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Geometric Transitions, del Pezzo Surfaces and Open String Instantons

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We continue the study of a class of geometric transitions proposed by Aganagic and Vafa which exhibit open string instanton corrections to Chern-Simons theory. In this paper we consider an extremal transition for a local del Pezzo model which predicts a highly nontrivial relation between topological open and closed string amplitudes. We show that the open string amplitudes can be computed exactly using a combination of enumerative techniques and Chern-Simons theory proposed by Witten some time ago. This yields a striking conjecture relating all genus topological amplitudes of the local del Pezzo model to a system of coupled Chern-Simons theories.

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1 Introduction

In the original formulation [20], geometric transitions have predicted a remarkable relation between Chern-Simons theory on S^3 and closed topological strings on the small resolution of a conifold singularity. This correspondence has been extended to knots and links in [32, 33, 34, 42, 45] and it has been recently proven from a linear sigma model perspective in [43]. A different generalization has been proposed in [5], where the Chern-Simons theory was corrected by open string instanton effects. This new class of dualities yields very interesting predictions relating topological open string amplitudes in various toric backgrounds to certain open string expansions. The new feature of these transitions is a fascinating interplay of open string enumerative geometry and Chern-Simons theory proposed by Witten in [48]. Open string enumerative techniques have been developed in [1]-[5],[8, 21, 22],[24]-[29],[32]-[35],[38]-[42],[45]-[47]. Applying some of these results, we have successfully tested this approach for a simple exactly soluble model in [12].

The question we would like to address in this paper is if one can perform similar high precision tests of the duality in more general toric backgrounds. In particular, we consider the local dP_2 model, which is a toric noncompact Calabi-Yau threefold fibered over the del Pezzo surface of degree two. This is the simplest local model containing a compact divisor which exhibits extremal transitions. Since there is an abundance of holomorphic curves on the del Pezzo surface, the topological closed string amplitudes are quite complicated. So far, concrete computations have been performed only for genus zero Gromov-Witten invariants [10]. The extremal transition in question is obtained by contracting two $(-1,-1)$ curves on the noncompact threefold, and then smoothing out the conifold singularities. After a somewhat technical analysis, one can show that the resulting open string theory consists of two Chern-Simons theories supported on two disjoint 3-spheres which are coupled by instanton effects. Systems of this kind have been predicted by Witten in [48].

The main result of this paper is that the open string instanton corrections can be summed exactly using the techniques developed in [22, 29, 38]. This yields a fairly simple system of Chern-Simons theories by interpreting the instanton corrections as Wilson loop perturbations of the Chern-Simons theories [48]. Then large N duality predicts that the 't Hooft expansion of these coupled Chern-Simons theories computes all topological closed string amplitudes of the local dP_2 model! We show by direct computations that this conjecture is valid up to degree four in the expansion in terms of Kähler parameters. This is very strong evidence that the conjecture is true to all orders, but we do not have a general proof.

This paper is structured as follows. In section two we study the geometry of the extremal transition and construct the primitive open string instantons after deformation. Section three consists of a review of the topological closed string theory for the local dP_2 model following [10]. In section four we present the main results, namely the open string instanton expansion accompanied by Chern-Simons computations. Here we find a precise agreement with the known genus zero Gromov-Witten invariants, and make some higher genus predictions. Sections five and six are devoted to open string enumerative computations based on localization techniques as in [22, 29, 38]. Finally, some technical details and calculations are presented in the two appendixes.

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2 Geometric Transitions for Local dP_2 Model

The local dP_2 model is a toric Calabi-Yau threefold X isomorphic to the total space of the canonical bundle $\mathcal{O}(K_{dP_2})$. We have $X = \left(\mathbb{C}^5 \setminus F\right)/(\mathbb{C}^*)^3$ defined by the following toric data

$$\begin{array}{ccccccc} & X_0 & X_1 & X_2 & X_3 & X_4 & X_5 \\ l_1 & -1 & 1 & -1 & 1 & 0 & 0 \\ l_2 & -1 & 0 & 1 & -1 & 1 & 0 \\ l_3 & -1 & 0 & 0 & 1 & -1 & 1 \end{array} \tag{1}$$

with disallowed locus $F = \{X_1 = X_3 = 0\} \cup \{X_2 = X_4 = 0\} \cup \{X_3 = X_5 = 0\}$. This toric quotient can be equivalently described as a symplectic quotient $\mathbb{C}^6//U(1)^3$ with moment maps

$$\begin{aligned} |X_1|^2 + |X_3|^2 - |X_2|^2 - |X_0|^2 &= \xi_1 \\ |X_2|^2 + |X_4|^2 - |X_3|^2 - |X_0|^2 &= \xi_2 \\ |X_3|^2 + |X_5|^2 - |X_4|^2 - |X_0|^2 &= \xi_3 \end{aligned} \tag{2}$$

where $\xi_1, \xi_2, \xi_3 > 0$. The toric fan of X is a cone over the two dimensional polytope represented below.

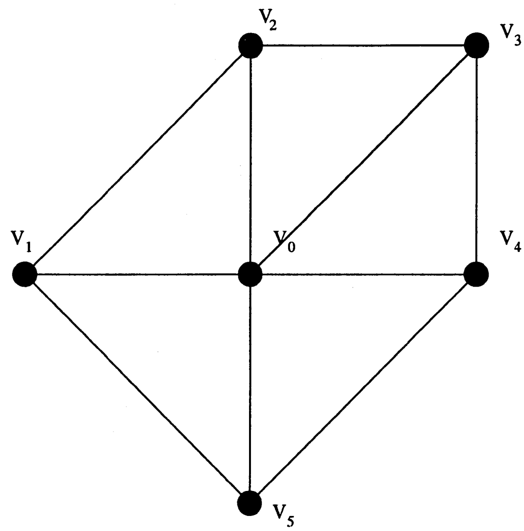

Figure 1: A section in the toric fan of X. The resulting polytope describes $\mathbb{P}^2$ blown-up at two points.

There is a single compact divisor S on X which is the zero section of $\pi : X \longrightarrow dP_2$ defined by $X_0 = 0$. The Mori cone of X is generated by the curve classes $e_1, h - e_1 - e_2, e_2$ corresponding the cones over v_0v_2, v_0v_3 and respectively v_0v_4. One can check that these are rigid rational curves with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Since X is a toric manifold, it can be represented as a topological $T^2 \times \mathbb{R}$ fibration over $\mathbb{R}^3$ [23, 36] whose discriminant is the two dimensional planar graph represented in fig. 1. Then the curves $e_1, h - e_1 - e_2, e_2$ can be represented as S^1 fibrations over certain edges of the graph as shown there.

The moment maps (2) yield the following parameterization of the Kähler cone

$$J = \xi_1(h - e_1) + \xi_2 h + \xi_3(h - e_2) \tag{3}$$

where J represents the restriction of the Kähler class to S. In the following we will use alternative Kähler parameters (s_1, t, s_2) defined by

$$J = -s_1 e_1 + th - s_2 e_2. \tag{4}$$

We are interested in a extremal transition consisting of a contraction of e_1, e_2 on X followed by a smoothing of the two nodal singularities. It turns out that the resulting singular threefold $\widehat{X}$ can be described as a nodal hypersurface in a toric variety $Z = \left(\mathbb{C}^3 \setminus \{0\}\right) \times \mathbb{C}^2/\mathbb{C}^*$. The $(\mathbb{C}^*)$ action is defined by

$$\begin{array}{cccccc} & Z_1 & Z_2 & Z_3 & U & V \\ \mathbb{C}^* & 1 & 1 & 1 & -1 & -2. \end{array} \tag{5}$$

Obviously, Z is isomorphic to the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-2)$ over $\mathbb{P}^2$. The embedding of $i : \widehat{X} \hookrightarrow Z$ is given terms of homogeneous coordinates by

$$Z_1 = X_2X_3X_4, \quad Z_2 = X_1X_2, \quad Z_3 = X_4X_5, \quad U = X_0X_1X_5, \quad V = -X_0X_3. \tag{6}$$

In order to make sure this is a regular map, one has to check that (6) is compatible with the toric actions (1), (5) and that the disallowed loci agree. We have included the details in appendix A.1. The image of $\widehat{X}$ in Z is given by

$$UZ_1 + VZ_2Z_3 = 0. \tag{7}$$

One can easily check that this hypersurface has exactly two nodal singularities, as expected. The singular locus is described by

$$U = 0, \quad Z_1 = 0, \quad Z_2Z_3 = 0, \quad VZ_2 = 0, \quad VZ_3 = 0. \tag{8}$$

Since Z_1, Z_2, Z_3 do not vanish simultaneously, the singular points are $\{U = V = 0, Z_1 = Z_2 = 0\} \cup \{U = V = 0, Z_1 = Z_3 = 0\}$. Therefore we obtain the two expected conifold singularities.

This representation of $\widehat{X}$ allows a concrete description of the extremal transition. We can resolve the singularities by blowing up Z along the zero section $U = V = 0$. The proper transform of $\widehat{X}$ will then be isomorphic to the local dP_2 model X discussed before. Alternatively, we can smooth out (7) by deforming the polynomial equation to

$$UZ_1 + VZ_2Z_3 = \mu. \tag{9}$$

We obtain a family of hypersurfaces $\mathcal{Y}/\Delta$ parameterized by a complex parameter $\mu \in \Delta$, where Δ is the unit disc. Without loss of generality we can take μ to be real and positive, and we denote by $Y \equiv Y_\mu$ the corresponding fiber. The transition can be very conveniently described in terms of the T^2 fibration structure represented in fig. 1. The discriminant of the torus fibration undergoes the following sequence of transformations. As a differential manifold, Y is obtained from X by performing surgery along the links of the two nodes, which are both isomorphic to $S^2 \times S^3$ [11]. The third homology is generated by the two vanishing cycles L_1, L_2 associated to the nodal singularities, subject to the relation $[L_1] - [L_2] = 0$. Topologically, these cycles are 3-spheres which can be locally described as fixed point sets of local antiholomorphic involutions. We will give some details below, since this is an important point for the rest of the paper. Let us cover Z with three coordinate

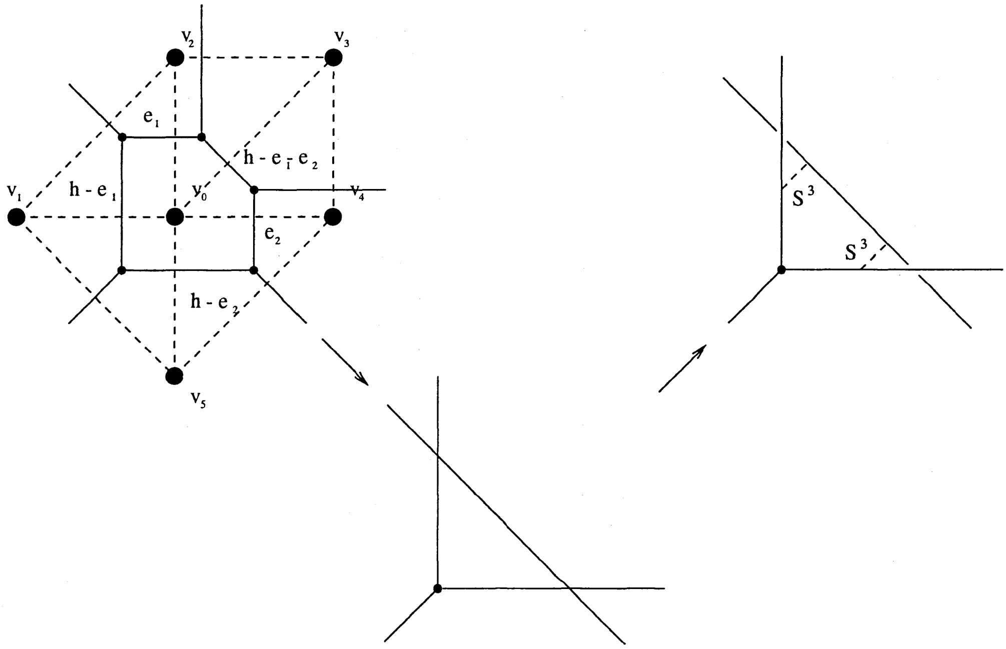

Figure 2: Description of the extremal transition in terms of T^2 fibration.

patches

$$\begin{aligned}
U_1 = \{Z_1 \neq 0\} : \qquad & x_1 = \frac{Z_2}{Z_1}, \quad y_1 = \frac{Z_3}{Z_1}, \quad u_1 = UZ_1, \quad v_1 = VZ_1^2 \\
U_2 = \{Z_2 \neq 0\} : \qquad & x_2 = \frac{Z_1}{Z_2}, \quad y_2 = \frac{Z_3}{Z_2}, \quad u_2 = UZ_2, \quad v_2 = VZ_2^2 \\
U_3 = \{Z_3 \neq 0\} : \qquad & x_3 = \frac{Z_1}{Z_3}, \quad y_3 = \frac{Z_2}{Z_3}, \quad u_3 = UZ_3, \quad v_3 = VZ_3^2.
\end{aligned} \tag{10}$$

In local coordinates, the hypersurface equation (7) can be written

$$\begin{aligned}
U_1 : \qquad & u_1 + v_1 x_1 y_1 = \mu \\
U_2 : \qquad & u_2 x_2 + v_2 y_2 = \mu \\
U_3 : \qquad & u_3 x_3 + v_3 y_3 = \mu.
\end{aligned} \tag{11}$$

One can then see two conifold singularities at $\mu = 0$ in the patches U_2, U_3. The corresponding vanishing cycles are defined by the real sections

$$\begin{aligned}
U_2 : \qquad & u_2 = \overline{x}_2, \qquad & v_2 = \overline{y}_2 \\
U_3 : \qquad & u_3 = \overline{x}_3, \qquad & v_3 = \overline{y}_3.
\end{aligned} \tag{12}$$

We show in appendix A.2. that one can choose a symplectic Kähler form ω on Y so that L_1, L_2 are lagrangian cycles. More precisely, one can construct ω so that it is locally isomorphic to the standard symplectic form on a deformed conifold near L_1, L_2. To conclude this section, let us discuss the second homology of Y. An exact sequence argument shows that $H_2(Y, L_1 \cup L_2, \mathbb{Z}) \simeq H_2(Y, \mathbb{Z}) = \mathbb{Z}$. One can construct certain nontrivial relative 2-cycles on Y as holomorphic discs D_1, D_2 with boundaries on L_1, L_2. Let $\Sigma_{0,1}$ be the disc $\{|t| \leq \mu^{-1/2}\} = \{\mu^{1/2} \leq |t'|\}$ in a projective line $\mathbb{P}^1$ with affine coordinates t, t'. We construct a holomorphic embedding $f_1 : \Sigma_{0,1} \longrightarrow Y$ given in local coordinates by

$$\begin{array}{lllll} U_1 : & x_1(t) = t, & y_1(t) = 0, & u_1(t) = \mu, & v_1(t) = 0 \\ U_2 : & x_2(t') = t', & y_2(t') = 0, & u_2(t') = \frac{\mu}{t'}, & v_2(t') = 0. \end{array} \tag{13}$$

It easy to check that f is well defined and it maps the boundary $|t'| = \mu^{1/2}$ of $\Sigma_{0,1}$ to an unknot Γ_1 in L_1. We will denote the image of $\Sigma_{0,1}$ in Y by D_1. We can construct similarly a disc ending on L_2. The embedding map is locally given by

$$\begin{array}{lllll} U_1 : & x_1(t) = 0, & y_1(t) = t, & u_1(t) = \mu, & v_1(t) = 0 \\ U_3 : & x_3(t') = t', & y_3(t') = 0, & u_3(t') = \frac{\mu}{t'}, & v_3(t') = 0. \end{array} \tag{14}$$

To complete this discussion, note that one can also embed a holomorphic annulus C in Y, the boundary components being mapped to L_1, L_2. For this we have to use the coordinate patches U_2, U_3. Let $\Sigma_{0,2}$ be the cylinder $\{\mu^{1/2} \leq |t| \leq \mu^{-1/2}\} = \{\mu^{-1/2} \geq |t'|^2 \geq \mu^{1/2}\}$ in $\mathbb{P}^1$ with affine coordinates (t, t') (recall that μ is a positive real number inside the unit disc, hence $\mu < 1$.) We define a map $f : \Sigma_{0,2} \longrightarrow Y$ by

$$\begin{array}{lllll} U_2 : & x_2(t) = 0, & y_2(t) = t, & u_2(t) = 0, & v_2(t) = \frac{\mu}{t} \\ U_3 : & x_3(t') = 0, & y_3(t') = t', & u_3(t') = 0, & v_3(t') = \frac{\mu}{t'}. \end{array} \tag{15}$$

Then the boundary component $|t| = \mu^{1/2}$ is mapped to L_1, while the boundary $|t| = \mu^{-1/2}$ is mapped to L_2. Note that the discs D_1, D_2 and the cylinder C can be in principle covered by a single coordinate patch. We have used two coordinate patches for reasons that will be clear in section six. Let us denote the two boundary components of C by Ξ_1, Ξ_2, which are again to be regarded as knots in L_1, L_2. An important point for Chern-Simons computations is that $\{\Gamma_1, \Xi_1\}$ and respectively $\{\Gamma_2, \Xi_2\}$ are algebraic links in L_1, L_2. This can be seen by noting that locally we can identify for example L_1 to the sphere

$$|x_2|^2 + |y_2|^2 = \mu \tag{16}$$

in $\mathbb{C}^2$ with coordinates (x_2, y_2). Then the disc D_1 and C can be locally described by the equation $x_2 y_2 = 0$ in $\mathbb{C}^2$. It is a well known fact that the intersection of this singular curve with the sphere surrounding the origin is an algebraic link with linking number one in the orientations induced by the complex structure. More precisely, if we parameterize the two boundary components as $x_2 = \mu^{1/2}e^{i\theta_x}$, $y_2 = \mu^{1/2}e^{i\theta_y}$, the 1-forms $d\theta_x, d\theta_y$ define orientations of Γ_1, Ξ_1 such that the linking number is 1 [9]. The same is true for Γ_2, Ξ_2 in L_2. In particular this shows that D_1, C and D_2, C are disconnected. Note that from now on we will fix the above orientations for $\Gamma_1, \Xi_1, \Gamma_2, \Xi_2$. In terms of the T^2 fibrations, D_1, D_2, C can be represented as in fig. 3.

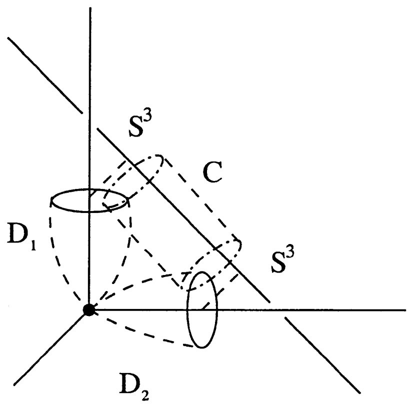

Figure 3: Primitive open string instantons on Y: the linking number of the linked boundaries is 1.

Using this picture it is easy to see that there is a continuous family of 2-spheres interpolating between D_1 and D_2, hence the relation $[D_1] - [D_2] = 0$. Similarly, we have $[C] = [D_1] = [D_2]$, and we will denote this relative homology class by β. However note that these 2-spheres are not holomorphically embedded in Y. In fact we will show in section five that there are no holomorphic curves on Y and that β is a generator of $H_2(Y, L_1 \cup L_2; \mathbb{Z})$. Therefore D_1, D_2, C constructed above are primitive open string instantons. Since L_1, L_2 are lagrangian, this shows that D_1, D_2, C have the same symplectic area

$$t_{op} = \int_{D_1} \omega = \int_{D_2} \omega = \int_C \omega. \tag{17}$$

By deforming the discs as topological spheres away from the vanishing cycles, one can show that classically $t = t_{op}$. This is so because the transition leaves the symplectic form essentially unchanged away from the singular locus. This concludes our discussion of the extremal transitions for the local dP_2 model from a geometric point of view. The physics of the transitions will be explored in the next sections.

3 Closed String Amplitudes and Duality Predictions

In the context of geometric transitions we are interested in a relation between the topological closed string A model on X and a topological open string A model on the deformation space Y. The topological open string theory on Y is defined by wrapping N_1 and respectively N_2 D-branes on the lagrangian cycles L_1, L_2. The target space action of this theory consists of two Chern-Simons theories with gauge groups $U(N_1), U(N_2)$ supported on the two cycles [48]. We will see later that these theories are coupled by open string instanton effects.

In order to have the right integrality properties, the topological amplitudes must be written in terms of flat coordinates. On the closed string side we have flat coordinates $(\widehat{s}_1, \widehat{s}_2, \widehat{t})$ corresponding to the classical coordinates (s_1, s_2, t). For simplicity, we will drop the notation $\widehat{\ }$, keeping in mind that topological amplitudes will always be written in terms of flat coordinates. The open string theory contains a classical geometric parameter defined in (17). Accordingly we have a flat coordinate $\widehat{t}_{op}$, which will also be denoted by t_{op} from now on. As discussed above, classically, one would predict a relation of the form $t = \widehat{t}_{op}$, but this has to be refined at quantum level, as discussed in [12]. Moreover, we will see later in section four that the open string amplitudes depend in fact on three flat coordinates corresponding to the three primitive instantons constructed in section two.

Without going into details for the moment, note that large N duality predicts a relation between closed and open string amplitudes of the form

$$\mathcal{F}_{cl}(g_s, s_1, s_2, t) = \mathcal{F}_{op}(g_s, \lambda_1, \lambda_2, t_{op}). \tag{18}$$

Here $\lambda_1 = N_1 g_s$, $\lambda_2 = N_2 g_s$ are the 't Hooft coupling constants of the two Chern-Simons theories on L_1, L_2 which should be related to the closed string parameters (s_1, s_2). We will discuss the precise relation in section four.

According to [19], the closed string free energy on X has the following structure[1]

$$\mathcal{F}_{cl}(t, s_1, s_2, g_s) = \sum_{r=0}^{\infty}\sum_{n=1}^{\infty}\sum_{C\in H_2(X,\mathbb{Z})} N_C^r \frac{1}{n}\left(2\sin\frac{ng_s}{2}\right)^{2r-2} e^{-n<J,C>}. \tag{19}$$

In this expression, N_C^r are the Gopakumar-Vafa invariants of X which count the number of BPS states of charge C and spin quantum number r in M-theory compactified on X; n counts multicovers. In the following we will refer to them as GV invariants. In terms of the generators $(e_1, h-e_1-e_2, e_2)$ of the Mori cone we have

$$C = d_1 e_1 + d(h - e_1 - e_2) + d_2 e_2, \qquad d, d_1, d_2 \in \mathbb{Z} \tag{20}$$

[1]Throughout this paper, we will consider truncated expressions for the free energy, that is we will omit the polynomial terms.

Note that C is representable by an irreducible reduced curve only if $0 \leq d_1, d_2 \leq d$ or $d = 0$ and $d_1 = d_2 = 1$. Recall that the Kähler class J is given by (4), $J = -s_1e_1 + th - s_2e_2$. Then we can rewrite (19) as

$$\begin{gathered}\mathcal{F}_{cl}(t, s_1, s_2, g_s) = \sum_{n=1}^{\infty} \frac{1}{n\left(2\sin\frac{ng_s}{2}\right)^2}\left(e^{-ns_1} + e^{-ns_2}\right) \\ + \sum_{r=0}^{\infty}\sum_{n=1}^{\infty}\sum_{d>0}\sum_{0\leq d_1,d_2\leq d} N^r_{d,d_1,d_2}\frac{1}{n}\left(2\sin\frac{ng_s}{2}\right)^{2r-2} e^{-n(dt-(d-d_1)s_1-(d-d_2)s_2)}. \\ (21)\end{gathered}$$

Note that the first two terms represent the universal contributions of the two exceptional curves, whose open string interpretation is well understood [20]. In the second series, the GV invariants are not known for the local dP_2 model, except for $r = 0$, when they can be computed using mirror symmetry. This calculation has been performed in [10].

4 Open String Amplitudes and The Duality Map

We now consider the open string A model defined by wrapping N_1 and respectively N_2 branes on the lagrangian cycles L_1, L_2 in Y. According to [48], the target space physics of this model is captured by two Chern-Simons theories with gauge groups $U(N_1)$ and respectively $U(N_2)$ supported on the cycles L_1, L_2. As explained in [48], the Chern-Simons theory is in general corrected by open string instantons which give rise to Wilson loop operators in the target space action. The complete action can then be schematically written in the form

$$S(A_1, A_2) = S_{CS}(A_1) + S_{CS}(A_2) + F_{inst}(g_s, t_{op}, A_1, A_2) \tag{22}$$

where A_1, A_2 denote the two gauge fields on L_1, L_2. The concrete form of the instanton expansion depends on the details of the model. In general A_1, A_2 enter $F_{inst}(g_s, t_{op}, A_1, A_2)$ via holonomy operators associated to the boundary components of open string instantons interpreted as knots in L_1, L_2. For large volume, the instanton corrections can be treated perturbatively from the Chern-Simons point of view. Therefore in the large N limit, the open string free energy can be written as

$$\mathcal{F}_{op}(t_{op}, \lambda_1, \lambda_2, g_s) = \mathcal{F}_1^{CS}(\lambda_1, g_s) + \mathcal{F}_2^{CS}(\lambda_2, g_s) + \ln\left\langle e^{F_{inst}(g_s, t_{op}, A_1, A_2)}\right\rangle \tag{23}$$

where $\lambda_1 = N_1g_s$ and $\lambda_2 = N_2g_s$ denote the 't Hooft coupling constants of the two Chern-Simons theories. In the last term of (23) we have a double functional integral over both gauge fields. Therefore the computation of the free energy consists of two steps. First we have to find $F_{inst}(g_s, t_{op}, A_1, A_2)$ using open string enumerative techniques, and then compute the Wilson line expectation values in

Chern-Simons theory. For convenience, we will denote the last term in (23) by $\mathcal{F}_{inst}(g_s, t_{op}, \lambda_1, \lambda_2)$ so that (23) becomes

$$\mathcal{F}_{op}(t_{op}, \lambda_1, \lambda_2, g_s) = \mathcal{F}_1^{CS}(\lambda_1, g_s) + \mathcal{F}_2^{CS}(\lambda_2, g_s) + \mathcal{F}_{inst}(g_s, t_{op}, \lambda_1, \lambda_2). \quad (24)$$

The above discussion is quite schematical since the interaction between Chern-Simons theory and open string instantons is more subtle. According to [48], the perturbative Chern-Simons expansion should be interpreted as a sum over degenerate open string Riemann surfaces which develop infinitely thin ribbons. The ribbons, which are mapped to the spheres L_1, L_2 as geodesic graphs, have been interpreted in [48] as virtual instantons at infinity. So far there is no rigorous mathematical treatment of this type of degenerate behavior at infinity. In particular it is not known how to actually write down the open string amplitudes as finite dimensional integrals on a well defined moduli space. The prescription outlined above following [48] is to first sum over nondegenerate instantons, i.e. Riemann surfaces which have at worst double node singularities, and then sum over degenerate instantons by performing the Chern-Simons path integral.

The sum over non-degenerate instantons should be in principle defined in terms of intersection theory on some moduli space of stable open string maps[2] $\overline{M}_{g,h}(Y, L, d\beta)$, where $L = L_1 \cup L_2$. This theory has not been rigorously developed so far, but at the level of rigor of [22, 29, 38], one can give a computational definition of open string amplitudes. The main idea is to proceed by localization with respect to a torus action induced by a torus action on Y preserving L_1, L_2. Although the structure of the moduli space is unknown, one can describe in detail the structure of the fixed point loci. From this data, we can obtain enumerative invariants essentially by adapting on spot the known closed string techniques [7, 17, 31, 37] in order to evaluate the contribution of each fixed point locus. This approach has been successfully implemented for noncompact lagrangian cycles in [22, 29, 38, 41]. In that case one can fix a flat unitary connection on the lagrangian cycles as a background field.

In the present context, since the cycles are compact, the unitary connections become dynamical variables and one should integrate over all (gauge equivalence classes) of such fields [48]. This is achieved by performing the Chern-Simons theory path integral. The coupling between finite area instantons and Chern-Simons theory is quite subtle [48]. As explained there, if one had only isolated open string instantons, their effects would be encoded in a series of holonomy (Wilson loop) operators added to the Chern-Simons action. For each rigid isolated instanton $D \subset Y$, one adds a term of the form $e^{-t}\mathrm{Tr}V$, where V is the holonomy around the boundary of D, which is a knot in L. It is important to note that in this formula V

[2]This is a schematic discussion. Since Y is noncompact, the compactification of this moduli space is a very subtle issue. Some aspects will be mentioned in section six.

is not a flat gauge field, therefore this operator is not invariant under deformations of the knot. If D is rigid and isolated, this is not a problem. However, what happens if we have families of such instantons? Then the holonomy V depends on the particular member in the family, and it is not clear how one should write the associated corrections. A general answer to this question is not known at the present stage, but we would like to propose an answer for situations in which there is a torus action on Y preserving the lagrangian cycles. In such cases, one can simply use localization arguments to argue that all nontrivial contributions to the instanton sum come from fixed maps under this action. Then, to each component of the fixed point locus we associate a certain series of holonomy operators in Chern-Simons theory, as detailed below. Because of this coupling with Chern-Simons theory, the procedure described above should not be thought as localization of a virtual cycle on a moduli space of maps in the usual sense. It would certainly be desirable to have a more precise mathematical formulation of this construction, but this is not known at the present stage.

The final step is to perform the Chern-Simons functional integral with the instanton corrections included in order to compute the open string free energy. This approach has been successfully tested in a simple geometric situation in [12]. Note that this last step requires the choice of a framing for each boundary ∂D in order to regulate the divergences in Chern-Simons perturbation theory. We will comment more in this point below.

In the present situation, we define an S^1 action on Z by

$$\begin{array}{cccccc} & Z_1 & Z_2 & Z_3 & U & V \\ \mathbb{C}^* & \lambda_1 & \lambda_2 & 0 & -\lambda_1 & -\lambda_2. \end{array} \tag{25}$$

Obviously, this action preserves Y and L_1, L_2. Then, a somewhat technical analysis shows that the only primitive open string instantons left invariant by this action are the discs D_1, D_2 and the annulus C constructed in the previous section. The proof follows the lines of [12] (section 5); one has to take a projective completion $\overline{Y}$ of Y and show that the problem reduces to finding invariant curves on $\overline{Y}$ subject to certain homology constraints. We leave the details for the next section.

Note that the two discs D_1, D_2 have a common origin, therefore they form a nodal (or pinched) cylinder $D_1 \cup D_2$. As discussed at the end of section two, $D_1 \cup D_2$ and C are disconnected. Therefore the fixed locus of the torus action on $\overline{M}_{g,h}(Y, L, d\beta)$ consists of two disconnected components: multicovers of $D_1 \cup D_2$ and respectively multicovers of C. The precise structure of these components will be discussed in detail in section five. For now, let us note that on general grounds, an open string map to the pinched cylinder $D_1 \cup D_2$ is characterized by two degrees d_1, d_2 and two sets of winding numbers m_i, $i = 1, \ldots, h_1$, n_j, $j = 1, \ldots, h_2$, where h_1, h_2 are the numbers of boundary components mapped to Γ_1 and respectively

Γ_2. We have the constraints $\sum_{i=1}^{h_1} = d_1$, $\sum_{j=1}^{h_2} n_j = d_2$. Similarly, a generic map to the cylinder C is characterized by a degree d and two sets of winding numbers m_i, n_j, $i = 1, \ldots, h_1$, $j = 1, \ldots, h_2$ with $\sum_{i=1}^{h_1} m_i = \sum_{j=1}^{h_2} n_j = d$, where h_1, h_2 are the numbers of boundary components mapped to Ξ_1, Ξ_2.

Based on these elements, we can write down the general form of the open string instanton expansion on Y. We noticed earlier that D_1, D_2 and C have the same symplectic area, therefore one would be tempted to write down this expansion in terms of a single open string Kähler modulus t_{op}. However, things are more subtle here since the instanton expansion should be written in terms of flat coordinates rather than classical moduli. One of the lessons of [12] was that in the presence of dynamical A branes, the open string flat coordinates can receive nonperturbative corrections generated by virtual instantons at infinity. These corrections can be different for different open string instantons, depending on the lagrangian cycle they end on. For this reason, we will refine (23) by writing the instanton expansion in terms of three distinct flat Kähler moduli t_1, t_2, t_c corresponding to D_1, D_2, C. We will show later that this is in precise agreement with the duality predictions from the closed string side. We also introduce the following holonomy variables

$$\begin{array}{ll} U_1 = \mathrm{Pexp} \int_{\Xi_1} A^{(1)}, & U_2 = \mathrm{Pexp} \int_{\Xi_2} A^{(2)}, \\ V_1 = \mathrm{Pexp} \int_{\Gamma_1} A^{(1)}, & V_2 = \mathrm{Pexp} \int_{\Gamma_2} A^{(2)} \end{array} \tag{26}$$

corresponding to the boundary components of D_1, D_2 and C. Then the open string expansion takes the form

$$F_{inst}(g_s, t_1, t_2, t_c, U_1, U_2, V_1, V_2) = F^{(1)}_{inst}(g_s, t_c, U_1, U_2) + F^{(2)}_{inst}(g_s, t_1, t_2, V_1, V_2) \tag{27}$$

where

$$\begin{aligned} F^{(1)}_{inst}(g_s, t_c, U_1, U_2) = & \sum_{g=0}^{\infty} \sum_{h_1, h_2 = 0}^{\infty} \sum_{d=0}^{\infty} \sum_{m_i \geq 0, n_j \geq 0} i^{h_1+h_2} g_s^{2g-2+h_1+h_2} \\ & \times C_{g,h_1,h_2}(d; m_i, n_j) e^{-dt_c} \prod_{i=1}^{h_1} \mathrm{Tr} U_1^{m_i} \prod_{j=1}^{h_2} \mathrm{Tr} U_2^{n_j} \end{aligned} \tag{28}$$

$$\begin{aligned} F^{(2)}_{inst}(g_s, t_1, t_2, V_1, V_2) = & \sum_{g=0}^{\infty} \sum_{h_1, h_2 = 0}^{\infty} \sum_{d_1, d_2 = 0}^{\infty} \sum_{m_i \geq 0, n_j \geq 0} i^{h_1+h_2} g_s^{2g-2+h_1+h_2} \\ & \times F_{g,h_1,h_2}(d_1, d_2; m_i, n_j) e^{-d_1 t_1 - d_2 t_2} \prod_{i=1}^{h_1} \mathrm{Tr} V_1^{m_i} \prod_{j=1}^{h_2} \mathrm{Tr} V_2^{n_j}. \end{aligned} \tag{29}$$

In (28) we have a sum over multicovers of the annulus C, while (29) represents a sum over multicovers of the two discs D_1, D_2, which form a nodal cylinder. Note that the winding numbers are subject to the constraints mentioned above, that is $\sum_{i=1}^{h_1} m_i = \sum_{j=1}^{h_2} n_j = d$ for C, and $\sum_{i=1}^{h_1} m_i = d_1$, $\sum_{j=1}^{h_2} n_j = d_2$ for $D_1 \cup D_2$.

As discussed earlier in this section, the coefficients $C_{g,h_1,h_2}(d; m_i, n_j)$ as well as $F_{g,h_1,h_2}(d_1, d_2; m_i, n_j)$ can be computed by evaluating the contribution of the fixed loci in $\overline{M}_{g,h}(Y, L, d\beta)$. Note however, that to each component of the fixed locus we assign a certain holonomy operator in the Chern-Simons theory. Therefore, one does not simply sum over all fixed loci as in standard localization computations. This means that the contribution of each fixed component depends on the weights of the toric action used in the localization process. In order to obtain a physically sensible answer, we have to make a certain choice of weights similar to the choices made in [22, 29, 41]. Moreover, in our case the situation is more complicated since we also have to make a choice of framing in Chern-Simons theory. The two choices are in fact related, as discussed below and in more detail in section 6.3.

Before giving the details, note that given the geometric context, there may be many choices of weights and/or framings that result in distinct open string expansions. At this stage we do not know if there is a preferred choice based on certain intrinsic consistency criteria of the open string theory. This problem is very hard, and it cannot be answered without a better development of the mathematical formalism. In the following we will pursue a more modest goal, namely we will try to find a set of choices which leads to an agreement with the dual closed string expansion. Formulated differently, we will try to find the correct prescriptions for the duality map in this geometric situation.

For a single disc $D \subset \mathbb{C}^3$ with boundary on a noncompact lagrangian cycle L, it was shown in [29] that the choice of weights is equivalent to the choice of an equivariant section of the normal bundle $N_{\partial D/L}$. This prescription formalizes the relation between framing and toric action found for the first time in [3]. In particular, the instanton expansion for D depends on an integer ambiguity a which parameterizes isomorphism classes of S^1 equivariant sections of the normal bundle $N_{\partial D/L}$. For the discs D_1, D_2 embedded in Y, one can still choose the boundary data in the form of two sections to $N_{\Gamma_1/L_1}, N_{\Gamma_2/L_2}$ which are labeled by two integers a, b. Generalizing the strategy proposed by [29], we assume that these choices have to be compatible with the global S^1 action on Y. By explicitly writing these conditions in local coordinates, we show in section 6.3. that we are left with only two consistent choices, namely $(a, b) = (0, 0)$ or $(a, b) = (2, 2)$. In the following we will choose $(a, b) = (0, 0)$ since in this case the instanton expansion takes a very simple compact form. The second choice is not logically ruled out, but leads to a very complicated formula for the instanton corrections. We leave it for future work.

Having made this choice, the open string topological theory is still not completely determined, since we also have to specify the framing of the knots $\Gamma_1, \Gamma_2, \Xi_1$ and Ξ_2. In principle, the equivariant sections introduced in the previous paragraph should determine the framings of $\Gamma_1 = \partial D_1$ and $\Gamma_2 = \partial D_2$. However, there is a subtlety at this point explained in detail in section 6.3. Briefly, the choice of a single section does not determine the framing as an integer number; one also needs a reference section which is typically provided by the geometric context. In our case we have a natural reference section since Γ_1, Γ_2 are algebraic knots. Then a short local computation shows that the framings of Γ_1, Γ_2 are $(2-a, 2-b)$. Therefore for $(a, b) = (0, 0)$ we obtain framings $(2, 2)$.

For the annulus C, the choice of framing is more subtle since the localization computation does not require the choice of special values of toric weights. Therefore one does not have to choose equivariant sections on the two boundaries components Ξ_1, Ξ_2. In the present context, this framing can be related to the framings of the discs by a deformation argument detailed in section six. The resulting values are $(1 - \frac{a}{2}, 1 - \frac{b}{2}) = (1, 1)$ for the two boundaries of C. Moreover, it is shown in [6] that for an annulus with framings (p, p), the p-dependence of the amplitudes can be absorbed in a simple shift of the open string Kähler parameters, leaving the amplitudes otherwise unchanged. This allows us to choose canonical framing without loss of generality. We are very grateful to the authors of [6] for explaining this to us.

The open string instanton expansions (28),(29) can be determined by adding the contributions of all fixed points of the S^1 action. These are computed in section six, equations (6.30), (6.100), (6.101) and (6.102). There is one subtle aspect at this point, namely given the choices made so far, one has to count the contribution of the pinched cylinder, eqn (6.102), twice in order to match the predictions of large N duality. This factor of two does not follow directly from localization computations, and it cannot be satisfactorily explained using our present knowledge of moduli spaces of open string maps. In fact, since the only criterion for introducing this factor is agreement with the closed string dual, we should think of it as a prescription of the duality map. A more conceptual explanation would require a much deeper mathematical understanding of the open/closed string duality, which is beyond the purpose of this work. We hope to report on this aspect in the future.

To summarize this discussion, we propose the following large N Chern-Simons dual to the local dP_2 model

$$F^{(1)}_{inst}(g_s, t_c, U_1, U_2) = -\sum_{d=1}^{\infty} \frac{e^{-dt_c}}{d} \mathrm{Tr} U_1^d \mathrm{Tr} U_2^d \tag{30}$$

$$
\begin{aligned}
F^{(2)}_{inst}(g_s, t_1, t_2, V_1, V_2) = & \sum_{d=1}^{\infty} \frac{ie^{-dt_1}}{2d \sin \frac{dg_s}{2}} \mathrm{Tr} V_1^d + \sum_{d=1}^{\infty} \frac{ie^{-dt_2}}{2d \sin \frac{dg_s}{2}} \mathrm{Tr} V_2^d \\
& + 2 \sum_{d=1}^{\infty} \frac{e^{-d(t_1+t_2)}}{d} \mathrm{Tr} V_1^d \mathrm{Tr} V_2^d
\end{aligned} \tag{31}
$$

where the framings of the knots $\Gamma_1, \Gamma_2, \Xi_1, \Xi_2$ are $(2, 2, 0, 0)$. In the next subsection, we will present very convincing evidence for this conjecture by computing the open string free energy up to degree four in $e^{-t_1}, e^{-t_2}, e^{-t_c}$ and finding perfect agreement with the closed string results.

4.1 Chern-Simons Computations

As discussed above, we have to evaluate

$$
\mathcal{F}_{op}(t_1, t_2, t_c, \lambda_1, \lambda_2) = \mathcal{F}_1^{CS}(\lambda_1, g_s) + \mathcal{F}_2^{CS}(\lambda_2, g_s) + \mathcal{F}_{inst}(g_s, t_1, t_2, t_c, \lambda_1, \lambda_2) \tag{32}
$$

where

$$
\mathcal{F}_{inst}(g_s, t_1, t_2, t_c, \lambda_1, \lambda_2) = \ln \left\langle e^{F_{inst}(g_s, t_1, t_2, t_c, U_1, U_2, V_1, V_2)} \right\rangle . \tag{33}
$$

Moreover, the instanton expansion is obtained by adding (30) and (31)

$$
F_{inst}(g_s, t_1, t_2, t_c, U_1, U_2, V_1, V_2) = F^{(1)}_{inst}(g_s, t_c, U_1, U_2) + F^{(2)}_{inst}(g_s, t_1, t_2, V_1, V_2). \tag{34}
$$

The holonomy variables (U_1, V_1),(U_2, V_2) are associated to the unknots (Ξ_1, Γ_1), (Ξ_2, Γ_2) in L_1 and respectively L_2. Each pair of knots form a link with linking number one in the present choice of orientations. Moreover, (Ξ_1, Ξ_2) are canonically framed while (Γ_1, Γ_2) have framings $(2, 2)$. This completely specifies the Chern-Simons system. The first two terms in (32) are well understood. Performing an analytic continuation as in [13, 19, 20, 44], we can write them in the form

$$
\mathcal{F}_1^{CS}(\lambda_1, g_s) + \mathcal{F}_2^{CS}(\lambda_2, g_s) = \sum_{n=1}^{\infty} \frac{1}{n\left(2 \sin \frac{ng_s}{2}\right)^2} (e^{in\lambda_1} + e^{in\lambda_2}). \tag{35}
$$

The third term is more complicated. We will evaluate it perturbatively up to terms of order three in $e^{-t_1}, e^{-t_2}, e^{-t_c}$.

For a systematic approach, let us write the instanton corrections in the form

$$
\begin{aligned}
& F_{inst}(g_s, t_1, t_2, t_c, U_1, U_2, V_1, V_2) \\
& \qquad = \sum_{n=1}^{\infty} \left[ia_n e^{-nt_1} + ib_n e^{-nt_2} - c_n e^{-nt_c} + 2d_n e^{-n(t_1+t_2)} \right]
\end{aligned} \tag{36}
$$

where

$$\begin{array}{ll} a_n = \frac{1}{2n \sin \frac{n g_s}{2}} \mathrm{Tr} V_1^n, & b_n = \frac{1}{2n \sin \frac{n g_s}{2}} \mathrm{Tr} V_2^n \\ c_n = \frac{1}{n} \mathrm{Tr} U_1^n \mathrm{Tr} U_2^n, & d_n = \frac{1}{n} \mathrm{Tr} V_1^n \mathrm{Tr} V_2^n. \end{array} \tag{37}$$

Then the first order terms are

$$\mathcal{F}_{inst}(g_s, t_1, t_2, t_c, \lambda_1, \lambda_2)^{(1)} = e^{-t_1} X_{(t_1)} + e^{-t_2} X_{(t_2)} + e^{-t_c} X_{(t_c)}, \tag{38}$$

where we have introduced the notation

$$X_{(t_1)} = x_{(t_1)} = i\langle a_1 \rangle, \quad X_{(t_2)} = x_{(t_2)} = i\langle b_1 \rangle, \quad X_{(t_c)} = x_{(t_c)} = -\langle c_1 \rangle. \tag{39}$$

The expectation values in (39) can be evaluated using the Chern-Simons techniques developed in [32, 33, 34, 39, 45]. Some extra care is needed since we have to work with $U(N)$ Chern-Simons theory, and not $SU(N)$ [39]. Let us consider a link $\mathcal{L}$ with c components with representations R_α and framings p_α, $\alpha = 1, \dots, c$. The framing dependence of the expectation value $\langle W_{R_\alpha}(\mathcal{L}) \rangle$ is of the form

$$\langle W_{R_\alpha}(\mathcal{L}) \rangle_{(p_1, \dots, p_c)} = e^{\frac{i g_s}{2} \sum_{\alpha=1}^{c} \kappa_{R_\alpha} p_\alpha} e^{\frac{i\lambda}{2} \sum_{\alpha=1}^{c} l_\alpha p_\alpha} \langle W_{R_\alpha}(\mathcal{L}) \rangle_{(0, \dots, 0)} \tag{40}$$

where l_α is the total number of boxes in the Young tableau of R_α, and κ_{R_α} is a group theoretic quantity defined as follows. Let $v = 1, \dots r$ label the rows of the Young tableau of a representation R, and l_v denote the length of the v-th row. Then we have [39]

$$\kappa_R = l + \sum_{v=1}^{r} (l_v^2 - 2v l_v) \tag{41}$$

where $l = \sum_{v=1}^{r} l_v$ is the total number of boxes. Applying this formula, and taking into account the framings specified below (34), we have

$$\begin{aligned} \langle a_1 \rangle &= (-i) e^{i\lambda_1} \frac{e^{\frac{1}{2} i\lambda_1} - e^{-\frac{1}{2} i\lambda_1}}{\left(2 \sin \frac{g_s}{2}\right)^2} \\ \langle b_1 \rangle &= (-i) e^{i\lambda_2} \frac{e^{\frac{1}{2} i\lambda_2} - e^{-\frac{1}{2} i\lambda_2}}{\left(2 \sin \frac{g_s}{2}\right)^2} \\ \langle c_1 \rangle &= -\frac{(e^{\frac{1}{2} i\lambda_1} - e^{-\frac{1}{2} i\lambda_1})(e^{\frac{1}{2} i\lambda_2} - e^{-\frac{1}{2} i\lambda_2})}{\left(2 \sin \frac{g_s}{2}\right)^2}. \end{aligned} \tag{42}$$

By direct substitution in (36), we find

$$\begin{aligned} &\mathcal{F}_{inst}(g_s, t_1, t_2, t_c, \lambda_1, \lambda_2)^{(1)} \\ &= \frac{1}{\left(2 \sin \frac{g_s}{2}\right)^2} \Big[e^{-t_1} (-e^{\frac{1}{2} i\lambda_1} + e^{\frac{3}{2} i\lambda_1}) + e^{-t_2} (-e^{\frac{1}{2} i\lambda_2} + e^{\frac{3}{2} i\lambda_2}) \\ &\quad + e^{-t_c} (e^{\frac{1}{2} i(\lambda_1 + \lambda_2)} - e^{\frac{1}{2} i(-\lambda_1 + \lambda_2)} - e^{\frac{1}{2} i(\lambda_1 - \lambda_2)} + e^{-\frac{1}{2} i(\lambda_1 + \lambda_2)}) \Big]. \end{aligned} \tag{43}$$

Let us consider the second order terms in (33). By successively expanding the exponential and the logarithm, we obtain

$$\begin{aligned}\mathcal{F}_{inst}(g_s, t_1, t_2, t_c, \lambda_1, \lambda_2)^{(2)} = {} & e^{-2t_1} X_{(2t_1)} + e^{-2t_2} X_{(2t_2)} + e^{-2t_c} X_{(2t_c)} \\ & + e^{-t_1-t_2} X_{(t_1,t_2)} + e^{-t_1-t_c} X_{(t_1,t_c)} + e^{-t_2-t_c} X_{(t_2,t_c)},\end{aligned} \tag{44}$$

where

$$\begin{aligned} X_{(2t_i)} &= x_{(2t_i)} - \tfrac{1}{2} x_{(t_i)}^2, \qquad & i = 1, 2, c, \\ X_{(t_i,t_j)} &= x_{(t_i,t_j)} - x_{(t_i)} x_{(t_j)}, \qquad & i, j = 1, 2, c,\ i \neq j, \end{aligned} \tag{45}$$

with

$$\begin{aligned} x_{(2t_1)} &= i\langle a_2 \rangle - \tfrac{1}{2}\langle a_1^2 \rangle, \quad & x_{(2t_2)} &= i\langle b_2 \rangle - \tfrac{1}{2}\langle b_1^2 \rangle, \quad & x_{(2t_c)} &= -\langle c_2 \rangle + \tfrac{1}{2}\langle b_1^2 \rangle, \\ x_{(t_1,t_2)} &= -\langle a_1 b_1 \rangle + 2\langle d_1 \rangle, \quad & x_{(t_1,t_c)} &= -i\langle a_1 c_1 \rangle, \quad & x_{(t_2,t_c)} &= -i\langle b_1 c_1 \rangle. \end{aligned} \tag{46}$$

In order to compute the relevant expectation values, we have to use the Frobenius formula in order to linearize quadratic expressions in the holonomy variables. Since the formula (44) is symmetric under the exchange of Γ_1 and Γ_2, it suffices to consider only V_1

$$\begin{aligned} (\mathrm{Tr} V_1)^2 &= \mathrm{Tr}_{\square\!\square} V_1 + \mathrm{Tr}_{\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}} V_1 \\ \mathrm{Tr} V_1^2 &= \mathrm{Tr}_{\square\!\square} V_1 - \mathrm{Tr}_{\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}} V_1 \end{aligned} \tag{47}$$

where $\square\!\square$, $\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}$ are the Young tableaux corresponding to the symmetric and antisymmetric representations of $U(N_1)$, respectively. Applying (40) we have

$$\begin{aligned} \langle (\mathrm{Tr} V_1)^2 \rangle &= e^{2ig_s} e^{2i\lambda_1} \langle \mathrm{Tr}_{\square\!\square} V_1 \rangle_0 + e^{-2ig_s} e^{2i\lambda_1} \langle \mathrm{Tr}_{\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}} V_1 \rangle_0 \\ \langle \mathrm{Tr} V_1^2 \rangle &= e^{2ig_s} e^{2i\lambda_1} \langle \mathrm{Tr}_{\square\!\square} V_1 \rangle_0 - e^{-2ig_s} e^{2i\lambda_1} \langle \mathrm{Tr}_{\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}} V_1 \rangle_0 \end{aligned} \tag{48}$$

where the subscript zero means canonical framing. Now, the expectation value of $\mathrm{Tr}_R V_1$ for the unknot with the canonical framing is given by the quantum dimension of R with quantum parameter e^{ig_s} [39]. Therefore we have

$$\begin{aligned} \langle \mathrm{Tr} V_1 \rangle_0 &= \frac{e^{\frac{1}{2}i\lambda_1} - e^{-\frac{1}{2}i\lambda_1}}{e^{\frac{1}{2}ig_s\lambda} - e^{-\frac{1}{2}ig_s\lambda}} \\ \langle \mathrm{Tr}_{\square\!\square} V_1 \rangle_0 &= \frac{(e^{\frac{1}{2}i\lambda_1} - e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_1} e^{\frac{1}{2}ig_s} - e^{-\frac{1}{2}i\lambda_1} e^{-\frac{1}{2}ig_s})}{(e^{\frac{1}{2}ig_s} - e^{-\frac{1}{2}ig_s})(e^{ig_s} - e^{-ig_s})} \\ \langle \mathrm{Tr}_{\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}} V_1 \rangle_0 &= \frac{(e^{\frac{1}{2}i\lambda_1} - e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_1} e^{-\frac{1}{2}ig_s} - e^{-\frac{1}{2}i\lambda_1} e^{\frac{1}{2}ig_s})}{(e^{\frac{1}{2}ig_s} - e^{-\frac{1}{2}ig_s})(e^{ig_s} - e^{-ig_s})}. \end{aligned} \tag{49}$$

Taking also into account (37), after some elementary computations, we arrive at

$$X_{(2t_1)} = \frac{1}{\left(2\sin\frac{g_s}{2}\right)^2}(e^{2i\lambda_1} - e^{3i\lambda_1}) + \frac{1}{2(2\sin g_s)^2}(-e^{i\lambda_1} + e^{3i\lambda_1}). \quad (50)$$

The result for $X_{(2t_2)}$ can be obtained by substituting $t_1 \longrightarrow t_2$ and $\lambda_1 \longrightarrow \lambda_2$ in (50)

$$X_{(2t_2)} = \frac{1}{\left(2\sin\frac{g_s}{2}\right)^2}(e^{2i\lambda_2} - e^{3i\lambda_2}) + \frac{1}{2(2\sin g_s)^2}(-e^{i\lambda_2} + e^{3i\lambda_2}). \quad (51)$$

The term $X_{(3t_c)}$ can be evaluated analogously by linearizing the quadratic expressions in U_1, U_2. The computation is straightforward since the unknots Ξ_1, Ξ_2 are canonically framed. We obtain

$$X_{(2t_c)} = \frac{1}{2(2\sin g_s)^2}(e^{i\lambda_1} - e^{-i\lambda_1})(e^{i\lambda_2} - e^{-i\lambda_2}). \quad (52)$$

We now compute $X_{(t_1,t_c)}$ and $X_{(t_2,t_c)}$. These terms are more interesting since they involve expectation values of linked Wilson loops. We can exploit again the $\mathbb{Z}/2$ symmetry which exchanges the links (Γ_1, Ξ_1) and (Γ_2, Ξ_2). This means it suffices to consider

$$X_{(t_1,t_c)} = \frac{i}{2\sin\frac{g_s}{2}}\Big[- \langle \mathrm{Tr}V_1 \mathrm{Tr}U_1\rangle + \langle \mathrm{Tr}V_1\rangle\langle \mathrm{Tr}U_1\rangle\Big]\langle \mathrm{Tr}U_2\rangle. \quad (53)$$

As discussed at length in [33, 34], in order for this expression to have the correct integrality properties, the expectation value of the link $\langle \mathrm{Tr}V_1 \mathrm{Tr}U_1\rangle$ has to be taken with a particular normalization. For $U(N_1)$ Chern-Simons theory, this amounts to writing the expectation value of a Hopf link $\mathcal{L}$ with linking number -1 as [33, 34]

$$\langle W(\mathcal{L})\rangle_{(0,0)} = \left(\frac{e^{\frac{1}{2}i\lambda_1} - e^{-\frac{1}{2}i\lambda_1}}{e^{\frac{1}{2}ig_s} - e^{-\frac{1}{2}ig_s}}\right)^2 - (1 - e^{-i\lambda_1}) \quad (54)$$

where the subscript $(0,0)$ means that both components are in canonical framing. Up to normalization, this is the HOMFLY polynomial of the Hopf link. In our case, we have a link with linking number $+1$, which is in fact the mirror link $\mathcal{L}^*$ of $\mathcal{L}$. The expectation value of the mirror link $\mathcal{L}^*$ is related to that of $\mathcal{L}$ by sending $g_s \longrightarrow -g_s, \lambda \longrightarrow -\lambda$ [39]. Therefore, taking into account the framing correction, the expression we need in (53) is

$$\langle \mathrm{Tr}V_1 \mathrm{Tr}U_1\rangle = e^{i\lambda_1}\left[\left(\frac{e^{\frac{1}{2}i\lambda_1} - e^{-\frac{1}{2}i\lambda_1}}{e^{\frac{1}{2}ig_s} - e^{-\frac{1}{2}ig_s}}\right)^2 - (1 - e^{i\lambda_1})\right]. \quad (55)$$

Then, a direct computation yields

$$X_{(t_1,t_c)} = \frac{1}{\left(2\sin\frac{g_s}{2}\right)^2}(e^{i\lambda_1} - e^{2i\lambda_1})(e^{\frac{1}{2}i\lambda_2} - e^{-\frac{1}{2}i\lambda_2}). \quad (56)$$

Next, $X_{(t_2,t_c)}$ can be obtained using the exchange symmetry as before

$$X_{(t_2,t_c)} = \frac{1}{\left(2\sin\frac{g_s}{2}\right)^2}(e^{i\lambda_2} - e^{2i\lambda_2})(e^{\frac{1}{2}i\lambda_1} - e^{-\frac{1}{2}i\lambda_1}). \tag{57}$$

Finally, we have to evaluate $X_{(t_1,t_2)}$. Given what has been said so far, and using the observation that

$$-\langle a_1 b_1\rangle + \langle a_1\rangle\langle b_1\rangle = \frac{1}{2\sin\frac{g_s}{2}}\Big[\langle \mathrm{Tr}V_1 \mathrm{Tr}V_2\rangle - \langle \mathrm{Tr}V_1\rangle\langle \mathrm{Tr}V_2\rangle\Big] = 0, \tag{58}$$

a straightforward computation leads to

$$X_{(t_1,t_2)} = -\frac{2}{\left(2\sin\frac{g_s}{2}\right)^2}e^{i(\lambda_1+\lambda_2)}(e^{\frac{1}{2}i\lambda_1} - e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2} - e^{-\frac{1}{2}i\lambda_2}). \tag{59}$$

This concludes the evaluation of (44). The last step is to collect all the intermediate results (35),(43),(50)-(52) and (56)-(59) in a single formula. Writing only terms up to order two, we have

$$\begin{aligned}
&\mathcal{F}_{op}(t_1,t_2,t_c,\lambda_1,\lambda_2)\\
&= \frac{1}{\left(2\sin\frac{g_s}{2}\right)^2}\Big[e^{i\lambda_1} + e^{i\lambda_2} + e^{-t_1}(-e^{\frac{1}{2}i\lambda_1} + e^{\frac{3}{2}i\lambda_1}) + e^{-t_2}(-e^{\frac{1}{2}i\lambda_2} + e^{\frac{3}{2}i\lambda_2})\\
&+e^{-t_c}(e^{\frac{1}{2}i(\lambda_1+\lambda_2)} - e^{\frac{1}{2}i(-\lambda_1+\lambda_2)} - e^{\frac{1}{2}i(\lambda_1-\lambda_2)} + e^{-\frac{1}{2}i(\lambda_1+\lambda_2)}) + e^{-2t_1}(e^{2i\lambda_1} - e^{3i\lambda_1})\\
&+e^{-2t_2}(e^{2i\lambda_2} - e^{3i\lambda_2}) - 2e^{-t_1-t_2}e^{i(\lambda_1+\lambda_2)}(e^{\frac{1}{2}i\lambda_1} - e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2} - e^{-\frac{1}{2}i\lambda_2})\\
&+e^{-t_1-t_c}(e^{i\lambda_1} - e^{2i\lambda_1})(e^{\frac{1}{2}i\lambda_2} - e^{-\frac{1}{2}i\lambda_2}) + e^{-t_2-t_c}(e^{i\lambda_2} - e^{2i\lambda_2})(e^{\frac{1}{2}i\lambda_1} - e^{-\frac{1}{2}i\lambda_1})\Big]\\
&+\frac{1}{2(2\sin g_s)^2}\Big[e^{2i\lambda_1} + e^{2i\lambda_2} + e^{-2t_1}(-e^{i\lambda_1} + e^{3i\lambda_1}) + e^{-2t_2}(-e^{i\lambda_2} + e^{3i\lambda_2})\\
&+e^{-2t_c}(e^{i(\lambda_1+\lambda_2)} - e^{i(-\lambda_1+\lambda_2)} - e^{i(\lambda_1-\lambda_2)} + e^{-i(\lambda_1+\lambda_2)})\Big].
\end{aligned} \tag{60}$$

We have also computed the terms of degree three and four in Appendix B, but we will not reproduce them here.

4.2 Comparison with Duality Predictions

In order to test large N duality, the free energy (60) must be compared with its closed string counterpart. For convenience we write below the $r = 0$ (see (19)) closed string result of [10]

$$
\begin{aligned}
\mathcal{F}_{cl}^{(0)}(q,q_1,q_2,g_s) = {} & \frac{1}{\left(2\sin\frac{g_s}{2}\right)^2}\Big[q_1+q_2+q(q_1^{-1}q_2^{-1}-2q_1^{-1}-2q_2^{-1}+3)+q^2(-4q_1^{-1}q_2^{-1} \\
& +5q_1^{-1}+5q_2^{-1}-6)+q^3(-6q_1^{-2}q_2^{-1}-6q_1^{-1}q_2^{-2}+7q_1^{-2} \\
& +7q_2^{-2}+35q_1^{-1}q_2^{-1}-32q_1^{-1}-32q_2^{-1}+27)+q^4(-8q_1^{-3}q_2^{-1} \\
& -32q_1^{-2}q_2^{-2}-8q_1^{-1}q_2^{-3}+9q_1^{-3}+135q_1^{-2}q_2^{-1}+135q_1^{-1}q_2^{-2} \\
& +9q_2^{-3}-110q_1^{-2}-400q_1^{-1}q_2^{-1}-110q_2^{-2}+286q_1^{-1}+286q_2^{-1} \\
& -192)+\ldots\Big] \\
& +\frac{1}{2\left(2\sin g_s\right)^2}\Big[q_1^2+q_2^2+q^2(q_1^{-2}q_2^{-2}-2q_1^{-2}-2q_2^{-2}+3) \\
& +q^4(-4q_1^{-2}q_2^{-2}+5q_1^{-2}+5q_2^{-2}-6)+\ldots\Big] \\
& +\frac{1}{3\left(2\sin\frac{3g_s}{2}\right)^2}\Big[q_1^3+q_2^3+q^3(q_1^{-3}q_2^{-3}-2q_1^{-3}-2q_2^{-3}+3)+\ldots\Big] \\
& +\frac{1}{4\left(2\sin 2g_s\right)^2}\Big[q_1^4+q_2^4+q^4(q_1^{-4}q_2^{-4}-2q_1^{-4}-2q_2^{-4}+3)+\ldots\Big],
\end{aligned}
\tag{61}
$$

where we have introduced the notation $q=e^{-t}, q_1=e^{-s_1}, q_2=e^{-s_2}$. Note that we have written only terms up to degree 4 in q in (61).

As a first qualitative test, note that the terms in the open string expression (60) satisfy the integrality constraints of a closed string expansion [19]. Namely, the terms weighted by $\frac{1}{2(2\sin g_s)^2}$ have the structure of degree two multicover contributions of the terms of degree one. In order to perform a precise test, we have to find a duality map relating the open string parameters $(t_1,t_2,t_c,\lambda_1,\lambda_2)$ to the closed string parameters (t,s_1,s_2). We conjecture the following relations

$$
\begin{array}{lll}
\lambda_1 = is_1, & \lambda_2 = is_2 & \\
t_1 = t-\frac{3}{2}s_1, & t_2 = t-\frac{3}{2}s_2, & t_c = t-\frac{1}{2}(s_1+s_2).
\end{array}
\tag{62}
$$

Note that the flat coordinates t_1,t_2,t_c associated to the three primitive instantons receive indeed different quantum corrections, as anticipated in the paragraph preceding equation (26). This is quite sensible, given the interpretation of these corrections in terms of degenerate instantons proposed in [12].

Collecting (60) and the degree three and four terms computed in Appendix B, we can write the final expression for the free energy up to degree four

$$
\mathcal{F}_{op}(t_1,t_2,t_c,\lambda_1,\lambda_2)=\mathcal{F}_{op}^{(0)}+\mathcal{F}_{op}^{(1)}+\mathcal{F}_{op}^{(2)}+\mathcal{F}_{op}^{(3)},
\tag{63}
$$

where

$$\begin{aligned}
\mathcal{F}_{op}^{(0)} = \frac{1}{\left(2\sin\frac{g_s}{2}\right)^2}\Big[& e^{-s_1}+e^{-s_2}+e^{-t}(e^{s_1+s_2}-2e^{s_1}-2e^{s_2}+3)+e^{-2t}(-4e^{s_1+s_2}\\
&+5e^{s_1}+5e^{s_2}-6)+e^{-3t}(-6e^{2s_1+s_2}-6e^{s_1+2s_2}+7e^{2s_1}+7e^{2s_2}\\
&+35e^{s_1+s_2}-32e^{s_1}-32e^{s_2}+27)+e^{-4t}(-8e^{3s_1+s_2}-32e^{2s_1+2s_2}\\
&-8e^{s_1+3s_2}+9e^{3s_1}+135e^{2s_1+s_2}+135e^{s_1+2s_2}+9e^{3s_2}-110e^{2s_1}\\
&-400e^{s_1+s_2}-110e^{2s_2}+286e^{s_1}+286e^{s_2}-192)\Big]\\
+\frac{1}{2(2\sin g_s)^2}\Big[& e^{-2s_1}+e^{-2s_2}+e^{-2t}(e^{2s_1+2s_2}-2e^{2s_1}-2e^{2s_2}+3)\\
&+e^{-4t}(-4e^{2s_1+2s_2}+5e^{2s_1}+5e^{2s_2}-6)\Big]\\
+\frac{1}{3\left(2\sin\frac{3g_s}{2}\right)^2}\Big[& e^{-3s_1}+e^{-3s_2}+e^{-3t}(e^{3s_1+3s_2}-2e^{3s_1}-2e^{3s_2}+3)\Big]\\
+\frac{1}{4(2\sin 2g_s)^2}\Big[& e^{-4s_1}+e^{-4s_2}+e^{-4t}(e^{4s_1+4s_2}-2e^{4s_1}-2e^{4s_2}+3)\Big],
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_{op}^{(1)} &= e^{-3t}\big[-8e^{s_1+s_2}+9e^{s_1}+9e^{s_2}-10\big]+e^{-4t}\big[9e^{2s_1+2s_2}-72e^{2s_1+s_2}\\
&\quad -72e^{s_1+2s_2}+68e^{2s_1}+344e^{s_1+s_2}+68e^{2s_2}-288e^{s_1}-288e^{s_2}+231\big],\\
\mathcal{F}_{op}^{(2)} &= \left(2\sin\tfrac{g_s}{2}\right)^2 e^{-4t}\big[11e^{2s_1+s_2}+11e^{s_1+2s_2}-12e^{2s_1}-112e^{s_1+s_2}-12e^{2s_2}\\
&\quad +108e^{s_1}+108e^{s_2}-102\big],\\
\mathcal{F}_{op}^{(3)} &= \left(2\sin\tfrac{g_s}{2}\right)^4 e^{-4t}\big[13e^{s_1+s_2}-14e^{s_1}-14e^{s_2}+15\big].
\end{aligned} \tag{64}$$

Since $q=e^{-t}, q_1=e^{-s_1}, q_2=e^{-s_2}$, we see that all the terms in $\mathcal{F}_{op}^{(0)}$ are in perfect agreement with (61). This is a highly nontrivial test of our conjecture. The other three expressions contain terms that have the g_s dependence of GV invariants with $r=1$, $r=2$ and $r=3$ respectively and we interpret them as duality predictions. Note that some of these higher genus invariants can be computed using the methods of [30]. For example all degree three $r=1$ invariants can be computed this way obtaining perfect agreement with (64). The successful comparison obtained so far leads us to conjecture that the free energy of the Chern-Simons system considered in this section captures all genus topological amplitudes of the local dP_2 model.

It would be very interesting to test this conjecture at higher order and eventually develop a more conceptual understanding of this correspondence.

5 Localization of Open String Maps – Geometric Considerations

As discussed in the previous section, the instanton expansions (30) and (31) can be derived using the open string localization techniques developed in [12]. In this section we carry out the first part of this program, by finding the general structure of invariant open string maps $f : \Sigma_{g,h} \longrightarrow Y$ subject to the homology constraint $f_*[\Sigma_{g,h}] = d\widetilde{\beta}$, where $\widetilde{\beta}$ is an (yet undetermined) integral generator of $H_2(Y, L_1 \cup L_2; \mathbb{Z})$. The main result is that $\widetilde{\beta} = [D_1] = [D_2] = [C] = \beta$, where D_1, D_2, C are the holomorphic cycles constructed in section two. The sought maps are then multicovers of D_1, D_2, C, as promised in section two.

We first compactify the hypersurfaces Y_μ by taking a projective closure of the ambient variety Z. Recall that Z is isomorphic to the total space of $\mathcal{O}(-1)\oplus\mathcal{O}(-2)$ over $\mathbb{P}^2$, which can be represented as a toric variety

$$\begin{array}{cccccc} & Z_1 & Z_2 & Z_3 & U & V \\ \mathbb{C}^* & 1 & 1 & 1 & -1 & -2. \end{array} \tag{65}$$

The family $\mathcal{Y}/\Delta$ can be described as a family of hypersurfaces in Z determined by

$$Z_1U + VZ_2Z_3 = \mu. \tag{66}$$

Recall also that $Y_\mu \to \mathbb{P}^2$ is a fibration with non-compact fibers and no holomorphic section (nor multi-section).

The (relative) projective closure of Z is the compact toric variety $\overline{Z} = \left(\mathbb{C}^3 \setminus \{0\}\right) \times \left(\mathbb{C}^3 \setminus \{0\}\right) / (\mathbb{C}^*)^2$ determined by

$$\begin{array}{ccccccc} & Z_1 & Z_2 & Z_3 & U & V & W \\ \mathbb{C}^* & 1 & 1 & 1 & -1 & -2 & 0 \\ \mathbb{C}^* & 0 & 0 & 0 & 1 & 1 & 1. \end{array} \tag{67}$$

Note that $\overline{Z} \simeq \mathbb{P}\left(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2)\right)$ over $\mathbb{P}^2$. The Picard group of $\overline{Z}$ has rank two and is generated by the divisor classes

$$\zeta_1 : \quad Z_1 = 0, \qquad \zeta_2 : \quad W = 0. \tag{68}$$

By constructing the toric fan associated to (67), one can check that the Mori cone of $\overline{Z}$ is generated by the curve classes $\eta_1 = \zeta_1(\zeta_2 - \zeta_1)(\zeta_2 - 2\zeta_1)$, $\eta_2 = \zeta_1^2(\zeta_2 - 2\zeta_1)$.

One can also choose concrete representatives of the form

$$\eta_1 : \quad Z_1 = U = V = 0, \qquad \eta_2 : \quad Z_1 = Z_2 = V = 0. \tag{69}$$

The projective completion of the family (66) is a family $\overline{\mathcal{Y}}/\Delta$ of compact hypersurfaces in $\overline{Z}$ given by

$$Z_1 U + V Z_2 Z_3 = \mu W. \tag{70}$$

Let $\overline{Y}$ denote a generic fiber of this family (as noted before, we drop the subscript μ with the understanding that μ is fixed at some real positive value). We denote by $Y \overset{i}{\hookrightarrow} \overline{Y} \overset{j}{\hookrightarrow} \overline{Z}$ the obvious embedding maps. The induced fibration $\pi : \overline{Y} \to \mathbb{P}^2$ is a $\mathbb{P}^1$ fibration; the inverse image of a line L in $\mathbb{P}^2$ is a complex surface $\pi^{-1}(L) \simeq \mathbb{F}_1$. Moreover, $\overline{Y} \simeq \mathbb{P}\left(\mathcal{O}(-1) \oplus \mathcal{O}(-2)\right)$, and $\overline{Y}$ can be identified with $\mathbb{P}^3$ blown up at a point. It follows that $H_2(\overline{Y}, \mathbb{Z}) \simeq H^2(\overline{Y}, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}$. As generators of $H^2(\overline{Y}, \mathbb{Z})$ we can take the section σ defined by $V = 0$, and $\pi^*(L)$ (defined, say, by $Z_1 = 0$). The second homology of $\overline{Y}$ is similarly generated by algebraic cycles. For example, one can choose a basis consisting of the curve classes $\eta_2, \eta_1 + \eta_2$ restricted to $\overline{Y}$. In particular this shows that the pushforward map $j_* : H_2(\overline{Y}, \mathbb{Z}) \longrightarrow H_2(\overline{Z}, \mathbb{Z})$ is an isomorphism.

The divisor at infinity in $\overline{Y}$ is defined as the pull back of the Cartier divisor $W = 0$ on $\overline{Z}$: $\zeta_\infty = \overline{Y} \backslash Y$. Note that $\zeta_\infty \in |\sigma + \pi^*(L)|$, which is an ample divisor on $\overline{Y}$. In particular any holomorphic curve on $\overline{Y}$ must intersect ζ_∞, and Y contains no holomorphic curve classes, as mentioned in section two.

From this point, our analysis follows [12]. We restrict our considerations to open string morphisms which are fixed points of a certain torus action. Recall that there is a natural S^1 action (25) on Z, which can be extended such that it preserves ζ_∞[3]:

$$\begin{array}{cccccc} Z_1 & Z_2 & Z_3 & U & V & W \\ \lambda_1 & \lambda_2 & 0 & -\lambda_1 & -\lambda_2 & 0. \end{array} \tag{71}$$

As in [12], we restrict our search to T-invariant open string maps to the pair $(\overline{Y}, L_1 \cap L_2)$ subject to the homology constraint $f_*[\Sigma_{g,h}] = d\widetilde{\beta}$. There is a subtlety here, since one might think it suffices to consider open string maps to Y instead of $\overline{Y}$. The problem is related to the compactification of $M_{g,h}(Y, d\widetilde{\beta})$. Since Y is not compact, the correct point of view is to consider open string maps to $(\overline{Y}, L_1 \cup L_2)$ subject to certain contact conditions along ζ_∞. In our case, the order of contact should be zero, but we cannot automatically exclude eventual compactification effects. Therefore we will consider open string morphisms to $\overline{Y}$, and in the end show that such effects are ruled out by homology constraints.

[3]This is not the most general action, but it suffices for localization purposes on $\overline{Y}$.

According to [22, 29], the domain of an S^1-invariant map $f : \Sigma_{g,h} \longrightarrow \overline{Y}$ has to be a nodal bordered Riemann surface whose irreducible components are either closed compact Riemann surfaces or holomorphic discs. In the present situation there is an extra possibility, namely the domain can also be an annulus. Therefore, we can divide the problem into two parts. The compact components of $\Sigma_{g,h}$ have to be mapped to invariant closed curves on $\overline{Y}$, which can be found using simple toric considerations. The second part reduces to finding T-invariant maps $f : \Sigma_{0,1} \longrightarrow \overline{Y}$ and $f : \Sigma_{0,2} \longrightarrow \overline{Y}$ with lagrangian boundary conditions along L_1 and L_2. As in [12], we can show that any such f can be extended to a T-invariant map $f : \Sigma_0 \longrightarrow \overline{Y}$ from a smooth rational curve to $\overline{Y}$. Proceeding as in section five of [12], we see that the only curves on $\overline{Y}$ satisfying these conditions are

$$d_1 : \qquad Z_1 U = \mu W, \qquad Z_3 = V = 0, \tag{72}$$

$$d_2 : \qquad Z_1 U = \mu W, \qquad Z_2 = V = 0, \tag{73}$$

$$c : \qquad Z_2 Z_3 V = \mu W, \qquad Z_1 = U = 0. \tag{74}$$

By writing (72) (resp. (73)) in local coordinates, it follows that d_1 (resp. d_2) intersects L_1 (resp. L_2) along Γ_1 (Γ_2) which divides it into two discs D_1 (resp. D_2) and D_1' (D_2') with boundary on L_1 (L_2). Similarly c intersects both L_1 and L_2 in Ξ_1 and Ξ_2, which divide c in the cylinder C and two other discs D_3 and D_4.

For future reference, let us note at this point that there is a family of degree two curves on $\overline{Y}$ which induces similarly a family of degree two holomorphic annuli on Y with boundaries on L_1, L_2. In terms of homogeneous coordinates, this family is described by

$$\rho Z_1^2 - \eta Z_2 Z_3 = 0, \qquad \overline{\rho} U^2 - \overline{\eta} V = 0, \qquad U Z_1 + V Z_2 Z_3 = \mu W \tag{75}$$

where (ρ, η) are projective moduli. By writing these equations in local coordinates, one can check that (75) indeed intersects the two spheres L_1, L_2 along two circles. Moreover, if $(\rho, \eta) = (1, 0)$, (75) describes the cylinder C with multiplicity 2 whereas if $(\rho, \eta) = (0, 1)$ we obtain the nodal cylinder $D_1 \cup D_2$ with multiplicity 1. Therefore $2C$ and $D_1 \cup D_2$ belong to the same moduli space of bordered Riemann surfaces in Y with boundaries on L_1, L_2.

Returning to our problem, we next show as in [12] that the discs D_1', D_2', D_3 and D_4 are in fact ruled out by the homology constraint $f_*[\Sigma_{g,h}] = d[D]$ by computing the homology classes in $H_2(\overline{Y}, L_1 \cap L_2; \mathbb{Z}) \simeq H_2(\overline{Y}, \mathbb{Z})$. We have a commutative diagram of homology groups

$$\begin{tikzcd} H_2(\overline{Y}, L; \mathbb{Z}) \arrow[r, "j_*"] & H_2(\overline{Z}, L; \mathbb{Z}) \\ H_2(\overline{Y}, \mathbb{Z}) \arrow[u] \arrow[r, "j_*"] & H_2(\overline{Z}, \mathbb{Z}) \arrow[u] \end{tikzcd} \tag{76}$$

in which all the four arrows are isomorphisms. Therefore we can reduce the problem to computing the homology classes of the discs in question as relative cycles for $\overline{Z}$ rather than $(\overline{Y}, L)$. In fact L_1 and L_2 are fillable in $\overline{Z}$; this is realized geometrically by deforming to $\mu = 0$. In this limit, the discs become holomorphic curves on the singular fiber $\overline{Y}_0$ whose homology classes in $H_2(\overline{Z}, \mathbb{Z})$ can be easily determined from the algebraic equations. Note that the singularities of $\overline{Y}_0$ have no effect on this computation. In the limit $\mu = 0$, d_1 specializes to a reducible curve with components $Z_1 = U = V = 0$, $V = Z_1 = Z_3 = 0$ (class of a fiber) and $V = Z_1 = Z_3 = 0$, which are precisely the generators η_1 and η_2 of $H_2(\overline{Z}, \mathbb{Z})$. The two discs D_1', D_1 are deformed in this limit to these two components of d_1; therefore we find

$$j_*[D_1] = \eta_1, \qquad j_*[D_1'] = \eta_2. \tag{77}$$

By a similar reasoning we also find

$$j_*[D_2] = j_*[C] = \eta_1, \qquad j_*[D_2'] = j_*[D_3] = j_*[D_4] = \eta_2. \tag{78}$$

Since η_1, η_2 are generators of the Mori cone of $\overline{Z}$, using the commutative diagram (76), we conclude that $[D_1] = [D_2] = [C] = \widetilde{\beta}$ are integral generators of $H_2(Y, L_1 \cup L_2; \mathbb{Z})$. Therefore we can identify $\widetilde{\beta} = \beta$ from now on.

Now we can complete the description of a general S^1-invariant map $f : \Sigma_{g,h} \longrightarrow \overline{Y}$ subject to the homology constraint $f_*[\Sigma_{g,h}] = d\beta$. If the domain of the map is an annulus $\Sigma_{0,2}$, the above analysis shows that $f : \Sigma_{0,2} \longrightarrow \overline{Y}$ has to be an invariant $d:1$ cover of C. If the domain is a nodal bordered Riemann surface, the disc components have to be mapped either to D_1 or D_2 in an invariant manner. This leaves us with question of finding the images of the closed compact components of $\Sigma_{g,h}$. This is a simple exercise, taking into account the homology constraints. Note that $j_*\beta$ is identified with η_1 under the isomorphism $H_2(\overline{Z}, L_1 \cup L_2; \mathbb{Z}) \simeq H_2(\overline{Z}, \mathbb{Z})$. Since η_1 is a generator of the Mori cone of $\overline{Z}$, it follows that any closed component of $\Sigma_{g,h}$ has to be mapped to an invariant curve in the class η_1 lying on $\overline{Y}$. Moreover, since the image of f must be connected, this curve would have to pass through the common origin of D_1, D_2 i.e. the point $P: \ Z_2 = Z_3 = U = V = 0$. It is easy to check that the only invariant curves on $\overline{Y}$ passing through this point are in the fiber class η_2, which is a distinct generator of the Mori cone of $\overline{Z}$. Therefore we conclude that there are no curves on $\overline{Y}$ satisfying all the required conditions; all closed components of $\Sigma_{g,h}$ have to be mapped to the point P. This completes the description of the fixed loci. In the next section we will evaluate the contributions all these fixed points using localization techniques.

6 Localization of Open String Maps – Explicit Computations

In this section we conclude the localization computation for open string maps with an explicit evaluation of the contributions of all fixed loci. We will be using the tangent–obstruction complex technique [7, 17, 31, 37] generalized to open string morphisms in [22, 29]. Based on the results obtained so far, the fixed locus of the induced toric action on $\overline{M}_{g,h}(Y, L, d\beta)$ consists of two disconnected components. The first component consists of invariant maps to the annulus C, while the second component consists of invariant maps to the pinched cylinder $D_1 \cup D_2$. Given the structure of the fixed locus in the target space Y, there are no other components. Let us evaluate their contributions.

6.1 Multicovers of C

Recall that the annulus C is defined by an embedding $f : \Sigma_{0,2} \longrightarrow Y$ given locally by

$$\begin{array}{lllll} U_2: & x_2(t) = 0, & y_2(t) = t, & u_2(t) = 0, & v_2(t) = \frac{\mu}{t} \\ U_3: & x_3(t') = 0, & y_3(t') = t', & u_3(t') = 0, & v_3(t') = \frac{\mu}{t'}. \end{array} \tag{79}$$

Moreover, the boundary conditions at the two ends are imposed by the local equations of L_1, L_2, which are

$$\begin{array}{lll} L_1: & u_2 = \overline{x}_2, & v_2 = \overline{y}_2 \\ L_2: & u_3 = \overline{x}_3, & v_3 = \overline{y}_3. \end{array} \tag{80}$$

Up to reparameterizations, there is a single invariant multicover of C of degree d, which is the Galois cover. In local coordinates we have $f : \Sigma_{0,2} \longrightarrow Y$ given by[4]

$$\begin{array}{lllll} U_2: & x_2(t) = 0, & y_2(t) = t^d, & u_2(t) = 0, & v_2(t) = \frac{\mu}{t^d} \\ U_3: & x_3(t') = 0, & y_3(t') = t'^d, & u_3(t') = 0, & v_3(t') = \frac{\mu}{t'^d}. \end{array} \tag{81}$$

where t, t' are local coordinates on $\Sigma_{0,2} = \{\mu^{1/2d} \leq |t| \leq \mu^{-1/2d}\}$. This means that the only nontrivial coefficients in (28) are $C_{0,1,1}(d; d, d)$, the rest being zero. These coefficients can be computed by evaluating the contributions of the fixed loci in the moduli space of stable open string maps along the lines of [22, 29].

[4]In order to keep the notation simple, we will denote all open string maps generically by f. The meaning should be clear from the context.

Let us denote $\Sigma_{0,2}$ by Σ in order to simplify the formulas. Note that the automorphism group of this map has an automorphism group of order d generated by $(t \longrightarrow \zeta t, t' \longrightarrow \zeta^{-1} t')$ where ζ is a d-th root of unity. In local coordinates, the torus action (25) reads

$$\begin{array}{lllll} U_2: & x_2 \longrightarrow e^{i\phi(\lambda_1-\lambda_2)} x_2, & y_2 \longrightarrow e^{-i\phi\lambda_2} y_2, & u_2 \longrightarrow e^{i\phi(-\lambda_1+\lambda_2)} u_2, & v_2 \longrightarrow e^{i\phi\lambda_2} v_2 \\ U_3: & x_3 \longrightarrow e^{i\phi\lambda_1} x_3, & y_3 \longrightarrow e^{i\phi\lambda_2} y_3, & u_3 \longrightarrow e^{-i\lambda_1} u_3, & v_3 \longrightarrow e^{-i\phi\lambda_2} v_3. \end{array} \tag{82}$$

The map (81) is left invariant if we let S^1 act on Σ by

$$t \longrightarrow e^{-i\phi\lambda_2/d} t, \qquad t' \longrightarrow e^{i\phi\lambda_2/d} t'. \tag{83}$$

In order to write down the tangent-obstruction complex for the above map we need to introduce some more notation. Note that the tangent bundles $T_{L_{1,2}}$ of the lagrangian cycles L_1, L_2 are real subbundles of the holomorphic tangent bundle T_Y restricted to L_1 and respectively L_2. Pulling back to Σ we obtain a triple[5] $(f^*(T_Y), f_\partial^*(T_{L_1}), f_\partial^*(T_{L_2}))$ which defines a Riemann-Hilbert bundle on Σ. We let $\mathcal{T}_Y$ denote the sheaf of germs of holomorphic sections of $f^*(T_Y)$ with boundary values in $(f_\partial^*(T_{L_1}), f_\partial^*(T_{L_2}))$. Similarly, the holomorphic tangent bundle of Σ together with the tangent bundles to the two boundary components form a Riemann-Hilbert bundle whose associated sheaf of holomorphic sections will be denoted by $\mathcal{T}_\Sigma$. Then the tangent-obstruction complex of f takes the form

$$0 \longrightarrow H^0(\Sigma, \mathcal{T}_\Sigma) \longrightarrow H^0(\Sigma, \mathcal{T}_Y) \longrightarrow \mathbf{T}^1 \longrightarrow H^1(\Sigma, \mathcal{T}_\Sigma) \longrightarrow H^1(\Sigma, \mathcal{T}_Y) \longrightarrow \mathbf{T}^2 \longrightarrow 0. \tag{84}$$

The torus actions (82), (83) induce an action on all terms in (84), so that we obtain a complex of equivariant vector spaces. The contribution of the fixed point (81) is given by

$$\frac{1}{|Aut(f)|} \int_{pt_{S^1}} \frac{e(\mathbf{T}^2)}{e(\mathbf{T}^1)} = \frac{1}{d} \int_{pt_{S^1}} \frac{e(B_5^m) e(B_1^m)}{e(B_4^m) e(B_2^m)} \tag{85}$$

where B_i^m denotes the moving part of the i-th term in the complex (84) and $e(\ )$ is the S^1- equivariant Euler class. The integration is S^1-equivariant integration over the fixed point locus, which is a point.

In order to evaluate (85) we have to compute the relevant cohomology groups and take the moving parts [17]. We will do this computation in Čech cohomology, as in [38]. The cylinder Σ is covered by two open sets

$$\mathcal{U}_1 = \{\mu^{1/2d} < t \leq \mu^{-1/2d}\}, \qquad \mathcal{U}_2 = \{\mu^{1/2d} < t' \leq \mu^{-1/2d}\}. \tag{86}$$

[5] We denote by f_∂ the restriction of $f: \Sigma \longrightarrow Y$ to the boundary of Σ.

The Čech complex we are interested in is

$$0 \longrightarrow \mathcal{T}_Y(\mathcal{U}_1) \oplus \mathcal{T}_Y(\mathcal{U}_2) \xrightarrow{\kappa} \mathcal{T}_Y(\mathcal{U}_{12}) \longrightarrow 0 \tag{87}$$

where $\kappa(s_1, s_2) = s_1|_{\mathcal{U}_{12}} - s_2|_{\mathcal{U}_{12}}$. The local sections of $\mathcal{T}_Y$ over $\mathcal{U}_1$, $\mathcal{U}_2$ have the form

$$\begin{gathered} s_1 = \left(\sum_{n\in\mathbb{Z}} \alpha_n t^n\right) \partial_{x_2} + \left(\sum_{n\in\mathbb{Z}} \beta_n t^n\right) \partial_{y_2} + \left(\sum_{n\in\mathbb{Z}} \gamma_n t^n\right) \partial_{u_2} + \left(\mu \sum_{n\in\mathbb{Z}} \delta_n t^n\right) \partial_{v_2} \\ s_2 = \left(\sum_{n\in\mathbb{Z}} \alpha'_n t'^n\right) \partial_{x_3} + \left(\sum_{n\in\mathbb{Z}} \beta'_n t'^n\right) \partial_{y_3} + \left(\sum_{n\in\mathbb{Z}} \gamma'_n t'^n\right) \partial_{u_3} + \left(\mu \sum_{n\in\mathbb{Z}} \delta'_n t'^n\right) \partial_{v_3} \end{gathered} \tag{88}$$

where the coefficients $\alpha_n, \beta_n, \gamma_n, \delta_n, \alpha'_n, \beta'_n, \gamma'_n, \delta'_n$ are subject to two types of constraints. We have constraints imposed by the boundary conditions (80) which can be written in the form

$$\sum_{n\in\mathbb{Z}} \alpha_n r^n e^{in\theta} = \sum_{n\in\mathbb{Z}} \overline{\gamma}_n r^n e^{-in\theta}, \qquad \sum_{n\in\mathbb{Z}} \beta_n r^n e^{in\theta} = \mu \sum_{n\in\mathbb{Z}} \overline{\delta}_n r^n e^{-in\theta} \tag{89}$$

and identical conditions for primed coefficients. At the same time, one must impose the condition that the infinitesimal deformations parameterized by (88) be tangent to Y. Recall that the local equation of Y in U_2 is

$$x_2 u_2 + y_2 v_2 = \mu. \tag{90}$$

Therefore the sheaf of holomorphic differentials on $Y \cap U_2$ is generated by (dx_2, dy_2, du_2, dv_2) subject to the condition

$$x_2 du_2 + u_2 dx_2 + y_2 dv_2 + v_2 dy_2 = 0. \tag{91}$$

By contracting (91) with (88) and taking into account the fact that $x_2 = u_2 = 0$ along C, we find

$$\left(\sum_{n\in\mathbb{Z}} \beta_n t^n\right) + t^{2d} \left(\sum_{n\in\mathbb{Z}} \delta_n t^n\right) = 0. \tag{92}$$

The conditions for local sections over $\mathcal{U}_2$ are very similar. In terms of coefficients, (89) and (92) yield

$$\alpha_n = \mu^{-n/d} \overline{\gamma}_{-n}, \qquad \beta_n = \mu^{(d-n)/d} \overline{\delta}_{-n}, \qquad \beta_n + \delta_{n-2d} = 0 \tag{93}$$

and, similarly,

$$\alpha'_n = \mu^{-n/d} \overline{\gamma}'_{-n}, \qquad \beta'_n = \mu^{(d-n)/d} \overline{\delta}'_{-n}, \qquad \beta'_n + \delta'_{n-2d} = 0. \tag{94}$$

In order to be able to compare the two sections upon restriction, we have to write say s_2 in terms of $t = 1/t'$ and $\partial_{x_2}, \ldots, \partial_{v_2}$. The relevant linear transformations are

$$\begin{aligned} \partial_{x_3} &= y_2 \partial_{x_2}, & \partial_{y_3} &= -x_2 y_2 \partial_{x_2} - y_2^2 \partial_{y_2} + y_2 u_2 \partial_{u_2} + 2 y_2 v_2 \partial_{v_2}, \\ \partial_{u_3} &= \frac{1}{y_2} \partial_{u_2}, & \partial_{v_3} &= \frac{1}{{y_2}^2} \partial_{v_2}. \end{aligned} \tag{95}$$

Using these transformations, a straightforward computation shows that the map κ is given by

$$\begin{aligned}\kappa(s_1,s_2) = \quad & \left(\textstyle\sum_{n\in\mathbb{Z}}(\alpha_n-\alpha'_{-n+d})t^n\right)\partial_{x_2}+\left(\textstyle\sum_{n\in\mathbb{Z}}(\beta_n+\beta'_{-n+2d})t^n\right)\partial_{y_2}\\ &+\left(\textstyle\sum_{n\in\mathbb{Z}}(\gamma_n-\gamma'_{-n-d})t^n\right)\partial_{u_2}+\left(\mu\textstyle\sum_{n\in\mathbb{Z}}(\delta_n+\delta'_{-n-2d})t^n\right)\partial_{v_2}.\end{aligned} \tag{96}$$

Let us determine the kernel and cokernel of this map. For this, let us consider the following system of equations

$$\begin{aligned}\alpha_n-\alpha'_{-n+d}&=a_n, & \beta_n+\beta'_{-n+2d}&=b_n\\ \gamma_n-\gamma'_{-n-d}&=c_n, & \delta_n+\delta'_{-n-2d}&=d_n\end{aligned} \tag{97}$$

where a_n,b_n,c_n,d_n are coefficients of a generic section of $\mathcal{T}_Y(\mathcal{U}_{12})$. Hence we have

$$b_n+d_{n-2d}=0. \tag{98}$$

Using the constraints (93) and (94) we can rewrite (97) in the following form

$$\begin{aligned}\alpha_n-\alpha'_{-n+d}&=a_n, & \beta_n+\beta'_{-n+2d}&=b_n\\ \mu^{n/d}\alpha_n-\mu^{(d-n)/d}\alpha'_{-n+d}&=\overline{c}_{-n}, & \mu^{(n-d)/d}\beta_n+\mu^{(d-n)/d}\beta'_{-n+2d}&=\overline{d}_{-n}.\end{aligned} \tag{99}$$

In order to find the kernel of κ we have to set all $a_n,\ldots,d_n$ to zero, and solve the resulting homogeneous equations. Then it is easy to check that there are no nonzero solutions except for the following cases

$$\begin{aligned}2n&=d, & \alpha_{d/2}&=\alpha'_{d/2}\\ n&=d, & \beta_d&=-\beta'_d.\end{aligned} \tag{100}$$

In all other cases, the solutions are identically zero. Note that the first case cannot be realized unless d is even. Therefore we find that $\mathrm{Ker}(\kappa)\simeq H^0(\Sigma,\mathcal{T}_Y)$ is generated by

$$\begin{aligned}&\alpha_{d/2}t^{d/2}\partial_{x_2}+\beta_d t^d\partial_{y_2}+\mu^{1/2}\overline{\alpha}_{d/2}t^{-d/2}\partial_{u_2}+\mu\overline{\beta}_d t^{-d}\partial_{v_2}, &&\text{for } d \text{ even}\\ &\beta_d t^d\partial_{y_2}+\mu\overline{\beta}_d t^{-d}\partial_{v_2}, &&\text{for } d \text{ odd}\end{aligned} \tag{101}$$

where $\beta_d+\overline{\beta}_d=0$. We can similarly compute the cokernel of κ. The equations (99) have solutions for any values of $a_n,\ldots,d_n$ subject to the constraints

$$\overline{c}_{-d/2}=\mu^{1/2}a_{d/2},\qquad \overline{d}_{-d}=b_d,\qquad b_d+\overline{b}_d=0. \tag{102}$$

If these constraints are not satisfied, the equations (99) have no solutions.
Taking also into account (98), it follows that $\mathrm{Coker}(\kappa)\simeq H^1(\Sigma,\mathcal{T}_Y)$ is generated by sections of the form

$$\begin{aligned}&a_{d/2}t^{d/2}\partial_{x_2}+b_d t^d\partial_{y_2}, &&\text{for } d \text{ even}\\ &b_d t^d\partial_{y_2}, &&\text{for } d \text{ odd}\end{aligned} \tag{103}$$

where $b_d - \bar{b}_d = 0$. Note that for d even we obtain one extra deformation and one extra obstruction compared to d odd. This reflects the fact that there is a family of holomorphic cylinders of degree two on Y interpolating between $2C$ and $D_1 \cup D_2$ as shown in (75). This means that for d even the invariant map (81) can be deformed to a $d/2:1$ cover of the holomorphic cylinders described in (75).

In order to finish the computation of the integral in (85), we have to determine the moving parts of $H^0(\Sigma, \mathcal{T}_Y)$ and $H^1(\Sigma, \mathcal{T}_Y)$. The action of S^1 on $H^0(\Sigma, \mathcal{T}_Y)$ is determined by (1) and the explicit form of the generators (101), (103). We adopt the notation conventions of [29] for representations of S^1, namely the representation $z \longrightarrow e^{iw\phi}z$, where $z \in \mathbb{C}$, will be denoted by (w). The trivial real representation will be denoted by $(0)_{\mathbb{R}}$. One can define an S^1-equivariant isomorphism $H^0(\Sigma, \mathcal{T}_Y) \longrightarrow H^1(\Sigma, \mathcal{T}_Y)$ by

$$\begin{gathered}\alpha_{d/2}t^{d/2}\partial_{x_2} + \beta_d t^d \partial_{y_2} + \mu^{1/2}\overline{\alpha}_{d/2}t^{-d/2}\partial_{u_2} + \mu\overline{\beta}_d t^{-d}\partial_{v_2} \longrightarrow \alpha_{d/2}t^{d/2}\partial_{x_2} + \beta_d t^d \partial_{y_2} \\ \beta_d t^d \partial_{y_2} + \mu\overline{\beta}_d t^{-d}\partial_{v_2} \longrightarrow i\beta_d t^d \partial_{y_2},\end{gathered} \tag{104}$$

for d even and respectively odd. This shows that both $H^0(\Sigma, \mathcal{T}_Y)$ and $H^1(\Sigma, \mathcal{T}_Y)$ are isomorphic to

$$\begin{array}{ll}\left(\frac{\lambda_2}{2} - \lambda_1\right) \oplus (0)_{\mathbb{R}}, & \text{for } d \text{ even} \\ (0)_{\mathbb{R}}, & \text{for } d \text{ odd.}\end{array} \tag{105}$$

Therefore we have

$$\frac{e(B_5^m)}{e(B_2^m)} = 1 \tag{106}$$

in (85). A similar computation shows that B_1^m and B_4^m are trivial, hence

$$\frac{e(B_1^m)}{e(B_4^m)} = 1. \tag{107}$$

We conclude that contribution of the fixed point (81) to the open string topological amplitudes is simply

$$C_{0,1,1}(d;d,d) = \frac{1}{d}. \tag{108}$$

6.2 Multicovers of $D_1 \cup D_2$

We now perform a similar computation for the components of the fixed locus which map to the pinched cylinder $D_1 \cup D_2$. In this case the structure of S^1 invariant maps $f : \Sigma_{g,h} \longrightarrow Y$ is more complicated [22, 29, 38]. Recall that the two discs are defined locally by (13) and (14). For convenience let us reproduce the relevant formulas below. The common domain of the maps $f_{1,2} : \Sigma_{0,1} \longrightarrow Y$ is the disc $\{|t| \leq \mu^{-1/2}\} = \{\mu^{1/2} \leq |t'|\}$ in $\mathbb{P}^1$ with affine coordinates (t, t'). The local form of the holomorphic embeddings is

$$\begin{array}{lllll} U_1 : & x_1(t) = t, & y_1(t) = 0, & u_1(t) = \mu, & v_1(t) = 0 \\ U_2 : & x_2(t') = t', & y_2(t') = 0, & u_2(t') = \frac{\mu}{t'}, & v_2(t') = 0. \end{array} \tag{109}$$

and respectively

$$\begin{array}{lllll} U_1 : & x_1(t) = 0, & y_1(t) = t, & u_1(t) = \mu, & v_1(t) = 0 \\ U_3 : & x_3(t') = t', & y_3(t') = 0, & u_3(t') = \frac{\mu}{t'}, & v_3(t') = 0. \end{array} \tag{110}$$

We will also need the defining equations for the two spheres in all coordinate patches. For L_1 we have

$$\begin{array}{rcc} U_1 : & u_1 x_1 \overline{x}_1 = 1, & v_1 x_1^2 \overline{x}_1 = \overline{y}_1 \\ U_2 : & u_2 = \overline{x}_2, & v_2 = \overline{y}_2 \end{array} \tag{111}$$

and for L_2

$$\begin{array}{rcc} U_1 : & u_1 y_1 \overline{y}_1 = 1, & v_1 y_1^2 \overline{y}_1 = \overline{x}_1 \\ U_3 : & u_3 = \overline{x}_3, & v_3 = \overline{y}_3. \end{array} \tag{112}$$

Recall also that the open string maps are labelled by three integers (g, h_1, h_2), where h_1, h_2 denote the number of holes mapped to $\Gamma_1 = \partial D_1$ and respectively $\Gamma_2 = \partial D_2$. The homotopy class of such a map is determined by the degrees d_1, d_2 and the winding numbers m_i, $i = 1, \ldots, h_1$, n_j, $j = 1, \ldots, h_2$ subject to the constraints $\sum_{i=1}^{h_1} m_i = d_1$, $\sum_{j=1}^{h_2} n_j = d_2$. In order to describe the structure of the corresponding fixed loci, we need to distinguish several cases.

i) $(g, h_1, h_2) = (0, 1, 0)$. The invariant maps are Galois covers $f : \Delta^1 \longrightarrow D_1$ of degree $m = d_1$ with automorphism group $\mathbb{Z}/d_1$.

ii) $(g, h_1, h_2) = (0, 0, 1)$. The invariant maps are Galois covers $f : \Delta^2 \longrightarrow D_2$ of degree $n = d_2$ with automorphism group $\mathbb{Z}/d_2$.

iii) $(g, h_1, h_2) = (0, 2, 0)$. The domain of an invariant map has to be a pinched cylinder $\Delta_1^1 \cup \Delta_2^1$. The components are mapped as Galois covers of degrees m_1, m_2 to D_1. In this case the automorphism group is a product of $\mathbb{Z}/m_1 \times \mathbb{Z}/m_2$ with a permutation group $\mathbf{P}$ which is trivial in $m_1 \neq m_2$ and it permutes the two boundary components if $m_1 = m_2$. $|Aut(f)| = |\mathbf{P}| m_1 m_2$ where $|\mathbf{P}| = 1$ if $m_1 \neq m_2$, and $|\mathbf{P}| = 2$ if $m_1 = m_2$.

iv) $(g,h_1,h_2) = (0,0,2)$. The domain is similarly a pinched cylinder $\Delta_1^2 \cup \Delta_2^2$ which are again mapped as Galois covers of degrees (n_1,n_2) to D_2. By analogy with the previous case we have $|Aut(f)| = |\mathbf{P}|n_1n_2$, where $\mathbf{P}$ is defined similarly.

v) $(g,h_1,h_2) = (0,1,1)$. The domain is of the form $\Delta_1^1 \cup \Delta_1^2$, and the two components are mapped to D_1, D_2 as Galois covers of degrees $(m_1,n_1) = (d_1,d_2)$. The automorphism group is simply $\mathbb{Z}/m_1 \times \mathbb{Z}/n_1$ since permuting the two boundary components does not preserve f in this case. Therefore $|Aut(f)| = m_1n_1$.

vi) In all other cases, the domain of the map must be of the form $\Sigma_g^0 \cup \Delta_1^1 \cup \ldots \cup \Delta_{h_1}^1 \cup \Delta_1^2 \cup \ldots \cup \Delta_{h_2}^2$ where $(\Sigma_g^0, p_1, \ldots, p_{h_1}, q_1, \ldots, q_{h_2})$ is a stable h-punctured curve of genus g and $\Delta_1^1, \ldots, \Delta_{h_1}^1, \Delta_1^2, \ldots, \Delta_{h_2}^2$ are holomorphic discs whose origins are attached to Σ_g at the punctures $(p_1, \ldots, p_{h_1})$ and respectively $(q_1, \ldots, q_{h_2})$. The map f sends Σ_g^0 to the point P which is the common origin of the discs D_1, D_2 in Y, and maps the discs $\Delta_1^1, \ldots, \Delta_{h_1}^1$, $\Delta_1^2, \ldots, \Delta_{h_2}^2$ to D_1 and respectively D_2. By S^1 invariance, all these maps must be Galois covers of some degrees $m_1, \ldots, m_{h_1}$ and $n_1, \ldots, n_{h_2}$. In this case, the automorphism group is generated by a) automorphisms of the Galois covers and b) permutations of $(p_1, \ldots, p_{h_1})$ and respectively $(q_1, \ldots, q_{h_2})$ which leave the orders unchanged. Note that only permutations which act on $(p_1, \ldots, p_{h_1})$ and $(q_1, \ldots, q_{h_2})$ separately are allowed. The permutations which exchange p_i with q_j are not automorphisms of f even if $m_i = n_j$ for some values of i and j. Therefore $Aut(f)$ is a product between $\prod_{i=1}^{h_1} \mathbb{Z}/m_i \times \prod_{j=1}^{h_2} \mathbb{Z}/n_j \times \mathcal{S}_{h_1} \times \mathcal{S}_{h_2}$ We have accordingly $|Aut(f)| = \prod_{i=1}^{h_1} m_i \prod_{j=1}^{h_2} n_j h_1! h_2!$.

In the following we will denote $\Sigma_{g,h}$ by Σ and Σ_g^0 by Σ^0 in order to simplify the equations. The tangent-obstruction complex has the form

$$0 \longrightarrow Aut(\Sigma) \longrightarrow H^0(\Sigma, \mathcal{T}_Y) \longrightarrow \mathbf{T}^1 \longrightarrow Def(\Sigma) \longrightarrow H^1(\Sigma, \mathcal{T}_Y) \longrightarrow \mathbf{T}_2 \longrightarrow 0 \qquad (113)$$

where $Aut(\Sigma)$, $Def(\Sigma)$ are the groups of infinitesimal automorphisms and respectively infinitesimal deformations of Σ. In the generic case, which is number six on the above list, we have to evaluate

$$\frac{1}{|Aut(f)|} \int_{[\overline{M}_{g,h}]_{S^1}} \frac{e(B_5^m)e(B_1^m)}{e(B_4^m)e(B_2^m)} \qquad (114)$$

where the notation is as before. In the special cases (i)–(v), the fixed locus is just a point, hence the integral should be modified accordingly. Note that in the instanton series we have to sum over ordered h_1-uples $(m_1, \ldots, m_{h_1})$ and respectively h_2-uples $(n_1, \ldots, n_{h_2})$. Since the value of (114) is invariant under a permutation of the winding numbers, we can alternatively sum over unordered $(m_1, \ldots, m_{h_1})$, $(n_1, \ldots, n_{h_2})$ and divide only by $\prod_{i=1}^{h_1} m_i \prod_{j=1}^{h_2} n_j$ in (114). This is the point of view we will take in this section.

Let us start with the generic case. Since the domain Riemann surface is nodal, we have to use a normalization exact sequence in order to compute the terms in (113). Let $\mathcal{T}^1_{Y_i}$, $i = 1, \ldots, h_1$, $\mathcal{T}^2_{Y_j}$, $j = 1, \ldots, h_2$ denote the restrictions of $\mathcal{T}_Y$ to the discs Δ^1_i and respectively Δ^j_2. In other words, $\mathcal{T}^1_{Y_i}$ is the sheaf associated to the Riemann-Hilbert bundle obtained by pulling back the pair (T_Y, T_{L_1}) to the disc Δ^1_i, and similarly for $\mathcal{T}^2_{Y_j}$. In the following we will always label the boundary components mapped to L_1 by $i = 1, \ldots, h_1$, and the boundary components mapped to L_2 by $j = 1, \ldots, h_2$. Then we have the following exact sequence

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \oplus_{i=1}^{h_1} \mathcal{T}^1_{Y_i} \oplus \oplus_{j=1}^{h_2} \mathcal{T}^2_{Y_j} \oplus f_0^*(T_Y) \longrightarrow \oplus_{i=1}^{h_1} (T_Y)_P \oplus \oplus_{j=1}^{h_2} (T_Y)_P \longrightarrow 0 \tag{115}$$

where $f_0 : \Sigma^0 \longrightarrow Y$ denotes the restriction of f to Σ^0. This yields the following long exact sequence

$$\begin{aligned}&0 \longrightarrow H^0(\Sigma, \mathcal{T}_Y) \longrightarrow \oplus_{i=1}^{h_1} H^0(\Delta^1_i, \mathcal{T}^1_{Y_i}) \oplus \oplus_{j=1}^{h_2} H^0(\Delta^2_j, \mathcal{T}^2_{Y_j}) \oplus H^0(\Sigma^0, f_0^*(T_Y)) \\ &\longrightarrow \oplus_{i=1}^{h_1} (T_Y)_P \oplus \oplus_{j=1}^{h_2} (T_Y)_P \longrightarrow H^1(\Sigma, \mathcal{T}_Y) \\ &\longrightarrow \oplus_{i=1}^{h_1} H^1(\Delta^1_i, \mathcal{T}^1_{Y_i}) \oplus \oplus_{j=1}^{h_2} H^1(\Delta^2_j, \mathcal{T}^2_{Y_j}) \oplus H^1(\Sigma^0, f_0^*(T_Y)) \longrightarrow 0.\end{aligned} \tag{116}$$

We denote by F^m_k, $k = 1, \ldots, 5$ the moving parts of the terms in (116). Then we have

$$\frac{e(B^m_5)}{e(B^m_2)} = \frac{e(F^m_5)e(F^m_3)}{e(F^m_2)}. \tag{117}$$

The other factors in (114) are

$$\frac{e(B^m_1)}{e(B^m_4)} = \frac{e(Aut(\Sigma)^m)}{e(Def(\Sigma)^m)}. \tag{118}$$

Since we are in the generic case, $Aut(\Sigma)$ consists only of rotations of the h discs, which have S^1-weight zero, therefore we have $e(Aut(\Sigma)^m) = 1$. The moving part of $Def(\Sigma)$ consists only of deformations of the h nodes, namely

$$Def(\Sigma)^m = \oplus_{i=1}^{h_1} \left(T_{p_i}(\Sigma^0) \otimes T_0(\Delta^1_i)\right) \oplus \oplus_{j=1}^{h_2} \left(T_{q_j}(\Sigma^0) \otimes T_0(\Delta^2_j)\right). \tag{119}$$

Therefore, in order to finish the computation we have to determine F^m_2, F^m_3, F^m_5. In order to do that, let us begin with some local considerations. We will focus on the disc D_1, since D_2 is entirely analogous. In the following we will need the local form of the torus action in the coordinate patches U_1, U_2

$$\begin{array}{lllll} U_1: & x_1 \longrightarrow e^{i\phi(\lambda_2-\lambda_1)}x_1, & y_1 \longrightarrow e^{-i\phi\lambda_1}y_1, & u_1 \longrightarrow u_1, & v_1 \longrightarrow e^{i\phi(2\lambda_1-\lambda_2)} \\ U_2: & x_2 \longrightarrow e^{i\phi(\lambda_1-\lambda_2)}x_2, & y_2 \longrightarrow e^{-i\phi\lambda_2}y_2, & u_2 \longrightarrow e^{i\phi(-\lambda_1+\lambda_2)}u_2, & v_2 \longrightarrow e^{i\phi\lambda_2}v_2 \end{array} \tag{120}$$

We will also need the local form of the sheaf of Kähler differentials on U_1, U_2. Given the local equations (11) we find the following relations

$$\begin{array}{rc} U_1: & du_1 + y_1 v_1 dx_1 + x_1 v_1 dy_1 + x_1 y_1 dv_1 = 0 \\ U_2: & x_2 du_2 + u_2 dx_2 + y_2 dv_2 + v_2 dy_2 = 0. \end{array} \tag{121}$$

Let us now compute the Čech cohomology groups $H^*(\Delta_i^1, \mathcal{T}_{Y_i}^1)$ and $H^*(\Delta_j^2, \mathcal{T}_{Y_j}^2)$. We consider as before a disc Δ of the form $\{|t| \leq \mu^{-1/2d}\} = \{\mu^{1/2d} \leq |t'|\}$ in $\mathbb{P}^1$ with affine coordinates (t, t'). A Galois cover of degree d $f : \Delta \longrightarrow D_1$ is given in local coordinates by

$$\begin{array}{lllll} U_1: & x_1(t) = t^d, & y_1(t) = 0, & u_1(t) = \mu, & v_1(t) = 0 \\ U_2: & x_2(t') = t'^d, & y_2(t') = 0, & u_2(t') = \frac{\mu}{t'^d}, & v_2(t') = 0. \end{array} \tag{122}$$

Note that (122) is left invariant if we let S^1 act on the domain Δ by

$$t \longrightarrow e^{i\phi(\lambda_1 - \lambda_2)/d}, \qquad t' \longrightarrow e^{i\phi(\lambda_2 - \lambda_1)/d}. \tag{123}$$

We cover the disc Δ by two open sets

$$\mathcal{U}_1 = \{0 \leq |t| < \mu^{-1/2d}\}, \qquad \mathcal{U}_2 = \{\mu^{1/2d} \leq |t| < (\mu + \epsilon^2)^{1/2d}\} \tag{124}$$

and construct the Čech complex

$$0 \longrightarrow \mathcal{T}_Y(\mathcal{U}_1) \oplus \mathcal{T}_Y(\mathcal{U}_2) \xrightarrow{\kappa} \mathcal{T}_Y(\mathcal{U}_{12}) \longrightarrow 0. \tag{125}$$

The generic sections in $\mathcal{T}_Y(\mathcal{U}_1), \mathcal{T}_Y(\mathcal{U}_2)$ have the form

$$\begin{array}{l} s_1 = \left(\sum_{n=0}^{\infty} \alpha_n t^n\right) \partial_{x_1} + \left(\sum_{n=0}^{\infty} \beta_n t^n\right) \partial_{y_1} + \left(\mu \sum_{n=0}^{\infty} \gamma_n t^n\right) \partial_{u_1} + \left(\sum_{n=0}^{\infty} \delta_n t^n\right) \partial_{v_1} \\ s_2 = \left(\sum_{n\in\mathbb{Z}} \alpha'_n t'^n\right) \partial_{x_2} + \left(\sum_{n\in\mathbb{Z}} \beta'_n t'^n\right) \partial_{y_2} + \left(\mu \sum_{n\in\mathbb{Z}} \gamma'_n t'^n\right) \partial_{u_2} + \left(\sum_{n\in\mathbb{Z}} \delta'_n t'^n\right) \partial_{v_2}. \end{array} \tag{126}$$

Note that we sum over $n \geq 0$ for sections in $\mathcal{T}(\mathcal{U}_1)$. In order to have a uniform notation, we can extend these sums to $n \in \mathbb{Z}$, with the convention that $\alpha_n, \ldots, \delta_n$ are zero for $n < 0$.

The coefficients $\alpha_n, \beta_n, \ldots, \delta'_n$ are again subject to two types of constraints. The boundary conditions at $|t'| = \mu^{1/2d}$ take the form

$$\sum_{n\in\mathbb{Z}} \alpha'_n r^n e^{in\theta'} = \mu \sum_{n\in\mathbb{Z}} \overline{\gamma}'_n r^n e^{-in\theta'}, \qquad \sum_{n\in\mathbb{Z}} \beta'_n r^n e^{in\theta'} = \sum_{n\in\mathbb{Z}} \overline{\delta}'_n r^n e^{-in\theta'} \tag{127}$$

where $r = \mu^{1/2d}$. This yields the following relations between coefficients

$$\alpha'_n = \mu^{(d-n)/d} \overline{\gamma}'_{-n}, \qquad \beta'_n = \mu^{-n/d} \overline{\delta}'_{-n}. \tag{128}$$

Obviously, the un-primed coefficients are not subject to boundary conditions. Next, we want the infinitesimal deformations (126) to be tangent to Y. By evaluating (121) along D_1, and contracting with (88), we obtain

$$\sum_{n\in\mathbb{Z}} \alpha'_n t'^n + t'^{2d} \sum_{n\in\mathbb{Z}} \gamma'_n t'^n = 0, \qquad \sum_{n=0}^{\infty} \gamma_n t^n = 0. \tag{129}$$

The resulting relations between coefficients can be written as

$$\alpha'_n + \gamma'_{n-2d} = 0, \qquad \gamma_n = 0. \tag{130}$$

In order to compute the kernel and cokernel of κ, we have to rewrite s_2 in terms of t and $\partial_{x_1}, \ldots, \partial_{v_1}$. The relevant linear transformations are

$$\begin{aligned} \partial_{x_2} &= -x_1^2\partial_{x_1} - x_1y_1\partial_{y_1} + u_1x_1\partial_{u_1} + 2x_1v_1\partial_{v_1}, & \partial_{y_2} &= x_1\partial_{y_1}, \\ \partial_{u_2} &= \tfrac{1}{x_1}\partial_{u_1}, & \partial_{v_2} &= \tfrac{1}{x_1^2}\partial_{v_1}. \end{aligned} \tag{131}$$

Then, by direct computations, one can check that the map κ takes the form

$$\begin{aligned} \kappa(s_1, s_2) = &\left(\sum_{n\in\mathbb{Z}} (\alpha_n + \alpha'_{-n+2d})t^n\right)\partial_{x_1} + \left(\sum_{n\in\mathbb{Z}} (\beta_n - \beta'_{-n+d})t^n\right)\partial_{y_1} \\ &+ \left(\sum_{n\in\mathbb{Z}} (\delta_n - \delta'_{-n-2d})t^n\right)\partial_{v_1}. \end{aligned} \tag{132}$$

Again, in order to determine the kernel and cokernel of κ we have to consider the following system of equations

$$\begin{aligned} \alpha_n + \alpha'_{-n+2d} &= a_n, & \beta_n - \beta'_{-n+d} &= b_n \\ & & \delta_n - \delta'_{-n-2d} &= d_n \end{aligned} \tag{133}$$

where a_n, b_n, c_n, d_n are coefficients in the Laurent expansion of an arbitrary section of $\mathcal{T}_Y(\mathcal{U}_{12})$. Therefore they are subject to the constraints

$$c_n = 0. \tag{134}$$

In order to find the kernel, we set $a_n, \ldots, d_n$ to zero and solve for the coefficients in the left hand side of equations (98). Combining (128) and (130) we have

$$\alpha'_n + \mu^{(d-n)/d}\overline{\alpha}'_{-n+2d} = 0, \qquad \beta'_n - \mu^{-n/d}\overline{\delta}'_{-n} = 0. \tag{135}$$

Consider the equation in the first column of (98). Since $\alpha_n = 0$ for $n < 0$, using the first equation in (135), we find that $\alpha'_n = 0$ for $n < 0$ or $2d < n$. This leaves only $2d+1$ nonzero coefficients $\alpha_n = -\alpha'_{-n+2d}$, $0 \le n \le 2d$ subject to the relations

$$\alpha_n + \mu^{(d-n)/d}\overline{\alpha}_{-n+2d} = 0. \tag{136}$$

Next, using (133) and the second equation in (135) we can find the relations

$$\beta_n = \mu^{(n-d)/d}\overline{\delta}_{-n-d}, \qquad \beta_{-n-d} = \mu^{-(n+2d)/2}\overline{\delta}_n. \tag{137}$$

Since $\beta_n, \delta_n = 0$ for $n < 0$, it follows that the only solutions are $\beta_n = \delta_n = 0$. Therefore the kernel of κ is generated by sections of the form

$$\left[\sum_{n=0}^{d-1}(\alpha_n t^n - \mu^{(n-d)/d}\overline{\alpha}_n t^{2d-n}) + \alpha_d t^d\right]\partial_{x_1} \tag{138}$$

where $\alpha_d + \overline{\alpha}_d = 0$.

Now let us determine the cokernel of κ, which is generated by local sections with coefficients $a_n, \ldots, d_n$ for which the equations (133) have no solutions. Let us first analyze the equation in the first column of (133). Using (135), we find that

$$\begin{array}{lll} \alpha_n = 0, & \alpha'_n = -\mu^{(d-n)/d}\overline{a}_n, & \text{for } n < 0 \\ \alpha_n + \alpha'_{-n+2d} = a_n, & \alpha'_n - \mu^{(d-n)/d}\overline{\alpha}'_{-n+2d} = 0, & \text{for } 0 \leq n \leq 2d \\ \alpha_n = \mu^{(n-d)/d}\overline{a}_{2d-n} - a_n, & \alpha'_n = a_{2d-n}, & \text{for } 2d < n. \end{array} \tag{139}$$

It is clear that these equations have solutions for any values of a_n therefore we do not obtain any obstructions from the first equation in (98). We have to perform a similar analysis for the second set of equations in (98). Exploiting again the fact that $\beta_n, \delta_n = 0$ for $n < 0$, one can show that

$$\beta'_n = -b_{-n+d}, \qquad \delta'_{-n} = -d_{n-2d} \tag{140}$$

for $d < n < 2d$. Substituting (140) in the second equation of (135) we obtain the following condition on b_n, d_n

$$b_{-n-d} = \mu^{-(n+2d)/d}\overline{d}_n \tag{141}$$

for $-d < n < 0$. Therefore the cokernel of κ is $(d-1)$-dimensional and generated by sections of the form

$$\left(\sum_{n=-d+1}^{-1} d_n t^n\right)\partial_{v_1}. \tag{142}$$

In particular, if $d = 1$ there are no obstructions.

Collecting the results obtained so far, we can determine the contribution of a single disc to the integrand in (114). Recall that we are using the notation conventions of [29] for representations of S^1, namely the representation $z \longrightarrow e^{iw\phi}z$ is denoted by (w). The real trivial representation is denoted by $(0)_{\mathbb{R}}$. Then, taking into account (120) and (123), $H^0(\Delta, \mathcal{T}_Y)$ computed in (138) is S^1-isomorphic to[6]

$$(\lambda_1 - \lambda_2) \oplus \left((\lambda_1 - \lambda_2)\frac{d-1}{d}\right) \oplus \ldots \oplus \left((\lambda_1 - \lambda_2)\frac{1}{d}\right) \oplus (0)_{\mathbb{R}}. \tag{143}$$

[6]There is a subtlety here related to the choice of signs for the weights of the toric action on $H^{0,1}(\Delta, \mathcal{T}_Y)$, which can be traced to the choice of orientations. We made this choice so that the signs agree with [29].

Similarly, the obstruction group $H^1(\Delta, \mathcal{T}_Y)$ computed in (142) is S^1-isomorphic to

$$\begin{aligned}\left(\lambda_2 - 2\lambda_1 + \frac{1}{d}(\lambda_1 - \lambda_2)\right) \oplus \left(\lambda_2 - 2\lambda_1 + \frac{2}{d}(\lambda_1 - \lambda_2)\right) \oplus \ldots \\ \oplus \left(\lambda_2 - 2\lambda_1 + \frac{d-1}{d}(\lambda_1 - \lambda_2)\right).\end{aligned} \tag{144}$$

The above formulas have been derived for $f : \Delta \longrightarrow D_1$ a Galois cover of D_1. We can perform entirely analogous computations for Galois covers of D_2. In fact, one can simply find the S^1 action on the corresponding deformation and obstruction groups by exchanging $x_2 \leftrightarrow x_3$ and $y_2 \leftrightarrow y_3$ in the above computations. In that case, $H^0(\Delta, \mathcal{T}_Y)$ is S^1-isomorphic to

$$(\lambda_1) \oplus \left(\frac{d-1}{d}\lambda_1\right) \oplus \ldots \oplus \left(\frac{1}{d}\lambda_1\right) \oplus (0)_{\mathbb{R}} \tag{145}$$

and $H^1(\Delta, \mathcal{T}_Y)$ is S^1-isomorphic to

$$\left(\lambda_2 - 2\lambda_1 + \frac{1}{d}\lambda_1\right) \oplus \left(\lambda_2 - 2\lambda_1 + \frac{2}{d}\lambda_1\right) \oplus \ldots \oplus \left(\lambda_2 - 2\lambda_1 + \frac{d-1}{d}\lambda_1\right). \tag{146}$$

Using these formulas, we can complete the computation of the moving parts of the terms F_2, F_3, F_5 in (116). Recall that $\sum_{i=1}^{h_1} m_i = d_1$, $\sum_{j=1}^{h_2} n_j = d_2$, where (d_1, d_2) are the degrees of the map $f : \Sigma^0 \cup \cup_{i=1}^{h_1} \Delta_i^1 \cup \cup_{j=1}^{h_2} \Delta_j^2 \longrightarrow Y$ with respect to the two discs D_1, D_2. Moreover, Σ^0 is mapped to the point $P : Z_2 = Z_3 = 0$ on Y, hence $H^0(\Sigma^0, f_0^*(T_Y)) \simeq H^0(\Sigma_0, \mathcal{O}_{\Sigma^0}) \otimes T_P(Y)$. Then we have

$$\begin{aligned}
e(F_2^m) &= H^{d_1+d_2+3}\lambda_1(\lambda_1 - \lambda_2)(\lambda_2 - 2\lambda_1)(\lambda_1 - \lambda_2)^{d_1}\lambda_1^{d_2} \\
&\quad \times \textstyle\prod_{i=1}^{h_1}\prod_{k=0}^{m_i-1}\left(\frac{m_i-k}{m_i}\right)\prod_{j=1}^{h_2}\prod_{l=0}^{n_j-1}\left(\frac{n_j-l}{n_j}\right) \\
e(F_3^m) &= H^{3(h_1+h_2)}(\lambda_1(\lambda_1 - \lambda_2)(\lambda_2 - 2\lambda_1))^{h_1+h_2} \\
e(F_5^m) &= c_g(\mathbb{E}^*((\lambda_1 - \lambda_2)H))c_g(\mathbb{E}^*(\lambda_1 H))c_g(\mathbb{E}^*((\lambda_2 - 2\lambda_1)H))H^{d_1+d_2-h_1-h_2} \\
&\quad \times \textstyle\prod_{i=1}^{h_1}\prod_{k=1}^{m_i-1}\left(\lambda_2 - 2\lambda_1 + \frac{k}{m_i}(\lambda_1 - \lambda_2)\right)\prod_{j=1}^{h_2}\prod_{l=1}^{n_j-1}\left(\lambda_2 - 2\lambda_1 + \frac{l}{n_j}\lambda_1\right)
\end{aligned} \tag{147}$$

where $H \in H^2(BS^1) \simeq H^2_{S^1}(pt)$ is the generator of the equivariant cohomology ring of a point, and $\mathbb{E}$ denotes the Hodge bundle on the moduli space of stable pointed curves $\overline{M}_{g,h}$. Note that we are using the orientation conventions of [29]. In the second equation in (147), the expressions of the form $c_g(\mathbb{E}^*(\eta H))$ with $\eta = \lambda_1 - \lambda_2, \lambda_1, \lambda_2 - 2\lambda_1$ denote

$$c_g(\mathbb{E}^*(\eta H)) = (\eta H)^g - c_1(\mathbb{E})(\eta H)^{g-1} + c_2(\mathbb{E})(\eta H)^{g-2} + \ldots + (-1)^g c_g(\mathbb{E}) \tag{148}$$

in the equivariant cohomology ring $H^*_{S^1}(\overline{M}_{g,h})$. The last ingredients we need are the terms in equation (118). Since we are in the generic case, $Aut(\Sigma)$ consists only of rotations of the h discs $\Delta^1_1, \ldots, \Delta^1_{h_1}, \Delta^2_1, \ldots, \Delta^2_{h_2}$ which are generated by $t\partial t$ over $\mathbb{R}$. Therefore $Aut(\Sigma) \simeq (0)_{\mathbb{R}}$, and the moving part is trivial. We are left with the moving part of $Def(\Sigma)$, which is given by (119). This yields

$$e(Def(\Sigma)^m) = \prod_{i=1}^{h_1} \left(\frac{\lambda_1 - \lambda_2}{m_i} H - \psi_i \right) \prod_{j=1}^{h_2} \left(\frac{\lambda_1}{n_j} H - \psi_j \right) \tag{149}$$

where $\psi_i = c_1(\mathbb{L}_i)$, $i = 1, \ldots, h_1$, $\psi_j = c_1(\mathbb{L}_j)$, $j = 1, \ldots, h_2$ are the first Chern classes of the tautological line bundles $\mathbb{L}_i, \mathbb{L}_j$ over $\overline{M}_{g,h}$ associated to the marked points p_i, q_j.

Now we can collect all the results obtained so far, and write down an integral expression for the coefficients F_{g,h_1,h_2} in (29)

$$\begin{aligned} F_{g,h_1,h_2}(d_1, d_2; m_i, n_j) &= \frac{1}{\prod_{i=1}^{h_1} m_i \prod_{j=1}^{h_2} n_j} \int_{[\overline{M}_{g,h}]_{S^1}} \frac{e(F_5^m) e(F_3^m)}{e(F_2^m) e(Def(\Sigma)^m)} \\ &= \frac{[\lambda_1(\lambda_1-\lambda_2)(\lambda_2-2\lambda_1)]^{h_1+h_2-1}}{(\lambda_1-\lambda_2)^{d_1}\lambda_1^{d_2}} \prod_{i=1}^{h_1} \frac{\prod_{k=1}^{m_i-1}((\lambda_2-2\lambda_1)m_i+k(\lambda_1-\lambda_2))}{(m_i-1)!} \\ &\quad \times \prod_{j=1}^{h_2} \frac{\prod_{l=1}^{n_j-1}((\lambda_2-2\lambda_1)n_j+l\lambda_1)}{(n_j-1)!} \\ &\times \int_{[\overline{M}_{g,h}]_{S^1}} \frac{c_g(\mathbb{E}^*((\lambda_1-\lambda_2)H)) c_g(\mathbb{E}^*(\lambda_1 H)) c_g(\mathbb{E}^*((\lambda_2-2\lambda_1)H)) H^{2(h_1+h_2)-3}}{\prod_{i=1}^{h_1}((\lambda_1-\lambda_2)H - m_i\psi_i) \prod_{j=1}^{h_2}(\lambda_1 H - n_j\psi_j)}. \end{aligned} \tag{150}$$

This is the result for the generic case. Note that this formula takes values in the fraction field of the cohomology ring $H^*(BS^1)$ and in general it need not be a multiple the unit element. In order to obtain a physically meaningful answer, we have to impose certain conditions on the weights λ_1, λ_2 whose origin has been explained in [29]. We will discuss this specialization of the toric action after treating the special cases (i)-(v).

i) $(g, h_1, h_2) = (0, 1, 0)$. In this case, the map is of the form $f : \Delta^1 \longrightarrow D_1$ given in local coordinates by (122) with $d = d_1$. Therefore we can directly use equations (143) and (144) obtaining

$$\frac{e(B_5^m)}{e(B_2^m)} = \frac{d_1^{d_1-1} H^{-1}}{(d_1 - 1)!(\lambda_1 - \lambda_2)^{d_1}} \prod_{k=1}^{d_1-1} \left(\lambda_2 - 2\lambda_1 + \frac{k}{d_1}(\lambda_1 - \lambda_2) \right). \tag{151}$$

The automorphism group $Aut(\Delta^1)$ is in this case nontrivial and generated by $t'\partial_{t'}, \partial_{t'}$ which span $(0)_{\mathbb{R}} \oplus \left(\frac{\lambda_1 - \lambda_2}{d_1} \right)$. This yields

$$e(B_1^m) = \frac{\lambda_1 - \lambda_2}{d_1} H. \tag{152}$$

The overall contribution to the multicover formula is

$$F_{0,1,0}(d_1,0;d_1,0) = \frac{1}{d_1}\int_{pt_{S^1}} \frac{e(B_5^m)e(B_1^m)}{e(B_2^m)} = \frac{1}{d_1^2(d_1-1)!}\frac{1}{(\lambda_1-\lambda_2)^{d_1-1}}\prod_{k=1}^{d_1-1}\left((\lambda_2-2\lambda_1)d_1+k(\lambda_1-\lambda_2)\right). \tag{153}$$

ii) $(g,h_1,h_2)=(0,0,1)$. This case is analogous to the previous one. We just have to exchange $D_1 \leftrightarrow D_2$, and the final result is

$$F_{0,0,1}(0,d_2;0,d_2) = \frac{1}{d_2^2(d_2-1)!}\frac{1}{\lambda_1^{d_2-1}}\prod_{l=1}^{d_2-1}\left((\lambda_2-2\lambda_1)d_2+l\lambda_1\right). \tag{154}$$

iii) $(g,h_1,h_2)=(0,2,0)$. In this case we have a map $f:\Delta_1^1\cup\Delta_2^1 \longrightarrow D_1$ which maps the two components of the domain to D_1 with degrees m_1, m_2, $m_1+m_2=d_1$. Since the domain is nodal, we have to consider again a normalization exact sequence of the form

$$0\longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_{Y_1}^1 \oplus \mathcal{T}_{Y_2}^1 \longrightarrow (T_Y)_P \longrightarrow 0 \tag{155}$$

which yields a long exact sequence

$$\begin{aligned} 0&\longrightarrow H^0(\Sigma,\mathcal{T}_Y)\longrightarrow H^0(\Delta_1^1,\mathcal{T}_{Y_1}^1)\oplus H^0(\Delta_2^1,\mathcal{T}_{Y_2}^1)\longrightarrow (T_Y)_P \\ &\longrightarrow H^1(\Sigma,\mathcal{T}_Y)\longrightarrow H^1(\Delta_1^1,\mathcal{T}_{Y_1}^1)\oplus H^1(\Delta_2^1,\mathcal{T}_{Y_2}^1)\longrightarrow 0. \end{aligned} \tag{156}$$

We denote the terms of (156) by $F_1,\ldots,F_5$ as before, so that

$$F_{0,2,0}(d_1,0;m_1,m_2,0,0) = \frac{1}{m_1m_2}\int_{pt_{S^1}} \frac{e(F_5^m)e(F_3^m)}{e(F_2^m)}\frac{e(Aut(\Sigma)^m)}{e(Def(\Sigma)^m)}. \tag{157}$$

The contributions of the terms in (156) have been evaluated before. The automorphism group is again trivial from an equivariant point of view since we have only rotations of the two discs. The moving part of $Def(\Sigma)$ consists of deformations of the node, which are parameterized by $T_0(\Delta_1^1)\otimes T_0(\Delta_2^1)$. Therefore

$$e(Def(\Sigma)^m) = \left(\frac{1}{m_1}+\frac{1}{m_2}\right)(\lambda_1-\lambda_2)H. \tag{158}$$

Collecting all the factors, we arrive at the following expression

$$F_{0,2,0}(d_1,0;m_1,m_2,0,0) = \frac{1}{d_1}\frac{\lambda_1(\lambda_2-2\lambda_1)}{(\lambda_1-\lambda_2)^{d_1}}\prod_{i=1}^{2}\frac{\prod_{k=1}^{m_i-1}\left((\lambda_2-2\lambda_1)m_i+k(\lambda_1-\lambda_2)\right)}{(m_i-1)}. \tag{159}$$

iv) $(g, h_1, h_2) = (0, 0, 2)$. This case can be obtained from the previous one by exchanging $D_1 \leftrightarrow D_3$. We have a map $f : \Delta_1^2 \cup \Delta_2^2 \longrightarrow D_2$ with degrees n_1, n_2, $n_1 + n_2 = d_2$. Going through the steps (155)-(159) yields the following formula

$$F_{0,0,2}(0, d_2; 0, 0, n_1, n_2) = \frac{1}{d_2} \frac{(\lambda_1 - \lambda_2)(\lambda_2 - 2\lambda_1)}{\lambda_1^{d_2}} \prod_{j=1}^{2} \frac{\prod_{l=1}^{n_j - 1} ((\lambda_2 - 2\lambda_1)n_j + l\lambda_1)}{(n_j - 1)!}. \tag{160}$$

v) $(g, h_1, h_2) = (0, 1, 1)$. We have a map $f : \Delta_1^1 \cup \Delta_1^2 \longrightarrow Y$ mapping the two components of the domain onto D_1, D_2 with degrees d_1, d_2. Then one has to use a normalization sequence similar to (155) and evaluate the contributions of the terms as before. Therefore we have

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_{Y_1}^1 \oplus \mathcal{T}_{Y_2}^2 \longrightarrow (\mathcal{T}_Y)_P \longrightarrow 0 \tag{161}$$

which yields a long exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0(\Sigma, \mathcal{T}_Y) \longrightarrow H^0(\Delta_1^1, \mathcal{T}_{Y_1}^1) \oplus H^0(\Delta_1^2, \mathcal{T}_{Y_2}^2) \longrightarrow (\mathcal{T}_Y)_P \\ &\longrightarrow H^1(\Sigma, \mathcal{T}_Y) \longrightarrow H^1(\Delta_1^1, \mathcal{T}_{Y_1}^1) \oplus H^1(\Delta_1^2, \mathcal{T}_{Y_2}^2) \longrightarrow 0. \end{aligned} \tag{162}$$

Denoting again the terms of (162) by $F_1, \ldots, F_5$, we have

$$\frac{e(B_5^m)}{e(B_m^2)} = \frac{e(F_5^m)e(F_3^m)}{e(F_2^m)}. \tag{163}$$

In order to complete the computation, we also need to evaluate

$$\frac{e(B_1^m)}{e(B_4^m)} = \frac{e(Aut(\Sigma)^m)}{e(Def(\Sigma)^m)}. \tag{164}$$

This is analogous to cases (iii) and (iv) above. The only nontrivial moving part consists of deformations of the node of the domain, which are parameterized by $T_0(\Delta_1^1) \otimes T_0(\Delta_1^2)$. Hence we are left with

$$e(Def(\Sigma)^m) = \frac{\lambda_1 - \lambda_2}{d_1} + \frac{\lambda_1}{d_2}. \tag{165}$$

Collecting all the intermediate results, we obtain

$$\begin{aligned} F_{0,1,1}(d_1, d_2; d_1, d_2) &= \frac{1}{d_1 d_2} \int_{pt_{S^1}} \frac{e(F_5^m)e(F_3^m)}{e(F_2^m)e(Def(\Sigma)^m)} \\ &= \frac{1}{(d_1 - 1)!(d_2 - 1)!} \frac{\lambda_2 - 2\lambda_1}{(d_2(\lambda_1 - \lambda_2) + d_1\lambda_1)(\lambda_1 - \lambda_2)^{d_1 - 1}\lambda_1^{d_2 - 1}} \\ &\times \prod_{k=1}^{d_1 - 1} ((\lambda_2 - 2\lambda_1)d_1 + k(\lambda_1 - \lambda_2)) \prod_{l=1}^{d_2 - 1} ((\lambda_2 - 2\lambda_1)d_2 + l\lambda_1). \end{aligned} \tag{166}$$

This concludes our localization computations for open string morphisms. In principle, instanton corrections (29) can be obtained by adding add the contributions (150), (153), (154), (159), (160), and (166). Besides being very complicated, the resulting expression would be a homogeneous rational function of λ_1, λ_2 rather than a number. This is a common problem with open string localization computations [29, 22] which can be solved by making a specific choice of the toric action.

6.3 Choice of Toric Action

As discussed in detail in section four, in order to obtain a physically meaningful answer, one has to choose specific values for the weights λ_1, λ_2. This approach is similar to that of [22, 29, 41], except that in our case the motivation for these choices is somewhat different. In those references, there is an ambiguity in the definition of the virtual fundamental class due to the fact that the open string moduli spaces have boundaries. In our situation, we do not really have a virtual class in the standard sense since the contribution of each fixed locus is a formal series of operators in Chern-Simons theory. Therefore we have to choose weights in order to make sense of this series of instanton corrections for each fixed component. Moreover, the choice of weights has to be correlated to the choice of framings in Chern-Simons theory.

We will follow the approach of [29], choosing two sections of the real normal bundles $N^1_{\mathbb{R}}$, $N^2_{\mathbb{R}}$. We will work in the coordinate patch U_1, which covers both discs. The boundaries Γ_1, Γ_2 of the two discs can be parameterized by

$$\begin{array}{lllll} \Gamma_1: & x_1 = \mu^{-1/2}e^{i\theta_1}, & y_1 = 0, & u_1 = \mu, & v_1 = 0 \\ \Gamma_2: & x_1 = 0, & y_1 = \mu^{-1/2}e^{i\theta_2}, & u_1 = \mu, & v_1 = 0. \end{array} \tag{167}$$

The sections of the real normal bundles must be of the form

$$\begin{array}{ll} N^1_{\mathbb{R}}: & (y_1, v_1) = \left(e^{i(a-1)\theta_1}, \mu^{3/2}e^{-ia\theta_1}\right) \\ N^2_{\mathbb{R}}: & (x_1, v_1) = \left(e^{i(b-1)\theta_2}, \mu^{3/2}e^{-ib\theta_2}\right) \end{array} \tag{168}$$

where $a, b \in \mathbb{Z}$. At this point, one might be tempted to conclude that the sections (168) determine the framing of the two knots Γ_1, Γ_2 in L_1, L_2. In fact, we have to be more careful here, since we need more data in order to determine the framing as an integer number. Note that the sections (168) can be regarded as sections of the circle bundles associated to $N^1_{\mathbb{R}}, N^2_{\mathbb{R}}$, which are topologically trivial. The group of homotopy classes of sections of an S^1 bundle over S^1 is isomorphic to $\mathbb{Z}$, but one has to choose an isomorphism in order to associate an integer number to such a class. More concretely, the choice of such an isomorphism is equivalent to the choice of a trivialization of the circle bundle which plays the role of reference section. In our case, we can obtain natural trivializations of $N^1_{\mathbb{R}}, N^2_{\mathbb{R}}$ exploiting the fact that Γ_1, Γ_2 are algebraic knots. By changing coordinates to (x_2, y_2, u_2, v_2) and respectively (x_3, y_3, u_3, v_3) the spheres L_1, L_2 can be identified with the canonical sphere in $\mathbb{C}^2$, as in equation (16). Moreover, the boundary components of D_1, D_2 are now parameterized by

$$\begin{array}{lllll} \Gamma_1: & x_2 = \mu^{1/2}e^{i\theta_1'}, & y_2 = 0, & u_2 = \mu^{1/2}e^{-i\theta_1'}, & v_2 = 0 \\ \Gamma_2: & x_3 = \mu^{1/2}e^{i\theta_2'}, & y_3 = 0, & u_3 = \mu^{1/2}e^{-i\theta_2'}, & v_3 = 0 \end{array} \tag{169}$$

where $\theta_1' = -\theta_1$, $\theta_2' = -\theta_2$. In the new coordinates, the sections (168) read

$$\begin{array}{ll} N^1_{\mathbb{R}}: & (y_2, v_2) = \left(\mu^{1/2}e^{i(2-a)\theta_1'}, \mu^{1/2}e^{-i(2-a)\theta_1'}\right) \\ N^2_{\mathbb{R}}: & (x_2, v_2) = \left(\mu^{1/2}e^{i(2-b)\theta_2'}, \mu^{1/2}e^{-i(2-b)\theta_2'}\right). \end{array} \tag{170}$$

In this form, one can choose canonical reference sections for $N^1_{\mathbb{R}}, N^2_{\mathbb{R}}$ determined by ∂_{y_2} and respectively ∂_{x_2}. With respect to these sections, the framings of the two knots are $(2-a, 2-b)$.

Following the strategy proposed in [29], we require that the sections (168) be equivariant. Then the toric action is fixed, since we have the following constraints on the weights

$$\begin{array}{ll} -\lambda_1 = (a-1)(\lambda_2 - \lambda_1), & 2\lambda_1 - \lambda_2 = -a(\lambda_2 - \lambda_1) \\ \lambda_2 - \lambda_1 = -(b-1)\lambda_1, & 2\lambda_1 - \lambda_2 = b\lambda_1. \end{array} \tag{171}$$

Using the equations in the first column, we easily find

$$\lambda_1(\lambda_2 - \lambda_1)(ab - a - b) = 0. \tag{172}$$

If we choose either $\lambda_1 = 0$ or $\lambda_2 - \lambda_1 = 0$, one can check that the remaining equations imply $\lambda_1 = \lambda_2 = 0$. This is not an acceptable solution, therefore we are left with the equation

$$ab = a + b, \qquad a, b \in \mathbb{Z}. \tag{173}$$

This has two solutions, namely $(a,b) = (0,0)$ or $(a,b) = (2,2)$. The first case, $(a,b) = (0,0)$ implies $2\lambda_1 - \lambda_2 = 0$, while the second case implies $\lambda_2 = 0$.

At this point, we do not have any further selection criteria, hence we cannot rule any solution out. However, the first solution, $(a,b) = (0,0)$ is more convenient since it yields the simple closed form for the instanton expansion (30), (31). In the following we will make this choice, and finish the derivation of (30), (31). The second solution should also lead to consistent results, but it is much harder to do explicit calculations. We will not pursue this problem here.

If $(a,b)=(0,0)$, we have to impose the relation $2\lambda_1-\lambda_2=0$ in (150), (153), (154), (159), (160), (166) and collect the results. Let us start with (150). Because of the factor $(\lambda_2-2\lambda_1)^{h_1-h_2-1}$ we can obtain a nonzero answer only if $h_1+h_2=1$, that is from bordered Riemann surfaces with one boundary component. Some care is needed in this argument, since in principle we could get other contributions if the expansion of the integrand in (150) yields terms with negative powers of $(\lambda_2-2\lambda_1)$. In order to show that such terms are absent, we can rewrite (150) as

$$\begin{aligned}F_{g,h_1,h_2}(d_1,d_2;m_i,n_j)&=\frac{(\lambda_2-2\lambda_1)^{h_1+h_2-1}}{(\lambda_1-\lambda_2)^{d_1+1}\lambda_1^{d_2+1}}\prod_{i=1}^{h_1}\frac{\prod_{k=1}^{m_i-1}((\lambda_2-2\lambda_1)m_i+k(\lambda_1-\lambda_2))}{(m_i-1)!}\\&\times\prod_{j=1}^{h_2}\frac{\prod_{l=1}^{n_j-1}((\lambda_2-2\lambda_1)n_j+l\lambda_1)}{(n_j-1)!}\\&\times\int_{[\overline{M}_{g,h}]_{S^1}}c_g(\mathbb{E}^*((\lambda_1-\lambda_2)H))c_g(\mathbb{E}^*(\lambda_1H))c_g(\mathbb{E}^*((\lambda_2-2\lambda_1)H))H^{(h_1+h_2)-3}\\&\times\prod_{i=1}^{h_1}\left[\sum_{k=0}^{\infty}\left(\frac{m_i\psi_iH^{-1}}{\lambda_1-\lambda_2}\right)^k\right]\prod_{j=1}^{h_2}\left[\sum_{l=0}^{\infty}\left(\frac{n_j\psi_jH^{-1}}{\lambda_1}\right)^l\right].\end{aligned}\tag{174}$$

All terms in this expressions contain negative powers of $\lambda_1,(\lambda_1-\lambda_2)$ but not $(\lambda_2-2\lambda_1)$. Therefore we can conclude that imposing the relation $\lambda_2-2\lambda_1=0$ leaves only terms with $h_1+h_2=1$. This leads to a significant simplification of the instanton expansion. The nontrivial contributions are

$$\begin{aligned}F_{g,1,0}(d_1,0;d_1,0)=\tfrac{1}{\lambda_1^2}\int_{[\overline{M}_{g,h}]_{S^1}}&(-1)^gc_g(\mathbb{E}^*(-\lambda_1H))c_g(\mathbb{E}^*(\lambda_1H))c_g(\mathbb{E})H^{-2}\\&\times\left[\sum_{k=0}^{\infty}\left(-\tfrac{d_1\psi H^{-1}}{\lambda_1}\right)^k\right].\end{aligned}\tag{175}$$

$$\begin{aligned}F_{g,0,1}(0,d_2;0,d_2)=\tfrac{1}{\lambda_1^2}\int_{[\overline{M}_{g,h}]_{S^1}}&(-1)^gc_g(\mathbb{E}^*(-\lambda_1H))c_g(\mathbb{E}^*(\lambda_1H))c_g(\mathbb{E})H^{-2}\\&\times\left[\sum_{l=0}^{\infty}\left(\tfrac{d_2\psi H^{-1}}{\lambda_1}\right)^l\right].\end{aligned}\tag{176}$$

Now we can finish the computation as in [29] using the relation in [16]

$$c_g(\mathbb{E}^*(\lambda_1H))c_g(\mathbb{E}^*(-\lambda_1H))=(-1)^g(\lambda_1H)^g.\tag{177}$$

Then the final result reads

$$F_{g,1,0}(d_1,0;d_1,0)=d_1^{2g-2}b_g,\qquad F_{g,0,1}(0,d_2;0,d_2)=d_2^{2g-2}b_g\tag{178}$$

where b_g are the Bernoulli numbers, and $g \geq 1$.

The contributions of special cases $(i)-(iv)$ be computed easily by direct evaluation

$$\begin{aligned} F_{0,1,0}(d_1,0;d_1,0) &= \tfrac{1}{d_1^2} \\ F_{0,0,1}(0,d_2;0,d_2) &= \tfrac{1}{d_2^2} \\ F_{0,2,0}(d_1,0;m_1,m_2,0,0) = F_{0,0,2}(0,d_2;0,0,n_1,n_2) &= 0. \end{aligned} \tag{179}$$

This leaves us with (166) corresponding to the *fifth* case. Here we have again a factor of $(\lambda_2-2\lambda_1)$ which gives a zero answer unless we have similar factors in the denominator. The only monomial in the denominator which can produce powers of $(\lambda_2-2\lambda_1)$ is $d_2(\lambda_1-\lambda_2)+d_1\lambda_1$. If $d_1=d_2=d$ this reduces to $d(2\lambda_1-\lambda_2)$ cancelling the effect of the similar factor in the denominator. Therefore we obtain

$$F_{0,1,1}(d,d;d,d) = -\frac{1}{d}. \tag{180}$$

It is easy to check that all other amplitudes are zero.

In order to complete the description of the Chern-Simons system, we have to specify the framing of the knots. The framing of Γ_1,Γ_2 has been discussed below equation (170). Since we have fixed $a=b=0$, we have framings $(2,2)$. On the other hand, the framing of Ξ_1,Ξ_2 is not fixed by the localization computation. The result (108) is true for any values of (λ_1,λ_2). This is actually an important consistency check of the formalism, since one can see from (82) that once we fix $(a,b)=(0,0)$ or $(a,b)=(2,2)$, there is no S^1-equivariant choice of framings of Ξ_1,Ξ_2. Had such a choice been necessary, we would have found an inconsistency between the localization computations for C and $D_1\cup D_2$. However, we can determine the framing of Ξ_1,Ξ_2 using the following deformation argument. As noted in the paragraph containing equation (5.11) and also below (6.25), if d is even, the $d:1$ cover of C and the $d/2:1$ cover of D_1,D_2 belong to the same component of the moduli space $\overline{M}_{0,2}(Y,L,d\beta)$. A natural assumption is that the homotopy class of the framing is preserved under deformations of maps which do not change the isotopy type of the knots in L_1,L_2. In the present case, this shows that the framing of Ξ_1,Ξ_2 have to be equal half the framing of Γ_1,Γ_2, that is $\left(1-\frac{a}{2},1-\frac{b}{2}\right)$. This concludes the computation of open string instanton corrections.

A Some Geometric Facts

In this appendix we elaborate on some technical points which have been used without proof in section two. First, we would like to check that the map $i:\widehat{X}\longrightarrow Z$

defined in (6) is a well defined toric morphism. The second issue we would like to address is the construction of a suitable symplectic Kähler form of the deformed hypersurface Y_μ so that L_1, L_2 are lagrangian cycles.

A.1 The Toric Embedding

Recall that the map $i : \widehat{X} \longrightarrow Z$ is defined in terms of homogeneous coordinates by

$$Z_1 = X_2X_3X_4, \quad Z_2 = X_1X_2, \quad Z_3 = X_4X_5, \quad U = X_0X_1X_5, \quad V = -X_0X_3. \tag{181}$$

In order to check that this is a well defined toric morphism, we have to show that it is compatible with the toric actions and with the disallowed loci. Note first that the monomials (181) transform under the toric action (1) as

$$\begin{array}{ccccc} Z_1 & Z_2 & Z_3 & U & V \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0, \end{array} \tag{182}$$

which is indeed compatible with (5). In order to check the disallowed loci, note that $\widehat{X}$ is defined by the toric data (1), the Kähler parameters ξ_1, ξ_3 being set to zero in (2). This corresponds to a partial triangulation of the polytope in fig. 1. obtained by erasing the simplexes v_0v_2 and v_0v_4. Therefore we have to show that given the moment map equations (2) with $\xi_1 = \xi_3 = 0$ and $\xi_2 > 0$, the monomials Z_1, Z_2, Z_3 cannot vanish simultaneously. This is elementary, but we include the details below for completeness. Let us first rewrite the equations (2) in the form

$$\begin{array}{lcl} |X_2|^2 + |X_4|^2 - |X_3|^2 - |X_0|^2 & = & \xi_2 \\ |X_1|^2 + |X_4|^2 - 2|X_0|^2 & = & \xi_2 \\ |X_2|^2 + |X_5|^2 - 2|X_0|^2 & = & \xi_2 \\ |X_1|^2 + |X_3|^2 + |X_5|^2 - 3|X_0|^2 & = & \xi_2. \end{array} \tag{183}$$

The last equation is a linear combination of the first three, but it will be needed in this particular form below. Next, note that for Z_2 and Z_3 to vanish simultaneously, one of the following conditions must be realized

$$\begin{array}{l} a)\ X_1 = X_4 = 0 \\ b)\ X_1 = X_5 = 0 \\ c)\ X_2 = X_4 = 0 \\ d)\ X_2 = X_5 = 0 \end{array}$$

It suffices to prove on a case by case basis that if any of these conditions is realized, Z_1 cannot vanish. Note that $a)$, $d)$ and $c)$ are immediately excluded by

(183). We are left with b). If $X_1 = X_5 = 0$, it follows from (183) that X_2, X_3, X_4 are not allowed to vanish, therefore Z_1 is not allowed to vanish. It is also easy to check that any two of Z_1, Z_2, Z_3 are allowed to vanish at the same time. The monomials U, V are allowed to vanish independently of Z_1, Z_2, Z_3 since they are multiples of X_0. We conclude that the map (181) is well defined.

A.2 The Symplectic Form

Another claim made in section two is that we can choose a symplectic Kähler form ω on Y which agrees locally near the cycles L_1, L_2 with the standard symplectic form on a deformed conifold. The following construction is a refinement of standard symplectic surgery techniques [14, 15].

Since the argument is local, it suffices to consider only one cycle, say L_1. Since Z is toric, it can be represented as a symplectic quotient $\mathbb{C}^5//U(1)$ with level sets

$$|Z_1|^2 + |Z_2|^2 + |Z_3|^2 + |U|^2 + |V|^2 = \xi \tag{184}$$

where $\xi \in \mathbb{R}_+$. Therefore Z is endowed with a symplectic Kähler form ω_0 obtained by descent from the invariant form

$$\Omega = \frac{i}{2}\left(dZ_1 \wedge d\overline{Z}_1 + dZ_2 \wedge d\overline{Z}_2 + dZ_3 \wedge d\overline{Z}_3 + dU \wedge d\overline{U} + dV \wedge d\overline{V}\right). \tag{185}$$

In the patch $Z_3 \neq 0$, we can rewrite (183) in terms of local coordinates as

$$(|x_2|^2 + |y_2|^2 + 1)|Z_3|^6 + |u_2|^2|Z_3|^2 + |v_2|^2 = \xi|Z_3|^4, \tag{186}$$

which can be interpreted as an equation of degree three in $|Z_3|^2$. At this point, it is convenient to introduce polar coordinates

$$x_2 = r_x e^{i\theta_x}, \qquad y_2 = r_y e^{i\theta_y}, \qquad u_2 = r_u e^{i\theta_u}, \qquad v_2 = r_v e^{i\theta_v}. \tag{187}$$

By solving for $|Z_3|^2$ in (186), we obtain $|Z_3|^2 = F(r_x, r_y, r_u, r_v)$ for some real positive function F. Then we can take a local transversal slice for the $U(1)$ action on $\mathbb{C}^5$ of the form

$$\begin{gathered} Z_1 = F(r_x, r_y, r_u, r_v) r_x e^{i\theta_x}, \quad Z_2 = F(r_x, r_y, r_u, r_v) r_y e^{i\theta_y}, \quad Z_3 = F(r_x, r_y, r_u, r_v) r_x, \\ U = F(r_x, r_y, r_u, r_v) r_u e^{i\theta_u}, \quad V = F(r_x, r_y, r_u, r_v) r_v e^{i\theta_v}. \end{gathered} \tag{188}$$

Substituting (188) in (185) we find that ω_0 has the local form

$$\omega_0|_{U_2} = \frac{1}{2} d\left[F(r_x, r_y, r_u, r_v)(r_x^2 d\theta_x + r_y^2 d\theta_y + r_u^2 d\theta_u + r_v^2 d\theta_v)\right]. \tag{189}$$

On the other hand the standard symplectic form ω_c can be written

$$\omega_c = \frac{c}{2} d\left(r_x^2 d\theta_x + r_y^2 d\theta_y + r_u^2 d\theta_u + r_v^2 d\theta_v\right) \tag{190}$$

where c is a positive real constant. Note that in polar coordinates the cycle L_2 is determined by

$$r_x = r_u, \quad \theta_x + \theta_u = 0, \qquad r_y = r_v, \quad \theta_y + \theta_v = 0, \qquad r_x^2 + r_u^2 = \mu. \tag{191}$$

Now let us consider the polycylinder $C_\epsilon(r)$ defined by

$$\mu^{1/2} + r - \epsilon \leq r_x, r_y, r_u, r_v \leq \mu^{1/2} + r + \epsilon. \tag{192}$$

In the following argument, we will also need the polydiscs $\Delta(r-\epsilon) = \{0 \leq r_x, r_y, r_u, r_v \leq \mu^{1/2} + r - \epsilon\}$, $\Delta(r+\epsilon) = \{0 \leq r_x, r_y, r_u, r_v \leq \mu^{1/2} + r + \epsilon\}$. The main idea is to construct a symplectic Kähler form ω on Z which interpolates smoothly between ω_c and ω_0 over $C_\epsilon(r)$. Let $\rho : \mathbb{R}_+ \longrightarrow [0,1]$ be a decreasing interpolating smooth function such that

$$\rho(t) = \begin{cases} 1, & \text{for } 0 \leq t \leq \mu^{1/2} + r - \epsilon \\ 0, & \text{for } \mu^{1/2} + r + \epsilon \leq t. \end{cases}$$

Then we define the form

$$\begin{gathered} \omega = \omega_0 \\ +\tfrac{1}{2} d\left[(c - F(r_x, r_y, r_u, r_v))(\rho(r_x) r_x^2 d\theta_x + \rho(r_y) r_y^2 d\theta_y + \rho(r_u) r_u^2 d\theta_u + \rho(r_v) r_v^2 d\theta_v)\right]. \end{gathered} \tag{193}$$

It is straightforward to check that ω agrees with ω_c in the polydisc $\Delta(r-\epsilon)$ and with ω_0 on the complement of $\Delta(r+\epsilon)$. Since ω is also closed, in order to complete the argument, we have to check it is a Kähler form, that is $\omega(\psi, J\psi) \geq 0$ for any tangent vector ψ to Z. It is clear that it suffices to check that ω is positive definite over $C_\epsilon(r)$. A straightforward computation shows that

$$\begin{aligned} \omega|_{C_\epsilon(r)} = {} & \rho(r_x) r_x dr_x \wedge d\theta_x + \rho(r_y) r_y dr_y \wedge d\theta_y + \rho(r_u) r_u dr_u \wedge d\theta_u + \rho(r_v) r_v dr_v \wedge d\theta_v \\ & + (1 - \rho(r_x)) d(r_x^2 F) \wedge d\theta_x + (1 - \rho(r_y)) d(r_y^2 F) \wedge d\theta_y + (1 - \rho(r_u)) d(r_u^2 F) \wedge d\theta_u \\ & + (1 - \rho(r_v)) d(r_v^2 F) \wedge d\theta_v + (c - F)\left[\rho'(r_x) r_x^2 dr_x \wedge d\theta_x + \rho'(r_y) r_y^2 dr_y \wedge d\theta_y\right. \\ & \left. + \rho'(r_u) r_u^2 dr_u \wedge d\theta_u + \rho'(r_v) r_v^2 dr_v \wedge d\theta_v\right]. \end{aligned} \tag{194}$$

Moreover, the complex structure J on U_2 is given locally by

$$J(\partial_{r_x}) = \frac{1}{r_x} \partial_{\theta_x}, \qquad J(\partial_{\theta_x}) = -r_x \partial_{r_x} \tag{195}$$

and similar expressions for the other coordinates. Then one can check that the terms in the first three lines of (194) are positive definite since ω_0, ω_c must be positive definite. Therefore it suffices to check that the form defined by the last two lines of (194) is positive semi-definite. At this point, recall that we have chosen $\rho(t)$ a decreasing function, hence $\rho'(t) \leq 0$. Given the explicit form of the complex structure, it is easy to check that the form in question is positive semi-definite if $F(r_x, r_y, r_u, r_v) - c \leq 0$. Since the polycylinder is compact, this can be achieved throughout $C_\epsilon(r)$ if we choose c sufficiently small. Therefore, after a rescaling of the standard symplectic form ω_c by a sufficiently small positive constant, ω is a symplectic Kähler form on Z. By construction ω agrees with ω_c in a neighborhood of L_1, hence L_1 is a lagrangian cycle. It is now obvious that we can perform the same construction for L_2.

B Degree 3 and 4 Chern-Simons Computation

Let us consider now the third order terms in (33). By successively expanding the exponential and the logarithm, we obtain

$$\begin{aligned}\mathcal{F}_{inst} \qquad & (g_s, t_1, t_2, t_c, \lambda_1, \lambda_2)^{(3)} = e^{-3t_1}X_{(3t_1)} + e^{-3t_2}X_{(3t_2)} + e^{-3t_c}X_{(3t_c)} \\ & + e^{-2t_1-t_2}X_{(2t_1,t_2)} + e^{-t_1-2t_2}X_{(t_1,2t_2)} + e^{-t_1-t_2-t_c}X_{(t_1,t_2,t_c)} + e^{-2t_1-t_c}X_{(2t_1,t_c)} \\ & + e^{-2t_2-t_c}X_{(2t_2,t_c)} + e^{-t_1-2t_c}X_{(t_1,2t_c)} + e^{-t_2-2t_c}X_{(t_2,2t_c)},\end{aligned} \tag{196}$$

where

$$\begin{aligned} X_{(3t_i)} &= x_{(3t_i)} - x_{(2t_i)}x_{(t_i)} + \tfrac{1}{3}x^3_{(t_i)}, \quad i = 1, 2, c, \\ X_{(2t_i,t_j)} &= x_{(2t_i,t_j)} - x_{(2t_i)}x_{(t_j)} - x_{(t_i,t_j)}x_{(t_i)} + x^2_{(t_i)}x_{(t_j)} \quad i, j = 1, 2, c,\ i \neq j, \\ X_{(t_1,t_2,t_c)} &= x_{(t_1,t_2,t_c)} - x_{(t_1,t_2)}x_{(t_c)} - x_{(t_1,t_c)}x_{(t_2)} - x_{(t_2,t_c)}x_{(t_1)} + 2x_{(t_1)}x_{(t_2)}x_{(t_c)}, \end{aligned} \tag{197}$$

with $x_{(\ldots)}$ defined as in (39), (46) and

$$\begin{aligned} x_{(3t_1)} &= i\langle a_3\rangle - \langle a_1a_2\rangle - \tfrac{i}{6}\langle a_1^3\rangle, \quad x_{(3t_2)} = i\langle b_3\rangle - \langle b_1b_2\rangle - \tfrac{i}{6}\langle b_1^3\rangle, \\ x_{(3t_c)} &= -\langle c_3\rangle + \langle c_1c_2\rangle - \tfrac{1}{6}\langle c_1^3\rangle, \quad x_{(2t_1,t_2)} = -\langle a_2b_1\rangle + 2i\langle a_1d_1\rangle - \tfrac{i}{2}\langle a_1^2b_1\rangle, \\ x_{(t_1,2t_2)} &= -\langle a_1b_2\rangle + 2i\langle b_1d_1\rangle - \tfrac{i}{2}\langle a_1b_1^2\rangle, \quad x_{(t_1,t_2,t_c)} = \langle a_1b_1c_1\rangle - 2\langle c_1d_1\rangle, \\ x_{(2t_1,t_c)} &= -i\langle a_2c_1\rangle + \tfrac{1}{2}\langle a_1^2c_1\rangle, \quad x_{(2t_2,t_c)} = -i\langle b_2c_1\rangle + \tfrac{1}{2}\langle b_1^2c_1\rangle, \\ x_{(t_1,2t_c)} &= -i\langle a_1c_2\rangle + \tfrac{i}{2}\langle a_1c_1^2\rangle, \quad x_{(t_2,2t_c)} = -i\langle b_1c_2\rangle + \tfrac{i}{2}\langle b_1c_1^2\rangle. \end{aligned} \tag{198}$$

First, we evaluate $X_{(3t_1)}$. In doing so, we have to use again the Frobenius formula in order to linearize cubic expressions in the holonomy variables. We have

$$\begin{aligned} (\mathrm{Tr}V_1)^3 &= \mathrm{Tr}_{\yng(3)}V_1 + 2\mathrm{Tr}_{\yng(2,1)}V_1 + \mathrm{Tr}_{\yng(1,1,1)}V_1 \\ \mathrm{Tr}V_1\mathrm{Tr}V_1^2 &= \mathrm{Tr}_{\yng(3)}V_1 - \mathrm{Tr}_{\yng(1,1,1)}V_1 \\ \mathrm{Tr}V_1^3 &= \mathrm{Tr}_{\yng(3)}V_1 - \mathrm{Tr}_{\yng(2,1)}V_1 + \mathrm{Tr}_{\yng(1,1,1)}V_1. \end{aligned} \tag{199}$$

Applying (40) we have

$$\begin{aligned}
\left\langle (\mathrm{Tr}V_1)^3 \right\rangle &= e^{\frac{3}{2}i\lambda_1} e^{3ig_s} \left\langle \mathrm{Tr}_{\yng(3)} V_1 \right\rangle_0 + 2e^{3ig_s} \left\langle \mathrm{Tr}_{\yng(2,1)} V_1 \right\rangle_0 + e^{-\frac{3}{2}i\lambda_1} e^{3ig_s} \left\langle \mathrm{Tr}_{\yng(1,1,1)} V_1 \right\rangle_0 \\
\left\langle \mathrm{Tr}V_1 \mathrm{Tr}V_1^2 \right\rangle &= e^{\frac{3}{2}i\lambda_1} e^{3ig_s} \left\langle \mathrm{Tr}_{\yng(3)} V_1 \right\rangle_0 - e^{-\frac{3}{2}i\lambda_1} e^{3ig_s} \left\langle \mathrm{Tr}_{\yng(1,1,1)} V_1 \right\rangle_0 \\
\left\langle \mathrm{Tr}V_1^3 \right\rangle &= e^{\frac{3}{2}i\lambda_1} e^{3ig_s} \left\langle \mathrm{Tr}_{\yng(3)} V_1 \right\rangle_0 - e^{3ig_s} \left\langle \mathrm{Tr}_{\yng(2,1)} V_1 \right\rangle_0 + e^{-\frac{3}{2}i\lambda_1} e^{3ig_s} \left\langle \mathrm{Tr}_{\yng(1,1,1)} V_1 \right\rangle_0 .
\end{aligned} \tag{200}$$

The expectation values in the canonical framing are given by

$$\begin{aligned}
\left\langle \mathrm{Tr}_{\yng(3)} V_1 \right\rangle_0 &= \frac{(y-y^{-1})(yx-y^{-1}x^{-1})(yx^2-y^{-1}x^{-2})}{(x-x^{-1})(x^2-x^{-2})(x^3-x^{-3})}, \\
\left\langle \mathrm{Tr}_{\yng(2,1)} V_1 \right\rangle_0 &= \frac{(x+x^{-1})(y-y^{-1})(yx-y^{-1}x^{-1})(yx^{-1}-y^{-1}x)}{(x-x^{-1})(x^2-x^{-2})(x^3-x^{-3})}, \\
\left\langle \mathrm{Tr}_{\yng(1,1,1)} V_1 \right\rangle_0 &= \frac{(y-y^{-1})(yx^{-1}-y^{-1}x)(yx^{-2}-y^{-1}x^2)}{(x-x^{-1})(x^2-x^{-2})(x^3-x^{-3})},
\end{aligned} \tag{201}$$

where $x = e^{\frac{1}{2}ig_s}$ and $y = e^{\frac{1}{2}i\lambda_1}$. Using (199)-(201), a straightforward computation gives

$$X_{(3t_1)} = \frac{e^{\frac{5}{2}i\lambda_1}}{(2\sin\frac{g_s}{2})^2}(e^{i\lambda_1}-1)(2e^{i\lambda_1}-1) + \frac{1}{3(2\sin\frac{3g_s}{2})^2}(-e^{\frac{3}{2}i\lambda_1}+e^{\frac{9}{2}i\lambda_1}) + e^{\frac{7}{2}i\lambda_1} - e^{\frac{9}{2}i\lambda_1}. \tag{202}$$

By symmetry, we can therefore write

$$X_{(3t_2)} = \frac{e^{\frac{5}{2}i\lambda_2}}{(2\sin\frac{g_s}{2})^2}(e^{i\lambda_2}-1)(2e^{i\lambda_2}-1) + \frac{1}{3(2\sin\frac{3g_s}{2})^2}(-e^{\frac{3}{2}i\lambda_2}+e^{\frac{9}{2}i\lambda_2}) + e^{\frac{7}{2}i\lambda_2} - e^{\frac{9}{2}i\lambda_2}. \tag{203}$$

A similar computation gives

$$X_{(3t_c)} = \frac{1}{3(2\sin\frac{3g_s}{2})^2}(e^{\frac{3}{2}i\lambda_1}-e^{-\frac{3}{2}i\lambda_1})(e^{\frac{3}{2}i\lambda_2}-e^{-\frac{3}{2}i\lambda_2}). \tag{204}$$

Next, $X_{(2t_1,t_2)}$, $X_{(t_1,2t_2)}$ and $X_{(t_1,t_2,t_c)}$ can be easily computed using (42), (48), (49), (55) and the exchange symmetry. The results read

$$\begin{aligned}
X_{(2t_1,t_2)} &= 2\frac{e^{i(\frac{3}{2}\lambda_1+\lambda_2)}}{(2\sin\frac{g_s}{2})^2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(3e^{i\lambda_1}-1) \\
&\quad - 2e^{\frac{5}{2}i\lambda_1+i\lambda_2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2}), \\
X_{(t_1,2t_2)} &= 2\frac{e^{i(\lambda_1+\frac{3}{2}\lambda_2)}}{(2\sin\frac{g_s}{2})^2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(3e^{i\lambda_2}-1) \\
&\quad - 2e^{\frac{5}{2}i\lambda_2+i\lambda_1}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2}), \\
X_{(t_1,t_2,t_c)} &= \frac{e^{\frac{1}{2}i(\lambda_1+\lambda_2)}}{(2\sin\frac{g_s}{2})^2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(5e^{i\lambda_1+i\lambda_2}-2e^{i\lambda_1}-2e^{i\lambda_2}) \\
&\quad - 2e^{\frac{3}{2}i\lambda_1+\frac{3}{2}i\lambda_2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2}).
\end{aligned} \tag{205}$$

Next, in order to evaluate $X_{(2t_1,t_c)}$ and $X_{(2t_2,t_c)}$ we need to compute expectation values of the form $\langle(\mathrm{Tr}V_i)^2\mathrm{Tr}U_i\rangle$, $\langle\mathrm{Tr}V_i^2\mathrm{Tr}U_i\rangle$ for the Hopf links (Γ_i, Ξ_i) with $i = 1, 2$. Again, due to the exchange symmetry, it is sufficient to compute the above expectation values for the first link only. After taking into account the framing correction we have

$$\begin{aligned}\langle(\mathrm{Tr}V_1)^2\mathrm{Tr}U_1\rangle &= e^{2ig_s}e^{2i\lambda_1}\langle\mathrm{Tr}_{\square\square}V_1\mathrm{Tr}U_1\rangle_0 + e^{-2ig_s}e^{2i\lambda_1}\langle\mathrm{Tr}_{\substack{\square\\\square}}V_1\mathrm{Tr}U_1\rangle_0,\\ \langle\mathrm{Tr}V_1^2\mathrm{Tr}U_1\rangle &= e^{2ig_s}e^{2i\lambda_1}\langle\mathrm{Tr}_{\square\square}V_1\mathrm{Tr}U_1\rangle_0 - e^{-2ig_s}e^{2i\lambda_1}\langle\mathrm{Tr}_{\substack{\square\\\square}}V_1\mathrm{Tr}U_1\rangle_0.\end{aligned} \tag{206}$$

Now, for a Hopf link with linking number -1 we have

$$\langle\mathrm{Tr}_{R_1}V_1\mathrm{Tr}_{R_2}V_2\rangle_0 = e^{\frac{ig_s}{2}(k_{R_1}+k_{R_2})}\sum_{\rho\in R_1\otimes R_2} e^{-\frac{ig_s}{2}k_\rho}\dim_q\rho, \tag{207}$$

where the sum is after all the representations ρ occurring in the decomposition of the tensor product of R_1 and R_2 and $\dim_q\rho$ is the quantum dimension of the representation ρ [39] . Now, using (42), (48), (49), (55), (206) and (207) and keeping in mind that in our case the linking number is $+1$ we obtain after a simple computation

$$\begin{aligned}X_{(2t_1,t_c)} = \frac{e^{2i\lambda_1}}{(2\sin\frac{g_s}{2})^2}&(3e^{i\lambda_1}-1)(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})\\ &- e^{3i\lambda_1}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2}).\end{aligned} \tag{208}$$

By the exchange symmetry, we have

$$\begin{aligned}X_{(2t_2,t_c)} = \frac{e^{2i\lambda_2}}{(2\sin\frac{g_s}{2})^2}&(3e^{i\lambda_2}-1)(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})\\ &- e^{3i\lambda_2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2}).\end{aligned} \tag{209}$$

Finally, in order to compute $X_{(t_1,2t_c)}$ and $X_{(t_2,2t_c)}$ we need to evaluate expectation values of the form $\langle\mathrm{Tr}V_i(\mathrm{Tr}U_i)^2\rangle$, $\langle\mathrm{Tr}V_i\mathrm{Tr}U_i^2\rangle$ for the Hopf links (Γ_i, Ξ_i) with $i = 1, 2$. Taking into account the framing correction we have, for one of the links,

$$\begin{aligned}\langle\mathrm{Tr}V_1(\mathrm{Tr}U_1)^2\rangle &= e^{i\lambda_1}(\langle\mathrm{Tr}V_1\mathrm{Tr}_{\square\square}U_1\rangle_0 + \langle\mathrm{Tr}V_1\mathrm{Tr}_{\substack{\square\\\square}}U_1\rangle_0),\\ \langle\mathrm{Tr}V_1\mathrm{Tr}U_1^2\rangle &= e^{i\lambda_1}(\langle\mathrm{Tr}V_1\mathrm{Tr}_{\square\square}U_1\rangle_0 - \langle\mathrm{Tr}V_1\mathrm{Tr}_{\substack{\square\\\square}}U_1\rangle_0).\end{aligned} \tag{210}$$

Now, using (42), (48), (49), (55), (207) and (210) and the exchange symmetry, we obtain after another direct computation

$$\begin{aligned}X_{(t_1,2t_c)} &= -\frac{e^{i(2\lambda_1-\frac{1}{2}\lambda_2)}}{(2\sin\frac{g_s}{2})^2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2}),\\ X_{(t_2,2t_c)} &= -\frac{e^{i(-\frac{1}{2}\lambda_1+2\lambda_2)}}{(2\sin\frac{g_s}{2})^2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2}).\end{aligned} \tag{211}$$

Let us consider now the fourth order terms in (33). By successively expanding the exponential and the logarithm, we obtain

$$\begin{aligned}\mathcal{F}_{inst} \quad &(g_s, t_1, t_2, t_c, \lambda_1, \lambda_2)^{(4)} = e^{-4t_1} X_{(4t_1)} + e^{-4t_2} X_{(4t_2)} + e^{-4t_c} X_{(4t_c)} \\ &+e^{-3t_1-t_2} X_{(3t_1,t_2)} + e^{-2t_1-2t_2} X_{(2t_1,2t_2)} + e^{-t_1-3t_2} X_{(t_1,3t_2)} + e^{-3t_1-t_c} X_{(3t_1,t_c)} \\ &+e^{-2t_1-t_2-t_c} X_{(2t_1,t_2,t_c)} + e^{-t_1-2t_2-t_c} X_{(t_1,2t_2,t_c)} + e^{-3t_2-t_c} X_{(3t_2,t_c)} \\ &+e^{-2t_1-2t_c} X_{(2t_1,2t_c)} + e^{-t_1-t_2-2t_c} X_{(t_1,t_2,2t_c)} + e^{-2t_2-2t_c} X_{(2t_2,2t_c)} \\ &+e^{-t_1-3t_c} X_{(t_1,3t_c)} + e^{-t_2-3t_c} X_{(t_2,3t_c)},\end{aligned} \tag{212}$$

where

$$\begin{aligned}X_{(4t_i)} &= x_{(4t_i)} - x_{(3t_i)}x_{(t_i)} - \tfrac{1}{2}x^2_{(2t_i)} + x_{(2t_i)}x^2_{(t_i)} - \tfrac{1}{4}x^4_{(t_i)}, \quad i = 1, 2, c, \\ X_{(3t_i,t_j)} &= x_{(3t_i,t_j)} - x_{(3t_i)}x_{(t_j)} - x_{(2t_i,t_j)}x_{(t_i)} - x_{(2t_i)}x_{(t_i,t_j)} + 2x_{(2t_i)}x_{(t_i)}x_{(t_j)} \\ &\quad - x^3_{(t_i)}x_{(t_j)} + x^2_{(t_i)}x_{(t_i,t_j)}, \quad i, j = 1, 2, c, \ i \neq j, \\ X_{(2t_i,2t_j)} &= x_{(2t_i,2t_j)} - x_{(2t_i)}x_{(2t_j)} - x_{(2t_i,t_j)}x_{(t_j)} - x_{(t_i,2t_j)}x_{(t_i)} + x_{(2t_i)}x^2_{(t_j)} \\ &\quad +x_{(2t_j)}x^2_{(t_i)} - \tfrac{1}{2}x^2_{(t_i,t_j)} - \tfrac{3}{2}x^2_{(t_i)}x^2_{(t_j)} + 2x_{(t_i,t_j)}x_{(t_i)}x_{(t_j)}, \\ &\quad \text{for } \ i, j = 1, 2, c, \ i \neq j, \\ X_{(2t_i,t_j,t_k)} &= x_{(2t_i,t_j,t_k)} - x_{(2t_i,t_j)}x_{(t_k)} - x_{(2t_i,t_k)}x_{(t_j)} - x_{(2t_i)}x_{(t_j,t_k)} + 2x_{(2t_i)}x_{(t_j)}x_{(t_k)} \\ &\quad - x_{(t_i,t_j,t_k)}x_{(t_i)} + x^2_{(t_i)}x_{(t_j,t_k)} - 3x^2_{(t_i)}x_{(t_j)}x_{(t_k)} - x_{(t_i,t_j)}x_{(t_i,t_k)} \\ &\quad + 2x_{(t_i,t_j)}x_{(t_i)}x_{(t_k)} + 2x_{(t_i,t_k)}x_{(t_i)}x_{(t_j)}, \\ &\quad \text{for } \ i, j, k = 1, 2, c, \ i \neq j, \ j \neq k, \ k \neq i.\end{aligned} \tag{213}$$

with

$$\begin{aligned}x_{(4t_1)} &= i\langle a_4\rangle - \langle a_1a_3\rangle - \tfrac{1}{2}\langle a_2^2\rangle - \tfrac{i}{2}\langle a_1^2a_2\rangle + \tfrac{1}{24}\langle a_1^4\rangle, \\ x_{(4t_2)} &= i\langle b_4\rangle - \langle b_1b_3\rangle - \tfrac{1}{2}\langle b_2^2\rangle - \tfrac{i}{2}\langle b_1^2b_2\rangle + \tfrac{1}{24}\langle b_1^4\rangle, \\ x_{(4t_c)} &= -\langle c_4\rangle + \langle c_1c_3\rangle + \tfrac{1}{2}\langle c_2^2\rangle - \tfrac{1}{2}\langle c_1^2c_2\rangle + \tfrac{1}{24}\langle c_1^4\rangle, \\ x_{(3t_1,t_2)} &= -\langle a_3b_1\rangle + 2i\langle a_2d_1\rangle - i\langle a_1a_2b_1\rangle - \langle a_1^2d_1\rangle + \tfrac{1}{6}\langle a_1^3b_1\rangle, \\ x_{(2t_1,2t_2)} &= -\langle a_2b_2\rangle + 2\langle d_2\rangle - \tfrac{i}{2}\langle a_1^2b_2\rangle - \tfrac{i}{2}\langle a_2b_1^2\rangle + 2\langle d_1^2\rangle + \tfrac{1}{4}\langle a_1^2b_1^2\rangle - 2\langle a_1b_1d_1\rangle, \\ x_{(t_1,3t_2)} &= -\langle a_1b_3\rangle + 2i\langle b_2d_1\rangle - i\langle a_1b_1b_2\rangle - \langle b_1^2d_1\rangle + \tfrac{1}{6}\langle a_1b_1^3\rangle, \\ x_{(3t_1,t_c)} &= -i\langle a_3c_1\rangle + \langle a_1a_2c_1\rangle + \tfrac{i}{6}\langle a_1^3c_1\rangle, \\ x_{(3t_2,t_c)} &= -i\langle b_3c_1\rangle + \langle b_1b_2c_1\rangle + \tfrac{i}{6}\langle b_1^3c_1\rangle, \\ x_{(2t_1,t_2,t_c)} &= \langle a_2b_1c_1\rangle - 2i\langle a_1c_1d_1\rangle + \tfrac{i}{2}\langle a_1^2b_1c_1\rangle, \\ x_{(t_1,2t_2,t_c)} &= \langle a_1b_2c_1\rangle - 2i\langle b_1c_1d_1\rangle + \tfrac{i}{2}\langle a_1b_1^2c_1\rangle, \\ x_{(2t_1,2t_c)} &= -i\langle a_2c_2\rangle + \tfrac{i}{2}\langle a_2c_1^2\rangle + \tfrac{1}{2}\langle a_1^2c_2\rangle - \tfrac{1}{4}\langle a_1^2c_1^2\rangle, \\ x_{(t_1,t_2,2t_c)} &= \langle a_1b_1c_2\rangle - 2\langle c_2d_1\rangle - \tfrac{1}{2}\langle a_1b_1c_1^2\rangle + \langle c_1^2d_1\rangle, \\ x_{(2t_2,2t_c)} &= -i\langle b_2c_2\rangle + \tfrac{i}{2}\langle b_2c_1^2\rangle + \tfrac{1}{2}\langle b_1^2c_2\rangle - \tfrac{1}{4}\langle b_1^2c_1^2\rangle, \\ x_{(t_1,3t_c)} &= -i\langle a_1c_3\rangle + i\langle a_1c_1c_2\rangle - \tfrac{i}{6}\langle a_1c_1^3\rangle, \\ x_{(t_2,3t_c)} &= -i\langle b_1c_3\rangle + i\langle b_1c_1c_2\rangle - \tfrac{i}{6}\langle b_1c_1^3\rangle\end{aligned} \tag{214}$$

and the other $x_{(\dots)}$ defined as in (39), (46) and (198). First, we evaluate $X_{(4t_1)}$. In doing so, we have to use again the Frobenius formula in order to linearize quartic expressions in the holonomy variables. We have

$$\begin{aligned}
(\mathrm{Tr}V)^4 &= \mathrm{Tr}_{\tiny\yng(4)}V + 3\mathrm{Tr}_{\tiny\yng(3,1)}V + 2\mathrm{Tr}_{\tiny\yng(2,2)}V + 3\mathrm{Tr}_{\tiny\yng(2,1,1)}V + \mathrm{Tr}_{\tiny\yng(1,1,1,1)}V,\\
(\mathrm{Tr}V)^2\,\mathrm{Tr}V^2 &= \mathrm{Tr}_{\tiny\yng(4)}V + \mathrm{Tr}_{\tiny\yng(3,1)}V - \mathrm{Tr}_{\tiny\yng(2,1,1)}V - \mathrm{Tr}_{\tiny\yng(1,1,1,1)}V,\\
\mathrm{Tr}V^2\mathrm{Tr}V^2 &= \mathrm{Tr}_{\tiny\yng(4)}V - \mathrm{Tr}_{\tiny\yng(3,1)}V + 2\mathrm{Tr}_{\tiny\yng(2,2)}V - \mathrm{Tr}_{\tiny\yng(2,1,1)}V + \mathrm{Tr}_{\tiny\yng(1,1,1,1)}V,\\
\mathrm{Tr}V\mathrm{Tr}V^3 &= \mathrm{Tr}_{\tiny\yng(4)}V - \mathrm{Tr}_{\tiny\yng(2,2)}V + \mathrm{Tr}_{\tiny\yng(1,1,1,1)}V,\\
\mathrm{Tr}V^4 &= \mathrm{Tr}_{\tiny\yng(4)}V - \mathrm{Tr}_{\tiny\yng(3,1)}V + \mathrm{Tr}_{\tiny\yng(2,1,1)}V - \mathrm{Tr}_{\tiny\yng(1,1,1,1)}V.
\end{aligned} \tag{215}$$

The expectation values in the canonical framing are given by

$$\begin{aligned}
\left\langle \mathrm{Tr}_{\tiny\yng(4)}V_1\right\rangle_0 &= \frac{(y-y^{-1})(yx-y^{-1}x^{-1})(yx^2-y^{-1}x^{-2})(yx^3-y^{-1}x^{-3})}{(x-x^{-1})(x^2-x^{-2})(x^3-x^{-3})(x^4-x^{-4})},\\
\left\langle \mathrm{Tr}_{\tiny\yng(3,1)}V_1\right\rangle_0 &= \frac{(y-y^{-1})(yx-y^{-1}x^{-1})(yx^2-y^{-1}x^{-2})(yx^{-1}-y^{-1}x)}{(x-x^{-1})^2(x^2-x^{-2})(x^4-x^{-4})},\\
\left\langle \mathrm{Tr}_{\tiny\yng(2,2)}V_1\right\rangle_0 &= \frac{(y-y^{-1})^2(yx-y^{-1}x^{-1})(yx^{-1}-y^{-1}x)}{(x-x^{-1})(x^2-x^{-2})^2(x^3-x^{-3})},\\
\left\langle \mathrm{Tr}_{\tiny\yng(2,1,1)}V_1\right\rangle_0 &= \frac{(y-y^{-1})(yx-y^{-1}x^{-1})(yx^{-1}-y^{-1}x)(yx^{-2}-y^{-1}x^2)}{(x-x^{-1})^2(x^2-x^{-2})(x^4-x^{-4})},\\
\left\langle \mathrm{Tr}_{\tiny\yng(1,1,1,1)}V_1\right\rangle_0 &= \frac{(y-y^{-1})(yx^{-1}-y^{-1}x)(yx^{-2}-y^{-1}x^2)(yx^{-3}-y^{-1}x^3)}{(x-x^{-1})(x^2-x^{-2})(x^3-x^{-3})(x^4-x^{-4})},
\end{aligned} \tag{216}$$

where $x=e^{\frac{1}{2}ig_s}$ and $y=e^{\frac{1}{2}i\lambda_1}$. We also have

$$k_{\tiny\yng(4)}=12,\quad k_{\tiny\yng(3,1)}=4,\quad k_{\tiny\yng(2,2)}=0,\quad k_{\tiny\yng(2,1,1)}=-4,\quad k_{\tiny\yng(1,1,1,1)}=-12. \tag{217}$$

Now, using (215) , (216) and (217) we obtain

$$\begin{aligned}
X_{(4t_1)} = {}& \frac{e^{3i\lambda_1}}{(2\sin\frac{g_s}{2})^2}(e^{i\lambda_1}-1)(-7e^{2i\lambda_1}+6e^{i\lambda_1}-1)+\frac{1}{2(2\sin g_s)^2}(e^{4i\lambda_1}-e^{6i\lambda_1})\\
&+\frac{1}{4(2\sin 2g_s)^2}(-e^{2i\lambda_1}+e^{6i\lambda_1})+e^{4i\lambda_1}(e^{i\lambda_1}-1)(11e^{i\lambda_1}-5)\\
&+\left(2\sin\tfrac{g_s}{2}\right)^2e^{4i\lambda_1}(e^{i\lambda_1}-1)(-6e^{i\lambda_1}+1)+\left(2\sin\tfrac{g_s}{2}\right)^4(-e^{5i\lambda_1}+e^{6i\lambda_1}),
\end{aligned} \tag{218}$$

and, by the exchange symmetry,

$$\begin{aligned}
X_{(4t_2)} = {}& \frac{e^{3i\lambda_2}}{(2\sin\frac{g_s}{2})^2}(e^{i\lambda_2}-1)(-7e^{2i\lambda_2}+6e^{i\lambda_2}-1)+\frac{1}{2(2\sin g_s)^2}(e^{4i\lambda_2}-e^{6i\lambda_2})\\
&+\frac{1}{4(2\sin 2g_s)^2}(-e^{2i\lambda_2}+e^{6i\lambda_2})+e^{4i\lambda_2}(e^{i\lambda_2}-1)(11e^{i\lambda_2}-5)\\
&+\left(2\sin\tfrac{g_s}{2}\right)^2e^{4i\lambda_2}(e^{i\lambda_2}-1)(-6e^{i\lambda_2}+1)+\left(2\sin\tfrac{g_s}{2}\right)^4(-e^{5i\lambda_2}+e^{6i\lambda_2}).
\end{aligned} \tag{219}$$

Another straightforward calculation yields

$$X_{(4t_c)} = \frac{1}{4(2\mathrm{sin}2g_s)^2}(e^{2i\lambda_1} - e^{-2i\lambda_1})(e^{2i\lambda_2} - e^{-2i\lambda_2}). \tag{220}$$

Now, using (42), (48), (49) (55), (200) and (201), we get

$$\begin{aligned} X_{(3t_1,t_2)} = \frac{2e^{i(2\lambda_1+\lambda_2)}}{(2\mathrm{sin}\frac{g_s}{2})^2}&(e^{\frac{1}{2}i\lambda_1} - e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2} - e^{-\frac{1}{2}i\lambda_2})(-13e^{2i\lambda_1} + 9e^{i\lambda_1} - 1) \\ &+ 4e^{i(3\lambda_1+\lambda_2)}(e^{\frac{1}{2}i\lambda_1} - e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2} - e^{-\frac{1}{2}i\lambda_2})(8e^{i\lambda_1} - 1) \\ &+ 2\left(2\mathrm{sin}\tfrac{g_s}{2}\right)^2 e^{i(3\lambda_1+\lambda_2)}(e^{\frac{1}{2}i\lambda_1} - e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2} - e^{-\frac{1}{2}i\lambda_2})(-7e^{i\lambda_1} + 1) \\ &+ 2\left(2\mathrm{sin}\tfrac{g_s}{2}\right)^4 e^{i(4\lambda_1+\lambda_2)}(e^{\frac{1}{2}i\lambda_1} - e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2} - e^{-\frac{1}{2}i\lambda_2}), \end{aligned} \tag{221}$$

and, by the exchange symmetry,

$$\begin{aligned} X_{(t_1,3t_2)} = \frac{2e^{i(\lambda_1+2\lambda_2)}}{(2\mathrm{sin}\frac{g_s}{2})^2}&(e^{\frac{1}{2}i\lambda_1} - e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2} - e^{-\frac{1}{2}i\lambda_2})(-13e^{2i\lambda_2} + 9e^{i\lambda_2} - 1) \\ &+ 4e^{i(\lambda_1+3\lambda_2)}(e^{\frac{1}{2}i\lambda_1} - e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2} - e^{-\frac{1}{2}i\lambda_2})(8e^{i\lambda_2} - 1) \\ &+ 2\left(2\mathrm{sin}\tfrac{g_s}{2}\right)^2 e^{i(\lambda_1+3\lambda_2)}(e^{\frac{1}{2}i\lambda_1} - e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2} - e^{-\frac{1}{2}i\lambda_2})(-7e^{i\lambda_2} + 1) \\ &+ 2\left(2\mathrm{sin}\tfrac{g_s}{2}\right)^4 e^{i(\lambda_1+4\lambda_2)}(e^{\frac{1}{2}i\lambda_1} - e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2} - e^{-\frac{1}{2}i\lambda_2}). \end{aligned} \tag{222}$$

Next, a similar computation gives

$$\begin{aligned}
&X_{(2t_1,2t_2)}\\
&= \frac{2e^{\frac{3}{2}i(\lambda_1+\lambda_2)}}{(2\sin\frac{g_s}{2})^2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(-18e^{i(\lambda_1+\lambda_2)}+9e^{i\lambda_1}+9e^{i\lambda_2}-4)\\
&\quad -\frac{2e^{2i(\lambda_1+\lambda_2)}}{2(2\sin g_s)^2}(e^{i\lambda_1}-e^{-i\lambda_1})(e^{i\lambda_2}-e^{-i\lambda_2})\\
&\quad +e^{\frac{3}{2}i(\lambda_1+\lambda_2)}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(45e^{i(\lambda_1+\lambda_2)}-14e^{i\lambda_1}-14e^{i\lambda_2}+3)\\
&\quad +\left(2\sin\tfrac{g_s}{2}\right)^2 e^{\frac{3}{2}i(\lambda_1+\lambda_2)}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(-20e^{i(\lambda_1+\lambda_2)}+3e^{i\lambda_1}+3e^{i\lambda_2}-1)\\
&\quad +3\left(2\sin\tfrac{g_s}{2}\right)^4 e^{\frac{3}{2}i(\lambda_1+\lambda_2)}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2}).
\end{aligned}\tag{223}$$

Now we compute $X_{(3t_1,t_c)}$ and $X_{(3t_2,t_c)}$. Using (42), (48), (49), (55), (200), (201) and (207) and the fact that the linking number is $+1$ we obtain

$$\begin{aligned}
X_{(3t_1,t_c)} &= \frac{e^{\frac{5}{2}i\lambda_1}}{(2\sin\frac{g_s}{2})^2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(-13e^{2i\lambda_1}+9e^{i\lambda_1}-1)\\
&\quad +2e^{\frac{7}{2}i\lambda_1}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(8e^{i\lambda_1}-1)\\
&\quad +\left(2\sin\tfrac{g_s}{2}\right)^2 e^{\frac{7}{2}i\lambda_1}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(-7e^{i\lambda_1}+1)\\
&\quad +\left(2\sin\tfrac{g_s}{2}\right)^4 e^{\frac{9}{2}i\lambda_1}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2}),
\end{aligned}\tag{224}$$

and, by the exchange symmetry,

$$\begin{aligned}
X_{(3t_2,t_c)} &= \frac{e^{\frac{5}{2}i\lambda_2}}{(2\sin\frac{g_s}{2})^2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(-13e^{2i\lambda_2}+9e^{i\lambda_2}-1)\\
&\quad +2e^{\frac{7}{2}i\lambda_2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(8e^{i\lambda_2}-1)\\
&\quad +\left(2\sin\tfrac{g_s}{2}\right)^2 e^{\frac{7}{2}i\lambda_2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(-7e^{i\lambda_2}+1)\\
&\quad +\left(2\sin\tfrac{g_s}{2}\right)^4 e^{\frac{9}{2}i\lambda_2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2}).
\end{aligned}\tag{225}$$

Another direct computation gives

$$
\begin{aligned}
X_{(2t_1,t_2,t_c)} &= \frac{e^{i(\lambda_1+\frac{1}{2}\lambda_2)}}{(2\sin\frac{g_s}{2})^2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})\\
&\times(-29e^{i(2\lambda_1+\lambda_2)}+18e^{2i\lambda_1}+19e^{i(\lambda_1+\lambda_2)}-8e^{i\lambda_1}-2e^{i\lambda_2})\\
&+e^{i(2\lambda_1+\frac{1}{2}\lambda_2)}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(33e^{i(\lambda_1+\lambda_2)}-12e^{i\lambda_1}-12e^{i\lambda_2}+2)\\
&+2\left(2\sin\frac{g_s}{2}\right)^2 e^{i(2\lambda_1+\frac{1}{2}\lambda_2)}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(-7e^{i(\lambda_1+\lambda_2)}+e^{i\lambda_1}+e^{i\lambda_2})\\
&+2\left(2\sin\frac{g_s}{2}\right)^4 e^{i(3\lambda_1+\frac{3}{2}\lambda_2)}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2}),
\end{aligned}
\tag{226}
$$

and, by the exchange symmetry,

$$
\begin{aligned}
X_{(t_1,2t_2,t_c)} &= \frac{e^{i(\frac{1}{2}\lambda_1+\lambda_2)}}{(2\sin\frac{g_s}{2})^2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})\\
&\times(-29e^{i(\lambda_1+2\lambda_2)}+18e^{2i\lambda_2}+19e^{i(\lambda_1+\lambda_2)}-8e^{i\lambda_2}-2e^{i\lambda_1})\\
&+e^{i(\frac{1}{2}\lambda_1+2\lambda_2)}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(33e^{i(\lambda_1+\lambda_2)}-12e^{i\lambda_1}-12e^{i\lambda_2}+2)\\
&+2\left(2\sin\frac{g_s}{2}\right)^2 e^{i(\frac{1}{2}\lambda_1+2\lambda_2)}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(-7e^{i(\lambda_1+\lambda_2)}+e^{i\lambda_1}+e^{i\lambda_2})\\
&+2\left(2\sin\frac{g_s}{2}\right)^4 e^{i(\frac{3}{2}\lambda_1+3\lambda_2)}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2}).
\end{aligned}
\tag{227}
$$

Next, we compute $X_{(2t_1,2t_c)}$ and $X_{(2t_2,2t_c)}$. Using (42), (48), (49), (55), (206), (207) and (210) we obtain

$$
\begin{aligned}
&X_{(2t_1,2t_c)}\\
&= \frac{e^{\frac{1}{2}i(5\lambda_1-\lambda_2)}}{(2\sin\frac{g_s}{2})^2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(-2e^{i(\lambda_1+\lambda_2)}+7e^{i\lambda_1}+e^{i\lambda_2}-3)\\
&+\frac{1}{(2\sin g_s)^2}(e^{2i\lambda_1}-e^{4i\lambda_1})(e^{i\lambda_2}-e^{-i\lambda_2})\\
&+e^{\frac{1}{2}i(5\lambda_1-\lambda_2)}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(e^{i(\lambda_1+\lambda_2)}-5e^{i\lambda_1}+1)\\
&+\left(2\sin\frac{g_s}{2}\right)^2 e^{\frac{1}{2}i(7\lambda_1-\lambda_2)}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2}),
\end{aligned}
\tag{228}
$$

and, by the exchange symmetry,

$$\begin{aligned}
&X_{(2t_2,2t_c)}\\
&= \frac{e^{\frac{1}{2}i(-\lambda_1+5\lambda_2)}}{(2\sin\frac{g_s}{2})^2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(-2e^{i(\lambda_1+\lambda_2)}+7e^{i\lambda_2}+e^{i\lambda_1}-3)\\
&\quad+\frac{1}{(2\sin g_s)^2}(e^{2i\lambda_2}-e^{4i\lambda_2})(e^{i\lambda_1}-e^{-i\lambda_1})\\
&\quad+e^{\frac{1}{2}i(-\lambda_1+5\lambda_2)}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(e^{i(\lambda_1+\lambda_2)}-5e^{i\lambda_2}+1)\\
&\quad+\left(2\sin\tfrac{g_s}{2}\right)^2 e^{\frac{1}{2}i(-\lambda_1+7\lambda_2)}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2}).
\end{aligned} \tag{229}$$

After a computation similar to (226) we get

$$\begin{aligned}
X_{(t_1,t_2,2t_c)} &= \frac{e^{\frac{1}{2}i(\lambda_1+\lambda_2)}}{(2\sin\frac{g_s}{2})^2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})\\
&\quad\times(-2e^{i(\lambda_1+\lambda_2)}-2e^{i(\lambda_1-\lambda_2)}-2e^{i(-\lambda_1+\lambda_2)}+5e^{i\lambda_1}+5e^{i\lambda_2}-2)\\
&\quad-4e^{\frac{3}{2}i(\lambda_1+\lambda_2)}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2})(e^{\frac{1}{2}i(\lambda_1-\lambda_2)}+e^{\frac{1}{2}i(-\lambda_1+\lambda_2)}).
\end{aligned} \tag{230}$$

Finally, a computation similar to (224) yields

$$\begin{aligned}
X_{(t_1,3t_c)} &= -\frac{e^{i(\frac{5}{2}\lambda_1-\lambda_2)}}{(2\sin\frac{g_s}{2})^2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2}),\\
X_{(t_2,3t_c)} &= -\frac{e^{i(-\lambda_1+\frac{5}{2}\lambda_2)}}{(2\sin\frac{g_s}{2})^2}(e^{\frac{1}{2}i\lambda_1}-e^{-\frac{1}{2}i\lambda_1})(e^{\frac{1}{2}i\lambda_2}-e^{-\frac{1}{2}i\lambda_2}).
\end{aligned} \tag{231}$$

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