x_i(p)} C_p,$$

and denote by ψ_i the Miller classes ([Mil86, Wit91, Kir03, Mor99])

$$\psi_i := c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{C}).$$

Finally, denote by $\langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle_{g,n}$ the intersection number ([Wit91])

$$\langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle_{g,n} := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\nu_1} \cdots \psi_n^{\nu_n}.$$

The integral on the right hand side makes sense iff $\psi_1^{\nu_1} \cdots \psi_n^{\nu_n} \in H^{\mathrm{top}}(\overline{\mathcal{M}}_{g,n})$, i.e., if and only if $\nu_1 + \cdots + \nu_n = 3g-3+n$. Next, define

$$\langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle := \sum_g \langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle_{g,n}.$$

At most *one* contribution in this sum is non-zero, since $\langle \tau_{\nu_1} \cdots \tau_{\nu_s} \rangle_{g,n}$ can be non-null only for $g = 1 + \frac{1}{3}\left(\left(\sum_{i=1}^n \nu_i\right) - n\right)$.

2.1 The generating series and the partition function

It is convenient to arrange intersection numbers into some formal series.

Definition 2.1. The generating series of intersection numbers ("free energy functional" in physics literature) is the formal series

$$F(t_*) := \sum_{g,n} \Big(1/n! \sum_{\nu_1,\ldots,\nu_n} \langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle_{g,n} t_{\nu_1} \cdots t_{\nu_n} \Big), \tag{2.1}$$

The partition function is the formal series

$$Z(t_*) := \exp F(t_*). \tag{2.2}$$

As remarked by Witten [Wit91], algebraic relations among the intersection numbers $\langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle$ translate into differential equations satisfied by the formal series $F(t_*)$ and $Z(t_*)$. In fact, Witten conjectured [Wit91] and Kontsevich proved [Kon92] that $\partial^2 F(t_*)/\partial {t_0}^2$ satisfies the KdV hierarchy. Moreover, $F(t_*)$ satisfies the *string equation*

$$\frac{F(t_*)}{\partial t_0} = \frac{{t_0}^2}{2} + \sum_{i=0}^{\infty} t_{i+1} \frac{F(t_*)}{\partial t_i};$$

thus Kontsevich' result is equivalent to saying that $Z(t_*)$ is a weight 0 vector for a Virasoro algebra of differential operators. In this Virasoro algebra formulation, the Kontsevich-Witten result has been proven by Witten [Wit92].

A solution of the KdV hierarchy can be recursively computed, so the Kontsevich-Witten theorem allows one to recursively compute all intersection indices. Further details on integrable hierarchies related to intersection theory on the moduli spaces of stable curves and, more in general, of stable maps, can be found in [DZ01, EHX97, Get99, KM94, OP01].

2.1.1 Kontsevich' Matrix Integral

Both Kontsevich' and Witten's proofs are based on the integral representation for the partition function $Z(t_*)$, found by Kontsevich in [Kon92]. Let $\mathcal{H}(N)$ be the space of $N \times N$ Hermitian matrices, and let $\Lambda \in \mathcal{H}(N)$ be a diagonal matrix with positive real eigenvalues $\{\Lambda_i\}_{i=1,\ldots,N}$. Denote by $\mathrm{d}\mu_\Lambda$ the Gaussian measure on $\mathcal{H}(N)$ obtained by normalizing $\exp\{-\frac{1}{2} \operatorname{tr} \Lambda X^2\} \mathrm{d}X$, and let $t_k(\Lambda) = -(2k-1)!! \operatorname{tr} \Lambda^{-(2k+1)}$. Then:

$$Z(t_*)\big|_{t_*(\Lambda)} \asymp \int_{\mathcal{H}(N)} \exp\Big\{ \frac{\sqrt{-1}}{6} \operatorname{tr} X^3 \Big\} \mathrm{d}\mu_\Lambda(X), \tag{2.3}$$

which holds in the sense of asymptotic expansions as eigenvalues of Λ tend to ∞. The proof of this formula relies on the rôle played by ribbon graphs on both sides. Indeed, they arise on the left hand side as the cells of a combinatorial description of the moduli space of curves, and on the right hand side as Feynman diagrams in the asymptotic expansion of the integral. We will now briefly recall the definition of ribbon graphs and the combinatorial cellularization of the moduli spaces of curves; ribbon graphs as Feynman diagrams will be described in Section 3.

2.2 A Triangulation of the Moduli Space of Curves

A well-known construction (see [Har88, HZ86, Kon92, Loo95, MP98]), based on results of Jenkins-Strebel, leads to a combinatorial description of the moduli space $\mathcal{M}_{g,n}$. Let us recall its main points.

Definition 2.2. A ribbon graph is a 1-dimensional CW-complex such that any vertex is equipped with a cyclic order on the set of incident half-edges. Morphisms of ribbon graphs are morphisms of CW-complexes that preserve the cyclic ordering at every vertex.

Isomorphisms of ribbon graphs are, in particular, homeomorphisms of the underlying CW-complexes. Call $\operatorname{Aut}\Gamma$ the group of automorphisms of the ribbon graph Γ.

For any ribbon graph Γ, denote $\Gamma^{(0)}$ the set of its vertices and $\Gamma^{(1)}$ the set of its edges. Given any ribbon graph, one can use the cyclic order on the vertices to "fatten" edges into thin ribbons[1] (see Figure 1 on page 532). Therefore, a closed ribbon graph Γ is turned into a compact oriented surface with boundary $S(\Gamma)$. The boundary components of $S(\Gamma)$ retract onto particular 1-homology cycles on Γ, which we call "holes"; the set of holes of Γ is denoted $\Gamma^{(2)}$.

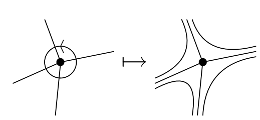


Figure 1: Fattening edges at a vertex with cyclic order.

The number of boundary components n and the genus g of the closed ribbon graph Γ are defined to be those of the surface $S(\Gamma)$.

Definition 2.3. A metric on a closed ribbon graph Γ is a function $\ell : \Gamma^{(1)} \to \mathbb{R}_{>0}$. A numbering on Γ is a map $h : \Gamma^{(2)} \to \{1, 2, \ldots, n\}$.

[1]Hence the name "ribbon graph".

Morphisms of numbered (resp. metric) ribbon graphs are morphisms of ribbon graphs that, in addition, preserve the numbering (resp. the metric).

Note that automorphisms of a numbered graph (Γ, h) act trivially on the set $\Gamma^{(2)}$.

Fix a connected numbered graph (Γ, h) with all vertices of valence $\geqslant 3$. The set $\Delta(\Gamma, h)$ of all metrics on Γ has a natural structure of a topological m-cell ($m = |\Gamma^{(1)}|$) equipped with a natural action of $\mathrm{Aut}(\Gamma, h)$; cells $\Delta(\Gamma, h)$, with (Γ, h) ranging over numbered ribbon graphs of given genus g and number of holes n, can be glued to form an orbi-cell-complex $\mathcal{M}_{g,n}^{\mathrm{comb}}$ (see, e.g., [Loo95], [MP98]), of which $\mathcal{M}_{g,n}$ is a deformation retract.

Proposition 2.1 ([Loo95]). *There is an orbifold isomorphism $\mathcal{M}_{g,n} \times \mathbb{R}_{>0}^n \simeq \mathcal{M}_{g,n}^{comb}$.*

It follows by the above proposition that any integral over $\mathcal{M}_{g,n} \times \mathbb{R}_{>0}^n$ can be written as a sum over numbered ribbon graphs:

$$\int_{\mathcal{M}_{g,n}^{\mathrm{comb}}} = \sum_{(\Gamma,h)} \frac{1}{|\mathrm{Aut}(\Gamma, h)|} \int_{\Delta(\Gamma,h)} \tag{2.4}$$

where (Γ, h) ranges over the set of isomorphism classes of closed connected numbered ribbon graphs of genus g with n holes, and the factors $1/|\mathrm{Aut}(\Gamma, h)|$ appear since we are integrating over an orbifold [Sat56]. In [Kon92], representative 2-forms ω_i for the cohomology classes ψ_i are found. In terms of ω_i's, the intersection indices are written

$$\langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle_{g,n} = \int_{\mathcal{M}_{g,n}^{\mathrm{comb}}} \omega_1^{\nu_1} \wedge \cdots \wedge \omega_n^{\nu_n} \wedge [\mathbb{R}_{>0}^n], \tag{2.5}$$

where $[\mathbb{R}_{>0}^n]$ is the fundamental class with compact support of $\mathbb{R}_{>0}^n$.

As a consequence of (2.4), Kontsevich finds the following remarkable identity.

Proposition 2.2 (Kontsevich' Main Identity [Kon92]). *For all n and g, the following formula holds:*

$$\sum_{\nu_1,\ldots,\nu_n} \langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle_{g,n} \prod_{i=1}^{n} \frac{(2\nu_i - 1)!!}{\lambda_i^{2\nu_i+1}} = \sum_{(\Gamma,h)} \frac{1}{|\mathrm{Aut}(\Gamma, h)|} \left(\frac{1}{2}\right)^{|\Gamma^{(0)}|} \prod_{l \in \Gamma^{(1)}} \frac{2}{\lambda_{h(l^+)} + \lambda_{h(l^-)}}$$

where: λ_i are positive real variables; for any edge l of a ribbon graph Γ, $l^+, l^- \in \Gamma^{(2)}$ denote the (not necessarily distinct) holes l belongs to; (Γ, h) ranges over the set of isomorphism classes of closed connected numbered ribbon graphs of genus g with n holes.

2.3 Intersection theory on combinatorial moduli spaces

Kontsevich described a natural generalization of these constructions.

Definition 2.4. Let $m_* := (m_0, m_1, \ldots, m_k, \ldots)$ be a sequence of non-negative integers such that $m_i \neq 0$ only for a *finite* number of indices i. A ribbon graph Γ is said to be of *combinatorial type* m_* if it has exactly m_i vertices of valence $2i+1$, for $i \geqslant 0$, and no vertices of even valence.

One can consider the set of all cells $\Delta(\Gamma, h)$, where Γ ranges over closed connected ribbon graphs of a given combinatorial type m_*, and $h : \Gamma^{(2)} \to \{1, \ldots, n\}$ is a numbering on the holes of Γ. It can be shown that these cells can be glued together into an orbifold $\mathcal{M}_{m_*,n}$. If $m_0 = 0$, then $\mathcal{M}_{m_*,n}$ is a sub-orbifold of $\cup_g \mathcal{M}_{g,n}^{\text{comb}}$ and the support of a homological cycle.

Equation (2.5) can be generalized to the following definition.

Definition 2.5. The combinatorial intersection index $\langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle_{m_*,n}$ is defined by

$$\langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle_{m_*,n} := \int_{\mathcal{M}_{m_*,n}^{\text{comb}}} \omega_1^{\nu_1} \wedge \cdots \wedge \omega_n^{\nu_n} \wedge [\mathbb{R}_{>0}^n].$$

Since $\bigcup_{m_1} \mathcal{M}_{(0,m_1,0,\ldots);n} = \bigcup_g \mathcal{M}_{g,n}^{\text{comb}}$, then one easily computes:

$$\sum_{m_1} \langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle_{0,m_1,0,0,\ldots;n} = \sum_g \langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle_{g,n} =: \langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle. \tag{2.6}$$

Also Kontsevich' Main Identity can be generalized to this combinatorial context.

Proposition 2.3 (Kontsevich' Main Identity [Kon92]). *For any n and any combinatorial type m_*, the following formula holds:*

$$\sum_{\nu_1,\ldots,\nu_n} s_*^{m_*} \langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle_{m_*;n} \prod_{i=1}^n \frac{(2\nu_i - 1)!!}{\lambda_i^{2\nu_i+1}}$$

$$= \sum_{(\Gamma,h)} \frac{1}{|\mathrm{Aut}(\Gamma, h)|} \prod_{j=0}^{\infty} \left(\frac{s_j}{2^j}\right)^{m_j} \prod_{l \in \Gamma^{(1)}} \frac{2}{\lambda_{h(l^+)} + \lambda_{h(l^-)}} \tag{2.7}$$

where (Γ, h) *ranges over the set of isomorphism classes of closed connected numbered ribbon graphs with* n *holes and combinatorial type* m_**, and the* s_* *are complex variables.*

The free energy and partition function make sense also in this broader setting:

$$F(s_*; t_*) = \sum_{m_*, n} F_{m_*,n}(s_*; t_*) := \sum_{m_*, n} \left(\frac{1}{n!} \sum_{\nu_1, \ldots, \nu_n} s_*^{m_*} \langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle_{m_*,n} t_{\nu_1} \cdots t_{\nu_n} \right), \tag{2.8}$$

where $s_*^{m_*} = \prod_{i=0}^{\infty} s_i^{m_i}$ (this is actually a *finite* product), and

$$Z(s_*; t_*) := \exp F(s_*; t_*). \tag{2.9}$$

Correspondingly, one has the integral representation:

$$Z(s_*; t_*)\Big|_{t_* = t_*(\Lambda)} \asymp \int_{\mathcal{H}(N)} \exp\Big\{ -\sqrt{-1} \sum_j s_j \frac{\operatorname{tr} X^{2j+1}}{2j+1} \Big\} \mathrm{d}\mu_\Lambda, \tag{2.10}$$

which will be proved in Proposition 4.1. We recover the usual relations (2.1), (2.2) and (2.3) by setting $s_* = (0, 1, 0, 0, \ldots)$ in (2.8), (2.9) and (2.10), because $\bigcup_{m_1} \mathcal{M}_{(0, m_1, 0, 0, \ldots); n} = \bigcup_g \mathcal{M}_{g,n}^{\mathrm{comb}}$.

Remark 2.1. The variables s_* and t_* are actually of *two* different kinds. Indeed, the t_* variables are free indeterminates, whereas the s_* variables are the structure constants of a 1-dimensional cyclic A_∞ algebra (see [Kon94]). Since there are no constraints on the structure constants of a 1-dimensional cyclic A_∞-algebra, the s_*'s are free in the context of this paper. However, when dealing with combinatorial classes arising from higher dimensional cyclic A_∞ algebras, the different nature of the two set of variables becomes evident.

3 Ribbon graphs as Feynman diagrams

In [RT90], Reshetikhin and Turaev defined graphical calculus as a functorial correspondence between certain sets of graphs and morphisms in suitable categories. In one of its incarnations, this graphical calculus is suitable for working on ribbon graphs: we follow our treatment [FM02] and refer the reader also to [Fio02, Oec01] for precise statements and proofs.

Definition 3.1. A ribbon graph with n legs is a ribbon graph with a distinguished subset of n univalent vertices, called "endpoints". An edge stemming from one endpoint is called a "leg". Edges which are not legs, and vertices which are not endpoints are called "internal". We shall divide internal vertices into two classes ("colors"): "ordinary" and "special" vertices. Morphisms of ribbon graphs with legs map endpoints into endpoints, and preserve vertex color.

The set of isomorphism classes of ribbon graphs with n legs is denoted by the symbol $\mathcal{R}(n)$; we also set

$$\mathcal{R} := \bigcup_{n=0}^{\infty} \mathcal{R}(n).$$

An element of $\mathcal{R}(0)$ is called a *closed ribbon graph*. Disjoint union gives a map $\mathcal{R}(m) \times \mathcal{R}(m) \to \mathcal{R}(m+n)$, hence a map $\mathcal{R} \times \mathcal{R} \to \mathcal{R}$.

We will say just "vertex" to mean "internal vertex". Moreover, by abuse of notation, a connected and simply connected ribbon graph with exactly one internal vertex will be called simply a "vertex" (ordinary or special depending on the color of the internal vertex). The n-valent *special* vertex will be denoted by the symbol $\mathsf{v}_n \in \mathcal{R}(n)$.

Definition 3.2. A ribbon graph of type (p, q) is a ribbon graph with $p + q$ legs, which are partitioned into two disjoint totally ordered subsets: p "inputs" and q "outputs" (see Figure 2 on page 537).

Morphisms of ribbon graphs of type (p, q) are morphisms of ribbon graphs with $p+q$ legs which send input legs into input legs, output legs into output legs, and preserve the total order on both.

The set isomorphism classes of ribbon graphs of type (p, q) is denoted by the symbol $\mathcal{R}(p, q)$ and its $\mathbb{C}$-linear span by the symbol $\mathscr{R}(p, q)$.

The sets of inputs and outputs of a ribbon graph $\hat{\Gamma}$ of type (p, q) are denoted, respectively, as $\mathrm{In}(\hat{\Gamma})$ and $\mathrm{Out}(\hat{\Gamma})$. The datum of the total order on the legs (of either kind) is equivalent to a numbering, i.e., to bijections

$$\mathrm{In}(\hat{\Gamma}) \leftrightarrow \{1, \dots, p\}, \qquad \mathrm{Out}(\hat{\Gamma}) \leftrightarrow \{1, \dots, q\}.$$

One can define a composition product $\mathcal{R}(p, q) \times \mathcal{R}(r, p) \ni (\hat{\Phi}, \hat{\Psi}) \to \hat{\Phi} \circ \hat{\Psi} \in \mathcal{R}(r, q)$ by gluing input edges of $\hat{\Phi}$ with corresponding output edges of $\hat{\Psi}$. It extends to a bilinear composition product $\mathscr{R}(p, q) \otimes \mathscr{R}(r, p) \to \mathscr{R}(r, q)$ which turns $\mathscr{R}$ into the Hom-functor of a category whose objects are natural numbers; denote this category by the same symbol $\mathscr{R}$. Juxtaposition defines a tensor product $\otimes$ on (isomorphism classes of) ribbon graphs; one can check that $\otimes$ makes the category $\mathscr{R}$ monoidal.

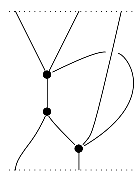


Figure 2: A ribbon graph of type (2,3).

Every ribbon graph can be assembled from elementary graphs: vertices (both ordinary and special) with k inputs and no output for each $k \geqslant 1$, and connecting "bent" edges with both legs outgoing (see Figure 3 on page 537). As a monoidal category, $\mathscr{R}$ is generated by elements corresponding to these pieces; therefore, one can define a monoidal functor just by assigning its values on generating elements.

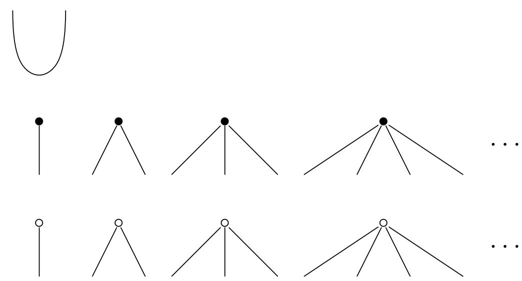


Figure 3: The generators of $\mathscr{R}$: the bent edge, ordinary vertices, special vertices

Definition 3.3. A cyclic algebra $A = (V, b, T_1, T_2, \ldots)$ over $\mathbb{C}$ is the data of a $\mathbb{C}$-linear space V, of a symmetric non-degenerate bilinear form $b : V \otimes V \to \mathbb{C}$, and of cyclically invariant tensors $T_r : V^{\otimes r} \to \mathbb{C}$:

$$T_r(v_1 \otimes \ldots \otimes v_{r-1} \otimes v_r) = T_r(v_r \otimes v_1 \otimes \ldots v_{r-1}).$$

Let $\langle V \rangle_{x_*}$ be the category having the tensor powers $V^{\otimes r}$, for $r \geqslant 0$, as objects, and

$$\mathrm{Hom}_{x_*}(V^{\otimes p}, V^{\otimes q}) := \mathrm{Hom}(V^{\otimes p}, V^{\otimes q}) \otimes \mathbb{C}[x_*]$$

as Hom-spaces. Since b is non-degenerate, it induces a canonical isomorphism between V and its dual, and we can give $\langle V \rangle_{x_*}$ a structure of a rigid monoidal category in which every object is self-dual.

Proposition 3.1. *Given a cyclic algebra A, there is a unique monoidal functor $Z_A : \mathscr{R} \to \langle V \rangle_{x_*}$ that maps:*

- *r-valent ordinary vertices to morphisms $x_r T_r$;*
- *r-valent special vertices to morphisms T_r;*
- *bent edges to the copairing $b^\vee : \mathbb{C} \to V \otimes V$ dual to the pairing $b : V \otimes V \to \mathbb{C}$.*

The graphical calculus functor Z_A defines (a family of) linear maps

$$Z_A : \mathscr{R}(p, q) \to \mathrm{Hom}(V^{\otimes p}, V^{\otimes q}) \otimes \mathbb{C}[x_*]$$

such that

$$\begin{aligned} Z_A(\hat{\Phi} \circ \hat{\Gamma}) &= Z_A(\hat{\Phi}) \circ Z_A(\hat{\Gamma}), \\ Z_A(\hat{\Phi} \otimes \hat{\Gamma}) &= Z_A(\hat{\Phi}) \otimes Z_A(\hat{\Gamma}). \end{aligned}$$

Note that, if $\hat{\Gamma}$ is a ribbon graph of type $(0, 0)$, then $Z_A(\hat{\Gamma})$ is an element of $\mathrm{Hom}(V^{\otimes 0}, V^{\otimes 0}) \otimes \mathbb{C}[x_*] \simeq \mathbb{C}[x_*]$, i.e., it is actually a polynomial with complex coefficients.

Remark 3.1. The vector space V is called the space of "fields". The tensor $Z_A(\hat{\Gamma})$ is the "amplitude" of the graph $\hat{\Gamma}$; in the graphical notation, structure constants of this tensor are denoted by the graph with indices attached to the legs, whereas the same graph with no indices will stand for the amplitude tensor itself. Amplitudes of vertices and bent edges are called, respectively, "interactions" and "propagators". The data of propagators and interactions are called the *Feynman rules* of Z_A.

3.1 Expectation values of graphs

The usual correspondence between Feynman diagrams and Gaussian integrals will play a key rôle in this paper; for our purposes, we can summarize it in the following.

Let $(V_\mathbb{R}, b)$ be a real Hilbert space and $(V_\mathbb{C}, b)$ its complexification; let $A := (V_\mathbb{C}, b, T_1, T_2, \ldots)$ be a fixed cyclic algebra structure on $V_\mathbb{C}$.

For any ribbon graph $\Psi \in \mathcal{R}$ (possibly with special vertices), we denote by $\mathcal{R}_\Psi$ the set of (isomorphism classes of) ribbon graphs containing Ψ as a distinguished sub-graph and having no special vertex outside Ψ. By saying that the sub-graph Ψ is distinguished, we require that any automorphism of an object $\Gamma \in \mathcal{R}_\Psi$ maps Ψ onto itself. It follows from the definition that $\mathcal{R}_\emptyset$ is the set of ribbon graphs having only ordinary vertices.

The amplitude of an element of $\mathcal{R}(n)$ is not a well-defined tensor, since there is no distinction between inputs and outputs and no ordering on the legs. On the other hand, forgetting this ordering and the distinction between "inputs" and "outputs" gives a natural map $\mathcal{R}(p,q) \to \mathcal{R}(p+q)$; in particular, if $\Gamma \in \mathcal{R}(n)$, then two pre-images of Γ in $\mathcal{R}(n,0)$ may only differ by a permutation of the indices on the inputs. Thus, we can regard all the legs of Γ as inputs, and define a linear map $Z_A(\Gamma)$ on the sub-space of $\mathfrak{S}_n$-invariant vectors of $V^{\otimes n}$:

$$Z_A(\Gamma)\colon \left(V^{\otimes n}\right)^{\mathfrak{S}_n} \to \mathbb{C}[x_*], \qquad \Gamma \in \mathcal{R}(n).$$

Note, in particular, that the amplitude of a closed ribbon graph is a well defined polynomial: indeed, the canonical map $\mathcal{R}(0,0) \to \mathcal{R}(0)$ is an identification.

Definition 3.4. Let Ψ be any ribbon graph. Its *expectation value* is the formal series in the variables x_*:

$$\langle\!\langle \Psi \rangle\!\rangle_A := \sum_{\Gamma \in \mathcal{R}_\Psi(0)} \frac{Z_A(\Gamma)}{|\mathrm{Aut}\,\Gamma|}\,.$$

If the graph Ψ has n legs, the function $v \mapsto Z_A(\Psi)\big(v^{\otimes n}\big)$ is a well defined polynomial map $V \to \mathbb{C}[x_*]$. Therefore, it is integrable on $V_\mathbb{R}$ with respect to the normalized Gaussian measure

$$\mathrm{d}\mu := \frac{\exp\left\{-\frac{1}{2}b(v,v)\right\}\mathrm{d}v}{\int_{V_\mathbb{R}} \exp\left\{-\frac{1}{2}b(v,v)\right\}\mathrm{d}v}.$$

Definition 3.5. The formal series

$$S_A(x_*) := \sum_{k=1}^{\infty} x_k \frac{T_k(v^{\otimes k})}{k}$$

is called the *potential* of the cyclic algebra A.

With the above notations, we have the following fundamental formula.

Proposition 3.2 (Feynman-Reshetikhin-Turaev). *For any ribbon graph $\Psi \in \mathcal{R}(n)$, the following asymptotic expansion holds:*

$$\langle\!\langle \Psi \rangle\!\rangle_A = \int_{V_\mathbb{R}} \frac{Z_A(\Psi)(v^{\otimes n})}{|\mathrm{Aut}\,\Psi|} \exp S_A(x_*) \mathrm{d}\mu(v) \tag{3.1}$$

For a proof, see [FM02, Theorem 3.6. and Formula 3.3] and [Fio02].

In particular, we have:

$$\langle\!\langle \emptyset \rangle\!\rangle_A = \int_{V_{\mathbb{R}}} \exp S_A(x_*) \mathrm{d}\mu(v)$$

$$\langle\!\langle \mathsf{v}_n \rangle\!\rangle_A = \int_{V_{\mathbb{R}}} \frac{T_n(v^{\otimes n})}{n} \exp S_A(x_*) \mathrm{d}\mu(v)$$

$$\langle\!\langle \coprod_{i=1}^{n} \mathsf{v}_i{}^{\amalg\, m_i} \rangle\!\rangle_A = \int_{V_{\mathbb{R}}} \frac{\prod_{i=1}^n \left(T_i(v^{\otimes i})\right)^{m_i}}{\prod_{i=1}^n i^{m_i} m_i!} \exp S_A(x_*) \mathrm{d}\mu(v)$$

where $\coprod_{i=1}^n \mathsf{v}_i{}^{\amalg\, m_i}$ denotes the disjoint union of m_1 copies of v_1, m_2 copies of v_2, ... and m_n copies of v_n.

Definition 3.6. The formal series

$$Z_A(x_*) := \langle\!\langle \emptyset \rangle\!\rangle_A, \qquad F_A(x_*) := \log Z_A(x_*),$$

are called, respectively, the *partition function* and the *free energy* of the cyclic algebra A.

Remark 3.2. By definition, the partition function is the weighted sum of the amplitudes of all closed ribbon graphs with only ordinary vertices; a standard combinatorial argument (see [BIZ80]) proves that the free energy can be written as the weighted sum of the amplitudes of all *connected* closed ribbon graphs with only ordinary vertices.

Remark 3.3. Note that

$$\frac{\partial}{\partial x_n} \langle\!\langle \emptyset \rangle\!\rangle_A = \langle\!\langle \mathsf{v}_n \rangle\!\rangle_A, \tag{3.2}$$

and, more in general,

$$\frac{\partial^{m_1+\cdots+m_n}}{\partial {x_1}^{m_1} \cdots \partial {x_n}^{m_n}} \langle\!\langle \emptyset \rangle\!\rangle_A = m_1! \cdots m_n! \cdot \langle\!\langle \coprod_{i=1}^{n} \mathsf{v}_i{}^{\amalg\, m_i} \rangle\!\rangle_A, \tag{3.3}$$

that is, derivatives of $\langle\!\langle \emptyset \rangle\!\rangle_A$ can be written as expectation values of (disjoint unions of) special vertices.

3.2 Ribbon graphs with colored edge-sides

Let $(V, b, T_1, T_2, \dots)$ be a cyclic algebra and assume that V has a decomposition $V = \bigoplus_{\xi,\eta\in I} V_{\xi,\eta}$ where $V_{\xi,\eta}$ and $V_{\eta,\xi}$ are dual subspaces with respect to the pairing b. Since any edge of a ribbon graph has two (distinct) sides, it is meaningful to consider ribbon graphs with edge sides colored by elements of I. We introduce the following new graphical calculus element on V:

$$Z_{A,I}\left(\begin{array}{c} 1_{\mathrm{out}} \\ \xi \,|\, \eta \\ 1_{\mathrm{in}} \end{array}\right) := \pi_{\xi,\eta} \colon V \to V\,,$$

with $\pi_{\xi,\eta}$ the orthogonal projection on the subspace $V_{\xi,\eta}$. Since

$$\begin{array}{c} \xi \,|\, \eta \; \frown \\ x \quad\; y \end{array} = b(\pi_{\xi,\eta}(x), y) = b(x, \pi_{\eta,\xi}(y)) = \begin{array}{c} \frown \; \eta \,|\, \xi \\ x \quad\; y \end{array},$$

then graphical calculus extended to ribbon graphs with colored edge sides is well defined; that is, a graphical calculus functor $Z_{A,I}$ is defined for any cyclic algebra A and a decomposition into subspaces indexed by I as above; moreover, $Z_{A,I}$ enjoys the properties listed in Proposition 3.1.

From $\mathrm{id}_V = \bigoplus_{\xi,\eta\in I} \pi_{\xi,\eta}$ we obtain the graphical identity

$$Z_A\left(\begin{array}{c} 1_{\mathrm{out}} \\ | \\ 1_{\mathrm{in}} \end{array}\right) = \bigoplus_{\xi,\eta\in I} Z_{A,I}\left(\begin{array}{c} 1_{\mathrm{out}} \\ \xi \,|\, \eta \\ 1_{\mathrm{in}} \end{array}\right),$$

so the amplitude of a ribbon graph Γ is expanded into the sum of amplitudes of ribbon graphs obtained by coloring the edge sides of Γ with colors in the set I, in all possible ways.

4 The 't Hooft-Kontsevich model

The space $M_N(\mathbb{C})$ of $N \times N$ complex matrices has a natural Hermitian inner product

$$(X|Y) := \mathrm{tr}(X^*Y),$$

which induces the standard Euclidean inner product $(X|Y) = \mathrm{tr}(XY)$ on the real subspace of Hermitian matrices

$$\mathcal{H}(N) := \{X \in M_N(\mathbb{C}) | X^* = X\}.$$

For any positive definite $N \times N$ Hermitian matrix Λ, we can define a new Euclidean inner product on $\mathcal{H}(N)$ as

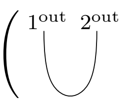
$$(X|Y)_\Lambda := \tfrac{1}{2}\big(\mathrm{tr}(X\Lambda Y) + \mathrm{tr}(Y\Lambda X)\big).$$

The complexification $\mathcal{H}(N) \otimes \mathbb{C}$ is canonically isomorphic to the space $M_N(\mathbb{C})$ of $N \times N$ complex matrices, so the pairing $(-|-)_\Lambda$ induces a non-degenerate symmetric bilinear form on $M_N(\mathbb{C})$. Define cyclic tensors $T_k : M_N(\mathbb{C})^{\otimes k} \to \mathbb{C}$ by

$$T_k(X_1 \otimes X_2 \otimes \cdots \otimes X_k) := \operatorname{tr}(X_1 \cdots X_k)$$

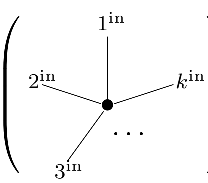
The tensors T_k together with the pairing $b := (-|-)_\Lambda$ give a cyclic algebra structure on the space of $N \times N$ complex matrices; denote Z_Λ the graphical calculus functor induced by this cyclic algebra structure on the category of ribbon graphs (see Proposition 3.1).

We ought to compute $Z_\Lambda(\Gamma)$ for all the generators, i.e., for k-valent vertices with incoming legs and for the bent edge with outgoing legs. Denote by $\{E_{ij}\}$ the canonical basis of $M_N(\mathbb{C})$; it is immediate to reckon:

$Z_\Lambda \Bigg($ 

$\Bigg)$

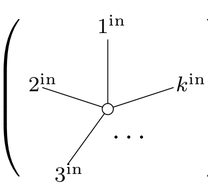
$$= \sum_{i,j} \frac{2}{\Lambda_i + \Lambda_j} E_{ij} \otimes E_{ji}, \tag{4.1}$$

$Z_\Lambda \Bigg($ 

$\Bigg)$

$$(E_{j_k i_1} \otimes E_{j_1 i_2} \otimes \cdots \otimes E_{j_{k-1} i_k})$$
$$= x_k T_k(E_{j_k i_1} \otimes E_{j_1 i_2} \otimes \cdots \otimes E_{j_{k-1} i_k})$$
$$= x_k \cdot \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_k j_k}, \tag{4.2}$$

and

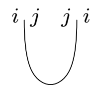
$Z_\Lambda \Bigg($ 

$\Bigg)$

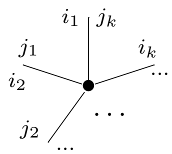
$$(E_{j_k i_1} \otimes E_{j_1 i_2} \otimes \cdots \otimes E_{j_{k-1} i_k})$$
$$= \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_k j_k}. \tag{4.3}$$

For any $i, j \in \{1, \ldots, N\}$, let $M_{ij} = \mathbb{C} \cdot E_{ij}$; then $M_N(\mathbb{C}) = \bigoplus_{i,j} M_{ij}$, and M_{ij} is the dual of M_{ji} with respect to the pairing $(-|-)_\Lambda$, so Z_Λ is actually a graphical calculus for ribbon graphs with edge sides colored with indices from $I = \{1, \ldots, N\}$.

Decorating the sides of a leg with the indices i, j from $\{1, \ldots, N\}$ is coherent with the convention of writing i, j near an endpoint to denote evaluation at the basis element E_{ij}. Formulas (4.1)–(4.3) can therefore be rewritten as:

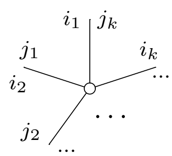


$$= \frac{2}{\Lambda_i + \Lambda_j} E_{ij} \otimes E_{ji},$$



$$= \begin{cases} x_k, & \text{if } i_l = j_l, \text{ for all } l, \\ 0, & \text{otherwise,} \end{cases}$$

and



$$= \begin{cases} 1, & \text{if } i_l = j_l, \text{ for all } l, \\ 0, & \text{otherwise.} \end{cases}$$

That is, according to (4.2), a vertex gives a non-zero contribution if and only if sides belonging to the same hole are decorated with the same index.

Summing up, for any closed ribbon graph Γ, one has:

$$Z_\Lambda(\Gamma) = \prod_{k=1}^{\infty} {x_k}^{m_k} \sum_c \prod_{l \in \Gamma^{(1)}} \frac{2}{\Lambda_{c(l^+)} + \Lambda_{c(l^-)}}, \tag{4.4}$$

where m_k is the number of ordinary k-valent vertices of Γ, c ranges in the set of all maps $\Gamma^{(2)} \to \{1, \dots, N\}$, and $l^\pm$ are the two (not necessarily distinct) holes l belongs to.

4.1 The 't Hooft-Kontsevich model

The right hand side of equation (4.4) is similar to the right hand side of the Kontsevich's Main Identity (2.7); indeed, we can tie graphical calculus to Kontsevich' results as follows.

Definition 4.1. Denote Z_{Λ,s_*} the functor obtained from the graphical calculus Z_Λ by taking:

$$x_k = \begin{cases} 0, & \text{if } k \text{ is even,} \\ -\sqrt{-1}\left(-\frac{1}{2}\right)^r s_r, & \text{if } k = 2r+1 \text{ is odd.} \end{cases} \tag{4.5}$$

The resulting graphical calculus Z_{Λ,s_*} is called the 't Hooft-Kontsevich model.

The partition function (see Definition 3.6) of the 't Hooft-Kontsevich model is the formal series in the variables s_*:

$$\langle\!\langle \emptyset \rangle\!\rangle_{\Lambda,s_*} := \sum_{\Gamma\in\mathcal{R}_\emptyset(0)} \frac{Z_{\Lambda,s_*}(\Gamma)}{|\operatorname{Aut}\Gamma|}. \tag{4.6}$$

It is an asymptotic expansion of the matrix integral

$$\int_{\mathcal{H}(N)} \exp S(s_*;X) \mathrm{d}\mu_\Lambda(X)\,,$$

where $S(s_*,X)$ is the potential of the 't Hooft-Kontsevich model, given by

$$S(s_*,X) = -\sqrt{-1}\sum_{j=0}^{\infty}(-1/2)^j s_j \frac{\operatorname{tr} X^{2j+1}}{2j+1},$$

and $\mathrm{d}\mu_\Lambda$ is the Gaussian measure on $\mathcal{H}(N)$ induced by the pairing $(-|-)_\Lambda$ — see Definition 3.5 and Proposition 3.2.

To compute the partition function $\langle\!\langle \emptyset \rangle\!\rangle_{\Lambda,s_*}$ we have to compute the amplitudes of closed ribbon graphs with only ordinary vertices.

Definition 4.2. We shall say that a closed ribbon graph has combinatorial type m_* if, for any i, it has exactly m_i ordinary vertices of valence $2i+1$, and no special vertices.

Lemma 4.1. *In the N-dimensional 't Hooft-Kontsevich model, for any closed ribbon graph Γ of combinatorial type m_* with n holes, the following formula holds:*

$$Z_{\Lambda,s_*}(\Gamma) = (-1)^n \prod_{r=0}^{\infty}\left(\frac{s_r}{2^r}\right)^{m_r} \sum_c \prod_{l\in\Gamma^{(1)}} \frac{2}{\Lambda_{c(l^+)}+\Lambda_{c(l^-)}}, \tag{4.7}$$

where c ranges in the set of all maps $\Gamma^{(2)} \to \{1,\dots,N\}$, and $l^\pm$ are the two (not necessarily distinct) holes l belongs to.

Proof. By formula (4.4) we immediately obtain:

$$Z_{\Lambda,s_*}(\Gamma) = (-\sqrt{-1})^{|\Gamma^{(0)}|}(-1)^{\sum_{j=0}^{\infty} jm_j} \prod_{r=0}^{\infty}\left(\frac{s_r}{2^r}\right)^{m_r} \sum_c \prod_{l\in\Gamma^{(1)}} \frac{2}{\Lambda_{c(l^+)}+\Lambda_{c(l^-)}}.$$

Thus, we just need to prove $(-\sqrt{-1})^{|\Gamma^{(0)}|}\cdot(-1)^{\sum_{j=0}^{\infty} jm_j} = (-1)^n$; the ribbon graph Γ satisfies the combinatorial relations:

$$\begin{aligned}
|\Gamma^{(0)}| &= \textstyle\sum_j m_j,\\
2|\Gamma^{(1)}| &= \textstyle\sum_j (2j+1)m_j = 2(\sum_j jm_j) + |\Gamma^{(0)}|,\\
|\Gamma^{(0)}| - |\Gamma^{(1)}| + n = |\Gamma^{(0)}| - |\Gamma^{(1)}| + |\Gamma^{(2)}| &= \chi\big(S(\Gamma)\big) \equiv 0 \pmod 2,
\end{aligned}$$

where $S(\Gamma)$ is the Riemann surface associated to Γ and χ denotes the Euler-Poincaré characteristic. Therefore,

$$(-1)^n = \sqrt{-1}^{2n} = \sqrt{-1}^{-|\Gamma^{(0)}|}\sqrt{-1}^{2\sum_j jm_j} = (-\sqrt{-1})^{|\Gamma^{(0)}|}(-1)^{\sum_j jm_j}.$$

$\square$

Proposition 4.1. *The partition function $Z(s_*;t_*)$ of combinatorial intersection numbers and the partition function $\langle\langle\emptyset\rangle\rangle_{\Lambda,s_*}$ of the 't Hooft-Kontsevich model are related by:*

$$Z(s_*;t_*)|_{t_*(\Lambda)} = \langle\langle\emptyset\rangle\rangle_{\Lambda,s_*}.$$

Proof. The statement is clearly equivalent to proving that the same relation holds between the free energies, i.e.,

$$F(s_*;t_*)|_{t_*(\Lambda)} = \log\langle\langle\emptyset\rangle\rangle_{\Lambda,s_*}.$$

By Remark 3.2 and formula (4.7) we immediately get:

$$\log\langle\langle\emptyset\rangle\rangle_{\Lambda,s_*} = \sum_{m_*,n}\sum_{\Gamma,c}\frac{(-1)^n}{|\mathrm{Aut}\,\Gamma|}\prod_{k=0}^{\infty}\left(\frac{s_k}{2^k}\right)^{m_k}\prod_{l\in\Gamma^{(1)}}\frac{2}{\Lambda_{c(l^+)}+\Lambda_{c(l^-)}},$$

where Γ ranges over *connected* closed numbered ribbon graphs of combinatorial type m_* with *only ordinary vertices*, and c is a coloring of $\Gamma^{(2)}$ with colors $\{1,\dots,N\}$. Now, any $c:\Gamma^{(2)}\to\{1,\dots,N\}$ factors in $n!$ ways as $j\circ h$ where h is a bijection $h:\Gamma^{(2)}\to\{1,\dots,n\}$ and j is a map $j:\{1,\dots,n\}\to\{1,\dots,N\}$, so we can rewrite the above equation as:

$$\log\langle\langle\emptyset\rangle\rangle_{\Lambda,s_*} = \sum_{m_*,n}\sum_{\Gamma,h,j}\frac{(-1)^n}{n!|\mathrm{Aut}\,\Gamma|}\prod_{k=0}^{\infty}\left(\frac{s_k}{2^k}\right)^{m_k}\prod_{l\in\Gamma^{(1)}}\frac{2}{\Lambda_{(j\circ h)(l^+)}+\Lambda_{(j\circ h)(l^-)}}.$$

The group $\mathrm{Aut}\,\Gamma$ acts on the sets $\Gamma^{(1)}$ and $\Gamma^{(2)}$; in particular, the second action induces an action of $\mathrm{Aut}\,\Gamma$ on the set

$$\mathrm{Num}(\Gamma) := \left\{h:\Gamma^{(2)}\xrightarrow{\sim}\{1,\dots,n\}\right\}.$$

It is immediate to check that, if h_1 and h_2 are in the same orbit with respect to the action of $\mathrm{Aut}\,\Gamma$, then

$$\prod_{l\in\Gamma^{(1)}}\frac{2}{\Lambda_{(j\circ h_1)(l^+)}+\Lambda_{(j\circ h_1)(l^-)}} = \prod_{l\in\Gamma^{(1)}}\frac{2}{\Lambda_{(j\circ h_2)(l^+)}+\Lambda_{(j\circ h_2)(l^-)}},$$

so that:

$$\log\langle\langle\emptyset\rangle\rangle_{\Lambda,s_*} = \sum_{m_*,n}\sum_{\Gamma,[h],j}\left\{\frac{(-1)^n|\text{orbit of } h|}{n!|\operatorname{Aut}\Gamma|}\prod_{k=0}^{\infty}\left(\frac{s_k}{2^k}\right)^{m_k}\times \prod_{l\in\Gamma^{(1)}}\frac{2}{\Lambda_{(j\circ h)(l^+)}+\Lambda_{(j\circ h)(l^-)}}\right\},$$

and this time we take one representative h from each orbit $[h]$ in $\operatorname{Num}(\Gamma)$.

The isotropy subgroup of any $h \in \operatorname{Num}(\Gamma)$ is $\operatorname{Aut}(\Gamma, h)$; therefore, the cardinality of the orbit of h is $|\operatorname{Aut}\Gamma|/|\operatorname{Aut}(\Gamma, h)|$. Moreover, the numbered ribbon graphs (Γ, h_1) and (Γ, h_2) are isomorphic if and only if h_1 and h_2 are in the same orbit with respect to the action of $\operatorname{Aut}\Gamma$ on $\operatorname{Num}(\Gamma)$, therefore:

$$\log\langle\langle\emptyset\rangle\rangle_{\Lambda,s_*} = \sum_{m_*,n}\sum_{(\Gamma,h),j}\left\{\frac{(-1)^n}{n!|\operatorname{Aut}(\Gamma,h)|}\prod_{k=0}^{\infty}\left(\frac{s_k}{2^k}\right)^{m_k}\times \prod_{l\in\Gamma^{(1)}}\frac{2}{\Lambda_{(j\circ h)(l^+)}+\Lambda_{(j\circ h)(l^-)}}\right\},$$

where (Γ, h) ranges over the set of isomorphism classes of *connected closed numbered* ribbon graphs of combinatorial type m_* with n holes.

Finally, by Kontsevich' Main Identity (2.7) we get:

$$\log\langle\langle\emptyset\rangle\rangle_{\Lambda,s_*} = \sum_{n,m_*,\nu_*}\frac{1}{n!}s_*^{m_*}\langle\tau_{\nu_1}\tau_{\nu_2}\cdots\tau_{\nu_n}\rangle_{m_*,n}\sum_j\prod_{i=1}^{n}\frac{-(2\nu_i-1)!!}{{\Lambda_{j(i)}}^{2\nu_i+1}}$$

where $j\colon \{1,\dots,n\} \to \{1,\dots,N\}$,

$$\begin{aligned}
&= \sum_{n;m_*;\nu_*}\frac{1}{n!}s_*^{m_*}\langle\tau_{\nu_1}\tau_{\nu_2}\cdots\tau_{\nu_n}\rangle_{m_*,n}\prod_{i=1}^{n}\left(-(2\nu_i+1)!!\operatorname{tr}\Lambda^{-(2\nu_i+1)}\right)\\
&= \sum_{n;m_*;\nu_*}\frac{1}{n!}s_*^{m_*}\langle\tau_{\nu_1}\tau_{\nu_2}\cdots\tau_{\nu_n}\rangle_{m_*,n}t_{\nu_1}(\Lambda)\cdots t_{\nu_n}(\Lambda)\\
&= F(s_*;t_*)\big|_{t_*(\Lambda)}.
\end{aligned}$$

$\square$

5 Witten's formula for derivatives

It has been remarked by Witten [Wit92] that first order derivatives of the partition function $Z(t_*)$ are related to expectation values in the $(N+1)$-dimensional 't Hooft-Kontsevich model of ribbon graph with one "distinguished" hole. We shall use graphical calculus for ribbon graphs with sides of two colors in order to present a version of Witten's argument suitable for application to the partition function $Z(s_*; t_*)$.

5.1 The $(N+1)$-dimensional 't Hooft-Kontsevich model

Let z be a *real* positive variable, and consider the $(N+1)$-dimensional 't Hooft-Kontsevich model $Z_{z\oplus\Lambda, s_*}$ defined by the diagonal matrix

$$z \oplus \Lambda = \begin{pmatrix} z & 0 \\ 0 & \Lambda \end{pmatrix}.$$

Let $\{E_{ij}\}_{i,j=0,\dots,N}$ be the canonical basis for $M_{N+1}(\mathbb{C})$, and define

$$\begin{aligned} M^{N+1}_{\Lambda,\Lambda} &:= \operatorname{span}(E_{ij})_{i,j>0} \sim M_N(\mathbb{C}), \qquad & M^{N+1}_{z,\Lambda} &:= \operatorname{span}(E_{0j})_{j>0}, \\ M^{N+1}_{\Lambda,z} &:= \operatorname{span}(E_{i0})_{i>0}, & M^{N+1}_{z,z} &:= \mathbb{C}\cdot E_{00}\,. \end{aligned}$$

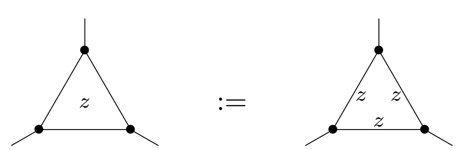
Then we have the following decomposition:

$$M_{N+1}(\mathbb{C}) = \bigoplus_{\xi,\eta\in\{\Lambda,z\}} M^{N+1}_{\xi,\eta}\,,$$

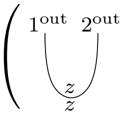
therefore a graphical calculus for ribbon graphs with sides colored with the two colors Λ and z is defined on $M_{N+1}(\mathbb{C})$, extending the Feynman rules for the 't Hooft-Kontsevich model.

Note that a ribbon graph with only Λ-decorated edges is naturally identified to a graphical element for the N-dimensional 't Hooft-Kontsevich model; for this reason, we will omit Λ from the decoration of edges in the displayed diagrams of this paper.

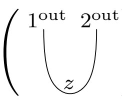
Moreover, we will put a z in the middle of an hole to mean that all the edge-sides of that hole are z-decorated, e.g.,



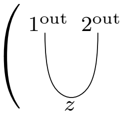
By the above definitions it is immediate to compute propagators in the $(N+1)$-dimensional 't Hooft-Kontsevich model:

$$Z_{z\oplus\Lambda,s_*}\left(\ \right)$$


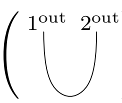
$$= \frac{1}{z}E_{00}\otimes E_{00}, \tag{5.1}$$

$$Z_{z\oplus\Lambda,s_*}\left(\ \right)$$


$$= \sum_{i=1}^{N}\frac{2}{z+\Lambda_i}E_{i0}\otimes E_{0i}, \tag{5.2}$$

$$Z_{z\oplus\Lambda,s_*}\left(\ \right)$$


$$= \sum_{i=1}^{N}\frac{2}{\Lambda_i+z}E_{0i}\otimes E_{i0}, \tag{5.3}$$

$$Z_{z\oplus\Lambda,s_*}\left(\ \right)$$


$$= \sum_{i,j=1}^{N}\frac{2}{\Lambda_i+\Lambda_j}E_{ij}\otimes E_{ji}. \tag{5.4}$$

Since amplitudes of vertices are null if two consecutive sides are not decorated with the same color, then, reasoning as in Section 4, one finds that the amplitude $Z_{z\oplus\Lambda,s_*}(\Gamma)$ of a ribbon graph Γ in the $(N+1)$-dimensional 't Hooft-Kontsevich model can be diagrammatically written as a sum of copies of Γ with some of the holes decorated by the variable z.

5.2 Witten's formula

The machinery is now in place to prove Witten's formula.

Proposition 5.1. *For any $k \geqslant 0$, the following identity holds:*

$$\left.\frac{\partial Z(s_*;t_*)}{\partial t_k}\right|_{t_*(\Lambda)} = -\frac{1}{(2k-1)!!}\,\mathrm{Coeff}_z^{-(2k+1)}\left(\sum_{\Gamma\in\mathcal{R}_\emptyset^{[1]}(0)}\frac{Z_{z\oplus\Lambda,s_*}(\Gamma)}{|\mathrm{Aut}\,\Gamma|}\right), \tag{5.5}$$

where $\mathcal{R}_\emptyset^{[1]}(0)$ denotes the set of isomorphism classes of closed ribbon graphs with only ordinary vertices and exactly one z-decorated hole.

Proof. Since a z-decorated hole lies in one connected component of the ribbon graph, by the usual combinatorial argument we can reduce to connected ribbon graphs, i.e., the statement is equivalent to:

$$\left.\frac{\partial F(s_*;t_*)}{\partial t_k}\right|_{t_*(\Lambda)} = -\frac{1}{(2k-1)!!}\,\mathrm{Coeff}_z^{-(2k+1)}\left(\sum_{\Gamma}\frac{Z_{z\oplus\Lambda,s_*}(\Gamma)}{|\mathrm{Aut}\,\Gamma|}\right), \tag{5.6}$$

with Γ ranging in the set of isomorphism classes of connected closed ribbon graphs with exactly one z-decorated hole and with only generic vertices. To prove equation (5.6), note that the Kontsevich' Main Identity (Proposition 2.3) for graphs with $n+1$ holes numbered from 0 to n gives:

$$\sum_k \left(s_*^{m_*} \sum_{\nu_1,\ldots,\nu_n} \langle \tau_k \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle_{m_*,n+1} \prod_{i=1}^{n} \frac{(2\nu_i - 1)!!}{\lambda_i^{2\nu_i+1}} \right) \frac{(2k-1)!!}{\lambda_0^{2k+1}}$$
$$= \sum_{(\Gamma,h)} \frac{1}{|\mathrm{Aut}\,(\Gamma,h)|} \prod_{r=0}^{\infty} \left(\frac{s_r}{2^r}\right)^{m_r} \prod_{l\in\Gamma^{(1)}} \frac{2}{\lambda_{h(l^+)} + \lambda_{h(l^-)}},$$

where (Γ, h) ranges in the set of isomorphism classes of closed connected ribbon graphs of combinatorial type m_*, with $n+1$ holes, numbered from 0 to n. Therefore,

$$\sum_{m_*,\nu_*} s_*^{m_*} \langle \tau_k \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle_{m_*,n+1} \prod_{i=1}^{n} \frac{(2\nu_i-1)!!}{\lambda_i^{2n_i+1}} = \frac{1}{(2k-1)!!} \times$$
$$\times \,\mathrm{Coeff}_{\lambda_0}^{-(2k+1)} \left(\sum_{(\Gamma,h)} \frac{1}{|\mathrm{Aut}\,(\Gamma,h)|} \prod_{r=0}^{\infty} \left(\frac{s_r}{2^r}\right)^{m_r} \prod_{l\in\Gamma^{(1)}} \frac{2}{\lambda_{h(l^+)} + \lambda_{h(l^-)}} \right).$$

Now set $\lambda_i = \Lambda_{j(i)}$, for $i = 0, 1, \ldots, n$; and sum over all $j : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, N\}$ such that $j(0) = 0$ and recall that $\Lambda_0 = z$ to get:

$$\sum_{m_*,\nu_*,j} s_*^{m_*} \langle \tau_k \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle_{m_*,n+1} \prod_{i=1}^{n} \frac{(2\nu_i-1)!!}{\Lambda_{j(i)}^{(2n_i+1)}} = \frac{1}{(2k-1)!!} \times$$
$$\times \,\mathrm{Coeff}_{z}^{-(2k+1)} \left(\sum_{(\Gamma,h),j} \frac{1}{|\mathrm{Aut}\,(\Gamma,h)|} \prod_{r=0}^{\infty} \left(\frac{s_r}{2^r}\right)^{m_r} \prod_{l\in\Gamma^{(1)}} \frac{2}{\Lambda_{j\circ h(l^+)} + \Lambda_{j\circ h(l^-)}} \right).$$

The proof can then be easily concluded by using the fact that

$$\frac{\partial F(s_*; t_*)}{\partial t_k} = \sum_{m_*,\nu_*} \frac{1}{n!} \langle \tau_k \tau_{\nu_1} \ldots \tau_{\nu_n} \rangle s_*^{m_*} t_{\nu_1} \cdots t_{\nu_n},$$

and reasoning as in the proof of Proposition 4.1. □

5.3 Hole types

If Γ is an element of $\mathcal{R}_{\emptyset}^{[1]}$, i.e., a closed ribbon graph with only ordinary vertices and exactly one hole decorated by the variable z, then its z-decorated hole can be regarded as a distinguished sub-diagram of Γ.

Definition 5.1. A (z-decorated) hole type is a ribbon graph with only ordinary vertices and exactly one z-decorated hole, which is minimal with respect to this property, i.e., such that none of its proper subgraphs contains the z-decorated hole. The set of isomorphism classes of hole types will be denoted by the symbol $\mathcal{S}$.

Having introduced this terminology, the previous remark can be restated as:

$$\mathcal{R}_{\emptyset}^{[1]}(0) = \bigcup_{\Gamma \in \mathcal{S}} \mathcal{R}_{\Gamma}^{[1]}(0),$$

where $\mathcal{R}_{\Gamma}^{[1]}$ denotes the set of isomorphism classes of closed ribbon graphs containing the hole type Γ as a distinguished subgraph and having no z-decorated hole apart from the hole of Γ. Therefore, we can rewrite (5.5) as

$$\left.\frac{\partial Z(s_*;t_*)}{\partial t_k}\right|_{t_*(\Lambda)} = -\frac{1}{(2k-1)!!} \sum_{\Gamma \in \mathcal{S}} \mathrm{Coeff}_z^{-(2k+1)} \left(\sum_{\Phi \in \mathcal{R}_{\Gamma}^{[1]}(0)} \frac{Z_{z\oplus\Lambda,s_*}(\Phi)}{|\mathrm{Aut}\,\Phi|} \right). \tag{5.7}$$

By introducing the shorthand notation

$$\langle\!\langle \Gamma \rangle\!\rangle_{z\oplus\Lambda,s_*}^{[1]} := \sum_{\Phi \in \mathcal{R}_{\Gamma}^{[1]}(0)} \frac{Z_{z\oplus\Lambda}(\Phi)}{|\mathrm{Aut}\,\Phi|}\,, \tag{5.8}$$

where Γ is an hole type, Witten's formula for derivatives finally becomes:

$$\left.\frac{\partial Z(s_*;t_*)}{\partial t_k}\right|_{t_*(\Lambda)} = -\frac{1}{(2k-1)!!} \sum_{\Gamma \in \mathcal{S}} \mathrm{Coeff}_z^{-(2k+1)} \langle\!\langle \Gamma \rangle\!\rangle_{z\oplus\Lambda,s_*}^{[1]} \tag{5.9}$$

6 Proof of the Main Theorem

By the correspondence between Gaussian integrals and expectation values of graphs (see Proposition 3.2), if Γ is an hole type with n legs, then

$$\langle\!\langle \Gamma \rangle\!\rangle_{z\oplus\Lambda,s_*}^{[1]} = \int_{\mathcal{H}(N)} \frac{Z_{z\oplus\Lambda}(\Gamma)}{|\mathrm{Aut}\,\Gamma|} \left(X^{\otimes n}\right) \exp S(s_*;X) \mathrm{d}\mu_{\Lambda}(X)\,. \tag{6.1}$$

Therefore, equation (5.9) is equivalent to

$$\left.\frac{\partial Z(s_*;t_*)}{\partial t_k}\right|_{t_*(\Lambda)} = -\frac{1}{(2k-1)!!}\times \sum_{\Gamma\in\mathcal{S}}\int_{\mathcal{H}(N)}\mathrm{Coeff}_z^{-(2k+1)}\left(\frac{Z_{z\oplus\Lambda,s_*}(\Gamma)}{|\mathrm{Aut}\,\Gamma|}(X^{\otimes n})\right)\times \exp S(s_*;X)\mathrm{d}\mu_\Lambda(X); \quad (6.2)$$

so we are interested in the tensors $\mathrm{Coeff}_z^{-(2k+1)}\, Z_{z\oplus\Lambda,\, s_*}(\Gamma)/|\mathrm{Aut}\,\Gamma|(X^{\otimes n})$.

6.1 Special vertices decorated by polynomials

If Γ is a z-hole type, then its amplitude $Z_{z\oplus\Lambda;s_*}$ has a Laurent expansion in powers of z^{-1} as $z\to\infty$, whose coefficients are polynomials in the Λ_i's. Indeed, by the Feynman rules for the $(N+1)$-dimensional 't Hooft-Kontsevich model, we have:

1. each $(2r+1)$-valent ordinary vertex brings a factor $-\sqrt{-1}(-1/2)^r\, s_r$;
2. each internal edge bordering the z-decorated hole on both sides contributes a factor $1/z$;
3. the other internal edges contribute factors of the form $2/(z+\Lambda_i)$ for $i=1,\ldots,N$.

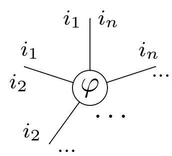
In other words, the structure constants of the tensors $\mathrm{Coeff}_z^{-k}\, Z_{z\oplus\Lambda;s_*}(\Gamma)$ are polynomials in the Λ_i's. We can therefore graphically represent these tensors enlarging the class of special vertices by adding special vertices decorated by polynomials. This is formally done as follows.

Let φ be a polynomial in $\mathbb{C}[\theta_1,\theta_2,\ldots,\theta_n]$; we say that the polynomial φ is cyclically invariant iff it is invariant with respect to the natural action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ on the coordinates. By the symbol v_n^φ we denote an n-valent special vertex decorated by the polynomial φ. We represent graphically these vertices as:

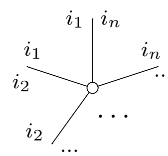
(n edges)

cyclically invariant φ — any polynomial φ

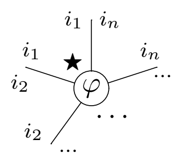
The rôle of the "★" mark is precisely to break the cyclical symmetry of the graphical element. We have to define the Feynman rules for these new vertices; set



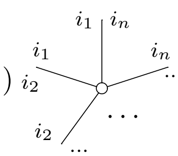
$$:= \varphi(\Lambda_{i_1}, \Lambda_{i_2}, \dots, \Lambda_{i_n}) \cdot$$



— this is well defined due to the cyclical invariance of φ —, and



$$:= \varphi(\Lambda_{i_1}, \Lambda_{i_2}, \dots, \Lambda_{i_n})$$

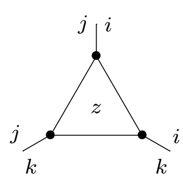


,

so that the "★" tells which indeterminate —among those corresponding to indices decorating holes around the vertex— comes first. Note that, if $\varphi \in \mathbb{C}[\theta_1, \dots, \theta_\nu]$ is non-cyclic and $\nu \neq n$, then we can nonetheless give v_n^φ a meaning: indeed, if $\nu < n$ then the above equation still makes sense; if $\nu > n$ then wrap around the vertex as many times as needed. Note that special vertices decorated by the constant polynomial 1 are identified with the non-decorated special vertices.

Using these notations, the Laurent expansions of $Z_{z\oplus\Lambda;s_*}(\Gamma)$, for an hole type Γ are easily written as sums over ribbon graphs; we give some illustrative examples here.

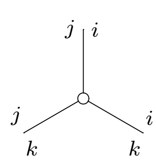
Example 6.1.



$$= -\sqrt{-1}{s_1}^3 \cdot \frac{1}{(z+\Lambda_i)\cdot(z+\Lambda_j)\cdot(z+\Lambda_k)}$$

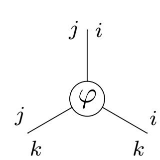
$$= -\sqrt{-1}{s_1}^3 \cdot 1/z^3 + \sqrt{-1}{s_1}^3(\Lambda_i+\Lambda_j+\Lambda_k)\cdot 1/z^4 + \cdots$$

$$= \Bigg(-\sqrt{-1}{s_1}^3 \cdot$$



$$\Bigg) \cdot \frac{1}{z^3}$$

$$+ \Bigg(\sqrt{-1}{s_1}^3 \cdot$$

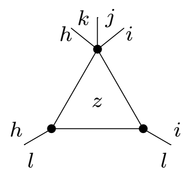


$$\Bigg) \cdot \frac{1}{z^4} + \cdots,$$

where

$$\varphi(\theta_1, \theta_2, \theta_3) := \theta_1 + \theta_2 + \theta_3.$$

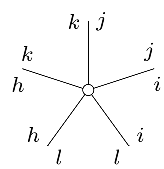
Example 6.2.



$$= \frac{\sqrt{-1}}{2} {s_1}^2 s_2 \cdot \frac{1}{(z+\Lambda_i)\cdot(z+\Lambda_l)\cdot(z+\Lambda_h)}$$

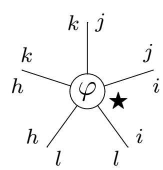
$$= \frac{\sqrt{-1}}{2} {s_1}^2 s_2 \cdot 1/z^3 - \frac{\sqrt{-1}}{2} {s_1}^2 s_2 (\Lambda_i + \Lambda_l + \Lambda_h)\cdot 1/z^4 + \cdots$$

$$= \left(\frac{\sqrt{-1}}{2} {s_1}^2 s_2 \cdot \right.$$



$$\left. \right) \cdot \frac{1}{z^3}$$

$$- \left(\frac{\sqrt{-1}}{2} {s_1}^2 s_2 \cdot \right.$$



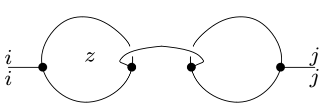
$$\left. \right) \cdot \frac{1}{z^4} + \cdots$$

where

$$\varphi(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) := \theta_1 + \theta_4 + \theta_5.$$

Note that the coefficient of z^{-4} is not a function of all the Λ_*'s around the z-decorated hole, so φ does not depend on θ_2 and θ_3, and does not exhibit the cyclical invariance found in Example 6.1. Because of this lack of cyclicity, we use the "★" mark.

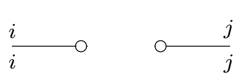
Example 6.3.



$$= {s_1}^4 \cdot \frac{1}{z(z+\Lambda_i)^2(z+\Lambda_j)^2}$$

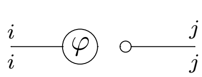
$$= {s_1}^4 \cdot 1/z^5 + {s_1}^4(-2\Lambda_i - 2\Lambda_j)\cdot 1/z^6 + \cdots$$

$$= {s_1}^4 \Big($$

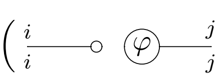


$$\Big) \cdot \frac{1}{z^5} +$$

$$+ \left[-2{s_1}^4 \Big(\right.$$



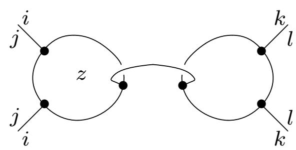
$$\Big) - 2{s_1}^4 \Big($$



$$\left. \Big) \right] \cdot \frac{1}{z^6} + \cdots$$

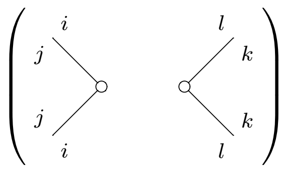
where $\varphi(\theta_1) = \theta_1$.

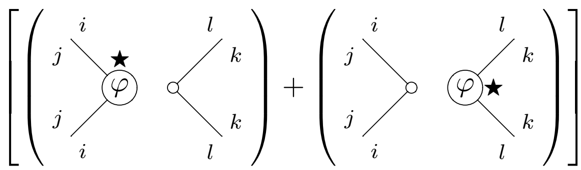
Example 6.4.



$$= -{s_1}^6 \cdot \frac{1}{z(z+\Lambda_i)^2(z+\Lambda_j)(z+\Lambda_k)^2(z+\Lambda_l)}$$

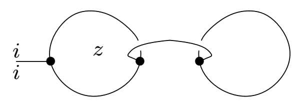
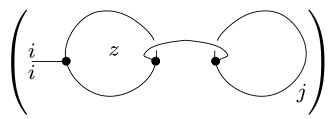
$$= -{s_1}^6 \cdot 1/z^7 + {s_1}^6(2\Lambda_i + \Lambda_j + 2\Lambda_k + \Lambda_l)\cdot 1/z^8 + \cdots$$

$= -{s_1}^6$  $\dfrac{1}{z^7}+$

$+ {s_1}^6$  $\cdot \dfrac{1}{z^8} + \cdots ,$

where $\varphi(\theta_1,\theta_2) = 2\theta_1 + \theta_2$. Note that, in contrast with Example 6.3, the polynomial φ is not cyclically invariant.

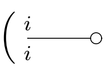
Example 6.5.

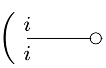
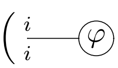
 $= \sum_{j=1}^{N}$ 

$$= -\sqrt{-1}{s_1}^3 \sum_{j=1}^{N} \cdot \frac{1}{z(z+\Lambda_i)^2(z+\Lambda_j)}$$

$$= \sqrt{-1}{s_1}^3 \sum_{j=1}^{N} \bigl(-1/z^4 + (2\Lambda_i + \Lambda_j)\cdot 1/z^5 + \cdots\bigr)$$

$$= -\sqrt{-1}{s_1}^3 \operatorname{tr}\Lambda^0 \cdot 1/z^4 + \sqrt{-1}{s_1}^3\bigl(\operatorname{tr}\Lambda + 2(\operatorname{tr}\Lambda^0)\Lambda_i\bigr)\cdot 1/z^5 + \cdots$$

$= -\sqrt{-1}{s_1}^3 \operatorname{tr}\Lambda^0 \Bigl($  $\Bigr)\dfrac{1}{z^4}+$

$+ \sqrt{-1}{s_1}^3 \Bigl[\operatorname{tr}\Lambda \Bigl($  $\Bigr) + 2\operatorname{tr}\Lambda^0\Bigl($  $\Bigr)\Bigr]\dfrac{1}{z^5} + \cdots ,$

where $\varphi(\theta_1) = \theta_1$. Note that in this last example traces of positive powers of Λ appear at the right-hand side.

All ribbon graphs at the right-hand side in the previous examples are of a peculiar kind, namely, they are disjoint union of special vertices.

Definition 6.1. A cluster of special vertices Ξ is a ribbon graph of the form

$$\Xi = \mathsf{v}_{n_1}^{\varphi_1} \coprod \cdots \coprod \mathsf{v}_{n_k}^{\varphi_k},$$

where the symbol $\coprod$ denotes disjoint union. The valence of a cluster Ξ is the sum of the valences of its vertices; it is denoted by $\operatorname{val}(\Xi)$. The degree of a cluster Ξ is the sum of degrees of the polynomials decorating its vertices; denote it by $\deg \Xi$.

With these notations, examples 6.1–6.5 show that, for any hole type Γ, we have

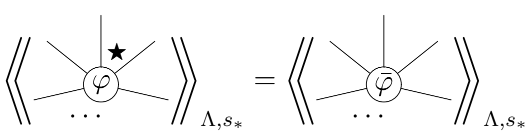
$$\operatorname{Coeff}_z^{-k} \frac{Z_{z\oplus\Lambda,s_*}(\Gamma)}{|\operatorname{Aut}\Gamma|} = \sum_{\Xi \in \mathfrak{X}_\Gamma^k} Q_\Xi^k(s_*, \operatorname{tr}\Lambda^*) \frac{Z_{\Lambda,s_*}(\Xi)}{|\operatorname{Aut}\Xi|},$$

where $\mathfrak{X}_\Gamma^k$ is a suitable set of clusters of special vertices, and $Q_\Xi^k \in \mathbb{C}[s_0, s_1, s_2, \ldots; \operatorname{tr}\Lambda^0, \operatorname{tr}\Lambda, \operatorname{tr}\Lambda^2, \ldots]$.

Moreover, we can assume that polynomials decorating special vertices of Ξ are cyclic; indeed, for any polynomial $\varphi(\theta_1, \ldots, \theta_n)$, the cyclic polynomial

$$\bar{\varphi}(\theta_1, \ldots, \theta_n) := \sum_{\sigma \in \mathbb{Z}/n\mathbb{Z}} \varphi(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(n)}) \tag{6.3}$$

satisfies



(6.4)

The above argument can be straightforwardly adapted to clusters made up by several vertices.

Finally, up to splitting polynomials $\bar{\varphi}$ into homogeneous components, we can further assume that polynomials decorating each Ξ are homogeneous.

The arguments used in this section lead to the following proposition, which summarizes the way Laurent coefficients $\operatorname{Coeff}_z^{-k}$ transform z-hole types into clusters of special vertices.

Proposition 6.1. *For any hole type Γ with only ordinary vertices and each $k \in \mathbb{N}$ there exist:*

1. *a set $\mathfrak{X}_\Gamma^k$ of clusters of vertices decorated by homogeneous cyclic polynomials $\varphi \in \mathbb{C}[\theta_1, \theta_2, \ldots]$,*
2. *polynomials $Q_\Xi^k \in \mathbb{C}[s_0, s_1, s_2, \ldots; \operatorname{tr}\Lambda^0, \operatorname{tr}\Lambda, \operatorname{tr}\Lambda^2, \ldots]$,*

such that:

$$\mathrm{Coeff}_z^{-k} \langle\!\langle \Gamma \rangle\!\rangle^{[1]}_{z\oplus\Lambda,s_*} = \sum_{\Xi\in\mathfrak{X}^k_\Gamma} Q^k_\Xi(s_*, \operatorname{tr}\Lambda^*)\cdot \langle\!\langle \Xi \rangle\!\rangle_{\Lambda,s_*}. \tag{6.5}$$

A more accurate description of the polynomials $Q^k_\Xi(s_*, \operatorname{tr}\Lambda^*)$ will be useful in the sequel of this paper.

Proposition 6.2. *If Γ is a z-hole type with only ordinary vertices, then, with the notations of Proposition 6.1 above, the polynomials $Q^k_\Xi(s_*, \operatorname{tr}\Lambda^*)$ have the form*

$$Q^k_\Xi(s_*, \operatorname{tr}\Lambda^*) = {s_*}^{m_*}\cdot q^k_\Xi(\operatorname{tr}\Lambda^*)\,, \tag{6.6}$$

where m_i is the number of $(2i+1)$-valent vertices in Γ. Moreover, the following inequalities hold:

1. $\operatorname{val}(\Xi) \leqslant \sum_{i=1}^{\infty}(2i-1)m_i$;
2. $\deg\Xi + \deg_{s_*} Q^k_\Xi \leqslant 2k$;
3. $\deg\Xi + \deg_{s_1} Q^k_\Xi \leqslant k$;
4. *if equality holds in 3), then Ξ consists of a single vertex of valence $(k-\deg\Xi)$.*

Proof. Let m_i be the number of $(2i+1)$-valent vertices of Γ; equation (6.6) follows by a straightforward application of the Feynman rules; so we only need to show bounds *1)–4)*.

The valence (i.e., the number of legs) of any cluster of vertices Ξ at right-hand side in (6.5) is exactly the number of half-edges which stem from the vertices of Γ and which do not border the z-decorated hole. If a half-edge of Γ stems from a 1-valent vertex, then it must border the z-decorated hole on both sides; when $i \geqslant 1$, at most $(2i-1)$ half-edges stemming from a $(2i+1)$-valent vertex may not border the z-decorated hole. This proves *1)*.

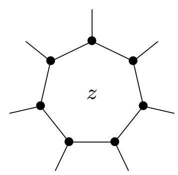
Let ν be the number of internal edges of Γ. Since an internal edge of Γ carries either a factor $1/z$ or a factor $2/(z+\Lambda_i) = (2/z)(1-\Lambda_i/z + {\Lambda_i}^2/z^2 - \cdots)$, then the Laurent coefficient of z^{-k} is a polynomial of degree at most $k-\nu$ in the Λ_i's. The graph Γ can have at most 2ν vertices, and the Laurent coefficient of z^{-k} can be non-zero only if $k \geqslant \nu$. Then $\deg\Xi + \deg_{s_*} Q^k_\Xi \leqslant (k-\nu)+2\nu \leqslant 2k$, which is *2)*.

If Γ has m_1 trivalent vertices, then it has at least m_1 internal edges, so $\deg\Xi + \deg_{s_1} Q^k_\Xi \leqslant (k-m_1)+m_1 = k$. Then *3)* is proven.

Assume now that $\deg \Xi + \deg_{s_1} Q_\Xi^k = k$. The number m_1 of 3-valent vertices of Γ is $\deg_{s_1} Q_\Xi^k$. Since m_1 is at most equal to the number ν of internal edges of Γ, we have:

$$k = \deg \Xi + \deg_{s_1} Q_\Xi^k \leqslant (k - \nu) + m_1 \leqslant k,$$

which forces $m_1 = \nu$ and $\deg \Xi = k - \nu$. Since $\deg \Xi = k - \nu$, no edge of Γ borders the z-decorated hole on both sides; this, together with $m_1 = \nu$, implies that all the vertices of Γ are trivalent and Γ must be the z-hole type



(ν legs)

which is changed into a ν-valent special vertex by the operation of taking a coefficient of the Laurent expansion of its amplitude with respect to $1/z$ at $z = \infty$. $\square$

6.2 Expectation values of polynomial vertices

The aim of this section is to find a canonical form to express expectation values of polynomial-decorated vertices.

Definition 6.2. Let $\varphi \in \mathbb{C}[\theta_1, \theta_2, \dots, \theta_n]$. We say that φ is cyclically decomposable iff there exists $\psi \in \mathbb{C}[\theta_1, \theta_2, \dots, \theta_n]$ such that

$$\varphi(\theta_1, \theta_2, \dots, \theta_n) = \sum_{\sigma \in \mathbb{Z}/n\mathbb{Z}} (\theta_{\sigma(n)} + \theta_{\sigma(1)}) \psi(\theta_{\sigma(1)}, \theta_{\sigma(2)}, \dots, \theta_{\sigma(n)}).$$

We say that φ is residual iff it is a degree zero polynomial or has the form:

$$\varphi(\theta_1, \dots, \theta_{2n}) = \text{const} \cdot \sum \theta_i^{2d}.$$

Lemma 6.1. *Every homogeneous cyclic polynomial $\varphi \in \mathbb{C}[\theta_1, \dots, \theta_n]$ can be split into a sum of a cyclically decomposable φ^{dec} and a residual φ^{res}:*

$$\varphi(\theta_1, \dots, \theta_n) = \varphi^{dec}(\theta_1, \dots, \theta_n) + \varphi^{res}(\theta_1, \dots, \theta_n),$$

Proof. Let d be the degree of φ. The statement is trivial if $d = 0$, so assume $d \geqslant 1$. Let I_n^d be the ideal in $\mathbb{C}[\theta_1, \dots, \theta_n]$ generated by $\theta_1 + \theta_2, \theta_2 + \theta_3, \dots, \theta_n + \theta_1$. If n is odd, then $I_n^d = \mathbb{C}[\theta_1, \dots, \theta_n]$, and from the cyclical invariance of φ it easily follows that it is cyclically decomposable. If n is even, then I_n^d is the ideal of polynomials that vanish at the point $(1, -1, 1, \dots, -1)$. Therefore, by adding to φ a suitable multiple of $\sum \theta_i^{2d}$ we get an element of I_n^d which is cyclically invariant, and so cyclically decomposable. $\square$

In particular, a polynomial of positive degree in an odd number of indeterminates is always cyclically decomposable.

Definition 6.3. We say that a cluster Ξ is *decomposable* if at least one vertex of Ξ is decorated by a cyclically decomposable polynomial, otherwise we say that Ξ is *residual*.

By linearity, each cluster Ξ can be split into a sum $\Xi = \Xi^{\mathrm{dec}} + \Xi^{\mathrm{res}}$ where Ξ^{dec} is decomposable and Ξ^{res} is residual.

Motivation for distinguishing between decomposable and residual clusters is given by Proposition 6.3; to prove it, we need first a technical result.

Lemma 6.2. *If Υ is a decomposable cluster, then its expectation value can be written as a linear combination (over $\mathbb{C}[s_*, \operatorname{tr}\Lambda^*]$) of expectation values of clusters of lower degree:*

$$\langle\!\langle \Upsilon \rangle\!\rangle_{\Lambda, s_*} = \sum_{\Xi \in \mathfrak{X}_\Upsilon} p_\Xi(s_*, \operatorname{tr}\Lambda^*) \langle\!\langle \Xi \rangle\!\rangle_{\Lambda, s_*},$$

with

$$\deg \Xi + \deg_{s_*} p_\Xi \leqslant \deg \Upsilon, \quad \forall \Xi \in \mathfrak{X}_\Upsilon.$$

Proof. We first give a proof for a cluster made up of a single vertex.

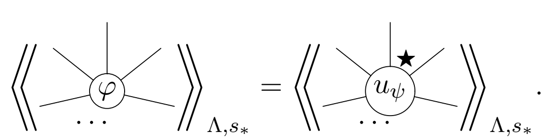
For any $\psi \in \mathbb{C}[\theta_1, \dots, \theta_n]$, let $u_\psi \in \mathbb{C}[\theta_1, \dots, \theta_n]$ be the polynomial

$$u_\psi(\theta_1, \theta_2, \dots, \theta_n) := (\theta_n + \theta_1) \cdot \psi(\theta_1, \theta_2, \dots, \theta_n).$$

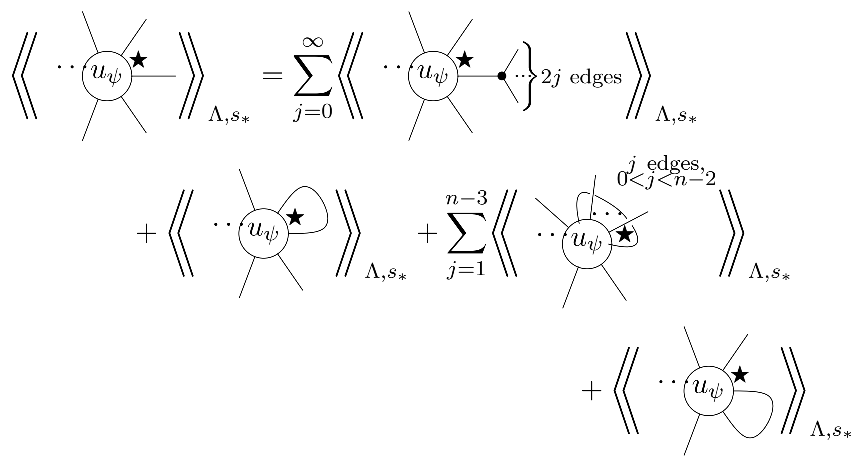
Then φ is cyclically decomposable iff, for some ψ:

$$\varphi(\theta_1, \dots, \theta_n) = \sum_{\sigma \in \mathbb{Z}/n\mathbb{Z}} u_\psi(\theta_{\sigma(1)}, \dots, \theta_{\sigma(n)});$$

this implies the graphical identity

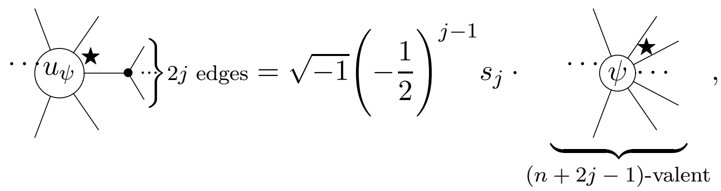


By definition of expectation value of a diagram, both sides are sums over ribbon graphs (with distinguished sub-diagrams); for any Γ in the sum at right-hand side, the edge stemming from the vertex just before the ciliation (in the cyclic order of the vertex) must *either* end at another —distinct— vertex *or* make a loop. Therefore, using the definition of expectation value again,

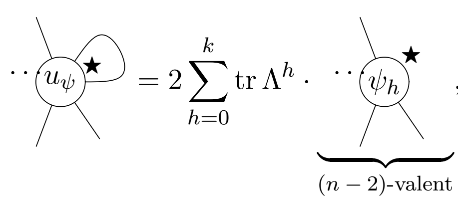


(6.7)

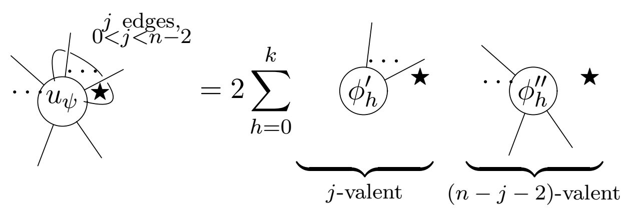
Now, each of the terms at right-hand side of (6.7) above, can be rewritten as the expectation value of a linear combination (over $\mathbb{C}[s_*; \operatorname{tr} \Lambda^*]$) of clusters; indeed, one can directly compute:



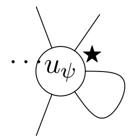
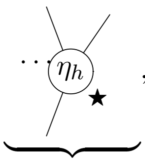
(C1)



(C2)



(C3)

 $= 2 \sum_{h=0}^{k} \operatorname{tr} \Lambda^h \cdot$  , (C4)

(n − 2)-valent

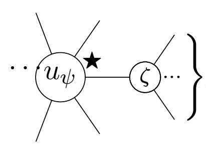
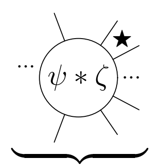
for polynomials ψ_h, ϕ'_h, ϕ''_h, η_h defined by:

$$\sum_{h=0}^{k} \theta_1^h \psi_h(\theta_2, \dots, \theta_{n-1}) = \psi(\theta_1, \dots, \theta_{n-1}, \theta_2),$$

$$\begin{aligned} \sum_{h=0}^{k} \phi'_h(\theta_1, \dots, \theta_j) \cdot \phi''_h(\theta_{j+2}, \dots, \theta_{n-1}) \\ = \psi(\theta_1, \dots, \theta_j, \theta_1, \theta_{j+2}, \dots, \theta_{n-1}, \theta_{j+2}), \end{aligned}$$

$$\sum_{h=0}^{k} \theta_n^h \eta_h(\theta_1, \dots, \theta_{n-2}) = \psi(\theta_1, \dots, \theta_{n-2}, \theta_1, \theta_n).$$

The general case of clusters made up of more than 1 vertex is done by picking a vertex out of the cluster and applying the above procedure to it. A new combination of vertices may appear, which is not listed in equations (C1)–(C4) above; namely, that the ciliated edge connects the chosen vertex to another one in the same cluster. Direct computation again gives:

 j edges $= 2 \cdot$  , (C1')

(n + j − 1)-valent

for a polynomial $\psi * \zeta$ given by

$$(\psi * \zeta)(\theta_1, \dots, \theta_{n+j}) = \psi(\theta_1, \dots, \theta_n) \cdot \zeta(\theta_n, \theta_{n+1}, \dots, \theta_{n+j}, \theta_1).$$

This proves the claim. $\square$

By repeatedly applying the edge-contraction procedure from Lemma 6.2 to the right hand side of equation (6.5), and by inequalities described in Proposition 6.2, one can prove the following.

Proposition 6.3. *For any hole type Γ with only ordinary vertices, and any positive integer k,*

$$\operatorname{Coeff}_z^{-k} \langle\!\langle \Gamma \rangle\!\rangle_{z \oplus \Lambda}^{[1]} = \sum_{\|m_*\| \leqslant 2k} \sum_{\Xi \in \mathfrak{X}_{m_*, \Gamma}} s_*^{m_*} q_{m_*, \Xi}(\operatorname{tr} \Lambda^*) \langle\!\langle \Xi \rangle\!\rangle_\Lambda$$

for suitable residual clusters Ξ *and polynomials* $q_{m*,\Xi} \in \mathbb{C}[\operatorname{tr}\Lambda^0, \operatorname{tr}\Lambda, \operatorname{tr}\Lambda^2, \ldots]$. *Moreover, for any cluster in* $\mathcal{X}_{m_*,\Gamma}$, *the following inequalities hold:*

1. $\operatorname{val}(\Xi) \leqslant \sum_i (2i-1)m_i$;
2. $\deg\Xi + |m_*| \leqslant 2k$;
3. $\deg\Xi + m_1 \leqslant k$ *and if* $\deg\Xi + m_1 = k$ *then* Ξ *consists of a single special vertex of valence* $(k - \deg\Xi)$.

6.3 Proof of the Main Theorem

To conclude the proof of the main result of this paper, we need to introduce an algebra of formal differential operators in the variables s_*. For any polyindex $m_* = (m_0, m_1, \ldots, m_l, 0, 0, \ldots)$ set:

$$|m_*| := \sum_{i=0}^{\infty} m_i, \quad \|m_*\|_- := \sum_{i=1}^{\infty}(2i-1)m_i, \quad \|m_*\|_+ := \sum_{i=0}^{\infty}(2i+1)m_i.$$

Definition 6.4. A formal triangular differential operator in the variables s_* is a formal series

$$D(s_*, \partial/\partial s_*) = \sum_{\|n_*\|_+ \leqslant \|m_*\|_-} a_{m_*,n_*} s_*^{m_*} \frac{\partial^{|n_*|}}{\partial s_*^{n_*}}, \qquad a_{m_*,n_*} \in \mathbb{C},$$

of bounded degree in s_*.

Formal triangular differential operators in the variables s_* form a (non commutative) algebra $\mathbb{C}\langle\langle s_*, \partial/\partial s_* \rangle\rangle$.

Theorem 1. *For any* $k \geqslant 0$ *there exists a formal triangular differential operator*

$$D_k = c_k s_1^{2k+1} \partial/\partial s_k + \textit{lower } s_1\textit{-degree terms}, \quad c_k \in \mathbb{C}.$$

such that

$$\frac{\partial Z(s_*; t_*)}{\partial t_k} = D_k(s_*, \partial/\partial s_*) Z(s_*; t_*).$$

Proof. By equation (5.9),

$$\left.\frac{\partial Z(s_*; t_*)}{\partial t_k}\right|_{t_*(\Lambda)} = -\frac{1}{(2k-1)!!} \sum_{\Gamma \in \mathcal{S}} \operatorname{Coeff}_z^{-(2k+1)} \langle\langle \Gamma \rangle\rangle^{[1]}_{z \oplus \Lambda, s_*}, \tag{6.8}$$

where $\mathcal{S}$ denotes the set of all the z-hole types.

By the Feynman rules for the $(N+1)$-dimensional 't Hooft-Kontsevich model, each edge with one or both sides decorated by the variable z corresponds to a factor of order $O(z^{-1})$ as $z \to \infty$. This implies that $Z_{z\oplus\Lambda;s_*}(\Gamma) = O(z^{-k})$ if the hole type Γ is such that the z-decorated hole is bounded by k edges, so that $\operatorname{Coeff}_z^{-(2k+1)}\big(Z_{z\oplus\Lambda;s_*}(\Gamma)\big) = 0$ if more than $2k+1$ edges border the z-decorated hole. Equation (6.8) is therefore equivalent to

$$\left.\frac{\partial Z(s_*;t_*)}{\partial t_k}\right|_{t_*(\Lambda)} = \sum_{h\leqslant 2k+1}\sum_{\Gamma\in\mathcal{S}_h} \operatorname{Coeff}_z^{-(2k+1)} \langle\!\langle\Gamma\rangle\!\rangle^{[1]}_{z\oplus\Lambda;s_*}, \tag{6.9}$$

where $\mathcal{S}_h$ denotes the set of hole-types whose z-decorated hole is bounded by exactly h edges.

By applying Proposition 6.3 to the right-hand side of equation (6.9), we find:

$$\left.\frac{\partial Z(s_*;t_*)}{\partial t_k}\right|_{t_*(\Lambda)} = \sum_{\|m_*\|\leqslant 4k+2}\left(\sum_{\Xi\in\mathcal{X}[m_*]} s_*^{m_*} q_{m_*,\Xi}(\operatorname{tr}\Lambda^*)\langle\!\langle\Xi\rangle\!\rangle_{\Lambda,s_*}\right), \tag{6.10}$$

for suitable residual clusters Ξ and polynomials $q_{m_*\Xi} \in \mathbb{C}[\operatorname{tr}\Lambda^0, \operatorname{tr}\Lambda^1, \operatorname{tr}\Lambda^2, \ldots]$. The behavior of the left hand side imposes strict constraints both on $q_{m_*,\Xi}$ and the clusters Ξ.

1) The polynomials $q_{m_*,\Xi}(\operatorname{tr}\Lambda^*)$ are constant with respect to Λ; indeed, since $Z(s_*;t_*)$ is a formal power series in the variables t_*, the left-hand side of (6.10) is a function of the traces of negative powers of Λ only; thus, terms in the right-hand side containing traces of positive powers of Λ must cancel out. Therefore, $q_{m_*,\Xi}(\operatorname{tr}\Lambda^*) = r_{m_*,\Xi} \in \mathbb{C}$, so that

$$\left.\frac{\partial Z(s_*;t_*)}{\partial t_k}\right|_{t_*(\Lambda)} = \sum_{\|m_*\|\leqslant 4k+2}\left(\sum_{\Xi\in\mathcal{X}[m_*]} r_{m_*,\Xi}\, s_*^{m_*}\langle\!\langle\Xi\rangle\!\rangle_{\Lambda,s_*}\right). \tag{6.11}$$

2) All the vertices appearing in the clusters on the right hand side of equation (6.11) are odd-valent. Indeed, since formula (6.11) holds for every N, a fortiori it holds for $N = 2$; both sides of (6.11) are real analytic for positive real Λ_1,Λ_2; their analytic prolongations coincide on the connected region $U = \mathbb{C}^2 \setminus \{\Lambda_1 = 0; \Lambda_2 = 0; \Lambda_1 + \Lambda_2 = 0\}$. For real positive ε, set

$$\Lambda_\varepsilon(\lambda) := \begin{pmatrix} -\lambda + 2\varepsilon & 0 \\ 0 & \lambda \end{pmatrix}.$$

For any $|\lambda| > 2\varepsilon$, the diagonal matrix $\Lambda_\varepsilon(\lambda)$ lies in U and we can consider (6.11) at $\Lambda = \Lambda_\varepsilon(\lambda)$.

Since $t_*\bigl(\Lambda_\varepsilon(\lambda)\bigr) \to 0$ as $\lambda \to +\infty$, the left-hand side of (6.11) has a finite limit, independent of ε, for $\lambda \to +\infty$; in particular, there exist some formal power series $\chi_k(s_*) \in \mathbb{C}[[s_*]]$, such that

$$\lim_{\varepsilon\to 0}\lim_{\lambda\to+\infty}\sum_{\|m_*\|\leqslant 4k+2}\left(\sum_{\Xi\in\mathfrak{X}[m_*]} r_{m_*,\Xi}\, s_*^{m_*}\langle\!\langle\Xi\rangle\!\rangle_{\Lambda_\varepsilon(\lambda),s_*}\right) = \chi_k(s_*)\,. \tag{6.12}$$

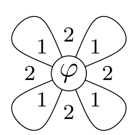
Assume an even-valent vertex $\mathsf{v}_{2n}^{\varphi}$ appears in a cluster Ξ_0 on the right-hand side of (6.11), and let d be the degree of the polynomial $\varphi(\theta_1\dots\theta_{2n})$. Since all clusters on the right-hand side of (6.11) are residual, the degree d is even and

$$\varphi(\theta_1\dots\theta_{2n}) = \mathrm{const}\cdot\sum_{i=1}^{2n}{\theta_i}^{d}\,.$$

According to the Feynman rules for the 't Hooft-Kontsevich model, the expectation value $\langle\!\langle\Xi_0\rangle\!\rangle_{\Lambda_\varepsilon(\lambda),s_*}$ expands into a sum over ribbon graphs whose holes are colored with the two colors 1, 2. The edges of such a graph fall within one of these kinds:

1. both sides of the edge are decorated by the color 1: this edge brings a factor $-1/(\lambda-2\varepsilon)$;
2. both sides of the edge are decorated by the color 2: this edge brings a factor $1/\lambda$;
3. one side of the edge is decorated by the color 1 and the other by the color 2: this edge brings a factor $1/\varepsilon$.

Since $\mathsf{v}_{2n}^{\varphi}$ is an even-valent vertex, in the expansion of $\langle\!\langle\Xi_0\rangle\!\rangle_{\Lambda_\varepsilon(\lambda),s_*}$ into ribbon graphs with holes decorated by the indices 1 and 2, we find terms with a connected component having only edges of the third type, e.g.,



which evaluate to

$$(1/\varepsilon^n)\cdot\varphi(-\lambda+\varepsilon,\lambda,\dots,-\lambda+\varepsilon,\lambda). \tag{6.13}$$

If $d>0$, this diverges as $\lambda\to+\infty$. If $d=0$, then in the limit $\lambda\to+\infty$, the term (6.13) has a polar behavior as $\varepsilon\to 0$. In either case we would have a divergent behavior contradicting equation (6.12). Therefore, no even-valent vertices can appear in the clusters on the right-hand side of (6.11).

Since a residual odd-valent vertex must have degree zero, and any cluster made up of degree zero odd-valent vertices is of the form

$$\mathsf{v}_1{}^{\coprod n_0} \coprod \cdots \coprod \mathsf{v}_{2l+1}{}^{\coprod n_l}$$

for some polyindex n_*, we have finally proven: in the large N limit,

$$\left.\frac{\partial Z(s_*;t_*)}{\partial t_k}\right|_{t_*(\Lambda)} = \sum_{\substack{\|m_*\| \leqslant 4k+2 \\ \|n_*\|_+ \leqslant \|m_*\|_-}} r_{m_*,n_*}\, s_*^{m_*} \langle\!\langle \mathsf{v}_1{}^{\coprod n_0} \coprod \cdots \coprod \mathsf{v}_{2l+1}{}^{\coprod n_l} \rangle\!\rangle_{\Lambda,s_*}.$$

By equation (3.3), expectation values of clusters of degree zero odd-valent special vertices can be expressed as derivatives of the partition function of the 't Hooft-Kontsevich model with respect to the s_* variables. Namely,

$$\langle\!\langle \mathsf{v}_1{}^{\coprod n_0} \coprod \cdots \coprod \mathsf{v}_{2l+1}{}^{\coprod n_l} \rangle\!\rangle_{\Lambda,s_*} = \frac{\sqrt{-1}^{|n_*|}(-2)^{\sum_j j n_j}}{n_0! \cdots n_l!} \frac{\partial^{|n_*|}}{\partial {s_0}^{n_0} \cdots \partial {s_l}^{n_l}} \langle\!\langle \emptyset \rangle\!\rangle_{\Lambda,s_*}.$$

Therefore, there exist a formal triangular differential operator $D_k(s_*, \partial/\partial s_*)$ such that, in the large N limit,

$$\left.\frac{\partial Z(s_*;t_*)}{\partial t_k}\right|_{t_*(\Lambda)} = \left. D_k(s_*, \partial/\partial s_*) Z(s_*;t_*)\right|_{t_*(\Lambda)}$$

Moreover, bounds *1)–3)* in Proposition 6.3 dictate that D_k has the form

$$D_k = c_k \cdot {s_1}^{2k+1} \partial/\partial s_k + \text{ lower } s_1\text{-degree terms.}$$

Since, in the large N limit, the $t_*(\Lambda)$ become independent coordinates, the statement follows. □

Let now $\mathbb{C}\langle \partial/\partial t_* \rangle$ be the free non-commutative algebra generated by the $\partial/\partial t_k$. It acts on the space of formal power series in the variables t_* through its abelianization $\mathbb{C}[\partial/\partial t_*]$.

Theorem 2. *The map*

$$\frac{\partial}{\partial t_k} \mapsto D_k(s_*, \partial/\partial s_*)$$

induces an algebra homomorphism

$$\begin{aligned} D \colon \mathbb{C}\langle \partial/\partial t_* \rangle &\to \mathbb{C}\langle\!\langle s_*, \partial/\partial s_* \rangle\!\rangle \\ P &\mapsto D_P \end{aligned}$$

such that:

$$P(\partial/\partial t_*) Z(s_*;t_*) = D_P(s_*, \partial/\partial s_*) Z(s_*;t_*). \tag{6.14}$$

Proof. The statement immediately follows by the fact that, when regarded as differential operators acting on the space of formal power series in the variables t_* and s_*, the elements $\partial/\partial t_k$ commute among themselves and with the operators $D_P(s_*, \partial/\partial s_*)$. For instance, to prove that $(\partial_{t_i} \cdot \partial_{t_j})Z(s_*; t_*) = D_{\partial_{t_i} \cdot \partial_{t_j}}(s_*, \partial_{s_*})Z(s_*; t_*)$, one computes:

$$\begin{aligned}
(\partial_{t_i} \cdot \partial_{t_j})Z(s_*; t_*) &= \frac{\partial^2 Z(s_*; t_*)}{\partial t_i \partial t_j} = \frac{\partial}{\partial t_j}\left(\frac{\partial}{\partial t_i} Z(s_*; t_*)\right) \\
&= \frac{\partial}{\partial t_j} D_i(s_*, \partial_{s_*})Z(s_*; t_*) \\
&= D_i(s_*, \partial_{s_*})\frac{\partial}{\partial t_j} Z(s_*; t_*) \\
&= D_i(s_*, \partial_{s_*})D_j(s_*, \partial_{s_*})Z(s_*; t_*) \\
&= D_{\partial_{t_i} \cdot \partial_{t_j}}(s_*, \partial_{s_*})Z(s_*; t_*).
\end{aligned}$$

The proof for a higher order monomial in the $\partial/\partial t_*$ goes along the same lines. $\square$

7 Examples and Applications

Let $s_0^\circ, \dots, s_r^\circ$ be complex constants, and set $s_*^\circ = (s_0^\circ, s_1^\circ, \dots, s_r^\circ, 0, 0, \dots)$. There is a well-defined evaluation map

$$\mathrm{ev}_{s_*^\circ} : \mathbb{C}\langle\langle s_*; \partial/\partial s_* \rangle\rangle \to \mathbb{C}[\partial/\partial s_*],$$

which is linear but *not* an algebra homomorphism.

Corollary 1. *For any $s_*^\circ = (s_0^\circ, \dots, s_r^\circ, 0, 0, \dots)$, there exists a linear map*

$$\begin{aligned} Q^{s_*^\circ} : \mathbb{C}[\partial/\partial t_*] &\to \mathbb{C}[\partial/\partial s_*], \\ P &\mapsto Q_P^{s_*^\circ}, \end{aligned} \tag{7.1}$$

such that

$$P(\partial/\partial t_*)Z(s_*^\circ; t_*) = \mathrm{ev}_{s_*^\circ}\left[Q_P^{s_*^\circ}(\partial/\partial s_*)Z(s_*; t_*)\right] \tag{7.2}$$

Proof. The basis $\{\partial^{|m_*|}t/\partial {t_0}^{m_0} \partial {t_1}^{m_1} \cdots \partial {t_q}^{m_q}\}$ defines a linear section ς to the projection $\mathbb{C}\langle \partial/\partial t_* \rangle \to \mathbb{C}[\partial/\partial t_*]$; take $D : \mathbb{C}\langle \partial/\partial t_* \rangle \to \mathbb{C}\langle\langle s_*; \partial/\partial s_* \rangle\rangle$ as in Theorem 2 and set:

$$Q^{s_*^\circ} = \mathrm{ev}_{s_*^\circ} \circ D \circ \varsigma.$$

Equation (7.2) now follows from (6.14). $\square$

Recall that $Z(t_*)$ is the partition function for intersection numbers on the moduli spaces of stable curves; it is a special case of $Z(s_*;t_*)$ when $s_* = (0,1,0,0,\dots)$. Thus, in particular, we get the following.

Corollary 2 (DFIZ Theorem). *There exists a linear isomorphism*

$$Q : \mathbb{C}[\partial/\partial t_*] \to \mathbb{C}[\partial/\partial s_*]$$

such that

$$P(\partial/\partial t_*)Z(t_*) = \mathrm{ev}_{(0,1,0,0,\dots)}\left[Q_P(\partial/\partial s_*)Z(s_*;t_*)\right].$$

Proof. We just need to prove that the map $Q := Q^{(0,1,0,0,\dots)}$ is a linear isomorphism. Indeed, from

$$D_{\partial/\partial t_k} = c_k {s_1}^{2k+1}\partial/\partial s_k + \text{lower } s_1\text{-degree terms},$$

we get, for a monomial $P = \partial^n/\partial t_{k_1}\cdots\partial t_{k_n}$,

$$D_P = c_{k_1,k_2,\dots,k_n} {s_1}^{\sum(2k_i+1)}\partial^n/\partial s_{k_1}\cdots\partial s_{k_n} + \text{lower } s_1\text{-degree terms}.$$

So that, evaluating at $(0,1,0,0,\dots)$,

$$D_P(0,1,0,0,\dots;\partial/\partial s_*) = c_{k_1,k_2,\dots,k_n}\partial^n/\partial s_{k_1}\cdots\partial s_{k_n} + \text{lower order terms},$$

where the omitted terms are differential operators $\partial_{s_*}^{m_*}$ such that $\|m_*\|_+ < \|k_*\|_+$.

Thus, in the bases $\{\partial^n/\partial t_{k_1}\cdots\partial t_{k_n}\}$ and $\{\partial^m/\partial s_{l_1}\cdots\partial s_{l_m}\}$, the linear map $P \mapsto D_P$ is triangular and, therefore, invertible. $\square$

7.1 A matrix integral interpretation

The Di Francesco-Itzykson-Zuber theorem first appeared as a statement about Hermitian matrix integrals; we recover the original formulation by translating Corollary 2 into the language of Gaussian integrals related to the 't Hooft-Kontsevich model.

Corollary 3 (DFIZ Theorem). *There exists a vector space isomorphism*

$$Q : \mathbb{C}[\partial/\partial t_*] \to \mathbb{C}[\operatorname{tr} X, \operatorname{tr} X^3, \operatorname{tr} X^5, \dots]$$

such that, for $N \gg 0$,

$$P(\partial_{t_*})\int_{\mathcal{H}(N)} \exp\left\{\frac{\sqrt{-1}}{6}\operatorname{tr} X^3\right\} d\mu_\Lambda(X) = \int_{\mathcal{H}(N)} Q_P(X)\times \\ \times\exp\left\{\frac{\sqrt{-1}}{6}\operatorname{tr} X^3\right\} d\mu_\Lambda(X)$$

in the sense of asymptotic expansions.

The more general statement of Corollary 1 corresponds to the following.

Corollary 4. *There exists a linear map*

$$Q^{s_*^\circ} : \mathbb{C}[\partial/\partial t_*] \to \mathbb{C}[s_*^\circ; \operatorname{tr} X, \operatorname{tr} X^3, \dots]$$

such that, for $N \gg 0$,

$$P(\partial/\partial t_*) \int_{\mathcal{H}(N)} \exp\left\{ -\sqrt{-1} \sum_{j=0}^{r} (-1/2)^j s_j^\circ \frac{\operatorname{tr}(X^{2j+1})}{2j+1} \right\} d\mu_\Lambda(X) =$$

$$= \int_{\mathcal{H}(N)} Q_P^{s_*^\circ}(s_*^\circ; X) \exp\left\{ -\sqrt{-1} \sum_{j=0}^{r} (-1/2)^j s_j^\circ \frac{\operatorname{tr}(X^{2j+1})}{2j+1} \right\} d\mu_\Lambda(X),$$

in the sense of asymptotic expansions.

7.2 A geometrical interpretation

As we have already remarked, differentiating the partition function $Z(s_*; t_*)$ with respect to the variable t_k corresponds to "inserting a τ_k in the coefficients". Then, evaluating at the point $s_*^\circ = (0, 1, 0, 0, \dots)$, one considers just the combinatorial class corresponding to ribbon graphs with only trivalent vertices, i.e., to the fundamental class in the moduli spaces $\overline{\mathcal{M}}_{g,n}$. This means that the action of $\operatorname{ev}_{s_*^\circ} \circ \partial/\partial t_k$ on $Z(s_*; t_*)$ describes the linear functionals

$$\int_{\overline{\mathcal{M}}_{g,n}} {\psi_1}^k \wedge - : \mathbb{C}[\psi_2, \dots, \psi_n]_{g,n} \to \mathbb{C}, \tag{7.3}$$

where $\mathbb{C}[\psi_2, \dots, \psi_n]_{g,n}$ is the subalgebra of $H^*(\overline{\mathcal{M}}_{g,n})$ generated by the classes $\psi_2, \dots, \psi_n$. More in general, the action of operators $\operatorname{ev}_{s_*^\circ} \circ P(\partial/\partial t_*)$ on the partition function $Z(s_*; t_*)$ describes the linear functionals given by

$$\int_{\overline{\mathcal{M}}_{g,n}} \tilde{P}(\psi_*) \wedge -$$

where $\tilde{P}(\psi_*)$ is a polynomial in the Miller classes.

On the other hand, for any polyindex m_*, acting on $Z(s_*; t_*)$ with the operator $\operatorname{ev}_{s_*^\circ} \circ \partial^{|m_*|}/\partial {s_0}^{m_0} \cdots \partial {s_r}^{m_r}$ corresponds to integrating the ψ classes on the combinatorial stratum $W_{m_*;n}$ described by ribbon graphs having exactly m_i distinguished $(2i+1)$-valent vertices, and all the other vertices of valence three. In other words, the action of $\operatorname{ev}_{s_*^\circ} \circ \partial^{|m_*|}/\partial {s_0}^{m_0} \cdots \partial {s_r}^{m_r}$ on $Z(s_*; t_*)$ describes the linear operators

$$\int_{W_{m_*;n}} : \mathbb{C}[\psi_*]_{g,n} \to \mathbb{C}. \tag{7.4}$$

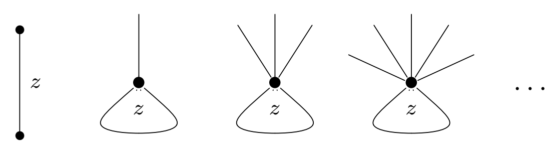
Therefore, Corollary 2 could be interpreted by saying that the combinatorial classes and the ψ classes define the same families of functionals on the subalgebra of the cohomology of the moduli spaces of stable curves generated by the Miller classes. So, in a certain sense (i.e., up to push-forwards and addition of classes supported in the boundary of moduli), one can say that the combinatorial classes are the Poincaré duals of the Mumford classes. For a precise statement and more details on this topic, see [Kon94, AC96, Igu02, Mon03]

7.3 Example computation: $\partial Z(t_*; s_*)/\partial t_0$

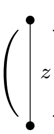
Equation (6.9) tells us

$$\frac{\partial}{\partial t_0}\langle\!\langle\emptyset\rangle\!\rangle_{\Lambda,s_*} = -\sum_{\Gamma\in\mathcal{S}_1}\mathrm{Coeff}_z^{-1}\langle\!\langle\Gamma\rangle\!\rangle^{[1]}_{z\oplus\Lambda,s_*}, \tag{7.5}$$

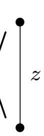
where $\mathcal{S}_1$ is the set of hole types with a z-decorated hole bounded only by one edge. It consists of elements



The first graph in the list above has exactly *two* automorphisms, while none of the other graphs has non-trivial automorphisms. According to the Feynman rules, one computes

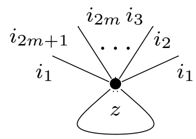
$$\mathrm{Coeff}_z^{-1}\,\frac{1}{2}Z_{z\oplus\Lambda,s_*}\Bigg($$

$$\Bigg) = -\frac{1}{2}{s_0}^2 = -\frac{1}{2}{s_0}^2 Z_\Lambda(\emptyset),$$

so that

$$\mathrm{Coeff}_z^{-1}\Bigg\langle\!\!\Bigg\langle$$

$$\Bigg\rangle\!\!\Bigg\rangle^{[1]}_{z\oplus\Lambda,s_*} = -\frac{1}{2}{s_0}^2\langle\!\langle\emptyset\rangle\!\rangle_{\Lambda,s_*}\,.$$

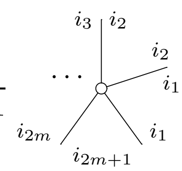
Moreover,

$$\mathrm{Coeff}_z^{-1}$$



$$= \mathrm{Coeff}_z^{-1} \left(-\sqrt{-1}(-1/2)^{m+1} s_{m+1} \cdot \frac{2}{z + \Lambda_{i_1}} \right)$$

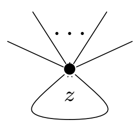
$$= \frac{(-1)^m \sqrt{-1}(2m+1)}{2^m} s_{m+1} \cdot \left(\frac{1}{2m+1} \right.$$



$$\left. \right),$$

so that

$$\mathrm{Coeff}_z^{-1} \left\langle\!\!\left\langle \right.\right.$$



$$\left.\left. \right\rangle\!\!\right\rangle^{[1]}_{z \oplus \Lambda, s_*} = \frac{(-1)^m \sqrt{-1}(2m+1)}{2^m} s_{m+1} \cdot \langle\!\langle \mathsf{v}_{2m+1} \rangle\!\rangle_{\Lambda, s_*}$$

$$= -(2m+1) s_{m+1} \cdot \frac{\partial}{\partial s_m} \langle\!\langle \emptyset \rangle\!\rangle_{\Lambda, s_*}$$

Therefore, equation (7.5) becomes:

$$\frac{\partial}{\partial t_0} \langle\!\langle \emptyset \rangle\!\rangle_{\Lambda, s_*} = \left(\frac{{s_0}^2}{2} + \sum_{m=0}^{\infty} (2m+1) s_{m+1} \frac{\partial}{\partial s_m} \right) \langle\!\langle \emptyset \rangle\!\rangle_{\Lambda, s_*};$$

we can rewrite it as:

$$\frac{\partial}{\partial t_0} Z(s_*; t_*) \Big|_{t_*(\Lambda)} = \left(\frac{{s_0}^2}{2} + \sum_{m=0}^{\infty} (2m+1) s_{m+1} \frac{\partial}{\partial s_m} \right) Z(s_*; t_*) \Big|_{t_*(\Lambda)}.$$

In the large N limit, the $t_*(\Lambda)$ are independent coordinates, thus

$$\frac{\partial}{\partial t_0} Z(s_*; t_*) = \left(\frac{{s_0}^2}{2} + \sum_{m=0}^{\infty} (2m+1) s_{m+1} \frac{\partial}{\partial s_m} \right) Z(s_*; t_*),$$

which can be rewritten as:

$$\frac{\partial}{\partial t_0} F(s_*; t_*) = \frac{{s_0}^2}{2} + \sum_{m=0}^{\infty} (2m+1) s_{m+1} \frac{\partial}{\partial s_m} F(s_*; t_*). \tag{7.6}$$

Now, by equation (7.6), one finds:

$$\frac{\partial^3}{\partial {t_0}^3} F(t_*) = 1 + \left(\frac{\partial^3}{\partial s_0^3} F(s_*; t_*) \right) \Big|_{s_* = (0,1,0,\dots)},$$

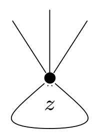
which implies

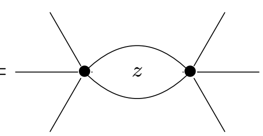
$$\langle \tau_0^3 \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle = 1 + \sum_{m_1=0}^{\infty} \langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle_{(3,m_1,0,\dots);n}.$$

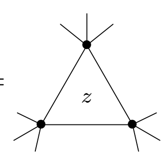
In particular, for $n = 0$, one recovers the well-known relation $\langle {\tau_0}^3 \rangle = 1$ [Kon92, Wit91].

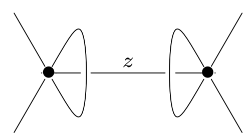
7.4 Example computation: $\partial \langle\!\langle \emptyset \rangle\!\rangle_{\Lambda,s_*} / \partial t_1$ at $s_* = (0, 0, s_2, 0, \dots)$

As an illustration of Corollary 4, we compute $\partial \langle\!\langle \emptyset \rangle\!\rangle_{\Lambda,s_*} / \partial t_1$ at $s_*^\circ = (0, 0, s_2, 0, \dots)$. Since $s_i = 0$ for $i \neq 2$ we need to consider graphs with 5-valent vertices only; moreover, since $k = 1$, we need to consider only holes made up of at most 3 edges. The relevant hole types therefore are:

$\Gamma_1 :=$ 

$\Gamma_2 :=$ 

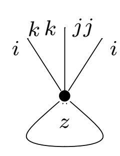
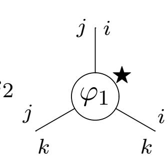
$\Gamma_3 :=$ 

$\Gamma_4 :=$ 

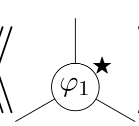
By equation (5.9),

$$\left. \frac{\partial \langle\!\langle \emptyset \rangle\!\rangle_{\Lambda,s_*}}{\partial t_1} \right|_{s_*=(0,0,s_2,0,\dots)} = - \sum_{i=1}^{4} \left. \operatorname{Coeff}_z^{-3} \langle\!\langle \Gamma_i \rangle\!\rangle_{z \oplus \Lambda, s_*}^{[1]} \right|_{s_*=(0,0,s_2,0,\dots)}. \quad (7.7)$$

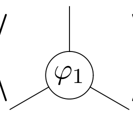
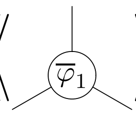
One computes:

$\operatorname{Coeff}_z^{-3} \Bigg($  $\Bigg) = -\frac{\sqrt{-1}}{2} s_2 \, {\Lambda_i}^2 = -\frac{\sqrt{-1}}{2} s_2$  ,

with $\varphi_1(\theta_1) = {\theta_1}^2$, so that

$\operatorname{Coeff}_z^{-3} \langle\!\langle \Gamma_1 \rangle\!\rangle_{z \oplus \Lambda, s_*}^{[1]} = -\frac{\sqrt{-1}}{2} s_2 \Bigg\langle\!\!\Bigg\langle$  $\Bigg\rangle\!\!\Bigg\rangle_{\Lambda,s_*}$.

The polynomial φ_1 is not cyclically invariant, but we can change it into a cyclically invariant one by means of formula (6.3):

$\left\langle\!\!\left\langle$  $\right\rangle\!\!\right\rangle_{\Lambda,s_*} = \left\langle\!\!\left\langle$  $\right\rangle\!\!\right\rangle_{\Lambda,s_*}$, $\qquad \overline{\varphi}_1(\theta_1,\theta_2,\theta_3) = {\theta_1}^2 + {\theta_2}^2 + {\theta_3}^2.$

To avoid cumbersome notations, let us write

$$\langle\!\langle\Gamma\rangle\!\rangle_{s_*^\circ} := \langle\!\langle\Gamma\rangle\!\rangle_{\Lambda,s_*}\Big|_{s_*=(0,0,s_2,0,\dots)},$$

so that, evaluating at $s_* = (0,0,s_2,0,\dots)$, we find:

$$\mathrm{Coeff}_z^{-3}\langle\!\langle\Gamma_1\rangle\!\rangle_{z\oplus\Lambda,s_*}^{[1]}\Big|_{s_*=(0,0,s_2,0,\dots)} = -\frac{\sqrt{-1}}{2}s_2\langle\!\langle \mathsf{v}_3^{\overline{\varphi}_1}\rangle\!\rangle_{s_*^\circ}.$$

In the same way (and accounting for the automorphism groups involved) we get:

$$\mathrm{Coeff}_z^{-3}\langle\!\langle\Gamma_2\rangle\!\rangle_{z\oplus\Lambda,s_*}^{[1]}\Big|_{s_*=(0,0,s_2,0,\dots)} = \frac{{s_2}^2}{4}\langle\!\langle \mathsf{v}_6^{\overline{\varphi}_2}\rangle\!\rangle_{s_*^\circ},$$

where $\overline{\varphi}_2(\theta_1,\theta_2,\dots,\theta_6) = \theta_1+\theta_2+\dots+\theta_6$;

$$\mathrm{Coeff}_z^{-3}\langle\!\langle\Gamma_3\rangle\!\rangle_{z\oplus\Lambda,s_*}^{[1]}\Big|_{s_*=(0,0,s_2,0,\dots)} = \frac{3\sqrt{-1}}{8}{s_2}^3\langle\!\langle \mathsf{v}_9\rangle\!\rangle_{s_*^\circ};$$
$$\mathrm{Coeff}_z^{-3}\langle\!\langle\Gamma_4\rangle\!\rangle_{z\oplus\Lambda,s_*}^{[1]}\Big|_{s_*=(0,0,s_2,0,\dots)} = -{s_2}^2\langle\!\langle \mathsf{v}_2\textstyle\coprod \mathsf{v}_2\rangle\!\rangle_{s_*^\circ}.$$

Now, equation (7.7) can be rewritten as:

$$\frac{\partial\langle\!\langle\emptyset\rangle\!\rangle_{\Lambda,s_*}}{\partial t_1}\Big|_{s_*=(0,0,s_2,0,\dots)} = \frac{\sqrt{-1}}{2}s_2\langle\!\langle \mathsf{v}_3^{\overline{\varphi}_1}\rangle\!\rangle_{s_*^\circ} - \frac{{s_2}^2}{4}\langle\!\langle \mathsf{v}_6^{\overline{\varphi}_2}\rangle\!\rangle_{s_*^\circ}$$
$$- \frac{3\sqrt{-1}}{8}{s_2}^3\langle\!\langle \mathsf{v}_9\rangle\!\rangle_{s_*^\circ} + {s_2}^2\langle\!\langle \mathsf{v}_2\textstyle\coprod \mathsf{v}_2\rangle\!\rangle_{s_*^\circ}. \quad (7.8)$$

According to the proof of Theorem 1, we could forget the contribution coming from the last term in the right-hand side, because it contains even-valent residual vertices. However, we will not do this, so to explicitly show how it gets canceled out.

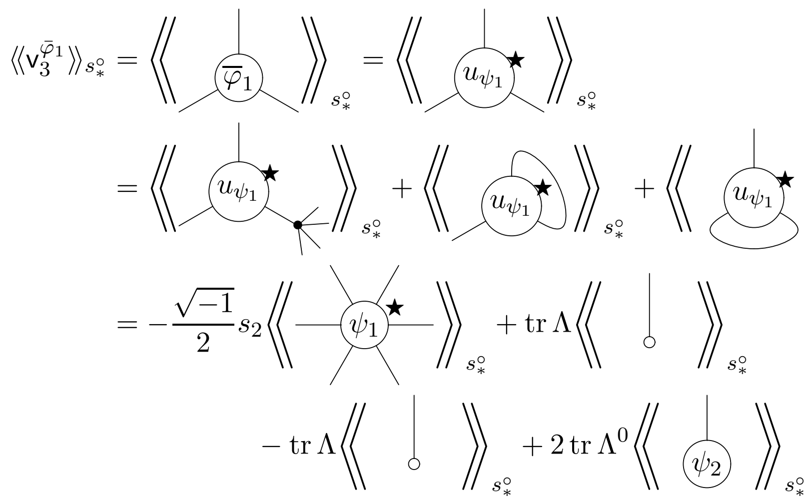
Let us proceed to lower the degree of the polynomials decorating the vertices in equation (7.8) by contraction of edges, starting with the trivalent vertex decorated by $\overline{\varphi}_1$. It is cyclically decomposable; a possible decomposition is:

$$\overline{\varphi}_1(\theta_1, \theta_2, \theta_3) = (\theta_1 + \theta_2) \cdot \psi_1(\theta_1, \theta_2, \theta_3) + \text{cyclic permutations},$$

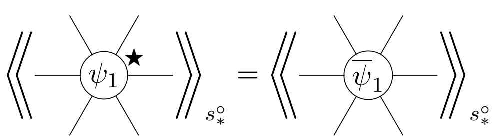
where

$$\psi_1(\theta_1, \theta_2, \theta_3) = \frac{\theta_1 + \theta_2 - \theta_3}{2}.$$

As in the proof of Lemma 6.2,



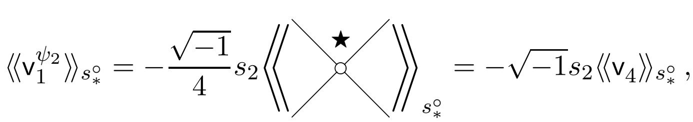
with $\psi_2(\theta_1) = \theta_1$. Find a cyclic equivalent of ψ_1, by applying (6.3) again:



Explicit computation shows that $\overline{\psi}_1 = (1/2)\overline{\varphi}_2$, so that:

$$\langle\!\langle \mathsf{v}_3^{\overline{\varphi}_1} \rangle\!\rangle_{s_*^\circ} = -\frac{\sqrt{-1}}{4} s_2 \langle\!\langle \mathsf{v}_6^{\overline{\varphi}_2} \rangle\!\rangle_{s_*^\circ} + 2\,\mathrm{tr}(\Lambda^0) \langle\!\langle \mathsf{v}_1^{\psi_2} \rangle\!\rangle_{s_*^\circ},$$

and both vertices on the right are decorated by cyclic polynomials. Since $\psi_2(\theta_1) = (\theta_1 + \theta_1) \cdot (1/2) = u_{1/2}$, then (C1) gives:

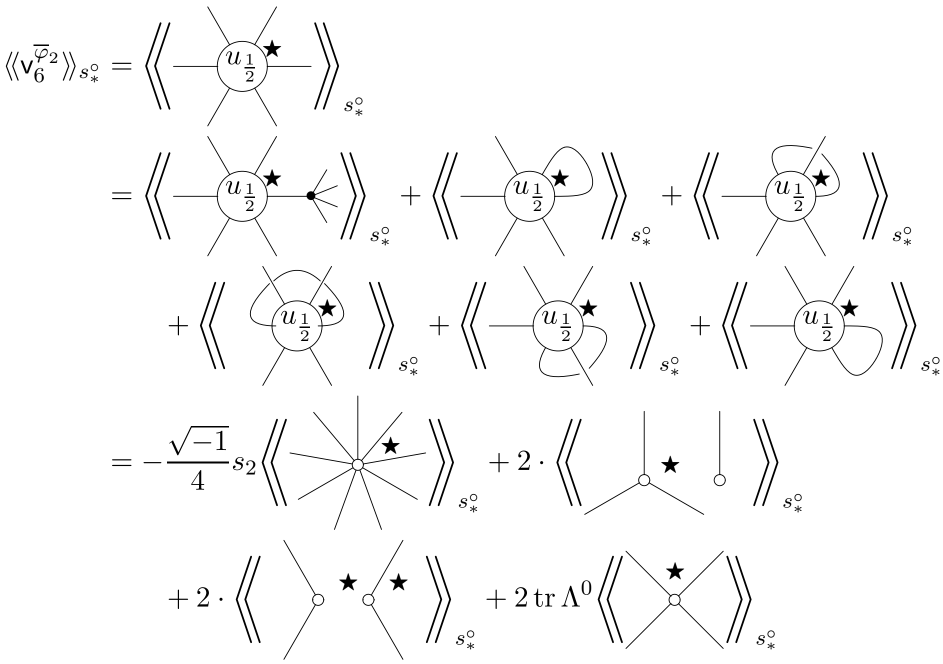
 (7.9)

which is residual.

The other positive degree polynomial appearing on the right hand side of equations (7.8) and (7.9) is $\overline{\varphi}_2$, which has a cyclic decomposition

$$\overline{\varphi}_2(\theta_1, \dots, \theta_6) = u_{1/2} + \text{cyclic permutations}.$$

Again as in the proof of Lemma 6.2, we get:



$$= -\frac{9\sqrt{-1}}{4} s_2 \langle\!\langle \mathsf{v}_9 \rangle\!\rangle_{s_*^\circ} + 8 \cdot \operatorname{tr} \Lambda^0 \langle\!\langle \mathsf{v}_4 \rangle\!\rangle_{s_*^\circ} + 6 \, \langle\!\langle \mathsf{v}_3 \coprod \mathsf{v}_1 \rangle\!\rangle_{s_*^\circ} + 8 \, \langle\!\langle \mathsf{v}_2 \coprod \mathsf{v}_2 \rangle\!\rangle_{s_*^\circ}.$$

In the end, we substitute back into (7.8):

$$\frac{\partial \langle\!\langle \emptyset \rangle\!\rangle_{\Lambda, s_*}}{\partial t_1} \bigg|_{s_* = (0,0,s_2,0,\dots)} = -\frac{3}{4} {s_2}^2 \langle\!\langle \mathsf{v}_3 \coprod \mathsf{v}_1 \rangle\!\rangle_{s_*^\circ} - \frac{3\sqrt{-1}}{32} {s_2}^3 \langle\!\langle \mathsf{v}_9 \rangle\!\rangle_{s_*^\circ}.$$

In terms of Hermitian matrix integrals this reads:

$$\frac{\partial}{\partial t_1} \int_{\mathcal{H}(N)} \exp\left\{ -\frac{\sqrt{-1}}{4} s_2 \frac{\operatorname{tr} X^5}{5} \right\} \mathrm{d}\mu_\Lambda(X) = \int_{\mathcal{H}(N)} \left(-\frac{1}{4} {s_2}^2 \operatorname{tr} X^3 \operatorname{tr} X + \right.$$
$$\left. - \frac{\sqrt{-1}}{96} {s_2}^3 \operatorname{tr} X^9 \right) \exp\left\{ -\frac{\sqrt{-1}}{4} s_2 \frac{\operatorname{tr} X^5}{5} \right\} \mathrm{d}\mu_\Lambda(X).$$

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