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The C-Deformation of Gluino and Non-planar Diagrams

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Abstract

We consider a deformation of $\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions, which we call the C-deformation, where the gluino field satisfies a Clifford-like algebra dictated by a self-dual two-form, instead of the standard Grassmannian algebra. The superpotential of the deformed gauge theory is computed by the full partition function of an associated matrix model (or more generally a bosonic gauge theory), including non-planar diagrams. In this identification, the strength of the two-form controls the genus expansion of the matrix model partition function. For the case of pure $\mathcal{N} = 1$ Yang-Mills this deformation leads to the identification of the all genus partition function of $c = 1$ non-critical bosonic string at self-dual radius as the glueball superpotential. Though the C-deformation violates Lorentz invariance, the deformed F-terms are Lorentz invariant and the Lorentz violation is screened in the IR.

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1 Introduction

Topological strings [1] and its connection to superstrings [2],[3] have proven to be rather important for a better understanding of the dynamics of $\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions. In particular, the open/closed topological string duality conjectured in [4] and proven in [5] leads to some non-perturbative predictions for $\mathcal{N} = 1$ gauge theories in 4 dimensions constructed as low energy limits of superstring theory [6]. Some of these predictions (coming from genus 0 computations on the closed string side) relate to the superpotential for the glueball fields [7]. This relation has recently been better understood and has led to a striking connection between a wide class of $\mathcal{N} = 1$ supersymmetric gauge theories and planar diagrams of matrix models (or more generally the associated bosonic gauge theories) [8].

However the open/closed string duality suggests an even more extensive insight into the dynamics of $\mathcal{N} = 1$ supersymmetric gauge theory. In particular the closed string side is an $\mathcal{N} = 2$ supersymmetric theory, deformed to $\mathcal{N} = 1$ by turning on fluxes. The topological string computes F-terms for the $\mathcal{N} = 2$ supersymmetric closed string dual of the form [2],[3]

$$\int d^4x d^4\theta (\mathcal{W}_{\alpha\beta}\mathcal{W}^{\alpha\beta})^g F_g(S_i) \tag{1.1}$$

where $\mathcal{W}_{\alpha\beta}$ denotes the $\mathcal{N} = 2$ graviphoton superfield, and the $d^4\theta$ denotes a superintegral over half of the 8 super-directions of the $\mathcal{N} = 2$ superspace, and S_i denote vector multiplets of $\mathcal{N} = 2$. Let us write the four $\mathcal{N} = 2$ super-directions of 1.1 by exhibiting its $\mathcal{N} = 1$ content as $(\theta^\alpha, \hat{\theta}^\alpha)$. As pointed out in [6] turning on fluxes deforms this to an $\mathcal{N} = 1$ supersymmetric theory by giving vev to an auxiliary field of S_i of the form

$$S_i(\theta, \hat{\theta}) = S_i(\theta) + N_i\hat{\theta}^2$$

where $S_i(\theta)$ can now be viewed as an $\mathcal{N} = 1$ chiral superfield, which in the gauge theory context will be interpreted as a glueball superfield. To write the content of 1.1 in purely $\mathcal{N} = 1$ terms we can do one of two things: We can absorb two $\hat{\theta}$'s by expanding $\mathcal{W}_{\alpha\beta}$ to obtain the $\mathcal{N} = 1$ gravitino multiplet $\mathcal{W}_{\alpha\beta\gamma}$, or we can use the auxiliary field vev of the S_i above to absorb them. Turning on the graviphoton field $F_{\alpha\beta}$ (now viewed as a parameter in the $\mathcal{N} = 1$ supersymmetric theory), leads to two terms in the action

$$\begin{aligned}
\Gamma_1 &= g\int d^4x d^2\theta \mathcal{W}_{\alpha\beta\gamma}\mathcal{W}^{\alpha\beta\gamma}(F_{\delta\zeta}F^{\delta\zeta})^{g-1}F_g(S_i), \\
\Gamma_2 &= \int d^4x d^2\theta (F_{\alpha\beta}F^{\alpha\beta})^g N_i \frac{\partial F_g}{\partial S_i}.
\end{aligned}$$

Note that the first term Γ_1 exists even if we do not break the supersymmetry from $\mathcal{N}=2$ to $\mathcal{N}=1$. In particular it is present even if $N_i=0$. The second term Γ_2 is more in tune with breaking supersymmetry to $\mathcal{N}=1$. If we turn off the Lorentz violating term $F_{\alpha\beta}=0$ then we only have the $g=1$ part of Γ_1, giving terms of the form $\int d^4xR^2$ (with appropriate index contractions), or the $g=0$ part of Γ_2, giving the superpotential term for the glueball field.

The main question is to give an interpretation of turning on the Lorentz violating graviphoton background $F_{\alpha\beta}$ in purely $\mathcal{N}=1$ gauge theoretic terms.[1] We will find a satisfactory answer to this question in this paper. In particular we find that deforming the classical anti-commutativity of the gluino fields by making it satisfy the Clifford algebra, dictated by the vev of $F_{\alpha\beta}$ of the form,

$$\{\psi_\alpha,\psi_\beta\}=2F_{\alpha\beta},$$

leads to the $\mathcal{N}=1$ realization of the string deformation. We will see how this arises in string theory and field theory context. The string theory derivation follows the setup of [2] and the more general field theory derivation follows the setup introduced in [9]. Even though the field theory argument is more general (and in particular includes field theories that are not known to be constructible in string theory context), the intuition and ideas coming from the string derivation are crucial for field theory derivation. We in particular find a simple map between the superspace part of these two computations. This leads to the statement that the gradient of the full partition function of the matrix model (not just its planar limit) with potential equal to the superpotential of the gauge theory, computes the superpotential of the associated supersymmetric gauge theory, where the $|F|$ gets identified with the genus counting parameter of the matrix model.[2] This completes the interpretation of the meaning of Γ_2 from the gauge theory side. The interpretation of Γ_1 should follow a similar derivation.

The organization of this paper is as follows: In section 2 we motivate and define C-deformation by studying string theory diagrams with graviphoton turned on. In section 3 we show how the relevant part of the topological string computation works with graviphoton turned on. In section 4 we discuss the field theory limit and how to obtain the same results using the more general field theory setup á la [9] . In section 5 we discuss the physical interpretation of this deformation.

[1] This question was raised in [6] where it was proposed that it may be related to making space non-commutative. Here we find a different interpretation.

[2] This can be generalized to the deformation of the $U(1)$ coupling constants in a straightforward manner.

2 The C-deformation

It has been shown in [2],[3] that the partition functions of topological closed string on a Calabi-Yau three-fold M compute the F-terms of the four dimensional theory obtained by compactifying Type II superstring on M. A similar statement holds when we add D branes [2],[10],[6]. The partition functions of topological string ending on D branes wrapping on n-dimensional cycles on M give the F-terms for the $\mathcal{N} = 1$ supersymmetric gauge theory in four dimensions which is defined as the low energy limit of Type II superstring with D$(n+3)$ branes wrapping on these cycles and extending in four dimensions. The F-terms take the form

$$N \sum_{g,h} \int d^4x d^2\theta \ (\mathcal{W}_{\alpha\beta}\mathcal{W}^{\alpha\beta})^g \ hS^{h-1} \ F_{g,h}, \tag{2.1}$$

where N is the rank of the $U(N)$ gauge group, $\mathcal{W}_{\alpha\beta}$ is the supergravity multiplet whose bottom component is the self-dual part of the graviphoton field strength

$$\mathcal{W}_{\alpha\beta} = F_{\alpha\beta} + \cdots , \tag{2.2}$$

with $\alpha, \beta = 1, 2$ being spinor indices in four dimensions, S is the glueball superfield,

$$S = \frac{1}{32\pi^2}\epsilon^{\alpha\beta}\mathrm{Tr}\ \mathcal{W}_\alpha\mathcal{W}_\beta, \tag{2.3}$$

where $\mathcal{W}_\alpha$ is the chiral superfield with gluino ψ_α as its bottom component,

$$\mathcal{W}_\alpha = \psi_\alpha + \cdots .$$

The topological string computes the coefficients $F_{g,h}$ as the partition function for genus g worldsheet with h boundaries.

In particular the terms in 2.3 with $g = 0$, namely the sum over the planar worldsheets, gives the effective superpotential for S as $Nd\mathcal{F}_0/dS$ where

$$\mathcal{F}_0(S) = \sum_h S^h \ F_{0,h}. \tag{2.4}$$

Combining this with the fact [1] that the partition function for the topological string on the D$(n+3)$ brane can be computed using the Chern-Simons theory (or its dimensional reduction) leads to the statement that the effective action is computed by a sum over planar diagrams of the Chern-Simons theory, or its reduction to the matrix model for specific class of D5 branes wrapping 2-cycles [8].

The purpose of this paper is to understand the meaning of the sum over non-planar diagrams. Among terms that are generated by the flux we have

$$\Gamma(S,F) = N \sum_g \int d^4x \, (F_{\alpha\beta}F^{\alpha\beta})^g \int d^2\theta \frac{\partial}{\partial S} F_g\left(S(\theta)\right), \tag{2.5}$$

where

$$F_g(S) = \sum_h S^h \, F_{g,h}. \tag{2.6}$$

This gives the effective action for the glueball superfield S when the graviphoton field strength is non-zero. The question is whether there is a purely $\mathcal{N}=1$ gauge theoretical interpretation of the same quantity without invoking the coupling to the $\mathcal{N}=2$ supergravity field. Does the graviphoton deform the gauge theory in a way similar to the Neveu-Schwarz two form $B_{\mu\nu}$ generating noncommutativity of coordinates on D branes [11],[12]?

The relation for the topological string amplitudes and the F-terms for the type II string compactification was originally derived using the NSR formalism, where the graviphoton vertex operator in the Ramond-Ramond sector is constructed in terms of the spin field. It was observed in [2] that it generates the topological twist on the worldsheet, which gives the connection between the type II string computation and the topological string computation. A precise derivation of the connection is rather nontrivial, involving a sum over spin structure and nontrivial identities of theta functions [3] . A more economical derivation was found in [13] making use of the covariant quantization of type II superstring compactified on a Calabi-Yau three-fold, which was developed in [14] .

Compared with the covariant quantization of superstring in ten dimensions [15] , the formalism is substantially simpler for superstring compactified on a Calabi-Yau three-fold since the supersymmetry we need to make manifest is smaller. In fact, the four-dimensional part of the worldsheet Lagrangian density that is relevant for our discussion is simply given by

$$\mathcal{L} = \frac{1}{2}\partial X^\mu \bar{\partial} X_\mu + p_\alpha \bar{\partial}\theta^\alpha + p_{\dot\alpha}\bar{\partial}\theta^{\dot\alpha} + \bar{p}_\alpha \partial\bar{\theta}^\alpha + \bar{p}_{\dot\alpha}\partial\bar{\theta}^{\dot\alpha}, \tag{2.7}$$

where p's are $(1,0)$-forms, $\bar{p}$'s are $(0,1)$-forms, and $\theta, \bar{\theta}$'s are 0-forms. The remainder of the Lagrangian density consists of the topologically twisted $\mathcal{N}=2$ supersymmetric sigma-model on the Calabi-Yau three-fold and a chiral boson which is needed to construct the R current. It is useful to note that the worldsheet theory defined by 2.7 (excluding the fermionic fields

with dotted indices) can be regarded as a topological B-model on R^4. The topological BRST symmetry is defined by

$$\begin{aligned} \delta X_{\alpha\dot{\alpha}} &= \epsilon_{\dot{\alpha}}\theta_\alpha + \bar{\epsilon}_{\dot{\alpha}}\bar{\theta}_\alpha, \\ \delta\theta_\alpha = 0, &\quad \delta\bar{\theta}_\alpha = 0, \\ \delta p_\alpha = \epsilon^{\dot{\alpha}}\partial X_{\alpha\dot{\alpha}}, &\quad \delta\bar{p}_\alpha = \bar{\epsilon}^{\dot{\alpha}}\bar{\partial} X_{\alpha\dot{\alpha}}, \end{aligned} \tag{2.8}$$

where $X_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} X_\mu$ with $\sigma^0 = -1$ and $\sigma^{1,2,3}$ being the Pauli matrices, and raising and lowering of spinor indices are done by using the anti-symmetric tensors $\epsilon_{\alpha\beta}, \epsilon_{\dot{\alpha}\dot{\beta}}$ as usual. We recognize that this topological symmetry is closely related to the spacetime supersymmetry. In fact, modulo some shift of $X_{\alpha\dot{\alpha}}$ by $\theta_\alpha\theta_{\dot{\alpha}}$ and $\bar{\theta}_\alpha\bar{\theta}_{\dot{\alpha}}$, 2.8 is identical to transformations generated by the anti-chiral components $Q_{\dot{\alpha}}, \bar{Q}_{\dot{\alpha}}$ of the $\mathcal{N} = 2$ supercharges in the bulk.

When the worldsheet is ending on D branes and extending in four dimensions, the boundary conditions for the worldsheet variables are given by

$$\begin{aligned} (\partial - \bar{\partial})X^\mu = 0, \\ \theta^\alpha = \bar{\theta}^\alpha, \quad p_\alpha = \bar{p}_\alpha. \end{aligned} \tag{2.9}$$

Here we assume that the boundary is located at Im $z = 0$. These boundary conditions preserve one half of the supersymmetry generated by $Q + \bar{Q}$.

Let us turn on the graviphoton field strength $F_{\alpha\beta}$ and the gluino superfield $\mathcal{W}_\alpha$, both of which we assume to be constant. They couple to the bulk and the boundaries of the string worldsheet as

$$\int F^{\alpha\beta}\mathcal{J}_\alpha\bar{\mathcal{J}}_\beta + \oint \mathcal{W}^\alpha\mathcal{J}_\alpha, \tag{2.10}$$

where $\mathcal{J}_\alpha, \bar{\mathcal{J}}_\beta$ are the worldsheet currents for the spacetime supercharges $Q_\alpha, \bar{Q}_\beta$ [16] . We find it convenient to work in the chiral representation of supersymmetry,[3] in which they are given by

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[3]Our convention in this paper is related to that of [13],[14] by redefinition of the worldsheet variables by

$$\begin{aligned} p_\alpha &\rightarrow p'_\alpha = p_\alpha - i\theta^{\dot{\alpha}}\partial X_{\alpha\dot{\alpha}} - \frac{1}{4}\theta^2\partial\theta_\alpha, \\ p_{\dot{\alpha}} &\rightarrow p'_{\dot{\alpha}} = p_{\dot{\alpha}} + i\theta^\alpha\partial X_{\alpha\dot{\alpha}} - \frac{1}{2}\theta^2\partial\theta_{\dot{\alpha}} + \frac{1}{4}\theta_{\dot{\alpha}}\partial\theta^2, \\ X_{\alpha\dot{\alpha}} &\rightarrow X'_{\alpha\dot{\alpha}} = X_{\alpha\dot{\alpha}} + i\theta_\alpha\theta_{\dot{\alpha}} + i\bar{\theta}_\alpha\bar{\theta}_{\dot{\alpha}}. \end{aligned}$$

See also [17] for a related discussion.

$$\begin{aligned} \mathcal{J}_\alpha &= p_\alpha, \\ \mathcal{J}_{\dot\alpha} &= p_{\dot\alpha} - 2i\theta^\alpha \partial X_{\alpha\dot\alpha} + \cdots , \end{aligned} \tag{2.11}$$

where $\cdots$ in the second line represents terms containing $\theta^{\dot\alpha}$ and $\theta^2 = \epsilon_{\alpha\beta}\theta^\alpha\theta^\beta$. The second set of supercharges $\bar{Q}_\alpha, \bar{Q}_{\dot\alpha}$ are defined by replacing p, θ by $\bar{p}, \bar{\theta}$. In this convention, the coupling 2.10 of the graviphoton and the gluino to the worldsheet becomes

$$\int F^{\alpha\beta} p_\alpha \bar{p}_\beta + \oint \mathcal{W}^\alpha p_\alpha. \tag{2.12}$$

This simplifies our analysis in this section. In the field theory limit, the supercharges in this convention take the form,

$$\begin{aligned} Q_\alpha &= \tfrac{\partial}{\partial\theta^\alpha}, \\ Q_{\dot\alpha} &= \tfrac{\partial}{\partial\theta^{\dot\alpha}} + 2i\theta^\alpha \tfrac{\partial}{\partial x^{\alpha\dot\alpha}}. \end{aligned} \tag{2.13}$$

The boundary conditions 2.9 identify the supercurrents $\mathcal{J} = \bar{\mathcal{J}}$, reducing the supersymmetry to $\mathcal{N} = 1$.

2.1 Deformed superspace

Let us analyze the effect of the graviphoton in the bulk. We will find it useful to keep track of mass dimensions of operators, so we introduce the string scale α', which has dimension -2. As usual θ^α has dimension $-1/2$ and its conjugate p_α has dimension $+1/2$. The gluino $\mathcal{W}_\alpha$ has dimension $+3/2$. For the graviphoton field strength $F_{\alpha\beta}$, we assign dimension $+3$. One might have thought that dimension $+2$ would be canonical for the field strength. Here we assign dimension $+3$ to $F_{\alpha\beta}$ so that the higher genus contributions to the superpotential 2.5 remain finite in the field theory limit $\alpha' \to 0$. For example, as we will see later, the genus g contribution to the superpotential in the pure $\mathcal{N} = 1$ super Yang-Mills theory is of the form $\sim N\ (F_{\alpha\beta}F^{\alpha\beta})^g\ S^{2-2g}$ and it has dimension 3 (which is the correct dimension for the superpotential in four dimensions) for all g only if we assign the same dimension to F and $S = \frac{1}{32\pi^2}\epsilon^{\alpha\beta}\mathrm{Tr}\ \mathcal{W}_\alpha\mathcal{W}_\beta$. With this assignment of mass dimensions, the relevant part of the Lagrangian density is expressed as

$$\mathcal{L} = \frac{1}{2\alpha'}\partial X^\mu \bar{\partial} X_\mu + p_\alpha \bar{\partial}\theta^\alpha + \bar{p}_\alpha \partial\bar{\theta}^\alpha + \alpha'^2 F^{\alpha\beta} p_\alpha \bar{p}_\beta. \tag{2.14}$$

As we mentioned, we are working in the chiral representation where supercharges are defined by 2.11 .

It is useful to note that the self-dual configuration of the graviphoton, namely

$$F_{\alpha\beta} \neq 0, \quad F_{\dot{\alpha}\dot{\beta}} = 0, \tag{2.15}$$

gives an exact solution to the string equation of motion. This can be seen clearly from the fact that the perturbed action 2.14 does not break the conformal invariance on the worldsheet. From the target space point of view, we see that the energy-momentum tensor for the graviphoton vanishes for 2.15 , so there is no back-reaction to the metric.[4]

Now let us add boundaries to the worldsheet. For the moment, we turn off the gluino field $\psi_\alpha = 0$ and discuss effects due to the graviphoton field strength. In the presence of graviphoton, the equations of motion for θ and $\bar{\theta}$ are deformed to

$$\begin{aligned} \bar{\partial}\theta^\alpha &= \alpha'^2 F^{\alpha\beta}\bar{p}_\beta \\ \partial\bar{\theta}^\alpha &= -\alpha'^2 F^{\alpha\beta} p_\beta. \end{aligned} \tag{2.16}$$

Before turning on $F^{\alpha\beta}$, the only nontrivial operator product is that between p_α and θ^α (and between $\bar{p}_\alpha$ and $\bar{\theta}^\alpha$) as

$$p_\alpha(z)\theta^\beta(w) \sim \frac{\delta_\alpha^\beta}{2\pi i(z-w)}. \tag{2.17}$$

The relation 2.16 modifies this. If we write

$$\theta^\alpha = \Theta^\alpha + \eta^\alpha, \quad \bar{\theta}^\alpha = \Theta^\alpha - \eta^\alpha,$$

so that the boundary condition is $\eta = 0$, the relation 2.16 together with the short distance singularity 2.17 imply

$$\begin{aligned} \Theta^\alpha(z)\Theta^\beta(w) &\sim \tfrac{1}{2\pi i}\alpha'^2 F^{\alpha\beta}\log\left[\tfrac{z-\bar{w}}{\bar{z}-w}\right], \\ \Theta^\alpha(z)\eta^\beta(w) &\sim -\tfrac{1}{4\pi i}\alpha'^2 F^{\alpha\beta}\log\left[\tfrac{(z-w)(\bar{z}-\bar{w})}{(z-\bar{w})(\bar{z}-w)}\right], \\ \eta^\alpha(z)\eta^\beta(w) &\sim 0. \end{aligned} \tag{2.18}$$

In particular, on the boundary we have

$$\theta^\alpha(\tau+\epsilon)\theta^\beta(\tau) + \theta^\beta(\tau)\theta^\alpha(\tau-\epsilon) = 2\alpha'^2 F^{\alpha\beta}.$$

[4]The energy-momentum tensor for the non-zero field strength can vanish since the self-dual field strength becomes complex valued when analytically continued to Minkowski space.

Therefore a correlation function of operators $\theta = \bar{\theta}$ on the boundary with the time-ordering along the boundary obey the Clifford algebra

$$\{\theta^\alpha, \theta^\beta\} = 2\alpha'^2 F^{\alpha\beta}, \tag{2.19}$$

rather than the standard Grassmannian algebra. Such deformation of the superspace has been studied earlier [18],[19],[20], and it is interesting that it is realized in the context of string theory.[5]

The presence of the factor α'^2 in 2.19 means that the deformation of the superspace does not survive the field theory limit $\alpha' \to 0$ unless we simultaneously take $F^{\alpha\beta} \to \infty$ so that $\alpha'^2 F^{\alpha\beta}$ remains finite. It may be possible to make sense of such a limit since the constant graviphoton field strength is an exact solution to the string equation of motion for any large value of $F_{\alpha\beta}$ as we explained earlier. It turns out, however, if one wants to preserve the $\mathcal{N} = 1$ supersymmetry, we will need to make another modification to the theory, which we call the C-deformation. We will find that this restores the anticommutativity of θ's and undoes the deformation of the superspace. These effects survive the field theory limit without taking $F^{\alpha\beta}$ to be large.

2.2 C-deformation of gluino and undeforming of superspace

Since we work in the chiral representation where $Q_\alpha = \oint p_\alpha$, the supercharges in the field theory limit takes the form,

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} \quad Q_{\dot{\alpha}} = \frac{\partial}{\partial\theta^{\dot{\alpha}}} + 2i\theta^\alpha \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}.$$

The deformation of the superspace by 2.19 would then modify the supersymmetry algebra as

[5] The result of this subsection has been generalized to other dimensions in a recent work [21].

$$\begin{aligned} \{Q_\alpha, Q_{\dot\beta}\} &= 2i\tfrac{\partial}{\partial x^{\alpha\dot\beta}}, \\ \{Q_\alpha, Q_\beta\} &= 0, \\ \{Q_{\dot\alpha}, Q_{\dot\beta}\} &= -8\alpha'^2 F^{\alpha\beta}\tfrac{\partial^2}{\partial x^{\alpha\dot\alpha}\partial x^{\beta\dot\beta}}. \end{aligned} \tag{2.20}$$

A closely related issue arises on the string worldsheet, where the constant graviphoton field strength breaks supersymmetry on the D branes. When $F_{\alpha\beta} = 0$, there are two sets of supercharges Q and $\bar{Q}$, which are identified on the boundary $Q = \bar{Q}$ by the boundary conditions 2.9 . It turns out that the graviphoton vertex operator $F^{\alpha\beta}p_\alpha\bar{p}_\beta$ is not invariant under the supersymmetry but transforms into a total derivative on the worldsheet. Let us consider the combination $\epsilon^{\dot\alpha}\left(Q_{\dot\alpha} + \bar{Q}_{\dot\alpha}\right)$, which preserves the boundary conditions. We find

$$\begin{aligned} \delta\left[\alpha'^2\int F^{\alpha\beta}p_\alpha\bar{p}_\beta\right] &= 2i\alpha'\epsilon^{\dot\alpha}F^{\alpha\beta}\int d\left[Y_{\alpha\dot\alpha}(p_\beta + \bar{p}_\beta)\right] \\ &= \textstyle\sum_{i=1}^{h} 4\alpha'\epsilon^{\dot\alpha}F^{\alpha\beta}\oint_{\gamma_i} Y_{\alpha\dot\alpha}\, p_\beta, \end{aligned} \tag{2.21}$$

where

$$Y_{\alpha\dot\beta} = X_{\alpha\dot\beta} + i\theta_\alpha\theta_{\dot\alpha} + i\bar{\theta}_\alpha\bar{\theta}_{\dot\alpha}. \tag{2.22}$$

and γ_i's are boundaries of the worldsheet. Therefore, as it is, the supersymmetry is broken on the boundaries of the worldsheet. Unlike the deformation of the superspace 2.19 , which disappears in the field theory limit $\alpha' \to 0$, the amount of supersymmetry breaking is comparable to the gluino coupling $\alpha'\oint \mathcal{W}^\alpha p_\alpha$ and therefore is not negligible in this limit.

On the other hand, if the gluino fields $\mathcal{W}_\alpha$ are constant Grassmannian variables taking value in the diagonal of the $U(N)$ gauge group, its coupling to the worldsheet does not break the topological invariance since

$$\delta\oint_\gamma \mathcal{W}^\alpha p_\alpha = 2i\epsilon^{\dot\alpha}\mathcal{W}^\alpha\oint_\gamma dY_{\alpha\dot\alpha} = 0. \tag{2.23}$$

It turns out that there is a natural way to modify this assumption so that the variation of the gluino coupling precisely cancels the boundary term generated by the graviphoton in the bulk. That is to assume that the gluino fields make the Clifford algebra

$$\{\psi_\alpha,\ \psi_\beta\} = 2F_{\alpha\beta}. \tag{2.24}$$

Note that the mass dimensions of the both sides of this equation match up without introducing the string scale α', so this relation survives the field theory limit $\alpha' \to 0$ without making F large. In the following computation, we continue to assume that ψ_α is constant and takes value in the diagonal of $U(N)$. In a more general situation, we interpret 2.24 as saying that

$$\{\mathcal{W}_\alpha(x),\ \mathcal{W}_\beta(x)\}^i{}_j = F_{\alpha\beta}\ \delta^i{}_j \mod \bar{D}, \tag{2.25}$$

where $i, j = 1, \cdots, N$ and the product $\mathcal{W}_\alpha \mathcal{W}_\beta$ in the left-hand side includes the matrix multiplication with respect to these $U(N)$ indices. Note that the identity is modulo $D_{\dot\alpha}$ since that is all we need to cancel the boundary term. Therefore 2.25 should be regarded as a relation in the chiral ring. We call this the C-deformation of the gluino.

This deformation also undoes the deformation of the superspace 2.19 . The analysis of the previous section changes because the gluino is turned on, and it affects the boundary condition of θ and p. One can easily show that the C-deformation of the gluino compensates the effect of the graviphoton on correlation functions of θ's on the boundaries and restores the anticommutativity of θ's there. Namely, θ's remain ordinary Grassmannian variables and the superspace is undeformed. This eliminates the $F^{\alpha\beta}$ dependent term in 2.20 and recovers the standard supersymmetry algebra. This is consistent with the fact that the C-deformation of the gluino restores the spacetime supersymmetry in the graviphoton background.

Let us show that the C-deformation cancels the boundary terms 2.21 and restores the supersymmetry. Since ψ's do not anti-commute with each other, we need to use the path-ordered exponential,

$$\mathcal{P} \exp\left(\alpha' \oint_\gamma \mathcal{W}^\alpha p_\alpha\right), \tag{2.26}$$

along each boundary to define the gluino coupling. As we will see below such a term makes sense, *i.e.* it does not depend on the origin of the path-ordering, as long as the $\oint p_\alpha$ through each boundary is zero. Let us evaluate the variation of the boundary factor 2.26 and find

$$\begin{aligned}
\delta & \left[\mathcal{P} \exp\left(\alpha' \oint_\gamma \mathcal{W}^\alpha p_\alpha\right)\right] \\
= & \ 2i\epsilon^{\dot\alpha} \mathcal{P}\left[\oint_\gamma \mathcal{W}^\alpha dY_{\alpha\dot\alpha} \exp\left(\alpha' \oint_\gamma \mathcal{W}^\alpha p_\alpha\right)\right] \\
= & \ -2i\alpha'\epsilon^{\dot\alpha} F^{\alpha\beta} \mathcal{P}\left[\left(\oint_\gamma Y_{\alpha\dot\alpha} p_\beta - Y_{\alpha\dot\alpha}(o) \oint_\gamma p_\beta\right) \exp\left(\alpha' \oint_\gamma \mathcal{W}^\alpha p_\gamma\right)\right].
\end{aligned} \tag{2.27}$$

Here o is an arbitrarily chosen base point on the boundary γ which is used to define the path-ordering.

This almost cancels the boundary terms 2.21 coming from the graviphoton variation, except for the term $Y_{\alpha\dot{\alpha}}(o) \oint p_\beta$, which depends on the choice of the base point o. If $\oint p_\beta$ through each boundary is zero, this definition of path-ordering is independent of the base point o, and its supersymmetry variation completely cancels the graviphoton variation. For the worldsheet with a single boundary, the condition that $\oint p_\beta$ vanish is automatic, as the boundary is homologically trivial. If there are more boundaries $h > 1$, we need to insert an operator which enforces $\oint p_\beta = 0$ to make the path-ordering well-defined. In particular, the dependence on the base point of path-ordering disappears if we compute a correlation function of $2(h-1)$ gluino fields, which inserts

$$\prod_{i=1}^{h-1} \left(\alpha' \oint_{\gamma_i} \mathcal{W}^\alpha p_\alpha \right)^2 = \left(\alpha'^2 \epsilon^{\alpha\beta} \mathcal{W}_\alpha \mathcal{W}_\beta \right)^{h-1} \prod_{i=1}^{h-1} \epsilon^{\alpha\beta} \oint_{\gamma_i} p_\alpha \oint_{\gamma_i} p_\beta.$$

Note that for these insertion, which are *not* path-ordered, the Grassmannian property of p_α projects the $\mathcal{W}$ contribution on the antisymmetric part via $\epsilon_{\alpha\beta}$. There are no contributions from the graviphoton $F^{\alpha\beta}$, because $\epsilon^{\alpha\beta} F_{\alpha\beta} = 0$. Note that $2(h-1)$ is the maximum number of gluino insertions we can make for a given number of boundaries if we take into account the global constraint

$$\sum_{i=1}^{h} \oint_{\gamma_i} p_\alpha(\tau) = \frac{1}{2} \int d(p_\alpha + \bar{p}_\alpha) = 0.$$

Since $p_\alpha, \bar{p}_\alpha$ are fermionic, inserting $2(h-1)$ gluino fields amounts to imposing a constraint

$$\oint_{\gamma_i} p_\alpha(\tau) = 0, \tag{2.28}$$

on each boundary and the o dependent terms in 2.27 vanishes in this case, completely cancelling 2.21 . Therefore we can compute the topological open string amplitude for worldsheet with h boundaries if and only if we compute the correlation function of $2(h-1)$ gluino superfields, consistently with the structure in 2.1 . In other words, the only F-terms that make sense in this context involve insertions of $(h-1)$ factors of S. As we will see in the next section, the path ordering of the gluino vertex on all the boundaries leads in the path-integral computation to a term involving $(F^2)^g$.

The fact that we can make sense of only such F-term amplitudes, which impose the vanishing of the fermionic momentum through each hole, strongly suggests that the rest of the amplitudes should be set to 0. In some sense, those would be the analog of trying to obtain a non-gauge invariant correlator in a gauge invariant theory and finding it to be zero after integrating over the gauge orbit.

We have found that the string theory computation in the presence of the constant graviphoton field strength preserves the topological invariance on the worldsheet and compute the F-terms 2.1 of the low energy effective theory on the D branes if we turn on the C-deformation 2.19 of the gluino fields. The C-deformation also restores the anticommutativity of θ's, and thereby undeforms the superspace.

3 Topological string amplitudes

We found that the constant graviphoton background by itself does not preserve the $\mathcal{N} = 1$ supersymmetry on the D branes. We need to turn on the C-deformation of the gluino 2.24 in order to restore the supersymmetry. We can then compute the F-terms for this background by evaluating the topological string amplitude. The mechanism to absorb the zero modes of the worldsheet fermions works essentially in the same way as described in [2],[13] in the case of the closed string. The only nontrivial part of the topological string computation is the one that involves the zero modes of $(p_\alpha, \theta^\alpha)$ system.

Before evaluating the zero mode integral, it is useful to establish the following formula on a genus-g worldsheet with h boundaries,

$$\begin{aligned} & \prod_{i=1}^{h-1} \alpha'^2 \epsilon^{\alpha\beta} \oint_{\gamma_i} p_\alpha \oint_{\gamma_i} p_\beta \ \times \exp\left[\alpha'^2 F^{\alpha\beta} \int p_\alpha \bar{p}_\beta + \alpha' \sum_{i=1}^{h} \oint_{\gamma_i} \mathcal{W}^\alpha p_\alpha\right] \\ = & \prod_{i=1}^{h-1} \alpha'^2 \epsilon^{\alpha\beta} \oint_{\gamma_i} p_\alpha \oint_{\gamma_i} p_\beta \ \times \exp\left[\alpha'^2 F^{\alpha\beta} \sum_{a,b=1}^{2g} c^{ab} \oint_a (p_\alpha + \bar{p}_\alpha) \oint_b (p_\alpha + \bar{p}_\alpha)\right]. \end{aligned} \tag{3.1}$$

Here we assume that p_α is holomorphic and $\bar{p}_\alpha$ is anti-holomorphic. The factor $\prod_i \epsilon^{\alpha\beta} \oint_{\gamma_i} p_\alpha \oint_{\gamma_i} p_\beta$ is due to the insertion of the glueball superfield S^{h-1}. The gluino fields $\mathcal{W}_\alpha$ in the left-hand side are C deformed as in 2.24 , and $\exp\left(\oint \mathcal{W}^\alpha p_\alpha\right)$ is defined by the path-ordering. We choose homology cycles on the surface as γ_i homologous to the boundaries and $a, b = 1, \cdots, 2g$

associated to the handles on the worldsheet, and c^{ab} is the intersection matrix of these cycles. Since $c^{ij} = 0, c^{ai} = 0$, we can add γ_i to a, b without changing the intersection number. This does not change the value of the exponent thanks to the constraint $\oint_{\gamma_i} p_\alpha = 0$ imposed by the insertion of $\epsilon^{\alpha\beta} \oint_{\gamma_i} p_\alpha \oint_{\gamma_i} p_\beta$.

The proof of the identity 3.1 consists of two parts. Let us first evaluate $\exp\left[\alpha' \oint_\gamma \psi^\alpha p_\alpha\right]$ with the path-ordering along a boundary γ. If ψ_α were ordinary Grassmannian variables, this would be equal to 1 due to the momentum constraint $\oint_\gamma p_\alpha = 0$ on the boundary. To obtain a nontrivial answer, we need to use the anti-commutation relation 2.24 . For example,

$$\mathcal{P} \oint \mathcal{W}^\alpha p_\alpha \oint \mathcal{W}^\beta p_\beta = 2F^{\alpha\beta} \oint p_\alpha(\tau) \int_o^\tau p_\beta,$$

where we used $\oint p_\alpha = 0$ in the last line. By iteratively using this identity, we can show

$$\mathcal{P} \exp\left(\alpha' \oint_\gamma \mathcal{W}^\alpha p_\alpha\right) = \exp\left[2\alpha'^2 F_{\alpha\beta} \oint_\gamma p_\alpha(\tau) \int_o^\tau p_\beta\right]. \tag{3.2}$$

The second part of the proof is essentially the same as the proof of the Riemann bilinear identity. We start by writing

$$F^{\alpha\beta} \int p_\alpha \bar{p}_\beta = -\frac{1}{2} F^{\alpha\beta} \int d\left[(p_\alpha(z) + \bar{p}_\alpha(\bar{z})) \int^z (p_\beta + \bar{p}_\beta)\right]. \tag{3.3}$$

We then cut the worldsheet open along the cycles a, b and also introduce cuts $\widehat{\gamma}_{i,i+1}$ between the boundaries γ_i and γ_{i+1} so that we can perform the integration-by-parts in the resulting contractible domain (see Fig. 1). The surface integral in 3.3 is then transformed into contour integrals along the homology cycles a, b, the boundaries γ_i and the cuts $\widehat{\gamma}_{i,i+1}$ connecting them. The integral along $\widehat{\gamma}_{i,i+1}$ vanishes since it intersects only with the boundaries γ_i and γ_{i+1} and $\oint p_\alpha$ vanishes for them. Thus we are left with

$$\begin{aligned} F^{\alpha\beta} \int p_\alpha \bar{p}_\beta \quad &= -\tfrac{1}{2} F^{\alpha\beta} \textstyle\sum_{a,b=1}^{2g} c^{ab} \oint_a (p_\alpha + \bar{p}_\alpha) \oint_b (p_\beta + \bar{p}_\beta) \\ &\quad - 2F^{\alpha\beta} \textstyle\sum_{i=1}^{h-1} \oint_{\gamma_i} p_\alpha(\tau) \int_{o_i}^\tau p_\beta. \end{aligned} \tag{3.4}$$

Combining 3.3 with 3.2 , we obtain 3.1 .

Let us now evaluate the right-hand side of 3.1 with an explicit parametrization of the fermion zero modes. A genus-g worldsheet Σ with h handles can

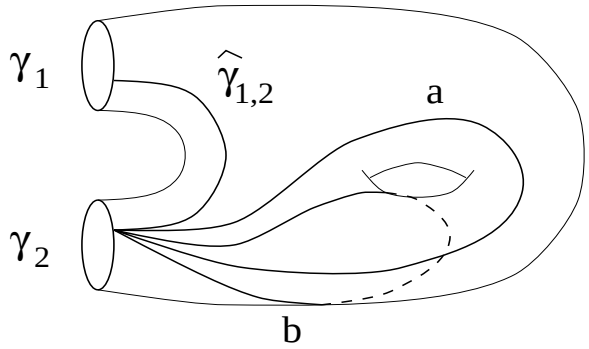

Figure 1: *The worldsheet can be made into a contractible region by cutting along cycles a, b and $\hat{\gamma}_{1,2}$.*

be constructed from a genus $(2g+h-1)$ surface $\widetilde{\Sigma}$ with a complex conjugation involution Z_2 as $\Sigma = \widetilde{\Sigma}/Z_2$, where the boundaries of Σ are made of Z_2 fixed points of $\widetilde{\Sigma}$. With respect to the Z_2 involution, we can choose the canonical basis of one forms on $\widetilde{\Sigma}$ as $\{\omega_a, \tilde{\omega}_i\}_{a=1,\cdots,2g;\ i=1,\cdots,h-1}$ so that

$$\omega_a(z) = \bar{\omega}_{a+g}(\bar{w})_{|\bar{w}=z}, \quad \tilde{\omega}_i(z) = \bar{\tilde{\omega}}_i(\bar{w})_{|\bar{w}=z},$$

and normalized as $\oint_{\gamma_i} \omega_j = \delta_{ij}$. We can then parametrize p_α as

$$\begin{aligned} p_\alpha &= \sum_{a=1}^{g} (\pi^a \omega_a + \bar{\pi}^a \omega_{a+g}) + \sum_i \tilde{\pi}^i \tilde{\omega}_i, \\ \bar{p}_\alpha &= \sum_{a=1}^{g} (\bar{\pi}^a \bar{\omega}_a + \pi^a \bar{\omega}_{a+g}) + \sum_i \tilde{\pi}^i \bar{\tilde{\omega}}_i. \end{aligned} \tag{3.5}$$

The right-hand side of 3.1 is then expressed as

$$\begin{aligned} &\prod_{i=1}^{h-1} \alpha'^2 \epsilon^{\alpha\beta} \oint_{\gamma_i} p_\alpha \oint_{\gamma_i} p_\beta \times \exp\left(\alpha'^2 F^{\alpha\beta} \int p_\alpha \bar{p}_\beta + \alpha' \sum_{i=1}^{h} \oint_{\gamma_i} \mathcal{W}^\alpha p_\alpha \right) \\ &= \prod_{i=1}^{h-1} \alpha'^2 \epsilon^{\alpha\beta} \tilde{\pi}_{i\alpha} \tilde{\pi}_{j\beta} \times \\ &\times \exp\left[2\alpha'^2 F^{\alpha\beta} (\pi^a_\alpha + \bar{\pi}^a_\alpha) \Big(\left(\Omega_{ab} - \bar{\Omega}_{a+g,b}\right) \pi^b_\beta + \left(\bar{\Omega}_{ab} - \Omega_{a+g,b}\right) \bar{\pi}^b_\beta \Big) \right]. \end{aligned} \tag{3.6}$$

We can then integrate over the zero modes $\pi_A, \tilde{\pi}_i$ and obtain

$$\prod_{i=1}^{h-1} \epsilon^{\alpha\beta} \oint_{\gamma_i} p_\alpha \oint_{\gamma_i} p_\beta \ \times \exp\left[\alpha'^2 F^{\alpha\beta} \int p_\alpha \bar{p}_\beta + \alpha' \sum_{i=1}^{h} \oint_{\gamma_i} \mathcal{W}^\alpha p_\alpha\right]$$
$$= \alpha'^{2(2g+h-1)} \ (F_{\alpha\beta}F^{\alpha\beta})^g \left[\det \mathrm{Im}\,(\Omega_{ab} + \Omega_{a+g,b})\right]^2. \qquad (3.7)$$

The factor $\alpha'^{2(2g+h-1)}$ and the determinant of the period matrix are cancelled by the integral over the bosonic zero modes of $X_{\alpha\dot{\alpha}}$, and we are simply left with $(F_{\alpha\beta}F^{\alpha\beta})^g$. We have found that the contribution from the four-dimensional part of the worldsheet theory is to supply the genus counting factor $(F_{\alpha\beta}F^{\alpha\beta})^g$ in addition to the standard S^{h-1} term. All the nontrivial g and h dependence of $F_{g,h}$ in the F-term should come from the topological string computation for the internal Calabi-Yau space described by a $\hat{c} = 3$, $\mathcal{N} = 2$ superconformal field theory. This is consistent with the general statement [2] about the correspondence between the topological string amplitudes for the $\hat{c} = 3$ superconformal field theory and the F-term computation for the Calab-Yau compactification. Here we have shown explicitly that it works perfectly in the case of the open string theory if we take into account the C-deformation of the gluino that is necessary to preserve the supersymmetry in the graviphoton background.

4 Field theory limit

The field theoretic computation of $\mathcal{N} = 1$ glueball superpotential was performed in [9] using a suitable chiral superspace diagram technique developed in [22]. Let us briefly recall the relevant part of the computation: As in [9] , we consider the computation in the context of an adjoint $U(N)$ matter, with some superpotential, though the generalization to arbitrary cases admitting large N description is straight-forward. One takes an anti-chiral superpotential $\overline{m}\bar{\Phi}^2$ and integrates the $\bar{\Phi}$ out to obtain a theory purely in terms of Φ, given by

$$S = \int d^4x d^2\theta \ \Phi \frac{1}{2\bar{m}} \left(\nabla^2 + \mathcal{W}^\alpha D_\alpha\right) \Phi + W(\Phi)$$

where $W(\Phi)$ is the superpotential, ∇^2 is the ordinary Laplacian. In the derivation of this result, it was assumed that $\mathcal{W}_\alpha$ is covariantly constant, *i.e.* constant in spacetime and in an Abelian configuration taking value in the Cartan subalgebra. Moreover $D_\alpha \mathcal{W}^\alpha = 0$. By integrating the Φ out, one obtains an effective superpotential for the glueball field $S = \frac{1}{32\pi^2}\epsilon_{\alpha\beta}\mathrm{Tr}\ \mathcal{W}^\alpha \mathcal{W}^\beta$.

The Feynman diagrams are dictated by the interaction of $W(\Phi)$, from which one extracts the $\frac{1}{2}m\Phi^2$ term and puts it in the propagator as usual. For each internal line I of the Feynman diagram, we have a propagator given by

$$\int_0^\infty ds_I \, \exp\left[-\frac{s_I}{2\bar{m}}(P_I^2 + \mathcal{W}^\alpha \pi_\alpha^I + m\bar{m})\right], \tag{4.1}$$

where s_I denotes the Schwinger time, P_I and π^I are the bosonic and fermionic momenta along the line. Moreover $\mathcal{W}^\alpha$ acts as an adjoint action on the boundaries of the 't Hooft diagram. We can remove the $\bar{m}$ dependence by rescaling $P_I \to (2\bar{m})^{\frac{1}{2}} P_I$ and $\pi^I \to 2\bar{m}\pi^I$ so that the propagator becomes

$$\int_0^\infty ds_I \, \exp\left[-s_I(P_I^2 + \mathcal{W}^\alpha \pi_\alpha^I + m)\right]. \tag{4.2}$$

This rescaling keeps invariant the measure $d^4P \, d^2\pi$ of the zero mode integral.

This piece of the Feynman diagram computation is exactly what one sees as the spacetime part of the superstring computation which we reviewed in the last section. In fact the propagator 4.2 is the zero slope limit of the open string propagator evaluated in the Hamiltonian formulation on the worldsheet, where the Schwinger parameters s_I are coordinates in scaling regions near the boundaries of the moduli space of open string worldsheet where open string propagators become infinitely elongated and worldsheets collapse to Feynman diagrams. As pointed out in [1] and elaborated in more detail in [2] , integrals over the moduli space of worldsheets which define topological string amplitudes localize to these regions. This is how the topological string amplitude computations discussed in the last section automatically give results in the field theory limit. In fact we saw explicitly the α' dependence cancels out in the final expression of the topological amplitudes. To make the dictionary complete, the bosonic and fermionic momenta, $P_{I\alpha\dot{\alpha}}$ and π_α^I, are the zero mode of $i\partial X_{\alpha\dot{\alpha}}$ and p_α on the open string propagator. In the exponent of 4.2, P_I^2 is the zero slope limit of the worldsheet Hamiltonian $L_0 + \bar{L}_0$, and and the term $s_I \mathcal{W}^\alpha \pi_\alpha^I$ comes from the gluino coupling $\oint \mathcal{W}^\alpha p_\alpha$ on the boundary of the string worldsheet. In this setup, the superpotential $W(\Phi)$ of the gauge theory encodes the information on the internal Calabi-Yau space.

For an l-loop Feynman diagram, the fermionic momenta π^I are parametrized by loop momenta π^A $(A = 1, \cdots, l)$ as

$$\pi_\alpha^I = \sum_{A=1}^{l} L_{IA} \pi_\alpha^A, \tag{4.3}$$

where $L_{IA} = \pm 1$ if the I-th propagator is part of the loop A (taking into account the relative orientation of I and A) and $L_{IA} = 0$ otherwise. Note

that, if we view the 't Hooft diagram as the zero slope limit of the open string worldsheet, we have the relation

$$l = 2g + h - 1,$$

where g and h are the numbers of handles and boundaries of the worldsheet. From the field theory point of view, h is also the number of 't Hooft index loops.

The computation in [9] proceeds by noting that, in order to absorb the fermion zero modes π_α^A, we need to bring down $2l$ gluino fields $\mathcal{W}_\alpha$. Moreover, for corrections involving $\epsilon^{\alpha\beta}\mathrm{tr}\,(\mathcal{W}_\alpha\mathcal{W}_\beta)$, each 't Hooft index loop can contain at most two $\mathcal{W}$ insertions. Therefore, if $\mathcal{W}$'s are Grassmannian, it immediately follows that we need the number h of index loops is $l+1$ or more in order to absorb the $2l$ fermion zero modes.[6] Since $l = 2g+h-1$, this is possible only when $g = 0$, namely the 't Hooft diagram must be planar. In this case, the product of the propagators 4.2 in the Feynman diagram gives the factor

$$\prod_{A=1}^{l} \exp\left(-\mathcal{W}^\alpha \sum_I s_I L_{IA} \pi_\alpha^I\right) = \prod_{A=1}^{l} \exp\left(-\mathcal{W}^\alpha \sum_B M_{AB}(s)\pi_\alpha^B\right), \quad (4.4)$$

where $M_{AB}(s)$ is an $l \times l$ matrix defined by

$$M_{AB}(s) = \sum_I s_I L_{IA} L_{IB}. \quad (4.5)$$

The integration over the fermionic momenta π^A produces the determinant $(\det M_{AB}(s))^2$. This s-dependent factor is cancelled out by the integral over the bosonic momenta P_I, which produces $(\det M_{AB}(s))^{-2}$. Similarly one can extract the contribution to the $U(1)$ coupling constants $\mathrm{Tr}\ \mathcal{W}_\alpha\ \mathrm{Tr}\ \mathcal{W}^\alpha$, and see that they also come only from the planar diagrams.

For non-planar diagrams, we have $l + 1 - h = 2g > 0$, and therefore we must have more than two $\mathcal{W}$'s on some loop in order to absorb all the fermion zero modes. This is not possible if $\mathcal{W}$'s are Grassmannian variables in the Abelian configuration relevant for [9] . Therefore non-planar amplitudes vanish in this case by the fermion zero mode integral. This is precisely the part of the story that is going to change when we consider the C-deformation 2.24 of $\mathcal{W}$.

[6] We need h to be $l+1$ or more rather than l since each propagator is associated to a pair of index loops going in opposite directions and a sum over $s_I\pi^I$ along all index loops vanish.

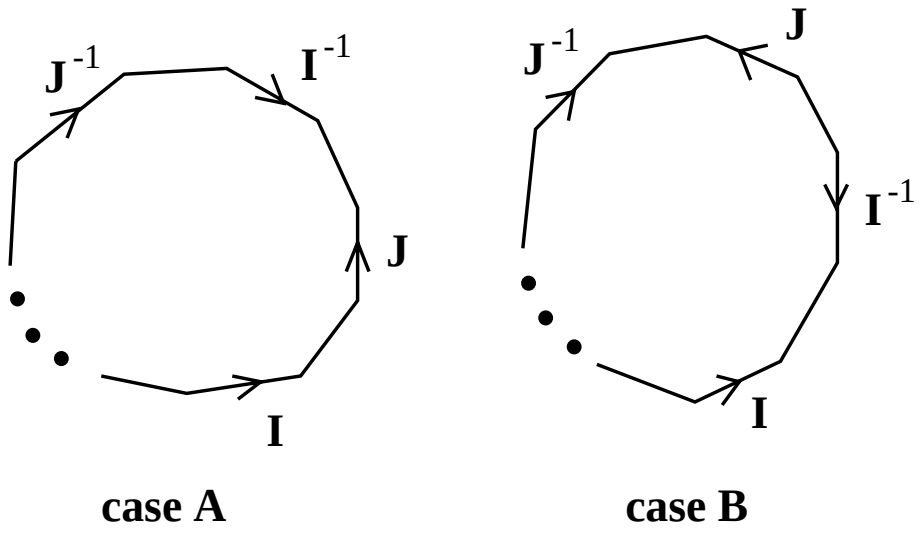

Figure 2: *The path-ordered gluino insertion receives contributions from pairs of edges I, J if they are oriented as in case A. On the other hand, the contributions cancel in case B.*

If $\mathcal{W}$ is not Grassmannian, we need to take into account their path-ordering along each index loop when we take a product of propagators as in 4.4 . In the last section, we saw that the path-ordered exponential of $\mathcal{W}^\alpha p_\alpha$ integrated around a boundary γ_i $(i = 1, \cdots, h)$ of the worldsheet gives the factor

$$\mathcal{P} \exp\left(\alpha' \oint_{\gamma_i} \mathcal{W}^\alpha p_\alpha\right) = \exp\left(-2\alpha'^2 F^{\alpha\beta} \oint_{\gamma_i} p_\alpha(\tau) \int_{o_i}^{\tau} p_\beta\right), \tag{4.6}$$

modulo $\oint_{\gamma_i} p_\alpha = 0$. In the field theory limit, we regard γ's as 't Hooft index loops and replace

$$\oint_{\gamma_i} p_\alpha \to \sum_I L_{Ii} s_I \pi_\alpha^I,$$

where L_{Ii} picks up internal lines I's along the i-th index loop taking into account the relative orientation of i and I. Thus we can write the exponent of 4.6 as

$$F^{\alpha\beta} \oint_{\gamma_i} p_\alpha(\tau) \int_{o_i}^{\tau} p_\beta \to F^{\alpha\beta} \sum_{I>J} s_I L_{Ii} \pi_\alpha^I \cdot s_J L_{Ji} \pi_\beta^J, \tag{4.7}$$

where the inequality $I > J$ is according to the path-ordering of the edges of the propagators I, J along the i-th index loop.

In fact the above expression can be evaluated by a simple set of rules when we have only one boundary as we will now discuss. If we have only

one boundary every edge appears twice with opposite orientation (this is in particular consistent with the fact that $\oint p_\alpha = \sum_I s_I \pi^I L_{Ii} = 0$). Note that the expression 4.7 involves pairs of distinct edges (for the same edge $F^{\alpha\beta}\pi_\alpha\pi_\beta$ vanishes). Let us consider two distinct edges I and J of the boundary (see Figure 2). The contribution to the exponent of 4.7 vanishes in the case depicted as B in the figure because the $\pi^J\pi^I$ terms appear twice with opposite sign, whereas in the case A they appear twice with the same sign and so it survives. Note that in case A if we had the ordering $IJ^{-1}I^{-1}J$ it would still survive, with an overall minus sign relative to the case depicted in the figure. Later we will use this rule to evaluate some examples.

Let us show that the product of exponential of 4.7 over all index loops, together with the usual $2(h-1)$ insertions of gluino fields, absorbs all the fermion zero modes π^A_α and the result of the zero mode integral cancels the s-dependent factor coming from the integral over the bosonic momenta. We have already seen that this is the case in the topological string computation in the last section. Here we will show how this works in the field theory limit. Two gluino fields inserted on each index loop γ_i enforce that the sum over momenta along the loop vanishes,

$$\sum_I L_{Ii} s_I \pi_I = 0. \tag{4.8}$$

Under this condition, we can prove the following identity.

$$\begin{aligned} \sum_{I>J} s_I L_{Ii}\pi^I_\alpha \cdot s_J L_{Ji}\pi^J_\beta &= 2c^{ab}F^{\alpha\beta} \sum_I s_I L_{Ia}\pi^I_\alpha \cdot \sum_J s_J L_{Jb}\pi^J_\beta \\ &= 2c^{ab}F^{\alpha\beta}\ M_{aA}(s)\pi^A_\alpha \cdot M_{bB}(s)\pi^B_\beta. \end{aligned} \tag{4.9}$$

To show this, it is most convenient to go back to the identity 3.3 on the string worldsheet and take the zero slope limit of string theory, where we can set $p_\alpha = \bar{p}_\beta$ everywhere on the string worldsheet. In this limit, 3.3 reduces to

$$-2F^{\alpha\beta}\sum_{i=1}^{h-1}\oint_{\gamma_i} p_\alpha(\tau)\int_{o_i}^{\tau} p_\beta = 2F^{\alpha\beta}c^{ab}\oint_a p_\alpha \oint_b p_\beta,$$

since $F^{\alpha\beta}\int p_\alpha\bar{p}_\beta = F^{\alpha\beta}\int p_\alpha p_\beta = 0$ by the symmetry of $F^{\alpha\beta}$ under exchange of α,β. The identity 4.9 can be obtained by expressing this in terms of the field theory quantities. [7]

[7] More careful computation at the leading order in α' shows

$$\alpha'^2 F^{\alpha\beta}\int p_\alpha\bar{p}_\beta = -\frac{1}{2}\alpha'^2 c^{ab}F^{\alpha\beta}\oint_a (p_\alpha - \bar{p}_\alpha)\oint_b (p_\beta - \bar{p}_\beta) = -2\alpha'^2 c^{ab}F^{\alpha\beta}\sum_{I,J} L_{Ia}L_{Jb}\pi^I_\alpha\pi^J_\beta, \tag{4.10}$$

Combining the exponential of 4.9 with the $2(h-1)$ insertions of the gluino fields, we find that the fermion zero mode integral is given by

$$\int d^{2l}\pi \prod_{i=1}^{h-1} \epsilon^{\alpha\beta} M_{iA}(s)\pi_\alpha^A M_{iB}\pi_\alpha^B \times \exp\left[2c^{ab}F^{\alpha\beta}M_{aA}(s)\pi_\alpha^A \cdot M_{bB}(s)\pi_\beta^B\right]. \tag{4.12}$$

To evaluate this, it is convenient to make the change of variables,

$$\pi_\alpha^A \to \hat{\pi}_{\alpha A} = M_{AB}(s)\pi_\alpha^B.$$

The integral 4.12 then becomes

$$\begin{aligned} (\quad & \det M_{AB}(s))^2 \int d^{2l}\hat{\pi}_A \prod_i \epsilon^{\alpha\beta}\hat{\pi}_{\alpha i}\hat{\pi}_{\beta i} \times \exp\left[2c^{ab}F^{\alpha\beta}\hat{\pi}_{\alpha a}\hat{\pi}_{\beta b}\right] \\ = \quad & (\det M_{AB}(s))^2 (F^{\alpha\beta}F_{\alpha\beta})^g. \end{aligned}$$

As in planar diagrams, the integral over bosonic momenta gives the factor of $(\det M(s))^{-2}$. So we are left with no s-dependent factor and we just have $(F^2)^g$ (with some factors of (2π) which can be absorbed into the definition of F), in addition to the S^{h-1} factor. Moreover we have the combinatoric factore of Nh. The factor of N comes from the loop with no glueball insertions and the factor of h comes from the choice of which of the h boundaries we choose not to put the glueball superfield on.

4.1 Examples

It is helpful to illustrate the general derivation in the field theory limit presented above by some examples. Here we will consider three examples. In the first example we show how the computation works for the case of a simple genus g diagram with one boundary, which in the field theory computation arises from a $2g$ loop Feynman diagram involving a single Tr Φ^{4g} interaction. In the second example we consider a genus 1 diagram with one boundary, involving two Tr Φ^3 vertices. In the third example we consider a diagram with $g=1$ and $h=2$ involving four Tr Φ^3 interactions. The last example

---

$$-\frac{1}{2}\alpha'^2 c^{ab}F^{\alpha\beta}\oint_a (p_\alpha + \bar{p}_\alpha)\oint_b (p_\beta + \bar{p}_\beta) = -2\alpha'^2 F^{\alpha\beta}\sum_{IJ} s_I s_J \cdot L_{Ia}L_{Jb}\pi_\alpha^I \pi_\beta^J. \tag{4.11}$$

In the zero slope limit $\alpha' \to 0$, the right-hand side of 4.11 remains finite if we rescale $s_I \to s_I/\alpha'$ (which infinitely elongate open string propagators) while 4.10 vanishes in this limit.

is the most interesting one, in the sense that it involves both the glueball superfield and the F^2 term.

Example 1:

From a single Tr Φ^{4g} vertex we can form a genus g surface with a single boundary. The Feynman diagram for this interaction involves $2g$ loops. The boundary consists of $2g$ pairs of oppositely oriented edges each of which forms one loop. Along the boundary of the Riemann surface they are ordered according to the usual opening up of a genus g surface in the form

$$I_1 I_2 I_1^{-1} I_2^{-1} I_3 I_4 I_3^{-1} I_4^{-1} ... I_{2g-1} I_{2g} I_{2g-1}^{-1} I_{2g}^{-1}$$

According to the rule discussed before, for the path-ordered gluino insertions we get

$$\exp\Big(\sum_{I=1}^{g} \langle s_{2I-1}\pi_{2I-1}, s_{2I}\pi_{2I} \rangle \Big)$$

where $\langle ., . \rangle$ denotes the contraction with $F^{\alpha\beta}$. Integration over the fermionic loop momenta is the same as integration over the π_I edge momenta as they are in one to one correspondence. To absorb the zero modes we have to bring down each term in the exponent exactly twice. This gives the factor

$$(F^2)^g \prod_{I=1}^{2g} s_I^2$$

The bosonic momentum integral (up to factors of 2π which can be absorbed into the definition of F) gives $1/s_I^2$ for each loop and so the product over all the loops cancels the s dependence, as expected, leading to $(F^2)^g$.

Example 2:

As our next example we consider a genus 1 diagram with one boundary formed from two trivalent vertices (see Fig. 3). The edges along the boundary are ordered as $1\ 2^{-1}\ 3\ 1^{-1}\ 2\ 3^{-1}$. Thus the path ordered contribution gives

$$\exp\Big(- \langle s_1\pi_1, s_2\pi_2 \rangle - \langle s_2\pi_2, s_3\pi_3 \rangle - \langle s_3\pi_3, s_1\pi_1 \rangle \Big) \tag{4.13}$$

Note that, as already explained, this factor can also be written as the product of integral of fermionic momenta around the two non-trivial cycles of the torus, denoted by A and B in Fig. 3. Namely

$$\pi_{\text{along } A} = s_1\pi_1 - s_2\pi_2$$

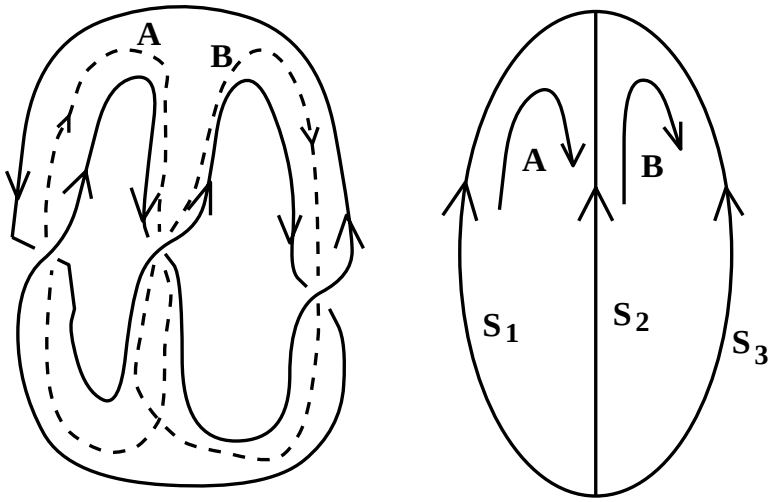

Figure 3: *The genus 1 Riemann surface with one boundary, constructed from two cubic interactions.*

$$\pi_{\text{along } B} = s_2\pi_2 - s_3\pi_3$$

and we have

$$\langle \pi_{\text{along } A}, \pi_{\text{along } B}\rangle = \langle s_1\pi_1, s_2\pi_2\rangle + \langle s_2\pi_2, s_3\pi_3\rangle + \langle s_3\pi_3, s_1\pi_1\rangle,$$

which is the same as 4.13 , up to choice of orientation of cycles. Here we have used the fact that $\langle \pi_2, \pi_2\rangle = 0$, *etc.* Writing these in terms of the fermionic loop momenta π_A and π_B we have

$$\pi_1 = \pi_A,\ \ \pi_2 = \pi_B - \pi_A,\ \ \pi_3 = -\pi_B$$

which leads to the path ordered contribution

$$\exp\Big[- (s_1s_2 + s_2s_3 + s_1s_3)\langle \pi_A, \pi_B\rangle\Big]$$

and integration over the π_A and π_B leads to the factor

$$(s_1s_2 + s_2s_3 + s_3s_1)^2F^2$$

The s dependence cancels the bosonic momentum, as can be readily checked by computation of $(\det M)^2$ where

$$M = \begin{pmatrix} s_1 + s_2 & -s_2 \\ -s_2 & s_2 + s_3 \end{pmatrix}.$$

Example 3:

For a more involved example consider the diagram in a theory with cubic interactions, drawn in Fig. 4. This corresponds to a diagram with genus 1 and 2 boundaries. Thus it will contribute a term $2N(F)^2 \times S$, times the amplitude of the matrix model, to the superpotential. The factor 2 comes from the fact that we can attach the two $\mathcal{W}$'s comprising the glueball field at either of the two holes, and the factor of N comes from the trace over the hole where there are no glueball fields. This diagram has 6 edges with Schwinger parameters s_I with $I = 1, ..., 6$. The three fermionic *loop* momenta we will denote by $\pi_{A,B,C}$. The two possible choices of the holes for attaching the glueball field both give the same contribution to the fermionic momentum integral, namely

$$(s_5(\pi_C - \pi_B) + s_6\pi_C)^2 = ((s_5 + s_6)\pi_C - s_5\pi_B)^2$$

where by square, we mean the $\epsilon^{\alpha\beta}$ contraction. Similarly the integral over the path ordered integral of $\mathcal{W}$ can be performed as follows: In this case only one of the two boundaries contribute because $s_5\pi_5 - s_6\pi_6 = 0$ (by the absorption of the fermion zero modes of the glueball insertion). The contribution for the larger boundary is given by the argument we outlined before, as we order the boundaries according to $1\ 2^{-1}\ 4\ 5\ 3\ 1^{-1}\ 2\ 3^{-1}\ 6^{-1}\ 4^{-1}$ and using the fact that $s_5\pi_5 = s_6\pi_6$, by

$$\exp\ \langle s_1\pi_1 - s_2\pi_2, s_2\pi_2 - s_3\pi_3 - s_5\pi_5 - s_4\pi_4\rangle.$$

This can also be viewed as the $\exp\langle \pi_{\text{along } A}, \pi_{\text{along } B}\rangle$, where A, B are the two cycles of the torus (see Fig. 4). Substituting fermionic loop momenta

$$\pi_1 = \pi_A,\ \pi_2 = \pi_B - \pi_A,\ \pi_3 = -\pi_B$$

$$\pi_4 = -\pi_B,\ \pi_5 = \pi_C - \pi_B,\ \pi_6 = -\pi_C$$

yields

$$\exp\Big(A(s)\langle\pi_A, \pi_B\rangle + B(s)\langle\pi_A, \pi_C\rangle + C(s)\langle\pi_B, \pi_C\rangle\Big)$$

and

$$A(s) = s_1s_2 + s_1s_3 + s_2s_3 + s_1s_5 + s_2s_5 + s_2s_4 + s_1s_4$$

$$B(s) = -s_1s_5 - s_2s_5$$

$$C(s) = s_2s_5$$

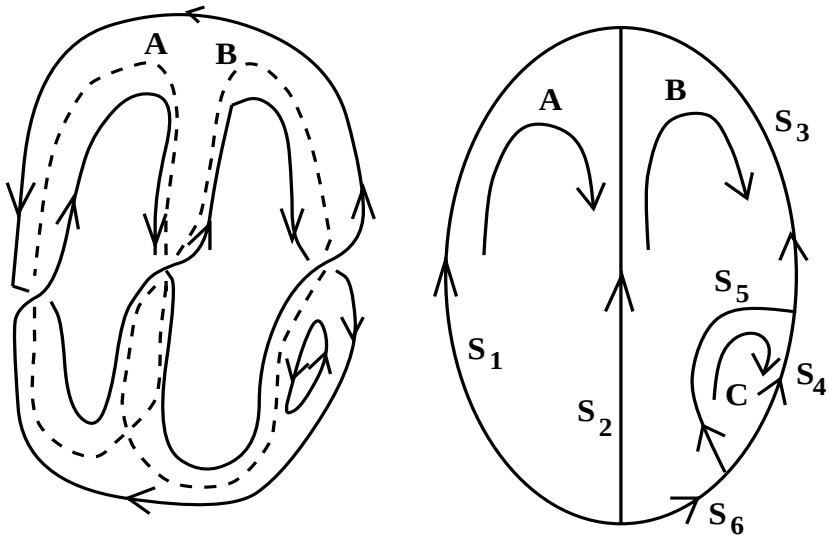

Figure 4: *A Riemann surface with genus 1 and two boundaries made out of four cubic interactions.*

We then need to integrate over the three fermionic loop momenta. To absorb the π_A fermions, we can bring down two A terms from the exponent, which also absorbs the π_B integral and so we will have to pair it up with the $(s_5+s_6)^2$ in the S contribution to absorb the π_C integral. Or we can bring two B terms which will have to be paired up with the s_5^2 term. Or we can bring down one A and one B term which will have to be paired with the cross term in the S contribution of the form $2(s_5+s_6)s_5$ in the $\pi_B\pi_C$ term. Putting all these together we find the s_i dependence is given by D^2, where

$$\begin{aligned} D &= A(s)(s_5+s_6)+B(s)s_5 \\ &= s_1s_2s_5+s_1s_3s_5+s_2s_3s_5+s_1s_4s_5 \\ &\quad +s_2s_4s_5+s_1s_2s_6+s_1s_3s_6+s_2s_3s_6 \\ &\quad +s_1s_4s_6+s_2s_4s_6+s_1s_5s_6+s_2s_5s_6. \end{aligned}$$

The integration over the bosonic momentum gives the inverse square of determinant of M where $M_{AB}=\sum_I s_I L_{IA}L_{IB}s_I$ is given by

$$M=\begin{pmatrix} s_1+s_2 & -s_2 & 0 \\ -s_2 & s_2+s_3+s_4+s_5 & -s_5 \\ 0 & -s_5 & s_5+s_6 \end{pmatrix}$$

and one easily checks that

$$D=\det M$$

as expected.

5 Physical interpretation

We have seen that the connection between matrix model and $\mathcal{N}=1$ supersymmetric gauge theories in four dimensions can be made more canonical, *i.e.* be extended to all higher genus matrix amplitudes if we make the gluino fields in the path-integral not to be purely Grassmannian. This lack of anticommutativity breaks Lorentz invariance, but preserves $\mathcal{N}=1$ supersymmetry. In this section we discuss possible physical implications of this idea.

Even though in this paper we have mainly concentrated on the chiral sector, corresponding to turning on $F_{\alpha\beta}$, we could also repeat this analysis for the anti-chiral sector, by turning on $F_{\dot\alpha\dot\beta}$.[8] In this Euclidean context these are independent real numbers, but in the Minkowski context these are complex quantities and the reality (*i.e.*, unitarity) conditions dictate

$$F_{\dot\alpha\dot\beta} = (F_{\alpha\beta})^*.$$

This in particular means that for the gluino fields ψ_α and $\psi_{\dot\alpha}$ in the path-integral we require,

$$\{\psi_\alpha, \psi_\beta\} = 2F_{\alpha\beta}$$

$$\{\psi_{\dot\alpha}, \psi_{\dot\beta}\} = 2F_{\dot\alpha\dot\beta}$$

In the case where $F=0$, *i.e.* the standard gauge theory context, we know that the glueball superfield

$$S = \frac{1}{32\pi^2}\epsilon_{\alpha\beta}\mathrm{Tr}\ \mathcal{W}^\alpha\mathcal{W}^\beta$$

is the right variable to describe the infrared physics. Similarly here, given the link between the full matrix model computation and gauge theory computation, it suggests that S again is the right field in the IR to capture the relevant physics. We will assume that to continue to be the case even after introducing the C-deformation. In particular in the IR we will have an effective superpotential

$$W(S) = N\frac{\partial F(S,\lambda^2)}{\partial S} + \tau S \tag{5.1}$$

where

$$\exp\left[\frac{1}{\lambda^2}F(S,\lambda^2)\right] = \frac{1}{\mathrm{vol}(U(M))}\int d\Phi\ \exp\left[-\frac{1}{\lambda}W(\Phi)\right] \tag{5.2}$$

[8] In the string theory context this would lead to a gravitational back-reaction, which is irrelevant in the field theory limit we are considering.

and $S = \lambda M$ in the above expression. Moreover

$$\lambda^2 = \epsilon^{\alpha\alpha'}\epsilon^{\beta\beta'}F_{\alpha\beta}F_{\alpha'\beta'},$$

in the gauge theory interpretation. In the corresponding low energy physics we are instructed to minimize the physical potential

$$V = g^{S\overline{S}}|\partial_S W|^2$$

where $g_{S\overline{S}} = \partial_S\partial_{\overline{S}}K(S,\overline{S})$ and K is the (as yet to be computed) potential coming from D-term.

There is a surprise here: The IR physics appears to be Lorentz invariant! Namely both S and $\lambda^2 = F_{\alpha\beta}F^{\alpha\beta}$ are Lorentz invariant, and so the $W(S)$ is Lorentz invariant. So there is no hint in the IR that we are dealing with a theory which intrinsically breaks Lorentz-invariance. In other words, it appears that Lorentz invariance has been restored in the IR! Even though there are examples where the theory in the IR has more symmetries than in the UV, for example theories which have higher dimension operators violating some symmetry which becomes irrelevant in the IR, it is amusing that this is appearing also in our case where the fundamental fields have Lorentz-violating rules for the path-integral. Note that turning on F^2 does change the expectation value of S at the critical point and the critical value of W, but in a Lorentz-invariant way. It is tempting to speculate about the potential realization of this idea in Nature. In particular this would be consistent with the macroscopic existence of Lorentz invariance, which could get violated at higher energies. This is even more tempting since from the viewpoint of the relation of $\mathcal{N} = 1$ supersymmetric gauge theories and matrix model, the C-deformation is forced on us! It would be interesting to explore the signature of the C-deformation for potential observations in the accelerator physics or cosmology.

5.1 Pure $\mathcal{N} = 1$ supersymmetric Yang-Mills revisited

Let us consider the special case of pure $\mathcal{N} = 1$ Yang-Mills, deformed by turning on $F_{\alpha\beta}$. This will be a leading piece of the superpotential of many other theories, in the limit where S is small and so higher powers of S can be ignored in the glueball superpotential. In this case, the partition function of the matrix model 5.2 is entirely given by the measure factor, $\log \mathrm{vol}(U(M))$, which has been shown [5] to give the partition function of $c = 1$ at self-dual radius [23]. We have (up to an addition of an irrelevant constant $\frac{1}{12}\lambda^2\log\lambda$)

$$F(S,\lambda^2) = \frac{1}{2}S^2\log S - \frac{1}{12}\lambda^2\log(S/\lambda) + \sum_{g>1}\frac{B_{2g}}{2g(2g-2)}\cdot\frac{\lambda^{2g}}{S^{2g-2}}$$

This can be written in a more unified form, up to an addition of $\frac{-1}{2}S^2\log\lambda$ which in the expression for the superpotential can be absorbed into redefining the coupling constant τ,

$$\frac{1}{\lambda^2}F(S,\lambda^2)=\frac{1}{2}\frac{S^2}{\lambda^2}\log\left(\frac{S}{\lambda}\right)-\frac{1}{12}\log\left(\frac{S}{\lambda}\right)+\sum_{g>1}\frac{B_{2g}}{2g(2g-2)}\cdot\left(\frac{\lambda}{S}\right)^{2g-2}.$$

It is natural to define a rescaled dimensionless glueball field $\mu = S/\lambda$. In terms of this we have

$$\frac{1}{\lambda^2}F=\frac{1}{2}\mu^2\log\mu-\frac{1}{12}\log\mu+\sum_{g>1}\frac{B_{2g}}{2g(2g-2)}\cdot\mu^{2-2g}$$

This leads to the superpotential

$$\frac{1}{\lambda}W=N\left(\mu\log\mu-\sum_{g>0}\frac{B_{2g}}{2g}\mu^{1-2g}\right)+\tau\mu \tag{5.3}$$

Note that this τ defers from the bare τ_0 in the gauge theory by

$$\tau=\tau_0+N\log\lambda/\Lambda_0^3$$

where Λ_0 is the cutoff where the bare coupling τ_0 is defined. Note that writing the superpotential in term of the new τ, undoes the dimensional transmutation. In other words, we now have gotten rid of λ and recaptured it in term of the coupling constant τ which does not run. Put differently, τ denotes the coupling constant of the gauge theory at the scale set by λ. It is interesting to note that 5.3 is the generating function for the Euler character of moduli space of Riemann surfaces with one puncture [24],[23].

Note that the superpotential 5.3 is a generalization of the Veneziano-Yankielowicz superpotential [25], taking the C-deformation into account. The fact that many different powers of μ enter is because F 'carries' an R charge and with respect to that S/λ is neutral and so in principle arbitrary powers of it can appear. Here we are predicting in addition very definite coefficients for these terms. We expect, as in the case of the Veneziano-Yankielowicz potential, the measure of the gauge theory should somehow dictate this structure, but we do not, at the present, have a direct gauge theory derivation of this.

Let us analyze the critical points of this superpotential. We need to solve $dW = 0$ (again we reabsorb a constant term in the shift of τ):

$$\frac{1}{N\lambda}\frac{dW}{d\mu}=\log\mu+\sum_{g>0}\frac{(2g-1)}{2g}B_{2g}\mu^{-2g}+\frac{\tau}{N}=0$$

(It is amusing to note that $dW/d\mu$ is the partition function of the Euler character of the moduli space of doubly punctured Riemann surfaces, up to addition of τ/N.) If $\tau \ll 0$ *i.e.* if F is much smaller than the physical scale of the original gauge theory, then we have

$$\mu \sim e^{-\tau/N}$$

This in particular is consistent with dropping the terms with negative powers of μ in 5.3 because $\mu \gg 1$ (this is self-consistent, *i.e.* $\langle S \rangle \gg F$). As we increase τ the correction terms to VY potential become more relevant. Let us define $q = e^{\tau/N}$ Then we expect, after minimization, to have an expansion

$$W(\mu)\big|_{\min} = N\lambda \left(q^{-1} + \sum_{n \geq 1} a_n q^{2n-1} \right)$$

for some computable a_n. It would be interesting to see if this function has any interesting modular properties.

5.2 Non-perturbative completion of W and baryons

From the definition of $F(S, \lambda^2)$ 5.2 as the full free energy of the matrix model, it is clear that we cannot stop to all orders in perturbation theory, and in particular we have to have a full definition of the matrix integral, including its non-perturbative completion. This is unlike [8] where we could restrict attention simply to planar diagrams of the matrix model defined by Feynman perturbation theory. Thus it is natural to ask how do we give the full non-perturbative definition of F or the associated physical superpotential 5.1 .

In order to get insight into this question it is useful to trace back, within string theory, what turning on the graviphoton does. On the dual gravity side, the theory is an $\mathcal{N} = 2$ supersymmetric theory broken down to $\mathcal{N} = 1$ by some flux. The topological string amplitudes do not depend on the choice of flux [6] . Thus we can ask how does turning on graviphoton field strength affect the $\mathcal{N} = 2$ theory.

This question was studied at length in [26]. The main idea is to relate the turning on of the graviphoton field strength, as giving rise to a correction to R^2 terms which are captured by Schwinger like computation. Recall that graviphoton couples to D-branes with charge proportional to their BPS mass. Motivated by this correspondence it is natural to write the superpotential W as coming from such a computation. For example for the case of pure

Yang-Mills we would be led to

$$\frac{1}{\lambda}W(\mu) = N\int_{\epsilon}^{\infty}\frac{ds}{s^2}\left(\frac{s/2}{\sinh(s/2)}\right)^2 e^{-s\mu} \tag{5.4}$$

This suggests a non-perturbative completion of the superpotential relevant for the cases with smaller values of μ, including terms of the form $\sim e^{2n\pi i\mu}$, as is familiar from the Schwinger computation. If $\mathrm{Im}\mu = \mathrm{Im}(S/|F|) \gg 1$ these effects are small. Note that for pure Yang-Mills, these corrections, on the matrix model side, would be invisible: Since $\mu = S/F = \lambda_s M/\lambda_s = M$ these terms correspond to $\exp(2\pi i M)$ which is 1 and do not depend on M. Thus the ambiguity of the map between the matrix model and gauge theory data allow for such additions.

Recall that in the string theory realization the wrapped brane corresponds to baryons [27], [28] as the corresponding brane is pierced by N units of RR flux. Even though in the $\mathcal{N} = 1$ theory these are not as part of the excitations of the theory (as one would have to supply the quarks as probes) nevertheless it is striking that they can be used to reproduce the superpotential. Therefore it is natural to interpret the full superpotential W 5.4 as obtainable from the baryon/anti-baryon pair production effect. It would be interesting to better understand this statement from the field theory side. It is amusing to note that this includes the Veneziano-Yankielowicz part of the superpotential as well, suggesting a new interpretation for it. For more general $\mathcal{N} = 1$ theories one would expect that there would be similar completions of the perturbative computation, similar to that studied in [4] in the context of A-model topological strings.

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