Boundary actions in

Ponzano-Regge discretization,

Quantum Groups and AdS_3

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Abstract

Boundary actions for three-dimensional quantum gravity in the discretized formalism of Ponzano-Regge are studied with a view towards understanding the boundary degrees of freedom. These degrees of freedom postulated in the holography hypothesis are supposed to be characteristic of quantum gravity theories. In particular it is expected that some of these degrees of freedom reside on black hole horizons. This paper is a study of these ideas in the context of a theory of quantum gravity that requires no additional structure such as supersymmetry or special gravitational backgrounds. Lorentzian as well as Euclidean regimes are examined. Some surprising relationships to Liouville theory and string theory in AdS_3 are found.

1   Introduction

This article presents the calculation, using continuum and lattice methods, of boundary terms in 3-dimensional gravity. The gravity theory is presented in first order Palatini form, this being a particular example of the general class of BF models [1] as this is the most convenient presentation for deriving the discretization. We find a variety of boundary conditions, and discuss the significance of these for different types of boundaries in space-time.

The bulk theory of three-dimensional gravity is well known to be a topological field theory, however it is also well known that three-dimensional topological field theories can give rise to non- topological boundary degrees of freedom, the classic example being the CS theory giving rise to a WZW model on the boundary [2]. In the case of three dimensional gravity with cosmological constant, one can utilize a trick that relates the action to the difference of two CS actions, and then use the standard CS-WZW relationship, however the actual boundary conditions are a little more subtle. In three dimensions this is relevant to the AdS_3 space, or more generally to BTZ black hole solutions.

In this paper we wish to understand in the context of discretization of quantum gravity the boundary degrees of freedom that correspond to black hole entropy. This paper is directed towards a longer study of boundary terms in gravity theories, ultimately in $3 + 1$ dimensions, with the hope of understanding directly in a theory of quantum gravity, the possibile origin of holographic phenomena, and of the microscopic details of black hole entropy, in particular well out of the supersymmetric and extremal limits which have been very well studied in the framework of string theory.

The actual type of discretization that we consider here is maybe at first sight a bit unusual. The approach is originally due to Ponzano and Regge [3] where they considered a simplicial decomposition of a three-manifold and the path-integral is then defined as a summation over the possible sets of lengths of the edges of the dual lattice. The alternative of course is to fix the size of the simplices and to form the path integral by summation over possible simplicial decompositions. For the major part of this paper, we will be discussing three dimensional models that have a topological invariance in the bulk and thus the fixed decomposition is somewhat innocuous but again the use of this simplicial decomposition also for the boundary where in general we believe there are physical degrees of freedom needs to be considered more cautiously. In addition we eventually need to extend our results to the realistic case of four-dimensional gravity where we do not even have topological invariance in the bulk making things more intricate though hopefully

still manageable.

We begin however, in the context of euclidean three-dimensional gravity where already we find some interesting results concerning the boundary theories. We will start off with a discussion of a discretization of the BF theory that corresponds to three-dimensional euclidean gravity in the framework of the Ponzano-Regge discretization, that is a discretization into tetrahedra with edges labelled by SO(3) spins, and each tetrahedron then weighted in the path integral (sum) by the corresponding $6j$ symbol. From this discretization we can then derive a boundary action and will compare this to what we may expect from the corresponding BF theory. We in fact find that there are two simple types of boundary conditions, one leads to a topological boundary theory and the other to a dynamical boundary theory. In addition we discuss mixed boundary conditions which are relevant for the boundary at infinity in AdS_3 for example. We discuss modifications to these boundary actions that arise when one replaces the group $SO(3)$ with $SO(2,1)$ which would correspond to gravity with lorentzian signature. We also discuss the regularization via quantum groups and find some interesting relationships to work on string theory and AdS_3/CFT duality.

Finally we make some suggestions for understanding black hole entropy in this context and we discuss briefly the extension of these methods to four-dimensional quantum gravity.

2   Ponzano Regge from BF-theory

We will now turn to a discretization of the BF representation of three-dimensional gravity and show how it leads to the Ponzano-Regge action. The BF action is a generic action for a certain class of topological field theories, [1]. For three-dimensional gravity it actually corresponds to the Palatini first order action. We will mostly use the BF variables which are related to the gravity variables via the dictionary; $B = e$ is the dreibein and $F = R = d\omega + \omega \wedge \omega = dA + A^2$ is the curvature of the spin-connection $\omega = A$.

The basic action for three-dimensional gravity in this first order formulation is then

$$S_{grav} = \int tr(e \wedge R), \tag{1}$$

where R is the curvature two form of a potential one-form ω, and e is the dreibein. These fields transform under the action of an $SO(3)$ gauge group.

The invariance of the action consists of a gauge transformation in ω, $\omega \to h^{-1}\omega h + h^{-1}dh$, coupled with a local gauge rotation $e \to h^{-1}eh$, and an additional invariance only acting on the dreibein (reparametrization) under which $\delta e = d\chi + [\omega, \chi]$. The parameters of these transformations are in $SO(3)$ and it's Lie algebra respectively. The first transformation expresses the local lorentz invariance, and the second the diffeomorphism invariance. The theory as formulated is diffeomorphism invariant with no explicit appearance of the metric in the action and thus topological. The constraint that the metric is torsion free, $de + \omega \wedge e = 0$, in this first order form, arises from the ω equation of motion. The total group of local symmetry is $ISO(3)$ [4].

We will proceed now to a discrete formulation of three-dimensional gravity. We will carry out the discretization as a means of studying the continuum theory, however we would like to point out that in [5] some arguments are given indicating that in three-dimensional gravity the space-time is necessarily discrete. Our study in fact also indicates another possibile method to prove that three-dimensional gravity is discrete.

One of the original motivations leading us to consider a discrete space-time approach to quantum gravity is the following. We will throughout this paper take the view that black holes in quantum gravity behave like quantum mechanical objects, and that this leads to unitarity in quantum gravity via some type of holographic mechanism [6]. If one considers the black hole horizon to be a quantum object capable of storing and retransmitting information, then one would imagine that this horizon follows a null or even time-like path in space-time and that the region inside the global horizon is not something that an outside observer can ever see or discuss. This is the view of black-hole complementarity developed to reconcile the apparent contradiction that unitary black hole evaporation implies that observers outside the black hole view the physics of the horizon in a very different way to freely falling observers who fall into the horizon of a large black hole [7]. As such one may view the formation of a black - hole as the expansion of a planckian bubble in space-time to become macroscopic. Inside such a bubble there is nothing. Thus it seems necessary to think of the microscopic structure of space-time to be a collection of bubbles. As such there is a discretization of space-time into units of size the Planck length.

There are a variety of ways to approach the discretization of the BF theory in three-dimensions, although all constructions give the same final result. For other discussions of the approach that we present here see [8, 9]. The simplest approach to discretization is to formally carry out the path integral over the B-field, as it is simply a Lagrange mutiplier for the Einstein
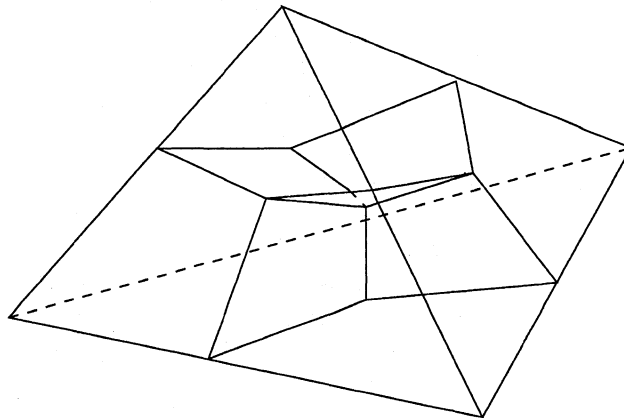
Figure 1: A tetrahedron and the the part of the dual lattice that it intersects.

equation. The result is,

$$Z[M] = \int \mathcal{D}A \prod_x \delta(e^{F(x)}) \tag{2}$$

where x are the co-ordinates on the closed manifold M, and the delta function is in the group manifold of $SO(3)$. The delta function can be rewritten using the identity

$$\delta(gh^{-1}) = \sum_R \chi_R(g)\chi_R(h^{-1}) \tag{3}$$

where $g, h \in G$ and the sum is over all representations of the group G. Using this identity we can write,

$$Z[M] = \int \mathcal{D}A \prod_x \sum_j (2j+1)\chi_j(e^{F(x)}). \tag{4}$$

To make this expression tractable we now discretize the manifold M by dividing it into tetrahedra. From this tetrahedral decomposition, we construct a dual discretization for which the vertices are at the centre of the tetrahedra [10], the edges pass between the centres of adjacent tetrahedra, and the faces are then bound by these edges and each dual face will be pierced by precisely one edge of the original tetrahedral decomposition. In Figure 1 we show the part of the dual lattice that will live inside one of the original tetrahedra.

We now assign to every face of the dual lattice (that is every edge of the original lattice) a representation and to every edge of the dual lattice a group element as shown in Figure 2. The product of the group elements around a dual face is then the holonomy of that cycle and thus represents
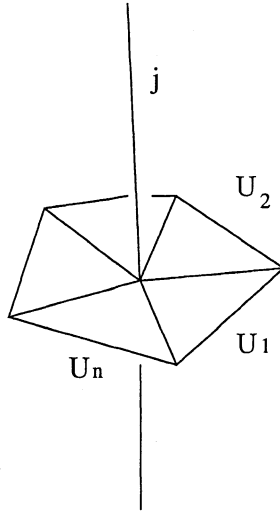
Figure 2: An edge of the lattice showing the corresponding dual face

a discretization of the curvature. Denoting the discretization of M as Δ we can finally write,

$$Z[M, \Delta] = \mathcal{N} \int \prod_{e \in \Delta} \sum_{j_e} dU_e (2j_e + 1) \chi_{j_e} (\prod_{\partial \tilde{e}} U) \qquad (5)$$

where $\mathcal{N}$ is a normalisation factor. In this expression, e is an edge of the tetrahedral decomposition Δ and $\tilde{e}$ is the face dual to the edge e.

To actually evaluate this expression we notice that the character can be written as a sum of products of the Wigner function $D^j_{mm'}(U)$ where U is the group element corresponding to an edge of the dual graph,

$$\chi_j(\prod_{i=1}^n U_i) = \sum_{m_i} \prod D^j_{m_i m_{i+1}}(U_i), \qquad (6)$$

where $m_{n+1} = m_n$. Then for each edge of the dual graph there will appear in the integral over the corresponding group elements three Wigner functions which can be evaluated immediately using,

$$\int dU D^{j_1}_{ll'}(U) D^{j_2}_{mm'}(U) D^{j_3}_{nn'}(U^{-1}) = \begin{pmatrix} j_1 & j_2 & j_3 \\ l & m & n \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ l' & m' & n' \end{pmatrix} \qquad (7)$$

(For these and other angular momentum identities that we use below and for the definitions of the various symbols that we use, we recommend that the reader refer to the very complete monograph [11]).

Each tetrahedron thus will contribute two $3jm$ symbols for every face, thus eight $3jm$ symbols for every tetrahedron. Half of these are summed over angular momentum projections in pairs, one of the pair coming from each of two tetrahedron with a common face and the orthogonality of the $3jm$ symbols ensures that this term becomes the identity. The remaining expression is such that summing over the projection quantum number of the angular momentum the four $3jm$ of a given tetrahedron gives a single $6j$ symbol using the identity,

$$
\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\} = \sum (-1)^{\sum j_i - \sum m_i} \left(\begin{matrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{matrix} \right) \times
$$
$$
\left(\begin{matrix} j_1 & j_5 & j_6 \\ -m_1 & m_5 & m_6 \end{matrix} \right) \left(\begin{matrix} j_5 & j_3 & j_4 \\ -m_5 & m_3 & m_4 \end{matrix} \right) \left(\begin{matrix} j_4 & j_2 & j_6 \\ -m_4 & -m_2 & -m_6 \end{matrix} \right) \quad (8)
$$

The final result for a closed manifold $\mathcal{M}$ and simplicial decomposition Δ is (see [8] or [9] for more details),

$$
Z(\mathcal{M}, \Delta) = \mathcal{N} \sum_{j_e} \prod_{e \in \Delta} (2j_e + 1) \prod_{t \epsilon \Delta} (-)^{\sum_{i=1}^{6} j_t^i} \left\{ \begin{matrix} j_t^1 & j_t^2 & j_t^3 \\ j_t^4 & j_t^5 & j_t^6 \end{matrix} \right\} \quad (9)
$$

This answer is the path sum proposed by Ponzano and Regge to be a discretization of three dimensional quantum gravity. In fact for a tetrahedron with edge lengths l_i, the corresponding weight is the 6j symbol with angular momenta $j_i = l_i + \frac{1}{2}$. In the semi-classical limit [3] (large angular momenta and vanishing Planck length such that the combination $l_i \ell_P$ remains constant), a 6j symbol actually becomes the cosine of the Regge action for the tetrahedron as a direct discretization of three dimensional gravity (the cosine arises as the BF theory path integral sums indiscriminately over positive, negative and degenerate values for B). The Regge action is the direct discretization of the Einstein-Hilbert action [12].

$$
S_{Regge} = \sum_{h,k=1}^{4} (j_{hk} + \frac{1}{2}) \theta_{hk} \quad (10)
$$

$$
\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\} \simeq \frac{1}{\sqrt{12\pi V}} \cos\left(S_{Regge} + \frac{\pi}{4} \right) \quad (11)
$$

θ_{hk} is the angle between the normals to adjoining faces and j_{hk} is the length of the edge common to the two faces labeled by h and k. The extra factor of half between the edge length and corresponding angular momentum is for consistency in this semi-classical limit and we can intuitively justify it by noting that the length of the angular momentum vector for the representation of spin j is actually $\sqrt{j(j+1)}$ which becomes $j + \frac{1}{2}$ in the limit of large angular momentum.

2.1   Symmetry and normalization

The above expression for the discretized path sum is not quite complete. We have ignored the fact that there could in principle be some normalization factor in front of the sum, and in fact one would hope that there is such a factor simply because the sum itself is divergent. One can simply choose a normalization factor to subtract the divergence, however it is interesting to see how the divergence arises. This was already analyzed in the original paper of Ponzano and Regge, and the reader should look there for the details. In short, one takes the Biedenharn-Elliot (BE) identity (Appendix A.), which relates a product of three $6j$ symbols summed over one angular momentum, to a product of two $6j$ symbols without summation. Geometrically this corresponds to taking three tetrahedra joined together along a common edge, and each with a face in common with two of the others. Removing the common edge (the sum in the BE identity) leaves one with two tetrahedra sharing one common face. Using the orthogonality of the $6j$ symbols, one can change this identity to one that relates a single tetrahedron to four tetrahedra formed by introducing an additional vertex at the centre of the original tetrahedron. The identity is in Appendix A for the interested reader. The important point is that there is an infinite factor

$$\Lambda(R) = \lim_{R \to \infty} \sum_{j=0}^{R} (2j + 1)^2, \tag{12}$$

for every vertex of the simplicial decomposition.

Therefore we see that an infinite factor of this form must be added to the denominator of the path sum to regularize it, and that there is one such factor for every vertex in the triangulation. The actual normalization factor is then

$$\mathcal{N} = \Lambda(R)^{-N_v}$$

where N_v is the number of vertices in the discretization.

In addition this discussion has shown us that the path sum is actually invariant under the two transformations derived from the BE identity. These two transformations are known as Pachner moves [13] and these are the discretized version of diffeomorphisms. We have therefore learnt that the path sum thus defined (in particular with the regularization discussed) is diffeomorphism invariant in discretized form just as the BF theory was before the discretization.

2.2 Regularization

The full path sum is then,

$$Z(\mathcal{M}, \Delta) = \tag{13}$$

$$\lim_{R \to \infty} \Lambda(R)^{-N_0} \sum_{j_e} \prod_{e \in \Delta} (2j_e + 1) \prod_{t \in \Delta} (-)^{\sum_{i=1}^6 j_t^i} \left\{ \begin{matrix} j_t^1 & j_t^2 & j_t^3 \\ j_t^4 & j_t^5 & j_t^6 \end{matrix} \right\}$$

where N_0 is the number of vertices in Δ. In this form however it is still not very practical for calculating. There exists a different regularization that involves a q-deformation of $so(3)$ due to Turaev and Viro [14].

The path sum is

$$Z_{TV}(\mathcal{M}, \Delta) = \tag{14}$$

$$\Lambda_q^{-N_0} \sum_{j_e=0}^{\frac{k-1}{2}} \prod_{e \in \Delta} [2j_e + 1]_q \prod_{t \in \Delta} (-)^{\sum_{i=1}^6 j_t^i} \left[\begin{matrix} j_t^1 & j_t^2 & j_t^3 \\ j_t^4 & j_t^5 & j_t^6 \end{matrix} \right]_q$$

where

$$\Lambda_q = -\frac{2k}{(q - q^{-1})^2} \tag{15}$$

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{16}$$

The parameter of the quantum deformation is a root of unity $q = e^{\pi i/k}$ and the sum is regularized as the representations of $U_q(so(3))$ involve angular momenta only in the range $0 \dots (k-1)/2$ so the path sum now involves all finite sums and $\Lambda(R)$ has been replaced by Λ_q which is clearly finite.

The semi-classical limit of the $q-6j$ symbol indicates that the q-deformed path sum is related to quantum gravity in three dimensions with a positive cosmological constant. The limit must be carried out in a way that as the angular momentum become large, correspondingly also k must go to infinity. The limit is [15],

$$\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}_q \simeq \frac{1}{\sqrt{12\pi V}} \cos \left(S_{Regge} - \frac{4\pi^2}{k^2} V + \frac{\pi}{4} \right) \tag{17}$$

and we see in particular that the limit which makes contact with the semi-classical physics is the limit in which the cosmological constant goes to zero. Note that this cannot be derived directly from the action

$$S_\Lambda = \int tr(BF + \Lambda B^3) \tag{18}$$

by any simple generalization of the discretization carried out above, as the non-linearity in B does not allow us to easily integrate over B to get a simple expression involving the curvature F. It would be very interesting to find a derivation of the TV path sum from the discretization of the path integral for S_Λ.

For simple manifolds this sum can actually be evaluated giving the Turaev-Viro invariants that are important for the understanding of the topology of three-manifolds. The restriction on angular momentum in the quantum group representations is the same as that which must be imposed on string states in AdS_3. We will take another look at the q-deformed action and limits thereof after we have discussed the boundary discretization and will find that in the context of gravity in AdS_3 there may indeed be a deeper meaning to this regularization.

It is also interesting to consider the relationship between this construction of three-dimensional gravity using a quantum deformation and studies of quantum doubles of groups [16]. In this article one has a different type of quantum group that does not have a fixed deformation parameter. It is used for the discussion of multi-particle states in three-dimensional gravity. Each particle, creates a localized source of curvature, and in general the space is conical at infinity. It is amusing to notice that for the Chern-Simons description of three-dimensional gravity, at zero cosmological constant one uses the group $ISO(3)$ [4], but at non-zero cosmological constant, one finds instead the group $SU(2) \times SU(2)$ with the level of the Chern-Simons theory related to the curvature. Going to the multi-particle Fock space in three-dimensions means that we are allowing variable localized curvature depending upon the location and mass of the particle sources. We find that the group $ISO(3)$ is replaced by $\mathcal{D}(SU(2))$ [16] but now with no additional parameter, indicating perhaps that all values of curvature are possibile depending on the number and mass of particles present. This relationship deserves to be studied in more detail as it indicates a possible second quantization that involves also the cosmological constant.

3   Boundaries

Let us consider the general variation of the BF action for a manifold with boundary, (other work on this subject can be found in the papers [17, 18, 19]).

$$\delta S_{BF} = \int_M tr(\delta BF + \delta A(dB + AB + BA)) - \int_{\partial M} tr(B\delta A) \qquad (19)$$

We see then that the field equations are not effected by the presence of the boundary provided that the variation of A is zero on the boundary. The path integral in the presence of the boundary will then be a function of the boundary value of the spin connection.

We have another choice, which corresponds to the BF theory with a boundary term

$$S = S_{BF} + \int_{\partial M} tr(BA) \tag{20}$$

The variation of S is now

$$\delta S = \int_M \text{"equations of motion"} + \int_{\partial M} tr(\delta BA) \tag{21}$$

and therefore the boundary condition must be that the variation of B is zero on the boundary, and the path integral will now be a function of the boundary metric.

The first boundary condition of fixed spin connection on the boundary actually gives rise to a topological field theory on the manifold plus boundary. The second boundary condition is Dirichlet on the metric, and this does not give rise to a topologically invariant boundary action.

In a study of asymptotic symmetries in three-dimensional gravity [20], it was shown that with appropriate boundary conditions one can also find a Liouville theory on the boundary at infinity of AdS_3 space. Such boundary conditions formulated in terms of the metric and connection are actually mixed boundary conditions, and we will give more details of how these work below.

The continuum boundary action can be easily derived by following a construction similar to that used in [2] where the WZW-CS relationship was discussed in some detail. First we consider the boundary condition $\delta\omega = 0$ for which there is no additional boundary term. To examine the boundary theory we will insert the solutions to the bulk equations and then examine the action of the gauge symmetries of the theory in the presence of the boundary. For the WZW-CS relationship the boundary degrees of freedom arise precisely because the bulk gauge symmetry is only a global symmetry on the boundary thus the breaking of the gauge symmetry by the presence of the boundary gives rise to new degrees of freedom.

The bulk equations of motion are solved by

$$A = -dUU^{-1}, \tag{22}$$

$$B = UdVU^{-1}. \tag{23}$$

Substituting these solutions into the action we find of course that it vanishes identically as $R = 0$. The gauge variation of the action is identically zero for the local lorentz invariance, but the diffeomorphism transformation has in principle an additional boundary term equal to

$$\delta_{diff} I = \int_{\partial M} tr(\chi R) \tag{24}$$

which we see also vanishes as $R = 0$ from the equations of motion (modulo some topological issues regarding the extension of a flat connection to a boundary of given topology). This boundary action is that of an obviously topological two-dimensional field theory in agreement with the proof by Ooguri and Sasakura [18] that with the $\delta\omega = 0$ boundary condition the path sum is a topological invariant not just of the bulk but of the bulk plus boundary theory.

For the $\delta e = 0$ boundary condition we must add to this result the boundary term $\int tr(e\omega)$. The boundary condition now seems to indicate that the boundary metric is important in the path sum, in fact the path sum will now be a function of the boundary triangulation. Again the solutions to the bulk equations will be inserted into the action, the bulk again giving zero contribution but the boundary now gives a non zero contribution equal to

$$S = -\int_{\partial M} tr(dVU^{-1}dU). \tag{25}$$

Furthermore the gauge transformations now give rise to non-trivial boundary terms,

$$\delta_{gauge} S = \int_{\partial M} tr(\Lambda dB) \tag{26}$$

$$\delta_{diff} S = -2\int_{\partial M} tr(\chi A^2) \tag{27}$$

We can see from these variations that the symmetry of the boundary theory is significantly smaller than that of the bulk theory. In fact we must have Λ constant for the gauge transformation to vanish and also $\chi = 0$. Therefore the boundary theory has no diffeomporphism invariance, and is invariant only under global lorentz transformations.

Finally we can consider the boundary conditions used in [20] which are related to three dimensional gravity with cosmological constant. To do this

we make a small deviation into the Chern-Simons representation of three-dimensional gravity with cosmological constant. Our action is then,

$$S_{BF} = \int_{\mathcal{M}} tr(BF + \Lambda B^3) \tag{28}$$

We make the change of variables,

$$A^{\pm} = \frac{1}{2}(B\sqrt{-3\Lambda} \pm A) \tag{29}$$

and we then find that S_{BF} becomes the difference of two Chern-Simons theories plus an additional boundary term.

$$S_{BF} = \frac{1}{\sqrt{-3\Lambda}} \int_{\mathcal{M}} (CS[A^+] - CS[A^-]) + \frac{1}{\sqrt{-3\Lambda}} \int_{\partial \mathcal{M}} tr(A^+ A^-) \tag{30}$$

We see here that the level of the Chern-Simons theory is inversely proportional to the square root of the cosmological constant, and also that if we started with a BF action with no boundary term, then after the change of variables we have a boundary term that is of a mixed form, rather than of the form $tr(AB)$. This is due to the fact that using the variables $A^{\pm}$ we can consider boundary conditions that would be mixed boundary conditions when expressed in terms of the variables A and B. Indeed, if we add $\frac{1}{2}\int tr(AB)$ to the BF action then following the construction of [2] one finds that in the Chern-Simons variables the action factorizes into two pieces that represent a pair of chiral WZW theories.

The boundary conditions now imply restrictions on a combination of the metric and connection. It is precisely this setup that was shown to arise for the boundary at infinity of AdS_3 in the work of Brown and Henneaux and afterwards Coussaert, Henneaux and van Driel [20]. The boundary theory is actually a Liouville theory. Note that to discuss this case in the discretized framework we really need to use the quantum group representations as it is only then that ones sees a cosmological constant in the semi-classical limit. The discretized boundary theory will turn out to be very similar to a discretization of Liouville theory. We will show how this relationship arises in more detail once we have set up the formalism for the quantum discrete boundaries. One may already worry here that we are trying to construct some triangulation of Liouville theory in the strongly coupled phase and it is well known that for Euclidean surfaces such theories have very non-continuum like phases. A discussion of these problems and arguments for better behaviour in the lorentzian case are in [21].
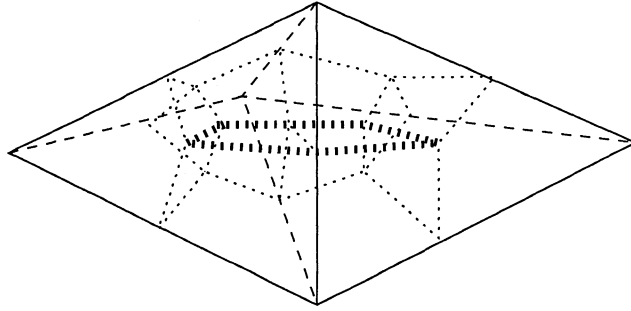
Figure 3: A pair of boundary triangles and their tetrahedra. Dotted lines are the dual lattice. Bold dotted lines highlight a dual face cut by the boundary.

3.1   Quantum discrete boundaries

From the bulk calculation of the discretized path sum, we saw that for every face of the simplicial decomposition, there are two $3jm$ symbols. Indeed if we consider a single tetrahedron as a discretization of the three-dimensional ball then it has a weight,

$$(-1)^{\sum_{i=1}^{6} j_i + (i_2+i_3+k_1+k_3+l_1+l_3)} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ i_1 & -i_2 & -i_3 \end{pmatrix} \times \quad (31)$$

$$\begin{pmatrix} j_1 & j_5 & j_6 \\ -k_3 & -k_1 & k_2 \end{pmatrix} \begin{pmatrix} j_4 & j_2 & j_6 \\ -l_1 & l_2 & -l_3 \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_3 \\ m_2 & m_3 & m_1 \end{pmatrix}$$

Joining now an additional tetrahedron to one of the faces of this tetrahedron, we get another decomposition of the three-ball. On the internal face there is now a $3jm$ symbol coming from each of the tetrahedra, but we must now sum over the angular momentum projections assigned to the internal faces (now identified of course). Using the orthogonality identity for a pair of $3jm$ symbols

$$\sum_{m_i} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 1 \qquad (32)$$

we see that the internal $3jm$ symbols vanish and we are left in the path sum with a $6j$ symbol for every bulk tetrahedron, and a $3jm$ symbol for every boundary face. In general the integral that gave rise to this pair of $3jm$ symbols was along the link of the dual lattice that passes from the centre of one tetrahedra to the centre of an adjacent one piercing one and only one face. In the presence of a boundary only half of this integral is carried out, from the centre of the tetrahedron to the face and this integral gives rise to a $3jm$ symbol for the bulk $6j$ symbol and additional single $3jm$ symbol

for the boundary face. The new feature that has given rise to the boundary weights for the boundary faces is that now we do not have an entire dual face, but rather the dual face is cut in half by the presence of the boundary as shown in figure 3.

We therefore need to also consider the group elements that live on the edge of the dual face that is exposed by the boundary. For the bulk path-sum described in the previous section the connection was integrated away. Now due to the exposed dual faces, we have a boundary dependence on the connection that we may or may not integrate over depending upon the boundary conditions chosen. In figure 4 we have labelled one such edge from X to Y with its weight $D^j_{mn}(U)$. For the boundary conditions that correspond to the action with no boundary term, that is the $\delta A = 0$ conditions, we are instructed to keep the connection fixed on the boundary, and thus we must not integrate over the boundary values of U. We thus find a network with trivalent vertices, each vertex is weighted by a $3jm$ symbol, and the vertices are tied together by the matrix elements of the corresponding group elements. The one and two tetrahedra path sums above easily generalize by gluing faces of tetrahedra together and using the orthogonality condition giving one the general expression for a simplicial decomposition with boundary.

$$
Z(\mathcal{M}, \partial\mathcal{M}, \Delta, \partial\Delta) =
$$

$$
\mathcal{N} \sum_{\{j_e\}} \prod_{e \in \Delta} (2j_e + 1) \prod_{t \in \Delta} (-)^{\sum_{i=1}^{6} j_t^i} \left\{ \begin{array}{ccc} j_t^1 & j_t^2 & j_t^3 \\ j_t^4 & j_t^5 & j_t^6 \end{array} \right\} \times \qquad (33)
$$

$$
\sum_{\{m_f^i\}} \prod_{f \in \partial\Delta} (-)^{\frac{1}{2}\sum m_f^i} \left(\begin{array}{ccc} j_f^1 & j_f^2 & j_f^3 \\ m_f^1 & m_f^2 & -m_f^3 \end{array} \right) \prod_{e \in \partial\tilde{\Delta}} D^{j_e}_{m_e,m_e'}(U_e)
$$

In this expression the normalization factor is the usual one mentioned above and $\tilde{\Delta}$ is the dual lattice. The summation is over the angular momenta assigned to edges in the bulk and the boundary, and over the angular momentum projections assigned to each triangular face of the boundary. For the situation where the group representations summed over are those of the quantum group this is precisely the bulk plus boundary action derived by Ooguri and Sasakura [18], where they show that the Hilbert space of the TV theory is equivalent to that of a pair of Chern-Simons theories for which the boundary state is described by Wilson lines joined by trivalent vertices with an identical structure to that derived above. We would also like to note that this path sum (for all boundary group elements U equal to the identity element) is the same as that derived in [23]. In contrast to our
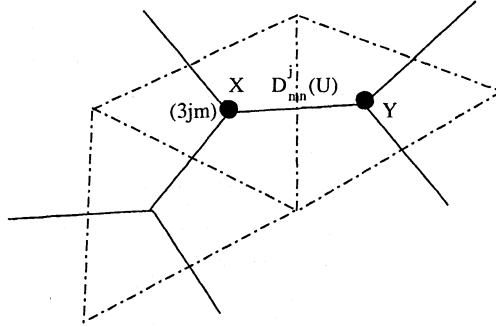
Figure 4: Boundary discretization arising from the boundary of bulk tetra-
hedra and the dual lattice (trivalent graph).

present approach, in that paper the boundary action was derived purely on
the grounds of topological invariance.

The boundary term $\int_{\partial M} tr(BA)$ required for the $\delta B = 0$ boundary con-
ditions when discretized becomes,

$$exp(\int_{\partial M} tr(\bar{B}A)) = \chi_j(U) = \sum_m D^j_{mm}(U) \qquad (34)$$

where $\bar{B}$ refers to the boundary value of B. In this expression the dreibein
$\bar{B}$ is replaced by its discretized representation that being the length of the
corresponding edge of the boundary of the original lattice, and the connec-
tion is represented by U which is the gauge field assigned to the link of the
boundary of the dual lattice that is dual to the edge where $\bar{B}$ resides. The
partition function is in this case a function only of $\bar{B}$. We must multiply
the path sum derived above for the $\delta A = 0$ boundary conditions by this
additional term, remove the sum over the boundary values of the spins j_f,
and integrate over U to derive the final path sum for Dirichlet boundary
conditions in the metric. The integral of importance is that over U and is

$$\int dU D^j_{mm'}(U)D^k_{nn}(U) = \frac{1}{2k+1}\delta_{jk}\delta_{mn}\delta_{m'n} \qquad (35)$$

Inserting this into the path sum gives the final result for fixed metric bound-
ary conditions,

$$Z(\mathcal{M}, \partial\mathcal{M}, \Delta, \partial\Delta) =$$

$$\mathcal{N} \sum_{\{j_e \in \Delta'\}} \prod_{e \in \Delta'} (2j_e + 1) \prod_{t \in \Delta} (-)^{\sum_{i=1}^6 j^i_t} \left\{ \begin{matrix} j^1_t & j^2_t & j^3_t \\ j^4_t & j^5_t & j^6_t \end{matrix} \right\} \times$$

$$\sum_{\{m^i_e\}} \prod_{f \in \partial\Delta} (-)^{\frac{1}{2}\sum m^i_e} \left(\begin{matrix} j^1_e & j^2_e & j^3_e \\ m^1_e & m^2_e & -m^3_e \end{matrix} \right)$$

where Δ' signifies the lattice without boundary components. The sum is now only over the angular momentum in the interior edges. The integral over U on the boundary has now fixed the angular momentum projections to be associated to edges of the boundary triangulations, rather than with faces as for $\delta A = 0$. Thus we see that the action is quite similar to that for the $\delta A = 0$ boundary conditions except that now the boundary values of the angular momenta are fixed corresponding to the fixed boundary metric. Note that in this path sum the factors of $(2j + 1)$ are absent for edges that lie in the boundary due to the restriction in the product over edges to $\Delta' = \Delta - \partial\Delta$. These factors are important in the angular momentum identities that one uses to prove topological invariance, thus indicating that for this choice of boundary conditions there is no topological invariance on the boundary agreeing with our continuum analysis.

In the path integral for a fixed boundary metric, one would expect that in the quantum gravity there would be a need to sum over all possible boundary configurations that give a discretization of the continuum boundary metric. A construction of such a type will be seen to be necessary for a calculation of black hole entropy in this discretized setup. In general before fixing boundary conditions we have the expression for $\delta A = 0$ without summation over angular momenta and without the integral over the boundary gauge connection. We need to understand what boundary conditions will allow calculations relevant to black hole physics, and also what representations of the (quantum) group one must include in this summation. The representations and boundary conditions will be discussed in the final section when we consider the construction for lorentzian metrics. Furthermore, we need to know how to implement the boundary conditions that give Liouville theory in our path sum construction.

3.2 Two-dimensional discrete path sums

We want to show a point of contact between our calculations and discrete TFT's in two dimensions. For $\delta A = 0$ everything is topological and there is an easy way to get a two dimensional TFT from this theory. In R^3 take a thickened wall and remove the bulk tetrahedra using the various Pachner moves in the bulk and on the boundary. The final result will be just two dimensional, but in some sense a double layer as the two faces will both carry their own $3jm$ symbols. The two dimensional action that one finds by

this procedure is,

$$Z(\Sigma, U) = \sum_{\{j_e\}} \prod_{e \in \Sigma} (2j_e + 1) \sum_{\{m_f^i, m_{f'}^i\}} \prod_{f \in \Sigma} (-)^{\frac{1}{2} \sum (m_f^i + m_{f'}^i)} \times$$

$$\begin{pmatrix} j_f^1 & j_f^2 & j_f^3 \\ m_f^1 & m_f^2 & -m_f^3 \end{pmatrix} \begin{pmatrix} j_f^1 & j_f^2 & j_f^3 \\ m_{f'}^1 & m_{f'}^2 & -m_{f'}^3 \end{pmatrix} \times \qquad (36)$$

$$\prod_{e \in \tilde{\Sigma}} D_{m_e, n_e}^{j_e}(U_e) D_{m_{e'}, n_{e'}}^{j_e}(U_e)$$

This is indeed a two-dimensional TFT, invariant under two-dimensional pachner moves and similar actions have been studied in a collection of works [22, 23].

For $\delta B = 0$ we cannot actually remove all the bulk tetrahedra, as the removal process that one uses for the totally topological situation of $\delta A = 0$ relies heavily on the topological invariance of the boundary theory and in particular on the elementary shelling operations. We can however take a limit that is inspired by the bulk boundary correspondence of the AdS/CFT conjecture [24]. To do this we imagine that we take a semi-classical limit of the bulk action leaving the boundary angular momentum fixed. The relevant limit of the bulk $6j$ symbols that have a face edge or vertex on the boundary were already studied in the original article of Ponzano and Regge. The interesting thing that we find is that the boundary answer depends crucially on the asymptotic properties of the manifold. This sort of behaviour is maybe not a surprise as it is precisely such a dependence in the AdS/CFT correspondence that accounts for the simplicity of the near horizon limit in the AdS case. For asymptotically flat spaces however the action is not expected to be similar to the CFT as it will live on a null surface rather than on a time-like surface and the asymptotic group of symmetries will be smaller.

The limits of $6j$ symbols in which only some of the angular momentum are taken to be large are of two basic types. The first involves removing one vertex to infinity, and thus the three edges connected to that vertex become large, while the three vertices that form the remaining face stay fixed, this face then is a triangle of the boundary configuration. The result is thus the $3jm$ symbol of the remaining face, where the pairwise differences between the large angular momenta make the m quantum numbers in this $3jm$ symbol and we thus find an answer similar to that which we derived

from the BF theory, a pair of $3jm$ symbols on the boundary. The answer is,

$$\lim_{R \to \infty} \left\{ \begin{array}{ccc} j^1 & j^2 & j^3 \\ j^4 + R & j^5 + R & j^6 + R \end{array} \right\}$$

$$\simeq (-1)^{\sum_{i=1}^3 j^i + 2 \sum_{i=4}^6 j^i} (2R)^{-\frac{1}{2}} \left(\begin{array}{ccc} j^1 & j^2 & j^3 \\ j^5 - j^6 & j^6 - j^4 & j^4 - j^5 \end{array} \right) \quad (37)$$

The other possibility corresponds to holding the length of one edge fixed, this representing a tetrahedra that has only an edge in contact with the boundary. In this case one still can do one of two things with the remaining angular momenta. One can take the angular momentum on the unique edge that does not touch our chosen edge to also be fixed, and the other four go to infinity. Or one can take all five to be large. This is where the dependence on the large scale asymptotics of the space have an effect. If for instance in the euclidean case we are considering a boundary that is a sphere in R^3, then clearly we must take the limit where all five other angular momenta become large. On the other hand, if the boundary is a plane in R^3 then one need take only four angular momenta to infinity, the other two corresponding to opposite edges of the tetrahedra remain fixed. The expressions for these limits contain additional dependence on parameters of the limiting process and can be found in appendix B.

The expressions for the path sums in these limits are relatively complicated. It is interesting to note that the answer for this "near-boundary" limit, is basically the two-dimensional double $3jm$ symbol action derived above for purely topological boundary conditions, however, with some additional structure depending upon the asymptotic behaviour of the space-time. In the next section for null surfaces in Lorentzian manifolds we will find that the semi-classical limit leads to an hypothesis that simplifies the boundary discretization considerably.

4 Lorentzian manifolds, Liouville theory and string theory on AdS_3

If we replace the $SO(3)$ of the euclidean construction with $SO(2,1)$ then the representation theory becomes somewhat more complicated, and all limits of the corresponding angular momentum coupling coefficients in the various representations have not been fully studied. However, the original large angular momentum limit of Ponzano and Regge has also been carried out for the discrete series of representations in the non-compact case [25]. The result is basically the same as for the compact group apart from the fact

that the angles are now hyperbolic given by the boost required to take the normal to a face into the normal of an adjoining face. In the limit of large angular momentum we have

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} \simeq \frac{1}{\sqrt{12\pi|V|}} \cos\phi \, exp\left(-\left|\sum_{h<k} j_{hk}\Theta_{hk}\right|\right) \tag{38}$$

In this expression, Θ_{hk} is the angle between the faces h and k,

$$\Theta_{hk} = \cosh^{-1}(n_h.n_k)$$

and n is the unit normal to the corresponding face. Thus as one face becomes null, the corresponding normal will also become null, and the angle that this face makes with the three neighbouring faces becomes infinite. The exponential in the weight for the tetrahedron implies that the corresponding angular momentum must be zero or that two of the sides of the triangle are of equal length and the corresponding angles are of opposite sign. Therefore the only configurations that can contribute have equilateral triangles and isosceles triangles where the short edge has length $1/2$ corresponding to zero angular momentum. The equilateral triangles must have all zero angular momentum labels and thus have all sides of length $1/2$. So in a path sum involving all discretizations with a given boundary metric the path sum is dominated by discretizations with boundary triangles that have all lengths equal to $1/2$. This is modified then by collective structures built from isosceles triangles. It is clear that the configurations involving isosceles triangles must be collective, as the presence of an isosceles triangle, implies also that neighbouring triangles are isosceles, and so on, until the structure closes again. An example of such a collective structure is shown in figure 5. These structures are reminiscent of macroscopic loop operators in the matrix models of dynamical triangulations [26].

We should note here that we have been a bit incautious regarding the order of limits. We took large j and then interpreted the expression for small j. There is indirect evidence that the result is sensible and we will discuss our reasoning below. The precise calculation that one needs to do is to take the null boundary limit of the quantum $6j$ symbol in a similar way to the original limits studied by Ponzano and Regge.

Thus we get a picture of horizon states in discretized quantum gravity and this is a positive step towards a micrpscopic understanding of black hole entropy. In 't Hooft's discussions of horizon states [27] one finds similarly a special role for the low angular momenta, $l = 0, \pm\frac{1}{2}$ when the horizon at fixed Rindler time is represented as a collection of discretized line segments labelled by angular momenta of $SO(2,1)$. Also in the Ashtekhar approach
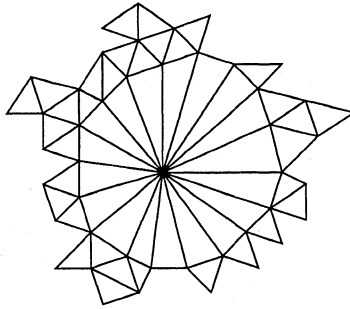
Figure 5: An example of part of a horizon configuration showing an isosceles "excitation".

to quantum gravity, the entropy calculations indicate that entropy is derived from contributions only from the lowest spin states on the horizon [28] and similarly in the paper [29].

It is interesting to reflect upon the meaning of the boundary action. If we assume that the semi-classical limit of the Clebsch-Gordon coefficients for $U_q(sl(2))$ for the discrete representations are an analytic continuation of those for $U_q(so(3))$ then we should find a negative cosmological constant. For the situation of $2 + 1$ gravity in a space of constant negative curvature, one finds as mentioned above that the boundary theory is a Liouville theory. Furthermore from recent work on Liouville theory [30, 31, 32] it is known that the representations of the Virasoro algebra that arise in the N-point functions, involve the quantum group $U_q(sl(2, R))$. In the string theory picture of AdS_3/CFT duality [33] the mass cut-off on angular momentum representations is also the same as that which arises in the discrete representations of $U_q(sl(2, R))$. Beginning as we did from the PRTV (Ponzano-Regge-Turaev-Viro) construction, it appears that we have arrived at almost the same conclusion. Note though that in the PRTV construction, after changing to a Lorentzian space-time signature it is not necessary that the representations are identical to those used for the Euclidean geometries. Maybe one should sum(integrate) over the continuous representations that arise in the Liouville approach for the boundary at infinity. On the other hand, for the null boundary at a global horizon, it is not so clear how to proceed, however some interesting insight will come from a comparison of our boundary action and string theory on AdS_3 [34]. If we had the Clebsch-Gordon coefficients for the discrete series of $U_q(sl(2))$ we could also explicitly calculate the weight of a "macroscopic loop" configuration and make a direct comparison with the macroscopic loop wavefunctions calculated for example in [26]. The Clebsch-Gordon coefficients are known for the continuous series [32] and for these one should be able to directly compute the null boundary

limit.

The representation theory of non-compact quantum groups is still very much under development, see [35, 32]. Also a discussion on the relationship between strings in AdS_3 and quantum groups can be found in [36]. There are representations of $U_q(sl(2))$ that are discrete, and agree basically with the discrete ones for the compact group, and give a cut-off in the path sum, see also [35] for a few more details on these. The other representations are those that arise from the quantum group representation of the Virasoro algebra of the Liouville cft at $c > 1$. These are similar to the continuous representations of $sl(2, R)$.

We can already make some speculative remarks derived from studies of string theory and continuum gravity in AdS_3 [33]. One can study the various physical excitations in this space-time both from the perspective of the space-time and that of the string theory. In the space-time picture, one finds a $c > 1$ Liouville theory, and indeed if one considers a non-critical string theory with target equal to AdS_3 then again one will find the world-sheet theory also to be Liouville with $c > 1$. The states that arise are classified by quantum group representations [30, 31, 32]. However the representations that arise are not those that we are using in the Turaev-Viro path sum. This strongly suggests that an extension of the PRTV (Ponzano-Regge-Turaev-Viro) construction to include the representations of $U_q(sl(2))$ that arise in the Liouville theory corresponds to extending the quantum gravity path sum, to a string field theory path sum (albeit with a fixed topology for the target manifold). Liouville theory at $c = 1$ also appears in the context of AdS_5 compactifications although in this case the theory appears as a consequence of $SU(2)$ group factors in the internal space [37]. It would be interesting to find connections between this structure and the Liouville theory that is naturally present for AdS_3 string compactifications.

The Liouville theory on the boundary cylinder at infinity for gravity in AdS_3 has a central charge

$$c = 1 + 6(b + 1/b)^2 \qquad (39)$$

where $b \in R$ or $|b| = 1$ and the correlation functions of this theory are constructed from the Clebsch-Gordon coefficients of the quantum group $U_q(sl(2))$ where

$$q = e^{(i\pi b^2)}. \qquad (40)$$

The cosmological constant of the AdS_3 space is proportional to b^4. In turn, the cosmological constant that arises in PRTV is proportional to $1/k^2$, where

the deformation for the Turaev-Viro quantum group is given by,

$$q = e^{\left(\frac{i\pi}{k}\right)}. \tag{41}$$

Clearly $b^2 \sim 1/k$ and thus the groups that arise in the two approaches are indeed identical deformations of $sl(2)$. Thus the group and deformation parameter agree in a manner which supports the conjecture that the quantum group that arises in the Regge calculus is the same as that of the Liouville theory on the boundary. However, the representations that arise in the Liouville theory have angular momentum in $\frac{Q}{2} + i\mathcal{R}$, while those in PRTV are identical to those that arose for $U_q(so(3))$ with angular momentum running from 0 to $\frac{k-1}{2}$. Of course once we changed from Euclidean to Lorentzian discretizations, the question already arose as to which representations one should sum over and now we see that the answer to this question may have deeper significance.

Actually one can make the relationship between our discrete boundary action involving the quantum group and the perturbation theory of the Liouville theory on the cylinder more concrete in a very geometrical manner by examining the perturbative expansion of the Liouville theory on a cylinder (corresponding to the boundary of AdS_3). Write the path integral with sources and charges for all Liouville vertex operators in selected representations. Use bootstrap to argue that all vertices can be reduced to cubic and recall that the cubic vertex for the Liouville theory is given precisely by the Clebsch-Gordon coefficient of the quantum group $U_q(sl(2))$ [32]. Furthermore the propogator of the perturbative expansion of the Liouville theory is the Wigner coefficient D^j_{mn} of the corresponding representations. Such Feynman diagrams correspond precisely to the dual lattice with weights as derived in the previous section and as shown in Figure 3. Geometrically all genus zero amplitudes correspond to one of our quantum group boundary terms in structure but with a sum over representations different from those used by Turaev-Viro.

If one considers the dual lattice to the boundary triangulation, one finds a trivalent graph that lives on the boundary of the manifold, being one of the Feynman diagrams discussed above. At any given time-slicing this will look like a collection of particles with mass given by their spin, and as this gas evolves there are interactions coming from the trivalent graph. Thus one can make a proposal for calculating the entropy using a system of particles making a gas. The XXZ spin chain is a possible starting point for such a calculation. This model is a chain of spins the solution to which involves the quantum group $U_q(sl(2))$ and which is related to Liouville theory for $c = 1$ and $c > 25$ and also possibly for all $c > 1$ [38]. Within this framework we should be able to formulate the explicit calculation that is necessary to
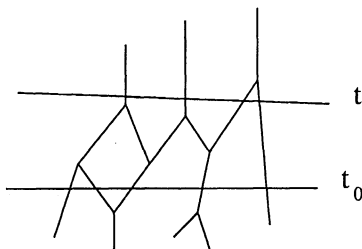
Figure 6: Two time slices of the boundary gas

calculate the entropy of the boundary theory and thus black hole entropy. The Feynman diagrams of the boundary action describe the time-evolution of the gas (Figure 6). For a null boundary the gas will be non-relativistic whereas for a time-like boundary the gas will be relativistic. In the case of a null boundary these representations will become more restricted and the three point interaction implies that during the evolution of the gas one has both creation and annihilation of particles. This may even imply some sort of dissipation in the null case. Other works arguing for dissipative behaviour for a theory describing a black hole horizon have appeared in [39, 40]. From a deeper understanding of this gas one should be able to directly calculate the entropy and thus the black hole horizon entropy.

Another consideration that we have not addressed directly but that has already arisen a few times in our discussions, and also one that is intimately related to the calculation of the entropy is the following. Without a boundary, it was clear that the prescription of Ponzano and Regge to hold fixed the simplicial decomposition was already sufficient due to the topological nature of the theory. Now in the presence of a boundary it is possible that one really needs to sum over the boundary triangulations. The bulk theory is topological and is insensitive to how one describes the sum in detail, however we expect some dynamics on the boundary. This indicates the possibility of extending the path-sum to dynamical triangulations. This sounds like trouble as such triangulations give rise to the matrix model of Liouville and for $c > 1$ these models are badly behaved with very rough surfaces dominating the path sum. However, discretizations for lorentzian manifolds have been studied in [21] where the authors have shown that when the simplicial decompositions are restricted by the requirement of a causal structure, the phases of the dynamical triangulations are well behaved involving smoother surfaces than in the Euclidean setup. The Haussdorf dimension in particular remaining $d_H = 2$ rather than becoming fractal and equal to 4 as it does in the Euclidean case. Indeed, our work also implies that there is another possibly interesting type of dynamical triangulation, where the "causal" structure

is that implied by the constraint that the surface be not Lorentzian, but null. It would be interesting to study in the context of [21] null dynamical triangulations, the results of which investigation would certainly shed light on the dynamics of black hole horizons in quantum gravity.

4.1   3 + 1 dimensions

For $3 + 1$ dimensions we now have some intuition for how to approach the discretization. The simplices will be labelled by $SO(3,1)$ representations. We can write the boundary path sum including boundaries following more or less the same philosophy as above. In this case from the beginning it seems that we probably need to consider dynamical triangulations as otherwise we will end up with a topological bulk theory rather than a theory containing also gravitational dynamics. Various versions of discretizations of four-dimensional Lorentzian manifolds have been studied as for example in [41, 42]. Furthermore the semi-classical limit of the $15j$ symbols that arise in these bulk path-sums, has been studied in [43] with results agreeing with the Regge discretization once more. We expect for null boundaries also in $3 + 1$ dimensions that some restrictions will be placed on the representations arising and that one will probably again find some sort of three-dimensional dynamical triangulation describing the behaviour of the horizon. From the work in [44] it has been shown that also for three-dimensional lorentzian dynamical triangulations, the branched polymer and crumpled phases, can not be reached leaving hope that such a system will have a nicely behaved continuum phase transition. It would be interesting to also look at three dimensional null dynamical triangulations to see if the causality restrictions on triangulations introduce some regulator of the geometries. The way to proceed is we believe clear. One must determine the representations that are important for the theory that is being investigated, and then one must look at various limits of the j-symbols.

Appendices

A   Angular momentum identities

The Biedenharn-Elliot identity relates the $6j$ symbols associated to two different ways of combining nine angular momenta. The sum on the right hand side is replaced by a product on the left. Geometrically this identity

is represented by the diagram shown.

$$
\begin{Bmatrix} j_7 & j_8 & j_9 \\ j_5 & j_1 & j_4 \end{Bmatrix} \begin{Bmatrix} j_7 & j_8 & j_9 \\ j_6 & j_2 & j_3 \end{Bmatrix} = \tag{42}
$$

$$
\sum_X (-1)^{(\sum_i j_i + X)} \begin{Bmatrix} j_1 & j_2 & X \\ j_3 & j_4 & j_7 \end{Bmatrix} \begin{Bmatrix} j_3 & j_4 & X \\ j_5 & j_6 & j_8 \end{Bmatrix} \begin{Bmatrix} j_5 & j_6 & X \\ j_2 & j_1 & j_9 \end{Bmatrix}
$$

Using the orthogonality for a pair of $6j$ symbols this identity can be rearranged as discussed in the text, up to an infinite multiplicative factor. The regularized version of this identity as first given in Ponzano and Regge [3], is

$$
\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \lim_{R \to \infty} \Lambda(R)^{-1} \sum_{j_7 \cdots j_{10}} \prod_{i=7 \cdots 10} (2j_i + 1) \times \tag{43}
$$

$$
\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_7 & j_8 & j_9 \end{Bmatrix} \begin{Bmatrix} j_6 & j_5 & j_1 \\ j_8 & j_9 & j_{10} \end{Bmatrix} \begin{Bmatrix} j_4 & j_2 & j_6 \\ j_9 & j_{10} & j_7 \end{Bmatrix} \begin{Bmatrix} j_3 & j_5 & j_4 \\ j_{10} & j_7 & j_8 \end{Bmatrix}
$$

Another useful identity for understanding the relationship between bulk and boundary transformations is,

$$
\sum_{m_3} (-1)^{j_3 - m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_3 & j_4 & j_5 \\ -m_3 & m_4 & m_5 \end{pmatrix} = \tag{44}
$$

$$
\sum_{j,m} (-1)^{j-m} \begin{pmatrix} j_1 & j_5 & j \\ m_1 & m_5 & m \end{pmatrix} \begin{pmatrix} j & j_4 & j_2 \\ -m & m_4 & m_2 \end{pmatrix} \begin{Bmatrix} j_2 & j_4 & j \\ j_5 & j_1 & j_3 \end{Bmatrix}
$$

The geometrical meaning of the left side is simply a pair of adjoining boundary faces. The right hand side involves the gluing of two faces of an additional tetrahedron to the original pair of faces resulting in a new pair of boundary triangles. This results in a $2 \leftrightarrow 2$ Pachner transformation in two-dimensions. Using the orthogonality of the $3jm$ symbols one can rewrite this equation to give the algebraic representation of the $3 \leftrightarrow 1$ transformation.

B   Limits of $6j$ symbols

Here are the $2 + 2$ and $2 + 1 + 1$ limits of Ponzano-Regge. We will use the following labelling for the tetrahedron (Figure 7). For the $2+2$ limit, we shift
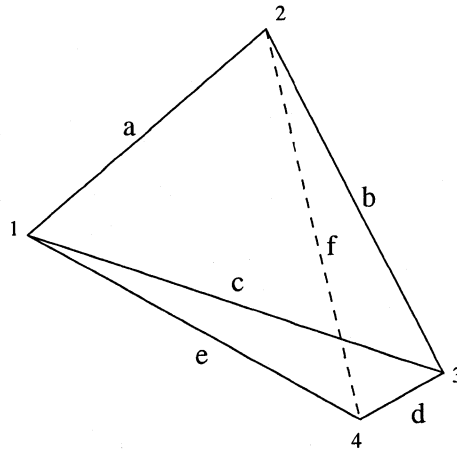
Figure 7: Some limits of tetrahedra

b, c, e, f by R and take R to be much larger than all of $a \dots f$. In the figure this limit corresponds to keeping the segments $[1, 2]$ and $[3, 4]$ of fixed length while all other edge lengths go to infinity. We can consider $a = \text{length}[1, 2]$ to be the edge of the tetrahedron that lies in the boundary.

The answer is then,

$$\left\{ \begin{array}{ccc} a & b+R & c+R \\ d & e+R & f+R \end{array} \right\} \simeq (-1)^{a+d+min(b+e,c+f)}$$

$$\left[\frac{(a-b+c)!(a-e+f)!(d-e+c)!(d-b+f)!}{(a+b-c)!(a+e-f)!(d+e-c)!(d+b-f)!} \right]^{\frac{1}{2}\text{sign}(c+f-b-e)} \times$$

$$\frac{(2R)^{-|b+e-c-f|-1}}{|b+e-c-f|!} \left[1 + O(R^{-2}) \right] \tag{45}$$

For the $2 + 1 + 1$ limit, we take $e = b + \delta$ and $f = c + \delta'$, where now d, δ, δ' are all large, though small with respect to a, b, c. This corresponds to keeping only the segment d of fixed length and all other edges to infinity. The one edge of small size is then d and in the text this is the edge that lies in the boundary of the manifold, all other edges in this case being internal. The final answer is,

$$\left\{ \begin{array}{ccc} a & b & c \\ d & b+\delta & c+\delta' \end{array} \right\} \simeq \frac{(-1)^{a+b+c+\delta+\delta'}}{[12\pi V]} \cos\left(t - \frac{1}{4}\pi\right), \tag{46}$$

where

$$t = \Omega - (a + b + c + \delta + \delta' - \frac{1}{4})\pi \tag{47}$$

and Ω is the Regge action for the tetrahedron.

In this limit, the dependence on the asymptotic structure of the space enters as the angle that remains in the final expression is the angle between the edges $[2,3]$ and $[1,3]$ or equivalently between $[1,4]$ and $[2,4]$. These angles enter the expression for the limit through the Regge action. If for instance the boundary is on a sphere of finite volume, then as one takes this limit a tetrahedron with one edge stuck on the sphere boundary, these angles will go to infinity. If the boundary is planar in flat space then the angles will go to zero and we go back to the $2 + 2$ result.

Acknowledgments

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