Adv. Theor. Math. Phys. 6 (2002) 575–591

Topological Quantum Field Theory for Calabi-Yau threefolds and G_2-manifolds

Naichung Conan Leung

School of Mathematics,
University of Minnesota,
Minneapolis, MN 55454,
USA.

LEUNG@MATH.UMN.EDU

1 Introduction

In the past two decades we have witnessed many fruitful interactions between mathematics and physics. One example is in the Donaldson-Floer theory for oriented four manifolds. Physical considerations lead to the discovery of the Seiberg-Witten theory which has profound impact to our understandings of four manifolds. Another example is in the mirror symmetry for Calabi-Yau manifolds. This duality transformation in the string theory leads to many surprising predictions in the enumerative geometry.

String theory in physics studies a ten dimensional space-time $X \times \mathbb{R}^{3,1}$. Here X a six dimensional Riemannian manifold with its holonomy group inside $SU(3)$, the so-called *Calabi-Yau threefold.* Certain parts of the mirror symmetry conjecture, as studied by Vafa's group, are specific for Calabi-Yau manifolds of complex dimension *three.* They include the Gopakumar-Vafa conjecture for the Gromov-Witten invariants of *arbitrary* genus, the Ooguri-Vafa conjecture on the relationships between knot invariants and enumerations of holomorphic disks and so on. The key reason is they belong to a duality theory for G_2-manifolds. G_2-manifolds can be naturally interpreted as special Octonion manifolds [23]. For any Calabi-Yau threefold X, the seven dimensional manifold $X \times S^1$ is automatically a G_2-manifold because of the natural inclusion $SU(3) \subset G_2$.

In recent years, there are many studies of G_2-manifolds in M-theory including works of Archaya, Atiyah, Gukov, Vafa, Witten, Yau, Zaslow and many others (e.g. [1], [5], [13], [2]).

In the studies of the symplectic geometry of a Calabi-Yau threefold X, we consider unitary flat bundles over three dimensional (special) Lagrangian submanifolds L in X. The corresponding geometry for a G_2-manifold M is called the *special* $\mathbb{H}$*-Lagrangian geometry* (or *C-geometry* in [19]). where we consider Anti-Self-Dual (abbrev. ASD) bundles over four dimensional coassociative submanifolds, or equivalently *special* $\mathbb{H}$*-Lagrangian submanifolds of type II* [23], (abbrev. $\mathbb{H}$-SLag) C in M.

Counting ASD bundles over a fixed four manifold C is the well-known theory of Donaldson differentiable invariants, $Don(C)$. Similarly, counting unitary flat bundles over a fixed three manifold L is Floer's Chern-Simons homology theory, $HF_{CS}(L)$. When C is a connected sum $C_1 \#_L C_2$ along a homology three sphere, the relative Donaldson invariants $Don(C_i)$'s take values in $HF_{CS}(L)$ and $Don(C)$ can be recovered from individual pieces by a gluing theorem, $Don(C) = \langle Don(C_1), Don(C_2)\rangle_{HF_{CS}(L)}$ (see e.g. [7]). Similarly when L has a handlebody decomposition $L = L_1 \#_\Sigma L_2$, each L_i determines a Lagrangian subspace $\mathcal{L}_i$ in the moduli space $\mathcal{M}^{flat}(\Sigma)$ of unitary flat bundles over the Riemann surface Σ and Atiyah conjectures that we can recover $HF_{CS}(L)$ from the Floer's Lagrangian intersection homology group of $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathcal{M}^{flat}(\Sigma)$, $HF_{CS}(L) = HF_{Lag}^{\mathcal{M}^{flat}(\Sigma)}(\mathcal{L}_1, \mathcal{L}_2)$. Such algebraic structures in the Donaldson-Floer theory can be formulated as a Topological Quantum Field Theory (abbrev. TQFT), as defined by Segal and Atiyah [3].

In this paper, we propose a construction of a TQFT by counting ASD bundles over four dimensional $\mathbb{H}$-SLag C in any closed (almost) G_2-manifold

M. We call these $\mathbb{H}$-*SLag cycles* and they can be identified as zeros of a naturally defined closed one form on the configuration space of topological cycles. We expect to obtain a homology theory $H_C(M)$ by applying the construction in the Witten's Morse theory. When M is non-compact with an asymptotically cylindrical end, $X \times [0, \infty)$, then the collection of boundary data of relative $\mathbb{H}$-SLag cycles determines a Lagrangian submanifold $\mathcal{L}_M$ in the moduli space $\mathcal{M}^{SLag}(X)$ of special Lagrangian cycles in the Calabi-Yau threefold X.

When we decompose $M = M_1 \#_X M_2$ along an infinite asymptotically cylindrical neck, it is reasonable to expect to have a gluing formula,

$$H_C(M) = HF_{Lag}^{\mathcal{M}^{SLag}(X)}(\mathcal{L}_{M_1}, \mathcal{L}_{M_2}).$$

The main technical difficulty in defining this TQFT rigorously is the *compactness* issue for the moduli space of $\mathbb{H}$-SLag cycles in M. We do not know how to resolve this problem and our homology groups are only defined in the *formal* sense (and physical sense?).

2 G_2-manifolds and $\mathbb{H}$-SLag geometry

We first review some basic definitions and properties of G_2-geometry, see [19] for more details.

Definition 1. *A seven dimensional Riemannian manifold M is called a G_2-manifold if the holonomy group of its Levi-Civita connection is inside $G_2 \subset SO(7)$.*

The simple Lie group G_2 can be identified as the subgroup of $SO(7)$ consisting of isomorphism $g : \mathbb{R}^7 \to \mathbb{R}^7$ preserving the linear three form Ω,

$$\Omega = f^1 f^2 f^3 - f^1 \left(e^1 e^0 + e^2 e^3\right) - f^2 \left(e^2 e^0 + e^3 e^1\right) - f^3 \left(e^3 e^0 + e^1 e^2\right),$$

where $e^0, e^1, e^2, e^3, f^1, f^2, f^3$ is any given orthonormal frame of $\mathbb{R}^7$. Such a three form, or up to conjugation by elements in $GL(7, \mathbb{R})$, is called *positive*, and it determines a unique compatible inner product on $\mathbb{R}^7$ [6].

Gray [12] shows that G_2-holonomy of M can be characterized by the existence of a positive harmonic three form Ω.

Definition 2. *A seven dimensional manifold M equipped with a positive closed three form Ω is called an almost G_2-manifold.*

Remark: The relationship between G_2-manifolds and almost G_2-manifolds is analogous to the relationship between Kahler manifolds and symplectic manifolds. Namely we replace a parallel non-degenerate form by a closed one.

For example, suppose that X is a complex three dimensional Kähler manifold with a trivial canonical line bundle, i.e. there exists a nonvanishing holomorphic three form Ω_X. Yau's celebrated theorem says that there is a Kähler form ω_X on X such that the corresponding Kahler metric has holonomy in $SU(3)$, i.e. a Calabi-Yau threefold. In particular both Ω_X and ω_X are parallel forms. Then the product $M = X \times S^1$ is a G_2-manifold with

$$\Omega = \operatorname{Re}\Omega_X + \omega_X \wedge d\theta.$$

Conversely, one can prove, using Bochner arguments, every G_2-metric on $X \times S^1$ must be of this form. More generally, if ω_X is a general Kähler form on X, then $\left(X \times S^1, \Omega\right)$ is an *almost* G_2-manifold and the converse is also true.

Next we quickly review the geometry of $\mathbb{H}$-SLag cycles in an almost G_2-manifold (see [19]).

Definition 3. *An orientable four dimensional submanifold C in an almost G_2-manifold (M, Ω) is called a coassociative submanifold, or simply a $\mathbb{H}$-SLag, if the restriction of Ω to C is identically zero,*

$$\Omega|_C = 0.$$

If M is a G_2-manifold, then any coassociative submanifold C in M is calibrated by $*\Omega$ in the sense of Harvey and Lawson [14], in particular, it is an absolute minimal submanifold in M. The normal bundle of any $\mathbb{H}$-SLag C can be naturally identified with the bundle of self-dual two forms on C. McLean [27] shows that infinitesimal deformations of any $\mathbb{H}$-SLag are unobstructed and they are parametrized by the space of harmonic self-dual two forms on C, i.e. $H^2_+(C, \mathbb{R})$.

For example, if S is a complex surface in a Calabi-Yau threefold X, then $S \times \{t\}$ is a $\mathbb{H}$-SLag in $M = X \times S^1$ for any $t \in S^1$. Notice that $H^2_+(S, \mathbb{R})$ is spanned by the Kahler form and the real and imaginary parts of holomorphic two forms on S, and the latter can be identified holomorphic normal vector fields along S because of the adjunction formula and the Calabi-Yau condition on X. Thus all deformations of $S \times \{t\}$ in M as $\mathbb{H}$-SLag submanifolds are of the same form. Similarly, if L is a three dimensional special

Lagrangian submanifold in X with phase $\pi/2$, i.e. $\omega|_L = \operatorname{Re}\Omega_X|_L = 0$, then $L \times S^1$ is also a $\mathbb{H}$-SLag in $M = X \times S^1$. Furthermore, all deformations of $L \times S^1$ in M as $\mathbb{H}$-SLag submanifolds are of the same form because $H^2_+ \left(L \times S^1\right) \cong H^1(L)$, which parametrizes infinitesimal deformations of special Lagrangian submanifolds in X.

Definition 4. *A $\mathbb{H}$-SLag cycle in an almost G_2-manifold (M, Ω) is a pair (C, D_E) with C a $\mathbb{H}$-SLag in M and D_E an ASD connection over C.*

Remark: $\mathbb{H}$-SLag cycles are supersymmetric cycles in physics as studied in [26]. Their moduli space admits a natural three form and a cubic tensor [19], which play the roles of the correlation function and the Yukawa coupling in physics.

We assume that the ASD connection D_E over C has rank one, i.e. a $U(1)$ connection. This avoids the occurrence of reducible connections, thus the moduli space $\mathcal{M}^{\mathbb{H}-SLag}(M)$ of $\mathbb{H}$-SLag cycles in M is a smooth manifold. It has a natural orientation and its expected dimension equals $b^1(C)$, the first Betti number of C. This is because the moduli space of $\mathbb{H}$-SLags has dimension equals $b^2_+(C)$ [27] and the existence of an ASD $U(1)$-connection over C is equivalent to $H^2_-(C, \mathbb{R}) \cap H^2(C, \mathbb{Z}) \neq \phi$. The number $b^1(C)$ is responsible for twisting by a flat $U(1)$-connection.

For simplicity, we assume that $b^1(C) = 0$, otherwise, one can cut down the dimension of $\mathcal{M}^{\mathbb{H}-SLag}(M)$ to zero by requiring the ASD connections over C to have trivial holonomy around loops $\gamma_1, ..., \gamma_{b^1(C)}$ in C representing an integral basis of $H_1(C, \mathbb{Z})$. We plan to count the algebraic number of points in this moduli space $\#\mathcal{M}^{\mathbb{H}-SLag}(M)$.

This number, in the case of $X \times S^1$, can be identified with a proposed invariant of Joyce [17] defined by counting rigid special Lagrangian submanifolds in any Calabi-Yau threefold. To explain this, we need the following proposition on the strong rigidity of product $\mathbb{H}$-SLags.

Proposition 5. *If $L \times S^1$ is a $\mathbb{H}$-SLag in $M = X \times S^1$ with X a Calabi-Yau threefold, then any $\mathbb{H}$-SLag representing the same homology class must also be a product.*

Proof: For simplicity we assume that the volume of the S^1 factor is unity, $Vol\left(S^1\right) = 1$. If $L \times S^1$ is a $\mathbb{H}$-SLag in M then L is special Lagrangian submanifold in X with phase $\pi/2$, i.e. $\operatorname{Re}\Omega_X|_L = \omega|_L = 0$. Suppose C is another $\mathbb{H}$-SLag in M representing the same homology class, we have $Vol(C) = Vol(L)$. If we write $C_\theta = C \cap (X \times \{\theta\})$ for any $\theta \in S^1$, then

$Vol\,(C_\theta) \geq Vol\,(L)$, as L is a calibrated submanifold in X. Furthermore the equality sign holds only if C_θ is also calibrated. In general we have

$$Vol\,(C) \geq \int_{S^1} Vol\,(C_\theta)\, d\theta,$$

with the equality sign holds if and only if C is a product with S^1. Combining these, we have

$$Vol\,(L) = Vol\,(C) \geq \int_{S^1} Vol\,(C_\theta)\, d\theta \geq \int_{S^1} Vol\,(L)\, d\theta = Vol\,(L)\,.$$

Thus both inequalities are indeed equal. Hence $C = L' \times S^1$ for some special Lagrangian submanifold L' in X. ∎

Suppose $M = X \times S^1$ is a product G_2-manifold and we consider product $\mathbb{H}$-SLag $C = L \times S^1$ in M. From the above proposition, every $\mathbb{H}$-SLag representing $[C]$ must also be a product. Since $b_+^2\,(C) = b^1\,(L)$, the rigidity of the $\mathbb{H}$-SLag C in M is equivalent to the rigidity of the special Lagrangian submanifold L in X. When this happens, i.e. L is a rational homology three sphere, we have $b^2\,(C) = 0$ and

$$\text{No. of ASD U(1)-bdl}/C = \#H^2\,(C, \mathbb{Z}) = \#H^2\,(L, \mathbb{Z}) = \#H_1\,(L, \mathbb{Z})\,.$$

Here we have used the fact that the first cohomology group is always torsion free. Thus the number of such $\mathbb{H}$-SLag cycles in $X \times S^1$ equals the number of special Lagrangian rational homology three spheres in a Calabi-Yau threefold X, weighted by $\#H_1\,(L, \mathbb{Z})$. Joyce [17] shows that with this particular weight, the numbers of special Lagrangians in any Calabi-Yau threefold behave well under various surgeries on X, and expects them to be invariants. Thus in this case, we have

$$\#\mathcal{M}^{\mathbb{H}-SLag}\left(X \times S^1\right) = \text{Joyce's proposed invariant for \#SLag. in } X.$$

In the next section, we will propose a homology theory, whose Euler characteristic gives $\#\mathcal{M}^{\mathbb{H}-SLag}\,(M)$.

3 Witten's Morse theory for $\mathbb{H}$-SLag cycles

We are going to use the parametrized version of $\mathbb{H}$-SLag cycles in any almost G_2-manifold M. We fix an oriented smooth four dimensional manifold C and a rank r Hermitian vector bundle E over C. We consider the *configuration space*

$$\mathcal{C} = Map\,(C, M) \times \mathcal{A}\,(E)\,,$$

where $\mathcal{A}\,(E)$ is the space of Hermitian connections on E.

Definition 6. *An element (f, D_E) in $\mathcal{C}$ is called a parametrized $\mathbb{H}$-SLag cycles in M if*

$$f^*\Omega = F_E^+ = 0,$$

where the self-duality is defined using the pullback metric from M.

Instead of $Aut\,(E)$, the symmetry group $\mathcal{G}$ in our situation consists of gauge transformations of E which cover *arbitrary* diffeomorphisms on M,

$$\begin{array}{ccc} E & \stackrel{g}{\to} & E \\ \downarrow & & \downarrow \\ M & \stackrel{g_M}{\to} & M. \end{array}$$

It fits into the following exact sequence,

$$1 \to Aut\,(E) \to \mathcal{G} \to Diff\,(C) \to 1.$$

The natural action of $\mathcal{G}$ on $\mathcal{C}$ is given by

$$g \cdot (f, D_E) = (f \circ g_M, g^* D_E)\,,$$

for any $(f, D_E) \in \mathcal{C} = Map\,(C, M) \times \mathcal{A}\,(E)$. Notice that $\mathcal{G}$ preserves the set of parametrized $\mathbb{H}$-SLag cycles in M.

The configuration space $\mathcal{C}$ has a natural one form Φ_0: At any $(f, D_E) \in \mathcal{C}$ we can identify the tangent space of $\mathcal{C}$ as

$$T_{(f,D_E)}\mathcal{C} = \Gamma\,(C, f^*T_M) \times \Omega^1\,(C, ad\,(E))\,.$$

We define

$$\Phi_0\,(f, D_E)\,(v, B) = \int_C Tr\,[f^*\,(\iota_v\Omega) \wedge F_E + f^*\Omega \wedge B]\,,$$

for any $(v, B) \in T_{(f,D_E)}\mathcal{C}$.

Proposition 7. *The one form Φ_0 on $\mathcal{C}$ is closed and invariant under the action by $\mathcal{G}$.*

Proof: Recall that there is a universal connection $\mathbb{D}_E$ over $C \times \mathcal{A}(E)$ whose curvature $\mathbb{F}_E$ at a point (x, D_E) equals,

$$\mathbb{F}_E|_{(x,D_E)} = \left(\mathbb{F}_E^{2,0}, \mathbb{F}_E^{1,1}, \mathbb{F}_E^{0,2}\right)$$
$$\in \Omega^2(C) \otimes \Omega^0(\mathcal{A}) + \Omega^1(C) \otimes \Omega^1(\mathcal{A}) + \Omega^0(C) \otimes \Omega^2(\mathcal{A})$$

with

$$\mathbb{F}_E^{2,0} = F_E,\ \mathbb{F}_E^{1,1}(v, B) = B(v),\ \mathbb{F}_E^{0,2} = 0,$$

where $v \in T_x C$ and $B \in \Omega^1(C, ad(E)) = T_{D_E}\mathcal{A}(E)$ (see e.g. [20]). The Bianchi identity implies that $Tr\mathbb{F}_E$ is a closed form on $C \times \mathcal{A}(E)$. We also consider the evaluation map,

$$ev : C \times Map(C, M) \to M$$
$$ev(x, f) = f(x).$$

It is not difficult to see that the pushforward of the differential form $ev^*(\Omega) \wedge Tr\mathbb{F}_E$ on $C \times Map(C, M) \times \mathcal{A}(E)$ to $Map(C, M) \times \mathcal{A}(E)$ equals Φ_0, i.e.

$$\Phi_0 = \int_C ev^*(\Omega) \wedge Tr\mathbb{F}_E.$$

Therefore the closedness of Φ_0 follows from the closedness of Ω. It is also clear from this description of Φ_0 that it is $\mathcal{G}$-invariant. ■

From this proposition, we know that $\Phi_0 = d\Psi_0$ locally for some function Ψ_0 on $\mathcal{C}$. As in the Chern-Simons theory, this function Ψ_0 can be obtained explicitly by integrating the closed one form Φ_0 along any path joining to a fixed element in $\mathcal{C}$. When $M = X \times S^1$ and $C = L \times S^1$, this is essentially the functional used by Thomas in [30].

From now on, we assume that E is a rank one bundle.

Lemma 8. *The zeros of Φ_0 are the same as parametrized $\mathbb{H}$-SLag cycles in M.*

Proof: Suppose (f, D_E) is a zero of Φ_0. By evaluating it on various $(0, B)$, we have $f^*\Omega = 0$, i.e. $f : C \to M$ is a parametrized $\mathbb{H}$-SLag. This implies that the map

$$\lrcorner\Omega : T_{f(x)}M \to \Lambda^2 T_x^* C$$

has image equals $\Lambda_+^2 T_x^* C$, for any $x \in C$. By evaluating Φ_0 on various $(v, 0)$, we have $F_E^+ = 0$, i.e. (f, D_E) is a parametrized $\mathbb{H}$-SLag cycle in M. The converse is obvious. ■

From above results, Φ_0 descends to a closed one form on $\mathcal{C}/\mathcal{G}$, called Φ. Locally we can write $\Phi = d\mathcal{F}$ for some function $\mathcal{F}$ whose critical points are precisely (unparametrized) $\mathbb{H}$-SLag cycles in M. Using the gradient flow lines of $\mathcal{F}$, we could formally define a Witten's Morse homology group, as in the famous Floer's theory. Roughly speaking one defines a complex $(\mathbf{C}_*, \partial)$, where $\mathbf{C}_*$ is the free Abelian group generated by critical points of $\mathcal{F}$ and ∂ is defined by counting the number of gradient flow lines between two critical points of relative index one.

Remark: The equations for the gradient flow are given by

$$\frac{\partial f}{\partial t} = *\left(f^*\xi \wedge F_E\right), \; \frac{\partial D_E}{\partial t} = *\left(f^*\Omega\right),$$

where $\xi \in \Omega^2\left(M, T_M\right)$ is defined by $\langle \xi\left(u,v\right), w\rangle = \Omega\left(u,v,w\right)$.

The equation

$$\partial^2 = 0$$

requires a good compactification of the moduli space of $\mathbb{H}$-SLag cycles in M, which we are lacking at this moment (see [31] however). We denote this proposed homology group as $H_C\left(M\right)$, or $H_C\left(M,\alpha\right)$ when $f_*\left[C\right] = \alpha \in H_4\left(M,\mathbb{Z}\right)$.

This homology group should be invariant under deformations of the almost G_2-metric on M and its Euler characteristic equals,

$$\chi\left(H_C\left(M\right)\right) = \#\mathcal{M}^{\mathbb{H}-SLag}\left(M\right).$$

Like Floer homology groups, they measure the *middle dimensional* topology of the configuration space $\mathcal{C}$ divided by $\mathcal{G}$.

4 TQFT of $\mathbb{H}$-SLag cycles

In this section we study complete almost G_2-manifold M_i with asymptotically cylindrical ends and the behavior of $H_C\left(M\right)$ when a closed almost G_2-manifold M decomposes into connected sum of two pieces, each with an asymptotically cylindrical end,

$$M = M_1 \underset{X}{\#} M_2.$$

Nontrivial examples of compact G_2-manifolds are constructed by Kovalev [18] using such connected sum approach. The boundary manifold X is necessary a Calabi-Yau threefold. We plan to discuss analytic aspects of M_i's in a future paper [24].

Each M_i's will define a Lagrangian subspace $\mathcal{L}_{M_i}$ in the moduli space of special Lagrangian cycles in X. Furthermore we expect to have a gluing formula expressing the above homology group for M in terms of the Floer Lagrangian intersection homology group for the two Lagrangian subspaces $\mathcal{L}_{M_1}$ and $\mathcal{L}_{M_2}$,

$$H_C(M) = HF_{Lag}^{\mathcal{M}^{SLag}(X)}(\mathcal{L}_{M_1}, \mathcal{L}_{M_2}).$$

These properties can be reformulated to give us a topological quantum field theory. To begin we have the following definition.

Definition 9. *An almost G_2-manifold M is called cylindrical if $M = X \times \mathbb{R}^1$ and its positive three form respect such product structure, i.e.*

$$\Omega_0 = \operatorname{Re}\Omega_X + \omega_X \wedge dt.$$

A complete almost G_2-manifold M with one end $X \times [0, \infty)$ is called asymptotically cylindrical if the restriction of its positive three form equals to the above one for large t, up to a possible error of order $O\left(e^{-t}\right)$. More precisely the positive three form Ω of M restricted to its end equals,

$$\Omega = \Omega_0 + d\zeta$$

for some two form ζ satisfying $|\zeta| + |\nabla\zeta| + \left|\nabla^2\zeta\right| + \left|\nabla^3\zeta\right| \le Ce^{-t}$.

Remark: If M is an almost G_2-manifold with an asymptotically cylindrical end $X \times [0, \infty)$, then (X, ω_X, Ω_X) is a complex threefold with a trivial canonical line bundle, but the Kähler form ω_X might not be Einstein. This is so, i.e. a Calabi-Yau threefold, provided that M is a G_2-manifold. We will simply write $\partial M = X$.

We consider $\mathbb{H}$-SLags C in M which satisfy a *Neumann condition* at infinity. That is, away from some compact set in M, the immersion $f : C \to M$ can be written as

$$f : L \times [0, \infty) \to X \times [0, \infty)$$

with $\partial f / \partial t$ vanishes at infinite [24]. A relative $\mathbb{H}$-SLag itself has asymptotically cylindrical end $L \times [0, \infty)$ with L a special Lagrangian submanifold in X. A *relative $\mathbb{H}$-SLag cycle* in M is a pair (C, D_E) with C a relative $\mathbb{H}$-SLag in M and D_E a unitary connection over C with finite energy,

$$\int_C |F_E|^2 \, dv < \infty.$$

Any finite energy connection D_E on C induces a unitary flat connection $D_{E'}$ on L [7].

Such a pair $(L, D_{E'})$ of a unitary flat connection $D_{E'}$ over a special Lagrangian submanifold L in a Calabi-Yau threefold X is called a *special Lagrangian cycle* in X. Their moduli space $\mathcal{M}^{SLag}(X)$ plays an important role in the Strominger-Yau-Zaslow Mirror Conjecture [29] or [22]. The tangent space to $\mathcal{M}^{SLag}(X)$ is naturally identified with $H^2(L, \mathbb{R}) \times H^1(L, ad(E'))$. For line bundles over L, the cup product

$$\cup : H^2(L, \mathbb{R}) \times H^1(L, \mathbb{R}) \to \mathbb{R},$$

induces a symplectic structure on $\mathcal{M}^{SLag}(X)$ [15]. Using analytic results from [24] about asymptotically cylindrical manifolds, we can prove the following theorem.

Claim 10. *Suppose M is an asymptotically cylindrical (almost) G_2-manifold with $\partial M = X$. Let $\mathcal{M}^{\mathbb{H}-SLag}(M)$ be the moduli space of rank one relative $\mathbb{H}$-SLag cycles in M. Then the map defined by the boundary values,*

$$b : \mathcal{M}^{\mathbb{H}-SLag}(M) \to \mathcal{M}^{SLag}(X),$$

is a Lagrangian immersion.

Sketch of the proof ([24]): For any closed Calabi-Yau threefold X (resp. G_2-manifold M), the moduli space of rank one special Lagrangian submanifolds L (resp. $\mathbb{H}$-SLags C) is smooth [27] and has dimension $b^2(L)$ (resp. $b^2_+(C)$). The same holds true for complete manifold M with a asymptotically cylindrical end $X \times [0, \infty)$, where $b^2_+(C)_{L^2}$ denote the dimension of L^2-harmonic self-dual two forms on a relative $\mathbb{H}$-SLag C in M.

The linearization of the boundary value map $\mathcal{M}^{\mathbb{H}-SLag}(M) \to \mathcal{M}^{SLag}(X)$ is given by $H^2_+(C)_{L^2} \overset{\alpha}{\to} H^2(L)$. Similar for the connection part, where the boundary value map is given by $H^1(C)_{L^2} \overset{\beta}{\to} H^1(L)$. We consider the following diagram where each row is a long exact sequence of L^2-cohomology groups for the pair (C, L) and each column in a perfect pairing.

$$\begin{array}{ccccccccc}
0 & \to & H^2_+(C,L) & \to & H^2_+(C) & \overset{\alpha}{\to} & H^2(L) & \to & H^3(C,L) & \to & \cdots \\
 & & \otimes & & \otimes & & \otimes & & \otimes & & \\
0 & \leftarrow & H^2_+(C) & \leftarrow & H^2_+(C,L) & \leftarrow & H^1(L) & \overset{\beta}{\leftarrow} & H^1(C) & \leftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} & &
\end{array}$$

Notice that $H_+^2(C,L)$, $H_+^2(C)$ and $H^2(L)$ parametrize infinitesimal deformation of C with fixed ∂C, deformation of C alone and deformation of L respectively.

By simple homological algebra, it is not difficult to see that $\operatorname{Im}\alpha \oplus \operatorname{Im}\beta$ is a Lagrangian subspace of $H^2(L) \oplus H^1(L)$ with the canonical symplectic structure. Hence the result. ■

Remark: The deformation theory of *conical* special Lagrangian submanifolds is developed by Pacini in [28].

We denote the immersed Lagrangian submanifold $b\left(\mathcal{M}^{\mathbb{H}-SLag}(M)\right)$ in $\mathcal{M}^{SLag}(X)$ by $\mathcal{L}_M$. When M decompose as a connected sum $M_1 \#_X M_2$ along a long neck, as in Atiyah's conjecture on Floer Chern-Simons homology group [3], we expect to have an isomorphism,

$$H_C(M) \cong HF_{Lag}^{\mathcal{M}^{SLag}(X)}(\mathcal{L}_{M_1}, \mathcal{L}_{M_2}).$$

More precisely, suppose Ω_t with $t \in [0,\infty)$, is a family of G_2-structure on $M_t = M$ such that as t goes to infinite, M decomposes into two components M_1 and M_2, each has an aymptotically cylindrical end $X \times [0,\infty)$. Then we expect that $\lim_{t\to\infty} H_C(M_t) \cong HF_{Lag}^{\mathcal{M}^{SLag}(X)}(\mathcal{L}_{M_1}, \mathcal{L}_{M_2})$. We summarize these structures in the following table:

Manifold:	(almost) G_2 -manifold, M^7	(almost) CY threefold, X^6
SUSY Cycles:	$\mathbb{H}$ -SLag. submfds.+ ASD bdl	SLag submfds.+ flat bdl
Invariant:	Homology group, $H_C(M)$	Fukaya category, $Fuk\left(\mathcal{M}^{SLag}(X)\right)$.

These associations can be formalized to form a TQFT [4]. Namely we associate an additive category $F(X) = Fuk\left(\mathcal{M}^{SLag}(X)\right)$ to a closed almost Calabi-Yau threefold X, a functor $F(M) : F(X_0) \to F(X_1)$ to an almost G_2-manifold M with asymptotically cylindrical ends $X_1 - X_0 = X_1 \cup \bar{X}_0$. They satisfy

(i) $F(\phi) =$ the additive tensor category of vector spaces $((Vec))$,
(ii) $F(X_1 \amalg X_2) = F(X_1) \otimes F(X_2)$.

For example, when M is a closed G_2-manifold, that is a cobordism between empty manifolds, then we have $F(M) : ((Vec)) \to ((Vec))$ and the image of the trivial bundle is our homology group $H_C(M)$.

5 More TQFTs

Notice that all TQFTs we propose in this paper are formal mathematical constructions. Besides the lack of compactness for the moduli spaces, the *obstruction* issue is also a big problem if we try to make these theories rigorous. This problem is explained to the author by a referee.

There are other TQFTs naturally associated to Calabi-Yau threefolds and G_2-manifolds but (1) they do not involve nontrivial coupling between submanifolds and bundles and (2) new difficulties arise because of corresponding moduli spaces for Calabi-Yau threefolds have virtual dimension zero and could be singular. They are essentially in the paper by Donaldson and Thomas [9].

TQFT of associative cycles

We assume that M is a G_2-manifold, i.e. Ω is parallel rather than closed. Three dimensional submanifolds A in M calibrated by Ω is called *associative submanifolds* and they can be characterized by $\chi|_A = 0$ ([14]) where $\chi \in \Omega^3(M, T_M)$ is defined by $\langle w, \chi(x,y,z)\rangle = *\Omega(w,x,y,z)$. We define a *parametrized A-cycle* to be a pair $(f, D_E) \in \mathcal{C}_A = Map(A, M) \times \mathcal{A}(E)$, with $f : A \to M$ a parametrized A-submanifold and D_E is a unitary flat connection on a Hermitian vector bundle E over A. There is also a natural $\mathcal{G}$-invariant closed one form Φ_A on $\mathcal{C}_A$ given by

$$\Phi_A(f, D_E)(v, B) = \int_A TrF_E \wedge B + \langle f^*\chi, v\rangle_{T_M},$$

for any $(v, B) \in \Gamma(A, f^*T_M) \times \Omega^1(A, ad(E)) = T_{(f,D_E)}\mathcal{C}_A$. Its zero set is the moduli space of A-cycles in M. As before, we could formally apply arguments in Witten's Morse theory to Φ_A and define a homology group $H_A(M)$.

The corresponding category associated to a Calabi-Yau threefold X would be the Fukaya-Floer category of the moduli space of unitary flat bundles over holomorphic curves in X, denote $\mathcal{M}^{curve}(X)$. We summarize these in the following table:

Manifold:	G_2 -manifold, M^7	CY threefold, X^6
SUSY Cycles:	A-submfds.+ flat bundles	Holomorphic curves+ flat bundles
Invariant:	Homology group, $H_A(M)$	Fukaya category, $Fuk(\mathcal{M}^{curve}(X))$.

TQFT of Donaldson-Thomas bundles

We assume that M is a seven manifold with a G_2-structure such that its positive three form Ω is co-closed, rather than closed, i.e. $d\Theta = 0$ with $\Theta = *\Omega$. In [9] Donaldson and Thomas introduce a first order Yang-Mills equation for G_2-manifolds,

$$F_E \wedge \Theta = 0.$$

Their solutions are the zeros of the following gauge invariant one form Φ_{DT} on $\mathcal{A}(E)$,

$$\Phi_{DT}(D_E)(B) = \int_M Tr[F_E \wedge B] \wedge \Theta,$$

for any $B \in \Omega^1(M, ad(E)) = T_{D_E}\mathcal{A}(E)$. This one form Φ_{DT} is closed because of $d\Theta = 0$. As before, we can formally define a homology group $H_{DT}(M)$. The corresponding category associated to a Calabi-Yau threefold X should be the Fukaya-Floer category of the moduli space of Hermitian Yang-Mills connections over X, denote $\mathcal{M}^{HYM}(X)$. Again we summarize these in a table:

Manifold:	G_2 -manifold, M^7	CY threefold, X^6
SUSY Cycles:	DT-bundles	Hermitian YM-bundles
Invariant:	Homology group, $H_{DT}(M)$	Fukaya category, $Fuk(\mathcal{M}^{HYM}(X))$.

It is an interesting problem to understand the transformations of these TQFTs under dualities in M-theory.

Acknowledgments: This paper is partially supported by NSF/DMS-0103355. The author expresses his gratitude to J.H. Lee, R. Thomas, A. Voronov and X.W. Wang for useful discussions. The author also thank the referee for many useful comments.

References

[1] B. S. Acharya, B. Spence, *Supersymmetry and M theory on 7-manifolds,* [hep-th/0007213].

[2] M. Aganagic, C. Vafa, *Mirror Symmetry and a G_2 Flop,* [hep-th/0105225].

[3] M. Atiyah, *New invariants of three and four dimensional manifolds,* in The Mathematical Heritage of Herman Weyl, Proc. Symp. Pure Math., **48**, A.M.S. (1988), 285-299.

[4] M. Atiyah, *Topological quantum field theories.* Inst. Hautes Études Sci. Publ. Math. No. 68 (1988), 175–186 (1989).

[5] M. Atiyah, E. Witten, *M-theory dynamics on a manifold of G_2 holonomy*, [hep-th/0107177].

[6] R. Bryant, *Metrics with exceptional holonomy*, Ann. of Math. 126 (1987) 525-576.

[7] S. Donaldson, *Floer homology group in Yang-Mills theory*, Cambridge Univ. Press (2002).

[8] S. Donaldson, P. Kronheimer, *The geometry of four-manifols,* Oxford University Press, (1990).

[9] S. Donaldson, R. Thomas, *Gauge theory in higher dimension,* The Geometric Universe: Science, Geometry and the work of Roger Penrose, S.A. Huggett et al edited, Oxford Univ. Press (1988).

[10] K. Fukaya, Y.G. Oh, H. Ohta, K. Ono, *Lagrangian intersection Floer theory - anomoly and obstruction*, to appear in International Press.

[11] R. Gopakumar, C. Vafa, *M-theory and topological strings - II,* [hep-th/9812127].

[12] A. Gray, *Vector cross products on manifolds*, Trans. Amer. Math. Soc. 141 (1969) 465-504.

[13] S. Gukov, S.-T. Yau, E. Zaslow, *Duality and Fibrations on G_2 Manifolds,* [hep-th/0203217].

[14] R. Harvey, B. Lawson, *Calibrated geometries*, Acta Math. 148 (1982), 47-157.

[15] N. Hitchin, *The moduli space of special Lagrangian submanifolds*. Dedicated to Ennio DeGiorgi. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 (1997), no. 3-4, 503–515 (1998). [dg-ga/9711002].

[16] N. Hitchin, *The geometry of three forms in 6 and 7 dimensions, J. Differential Geom. 55 (2000), no. 3,* 547–576. [math.DG/0010054].

[17] D. Joyce, *On counting special Lagrangian homology 3-spheres,* [hep-th/9907013].

[18] A. Kovalev, *Twisted connected sums and special Riemannian holonomy,* [math.DG/0012189].

[19] J.H. Lee, N.C. Leung, *Geometric structures on G_2 and Spin*(7)-*manifolds*, [math.DG/0202045].

[20] N.C. Leung, *Symplectic structures on gauge theory,* Comm. Math. Phys., 193 (1998) 47-67.

[21] N.C. Leung, *Mirror symmetry without corrections,* [math.DG/0009235].

[22] N.C. Leung, *Geometric aspects of mirror symmetry*, to appear in the proceeding of ICCM 2001, [math.DG/0204168].

[23] N.C. Leung, *Riemannian geometry over different normed division algebras*, preprint 2002.

[24] N.C. Leung, in preparation.

[25] N.C. Leung, S.Y. Yau, E. Zaslow, *From special Lagrangian to Hermitian-Yang-Mills via Fourier-Mukai transform*, to appear in Adv. Thero. Math. Phys.. [math.DG/0005118].

[26] M. Marino, R. Minasian, G. Moore, and A. Strominger, *Nonlinear Instantons from supersymmetric p-Branes*, [hep-th/9911206].

[27] R. McLean, *Deformations of calibrated submanifolds*, Comm. Analy. Geom., **6** (1998) 705-747.

[28] T. Pacini, *Deformations of Asymptotically Conical Special Lagrangian Submanifolds.* Preprint 2002. [math.DG/0207144].

[29] A. Strominger, S.-T. Yau, and E. Zaslow, *Mirror Symmetry is T-Duality*, Nuclear Physics **B479** (1996) 243-259; [hep-th/9606040].

[30] R. P. Thomas, *Moment maps, monodromy and mirror manifolds.* In "Symplectic geometry and mirror symmetry" , Proceedings of the 4th KIAS Annual International Conference, Seoul. Eds. K. Fukaya, Y.-G. Oh, K. Ono and G. Tian. World Scientific, 2001.

[31] G. Tian, *Gauge theory and calibrated geometry. I.* Ann. of Math. (2) 151 (2000), no. 1, 193–268.