

# HyperKähler Manifolds and Birational Transformations

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## 1 Introduction

The minimal model program has greatly enhanced our knowledge on birational geometry of varieties of dimension 3 and higher. About the same time, the last two decades have also witnessed increasing interests in HyperKähler manifolds, a particular class of Calabi-Yau manifolds. One interest in this area, which we hope to treat in the future, is to investigate the behavior of the SYZ mirror conjecture ([13]) under birational maps. The present paper focuses on the birational geometry of projective symplectic varieties and attempts to reconcile the two approaches altogether in *arbitrary* dimensions.

Let  $X$  be a smooth projective symplectic variety of dimension  $2n$  with a holomorphically symplectic form  $\omega$ . Hence  $X$  admits a HyperKähler structure by a theorem of Yau. For an arbitrary variety  $V$ , let  $V_{reg}$  denote the non-singular part of  $V$ . Also, by a small birational contraction, we will mean a non-divisorial birational contraction.

A striking and simple relation between minimal model program and symplectic geometry is that a small Mori contraction necessarily coincides (generically) with the null foliation of the symplectic form along the exceptional locus.

**Theorem A.** *Let  $\pi : X \rightarrow Z$  be a small birational contraction,  $B$  an arbitrary irreducible component of the degenerated locus and  $F$  a generic fiber of  $\pi : B \rightarrow S := \pi(B)$ . Then  $T_b F = (T_b B)^\perp$  for generic  $b \in F_{reg} \cap B_{reg}$ . In particular,*

1. *the inclusion  $j : B \hookrightarrow X$  is a coisotropic embedding;*
2. *the null foliation of  $\omega|_B$  through generic points of  $B$  coincides with generic fibers of  $\pi : B \rightarrow S$ .*

This bears an interesting consequence.

**Theorem B.** *Let  $\pi : X \rightarrow Z$  be a small birational contraction and  $B$  the exceptional locus. Assume that  $\pi : B \rightarrow S$  ( $:= \pi(B)$ ) is a smooth fibration. Then  $\pi : B \rightarrow S$  is a  $\mathbb{P}^r$ -bundle over  $S$  where  $r = \text{codim } B \geq 2$ .*

This shows, for example, that the dimensional condition that Mukai posed to define his elementary transformation is necessary.

Specializing to the case when  $S$  is a point, we obtain

**Theorem C.** *Let  $B$  be a smooth subvariety of a projective symplectic variety  $X$  of dimension  $2n$ . Assume that  $B$  can be contracted to a point. Then  $B$  is isomorphic to  $\mathbb{P}^n$ .*

In particular,  $B$  must be a Lagrangian subvariety. Indeed, this is a general phenomenon.

**Theorem D.** *Let  $B$  be an arbitrary subvariety of a projective symplectic variety  $X$ . Assume that  $B$  can be contracted to a point. Then  $B$  is Lagrangian.*

More generally, we prove (consult §6 for the explanation of the relevant technical terms)

**Proposition E.** *A small birational contraction  $\pi : X \rightarrow Z$  is always IH-semismall.*

Indeed, we expect that a stratum is IH-relevant to  $\pi$  if and only if it carries the reduced symplectic form  $\omega_{red}$ . In particular,  $\pi$  is strictly IH-semismall if and only if  $Z$  is symplectically stratified using reduced symplectic forms. In fact, we believe that all small symplectic birational contractions are strictly IH-semismall. (Conjecture 6.5).

The next relation between Mori theory and symplectic geometry is that a Mori contraction necessarily produces a large family of Hamiltonian flows around the exceptional locus, which makes the contraction behaving like a holomorphic moment map locally around the exceptional locus. This was observed jointly with Dan Burns. The “moment map” depends on the singularities. It becomes more complicated as the singularity becomes worse. In any case, the fiber of the contraction over a singular point is preserved by the corresponding Hamiltonian flows. Under some technical assumption we were able to see that the generic fiber of  $B \rightarrow S$  is a finite union of almost homogeneous spaces. In fact, we anticipate that every fiber should be a finite union of homogeneous spaces and a generic fiber is the projective space whose dimension coincides with the codimension of the exceptional locus.

However, in general, it is possible that these (almost) homogeneous spaces are not generated by the above Hamiltonian flows, some other vector fields are needed. The Hamiltonian approach should be useful in the absence of projective structure.

Finally we conjecture that any two smooth symplectic varieties that are birational to each other are related by a sequence of Mukai elementary transformations after removing closed subvarieties of codimension greater than 2 (Conjecture 7.3).

We end the introduction with the following question.

Let  $K_0$  be the Grothendieck ring of varieties over  $\mathbb{C}$ : this is the abelian group generated by isomorphism classes of complex varieties, subject to the relation  $[X - S] = [X] - [S]$ , where  $S$  is closed in  $X$ . All the explicitly known symplectic birational maps are of symmetric nature. We wonder whether the following is true: Let  $X$  and  $Y$  be smooth symplectic varieties. Then  $[X] = [Y]$  if and only if  $X$  and  $Y$  are birational.

In dimension 4, it follows from [1] and [15] that  $[X] = [Y]$  if  $X$  and  $Y$  are birational. The same is true for smooth Calabi-Yau 3-folds.

The current paper is sequel to [1] but does not depend on it. We men-

tion that we were aware of the work of Cho-Miyaoka-Shepherd-Barron part of which are parallel to ours; Only until August 2002, we were informed the existence of the paper of Cho-Miyaoka-Shepherd-Barron when the first author was visiting Taiwan. After that we learnt from the referee that that paper has now appeared as [3]. After our paper was circulated, the paper of Wierzba and Wisniewski ([15]) appeared in ArXiv (math.AG/0201028). Our paper was circulated in November 2001 (math.AG/0111089), and the results were reported earlier at Harvard differential geometry seminars in Fall 1999. There are also papers [12], [14] and [7] that deal with related topics with some overlaps. Similar results to Theorem D and Proposition E were also obtained in [12].

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## 2 Deformation of rational curves

Rational curves, occurring in the exceptional loci of our birational transformations, are studied in this section to prepare for what follows. The results are, however, of independent interest.

A rational curve  $C$  in a projective variety  $W$  is called incompressible if it can not be deformed to a reducible or non-reduced curve.

**Lemma 2.1.** ([2]) *Let  $C$  be an incompressible rational curve in a projective algebraic variety  $W$  of dimension  $n$ . Then  $C$  moves in a family of dimension at most  $2n - 2$ .*

*Proof.* Replacing  $W$  by its normalization if necessary, we may assume that  $W$  is normal. Then the lemma is equivalent to (1.4.4) of [2]. (One needs to observe that up to (1.4.4), the smooth assumption on the ambient variety is not used.)  $\square$

If there is a morphism  $f : \mathbb{P}^n \rightarrow W$  such that it is an isomorphism or a

normalization, then  $2n - 2$  is minimal and achieved.

**Theorem 2.2.** ([2]) *Let  $W$  be a smooth projective variety of dimension  $n$ . Assume that  $W$  contains a rational curve and every rational curve of  $W$  moves in a family of dimension at least  $2n - 2$ . Then  $W \cong \mathbb{P}^n$ .*

*Proof.* We borrow some proofs from [2].

If  $\text{Pic}(W) \geq 2$ , there is a non-trivial extremal contraction

$$\gamma : W \rightarrow W'$$

such that  $\dim W' > 0$ . Let  $S \subset \text{Chow}(W)$  be an irreducible component of the set of the contracted extremal rational curves of minimal degree and  $F_S$  the incident variety over  $S$ . Then for a general point  $x \in p_2(F_S)$ ,  $F_S(x)$ , the universal family of curves in  $F_S$  that pass the point  $x$ , has its image contained in a fiber of  $\gamma$ . That is,

$$\dim p_2(F_S(x)) \leq n - 1.$$

By (1.4.4) of [2],

$$\dim S \leq \dim p_2(F_S) + \dim p_2(F_S(x)) - 2 \leq n + n - 1 - 2 = 2n - 3.$$

This is a contradiction.

Hence  $\text{Pic}(W) = 1$ .  $W$  is therefore Fano. Then the result of [2] applies<sup>1</sup>.  
□

In fact, we expect

**Conjecture 2.3.** *Let  $W$  be an arbitrary projective variety of dimension  $n$ . Assume that  $W$  contains a rational curve and every rational curve of  $W$  moves in a family of dimension at least  $2n - 2$ . Then there is a birational dominating morphism  $\mathbb{P}^n \rightarrow W$  such that it is either an isomorphism or normalization.*

We learned that this would follow from a recent work by Cho-Miyaoka-Shepherd-Barron. When  $n = 2$ , it can be proved as follows.

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<sup>1</sup>In the summer of 1999. János Kollár informed YH of the paper [2] where the theorem was proved for  $W$  being smooth and Fano. We observed this theorem much later (although independently). Our assumption on the number of the family is by Riemann-Roch a consequence of their numerical assumption  $-K_W \cdot C \geq 2n - 2$ . We makes no mention of  $-K_W$  in the statement. Note that a nice rigorous proof of Cho-Miyaoka-Shepherd-Barron's theorem was recently provided by Kebekus [8]

**Proposition 2.4.** (Kollár.) *Let  $S$  be a normal surface, proper over  $\mathbb{C}$ . Then  $S$  satisfies exactly one of the following:*

1. *Every morphism  $f : \mathbb{P}^1 \rightarrow S$  is constant;*
2. *There is a morphism  $f : \mathbb{P}^1 \rightarrow S$  such that  $f$  is rigid.*
3.  *$S \cong \mathbb{P}^2$ ;*
4.  *$S \cong \mathbb{P}^1 \times \mathbb{P}^1$ , or  $S$  is isomorphic to a minimal ruled surface over a curve of positive genus or a minimal ruled surface with a negative section contracted.*

*Proof.* If we have either of (1) or (2), we are done. Otherwise, there is a morphism  $f : \mathbb{P}^1 \rightarrow S$  deforms in a 1-parameter family, thus  $S$  is uniruled.

Let  $p : \bar{S} \rightarrow S$  be the minimal desingularization with the exceptional curve  $E$ .  $\bar{S}$  is also uniruled, hence there is an extremal ray  $R$ . There are 3 possibilities for  $R$ .

1.  $\bar{S} \cong \mathbb{P}^2$ , thus also  $S \cong \mathbb{P}^2$  (which implies (3)).
2.  $\bar{S}$  is a minimal ruled surface (which implies (4)).
3.  $R$  is spanned by a  $(-1)$ -curve  $C_0$  in  $\bar{S}$ .

But (3) is impossible, because the image of  $C_0$  in  $S$  would have been rigid. The proof goes as follows. Assume the contrary that  $f_0 : \mathbb{P}^1 \cong C_0 \subset \bar{S} \rightarrow S$  is not rigid and let  $f_t : \mathbb{P}^1 \rightarrow S$  be a 1-parameter deformation. For general  $t$ ,  $f_t$  lifts to a family of morphisms  $\bar{f}_t : \mathbb{P}^1 \rightarrow \bar{S}$ . As  $t \rightarrow 0$ , the curves  $\bar{f}_t(\mathbb{P}^1)$  degenerate and we obtain a cycle

$$\lim_{t \rightarrow 0} \bar{f}_t(\mathbb{P}^1) = C_0 + F$$

where  $\text{Supp} F \subset \text{Supp} E$ .  $\bar{S}$  is the minimal resolution, thus  $K_{\bar{S}} \cdot F \geq 0$ . Therefore,

$$K_{\bar{S}} \cdot \bar{f}_t(\mathbb{P}^1) \geq K_{\bar{S}} \cdot C_0 = -1.$$

On the other hand, for a general  $t$  the morphism  $\bar{f}_t$  is free, thus

$$K_{\bar{S}} \cdot \bar{f}_t(\mathbb{P}^1) \leq -2$$

by II.3.13.1, [9]. This contradiction shows that  $f_0$  is rigid. □

**Lemma 2.5.** *Let  $B$  be a compact subvariety in a smooth complex variety. Assume that  $B$  is contractible and  $C$  a compact curve in  $B$  whose image is a point. Then  $C$  can not be moved out of  $B$ .*

*Proof.* (János Kollár.) Assume otherwise, there would be compact curves outside  $B$  which are arbitrarily close to the curve  $C$ . Let  $\pi : X \rightarrow Z$  be the contraction. Then images of these curves form a family of curves shrinking to the point  $\pi(C)$  in  $Z$ . This is impossible because it would imply that a Stein neighborhood of  $\pi(C)$  contains compact curves.  $\square$

### 3 Symplectic varieties

An arbitrary smooth variety is called (holomorphically) symplectic if it carries a closed holomorphic two form  $\omega$  such that it is non-degenerated.

Our objective is to understand relations between any two symplectic models within the same birational class. Hence we only consider *small* birational contractions. For simplicity, generic fiber of a birational contraction is sometimes assumed to be irreducible.

One of our first observations is

**Proposition 3.1.** <sup>2</sup> *Let  $\Phi : X \dashrightarrow X'$  be a birational map between two smooth symplectic varieties such that it is isomorphic over open subsets  $U \subset X$  and  $U' \subset X'$ , then for any symplectic form  $\omega$  over  $X$ , there is a unique symplectic form  $\omega'$  over  $X'$  such that  $\omega|_U = \Phi^*\omega'|_{U'}$ .*

*Proof.* Since  $\Phi$  is necessarily isomorphic in codimension one, every closed form over  $U'$  extends to a closed form over  $X'$ . Let  $\omega'$  be the extension to  $X'$  of  $(\Phi^{-1})^*(\omega|_U)$ . The question is whether this extension is necessarily non-degenerated. Let  $\dim X = 2n$ . Then  $(\omega')^n$  is a section of a line bundle over  $X'$  so that its zero locus is a divisor if not empty. Since  $\text{codim}(X' \setminus U') \geq 2$ ,  $\omega'$  has to be non-degenerated. The uniqueness is clear.  $\square$

Mukai discovered a class of well-behaved birational maps between symplectic varieties.

**Definition 3.2.** *Let  $P$  be a closed subvariety of  $X$ . Assume that*

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<sup>2</sup>Thanks are due to Conan Leung whose comments lead to this proposition.

1.  $P$  is  $\mathbb{P}^r$ -bundle over a symplectic manifold  $S$ ;
2.  $r = \text{codim } P$ .

Then  $X$  can be blown up along  $P$  and the resulting variety can be contracted along another ruling, yielding a symplectic variety  $X'$ . Moreover,  $X' \setminus U = P^*$  is the dual bundle of  $P$  and  $X'$  carries a symplectic form  $\omega'$  which agrees with  $\omega$  over  $U$ . This process is known as a Mukai's elementary transformation. Sometimes we abbreviate it as MET.

The following is a very important and useful property of a symplectic variety.

**Lemma 3.3.** (Ran) *Let  $W$  be an arbitrary smooth symplectic variety of dimension  $2n$ . Then every proper rational curve in  $W$  deforms in a family of dimension at least  $2n - 2$ .*

*Proof.* When the curve is smooth, it was proved by Ran. His proof can be extended to singular case by using graphs (see e.g., Lemma 2.3 of [1]).

Ran's original proof uses Block's semiregularity map. There is a second proof without using the Block semiregularity map. By Fujiki [4], there is a (general) deformation of  $X$  so that no curves (indeed no subvarieties) survive in the nearby members. Let  $\hat{X}$  be the total space of such a family and  $C$  a rational curve in the central fiber  $X$ . Then  $K_{\hat{X}} \cdot C = K_X \cdot C = 0$ . Hence by Riemann-Roch, we have

$$\dim \text{Hilb}_{[C]} \geq 2n + 1 - 3 = 2n - 2.$$

□

Combining this lemma and Lemma 2.1, we obtain that

**Proposition 3.4.** *Let  $\Phi : X \dashrightarrow X'$  be a birational map between two smooth projective symplectic varieties and  $E$  the exceptional locus. Then*

$$n \leq \dim E \leq 2n - 2.$$

**Example 3.5.** *Let  $S$  be an elliptic fibration K3 surface with a  $A_2$ -type singular fiber, that is, a singular fiber is a union of two smooth rational curves  $C_i, i = 1, 2$ , crossing at two distinct points  $P$  and  $Q$ . Let  $X = S^{[2]}$ . Then  $X$  contains two  $P^2$ s,  $C_i^{[2]}, i = 1, 2$ , meeting at the point  $P + Q \in X = S^{[2]}$ . We can flop  $C_i$  to get  $X_i, i = 1, 2$ , respectively. Then the exceptional locus of*

$$X_1 \dashrightarrow X_2$$



is a union of  $P^2$  and the Hirzebruch surface  $\mathbb{F}_1$  intersecting along the  $(-1)$ -section  $C$  of  $\mathbb{F}_1$ .

Note that  $C$  deforms in a 1-dimensional family in  $\mathbb{F}_1$ , hence must deform out of  $\mathbb{F}_1$  by Lemma 3.3. The strictly transform of this family in  $X$  under the birational map  $X_1 \dashrightarrow X$  is a deformation family of a nodal curve  $\ell_1 + \ell_2$  passing the point  $P + Q$  with a branch (line)  $\ell_i$  in each of  $C_i^{[2]}$ ,  $i = 1, 2$ . This nodal curve  $\ell_1 + \ell_2$  can be deformed out of  $C_1^{[2]} \cup C_2^{[2]}$ .

### 4 Mori contraction and null foliation

Let  $j : P \hookrightarrow X$  be a subvariety of  $X$ . Write  $\omega|_P := j^*\omega$ . When  $P$  is singular, by the form  $\omega|_P$  we mean the restriction to the nonsingular part  $P_{reg}$  of  $P$ .

**Lemma 4.1.** *Let  $\pi : X \rightarrow Z$  be a small birational contraction and  $F$  a fiber of  $\pi$ . Then  $H^{0,q}(F) = 0$  for all  $q > 0$ . In particular,  $F$  does not admit nontrivial global holomorphic forms.*

*Proof.* Since  $Z$  has (at worst) canonical singularities, it has rational singularities by Corollary 5.24 of [10]. That is,  $R^q\pi_*\mathcal{O}_X = 0$  for all  $q > 0$ . This implies that  $H^q(F, \mathcal{O}_F) = 0$  for all  $q > 0$ . Let  $F'$  be a smooth resolution of  $F$ . Then  $H^q(F', \mathcal{O}_{F'}) = 0$ , hence  $H^0(F', \Omega_{F'}^q) = 0$  for all  $q > 0$ . This implies that  $F$  does not admit global holomorphic forms because otherwise the pullback of a global form from  $F$  to  $F'$  yields a contradiction to the fact that  $H^0(F', \Omega_{F'}^q) = 0$  for all  $q > 0$ . □

Unless otherwise stated, we will use  $E$  to denote the degenerated locus of the map  $\pi$  and  $B$  to denote an arbitrary irreducible component of  $E$ . For a smooth point  $b$  of  $B$ ,  $(T_b B)^\perp$  denotes the orthogonal complement of the tangent space  $T_b B$  with respect to the form  $\omega_b$ .

**Theorem 4.2.** *Let  $\pi : X \rightarrow Z$  be a small birational contraction,  $B$  an arbitrary irreducible component of the degenerated locus and  $F$  a generic fiber of  $\pi : B \rightarrow S := \pi(B)$ . Then  $T_b F = (T_b B)^\perp$  for a generic  $b \in F_{reg} \cap B_{reg}$  such that  $s = \pi(b)$  is a smooth point in  $S$ . In particular,*

1. the inclusion  $j : B \hookrightarrow X$  is (generically) a coisotropic embedding;
2. the null foliation of  $\omega|_B$  through generic points of  $B$  coincides with generic fibers of  $\pi : B \rightarrow S$ .

*Proof.* First we have  $\omega|_F = 0$  because  $H^{0,2}(F) = 0$ .

Let  $s = \pi(b)$  for some  $b \in F$  as stated in the statement of the theorem. For any  $v \in T_s S$ , let  $v'$  be any lifting to  $T_b B$ . Define

$$\iota_v \omega(w) := \omega(w, v')$$

for any  $w \in T_b F$ . This is independent of the choice of the lifting because two choices are differed by an element of  $T_b F$  and  $\omega|_F = 0$ . Thus  $\iota_v \omega$  is well-defined holomorphic one form on  $F$ . By Lemma 4.1,  $\iota_v \omega$  is identical to zero. This shows that  $T_b F \subset (T_b B)^\perp$ . In particular,  $\dim F \leq \operatorname{codim} B$  because  $\omega$  is non-degenerate.

On the other hand, if  $\dim F < \operatorname{codim} B$ , that is,  $\dim B < 2n - \dim F$ , then  $\dim S = \dim B - \dim F < 2n - 2 \dim F$ , or  $\dim S + 2 \dim F - 2 < 2n - 2$ . By e.g. [10],  $F$  contains rational curves. Take a rational curve  $C$  in  $F$  that satisfies the statement of Lemma 2.1, then in a neighborhood of  $F$  in  $X$ ,  $C$  moves in a family of dimension  $\leq$

$$\dim S + 2 \dim F - 2 < 2n - 2,$$

contradicting to Lemma 3.3.

Hence  $\dim F = \operatorname{codim} B$ . This implies that  $T_b F = (T_b B)^\perp$ .

The rest of the statements follows immediately.  $\square$

**Corollary 4.3.** *Let  $F$  be a generic fiber of  $\pi : B \rightarrow S$ . Then*

$$\operatorname{codim} B = \dim F.$$

In particular, the assumption (2) in Definition 3.2 for Mukai's elementary transformations is not only sufficient but also necessary.

Note also that Corollary 4.3 implies that

**Corollary 4.4.** *Let the notation and assumptions be as before. Then*

$$\dim F \geq 2$$

for every fiber  $F$  of the projection  $\pi : B \rightarrow S$ .

**Corollary 4.5.** *Assume that the generic fiber  $F$  of the birational contraction is an irreducible surface. Then the normalization of  $F$  is isomorphic to  $\mathbb{P}^2$ .*

*Proof.* This follows by combining the fact that  $F$  contains a rational curve and every rational curve in  $F$  has to move in an (at least) 2-dimensional family and Proposition 2.4.  $\square$

We believe more is true but are not able to give a proof.

**Conjecture 4.6.** *Assume that the generic fiber  $F$  is a surface. Then  $F$  is normal and hence isomorphic to  $\mathbb{P}^2$ .*

**Theorem 4.7.** *Assume that the generic fiber  $F$  is smooth. Then  $F$  is isomorphic to  $\mathbb{P}^r$  where  $r = \dim F = \text{codim } B$ .*

*Proof.* Let  $r = \dim F$ . Then,  $F$  contains rational curve and every rational curve moves in a family of dimension at least  $2r - 2$  to satisfy the fact that every rational curve in  $B$  moves in a family of dimension at least  $2n - 2$ . By Theorem 2.2,  $F \cong \mathbb{P}^r$ , and by Theorem 4.3,  $r = \text{codim } B$ . □

**Theorem 4.8.** *Let  $S^0$  be the (largest) smooth open subset of  $S$  such that the restriction of  $\pi$  to*

$$B^0 = \pi^{-1}(S^0) \rightarrow S^0$$

*is a fibration. Then  $S^0$  carries a symplectic form  $\omega_{red}$  induced from  $\omega$  such that  $\pi^*\omega_{red} = \omega|_{B^0}$ .*

*Proof.* First note that the fibration  $B^0 = \pi^{-1}(S^0) \rightarrow S^0$  coincides with the null foliation of the form  $\omega|_{B^0}$  by Theorem 4.2. We can introduce a local coordinate  $z_1, \dots, z_m$  in  $B^0$  about any smooth point of  $B^0$  such that the leaves of the foliation are (locally) given by

$$z_1 = \text{const}, \dots, z_k = \text{const}$$

and so the tangent space to the foliation is spanned by  $\partial/\partial z_{k+1}, \dots, \partial/\partial z_r$  at each point in the coordinate neighborhood. If we now write

$$\omega|_B = \sum a_{ij} dz_i \wedge dz_j,$$

where  $a_{ij}$  are functions. The condition that

$$\iota_{\partial/\partial z_{k+1}} \omega|_B = \dots = \iota_{\partial/\partial z_r} \omega|_B = 0$$

implies that  $a_{ij} = 0$  if  $i$  or  $j$  is greater than  $k$ . The condition  $d\omega|_B = 0$  implies that  $a_{ij}$  are functions of  $z_1, \dots, z_k$  only. This exhibits the local neighborhood as a product and the 2-form splits according to the product as  $\omega|_B + 0$ . This implies that  $\omega$  descends to a non-degenerate closed two form  $\omega_{red}$  on  $S^0$  such that  $\pi^*\omega_{red} = \omega|_{B^0}$ . □

We will call  $\omega_{red}$  the reduced symplectic form over  $S^0$ .

The foliation is in general only locally trivial in étale or analytic topology. In the case that  $X$  is the moduli spaces  $M_H(\mathbf{v})$  of stable coherent sheaves over a fixed K3 surface with a polarization  $H$  and a Mukai vector  $\mathbf{v}$ , Markman has constructed such examples ([11]).

It is useful to record an immediate corollary.

**Corollary 4.9.** *If  $\pi : B \rightarrow S$  is a smooth fibration, then it coincides with the integrable null foliation of  $\omega|_B$ . In particular,  $\pi : B \rightarrow S$  is a  $\mathbb{P}^r$ -bundle where  $r = \text{codim } B$  and  $S$  is a smooth projective symplectic variety with the induced symplectic form  $\omega_{red}$ .*

**Remark 4.10.** *Let  $\pi : X \rightarrow Z$  be a small birational contraction. Let  $B_i$  ( $i \in I$ ) be all the irreducible component of the degenerate locus such that  $B_i \rightarrow S_i := \pi(S_i)$  is a smooth fibration for all  $i$ . Then we believe  $B_i$  ( $i \in I$ ) must be mutually disjoint. In particular, the Mukai elementary transformations can be performed along these components in any order. This observation is, however, not needed later.*

## 5 Lagrangian subvarieties and exceptional sets

**Lemma 5.1.** *Let  $B$  be a (possibly singular, reducible) subvariety of  $X$ . Assume that  $B$  is contractible. Then  $B$  is Lagrangian if and only if it is contractible to a point.*

*Proof.* Assume that  $B$  is Lagrangian. Let  $\pi : X \rightarrow Z$  be the contraction. If  $\dim \pi(B) > 1$ , choose a small neighborhood  $U$  in  $Z$  of a smooth point  $s$  in  $\pi(B)$ . Let  $g$  be a holomorphic function over  $U$  such that  $dg_s \neq 0$  along  $\pi(B)$ . Let  $f$  be the pullback of  $g$ , defined over a neighborhood of the fiber  $F = \pi^{-1}(s)$ . Then the Hamiltonian vector field  $\xi$  defined by  $\iota_{\zeta_f} \omega = df$  has the property that  $\omega(\zeta, T_b B) = df(T_b B) = dg(d\pi(T_b B)) \neq 0$  for a generic point  $b \in B$ . Since  $B$  is Lagrangian,  $\zeta_f$  is not tangent to  $B$ , hence deforms a generic fiber  $F$  around  $s \in S$  outside  $B$ , a contradiction.

Conversely, assume that  $B$  is contracted to a point. Then  $\omega|_B = 0$  by Lemma 4.1. That  $\dim B > n$  is impossible because  $\omega|_B = 0$  and  $\omega$  is non-degenerated. If  $\dim B = m < n$ , by Lemma 2.1, some rational curve will only move in a family of dimension bounded above by  $2m - 2$ , contradicting with Lemma 3.3.  $\square$

**Remark 5.2.** *Note that the proof of necessary part is purely analytic and hence the corresponding statement is valid in the analytic category as well. In the projective category and assuming that the contraction  $\pi : X \rightarrow Z$  is small, this lemma is a direct corollary of Theorem 4.2.*

**Theorem 5.3.** *Let  $P$  be a smooth closed subvariety of  $X$ . Then  $P$  can be contracted to a point if and only if  $P$  is isomorphic to  $\mathbb{P}^n$ .*

*Proof.* If  $P$  is isomorphic to  $\mathbb{P}^n$ , being rational, therefore Lagrangian, its normal bundle is isomorphic to  $T^*\mathbb{P}^n$ , hence negative. Thus  $P$  is contractible to a point. Conversely, if  $P$  can be contracted to a point, by Lemma 5.1 it is a Lagrangian. Combining Lemma 3.3 and Theorem 2.2, we conclude that  $P \cong \mathbb{P}^n$ . □

In general, we expect

**Conjecture 5.4.** *Let  $X^{2n} \rightarrow Z$  be a birational contraction such that the degenerate locus  $B$  is contracted to isolated points. Then  $B$  is a disjoint union of smooth subvarieties that are isomorphic to  $\mathbb{P}^n$ .*

See [1] for an affirmative answer when  $n = 2$  under the normality assumption. We were recently informed that this was proved for  $n = 2$  in [15]. Also [15] indicated that the general case has been treated by Cho-Miyaoka-Shepherd-Barron.

## 6 semismall and symplectic contractions

Let  $\pi : U \rightarrow V$  be a proper birational morphism. Assume that  $V = \bigcup_{\alpha} V_{\alpha}$  is a stratification of  $V$  by smooth strata such that  $\pi : U \rightarrow V$  is weakly stratified with respect to this stratification. That is,  $\pi^{-1}(V_{\alpha}) \rightarrow V_{\alpha}$  is a fibration for all  $\alpha$ . Such a stratification always exists.

**Definition 6.1.**  $\pi : U \rightarrow V$  is called *IH-semismall* if for any  $v \in V_{\alpha}$  we have

$$2 \dim \pi^{-1}(v) \leq \text{codim } V_{\alpha}.$$

The stratum  $V_{\alpha}$  is called *IH-relevant* to the map  $\pi$  if the equality holds.  $\pi$  is *strictly semismall* if all strata  $V_{\alpha}$  are relevant to  $\pi$ . It is known that the definition is independent of the choice of a stratification.

$\pi$  is *IH-small* if none of the strata is relevant to  $\pi$  except the open stratum. But by Corollary 4.3 *IH-small* maps do not exist for contractions of symplectic varieties.

**Proposition 6.2.** *Let  $\pi : X \rightarrow Z$  be a contraction and  $B$  an arbitrary irreducible component of the exceptional locus. Assume that  $S := \pi(B) = \bigcup_{\alpha} S_{\alpha}$  is a stratification such that  $\pi : B \rightarrow \bigcup_{\alpha} S_{\alpha}$  is weakly stratified. Let  $F_{\alpha} = \bigcup_i F_{\alpha,i}$  be the union of the irreducible components of a fiber  $F_{\alpha}$  over  $S_{\alpha}$ .  $B_{\alpha} = \pi^{-1}(S_{\alpha}) \cap B$ . Then we have*

$$T_b F_{\alpha,i} \subset (T_b B_{\alpha})^{\perp}$$

for generic  $b \in (F_{\alpha,i})_{reg} \cap (B_{\alpha})_{reg}$ . In particular,  $\pi : X \rightarrow Z$  is always IH-semismall.

*Proof.* Consider the fibration  $\pi : B_{\alpha} \rightarrow S_{\alpha}$ . Let  $F_{\alpha,i}$  be an irreducible component of the fiber  $F_{\alpha}$ . Since  $H^{0,2}(F_{\alpha,i}) = 0$  (Lemma 4.1), we have  $\omega|_{F_{\alpha,i}} = 0$ . Then the essentially same proof as in the proof of Theorem 4.2 will yield that

$$T_b F_{\alpha,i} \subset (T_b B_{\alpha})^{\perp}$$

for generic point  $b \in F_{\alpha,i}$ . This implies that

$$\dim F_{\alpha,i} \leq \dim(T_b B_{\alpha})^{\perp}$$

for all  $i$ . But

$$\dim(T_b B_{\alpha})^{\perp} = \dim X - \dim B_{\alpha} = \dim X - \dim F_{\alpha} - \dim S_{\alpha}$$

Hence we have

$$2 \dim F_{\alpha} \leq \operatorname{codim} S_{\alpha}$$

for all  $\alpha$ . This means that  $\pi : X \rightarrow Z$  is IH-semismall.

Here is an easy observation

**Proposition 6.3.** *Every contraction  $\pi : X \rightarrow Z$  can be made to be strictly semismall by removing subvarieties of codimension greater than 2, while still keeping the properness of the map.*

*Proof.* First note that for each irreducible component  $B$  of the exceptional locus we have  $\operatorname{codim} B \geq 2$ . By Theorem 4.3, over the generic part of each irreducible component  $B$ ,  $\pi$  is strictly semismall. Hence  $\pi$  is always strictly semismall after removing subvarieties of codimension greater than 2 and still keeping the properness of the map.  $\square$

**Definition 6.4.** *Let  $\pi : X \rightarrow Z$  be a birational contraction and  $\pi : B \rightarrow S$  be as before (cf. Theorem 4.2). Let  $S = \bigcup S_{\alpha}$  be a stratification such that  $B \rightarrow S$  is weakly stratified. If every  $S_{\alpha}$  admits a symplectic form  $\omega_{red}$  induced from  $\omega|_{B_{\alpha}}$  (cf. Theorem 4.8), we will say that  $Z$  admits a (reduced) symplectic stratification and that the contraction  $\pi$  is symplectically stratified.*

The two definitions 6.1 and 6.4 are conjecturally to be related as follows.

Let  $\pi : X \rightarrow Z$  be a contraction and  $B$  an arbitrary irreducible component of the exceptional locus. Assume that  $S := \pi(B) = \bigcup_{\alpha} S_{\alpha}$  is a stratification such that  $\pi : B \rightarrow \bigcup_{\alpha} S_{\alpha}$  is weakly stratified. Let  $F_{\alpha} = \bigcup_i F_{\alpha,i}$  be the union of the irreducible components of a fiber  $F_{\alpha}$  over  $S_{\alpha}$ .  $B_{\alpha} = \pi^{-1}(S_{\alpha}) \cap B$ . Then by Proposition 6.2 we have,

$$T_b F_{\alpha,i} \subset (T_b B_{\alpha})^{\perp}$$

for generic  $b \in (F_{\alpha,i})_{reg} \cap (B_{\alpha})_{reg}$ . This inclusion becomes equality for a generic fiber of  $\pi : B \rightarrow S := \pi(B)$  (Theorem 4.3). We expect the following is true: *the equality holds if and only if  $S_{\alpha}$  carries a reduced symplectic form. In particular,  $\pi$  is strictly semismall if and only if it can be symplectically stratified.*

In Markman’s paper ([11]), all the contractions he constructed are indeed strictly semismall. This re-enforces our conjecture

**Conjecture 6.5.** *Every birational contraction from a smooth projective symplectic variety is necessarily strictly semismall. Equivalently, it can always be symplectically stratified.*

## 7 Further Observation and speculations

In this section we will try to explain that Hamiltonian flows might be useful in the study of birational geometry of symplectic varieties in the purely complex context.

Consider and fix a generic fiber  $F := F_{\alpha}$  over a generic (smooth) point  $s$  of  $S_{\alpha}$ . Let  $B_{\alpha} := \pi^{-1}(S_{\alpha})$ . Take a sufficiently small open neighborhood  $U_s$  of  $s$  in  $Z$  such that it is embedded in  $\mathbb{C}^N$  where  $N$  is the minimal embedding dimension of  $Z$  at  $s$ . Let  $U_F = \pi^{-1}(U_s)$ . The composition of the maps

$$U_F \xrightarrow{\pi} U_s \hookrightarrow \mathbb{C}^N$$

is still denoted by  $\pi$  (confusion does not seem likely).

Let  $P$  be any subvariety of  $U_F$ . By saying that a vector field  $\xi$  on  $U_F$  is tangent to  $P$ , we mean  $\xi$  is tangent to the nonsingular part of  $P$ . For a point  $x \in U_F$ ,  $\xi(x)$  denotes the vector of the field at  $x$ .

**Proposition 7.1.** *For every function  $g$  about  $s$  in  $\mathbb{C}^N$ , let  $f = \pi^*g$ . Then the Hamiltonian vector field  $\zeta_f$  defined by  $\iota_{\zeta_f} \omega = df$  is tangent to  $B_{\beta} \cap U_F$  for all  $\beta$ .*

*Proof.* This is because otherwise we would be able to deform (an irreducible component of)  $B_\beta \cap U_F$  along the flow of  $\zeta_f$ , which is impossible by the weak stratification.  $\square$

**Proposition 7.2.** *For every function  $g$  about  $s$  in  $\mathbb{C}^N$  such that the differential  $dg$  vanishes along  $U_s \cap S$  at  $s$ , let  $f = \pi^*g$ . Assume that  $S_\alpha$  carries an induced symplectic form. Then the Hamiltonian vector field  $\zeta_f$  defined by  $\iota_{\zeta_f}\omega = df$  is tangent to  $B_\beta \cap U_F$  for all  $\beta$  and to the fiber  $F$  over  $s$ .*

*Proof.* That  $\zeta_f$  is tangent to  $B_\beta \cap U_F$  for all  $\beta$  is the previous proposition.

To prove the second statement, note that for every generic  $b \in (F_\alpha)_{reg} \cap (B_\alpha)_{reg}$  and every  $w \in T_b B_\alpha$ , we have

$$\omega_b(\zeta_f(b), w) = df_b(w) = dg_s(d\pi_b(w)) = 0$$

because  $d\pi_b(w)$  is tangent to  $S$  at  $s$ . Hence  $\zeta_f(b)$  belongs to the radical of  $\omega|_B$ . By Theorem 4.2 (2),  $\zeta_f(b)$  is tangent to  $F$ .  $\square$

Heuristically, the map  $\pi : U_F \rightarrow \mathbb{C}^N$  can be interpreted as a *generalized* moment map in the following sense.

There are many holomorphic Hamiltonian flows around the fiber  $F$ . Consider the coordinate functions  $\{z_1, \dots, z_N\}$ . They generate the Hamiltonian vector fields  $\{\zeta_1, \dots, \zeta_N\}$  which acts (infinitesimally) around a neighborhood of the fiber  $F$  over  $s$ . We can think of

$$\mathfrak{g} := \text{Span}\{\zeta_1, \dots, \zeta_N\}$$

as a “Lie algebra”. Its dual space  $\mathfrak{g}^* = \text{Span}\{dz_1, \dots, dz_N\}$  is naturally identified with  $\mathbb{C}^N \cong m_{s,Z}/m_{s,Z}^2$ . Then the map

$$\pi : U_F \rightarrow \mathbb{C}^N \cong m_{s,Z}/m_{s,Z}^2 \cong \mathfrak{g}^*$$

checks the conditions

$$dz_i = \iota_{\zeta_i}\omega, 1 \leq i \leq N$$

which characterize the most important feature of a holomorphic moment map. The presence of a genuine group action does not seem essential. What is important is that many consequences of a moment map stems only from the equations  $dz_i = \iota_{\zeta_i}\omega, 1 \leq i \leq N$ .

$\mathfrak{g}$  has a “Lie subalgebra”

$$\mathfrak{g}_s := \left\{ \sum_i \lambda_i \zeta_i : \sum_i \lambda_i (dz_i)_s = 0 \text{ along } S, \lambda_i \in \mathbb{C} \right\}$$



$$= \{g \in m_{s,Z}/m_{s,Z}^2 : (dg)_s = 0 \text{ along } S.\}$$

This can be interpreted as the isotropy subalgebra at the point  $s$ . Although  $\mathfrak{g}$  may not really be a Lie algebra, we suspect that  $\mathfrak{g}_s$  is indeed a Lie algebra.

We expect that the above Hamiltonian flows should play some important role in birational geometry of complex (not necessarily projective) symplectic geometry. For example, it is natural to speculate that every irreducible component of a fiber of the small contraction  $\pi : X \rightarrow Z$  is an almost homogeneous space. Here, an arbitrary variety  $V$  is said to be an almost homogeneous space of a complex Lie group  $G$  if  $G$  acts on  $V$  and has a dense open orbit.

Finally, we conjecture that

**Conjecture 7.3.** *Let  $\Phi : X \dashrightarrow X'$  be a birational transformation between two smooth projective holomorphic symplectic varieties of dimension  $2n$ . Then, after removing subvarieties of codimension greater than 2,  $X$  and  $X'$  are related by a sequence of Mukai's elementary transformations.*

An approach may go as follows. Let  $E$  be the exceptional locus of  $X$  and  $E = \bigcup_i B_i$  be the union of its irreducible component. We can arrange to have some components  $\{B_j : j \in J\}$  contracted through log-extremal contraction. Assume that  $B$  is an arbitrary one of the contracted components. If  $\text{codim } B > 2$ , we leave it alone. If  $\text{codim } B = 2$ , Theorem 4.5 indicates that after throwing away a subvariety of codimension greater than 2 from  $B$ , the resulting fibering  $B^0 \rightarrow S^0$  is a  $\mathbb{P}^2$ -bundle.

Now keep only these  $\mathbb{P}^2$ -bundles and remove all other subvarieties in the contracted components  $\cup\{B_j : j \in J\}$ . By Proposition 4.10, all the above  $\{B_j^0 : j \in J\}$  are disjoint.

The Mukai transformations can be performed along  $\{B_j : j \in J\}$  (in any order). Call the resulting symplectic variety  $X^1$ . One may repeat the above to the birational map  $X^1 \dashrightarrow X'$ . After a finite step, we will arrive  $X'$  from  $X$  provided the necessary surgeries are properly applied.

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