

# On the BRST Cohomology of Superstrings with/without Pure Spinors

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## Abstract

We replace our earlier condition that physical states of the superstring have non-negative grading by the requirement that they are analytic in a new real commuting constant  $t$  which we associate with the central charge of the underlying Kac-Moody superalgebra. The analogy with the twisted N=2 SYM theory suggests that our covariant superstring is a twisted version of another formulation with an equivariant cohomology. We prove that our vertex operators correspond in one-to-one fashion to the vertex operators in Berkovits' approach based on pure spinors. Also the zero-momentum cohomology is equal in both cases. Finally, we apply the methods of equivariant cohomology to the superstring, and obtain the same BRST charge as obtained earlier by relaxing the pure spinor constraints.

## 1 Introduction

Recently, we developed a new approach to the long-standing problem of the covariant quantization of the superstring [1]. The formulation of Berkovits of the super-Poincaré covariant superstring in 9+1 dimensions [2] is based on a *free* conformal field theory on the world-sheet and a nilpotent BRST charge which defines the physical vertices as representatives of its cohomology. In addition to the conventional variables  $x^m$  and  $\theta^\alpha$  of the Green-Schwarz formalism, a conjugate momentum  $p_\alpha$  for  $\theta^\alpha$  and a set of commuting ghost fields  $\lambda^\alpha$  are introduced. The latter are complex Weyl spinors satisfying the pure spinor conditions  $\lambda^\alpha \gamma_{\alpha\beta}^m \lambda^\beta = 0$  (cf. for example [3]). This equation can be solved by decomposing  $\lambda$  with respect to a non-compact  $U(5)$  subgroup of  $SO(9,1)$  into a singlet  $\underline{1}$ , a vector  $\underline{5}$ , and a tensor  $\underline{10}$ . The vector can be expressed in terms of the singlet and tensor, hence there are 11 independent complex variables in  $\lambda^\alpha$ .

Since the presence of the non-linear constraint  $\lambda^\alpha \gamma_{\alpha\beta}^m \lambda^\beta = 0$  makes the theory unsuitable for a path integral quantization and higher loop computations, we relaxed the pure spinor condition by adding further ghosts. We were naturally led to a finite set of extra fields, but the BRST charge  $Q$  of this system was not nilpotent, and the central charge of the conformal field theory did not vanish. The latter problem was solved by adding one more extra ghost system, which we denoted by  $\eta^m$  and  $\omega_z^m$ . The former problem was solved by introducing yet another new ghost pair,  $b$  and  $c_z$ , which we tentatively associated with the central charge generator in the affine superalgebra which plays an essential role in the superstring [4].

The BRST charge is linear in  $c_z$ , and without further conditions on physical states the theory would be trivial. We proposed that physical states belong not only to the BRST cohomology ( $Q|\psi\rangle = 0$ , but  $|\psi\rangle \neq Q|\phi\rangle$ ), but also that the deformed stress tensor  $T + \mathcal{V}^{(0)}$ , where  $\mathcal{V}^{(0)}$  is a vertex operator, satisfies the usual OPE of a conformal spin 2 tensor. (The latter condition is weaker than the requirement that vertex operators be primary fields with conformal spin equal to 1).

The definition proposed in [5] replaced the stress tensor condition by the requirement that the physical states belong to a subspace  $\mathcal{H}'$  of the entire linear space  $\mathcal{H}$  of vertex operator. The latter can be decomposed w.r.t. a grading naturally associated with the underlying affine algebra as  $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$ , with negative and non-negative grading, respectively. The BRST charge  $Q = \sum_{n \geq 0} Q_n$  contains only terms  $Q_n$  with non-negative grading, hence one can consistently consider the action of  $Q$  in  $\mathcal{H}_+$ . The physical

space is identified with the cohomology group  $H(Q, \mathcal{H}_+)$ , namely

$$\begin{aligned} Q|\psi\rangle &= 0, & |\psi\rangle &\in \mathcal{H}_+, \\ |\psi\rangle &\neq Q|\psi\rangle, & |\psi\rangle &\in \mathcal{H}_+. \end{aligned} \quad (1.1)$$

Furthermore, by rescaling the ghost fields with a parameter  $t$  to the power equal to the grading of the ghost field and assigning the grading  $-1/2$  to the parameter  $t$ , we restate the definition of physical states as the BRST cohomology of vertices with vanishing grading and analytical in the new parameter  $t$ .

The essential point is that the cohomology in the pure-spinor formulation [2] is a constrained cohomology and this translates in our formalism into an equivariant cohomology. This implies that the physical observables are identified not naively by the BRST cohomology, but with the classes of an equivariant cohomology. This is evident from the structure of our BRST operator and from the fact that on the complete functional space the BRST cohomology is trivial. Usually, in that situation one has to identify what is the functional space on which the BRST cohomology should be computed and, depending on the context, one has to determine an operator which defines such physical states.

At the time when we completed paper [5] we were not aware of the fact that the functional subspace characterized by the non-negative graded monomials was indeed the subspace on which the BRST cohomology becomes an equivariant cohomology, but we did observe that it gives the correct spectrum for the superstrings. In the present paper, we completely spell out the equivalence between the grading condition and the equivariant cohomology.

We also want to mention that the same situation can be found in the context of topological Yang-Mills, topological sigma models and RNS superstrings [6]. Essentially, also in those cases the BRST cohomology is not well defined due to the commuting character of superghosts unless a further condition is imposed. For example in [7], to avoid the ambiguities of the cohomology in presence of commuting ghosts, Siegel introduced non-minimal terms in the action and observed that suitable combinations of fields and constraints can be read as creation and annihilation operators acting in Hilbert space. The definition of the vacuum removes the ambiguities in the cohomology computations.<sup>1</sup>

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<sup>1</sup>Similar problems occur if one adds a BRST invariant field  $Y_\alpha$  such that  $Y_\alpha \lambda^\alpha \neq 0$  (see for example

[8]), but they can be solved by using our grading (restricted to  $\lambda^\alpha$ ) and our definition of physical states.

The paper is organized as follows: in section 2, we review the definition of the grading and of the decomposition of the BRST charge according to it. In section 3, we restate the condition on physical states and we show how the BRST charge presented in [1] can be reformulated in the context of the equivariant cohomology. This leads to the same result achieved in [1], but the interpretation is different. In section 4, as a pedagogical example and to underline the relation between the present formulation with the equivariant cohomology theories, we review the Donaldson-Witten model in  $D = (4, 0)$  and the relation with the twisted  $N = 2$  SYM. In section 5, we present a proof of the equivalence of the pure-spinor cohomology with our formulation and some examples. In section 6, we reproduce the results of [1] starting from yet another point of view, but which illustrates some of the details in the proof of the previous section. In section 7, as a last application, we compute the zero momentum cohomology.

## 2 Grading

Following [1], we review the definition of the grading, the construction of its worldsheet current and the decomposition of the BRST charge according to the grading.

We have based our approach on the following affine superalgebra [4]

$$\begin{aligned}
 d_\alpha(z)d_\beta(w) &\sim -\frac{\gamma_{\alpha\beta}^m \Pi^m(w)}{z-w}, & d_\alpha(z)\Pi^m(w) &\sim \frac{\gamma_{\alpha\beta}^m \partial\theta^\beta(w)}{z-w}, \\
 \Pi^m(z)\Pi^n(w) &\sim -\frac{1}{(z-w)^2} \eta^{mn} k, & d_\alpha(z)\partial_w\theta^\beta(w) &\sim \frac{1}{(z-w)^2} \delta_\alpha^\beta k, \\
 \Pi^m(z)\partial_w\theta^\beta(w) &\sim 0, & \partial_z\theta^\alpha(z)\partial_w\theta^\beta(w) &\sim 0,
 \end{aligned} \tag{2.1}$$

where  $\sim$  denotes the singular contributions to the OPE's.

This algebra has a natural grading defined as follows:  $d_\alpha(z)$  has grading  $1/2$ ,  $\Pi^m(z)$  has grading  $1$ ,  $\partial_z\theta^\alpha(z)$  has grading  $3/2$ , and the central charge  $k$  (which numerically is equal to unity) has grading  $2$ . The corresponding ghost systems are  $(\lambda^\alpha, \beta_{z\alpha})$ ,  $(\xi^m, \beta_{zm})$ ,  $(\chi_\alpha, \kappa_z^\alpha)$ , and  $(c_z, b)$ . We thus define the following grading for the ghosts and corresponding antighosts

$$\begin{aligned}
 \text{gr}(\lambda^\alpha) &= \frac{1}{2}, & \text{gr}(\xi^m) &= 1, & \text{gr}(\chi_\alpha) &= \frac{3}{2}, & \text{gr}(c_z) &= 2, \\
 \text{gr}(\beta_\alpha) &= -\frac{1}{2}, & \text{gr}(\beta_m) &= -1, & \text{gr}(\kappa^\alpha) &= -\frac{3}{2}, & \text{gr}(b) &= -2.
 \end{aligned} \tag{2.2}$$

We also need the ghost  $\omega^m$  and the antighost  $\eta_z^m$ , although this pair does not seem to correspond to a generator. We assign the grading  $\text{gr}(\eta_z^m) = -2$  and  $\text{gr}(\omega^m) = 2$  for the following reason. In [1], we relaxed the pure spinor

constraint by successively adding quartets starting from  $(\lambda_+, \lambda_{[ab]}; \beta^+, \beta^{[ab]})$  of [2] (the indices  $a, b$  belong to the fundamental representation of the  $U(5)$  subgroup of  $SO(1, 9)$ ), and adding the fields  $(\lambda^a, \beta_a; \xi^a, \beta'_a)$  with grading  $(1/2, -1/2, 1, -1)$ . This procedure yields the covariant spinors  $\lambda^\alpha$  and  $\beta_\alpha$ , but now the fields  $(\xi^a, \beta'_a)$  are non-covariant w.r.t.  $SO(9, 1)$ . Thus, we added the quartet  $(\xi_a, \beta'^a; \chi_a, \kappa^a)$  with grading  $(1, -1, 3/2, -3/2)$ . The spinors  $(\chi_a, \kappa^a)$  are part of a covariant spinor and the missing parts are introduced by adding the quartets  $(\chi^+, \kappa_+; c, b)$  and  $(\chi^{[ab]}, \kappa_{[ab]}; \omega^m, \eta^m)$ , both with grading  $(3/2, -3/2, 2, -2)$ . In this way, we obtain the covariant fields  $\lambda^\alpha = (\lambda_+, \lambda^a, \lambda_{[ab]})$ ;  $\beta_\alpha = (\beta^+, \beta_a, \beta^{[ab]})$ ;  $\xi^m = (\xi^a, \xi_a)$ ;  $\beta^m = (\beta'^a, \beta'_a)$ ;  $\chi_\alpha = (\chi^+, \chi_a, \chi^{[ab]})$ ;  $\kappa^\alpha = (\kappa_+, \kappa^a, \kappa_{ab})$ ;  $b, c$  and  $\eta_m, \omega^m$ .

As usual for a conformal field theory, it is natural to introduce a current whose OPE's with the ghost and antighosts reproduce the grading assignments in (2.2)

$$j_z^{grad} = -\frac{1}{2}\beta_{z,\alpha}\lambda^\alpha - \beta_{z,m}\xi^m - \frac{3}{2}\kappa_z^\alpha\chi_\alpha - 2b c_z - 2\eta_z^m\omega_m. \tag{2.3}$$

Independent confirmation that this current might be important comes from the cancellation of the anomaly (namely the terms with  $(z - w)^{-3}$ ) in the OPE of the stress energy tensor  $T_{zz}(z)$  (cf. eqs. (1-3) of ref. [1]) with  $j_z^{grad}$ . In fact, one finds

$$c^{grad} = \frac{1}{2} \times (+16)_{\lambda\beta} + 1 \times (-10)_{\xi\beta} + \frac{3}{2} \times (+16)_{\kappa\chi} + 2 \times (-1)_{bc} + 2 \times (-10)_{\eta\omega} = 0. \tag{2.4}$$

The requirement that the vertex operators contain only terms with non-negative grading leads to the correct massless spectrum [5]. It will also severely restrict the contribution of the vertex operators to correlation functions (in the usual RNS approach ghost insertions are needed to compensate the anomaly in the ghost current, whereas here we anticipate to need insertions of fields in  $\mathcal{H}_-$  to compensate the non-negative grading of vertex operators  $\mathcal{U}^{(1)} \in \mathcal{H}_+$ ).

All the terms in the stress tensor  $T_{zz}(z)$  and in the ghost current

$$\begin{aligned} T_{zz} &= -\frac{1}{2}\Pi_z^m\Pi_{mz} - d_{z\alpha}\partial_z\theta^\alpha - \beta_{zm}\partial_z\xi^m - \beta_{z\alpha}\partial_z\lambda^\alpha - \kappa_z^\alpha\partial_z\chi_\alpha + \partial_z b c_z - \eta_z^m\partial_z\omega_m, \\ J_z^{gh} &= -(\beta_{mz}\xi^m + \kappa_z^\alpha\chi_\alpha + \beta_{z\alpha}\lambda^\alpha + b c_z + \eta_z^m\omega_m), \end{aligned} \tag{2.5}$$

have grading zero, since they are sums of terms of ghost and antighost pairs with opposite grading. On the other hand, the terms in the current  $j_z^B(z)$  (cf. eq. (1.2) in [1]) and the field  $B_{zz}(z)$  have different grading<sup>2</sup>. For instance,

<sup>2</sup>In [1] we presented four different solutions  $B^i$  of the the equation  $T_{zz}(z) = \{Q, B_{zz}^i(z)\}$ . None of the solutions  $B^i$  have definite grading except  $B_{zz}^{IV}(z) = b\hat{T}_{zz}(z) + b\partial_z b c_z - \frac{1}{2}\partial_z^2 b$  which has grading equal to  $-2$  carried by the antighost  $b$ .

the BRST current can be decomposed into the following pieces  $j_z^B(z) = \sum_{n=0}^2 j_z^{B,(n)}(z)$

$$\begin{aligned} j_z^{B,(0)}(z) &= -\xi^m \kappa_z^\alpha \gamma_{m\alpha\beta} \lambda^\beta - \frac{1}{2} \lambda^\alpha \gamma_{\alpha\beta}^m \lambda^\beta \beta_{zm} + \\ &\quad - \frac{1}{2} b \left( \xi^m \partial_z \xi_m - \frac{3}{2} \chi_\alpha \partial_z \lambda^\alpha + \frac{1}{2} \partial_z \chi_\alpha \lambda^\alpha \right) - \frac{1}{2} \partial_z (b \chi_\alpha \lambda^\alpha), \\ j_z^{B,(\frac{1}{2})}(z) &= \lambda^\alpha d_{z\alpha}, & j_z^{B,(1)}(z) &= -\xi^m \Pi_{zm}, \\ j_z^{B,(\frac{3}{2})}(z) &= -\chi_\alpha \partial_z \theta^\alpha, & j_z^{B,(2)}(z) &= c_z. \end{aligned} \quad (2.6)$$

It is clear that all terms in  $j_z^B(z)$  have non-negative grading.

### 3 A New Definition of Physical States.

We begin with some notation that will be used in the following. We denote the quantities in Berkovits' formalism with pure spinor constraints with a lower index  $B$ . For example  $Q_B$  is his BRST charge and  $\mathcal{U}_B^{(1)}$  is his unintegrated vertex operator. The physical spectrum of superstrings is identified with the ghost number +1 elements of the cohomology  $H(Q_B|\mathcal{H}_{p.s.})$  where  $\mathcal{H}_{p.s.}$  is the linear vector space of vertex operators expressed as polynomials of the world-sheet fields  $x^m, \theta^\alpha$  and of the pure spinors  $\lambda^\alpha$ . The latter satisfy the pure spinor condition  $\lambda \gamma_m \lambda = 0$ . The group  $H(Q_B|\mathcal{H}_{p.s.})$  is an example of a constrained BRST cohomology, or equivalently, of equivariant cohomology [9]. In the latter case, the BRST cohomology is computed on the supermanifold  $x^m, \theta^\alpha$  on which the space-time translations  $x^m \rightarrow x^m + \frac{1}{2} \lambda \gamma^m \lambda$ , generated by unconstrained spinors  $\lambda^\alpha$ , act freely. One finds that  $Q_B^2 = -\mathcal{L}_V$  where  $V^m = \frac{1}{2} \lambda \gamma^m \lambda$ . (In Howe's work on pure spinors [3] a translation  $x^m \rightarrow x^m + \lambda \gamma^m \bar{\lambda}$  is considered where  $\lambda$  are pure spinors. The integrability condition for a covariantly constant field,  $\lambda^\alpha \nabla_\alpha \phi = 0$  lead to the SYM field equations).

In order to compare with our formalism [1], let us rescale the pure spinors with a constant commuting parameter  $t \in \mathbb{R}$ . One can interpret this constant as the quartic root of the central charge of the Kac-Moody algebra,  $t^4 = k$ . Using the gradings discussed in the previous section, we obtain  $Q_B = \oint \lambda^a d_\alpha \rightarrow t \oint \lambda^a d_\alpha$  and  $Q_B^2 = t^2 \oint \frac{1}{2} \lambda \gamma^m \lambda \Pi^m$ . Notice that the r.h.s. can be also written in term of the Lie derivative  $\mathcal{L}_V = d \iota_V + \iota_V d$ , where  $\iota_V$  is the contraction of a form with the vector  $V^m$ . One can represent  $\iota_V$  by the operator  $\oint dz V^m \beta_{zm}$ ; its action on (parity reversed forms)  $\xi^m$  is then given by the OPE of  $\beta_{zm}(z)$  with  $\xi^m(z)$ . The exterior differential  $d$  is  $\xi^m \partial_m$  where  $\xi^m$  are the parity-reversed coordinates of the cotangent bundle  $\Pi T^* \mathcal{M}$ . The usual exterior derivative  $d = dx^m \partial_m$  has been replaced by  $-\oint dz \xi^m \Pi_{zm}$ .

Since  $\Pi_z^m(z)\partial^l x^n(w) \sim (z-w)^{-l-1}$ , the operator  $-\oint dz \xi^m \Pi_{zm}$  represents the exterior derivative on the jet bundle  $\{x^m, \partial x^m, \partial^2 x^m, \dots\}$ . One may represent  $\Pi_z^m$  by the functional derivative  $\delta/\delta x^m(z)$ , but note that the latter operator has a central charge and the former has not. The definition in terms of  $\Pi_z^m$  is explicitly supersymmetric. Following the approach of equivariant cohomology [9], one can define a new BRST operator  $Q'$  by

$$Q' = t Q_B + t^2 d + \iota_V = Q - t^2 \oint \xi^m \Pi_{zm} - \oint \frac{1}{2} \lambda^\alpha \gamma_{\alpha\beta}^m \lambda^\beta \beta_{zm}. \quad (3.1)$$

Unfortunately, this operator fails to be nilpotent for two reasons: the operator  $d$  does not commute with  $Q_B$  and  $d^2 \neq 0$ . Notice that this is a quantum effect: in fact the Kac-Moody generator  $d_\alpha$  on the space of functions on the superspace  $\mathcal{M}$  acts like the covariant derivative  $\oint d_\alpha F(x, \theta) = D_\alpha F(x, \theta)$ , and in the same way  $\Pi_m$  acts like the ordinary space-time derivative. This is clearly true only on functions on the superspace  $\mathcal{M}$  and not on forms of  $\Omega^*(\mathcal{M})$ . In addition, one has to take into account that the OPE of  $\Pi^m$  with itself has a central term. Computing the square of  $Q'$  one finds

$$\begin{aligned} (Q')^2 &= t^2 (Q_B^2 + d \iota_V + \iota_V d) + t^3 \{Q_B, d\} + t^4 d^2 \\ &= t^3 \oint \xi_m \lambda^\alpha \gamma_{\alpha\beta}^m \partial_z \theta^\beta + t^4 \oint \xi^m \partial_z \xi_m. \end{aligned} \quad (3.2)$$

where we used  $Q_B^2 = -\mathcal{L}_V$  from [2] and we also used that  $Q_B = \oint \lambda^\alpha d_\alpha$  anticommutes with  $\iota_V$ , and  $\iota_V$  anticommutes with itself. According to the grading of [5],  $\xi_m \gamma_{\alpha\beta}^m \lambda^\beta$  has grading 3/2, and we associate the factor  $t^3$  to  $\partial_z \theta^\beta$  because then the whole expression for  $(Q')^2$  gets grading  $-3/2$  (we define  $\text{gr}(t) = -1/2$ ).

The  $t^3$  term generates fermionic translations in the extended superspace  $\mathcal{M}'$  parametrized by the coordinates  $(x^m, \theta^\alpha, \phi_\alpha)$  and constructed in [4]. However, as noticed by Siegel, since  $\{i \frac{\partial}{\partial \phi_\alpha}, i \frac{\partial}{\partial \phi_\beta}\} = 0$ , one can apply the first order constraint  $i \frac{\partial}{\partial \phi_\alpha} = 0$  to eliminate the variable  $\phi_\alpha$ , obtaining the usual superspace  $\mathcal{M}$ . Since  $\partial_z \theta^\beta$  generates translations of the variable  $\phi_\alpha$ , we can view it again as a Lie derivative and repeat the construction in (3.1). Namely, the first term in (3.2) can be seen as a Lie derivative  $\mathcal{L}_\psi$  along the fiber  $\phi_\alpha$  of the superspace  $\mathcal{M}'$  with respect the spinor  $\psi_\alpha = \xi_m \gamma_{\alpha\beta}^m \lambda^\beta$ . We have

$$Q'' = Q' + t^3 d_\phi + \iota_\psi \quad (3.3)$$

where  $d_\phi = \oint \chi_\alpha \partial_z \theta^\alpha$  and  $\iota_\psi = -\oint \kappa^\alpha \xi_m \gamma_{\alpha\beta}^m \lambda^\beta$ . One can again square this expression and study the terms on the right hand side. One finds only terms proportional to  $t^4$ , and these terms are  $Q''$  invariant. At first sight they seem not to contain any new translation generator. However, adding  $c_z(z)$  plus

$b(z)$  time the  $t^4$  terms yields the final BRST charge ([1],[10]). It coincides with the expression we derived in [1]

$$Q_0 = - \oint \left( \xi^m \kappa_z^\alpha \gamma_{m\alpha\beta} \lambda^\beta + \frac{1}{2} \lambda^\alpha \gamma_{\alpha\beta}^m \lambda^\beta \beta_{zm} + \frac{1}{2} b (\xi^m \partial_z \xi_m - \frac{3}{2} \chi_\alpha \partial_z \lambda^\alpha + \frac{1}{2} \partial_z \chi_\alpha \lambda^\alpha) \right),$$

$$Q = t \oint \lambda^\alpha d_{z\alpha} - t^2 \oint \xi^m \Pi_{zm} - t^3 \oint \chi_\alpha \partial_z \theta^\alpha + t^4 \oint c_z + Q_0. \quad (3.4)$$

First, we note that the BRST charge  $Q$  is a polynomial in the constant  $t$  and the ghost terms collected in  $Q_0$  are  $t$ -independent. As a consequence  $Q_0^2 = 0^3$ . This is a well-known fact in the Sugawara construction based on a super-Kac-Moody [12]. Extracting the ghosts  $(\lambda^\alpha, \xi^m, \chi_\alpha)$  from  $Q$  and  $Q_0$  leads to two representation of the generators of the same affine algebra, namely  $(d_\alpha, \Pi^m, \partial\theta^\alpha)$  and  $((-\beta_m \gamma^m \lambda - \xi_m \gamma^m \kappa - b \partial\chi - 3/4 \partial b \chi)_\alpha, -\kappa \gamma^m \lambda - b \partial\xi^m - 1/2 \partial b \xi^m, b \partial\lambda^\alpha + 1/4 \partial b \lambda^\alpha)$ . Next, we note that by assigning the grading to the fields discussed before and the grading  $-1/2$  to the parameter  $t$ , the BRST charge obtains zero grading<sup>4</sup>. Since the parameter  $t$  is constant the assignment of this grading does not spoil the cancellation of the anomaly of the grading current. It is interesting to compute the BRST transformations of the antighosts

$$\{Q, b\} = t^4,$$

$$\{Q, \kappa_z^\alpha\} = -t^3 \partial_z \theta^\alpha + b \partial_z \lambda^\alpha + \frac{1}{4} (\partial_z b) \lambda^\alpha, \quad (3.5)$$

$$\{Q, \beta_z^m\} = -t^2 \Pi_z^m - \kappa_z \gamma^m \lambda + b \partial_z \xi^m + \frac{1}{2} (\partial_z b) \xi^m,$$

$$\{Q, \beta_{z\alpha}\} = t d_{z\alpha} - \beta_z^m (\gamma_m \lambda)_\alpha - \xi^m (\gamma_m \kappa_z)_\alpha - b \partial_z \chi_\alpha - \frac{3}{4} (\partial_z b) \chi_\alpha.$$

From the  $t$ -dependent terms it becomes evident that the BRST transformation of  $b$  contains the central charge of the Kac-Moody algebra. Being a number, one can set it to 1. We refer to [13] (remark 17 on page 48) for a discussion of this point

The BRST charge  $Q$ , the stress tensor  $T_{zz}$ , the ghost current  $J_z^{gh}$  and the action  $S$  (see [1]) have grading zero. Thus, we require that physical observables have zero grading as well. A generic vertex operator  $\mathcal{U}$  can be expanded into power series of the parameter  $t$ ,  $\mathcal{U} = \sum_{n=-N_-}^{N_+} t^n \mathcal{U}_n$  where  $N_-$  and  $N_+$  are the lowest and the highest power of  $t$ . In general  $N_- \geq 0$ , and the numbers  $N_-$  and  $N_+$  are bounded for a fixed ghost number and at

<sup>3</sup>In [11] a non-nilpotent  $Q$  has been found, but it contained a  $Q_0$  which is nilpotent. This  $Q_0$  corresponds to our  $Q_0$ .

<sup>4</sup>In the case of topological field theories obtained from supersymmetric models by twisting, the grading corresponds to the  $\mathcal{R}$ -charge [13].



fixed world-sheet conformal weight (the latter is number of  $z$  indices in the expression for the unintegrated vertex  $\mathcal{U}$ ). The definition of physical states presented in [5] can be now reformulated by requiring that *the vertex operators are analytic functions of  $t$* , as earlier proposed for topological gauge theories [13] and for topological sigma models [14]. This is completely equivalent to our previous requirement that only  $\mathcal{H}_+$  with non-negative graded operators has to be taken into account [5]. In the following, the space  $\mathcal{H}_+$  is identified with the analytic functions of the parameter  $t$ . To justify the choice of functional space, we note that

$$Q_B = \lim_{t \rightarrow 0} t^{-1} Q(t), \tag{3.6}$$

if  $\xi_m = \chi_\alpha = c = \lambda \gamma^m \lambda = 0$ , namely if all additional ghost fields (except the pure spinors) are set to zero and  $\lambda$  satisfy the pure spinor constraints.

As an example, we consider the vertex operator massless states in the open string

$$\begin{aligned} \mathcal{U}^{(1)}(z) = & t \lambda^\alpha A_\alpha + t^2 \xi^m A_m + t^3 \chi_\alpha W^\alpha + t^4 \omega^m B_m \\ & + b \left( \frac{1}{t^2} \lambda^\alpha \lambda^\beta F_{\alpha\beta} + \frac{1}{t} \lambda^\alpha \xi^m F_{\alpha m} + t^0 \xi^m \xi^n F_{mn} \right. \\ & \left. + t^0 \lambda^\alpha \chi_\beta F_\alpha{}^\beta + t \chi_\alpha \xi^m F_m{}^\alpha + t^2 \chi_\alpha \chi_\beta F^{\alpha\beta} \right) \\ & + b \omega^m \left( t \lambda^\alpha G_{m\alpha} + t^2 \xi^n G_{mn} + t^3 \chi_\alpha G_m{}^\alpha \right) + t^4 b \omega^m \omega^n K_{mn}, \end{aligned} \tag{3.7}$$

where  $A_\alpha, \dots, K_{mn}$  are arbitrary superfields of  $x_m, \theta^\alpha$ . The analyticity w.r.t.  $t$  implies that the first two terms in the first bracket should be canceled. The rest of the vertex is polynomial in  $t$  and

$$\lim_{t \rightarrow 0} t^{-1} \mathcal{U}^{(1)}(z) \Big|_{\xi_m = \chi_\alpha = c = 0} = \mathcal{U}_B^{(1)}(z), \tag{3.8}$$

namely it coincides with pure-spinor unintegrated vertex. In fact, by identifying  $t = k^{\frac{1}{4}}$ , where  $k$  is the Kac-Moody central charge, setting  $k = t = 0$ , implies that the OPE of  $\Pi^m$  with itself vanishes, and the BRST charge is consequently nilpotent. There is a caveat in this argument:  $\partial_t Q|_{t=0}^2 \neq 0$  as we know from [2]. But if  $\lambda^\alpha$  satisfies the pure spinor constraint, it is nilpotent. This point will be clarified in the forthcoming sections.

## 4 N=2 D=4 SYM and Topological Yang-Mills

The introduction of grading by means of a constant parameter  $t$  and the requirement that the space of unintegrated vertex operator be restricted to

non-negative grading or to analytical functions of  $t$  is a common situation in so-called equivariant cohomology theories [9]. We believe that our covariant superstring is related to a worldsheet supersymmetric model by a suitable twisting. It may be illuminating to review the relation between N=2 SYM in D=(4,0) dimensions and the topological Donaldson-Witten model [15] because these models are also related by twisting, and the cohomology after the twisting is also restricted to the polynomials which are analytical in the constant twisted supersymmetry ghost  $t$ .

The N=2 supersymmetric theory is described by a gauge potential  $A_\mu$ , the gauginos  $\psi_\alpha^i, \bar{\psi}_{\dot{\alpha}}^i$  and a complex scalar  $\phi$ . The index  $i = 1, 2$  is the index of the R-symmetry group  $U(2)$ . The subgroup  $U(1)$  determines the  $R$ -charge. All fields carry an index  $a$  in the adjoint representation of the gauge group which we suppress. By twisting the R-symmetry with one of the  $SU(2)$  subgroup of  $SO(4)$ , one obtains fermions with Lorentz-vector indices, and the susy parameters become a Lorentz-scalar  $t$ , a vector and a self-dual antisymmetric tensor

$$\begin{aligned} \psi_\mu &= \bar{\sigma}_\mu^{i\dot{\alpha}} \psi_{i\dot{\alpha}}, & \chi_{\mu\nu} &= \sigma_{\mu\nu}^{i\alpha} \psi_{i\alpha}, & \eta &= \epsilon^{i\alpha} \psi_{i\alpha}, \\ \epsilon_\mu &= \bar{\sigma}_\mu^{i\dot{\alpha}} \zeta_{i\dot{\alpha}}, & t_{\mu\nu} &= \sigma_{\mu\nu}^{i\alpha} \zeta_{i\alpha}, & t &= \epsilon^{i\alpha} \zeta_{i\alpha}. \end{aligned} \quad (4.1)$$

With the gauge potential  $A_\mu$  and the complex scalar  $\phi$ , these are the fields of the Donaldson-Witten model. To compare the fields of the two different models, the Wess-Zumino gauge has been chosen in superspace, and susy auxiliary fields have been eliminated. In this particular case, the susy transformations generated by  $q_\alpha^i$  and  $\bar{q}_{\dot{\alpha}}^i$  close only up to gauge transformations and up to equation of motions

$$\begin{aligned} \{q_\alpha^i, \bar{q}_{j\dot{\alpha}}\} &= \delta_j^i \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu + \text{gauge transf.} + \text{eqs. of motion}, \\ \{q_\alpha^i, q_\beta^j\} &= \{\bar{q}_{\dot{\alpha}}^i, \bar{q}_{\dot{\beta}}^j\} = \text{gauge transf.} + \text{eqs. of motion}. \end{aligned} \quad (4.2)$$

To define the supersymmetric and gauge invariant observables in the N=2 susy model, one needs to define a new BRST operator which is the sum of the usual BRST operator  $Q$ , the supersymmetry generators and the translation generator multiplied by their constant ghosts (the commuting  $\zeta^{i\alpha}$  and  $\bar{\zeta}^{i\dot{\alpha}}$  and the anticommuting  $\tau^\mu$ ) and a further term

$$Q_S = Q + \zeta^{i\alpha} q_{i\alpha} + \bar{\zeta}^{i\dot{\alpha}} \bar{q}_{i\dot{\alpha}} + \tau^\mu \partial_\mu - \zeta^{i\alpha} \sigma_{\alpha\dot{\alpha}}^\mu \bar{\zeta}_{\dot{\alpha}}^i \partial_{\tau^\mu}. \quad (4.3)$$

The last term is needed in order to make  $Q_S$  nilpotent on all classical fields and on ghost except the gauginos. Further,  $Q$  contains also terms which transform the Yang-Mills ghost  $c^a$  into two supersymmetry ghosts  $\zeta^{i\alpha}$  and  $\bar{\zeta}^{i\dot{\alpha}}$ . Nilpotency of  $Q_S$  on the gauginos can be achieved by adding to the

theory suitable antifields and constructing the corresponding BRST operator of the BV formalism.

Twisting the supersymmetry generators, we find Witten's fermionic symmetry  $\delta_W = \epsilon^{i\alpha} q_{i\alpha}$ , the vector supersymmetry  $\delta_\mu = \bar{\sigma}_\mu^{i\dot{\alpha}} \bar{q}_{i\dot{\alpha}}$  and the selfdual antisymmetric tensor supersymmetry  $\delta_{\mu\nu} = \bar{\sigma}_{\mu\nu}^{i\alpha} q_{i\alpha}$ . The corresponding BRST operator is given by

$$Q_T = Q + t \delta_W + \epsilon^\mu \delta_\mu + \tau^\mu \partial_\mu - t \epsilon^\mu \partial_{\tau^\mu}, \quad (4.4)$$

where the ghost  $t$  is associated to  $\delta_W$  and  $\epsilon^\mu$  to  $\delta_\mu$ .  $Q_T$  is again nilpotent on all fields except the selfdual antisymmetric tensor  $\chi_{\mu\nu}$ . We drop the antisymmetric generator  $\delta_{\mu\nu}$  since the observables are completely determined by the remaining symmetries. By twisting the fields of the supersymmetric action, the new fields will carry the same  $R$ -charge as before twisting and in particular  $t$  carries the charge  $-1$ . Explicitly, the transformations generated by  $Q_T$  are given by (we can set  $\epsilon^\mu = \tau^\mu = 0$  without affecting the conclusions<sup>5</sup>)

$$\begin{aligned} [Q_T, A_\mu] &= -\nabla_\mu c + t \psi_\mu, & \{Q_T, \psi^\mu\} &= \{c, \psi^\mu\} - t \nabla_\mu \phi, & (4.5) \\ \{Q_T, c\} &= c^2 - t^2 \phi, & [Q_T, \phi] &= [c, \phi], & [Q_T, \bar{\phi}] &= [c, \bar{\phi}] + 2t\eta, \\ \{Q_T, \eta\} &= \{c, \eta\} + \frac{t}{2}[\phi, \bar{\phi}], & \{Q_T, \chi_{\mu\nu}\} &= \{c, \chi_{\mu\nu}\} + t F_{\mu\nu}^+ + \frac{t^2}{2} \chi_{\mu\nu}^*, \\ [Q_T, \chi_{\mu\nu}^*] &= -2(\nabla_{[\mu} \psi_{\nu]})^+ + 2[\phi, \chi_{\mu\nu}] + [c, \chi_{\mu\nu}^*], \end{aligned}$$

where  $\chi_{\mu\nu}^*$  is the antifield of  $\chi_{\mu\nu}$ . Here the superscript  $+$  denotes the selfdual part of the tensor. For the purposes of the present section we will not describe the action of  $Q_T$  on the antifields. It can be shown that the cohomology of  $Q_T$  is independent from the antifields [16].

The crucial point is that the cohomology of  $Q_T$  is only non-trivial if one restricts the space of polynomials to those which are analytical in the global ghosts  $t, \epsilon^\mu$  and  $t^\mu$  [16]. In fact, the cohomological classes are generated by monomials  $\mathcal{P}_n(\phi)$  of the undifferentiated fields  $\phi$

$$\mathcal{P}_n(\phi) = \frac{1}{n} \text{tr}(\phi^n), \quad n \geq 2. \quad (4.6)$$

Thus the cohomology is not only restricted to monomials analytic in  $t$ , but it is even independent of  $t$ . Due to the commuting nature of  $\phi$ , the expressions  $\text{tr}(\phi^n)$  for  $n$  sufficiently large is related to higher order Casimir invariants of the gauge group.

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<sup>5</sup>By this we mean that  $\epsilon^\mu$  and  $\tau^\mu$  do not transform into terms without either  $\epsilon^\mu$  or  $\tau^\mu$ , implying that we can apply filtration methods.

The analysis of the proof in [16] is based on a filtration of the functional space (which contains the constant ghosts  $t, \epsilon^\mu, \tau^\mu$ ), and of the BRST operator with respect to the counting operator  $N = t\partial_t$ . One has  $Q_T = \sum_{n=0}^2 Q_n$ , where

$$Q_0^2 = 0, \quad \{Q_1, Q_0\} = 0, \quad Q_1^2 + \{Q_0, Q_1\} = 0, \quad \{Q_1, Q_2\} = 0, \quad Q_2^2 = 0. \quad (4.7)$$

The first term of the decomposition  $Q_0$  selects the pure gauge transformations in the BRST symmetry (4.5) whereas  $Q_1$  and  $Q_2$  lead to shift transformations.<sup>6</sup>

By relaxing the constraint of analyticity, it is easy to show that all monomials  $\mathcal{P}_n(\phi)$  become BRST trivial. For instance we have

$$tr(\phi^2) = \left\{ Q, tr\left(-\frac{1}{t^2}c\phi + \frac{1}{3t^4}c^3\right) \right\}. \quad (4.8)$$

In other words, working in the functional space whose elements are power series in the global ghosts (in particular  $t$ ), namely  $\mathcal{U} = \sum_{n \geq 0} t^n \mathcal{U}_n$ , the cohomology is non-trivial, but in the larger space with also negative powers of  $t$  the BRST cohomology becomes trivial, in agreement with the results of Labastida-Pernici and Baulieu-Singer [9].<sup>7</sup>

In terms of the cohomological representatives (4.6), one can construct the solution to the descent equations:  $\{Q, \Omega_p^n\} + d\Omega_{p-1}^{n+1} = 0$ , where  $d$  is the exterior differential and  $\Omega_p^n$  are  $p$ -forms with ghost number  $n$ . The generators of the equivariant cohomology of  $Q_T$  satisfy the descent equations

$$\begin{aligned} [Q_T, \frac{1}{2t^4}trF^2] &= -d\frac{1}{t^3}tr(F\psi), & \{Q_T, \frac{1}{t^3}tr(F\psi)\} &= -d\frac{1}{t^2}tr(\phi F + \frac{1}{2}\psi^2), \\ [Q_T, \frac{1}{t^2}tr(\phi F + \frac{1}{2}\psi^2)] &= -dtr(\frac{1}{t}\phi\psi), & \{Q_T, tr(\frac{1}{t}\phi\psi)\} &= -\frac{1}{2}dTr\phi^2, \\ [Q_T, \frac{1}{2}tr\phi^2] &= 0. \end{aligned} \quad (4.9)$$

Except the last element of the descent equations, namely the monomial  $tr\phi^2$ , all the other generators are explicitly non-analytical. The same situation will happen in the case of open superstrings: the descent equations are given by

<sup>6</sup>In the case of superstrings, the charge  $Q_0$  in (3.4) implements the pure spinor constraint at the level of cohomology (it generates the gauge transformations of the antighost fields). The charge  $Q - Q_0$  in (3.4) leads to shifts of the fields as in the topological model.

<sup>7</sup>It is interesting to note that in the string case, by imposing the restriction that  $\mathcal{U} = \sum_{n \geq 1} t^n \mathcal{U}_n$ . The cohomology is further restricted to the states of topological super-Yang-Mills in D=(9,1). This might lead to the construction of topological super-Yang-Mills model in higher dimensions where the action is given by  $S = \langle \Psi, Q_T \Psi \rangle$ . Clearly, one needs a definition of the inner product in order to have a gauge invariant and supersymmetric model.

$\{Q, \mathcal{U}\} = 0$  and  $[Q, \mathcal{V}_z] = \partial_z \mathcal{U}$ . Here  $\mathcal{U}$  corresponds to the so-called non-integrated vertex and  $\mathcal{V}_z$  to the integrand of the integrated vertex. Following the suggestions of topological models, one finds that  $\mathcal{U}$  is written in terms of a power series of  $t$ , but  $\mathcal{V}_z$  will contain also non-analytical terms. It turns out that those non-analytical pieces are irrelevant for computations of amplitudes.

## 5 Equivalence with Berkovits' formulation

In the case of massless states a direct comparison with the equation of motions obtained in [2] can be easily done, but, for massive states, the field equations in N=1 d=(9,1) superspace formulation are not known. Only recently, the equations of motion for the first massive state for open superstring has been derived in [17] using the pure spinor formulation.

Since the comohology  $H^{(1)}(Q_B|\mathcal{H}_{p.s.})$  has been proved in [2] to contain uniquely the spectrum of the RNS superstring, or equivalently of the Green-Schwarz string quantized in the light-cone gauge, it will be sufficient to prove the equivalence of our cohomology group  $H^{(1)}(Q, \mathcal{H}_+)$  with the pure spinor constrained cohomology  $H^{(1)}(Q_B|\mathcal{H}_{p.s.})$ .

Both the BRST operator (3.4) and the vertex operators are analytic functions of an indeterminate variable  $t$ . We are therefore studying a cohomology with values in a ring of analytic functions of  $t$ . However, as discussed in [18], we can work at a fixed value  $t = t_0$  as long as the multiplication by the monomial  $(t - t_0)$  is an injective map in the cohomology. In our case, the presence of a grading implies that this is true for any value of  $t$  except possibly for  $t = 0$ . In fact, the equation  $(t - t_0)\mathcal{U} = 0$  can be separated according to the grading in  $t\mathcal{U} = 0$  and  $t_0\mathcal{U} = 0$ . The latter is only satisfied for  $\mathcal{U} = 0$  unless  $t_0 = 0$ . This means that, in analyzing the cohomology, we can consider  $t$  as a given non-zero real parameter.

The next step is to prove that the cohomology is in fact independent of the value of  $t$ . This follows from the fact that one can change the value of  $t$  by applying a similarity transformation to the BRST operator. More precisely, defining  $Q_{grad}$  to be the grading charge,  $Q_{grad} = \oint j_z^{grad}$ , one has the following "evolution" equation

$$t \frac{\partial}{\partial t} Q(t) = [Q_{grad}, Q(t)]. \quad (5.1)$$

This equation is in fact the statement that  $Q(t)$  is an homogeneous function of grading zero in  $t$  and all the fields. Since  $Q_{grad}$  does not depend on  $t$ , the

equation is easily solved by  $Q(t) = e^{Q_{grad} \ln \frac{t}{t_1}} Q(t_1) e^{-Q_{grad} \ln \frac{t}{t_1}}$ . Thus  $Q(t_1)$  is related to  $Q(t)$  by a similarity transformation that is, however, singular at  $t = 0$ . Following the ideas of Witten [19], we will consider the limit  $t \rightarrow 0$ . For this purpose, it is more convenient to use the operator  $D_t \equiv \frac{1}{t} Q(t)$ , which has of course the same cohomology as  $Q$ . In eq. (5.1), the left-hand side is manifestly at least linear in  $t$ , but the right-hand side is also linear because the  $t$ -independent term  $Q_0$  commutes with  $Q_{grad}$ . We can then divide both sides by  $t$  and get  $\partial_t D_t = [Q_{grad}, D_t]$ . The main idea of our proof is that, in the limit  $t \rightarrow 0$ , the divergent term in  $D_t$ ,  $Q_0/t$ , has the effect of localizing the cohomology on the fixed points of the action of  $Q_0$ . The transformation properties of various fields under  $Q_0$  are given below:

$$\begin{aligned} \{Q_0, \xi^m\} &= -\frac{1}{2} \lambda \gamma^m \lambda, \\ \{Q_0, \chi_\alpha\} &= \xi^m (\gamma_m \lambda)_\alpha, \\ \{Q_0, c\} &= \xi^m \partial \xi_m + \lambda^\alpha \partial \chi_\alpha - \chi_\alpha \partial \lambda^\alpha, \end{aligned} \quad (5.2)$$

one can see that the fixed points of  $Q_0$  are  $\lambda \gamma^m \lambda = 0$ ,  $\xi^m = 0$ ,  $\chi_\alpha = 0$  and  $c_z = 0$ . The first of these conditions, of course, reproduces the pure-spinor constraints, and the other ghosts are set to zero. The BRST operator reduces to the  $Q_1$  term, that reproduces the Berkovits' one, and  $Q_0$ . The only difference with Berkovits' cohomology is that the vertices can still depend on  $\beta_{zm}$ , the antighost of  $\xi^m$ . We must recall that the pure-spinor constraint  $\lambda \gamma^m \lambda = 0$  implies that the antighost of  $\lambda$  has the gauge-invariance  $w_\alpha \rightarrow w_\alpha + \Lambda_m (\gamma^m \lambda)_\alpha$ , for an arbitrary parameter  $\Lambda_m$ . The vertex operators must then be restricted to be invariant with respect to this gauge transformations [17]. In our formalism we do not see this requirement even after the localization. But we must still consider the action of  $Q_0$ , that exactly reproduces the transformations:  $\{Q_0, w_\alpha\} = \beta_m (\gamma^m \lambda)_\alpha$ . Vertices that are not gauge-invariant are ruled out by the cohomology of  $Q_0$ . This completes the proof of the equivalence of our cohomology with the pure-spinor one.

As an illustration of this point, we consider explicitly the first massive level of the open superstring. After the localization, the most general form of the vertex, at ghost number 1, is

$$\begin{aligned} \mathcal{U}_z^{(1)} &= \partial \lambda^\alpha A_\alpha(x, \theta) + \partial \theta^\beta \lambda^\alpha B_{\alpha\beta}(x, \theta) + d_\beta \lambda^\alpha C_\alpha^\beta(x, \theta) + \Pi^m \lambda^\alpha H_{m\alpha}(x, \theta) + \\ &+ w_\alpha \lambda^\beta \lambda^\gamma E_{(\beta\gamma)}^\alpha(x, \theta) + \beta_m \lambda^\beta \lambda^\gamma F_{(\beta\gamma)}^m(x, \theta). \end{aligned} \quad (5.3)$$

Comparing with the vertex in [17], one can see that the only difference is in the second line, where the second term is absent and the first one only appears in the gauge-invariant combinations  $J \lambda^\alpha E_\alpha$  and  $N^{mn} \lambda^\alpha E_{[mn]\alpha}$ , where  $J = w_\alpha \lambda^\alpha$  and  $N^{mn} = w_\alpha (\gamma^{mn})_\beta^\alpha \lambda^\beta$ . Requiring that  $Q_0$  annihilates the vertex implies  $\beta_m \lambda^\beta \lambda^\gamma \lambda^\delta \gamma_{\alpha(\beta}^m E_{\gamma\delta)}^\alpha = 0$  and  $\kappa^\alpha \lambda^\beta \lambda^\gamma \lambda^\delta \gamma_{m\alpha(\beta} F_{\gamma\delta)}^m = 0$ . The

coefficient  $E_{(\gamma\delta)}^\alpha$ , considered as a matrix in the indices  $\alpha, \gamma$ , can be expanded on a basis of Dirac matrices, and the expansion contains terms with 0, 2 or 4 gamma matrices. The terms with 0 and 2 matrices reproduce the Berkovits' terms. The term with 4 matrices has to satisfy  $\beta_m \lambda^\alpha \lambda^\beta \lambda^\gamma \gamma_{(\alpha\beta}^{[mpqrs]} E_{\gamma][pqrs]} = 0$  and by decomposing  $\lambda^\alpha \lambda^\gamma \rightarrow \lambda \gamma^{tuvwx} \lambda \gamma_{tuvwx}^{\alpha\gamma}$ , one obtains the equation  $\gamma_{\beta\alpha}^{[mpqrs]} \gamma_{tuvwx}^{\alpha\gamma} E_{\gamma][pqrs]} = 0$  which implies that  $E_{\gamma][pqrs]} = 0$ . On the other hand, decomposing  $F_{(\beta\gamma)}^m$  as a 5-form  $F_{[npqrs]}^m \gamma_{\alpha\beta}^{[npqrs]}$ , one immediately obtains that  $F_{[npqrs]}^m = 0$ .

At this point, to construct the elements of the cohomology for  $t \neq 0$ , it is convenient to disentangle the vertices and the BRST charge into fixed grading numbers. We shall show that only four equations must be really solved: all the others give only algebraic relations among the different pieces of the vertices and can be easily solved.

As already mentioned, the BRST charge  $Q$  is an analytic functions of  $t$  up to power four:  $Q = \sum_{n=0}^4 t^n Q_n$  (in order to simplify the notation, we denote  $Q_B$  by  $Q_1$ ). The nilpotency of  $Q$  is translated into the relations

$$\sum_{n=0}^m \{Q_{m-n}, Q_n\} = 0, \quad m = 0, \dots, 8, \quad Q_n = 0, \quad n > 4. \quad (5.4)$$

However, due to the particular form of the various  $Q_n$ , the equations (5.4) reduce to

$$\begin{aligned} Q_0^2 = 0, \quad \{Q_0, Q_B\} = 0, \quad Q_B^2 + \{Q_0, Q_2\} = 0, \\ \{Q_0, Q_3\} + \{Q_2, Q_B\} = 0, \quad Q_2^2 + \{Q_B, Q_3\} + \{Q_0, Q_4\} = 0, \\ \{Q_2, Q_3\} = 0, \quad Q_3^2 = 0, \quad \{Q_i, Q_4\} = 0, \quad i = 1, \dots, 4, \end{aligned} \quad (5.5)$$

A generic vertex operator  $\mathcal{U}^{(1)}$  for the open superstring with ghost number 1 belongs to  $\mathcal{H}_+$  and it can be expressed in terms of a power series of the parameter  $t$ ,  $\mathcal{U}^{(1)} = \sum_{n \geq 0} t^n \mathcal{U}_n$ . This implies that expanding the equation  $\{Q, \mathcal{U}^{(1)}\} = 0$  in different powers we have the following equations

$$\begin{aligned} \{Q_0, \mathcal{U}_0\} &= 0, \\ \{Q_0, \mathcal{U}_1\} + \{Q_B, \mathcal{U}_0\} &= 0, \\ \{Q_0, \mathcal{U}_2\} + \{Q_B, \mathcal{U}_1\} + \{Q_2, \mathcal{U}_0\} &= 0, \\ \{Q_0, \mathcal{U}_3\} + \{Q_B, \mathcal{U}_2\} + \{Q_2, \mathcal{U}_1\} + \{Q_3, \mathcal{U}_0\} &= 0, \\ \{Q_0, \mathcal{U}_n\} + \{Q_B, \mathcal{U}_{n-1}\} + \{Q_2, \mathcal{U}_{n-2}\} + \{Q_3, \mathcal{U}_{n-3}\} + \{Q_4, \mathcal{U}_{n-4}\} &= 0, \quad n \geq 4. \end{aligned} \quad (5.6)$$

Using the fact that  $b^2 = 0$ , we can decompose any contribution  $\mathcal{U}_n$  into a  $b$ -dependent term and a  $b$ -independent one,  $\mathcal{U}_n = \mathcal{U}'_n + b\Delta_n$ . We therefore

decompose all the equations into a  $b$ -dependent part and a  $b$ -independent one. Since  $\{Q_i, b\} = 0$  for  $i = 1, \dots, 3$ , and  $\{Q_4, b\} = 1$ , the  $b$ -independent equations for  $n \geq 4$  can be easily solved. For example, let us consider the equation for  $n = 4$ ; we can solve it for  $\Delta_0$

$$-\Delta_0 = \{Q_0, \mathcal{U}'_4\} + \{Q_B, \mathcal{U}'_3\} + \{Q_2, \mathcal{U}'_2\} + \{Q_3, \mathcal{U}'_1\}. \quad (5.7)$$

In a similar way all  $\Delta_n$  with  $n > 0$  are solved by using (5.6) with  $n > 4$ . Note that  $\{Q, \mathcal{U}\} = 0$  can be decomposed into  $Q = Q' + Q_4$  and  $(Q')^2 + \{Q', Q_4\} = 0$  and  $\mathcal{U} = \mathcal{U}' + b\Delta$ , this implies  $\{Q', \mathcal{U}'\} + \Delta = 0$  and  $\{Q', \Delta\} = 0$  (as a consequence of (5.6)). Now, inserting  $\Delta = -\{Q', \mathcal{U}'\}$  in  $\{Q', \Delta\} = -\{\{Q', Q'\}, \mathcal{U}'\} = \{\{Q', Q_4\}, \mathcal{U}'\} = 0$  and  $\{Q', \{Q_4, \mathcal{U}'\}\} + \{Q_4, \{Q', \mathcal{U}'\}\} = 0$ , but  $\{Q_4, \mathcal{U}'\} = \{Q', \Delta\}$  and  $\{Q', \mathcal{U}'\} = -\Delta$  which is  $Q_4$  invariant. This fixes all the  $\Delta_n$ . However, from the first equation of (5.6), one gets the two equations  $\{Q_0, \mathcal{U}'_0\} = 0$  and  $\{Q_0, \Delta_0\} = 0$ . The second is a constraint on  $\Delta_0$  and the solution (5.7) should be compatible with it. This can easily be proved by using the commutation relations (5.5) and equations (5.6) for  $\mathcal{U}'_i, i = 1, \dots, 3$ . In the same way, one can solve all the equations for  $n > 4$  and the four remaining equations can be now expressed in terms of only the  $b$ -independent part of  $\mathcal{U}_n$ . Hence, at this point all the equations in (5.6) for  $n \geq 4$  have been solved.

As an example, we illustrate the construction in the case of massless vertex for the open superstring. This example will also provide some hints for constructing the massive states in the present formalism.

In the massless case, we consider only worldsheet scalar vertex operators. This implies that only the antighost  $b$  is allowed in the expression for the vertex. Moreover, this also implies that  $\mathcal{U}^{(1)} = \sum_{n=0}^3 t^n \mathcal{U}_n^{(1)}$ . Now, using the decomposition  $\mathcal{U}_n^{(1)} = \mathcal{U}_n'^{(1)} + b\Delta_n^{(2)}$  and by noting that  $\mathcal{U}_0'^{(1)}$  vanishes we can simplify eqs. (5.6). For  $n = 4, \dots, 7$  we have

$$\begin{aligned} n = 4 : & \quad \{Q_B, \mathcal{U}'_3^{(1)}\} + \{Q_2, \mathcal{U}'_2^{(1)}\} + \{Q_3, \mathcal{U}'_1^{(1)}\} + \Delta_0^{(2)} = 0, \\ n = 5 : & \quad \{Q_2, \mathcal{U}'_3^{(1)}\} + \{Q_3, \mathcal{U}'_2^{(1)}\} + \Delta_1^{(2)} = 0, \\ n = 6 : & \quad \{Q_3, \mathcal{U}'_3^{(1)}\} + \Delta_2^{(2)} = 0, \\ n = 7 : & \quad \Delta_3^{(2)} = 0 \end{aligned} \quad (5.8)$$

Observing that  $\{Q_3, \mathcal{U}'_i^{(1)}\} = 0$  for  $i = 1, 2, 3$  because the massless vertex  $\mathcal{U}^{(1)}$  cannot depend upon  $d_{z\alpha}$  (and upon the corresponding right-movers in the closed string case), we obtain  $\Delta_3^{(2)} = \Delta_2^{(2)} = 0$ . The remaining  $\Delta_0^{(2)}$  and  $\Delta_1^{(2)}$  depend only upon the variations of  $\mathcal{U}'_i^{(1)}$  with  $i = 1, 2, 3$ . Moreover,



$\Delta_0^{(2)}$  and  $\Delta_1^{(2)}$  should satisfy the following consistency conditions

$$\begin{aligned} \{Q_0, \Delta_0^{(2)}\} &= 0, & \{Q_1, \Delta_0^{(2)}\} + \{Q_0, \Delta_1^{(2)}\} &= 0, \\ \{Q_1, \Delta_1^{(2)}\} + \{Q_2, \Delta_0^{(2)}\} &= 0, & \{Q_2, \Delta_1^{(2)}\} &= 0, \end{aligned} \quad (5.9)$$

where we have already used  $\{Q_3, \Delta_i^{(2)}\} = 0$  for  $i = 0, 1$ . The  $b$ -independent terms  $\mathcal{U}_i^{(1)}$  with  $i = 1, 2, 3$  should satisfy the equations

$$\begin{aligned} \{Q_0, \mathcal{U}_1^{(1)}\} &= 0, \\ \{Q_1, \mathcal{U}_1^{(1)}\} + \{Q_0, \mathcal{U}_2^{(1)}\} &= 0, \\ \{Q_2, \mathcal{U}_1^{(1)}\} + \{Q_1, \mathcal{U}_2^{(1)}\} + \{Q_0, \mathcal{U}_{23}^{(1)}\} &= 0. \end{aligned} \quad (5.10)$$

From Lorentz invariance, ghost number and analyticity, we have that  $\mathcal{U}_1^{(1)} = \lambda^\alpha A_\alpha(x, \theta)$  where  $A_\alpha(x, \theta)$  is a generic superfield. It automatically satisfies the first equation of (5.10). Furthermore, we have that  $\mathcal{U}_2^{(1)} = \xi^m A_m(x, \theta)$  solves the second equation if the superfields  $A_\alpha(x, \theta)$  and  $A_m(x, \theta)$  satisfy

$$A_m = \frac{1}{8} \gamma_m^{\alpha\beta} D_\alpha A_\beta, \quad \gamma_{mnrpq}^{\alpha\beta} D_\alpha A_\beta = 0. \quad (5.11)$$

The third equation is solved by assuming  $\mathcal{U}_3^{(1)} = \chi_\alpha W^\alpha(x, \theta)$  if the superfield  $W^\alpha$  is related to  $A_\alpha(x, \theta)$  and  $A_m(x, \theta)$  by the usual equation  $W^\alpha = \frac{1}{10} \gamma_m^{\alpha\beta} (D_\beta A^m - \partial^m A_\alpha)$ . From eqs. (5.8), we have

$$\Delta_0^{(2)} = -\lambda^\alpha \chi_\beta D_\alpha W^\beta - \xi^m \xi^n F_{mn}, \quad \Delta_1^{(2)} = -\xi^m \chi_\alpha \partial_m W^\alpha, \quad (5.12)$$

where  $F_{mn} = \frac{1}{2}(\partial_m A - n - \partial_n A_m)$ . It is easy to verify that the equations (5.9) hold because the superfields  $A_\alpha, A_m$  and  $W^\alpha$  satisfy

$$F_{mn} = \gamma_{mn,\beta}^\alpha D_\alpha W^\beta, \quad D_\alpha F_{mn} = (\gamma_{[m} \partial_{n]} W)_\alpha. \quad (5.13)$$

This concludes the example for the massless vertex operator. The result coincides with that obtained in [2] and in [5].

In order to underline again the relevance of the analyticity (or of the grading) to select the correct physical spectrum, one can notice that at a given mass level<sup>8</sup> (or, equivalently, at a given conformal weight) one has the following structure for the vertex operators

$$\mathcal{U}_{z_1 \dots z_n} = \sum_{i=0}^{l \leq n} \sum_{(\{p_i\}, \{k_i\})} \prod_{j=0}^i (\partial_z^{p_j} b)^{k_j} \mathcal{U}_{z_1 \dots z_{n-l}}^{\{\{p_i\}\} \{k_i\}}, \quad (5.14)$$

<sup>8</sup>In the following formulae, we denote by the subscript  $z_1 \dots z_n$  the conformal weight  $n$ , in order not to be confused with the grading index  $p$  of the vertex  $\mathcal{U}_p$

where  $\sum_i p_i = l$ ,  $0 < p_1 < \dots < p_l$  and  $k_i = 0, 1$ . For example the first massive vertex operator can be decomposed into

$$\mathcal{U}_z = \mathcal{U}_z^{\{0,0\},\{0,0\}} + b\mathcal{U}_z^{\{1,0\},\{0,0\}} + \partial_z b\mathcal{U}^{\{0,1\},\{0,1\}} + b\partial_z b\mathcal{U}^{\{1,1\},\{0,1\}}. \quad (5.15)$$

Since  $\{Q, \partial_z^k b\} = 0$ , given a vertex operator of a lower level, for example, the massless vertex  $\mathcal{U}$ , one can construct an element of the BRST cohomology at the next level by  $\partial_z b\mathcal{U}$ . In the same way, at the conformal weight 2 level, one can have  $\partial_z^2 b\mathcal{U}$ . This phenomenon is unwanted since the total cohomology at a given mass level is not described by a single vertex operator. However, the minimum grading of the unwanted terms is  $-4, -8, \dots$  and therefore they are excluded, by choosing analytical (or, positive gradings) vertex operators.

Notice that relaxing the constraint on analyticity, one can find the massless vertex as a part of the massive vertex operator by selecting the  $-4$  grading part of the vertex. In order, to project out the unwanted terms one can multiply the vertex operator  $\mathcal{U}_{z_1 \dots z_n}$  by  $\partial_z b \partial_z^2 b \partial_z^3 b \dots \partial_z^n b$ .

## 6 Construction of a nilpotent covariant BRST charge based on equivariant cohomology.

In this section we present yet another derivation of our BRST charge, in addition to the derivation in [1] based on relaxing constraints by adding new ghosts, and the derivation of Sec. 3 based the BRST charge with operators  $d$  and  $\iota_V$  but not imposing the pure spinor constraints. We assume that the pure spinor constraints is imposed each time after performing the OPE's. This is different viewpoint of equivariant cohomology, but it clarifies the construction of vertex operators in our cohomology starting from those constructed in the pure spinor formulation.

We assume for the present discussion that the spinors  $\lambda^\alpha$  satisfy the pure spinor constraint:  $\lambda\gamma^m\lambda = 0$ . As shown in [2], in order to match correctly the degrees of freedom and to cancel the central charge also the conjugate momentum  $\beta_\alpha$  describe only 22 dof. This can be achieved by observing that the action is invariant under the symmetry  $\delta\beta_\alpha = \Lambda_m(\gamma^m\lambda)_\alpha$  and  $\delta X^m = \delta d_\alpha = \delta\theta^\alpha = \delta\lambda^\alpha = 0$ . The gauge parameters  $\Lambda_m$  removes 10 dof from  $\beta_\alpha$  matching the corresponding 22 dof of  $\lambda^\alpha$ .

This symmetry is encoded in the BRST  $Q_B = \oint \lambda^\alpha d_\alpha$ , by acting twice o

$\beta_\alpha$

$$\{Q_B^2, \beta_\alpha\} = \{Q_B, d_\alpha\} = -\Pi_m(\gamma^m \lambda)_\alpha. \quad (6.1)$$

This implies that, on  $\mathcal{H}_{p.s.}$ , the BRST charge is nilpotent up to gauge transformations with  $\Lambda_m = \Pi_m$ . However, to study its cohomology, it convenient to modify the BRST operator such that it squares to zero on  $\mathcal{H}_{p.s.}$ . This is equivalent to doing the standard Weyl complex procedure [20].

This can be done by extending the BRST transformation of  $\beta_\alpha$  by adding a gauge transformation

$$\{Q'_B, \beta_\alpha\} = \{Q_B, \beta_\alpha\} - \beta_m(\gamma^m \lambda)_\alpha, \quad \{Q_B, \text{other fields}\} = 0, \quad (6.2)$$

This approach leads to introduce a new field  $\beta_m$  with ghost number  $-1$ . On all the other fields the action of  $Q'_B$  is the same of that of  $Q_B$ .

The requirement of nilpotency implies that  $(\{Q'_B, \beta_m\} + \Pi_m)(\gamma^m \lambda)_\alpha = 0$ . The most general solution of this equation is given by  $\{Q'_B, \beta_m\} = -\Pi_m - \kappa^\alpha \gamma_{\alpha\beta}^m \lambda^\beta$  where  $\kappa^\alpha$  is a new spinorial field. In this way, the BRST charge is nilpotent, except on  $\beta_m$ . Requiring that  $Q'_B$  is nilpotent,

$$\{Q'_B, \kappa_z^\alpha\} = -\partial_z \theta^\alpha - b \partial_z \lambda^\alpha - h_z \lambda^\alpha, \quad (6.3)$$

where  $b$  is a scalar with ghost number  $-1$ , and  $h_z$  is a 1-form also with ghost number  $-1$ . Notice that the new terms are allowed because of the pure spinor condition. Finally, imposing that  $Q'_B$  is nilpotent on  $\kappa^\alpha$ , we obtain that  $\{Q_B, b\} = 1$  and  $\{Q_B, h\} = 0$ . A particular realization of  $h_z$  is  $h = x \partial b$  where  $x$  is a constant. At this point we obtain the same BRST charge as obtained earlier by other methods.

The BRST transformations obtained for the antighost fields  $\beta_\alpha, \beta_m, \kappa^\alpha$  and  $b$  coincide with (3.5) if we set  $\xi_m = \chi_\alpha$  to zero. In particular, we note that this is implied by the eq. (3.8) which relates the pure-spinor vertices with those of our cohomology.

The BRST transformation of the field  $b$  would render the BRST cohomology trivial, if we did not introduce further constraint to define physical states. Therefore, we restrict the space on which the BRST charge  $Q'_B$  acts on that part of the enlarged space  $\mathcal{H}'_{p.s.}$  (the pure spinor space which also contains the new fields  $\beta_m, \kappa^\alpha$  and  $b$ ), which has non-negative grading (using the same grading discussed in the previous sections). This simplifies the comparison with pure-spinor formalism as already discussed in the previous section.

## 7 Zero-Momentum Cohomology

Another good test of the physical equivalence of our covariant formulation with the pure spinor approach is the computation of the zero-momentum cohomology in one holomorphic sector.<sup>9</sup> We compute the cohomology at zero momentum for all ghost numbers. This computation yields all zero-momentum states which describe not only the gauge field and its supersymmetric partner, but also, for ghost numbers different from one, the target space ghosts and their antifields [21]. First, we briefly review the zero momentum cohomology in the pure spinor formulation, then we discuss the procedure which extends this result to our formulation and, finally, we present the result.

Using pure spinors, the zero-momentum cohomology is described by the string field  $\Psi$

$$\Psi = C\mathcal{U}_0^{(0)} + a^m\mathcal{U}_{1,m}^{(1)} + \psi^\alpha\mathcal{U}_{2,\alpha}^{(1)} + \psi_\alpha^*\mathcal{U}_3^{(2)\alpha} + a_m^*\mathcal{U}_4^{(2)m} + C^*\mathcal{U}_5^{(3)}, \quad (7.1)$$

where  $C, a_m, \psi^\alpha$  are the ghost, the gauge field, and the gaugino, while  $C^*, a_m^*, \psi_\alpha^*$  are their antifields. The transversal components of the gauge field and the gaugino at  $k^m = 0$  are the natural extension of the same physical states at  $k^m \neq 0$ , but at  $k^m = 0$  there are new “physical” states which in physical application are expected to cancel each other: the longitudinal and timelike components of the Yang-Mills gauge and the spacetime Yang-Mills ghost fields.<sup>10</sup> The vertices  $\mathcal{U}_i$  generate the cohomology  $H(Q_B|\mathcal{H}_{p.s.})$ ; they are constructed in [22] and are given by

$$\begin{aligned} \mathcal{U}_0^{(0)} &= 1, & \mathcal{U}_{1,m}^{(1)} &= \lambda\gamma^m\theta, & \mathcal{U}_{2,\alpha}^{(1)} &= \lambda\gamma^m\theta(\gamma_m\theta)_\alpha, & (7.2) \\ \mathcal{U}_3^{(2)\alpha} &= \lambda\gamma^m\theta\lambda\gamma^n\theta(\gamma_{mn}\theta)^\alpha, & \mathcal{U}_4^{(2)m} &= \lambda\gamma^n\theta\lambda\gamma^r\theta(\theta\gamma_{mnr}\theta), \\ \mathcal{U}_5^{(3)} &= \lambda\gamma^m\theta\lambda\gamma^n\theta\lambda\gamma^r\theta(\theta\gamma_{mnr}\theta). \end{aligned}$$

The superscripts refer to the ghost number, and all the vertices have vanishing conformal spin. An inner product  $\langle\Psi, \Psi\rangle$  is defined by assuming that the product of the ghost field  $C$  should have inner product only with its antifield  $C^*$ , the gauge field  $a^m$  with its antifield  $a_m^*$  and so on.<sup>11</sup> Therefore, this leads to the conclusion that

$$\langle\mathcal{U}_0^{(0)}, \mathcal{U}_5^{(3)}\rangle = \langle\mathcal{U}_{1,m}^{(1)}, \mathcal{U}_4^{(2)m}\rangle = \langle\mathcal{U}_{2,\alpha}^{(1)}, \mathcal{U}_3^{(2)\alpha}\rangle = \mathcal{N} \quad (7.3)$$

<sup>9</sup>To restore the complete superstring spectrum, the string field for the closed superstring  $\Psi_C$  is given by the tensorial product of the two sectors  $\Psi_C = \Psi_L \otimes \Psi_R$ .

<sup>10</sup>For the bosonic string the zero-momentum cohomology consists of the four states given by  $\oint b\mathcal{U}|0, k^m = 0\rangle$  where  $\mathcal{U} = 1, c^z\partial_z x^m, c^z\partial_z c^z\partial_z x^m, c^z\partial_z c^z\partial_z^2 c^z$ .

<sup>11</sup>The definition of an inner product leads to the symplectic BV measure given by  $\langle\delta\Psi, \delta\Psi\rangle = \int d^{10}x \delta\phi^i \wedge \delta\phi_i^*$  where  $\phi^i$  and the fields and  $\phi_i^*$  are the antifields.

where  $\mathcal{N}$  is a normalization factor. It is easy to check that the vertices (7.2) indeed satisfy the equations (7.3) and, in particular, by choosing  $\mathcal{N} = 1$  for simplicity, one obtains the condition

$$\langle \lambda \gamma^m \theta \lambda \gamma^n \theta \lambda \gamma^r \theta (\theta \gamma_{mnr} \theta) \rangle = 1. \tag{7.4}$$

This coincides with Berkovits' prescription for the zero-mode computations in tree level amplitudes [2] and it leads to the construction of the measure  $\mu(\theta, \lambda)$  for the zero modes at tree level, namely

$$\begin{aligned} \langle \lambda \gamma^m \theta \lambda \gamma^n \theta \lambda \gamma^r \theta (\theta \gamma_{mnr} \theta) \rangle &= \int \mu(\theta, \lambda) \left( \lambda \gamma^m \theta \lambda \gamma^n \theta \lambda \gamma^r \theta (\theta \gamma_{mnr} \theta) \right), \tag{7.5} \\ \mu(\theta, \lambda) &= d\Omega_\lambda \left( \lambda^* \gamma^m \frac{\partial}{\partial \theta} \right) \left( \lambda^* \gamma^n \frac{\partial}{\partial \theta} \right) \left( \lambda^* \gamma^r \frac{\partial}{\partial \theta} \right) \left( \frac{\partial}{\partial \theta} \gamma_{mnr} \frac{\partial}{\partial \theta} \right), \end{aligned}$$

where  $d\Omega_\lambda$  is the Haar measure for the pure spinor coset. All vertices in (7.2) carry a grading (namely the grading of  $\lambda$  is 1 and the grading of  $\theta$  is zero) and since only the ghost  $\lambda^\alpha$  appears, the total grading of each vertex is equal to the ghost number which is positive. Following the analysis of the previous sections, given a vertex  $\mathcal{U}_{B,i}^{(n)}$  with ghost number  $n$ , of the zero momentum cohomology  $H^{(n)}(Q_B | \mathcal{H}_{p.s.})$  (the subscript  $B$  stands for Berkovits), it can be lifted to our cohomology  $H^{(n)}(Q | \mathcal{H}_+)$  such that

$$\mathcal{U}_i^{(n)} = \mathcal{U}_{B,i}^{(n)} + \sum_{p \geq 0}^N \mathcal{U}_{(p),i}^{(n)} \tag{7.6}$$

where  $\mathcal{U}_{(p),i}^{(n)}$  is a vertex operator with ghost number  $n$  and grading number  $p$ . At zero momentum the charges  $Q_2 = 0$  and  $Q_3$  have no effect on a generic zero momentum vertex  $\mathcal{U}_i^{(n)}$ . The latter is a polynomial of  $\theta$  and of the ghost fields  $\lambda^\alpha, \xi^m, \chi_\alpha, \omega_m$  and the antighost  $b$ .

Note that  $\{Q_0, \mathcal{U}_{B,i}^{(n)}\} = 0$  because  $\mathcal{U}_{B,i}^{(n)}$  depends only upon  $\theta$  and  $\lambda$ . Acting with  $Q_B$  on  $\mathcal{U}_{B,i}^{(n)}$  (which coincides with the charge  $Q_1$  in (3.4)), we obtain that  $\{Q_B, \mathcal{U}_{B,i}^{(n)}\} = \lambda \gamma^m \lambda M_m(\theta)$  where  $M_m(\theta)$  is a polynomial in  $\theta$ . The right hand side is  $Q_0$ -exact term:  $\lambda \gamma^m \lambda M_m(\theta) = -2 \{Q_0, \xi^m M_m(\theta)\}$  since  $M_m(\theta)$  is  $Q_0$  invariant. The new vertex operator  $\mathcal{U}_{(n+1),i}^{(n)} = 2 \xi^m M_m(\theta)$  has the same ghost number as  $\mathcal{U}_{B,i}^{(n)}$ , but the grading is increased by one unit. The next step is to insert the two vertices in the next equations of the system (5.6), namely

$$\{Q_2, \mathcal{U}_{B,i}^{(n)}\} + \{Q_B, \mathcal{U}_{(n+1),i}^{(n)}\} + \{Q_0, \mathcal{U}_{(n+2),i}^{(n)}\} = 0. \tag{7.7}$$

As already pointed out, the first terms are zero, and therefore we have to repeat the previous sequence of operations: one has to find  $\mathcal{U}_{(n+2),i}^{(n)}$  which

compensates the  $Q_B$  variation of  $\mathcal{U}_{(n+1),i}^{(n)}$ . At the next level, we have a further equation to satisfy, namely we have

$$\{Q_3, \mathcal{U}_{B,i}^{(n)}\} + \{Q_2, \mathcal{U}_{(n+1),i}^{(n)}\} + \{Q_B, \mathcal{U}_{(n+2),i}^{(n)}\} + \{Q_0, \mathcal{U}_{(n+3),i}^{(n)}\} = 0. \quad (7.8)$$

Again, due to the vanishing of momentum, the action of  $Q_2$  and  $Q_3$  on the vertices  $\mathcal{U}_{B,i}^{(n)}$  and  $\mathcal{U}_{(n+1),i}^{(n)}$  vanishes. Therefore, we can solve for  $\mathcal{U}_{(n+3),i}^{(n)}$ . At the next level, we have the simplification that by inserting a  $b$  term we can easily solve the equation. The only limitation comes from the fact that the grading should be positive. This means that  $n+3 \geq 4$ . Finally, we have to take into account that the operation  $Q_B$  removes from  $\mathcal{U}_{(p),i}^{(n)}$  one fermion  $\theta$  and replaces it by a ghost  $\lambda$ . This means that the new vertex  $\mathcal{U}_{(p+1),i}^{(n)}$  has one less fermion  $\theta$  and therefore the sequence of new vertices stops when all the  $\theta$  are removed. This also implies that the highest-grading term  $\mathcal{U}_{(p),N}^{(n)}$  in the polynomial  $\mathcal{U}_i^{(n)}$  is given by the sum of the ghost number  $n$  plus the fermion number.

We give two examples. Starting from  $\mathcal{U}_{B,1,m}^{(1)} = \lambda\gamma^m\theta$ , we have

$$\{Q_B, \lambda\gamma^m\theta\} = \lambda\gamma^m\lambda = -\{Q_0, 2\xi^m\}.$$

Here, we have  $\mathcal{U}_{(2),1,m}^{(1)} = -2\xi^m$ . Furthermore,  $\{Q_B, -2\xi^m\} = 0$ . This implies that  $\mathcal{U}_{(p),1,m}^{(1)} = 0$  for all  $p \geq 3$ . Notice that the complete vertex of our cohomology  $\mathcal{U}_{1,m}^{(1)} = \lambda\gamma^m\theta + 2\xi^m$  is not a cohomological trivial term at zero momentum. The vertex  $\mathcal{U}_{1,m}^{(1)}$  is coupled to the gauge field  $a^m$ .

In the same way, starting from  $\mathcal{U}_{2,\alpha}^{(1)} = \lambda\gamma^m\theta(\gamma_m\theta)_\alpha$ , by using the Fierz identities, we have  $\{Q_B, \lambda\gamma^m\theta(\gamma_m\theta)_\alpha\} = \frac{3}{2}(\lambda\gamma^m\lambda)(\gamma_m\theta)_\alpha$ . This gives the new vertex  $\mathcal{U}_{(2),2,\alpha}^{(1)} = -3\xi^m(\gamma_m\theta)_\alpha$ . Reiterating the procedure, we find the new vertex  $\mathcal{U}_{(3),2,\alpha}^{(1)} = -3\chi_\alpha$  and  $\mathcal{U}_{(p),2,\alpha}^{(1)} = 0$  for all  $p \geq 4$ .

The final result is

$$\begin{aligned} \mathcal{U}_0^{(0)} &= 1, \\ \mathcal{U}_{1,m}^{(1)} &= \lambda\gamma^m\theta + 2\xi^m, \\ \mathcal{U}_{2,\alpha}^{(1)} &= \lambda\gamma^m\theta(\gamma_m\theta)_\alpha - 3\xi^m(\gamma_m\theta)_\alpha - 3\chi_\alpha, \\ \mathcal{U}_3^{(2)\alpha} &= \lambda\gamma^m\theta\lambda\gamma^n\theta(\gamma_{mn}\theta)^\alpha + 3\xi^m(\lambda\gamma^n\theta)(\gamma_{mn}\theta)^\alpha \\ &\quad + 3(\lambda\gamma^m\theta)(\gamma_m\chi)^\alpha + 6\xi^m\xi^n(\gamma_{mn}\theta)^\alpha - 6\xi^m(\gamma_m\chi)^\alpha, \end{aligned} \quad (7.9)$$

$$\begin{aligned}
\mathcal{U}_4^{(2)m} &= \lambda\gamma^n\theta\lambda\gamma^r\theta(\theta\gamma_{mnr}\theta) + 6\xi^n\lambda\gamma^r\theta(\theta\gamma_{rmn}\theta) \\
&\quad + 9\xi^n\xi^r(\theta\gamma_{rmn}\theta) + 6\lambda\gamma^r\theta(\theta\gamma_r\gamma_m\chi) + 18\xi^r(\theta\gamma_r\gamma_m\chi) + 9\chi\gamma_m\chi, \\
\mathcal{U}_5^{(3)} &= \lambda\gamma^m\theta\lambda\gamma^n\theta\lambda\gamma^r\theta(\theta\gamma_{mnr}\theta) + 8\xi^m(\lambda\gamma^n\theta)(\lambda\gamma^r\theta)(\theta\gamma_{mnr}\theta) \\
&\quad + 21\xi^m\xi^n(\lambda\gamma^r\theta)(\theta\gamma_{rmn}\theta) + 6(\lambda\gamma^m\theta)(\lambda\gamma^r\theta)(\theta\gamma_{rm}\chi) \\
&\quad + 2\xi^m\xi^n\xi^r(\theta\gamma_{rmn}\theta) + 18(\lambda\gamma^m\theta)\xi^r(\theta\gamma_{rm}\chi) + 18\xi^m\chi\gamma_m\chi.
\end{aligned}$$

Note that the first and the last vertex operator are spacetime scalars; this suggests that there are no operators with ghost number larger than 3.<sup>12</sup> All the operators have again vanishing conformal spin.

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<sup>12</sup>Notice that the term  $\xi^m\xi^n - \frac{1}{2}\chi\gamma^{mn}\lambda$  is BRST invariant, but due to its grading number it is also trivial in fact:  $\xi^m\xi^n - \frac{1}{2}\chi\gamma^{mn}\lambda = \{Q, b(\xi^m\xi^n - \frac{1}{2}\chi\gamma^{mn}\lambda)\}$ . Therefore, it does not belong to the BRST cohomology.

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