Adv. Theor. Math. Phys. **6** (2002) 357–401

Conformal modes in simplicial quantum gravity and the Weil-Petersson volume of moduli space

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Abstract

Our goal here is to present a detailed analysis connecting the anomalous scaling properties of 2D simplicial quantum gravity to the geometry of the moduli space $\overline{\mathfrak{M}}_{g,N_0}$ of genus g Riemann surfaces with N_0 punctures. In the case of pure gravity we prove that the scaling properties of the set of dynamical triangulations with N_0 vertices are directly provided by the large N_0 asymptotics of the Weil-Petersson volume of $\overline{\mathfrak{M}}_{g,N_0}$, recently discussed by Manin and Zograf. Such a geometrical characterization explains why dynamical triangulations automatically take into account the anomalous scaling properties of Liouville theory. In the case of coupling with conformal matter we briefly argue that the anomalous scaling of the resulting discretized theory should be related to the Gromov-Witten invariants of the moduli space $\overline{\mathfrak{M}}_{g,N_0}(X,\beta)$ of stable maps from (punctured Riemann surfaces associated with) dynamical triangulations to a (smooth projective) manifold X parameterizing the conformal matter configurations.

e-print archive: http://lanl.arXiv.org/abs/math-ph/0107028

1 Introduction

The starting point of the path quantization of 2D gravity is the observation that the space of Riemannian structures $Riem(M)$ on a 2-dimensional Riemannian manifold M of genus g can be decomposed by means of a local slice for the action of the group of confeomorphisms $W(M) \ltimes Diff(M)$. This yields for the possibility of parameterizing locally a generic metric $h \in Riem(M)$ by means of a diffeomorphism $f \in Diff(M)$, a Weyl rescaling $e^{2u} \in W(M)$, and $3g-3$ complex parameters $\{m_\alpha\}$ varying in the moduli space $\mathfrak{M}_g$ of genus g Riemann surfaces. Explicitly we can write

$$\begin{gathered} S : \mathfrak{M}_g \times (W(M) \ltimes Diff(M)) \to Riem(M) \\ (\widehat{h}_{ab}(\mu_\alpha), e^{2v}, \psi) \longmapsto S(\widehat{h}_{ab}(\mu_\alpha), e^{2v}, \psi) \doteq h_{ab} = e^{2v}\left(\psi^*\widehat{h}(m_\alpha)\right)_{ab}, \end{gathered} \tag{1}$$

where $\widehat{h}_{ab}(m_\alpha)$ are the components of the reference metric whose conformal class defines the point $\{m_\alpha\}$ in $\mathfrak{M}_g$ we are considering. A natural choice for $\widehat{h}_{ab}(m_\alpha)$ is associated with the slice defined by the metrics of constant curvature. Such a framework allows us to introduce formal functional measures $D[h]$, $D[\psi]$, and $D_h[v]$ respectively over (the tangent spaces to) $Riem(M)$, $Diff(M)$ and $W(M)$, and whose properties, under conformal transformations, are instrumental to the theory. To briefly review the main features of such an analysis, (see *e.g.* [1]), let us introduce on M local complex coordinates $(z, \overline{z})$ in which $h = 2h_{z\overline{z}}|dz|^2$ and define

$$\begin{array}{rcl} \nabla^z_{(n)}\phi & = & (h_{z\overline{z}})^{-1}\frac{\partial}{\partial \overline{z}}\phi \otimes (dz)^{-1} \\ \nabla^{(n)}_z\phi & = & (h_{z\overline{z}})^{n}\frac{\partial}{\partial z}(h_{z\overline{z}})^{-n}\phi \otimes dz \\ \Delta^+_{(n)} & = & -2\nabla^z_{(n+1)}\nabla^{(n)}_z \\ \Delta^-_{(n)} & = & -2\nabla^{(n-1)}_z\nabla^z_{(n)} \end{array}, \tag{2}$$

where $\phi = \phi_{z\ldots z}(dz)^n$ is a tensor field of weight $(n,0)$. The conformal properties of $D[h]$ are strictly connected to the behavior of the family of functional determinants (arising as jacobians of the slice map (1))

$$Z^{\pm}_{(n)}(h) \equiv \frac{\det{}'\Delta^{\pm}_{(n)}}{\det\langle\phi_j|\,\phi_r\rangle_h \det\langle\psi_a|\,\psi_b\rangle_h}. \tag{3}$$

where $\phi_j \in \ker\nabla^z_{(n+1)}$, $\psi_a \in \ker\nabla^{(n)}_z$ for $Z^+_{(n)}(h)$, $\phi_j \in \ker\nabla^{(n-1)}_z$, $\psi_a \in \ker\nabla^z_{(n)}$ for $Z^-_{(n)}(h)$, and where $\det{}'\Delta^{\pm}_{(n)}$ denotes the ζ-regularized determinant restricted to the non-zero modes. According to a well-known result, the behavior of $Z^{\pm}_{(n)}(h)$ under a conformal rescaling $h_{ab} \to e^{2v}(\psi^* \widehat{h})_{ab}$, is provided by

$$Z^{\pm}_{(n)}(e^{2v}(\psi^* \widehat{h})) = Z^{\pm}_{(n)}(\widehat{h}) e^{-2c^{\pm}_{(n)} S_L(\widehat{h}, v)} \tag{4}$$

where

$$c^{\pm}_{(n)} \doteq 6n^2 \pm 6n + 1 \tag{5}$$

and where

$$S_L(\widehat{h}, v) \doteq \frac{1}{12\pi} \int_M d^2x \sqrt{\widehat{h}} \left[\frac{1}{2} \widehat{h}^{ik} \partial_i v \partial_k v + R(\widehat{h}) v + \gamma^2 (e^{2v} - 1) \right], \tag{6}$$

is the Liouville action (the constant γ depends upon the procedure used for regularizing $Z^{\pm}_{(n)}(h)$). Such a behavior characterizes $Z^{\pm}_{(n)}(h)$ as defining on M a conformal field theory with central charge related to $\pm c^{\pm}_{(n)}$, and is the basic ingredient for discussing two-dimensional quantum gravity coupled to matter fields with central charge c_{mat}. An analogous though much less transparent situation holds for the behavior the Weil measure $D_h[v]$ under conformal rescalings $h_{ab} \to e^{2v}(\psi^* \widehat{h})_{ab}$. If we formally assume for $D_h[v]$ transformation properties structurally similar to (4) , (for $n = -1$, and with a Liouville-type term containing two tunable parameters), then we get the celebrated relation [2], [3], [4]

$$Z_g[A] = Z_g[1]\ A^{\frac{(1-g)}{12}[c_m - 25 - \sqrt{(25 - c_m)(1 - c_m)}] - 1}, \tag{7}$$

characterizing the scaling, with surface area A, of the fixed area partition function $Z_g[A]$ of 2D gravity coupled to a matter field of central charge c_m. The quantity

$$\gamma_{string} = \frac{(1-g)}{12} \left(c_m - 25 - \sqrt{(25 - c_m)(1 - c_m)} \right) + 2, \tag{8}$$

defines the string susceptibility exponent γ_{string} which, for pure gravity, reduces to the well-known value

$$\gamma_{string} = \frac{5g - 1}{2}. \tag{9}$$

The scaling ansatz on $D_h[v]$ is believed to be reliable for c_m small, a fact reflected in the expression for γ_{string} which is ambiguous for $c_m > 1$. From the point of view of field theory, we are entering a region of strong coupling regime between conformal matter and gravity, and its analysis requires the understanding, not yet achieved, of the dynamics of Liouville theory. It is an intriguing fact that such difficulties are absent (or appear in a different, much less problematic guise) when we approximate the set of Riemannian surfaces $Riem(M)$ with dynamical triangulations. As a matter of fact, dynamical triangulations (DT) provide one of the most powerful technique for analyzing two-dimensional quantum gravity in regimes which are not accessible to the standard field-theoretic formalism. This is basically due to the circumstance that in such a discretized setting the quantum measure of the theory, describing the gravitational dressing of conformal operators in the continuum theory, reduces to a suitably constrained enumeration of distinct triangulations admitted by a surface of given topology,(see *e.g.*, [5] for a review). And it has been argued, mainly as a consequence of a massive numerical evidence, that such a (counting) measure automatically accounts for the anomalous scaling properties of the measure $D[h]$ and $D_h[v]$ governing the continuum path-quantization of 2D gravity. The geometrical origin of such a property is rather elusive and it is not clear how the counting for dynamical triangulations factorizes, so to speak, in terms of a discrete analogous of a moduli space measure and of a Liouville measure over the conformal degrees of freedom of the theory. Some results in such a direction have been recently discussed by Catterall and Mottola with an emphasis on the numerical simulation, [6]. Menotti and Peirano [7] discussed a similar issue in connection with the Regge measure in simplicial quantum gravity. However in such a case the problem takes on a rather different flavor, being more directly connected with the issue of the diffeomorphism invariance of the resulting measure.

The geometrical explanation of the behavior of the DT measure is not obvious if we only follow the folklore which considers dynamical triangulations as a sort of approximating net in the space of Riemannian structures $\frac{Riem(M)}{Diff(M)}$. Rather, we show that the explanation is deeply connected with a geometrical mechanism which allow to describe a dynamically triangulated manifold with N_0 vertices as a Riemann surface with N_0 punctures dressed with a field whose charges describe discretized curvatures (connected with the deficit angles of the triangulation). Such a picture calls into play the (compactified) moduli space of genus g Riemann surfaces with N_0 punctures $\overline{\mathfrak{M}}_{g,N_0}$, and allow us to prove that the counting of distinct dynamical triangulations is directly related to the computation of the Weil-Petersson volume of $\overline{\mathfrak{M}}_{g,N_0}$. We then exploit the large N_0 asymptotics of the Weil-Petersson volume of $\overline{\mathfrak{M}}_{g,N_0}$ recently discussed (in relation with the Witten-Kontsevich model [8], [9]) by Manin and Zograf [10], [11] in order to prove that the anomalous scaling properties of the counting measure of dynamical triangulations is only due to the modular degrees of freedom which parametrizes in $\overline{\mathfrak{M}}_{g,N_0}$ the vertices of the triangulations. Since these degrees of freedom characterize also the conformal factor defining the metric geometry of the triangulation, one has a geometrical explanation of how dynamical triangulations describe the anomalous scaling of the Weyl measure $D_h[v]$. Such an analysis is discussed here in detail in the case of pure gravity, in the case of coupling with conformal matter we briefly argue that the role of $\overline{\mathfrak{M}}_{g,N_0}$ is taken over by the moduli space $\overline{\mathfrak{M}}_{g,N_0}(X,\beta)$ of stable maps from the punctured Riemann surface representing a dynamical triangulation and a (smooth projective) manifold (variety) X ,(see *e.g.*, [12]). Roughly speaking we may think of X as the space of possible conformal fields over M, and points in $\overline{\mathfrak{M}}_{g,N_0}(X,\beta)$ as representing distributions of matter fields over dynamical triangulations. According to our analysis in the pure gravity case, it is rather natural to conjecture that the measure describing the statistical distribution of such matter fields over the distinct triangulation is provided by a (generalized) Weil-Petersson volume of $\overline{\mathfrak{M}}_{g,N_0}(X,\beta)$. This calls into play the (descendent) Gromov-Witten invariants of X which describe intersection theory over $\overline{\mathfrak{M}}_{g,N_0}(X,\beta)$. This is interesting from various perspectives since it is already expected [13] that $\overline{\mathfrak{M}}_{g,N_0}(X,\beta)$, via its intersection theory, is governed by matrix models. Some aspects of the connection between the anomalous scaling of conformal matter interacting with 2D gravity and the theory of G-W invariants will be discussed in a forthcoming paper.

2 Triangulations as singular Euclidean structures

In order to fix the terminology to which we adhere in the rest of the paper, we need to discuss from a rather unusual perspective the geometry of triangulations and polytopal complexes in 2d simplicial quantum gravity. Let T denote a 2-dimensional simplicial complex with underlying polyhedron $|T|$ and f- vector $(N_0(T), N_1(T), N_2(T))$, where $N_i(T) \in \mathbb{N}$ is the number of i-dimensional sub- simplices σ^i of T. A Regge triangulation of a 2-dimensional PL manifold M, (without boundary), is a homeomorphism $|T_l| \to M$ where each face of T is geometrically realized by a rectilinear simplex of variable edge-lengths $l(\sigma^1(k))$ of the appropriate dimension. A dynamical triangulation $|T_{l=a}| \to M$ is a particular case of a Regge PL-manifold realized by rectilinear and equilateral simplices of edge-length $l(\sigma^1(k)) = a$. The metric structure of a Regge triangulation is locally Euclidean everywhere except at the vertices σ^0, (the *bones*), where the sum of the dihedral angles, $\theta(\sigma^2)$, of the incident triangles σ^2's is in excess (negative curvature) or in defect (positive curvature) with respect to the 2π flatness constraint. The corresponding deficit angle ε is defined by $\varepsilon = 2\pi - \sum_{\sigma^2} \theta(\sigma^2)$, where the summation is extended to all 2-dimensional simplices incident on the given bone σ^0. If K_T^0 denotes the (0)-skeleton of $|T_l| \to M$, (*i.e.*, the collection of vertices of the triangulation), then $M\backslash K_T^0$ is a flat Riemannian manifold, and any point in the interior of an r- simplex σ^r has a neighborhood homeomorphic to $B^r \times C(lk(\sigma^r))$, where B^r denotes the ball in $\mathbb{R}^n$ and $C(lk(\sigma^r))$ is the cone over the link $lk(\sigma^r)$, (the product $lk(\sigma^r) \times [0,1]$ with $lk(\sigma^r) \times \{1\}$ identified to a point), (recall that if we denote by $st(\sigma)$, (the star of σ), the union of all simplices of which σ is a face, then $lk(\sigma^r)$ is the union of all faces σ^f of the simplices in $st(\sigma)$ such that $\sigma^f \cap \sigma = \emptyset$). For dynamical triangulations, the deficit angles are generated by the string of integers, the *curvature assignments*, $\{q(k)\}_{k=1}^{N_0(T)} \in \mathbb{N}^{N_0(T)}$, *viz.*,

$$\varepsilon(i) = 2\pi - q(i)\arccos(1/2), \tag{10}$$

where $q(i) \doteq \#\{\sigma^2(h) \perp \sigma^0(i)\}$ provides the numbers of triangles incident on the $N_0(T)$ distinct vertices. For a regular triangulation we have $q(k) \geq 3$, and since each triangle has 3 vertices σ^0, the set of integers $\{q(k)\}_{k=1}^{N_0(T)}$ is constrained by

$$\sum_k^{N_0} q(k) = 3N_2 = 6\left[1 - \frac{\chi(M)}{N_0(T)}\right] N_0(T), \tag{11}$$

where $\chi(M)$ denotes the Euler-Poincaré characteristic of the surface, and where $6\left[1 - \frac{\chi(M)}{N_0(T)}\right]$, ($\simeq 6$ for $N_0(T) >> 1$), is the average value of the curvature assignments $\{q(k)\}_{k=1}^{N_0}$.

Remark 1. *Note that in what follows we shall consider semi-simplicial complexes for which the constraint $q(k) \geq 3$ is removed. Examples of such configurations are afforded by triangulations with pockets, where two triangles are incident on a vertex, or more generally by triangulations where the star of a vertex may contain just one triangle. We shall refer to such extended configurations as generalized (Regge and dynamical) triangulations .*

In this connection, it is well known that in dimension two regular triangulations and generalized triangulations simply provide different approximations of the same continuum quantum gravity theory [5]. Thus it would seem natural to restrict attention to the simpler ensemble of pure simplicial complexes. Such a restriction is however quite artificial, and in discussing the geometrical aspects of simplicial quantum gravity (for instance its connection with moduli space theory [8], [9]), the set of semi-simplicial complexes provide a more natural setup for our analysis.

As recalled, a (generalized) Regge triangulation $|T_l| \to M$ defines on the PL manifold M a polyhedral metric with conical singularities, associated with the vertices $\{\sigma^0(i)\}_{i=1}^{N_0(T)}$ of the triangulation, but which is otherwise flat and smooth everywhere else. Such a metric has important special features, in particular it induces on the PL manifold M a geometrical structure which turns out to be a particular case of the theory of *Singular Euclidean Structure* (in the sense of M. Troyanov [14], and W. Thurston [15]). This structure can be most conveniently described in terms of complex function theory. To this end, let us consider the (first) barycentric subdivision of T, then the closed stars, in such a subdivision, of the vertices of the original triangulation T_l form a collection of 2-cells $\{\rho^2(i)\}_{i=1}^{N_0(T)}$ characterizing a polytope P barycentrically dual to T.

Remark 2. *Note that here we are not considering a rectilinear presentation of the dual cell complex P (where the PL-polytope is realized by flat polygonal 2-cells $\{\rho^2(i)\}_{i=1}^{N_0(T)}$) but rather a geometrical presentation $|P_{T_L}| \to M$ of P where the 2-cells $\{\rho^2(i)\}_{i=1}^{N_0(T)}$ retain the conical geometry induced on the barycentric subdivision by the original metric structure of $|T_l| \to M$. Namely, if $(\lambda(k), \chi(k))$ denote polar coordinates (based at $\sigma^0(k)$) of $p \in \rho^2(k)$, then $\rho^2(k)$ is geometrically realized as the space*

$$\frac{\{(\lambda(k), \chi(k)) \ : \lambda(k) \geq 0; \chi(k) \in \mathbb{R}/(2\pi - \varepsilon(k))\mathbb{Z}\}}{(0, \chi(k))} \sim (0, \chi'(k)) \tag{12}$$

endowed with the metric

$$d\lambda(k)^2 + \lambda(k)^2 d\chi(k)^2. \tag{13}$$

This definition characterizes the conical *Regge polytope $|T_l| \to M$ (and its rigid equilateral specialization $|P_{T_a}| \to M$) barycentrically dual to $|T_l| \to M$.*

In order to switch back and forth between the geometry of $|T_l| \to M$ and the corresponding conical geometry of the barycentrically dual polytope $|P_{T_L}| \to M$ it is sufficient to relate the lengths of the sides of the generic triangle $\sigma^2(\alpha) \in |T_l| \to M$ to the lengths of the corresponding medians connecting the barycenter $\rho^0(\alpha)$ of $\sigma^2(\alpha)$ to the barycenters of the sides of $\sigma^2(\alpha)$. If we denote by $l_i(\alpha)$, $i = 1, 2, 3$, the lengths of such sides, and by $\widehat{L}_i(a)$ the lengths of the corresponding medians, (*e.g.*, $\widehat{L}_1(a)$ is the length of the median connecting $\rho^0(\alpha)$ with the barycenter of $l_1(\alpha)$), then a direct computation provides

$$
\begin{array}{rcl}
\widehat{L}_1^2(a) & = & \frac{1}{18}l_3^2(\alpha)+\frac{1}{18}l_2^2(\alpha)-\frac{1}{36}l_1^2(\alpha)\\
\widehat{L}_2^2(a) & = & \frac{1}{18}l_1^2(\alpha)+\frac{1}{18}l_3^2(\alpha)-\frac{1}{36}l_2^2(\alpha)\\
\widehat{L}_3^2(a) & = & \frac{1}{18}l_1^2(\alpha)+\frac{1}{18}l_2^2(\alpha)-\frac{1}{36}l_3^2(\alpha)\\
 & & \\
l_1^2(\alpha) & = & 8\widehat{L}_3^2(a)+8\widehat{L}_2^2(a)-4\widehat{L}_1^2(a)\\
l_2^2(\alpha) & = & 8\widehat{L}_1^2(a)+8\widehat{L}_3^2(a)-4\widehat{L}_2^2(a)\\
l_3^2(\alpha) & = & 8\widehat{L}_1^2(a)+8\widehat{L}_2^2(a)-4\widehat{L}_3^2(a)
\end{array}
. \tag{14}
$$

In particular, if we denote by $L(\alpha,\beta)$ the length of the edge $\rho^1(\alpha,\beta) \in |P_{T_L}| \to M$ connecting the barycenters $\rho^0(\alpha)$ and $\rho^0(\beta)$ of two triangles $\sigma^2(\alpha)$ and $\sigma^2(\beta)$ in $|T_l| \to M$ sharing the sides $l_3(\alpha)$ and $l_1(\beta)$, then $L(\alpha,\beta)=\widehat{L}_3(a)+\widehat{L}_1(\beta)$, and we get

$$
L(\alpha,\beta)=\frac{1}{6}\left(\sqrt{2l_1^2(\alpha)+2l_2^2(\alpha)-l_3^2(\alpha)}+\sqrt{2l_3^2(\beta)+2l_2^2(\beta)-l_1^2(\beta)}\right). \tag{15}
$$

In such a geometrical setup, let $\rho^2(k)$ be the generic two-cell $\in |P_{T_L}| \to M$ barycentrically dual to the vertex $\sigma^0(k) \in |T_l| \to M$ and let us denote by

$$
L(\partial(\rho^2(k)))=\sum_{h=1}^{q(k)} L(\rho^1(h)) \tag{16}
$$

the length of the boundary $\partial(\rho^2(k))$ of $\rho^2(k)$, where $L(\rho^1(h))$ are the lengths of the $q(k)$ ordered edges $\{\rho^1(j)\} \in \partial(\rho^2(k))$. If $\varepsilon(k)$ denotes the deficit angle corresponding to $\sigma^0(k) \in |T_l| \to M$, then we define the slant radius associated with the cell $\rho^2(k) \in |P_{T_L}| \to M$ according to

$$
r(k) \doteq \frac{L(\partial(\rho^2(k)))}{2\pi-\varepsilon(k)}. \tag{17}
$$

Let

$$
B^2(k) \doteq \{p \in \rho^2(k)\backslash\partial(\rho^2(k))\} \tag{18}
$$

the open ball (a Euclidean cone) associated with $\rho^2(k)$ and contained in the star $st(\sigma^0(k))$ of the given vertex $\sigma^0(k)$. Note that any two such balls, say $B^2(k)$ and $B^2(j)$, $k \neq j$, are pairwise disjoint, and that the complex

$$
(|T_l| \to M)\,/\bigcup_{k=1}^{N_0(T)} B^2(k) \tag{19}
$$

retracts on the 1-skeleton $K^1[P_{T_L}]$ of $|P_{T_L}| \to M$. To any vertex $\sigma^0(k) \in |T_l| \to M$ we associate a complex uniformizing coordinate $t_k \in \mathbb{C}$ defined in the open disk of radius $r(k)$, *viz.*

$$\begin{gathered} B^2(k) \xrightarrow{t_k} D_k(r(k)) \doteq \{t_k \in \mathbb{C}\,|0 \leq t_k < r(k)\} \\ B^2(k) \ni p \longmapsto t_k(p). \end{gathered} \tag{20}$$

In terms of t_k we can explicitly write down the singular Euclidean metric locally characterizing the singular Euclidean structure of $B^2(k)$, according to

$$ds^2_{(k)} \doteq e^{2u}\left|t_k - t_k(\sigma^0(k))\right|^{-2\left(\frac{\varepsilon(k)}{2\pi}\right)}|dt_k|^2, \tag{21}$$

where $\varepsilon(k)$ is given by (10), and $u : B^2 \to \mathbb{R}$ is a continuous function (C^2 on $B^2 - \{\sigma^0(k)\}$) such that, for $t_k \to t_k(\sigma^0(k))$,

$$\begin{gathered} \left|t_k - t_k(\sigma^0(k))\right| \frac{\partial u}{\partial t_k} \to 0, \\ \left|t_k - t_k(\sigma^0(k))\right| \frac{\partial u}{\partial \overline{t_k}} \to 0. \end{gathered} \tag{22}$$

Up to the presence of the normalizing conformal factor e^{2u}, one recognizes in (21) the metric of a Euclidean cone of total angle $\theta(k) = 2\pi - \varepsilon(k)$. We can glue together the uniformizations $\{D_k(r(k))\}_{k=1}^{N_0(T)}$ along the pattern defined by the 1-skeleton of $|P_{T_L}| \to M$ and generate on M the quasi-conformal structure

$$(M, \mathcal{C}_{sg}) \doteq \bigcup_{|P_{T_L}| \to M} \{D_k(r(k)); ds^2_{(k)}\}_{k=1}^{N_0(T)} \tag{23}$$

naturally associated with $|T_l| \to M$. If $|dt|^2$ is a smooth (conformally flat) metric on M, then $(M, \mathcal{C}_{sg})$ can be (locally) represented by the metric

$$ds^2_T = e^{2v}|dt|^2, \tag{24}$$

where the conformal factor v is given by

$$v \doteq u - \sum_{k=1}^{N_0(T)} \left(\frac{\varepsilon(k)}{2\pi}\right) \ln|t - t_k|. \tag{25}$$

Even if $(M, \mathcal{C}_{sg})$ is uniquely defined by such metric, the singular metric $e^{2v}|dt|^2$ characterizing a given $(M, \mathcal{C}_{sg})$ is only defined up to the conformal symmetry of $(M, \mathcal{C}_{sg})$. For instance for $M \approx \mathbb{S}^2$ we have a residual $SL(2, \mathbb{C})$ invariance, [7]. It is easily verified that the number of degrees of freedom associated with (24) corresponds indeed to the number $N_1(T)$ of edges of the Regge triangulation $|T_{l=a}| \to M$. The actual count follows by observing that (24) is parametrized by the $N_0(T)$ complex coordinates t_k of the conical singularities, by the $N_0(T)$ deficit angles $\varepsilon(k)$ constrained by $\sum_{k=1}^{N_0(T)} \left(-\frac{\varepsilon(k)}{2\pi}\right) = 2g - 2$, by the conformal factor u (see (25)), and by the $6g - 6$ moduli which parametrize the possible inequivalent choices of the smooth base metric $|dt|^2$ in (24), (assume that $g \geq 2$, otherwise in such a count we have also to take into account the dimension of the space of conformal Killing vector fields of $|dt|^2$). This sums up to

$$3N_0(T) + 6g - 6, \tag{26}$$

degrees of freedom which, according to the Dehn-Sommerville relations $N_0(T) - N_1(T) + N_2(T) = 2 - 2g$, and $2N_1(T) = 3N_2(T)$, actually equals $N_1(T)$.

The singular structure of the metric defined by (24) and (25) can be naturally summarized in a formal linear combination of the points $\{\sigma^0(k)\}$ with coefficients given by the corresponding deficit angles (normalized to 2π), *viz.*, in the *real divisor* [14]

$$Div(T) \doteq \sum_{k=1}^{N_0(T)} \left(-\frac{\varepsilon(k)}{2\pi}\right) \sigma^0(k) = \sum_{k=1}^{N_0(T)} \left(\frac{\theta(k)}{2\pi} - 1\right) \sigma^0(k) \tag{27}$$

supported on the set of bones $\{\sigma^0(i)\}_{i=1}^{N_0(T)}$. Note that the degree of such a divisor, defined by

$$|Div(T)| \doteq \sum_{k=1}^{N_0(T)} \left(\frac{\theta(k)}{2\pi} - 1\right) = -\chi(M)$$

is, for dynamical triangulations, a rewriting of the combinatorial constraint (11). In such a sense, the pair $(|T_l| \to M, Div(T))$, or shortly, $(T, Div(T))$, encodes the datum of the triangulation $|T_l| \to M$ and of a corresponding set of curvature assignments $\{q(k)\}$ on the bones $\{\sigma^0(i)\}_{i=1}^{N_0(T)}$. The real divisor $|Div(T)|$ characterizes the Euler class of the pair $(T, Div(T))$ and yields for a corresponding Gauss-Bonnet formula. Explicitly, the Euler number associated with $(T, Div(T))$ is defined, [14], by

$$e(T, Div(T)) \doteq \chi(M) + |Div(T)|. \tag{28}$$

and the Gauss-Bonnet formula reads [14]:

Lemma 3. *(Gauss-Bonnet for triangulated surfaces)*
Let $(T, Div(T))$ be a triangulated surface with divisor

$$Div(T) \doteq \sum_{k=1}^{N_0(T)} \left(\frac{\theta(k)}{2\pi} - 1 \right) \sigma^0(k), \tag{29}$$

associated with the vertices incidences $\{\sigma^0(k)\}_{k=1}^{N_0(T)}$. Let ds^2 be the conformal metric (24) representing the divisor $Div(T)$. Then

$$\frac{1}{2\pi} \int_M K dA = e(T, Div(T)), \tag{30}$$

where K and dA respectively are the curvature and the area element corresponding to the metric ds_T^2.

Note that such a theorem holds for any singular Riemann surface Σ described by a divisor $Div(\Sigma)$ and not just for triangulated surfaces [14]. Since for a Regge (dynamical) triangulation, we have $e(T_a, Div(T)) = 0$, the Gauss-Bonnet formula implies

$$\frac{1}{2\pi} \int_M K dA = 0. \tag{31}$$

Thus, a triangulation $|T_l| \to M$ naturally carries a conformally flat structure. Clearly this is a rather obvious result, (since the metric in $M - \{\sigma^0(i)\}_{i=1}^{N_0(T)}$ is flat). However, it admits a not-trivial converse (recently proved by M. Troyanov, but, in a sense, going back to E. Picard) [14], [16]:

Theorem 4. *(Troyanov-Picard) Let $((M, \mathcal{C}_{sg}), Div)$ be a singular Riemann surface with a divisor such that $e(M, Div) = 0$. Then there exists on M a unique (up to homothety) conformally flat metric representing the divisor Div.*

These results geometrically characterize metrical triangulations (and the associated conical Regge polytopal surfaces) as a particular case of the theory of singular Riemann surfaces, and provides the rationale for viewing two-dimensional simplicial quantum gravity in a more analytic spirit. In order to put this latter remark in a proper perspective, let us set

$$(M; N_0) \doteq M - \{\sigma^0(i)\}_{i=1}^{N_0(T)}, \tag{32}$$

with $2-2g-N_0(T) < 0$. Also, let us denote by $\mathcal{H}^2 = \{\zeta \in \mathbb{C} \mid \mathrm{Im}(\zeta) > 0\}$ the upper half-plane equipped with the metric $h(\zeta) |d\zeta|^2 \doteq \frac{|d\zeta|^2}{(\mathrm{Im}(\zeta))^2}$. As is well known, the set of fractional linear (Möbius) transformations $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I\}$ acts on $(\mathcal{H}^2, h(\zeta)\,|d\zeta|^2)$ by homographic transformations and coincides with the group of orientation preserving isometries of the hyperbolic upper half-plane $(\mathcal{H}^2, h(\zeta)\,|d\zeta|^2)$. According to the Poincaré-Klein-Koebe uniformization theorem any Riemann surface M (of genus $g \geq 2$) can be represented as a quotient $\mathcal{H}^2/\Gamma$, where Γ is a discrete subgroup of $PSL(2,\mathbb{R})$ acting freely on $(\mathcal{H}^2, h(\zeta)\,|d\zeta|^2)$, and the metric $h(\zeta)\,|d\zeta|^2$ descends to the quotient defining a complete metric of constant curvature -1 on $\mathcal{H}^2/\Gamma$. The discrete subgroup Γ is canonically isomorphic to the fundamental group $\pi_1(M)$ of the surface $M = \mathcal{H}^2/\Gamma$, and any homotopy class of closed curves on $M = (\mathcal{H}^2/\Gamma, h(\zeta)\,|d\zeta|^2)$ contains a unique geodesic. Any two surfaces $(\mathcal{H}^2/\Gamma)$ and $(\mathcal{H}^2/\Gamma')$ resulting from such a construction are isomorphic if and only if the subgroups Γ and Γ' are conjugated by an element of $PSL(2,\mathbb{R})$. Let us denote by $\mathfrak{T}_g(M)$ the Teichmüller space of all conformal structures on M under the equivalence relation given by pullback by diffeomorphisms isotopic to the identity map $id : M \to M$. The standard uniformization $M = (\mathcal{H}^2/\Gamma, h(\zeta)\,|d\zeta|^2)$ provides an embedding of $\mathfrak{T}_g(M)$ into a connected component of the representation variety

$$\frac{Hom(\pi_1(M), PSL(2,\mathbb{R}))}{PSL(2,\mathbb{R})}, \tag{33}$$

the set of conjugacy classes of homomorphisms

$$\Phi : \pi_1(M) \to PSL(2,\mathbb{R}).$$

The component characterized by $\mathfrak{T}_g(M)$ is the one associated with conjugacy classes of representations $[\Phi]$ for which $\Phi(\lambda)$ is an hyperbolic element of $PSL(2,\mathbb{R})$, (*i.e.*, such that the trace of the corresponding matrix is > 2), whenever λ is a non trivial homotopy class in $\pi_1(M)$. As stressed by W. Goldman, [17] the remaining components of the representation variety (33) are related to the theory of singular geometric structures developed by Troyanov and Thurston, and briefly described above. In particular, each component in (33) is classified by a corresponding Euler class (28). As we have seen, for dynamical (and Regge) triangulations $|T_l| \to M$ the Euler class is zero, and there exists on $(M; N_0)$ a unique non-singular Euclidean structure. The corresponding homomorphism

$$\pi_1\left((M; N_0)\right) \longrightarrow PSL(2,\mathbb{R}) \tag{34}$$

is defined by sending the link of $\sigma^0(i)$ into the elliptic element of $PSL(2,\mathbb{R})$ which describes the rotation of angle $\theta(i)$ around $\sigma^0(i)$, (recall that a matrix of $PSL(2,\mathbb{R})$ is elliptic if and only if it is conjugate in $SL(2,\mathbb{R})$ to a unique rotation matrix; note also that for any $x \in \mathcal{H}^2$, the stabilizer group of x in $PSL(2,\mathbb{R})$ is conjugate to the circle group $SO(2)$). Thus, by means of the holonomy map (34), the singular Riemann surface $((M, \mathcal{C}_{sg}), Div)$ describes a (generalized) triangulation $|T_l| \to M$ as a singular uniformization of the Euclidean surface $(M; N_0)$. The advantage of this approach is that it directly relates simplicial quantum gravity to the properties of the Teichmüller space associated with the Riemann surfaces $((M, \mathcal{C}_{sg}), Div)$ describing the inequivalent triangulations $|T_l| \to M$. This is quite appealing since brings simplicial quantum gravity even closer to continuum 2D gravity, where Riemann surface theory plays a prominent role. The disadvantage is that here we are not dealing with ordinary Teichmüller space theory, since the relevant component of (33) refers to Euclidean structures with conical singularities. As recalled, these latter are in the connected component of the representation variety

$$\frac{Hom(\pi_1((M; N_0)), PSL(2,\mathbb{R}))}{PSL(2,\mathbb{R})}, \tag{35}$$

with Euler class 0, and we need non-standard techniques from the theory of geometric structures in order to work in such a setting. A typical example in this direction is provided by Thurston analysis of the space of dynamically triangulated spheres with positive deficit angles [15]. In this connection it is also interesting to remark that moduli spaces of singular Euclidean structures (again on the 2-sphere) occurs in the classical analysis of the monodromy properties of hypergeometric functions carried out by Deligne and Mostow [18].

There is a way of circumventing the use of sophisticated representation theoretic techniques by relating the space of inequivalent singular Euclidean structures to the more standard theory of punctured Riemann surfaces. This boils down to the observation that on punctured Riemann surfaces we can introduce flat metrics by exploiting the connection between ribbon graphs and the theory of Jenkins-Strebel (JS) quadratic differentials [19], [20]. Such a correspondence is well-known in 2-D quantum gravity where it plays a crucial role in Kontsevich's proof [8] of the Witten conjecture; see Loijenga [21] for details. In the spirit of our paper, we follow a slightly different approach emphasizing more the analytic aspects of the theory.

2.1 Conical Regge polytopes and ribbon graphs

Let us start by recalling that the geometrical realization of the 1-skeleton of the conical Regge polytope $|P_{T_L}| \to M$ is a 3-valent graph

$$\Gamma = (\{\rho^0(k)\}, \{\rho^1(j)\}) \tag{36}$$

where the vertex set $\{\rho^0(k)\}_{k=1}^{N_2(T)}$ is identified with the barycenters of the triangles $\{\sigma^o(k)\}_{k=1}^{N_2(T)} \in |T_l| \to M$, whereas each edge $\rho^1(j) \in \{\rho^1(j)\}_{j=1}^{N_1(T)}$ is generated by two half-edges $\rho^1(j)^+$ and $\rho^1(j)^-$ joined through the barycenters $\{W(h)\}_{h=1}^{N_1(T)}$ of the edges $\{\sigma^1(h)\}$ belonging to the original triangulation $|T_l| \to M$. Thus, if we formally introduce a degree-2 vertex at each middle point $\{W(h)\}_{h=1}^{N_1(T)}$, the actual graph naturally associated to the 1-skeleton of $|P_{T_L}| \to M$ is

$$\Gamma_{ref} = \left(\{\rho^0(k)\} \bigsqcup_{h=1}^{N_1(T)} \{W(h)\}, \{\rho^1(j)^+\} \bigsqcup_{j=1}^{N_1(T)} \{\rho^1(j)^-\} \right), \tag{37}$$

the so called edge-refinement [20] of $\Gamma = (\{\rho^0(k)\}, \{\rho^1(j)\})$. The relevance of such a notion stems from the observation that the natural automorphism group $Aut(P_L)$ of $|P_{T_L}| \to M$, (*i.e.*, the set of bijective maps $\Gamma = (\{\rho^0(k)\}, \{\rho^1(j)\}) \to \widetilde{\Gamma} = (\{\widetilde{\rho^0(k)}\}, \{\widetilde{\rho^1(j)}\}$ preserving the incidence relations defining the graph structure), is not the automorphism group of Γ but rather the (larger) automorphism group of its edge refinement [20], *i.e.*,

$$Aut(P_L) \doteq Aut(\Gamma_{ref}). \tag{38}$$

The locally uniformizing complex coordinate $t_k \in \mathbb{C}$ in terms of which we can explicitly write down the singular Euclidean metric (24) around each vertex $\sigma^0(k) \in |T_l| \to M$, provides a (counterclockwise) orientation in the 2-cells of $|P_{T_L}| \to M$. Such an orientation gives rise to a cyclic ordering on the set of half-edges $\{\rho^1(j)^\pm\}_{j=1}^{N_1(T)}$ incident on the vertices $\{\rho^0(k)\}_{k=1}^{N_2(T)}$. According to these remarks, the 1-skeleton of $|P_{T_L}| \to M$ is a ribbon (or fat) graph [5], *viz.*, a graph Γ together with a cyclic ordering on the set of half-edges incident to each vertex of Γ. Conversely, any ribbon graph Γ characterizes an oriented surface $M(\Gamma)$ with boundary possessing Γ as a spine, (*i.e.*, the inclusion $\Gamma \hookrightarrow M(\Gamma)$ is a homotopy equivalence). This is an appropriate place to note that not all trivalent metric ribbon graphs are barycentrically dual to regular triangulations. We may have trivalent metric ribbon graphs with digons and loops. From the point of view of the theory of singular Euclidean structures, (which is the real geometrical category underlying the use of simplicial methods in gravity), there is no obvious reason to get rid, a priori, of such more general configurations. As already mentioned, this is the reason for which we must extend our analysis to generalized triangulations, (see remark 1), and to their associated barycentrically dual conical Regge polytopes . In this way (the edge-refinement of) the 1-skeleton of a generalized conical Regge polytope $|P_{T_L}| \to M$ is in a one-to-one correspondence with trivalent metric ribbon graphs.

We can associate with $|P_{T_L}| \to M$ a complex structure $((M; N_0), \mathcal{C})$ (a punctured Riemann surface) which is, in a well-defined sense, dual to the structure $(M, \mathcal{C}_{sg})$ generated by $|T_l| \to M$. Note that according to our characterization of $|P_{T_L}| \to M$, the 2-cells $\{\rho^2(k)\}$ have the conical geometry (21), and the strategy for defining $((M; N_0), \mathcal{C})$ is to desingularize $(M, \mathcal{C}_{sg})$ by transforming the conical singularities into a suitable decoration of $((M; N_0), \mathcal{C})$. In order to see in detail such a construction, let $\rho^2(k)$ be the generic two-cell $\in |P_{T_L}| \to M$ barycentrically dual to the vertex $\sigma^0(k) \in |T_l| \to M$. To the generic edge $\rho^1(h)$ of $\rho^2(k)$ we associate a complex uniformizing coordinate $z(h)$ defined in the strip

$$U_{\rho^1(h)} \doteq \{z(h) \in \mathbb{C} | 0 < \operatorname{Re} z(h) < L(\rho^1(h))\}, \tag{39}$$

$L(\rho^1(h))$ being the length of the edge considered. The uniformizing coordinate $w(j)$, corresponding to the generic 3-valent vertex $\rho^0(j) \in \rho^2(k)$, is defined in the open set

$$U_{\rho^0(j)} \doteq \{w(j) \in \mathbb{C} | \; |w(j)| < \delta, \; w(j)[\rho^0(j)] = 0\}, \tag{40}$$

where $\delta > 0$ is a suitably small constant. Finally, the two-cell $\rho^2(k)$ is uniformized in the unit disk

$$U_{\rho^2(k)} \doteq \{\zeta(k) \in \mathbb{C} | \; |\zeta(k)| < 1, \; \zeta(k)[\sigma^0(k)] = 0\}, \tag{41}$$

where $\sigma^0(k)$ is the vertex $\in |T_l| \to M$ corresponding to the given two-cell.In order to coherently glue together

$$\{w(j), U_{\rho^0(j)}\}_{j=1}^{N_2(T)}, \{z(h), U_{\rho^1(h)}\}_{h=1}^{N_1(T)} \text{ and } \{\zeta(k), U_{\rho^2(k)}\}_{k=1}^{N_0(T)}$$

one exploits the connection between ribbon graphs and quadratic differentials. An appropriate way to proceed is to note that to each edge $\rho^1(h) \in \rho^2(k)$ we can associate the standard quadratic differential on $U_{\rho^1(h)}$ given by

$$\psi(h)|_{\rho^1(h)} = dz(h) \otimes dz(h). \tag{42}$$

Such $\psi(h)|_{\rho^1(h)}$ can be extended to the remaining local uniformizations $U_{\rho^0(j)}$, and $U_{\rho^2(k)}$, by exploiting a classic result in Riemann surface theory according to which a quadratic differential ψ has a finite number of zeros $n_{zeros}(\psi)$ with orders k_i and a finite number of poles $n_{poles}(\psi)$ of order s_i such that

$$\sum_{i=1}^{n_{zero}(\psi)} k_i - \sum_{i=1}^{n_{pole}(\psi)} s_i = 4g - 4. \tag{43}$$

In our case we must have $n_{zeros}(\psi) = N_2(T)$ with $k_i = 1$, (corresponding to the fact that the 1-skeleton of $|P_L| \to M$ is a trivalent graph), and $n_{poles}(\psi) = N_0(T)$ with $s_i = s \forall i$, for a suitable positive integer s. According to such remarks (43) reduces to

$$N_2(T) - sN_0(T) = 4g - 4. \tag{44}$$

However, from the Euler relation $N_0(T) - N_1(T) + N_2(T) = 2 - 2g$, and $2N_1(T) = 3N_2(T)$ we get $N_2(T) - 2N_0(T) = 4g - 4$. This is consistent with (44) if and only if $s = 2$. Thus the extension ψ of $\psi(h)|_{\rho^1(h)}$ along the 1-skeleton of $|P_L| \to M$ must have $N_2(T)$ zeros of order 1 corresponding to the trivalent vertices $\{\rho^0(j)\}$ of $|P_L| \to M$ and $N_0(T)$ quadratic poles corresponding to the polygonal cells $\{\rho^2(k)\}$ of perimeter lengths $\{L(\partial(\rho^2(k)))\}$. Around a zero of order one and a pole of order two, every quadratic differential ψ has a canonical local structure which (along with (42)) is summarized in the following table [20]

$$(|P_{T_L}| \to M) \to \psi \doteq \begin{cases} \psi(h)|_{\rho^1(h)} = dz(h) \otimes dz(h), \\ \psi(j)|_{\rho^0(j)} = \frac{9}{4} w(j) dw(j) \otimes dw(j), \\ \psi(k)|_{\rho^2(k)} = -\frac{[L(\partial(\rho^2(k)))]^2}{4\pi^2 \zeta^2(k)} d\zeta(k) \otimes d\zeta(k), \end{cases} \tag{45}$$

where $\{\rho^0(j), \rho^1(h), \rho^2(k)\}$ runs over the set of vertices, edges, and 2-cells of $|P_L| \to M$. Since $\psi(h)|_{\rho^1(h)}$, $\psi(j)|_{\rho^0(j)}$, and $\psi(k)|_{\rho^2(k)}$ must be identified on the non-empty pairwise intersections $U_{\rho^0(j)} \cap U_{\rho^1(h)}$, $U_{\rho^1(h)} \cap U_{\rho^2(k)}$ we can associate to the polytope $|P_{T_L}| \to M$ a complex structure $(M; N_0, C)$ by coherently glueing, along the pattern associated with the ribbon graph Γ, the local uniformizations $\{U_{\rho^0(j)}\}_{j=1}^{N_2(T)}$, $\{U_{\rho^1(h)}\}_{h=1}^{N_1(T)}$, and $\{U_{\rho^2(k)}\}_{k=1}^{N_0(T)}$. Explicitly, let $\{U_{\rho^1(h_\alpha)}\}$, $\alpha = 1, 2, 3$ be the three generic open strips associated with the three cyclically oriented edges $\{\rho^1(h_\alpha)\}$ incident on the generic vertex $\rho^0(j)$. Then the uniformizing coordinates $\{z(h_\alpha)\}$ are related to $w(j)$ by the transition functions

$$w(j) = e^{2\pi i \frac{\alpha - 1}{3}} z(h_\alpha)^{\frac{2}{3}}, \qquad \alpha = 1, 2, 3. \tag{46}$$

Note that in such uniformization the vertices $\{\rho^0(j)\}$ do not support conical singularities since each strip $U_{\rho^1(h)}$ is mapped by (46) into a wedge of angular opening $\frac{2\pi}{3}$. This is consistent with the definition of $|P_{T_L}| \to M$ according to which the vertices $\{\rho^0(j)\} \in |P_{T_L}| \to M$ are the barycenters of the flat $\{\sigma^2(j)\} \in |T_l| \to M$. Similarly, if $\{U_{\rho^1(h_\beta)}\}$, $\beta = 1, 2, ..., q(k)$ are the open strips associated with the $q(k)$ (oriented) edges $\{\rho^1(h_\beta)\}$ boundary of the generic polygonal cell $\rho^2(k)$, then the transition functions between the corresponding uniformizing coordinate $\zeta(k)$ and the $\{z(h_\beta)\}$ are given by

[20]

$$\zeta(k) = \exp\left(\frac{2\pi i}{L(\partial(\rho^2(k)))}\left(\sum_{\beta=1}^{\nu-1} L(\rho^1(h_\beta)) + z(h_\nu)\right)\right)_{\nu=1}^{q(k)}, \tag{47}$$

with $\sum_{\beta=1}^{\nu-1} \cdot \doteq 0$, for $\nu = 1$.

Summing up, we have the following result which can be considered as a rather elementary consequence of the connection between ribbon graphs and the complex analytic theory of Teichmüller spaces (see [20]),

Proposition 5. *The mapping*

$$\Upsilon : (|P_{T_L}| \to M) \longrightarrow ((M; N_0), \mathcal{C}) \tag{48}$$
$$\Gamma \longmapsto \bigcup_{\{\rho^0(j)\}}^{N_2(T)} U_{\rho^0(j)} \bigcup_{\{\rho^1(h)\}}^{N_1(T)} U_{\rho^1(h)} \bigcup_{\{\rho^2(k)\}}^{N_0(T)} U_{\rho^2(k)},$$

where the glueing maps are given by (46) and (47), defines the (punctured) Riemann surface $((M; N_0), \mathcal{C})$ canonically associated with the conical Regge polytope $|P_{T_L}| \to M$.

Note that by construction such a Riemann surface carries the decoration provided by the meromorphic quadratic differential ψ. It is through such a decoration that the punctured Riemann surface $((M; N_0), \mathcal{C}, \psi)$ keeps track of the metric geometry of the conical Regge polytope $|P_{T_L}| \to M$ out of which $((M; N_0), \mathcal{C})$ has been generated. Explicitly, since the meromorphic quadratic differential (45) has a second order poles, the correspondence Υ defined by (48) associates with the generic two-cell $\rho^2(k) \in |P_{T_L}| \to M$, a punctured disk

$$\Delta_k^* \doteq \{\zeta(k) \in \mathbb{C} |\ 0 < |\zeta(k)| < 1\} \tag{49}$$

endowed with a flat metric

$$|\psi(k)_{\rho^2(k)}| = \frac{[L(\partial(\rho^2(k)))]^2}{4\pi^2 |\zeta(k)|^2} |d\zeta(k)|^2. \tag{50}$$

On $(\Delta_k^*, |\psi(k)_{\rho^2(k)}|)$ we can evaluate the length $L(\Upsilon(l))$ of any closed curve l, homotopic to $\partial(\rho^2(k))$ and contained in $\rho^2(k) - \sigma^0(k)$,

$$L(\Upsilon(l)) \doteq \oint_{l \sim \partial(\rho^2(k))} \sqrt{\psi(k)_{\rho^2(k)}} = L(\partial(\rho^2(k))). \tag{51}$$

Moreover, if we define $\Delta_k^{\varrho} \doteq \{\zeta(k) \in \mathbb{C} | \ \varrho < |\zeta(k)| < 1\}$, then in terms of the area element $|\psi(k)| d\,\mathrm{Re}(\zeta(k)) \wedge \, d\,\mathrm{Im}(\zeta(k))$ associated with the flat metric $|\psi(k)_{\rho^2(k)}|$ we get

$$\int_{\Delta_k^{\varrho}} |\psi(k)_{\rho^2(k)}| d\,\mathrm{Re}(\zeta(k)) \wedge \, d\,\mathrm{Im}(\zeta(k)) = \frac{\left[L(\partial(\rho^2(k)))\right]^2}{2\pi} \ln\left(\frac{1}{\varrho}\right), \tag{52}$$

which, as $\varrho \to 0^+$, diverges logarithmically. Thus, from a metrical point of view the punctured disk $(\Delta_k^*, |\psi(k)_{\rho^2(k)}|)$, endowed with the flat metric $|\psi(k)_{\rho^2(k)}|$, is isometric to a flat semi-infinite cylinder. In this connection, it is interesting to remark that the flat metric (50) formally corresponds to the limiting case of the conical metric (24) when the total angle of the cone $\theta(k) \to 0$, thus interpreting the flat semi-infinite cylinder as a degenerate cone.

The explicit connection between the singular uniformization $(M, \mathcal{C}_{sg})$ associated with the singular Euclidean structure (24), (23) and the decoration $((M; N_0), \mathcal{C}, \psi)$ of $((M; N_0), \mathcal{C})$ associated with the quadratic differential ψ can be easily worked out and is defined (up to a normalization) by the mapping

$$(t_k, D_k(r(k))) \longrightarrow (\zeta(k), U_{\rho^2(k)}) \tag{53}$$

$$t_k \longmapsto \zeta(k) = \exp \frac{2\pi}{L(\partial(\rho^2(k)))} \left[\frac{2\pi}{2\pi - \varepsilon(k)} \left(t_k - t_k(\sigma^0(k))\right)^{\frac{2\pi - \varepsilon(k)}{2\pi}}\right]$$

for each $k = 1, ..., N_0(T)$.

The picture of the correspondence (48) which emerges from such an analysis the following: the map Υ associates with a conical Regge polytope $|P_{T_L}| \to M$ the pair $(\mathcal{C}, \psi)$, where $\mathcal{C}$ is a complex structure on $(M; N_0)$. Note that since $2 - 2g - N_0(T) < 0$, the punctured Riemann surface $(M; N_0, \mathcal{C})$ can be endowed with a hyperbolic structure, *i.e.*, with a metric $ds^2_{-1}[\mathcal{C}]$ in the conformal class of $(M; N_0, \mathcal{C})$ which is complete on $(M; N_0, \mathcal{C})$ and has constant curvature -1. Explicitly, on each punctured disk Δ_k^* associated with two-cell $\rho^2(k) \in |P_{T_L}| \to M$, we can locally write

$$ds^2_{-1}[\mathcal{C}]|_{\Delta_k^*} = \left[\frac{|d\zeta(k)|}{|\zeta(k)| \ln |\zeta(k)|}\right]^2, \tag{54}$$

an expression which can be easily obtained by quotienting he upper half-plane equipped with the standard hyperbolic metric ($\mathcal{H}^2, h(\zeta)^2 \, |d\zeta|^2$) by the isometry $\zeta \to \zeta + 1$ and by identifying the resulting quotient with (the unit punctured disk) Δ_k^* via the map $\zeta \to \exp[2\pi\sqrt{-1}\zeta]$. According to these remarks, the natural complex structure $(M; N_0, \mathcal{C})$ dual to (24) and associated with the conical Regge polytope $|P_L| \to M$ can be obtained by glueing to the boundary components of the ribbon graph Γ a corresponding set of punctured disks $(\Delta_k^*, ds^2_{-1}[\mathcal{C}]|_{\Delta_k^*})$ endowed with the hyperbolic metric (54) and decorated with the quadratic differential $\psi(k)_{\rho^2(k)}$. The punctures are identified with the vertices $\{\sigma^0(j)\}$ of the Regge triangulation $|T_l| \to M$ which, upon barycentrical dualization, gives rise to $|P_{T_L}| \to M$. Thus, pictorially, the mapping Υ establishes the following correspondence

$$\left\{\begin{array}{c}\text{Space of singular}\\ \text{Euclidean Structures}\\ \text{on } M \text{ with } N_0(T)\\ \text{conical points}\end{array}\right\} \Longrightarrow \left\{\begin{array}{c}\text{Decorated}\\ \text{Punctured}\\ \text{Riemann Surfaces}\\ \text{on } (M, N_0)\end{array}\right\} \tag{55}$$

Not unexpectedly, the set of (generalized) conical Regge polytopes (trivalent ribbon graphs) can be used to combinatorially parametrize, by means of $(M; N_0, \mathcal{C})$, the moduli space $\mathfrak{M}_{g,N_0}$ of genus g Riemann surfaces with N_0 punctures, (see the Appendix). We are being a bit vague here, but we will comment on some aspect of this well-known parametrization shortly.

From a topological viewpoint, the set of all possible metrics $|\psi|$ on a (trivalent) ribbon graph Γ with given edge-set $e(\Gamma)$ can be characterized [20], [21] as a space homeomorphic to $\mathbb{R}_+^{|e(\Gamma)|}$, ($|e(\Gamma)|$ denoting the number of edges in $e(\Gamma)$), topologized by the standard ϵ-neighborhoods $U_\epsilon \subset \mathbb{R}_+^{|e(\Gamma)|}$. On such a space there is a natural action of $Aut(\Gamma)$, the automorphism group of Γ defined by the homomorphism $Aut(\Gamma) \to \mathfrak{S}_{e(\Gamma)}$ where $\mathfrak{S}_{e(\Gamma)}$ denotes the symmetric group over $|e(\Gamma)|$ elements. Thus, the resulting space $\mathbb{R}_+^{|e(\Gamma)|}/Aut(\Gamma)$ is a differentiable orbifold. Let

$$Aut_\partial(P_L) \subset Aut(P_L), \tag{56}$$

denote the subgroup of ribbon graph automorphisms of the (trivalent) 1-skeleton Γ of $|P_{T_L}| \to M$ that preserve the (labeling of the) boundary components of Γ. Then, the space of 1-skeletons of conical Regge polytopes $|P_{T_L}| \to M$, with $N_0(T)$ labelled boundary components, on a surface M of genus g can be defined by [20]

$$RGP^{met}_{g,N_0} = \bigsqcup_{\Gamma \in RGB_{g,N_0}} \frac{\mathbb{R}_+^{|e(\Gamma)|}}{Aut_\partial(P_L)}, \tag{57}$$

where the disjoint union is over the subset of all trivalent ribbon graphs (with labelled boundaries) satisfying the topological stability condition $2 - 2g - N_0(T) < 0$, and which are dual to generalized triangulations. It follows, (see [20] theorems 3.3, 3.4, and 3.5), that the set RGP^{met}_{g,N_0} is locally modelled on a stratified space constructed from the components (rational orbicells) $\mathbb{R}_+^{|e(\Gamma)|}/Aut_\partial(P_L)$ by means of a (Whitehead) expansion and collapse procedure for ribbon graphs, which basically amounts to collapsing edges and coalescing vertices, (the Whitehead move in $|P_{T_L}| \to M$ is the dual of the familiar flip move for triangulations). Explicitly, if $L(t) = tL$ is the length of an edge $\rho^1(j)$ of a ribbon graph $\Gamma_{L(t)} \in RGP^{met}_{g,N_0}$, then, as $t \to 0$, we get the metric ribbon graph $\widehat{\Gamma}$ which is obtained from $\Gamma_{L(t)}$ by collapsing the edge $\rho^1(j)$. By exploiting such construction, we can extend the space RGP^{met}_{g,N_0} to a suitable closure $\overline{RGP^{met}_{g,N_0}}$ [21]. Since the dual of any metric ribbon graph $\in RGP^{met}_{g,N_0}$ is a (generalized) triangulation $|T_L| \to M$ of the surface M, we have the following equivalent characterization [21]

Remark 6. *The rational orbicells* $\frac{R_+^{|e(\Gamma)|}}{Aut_\partial(P_L)}$ *of* RGP^{met}_{g,N_0} *can be labelled by the generalized triangulations associated with the generalized conical Regge polytopal surfaces* $|P_{T_L}| \to M$.

The open cells of RGP^{met}_{g,N_0}, being associated with trivalent graphs, have dimension provided by the number $N_1(T)$ of edges of $|P_{T_L}| \to M$. From the Euler relation $N_0(T) - N_1(T) + N_2(T) = 2 - 2g$, and the constraint $2N_1(T) = 3N_2(T)$ associated with the trivalency, we get

$$\dim \left[RGP^{met}_{g,N_0}\right] = N_1(T) = 3N_0(T) + 6g - 6. \tag{58}$$

There is a natural projection

$$\begin{aligned} p : RGP^{met}_{g,N_0} &\longrightarrow \mathbb{R}_+^{N_0(T)} \\ \Gamma &\longmapsto p(\Gamma) = (L_1, ..., L_{N_0(T)}), \end{aligned} \tag{59}$$

where $(L_1, ..., L_{N_0(T)})$ denote the perimeters of the polygonal 2-cells $\{\rho^2(j)\}$ of $|P_{T_L}| \to M$. With respect to the topology on the space of metric ribbon graphs, the orbifold RGP^{met}_{g,N_0} endowed with such a projection acquires the structure of a cellular bundle. For a given sequence $\{L(\partial(\rho^2(k)))\}$, the fiber

$$p^{-1}(\{L(\partial(\rho^2(k)))\}) =$$

$$= \left\{|P_{T_L}| \to M \in RGP^{met}_{g,N_0} : \{L_k\} = \{L(\partial(\rho^2(k)))\}\right\} \tag{60}$$

is the set of all generalized conical Regge polytopes with the given perimeters. In particular, the equilateral conical Regge polytopes dual to the set of distinct generalized dynamically triangulated surfaces with given curvature assignments $\{q(i),\sigma^0(i)\}_{i=1}^{N_0(T)}$, is contained in

$$p^{-1}\left\{\{L(\partial(\rho^2(k)))\}=\frac{\sqrt{3}}{3}a\{q(k)\}_{k=1}^{N_0}\right\}. \tag{61}$$

If we take into account the $N_0(T)$ constraints associated with the perimeters assignments, it follows that the fibers $p^{-1}(\{L(\partial(\rho^2(k)))\})$ have dimension provided by

$$\dim\left[p^{-1}(\{L(\partial(\rho^2(k)))\}\right]=2N_0(T)+6g-6, \tag{62}$$

which, as is well known, exactly corresponds to the real dimension of the moduli space $\mathfrak{M}_{g,N_0}$ of genus g Riemann surfaces with N_0 punctures.

We conclude this section by observing that corresponding to each marked polygonal 2-cells $\{\rho^2(k)\}$ of $|P_{T_L}|\to M$ there is a further (combinatorial) bundle map

$$\mathcal{CL}_k\to RGP_{g,N_0}^{met} \tag{63}$$

whose fiber over $(\Gamma,\rho^2(1),...,\rho^2(N_0))$ is provided by the boundary cycle $\partial\rho^2(k)$, (recall that each boundary $\partial\rho^2(k)$ comes with a positive orientation). To any such $\mathcal{CL}_k$ one associates [8], [21] the piecewise smooth 2-form defined by

$$\omega_k(\Gamma)=\sum_{1\le h_\alpha<h_\beta\le q(k)-1} d\left(\frac{L(\rho^1(h_\alpha))}{L(\partial\rho^2(k))}\right)\wedge d\left(\frac{L(\rho^1(h_\beta))}{L(\partial\rho^2(k))}\right), \tag{64}$$

which is invariant under rescaling and cyclic permutations of the $L(\rho^1(h_\mu))$ and represents the first Chern class of $\mathcal{CL}_k$.

3 DT and moduli spaces: the nature of the approximation in simplicial quantum gravity

As already stressed, the above analysis can be considered as a particular case of the well-known results which allows to define a bijective mapping (a homeomorphism of orbifolds) between the space of ribbon graphs RGB_{g,N_0}^{met} and the moduli space $\mathfrak{M}_{g,N_0}$ of genus g Riemann surfaces $((M;N_0),\mathcal{C})$ with $N_0(T)$ punctures [20], [21],

$$\begin{aligned} h:\mathfrak{M}_{g,N}\times R_+^N &\to RGB_{g,N_0}^{met} \\ [((M;N_0),\mathcal{C}),L_i] &\longmapsto \Gamma, \end{aligned} \tag{65}$$

where $(L_1, ..., L_{N_0})$ is an ordered n-tuple of positive real numbers and Γ is a metric ribbon graphs with $N_0(T)$ labelled boundary lengths $\{L_\imath\}$ defined by the corresponding JS quadratic differential. The bijection h extends to $\overline{\mathfrak{M}}_{g,N_0} \times R_+^{N_0} \to \overline{RGB}^{met}_{g,N_0}$ in such a way that two (stable) surfaces M_1 and M_2 with $N_0(T)$ punctures and given perimeters $\{L_i\}$ are mapped in the same ribbon graph $\Gamma \in \overline{RGB}^{met}_{g,N_0}$ if and only if there exists an homeomorphism between M_1 and M_2 preserving the (labelling of the) punctures, and is holomorphic on each irreducible component containing one of the punctures. If one looks at moduli spaces from this point of view, the bundles $\mathcal{CL}_k$ over $\overline{RGP}^{met}_{g,N_0}$ defined in the previous section are the natural combinatorial counterpart of the line bundles $\mathcal{L}_k \to \overline{\mathfrak{M}}_{g,N_0}$ whose fiber at the moduli point $((M; N_0), \mathcal{C})$ is defined by the cotangent space $T^*_{(M,p_k)}$ to $((M; N_0), \mathcal{C})$ at $p_k \doteq \sigma^0(k)$. In particular, the pull-back under (65) of the 2-form $\omega_k(\Gamma)$ defined by (64) is a (combinatorial) representative of the first Chern class $c_1(\mathcal{L}_k)$ of the line bundle $\mathcal{L}_k \to \overline{\mathfrak{M}}_{g,N_0}$.

In full generality, the construction in [20] (see §§4 and 5) to which we refer for details, gives rise to an explicit map from the whole space of metric ribbon graphs RGB^{met}_{g,N_0} to the decorated moduli space $\mathfrak{M}_{g,N} \times R_+^N$. The explicit expression for such a mapping in the case of the trivalent graph associated with a conical Regge polytope is provided by (48) and can be easily specialized to the DT framework according to

Proposition 7. *Let*

$$\mathcal{DT}[\{q(k)\}_{k=1}^{N_0}] \doteq$$

$$\doteq \left\{|T_{l=a}| \to M \ : q(\sigma^0(k)) = q(k) \geq 2, \ \ k = 1, ..., N_0(T)\right\}. \tag{66}$$

denote the set of distinct generalized dynamically triangulated surfaces of genus g, with a given set of ordered curvature assignments $\{q(i), \sigma^0(i)\}_{i=1}^{N_0(T)}$ over its $N_0(T)$ labelled vertices. Assume that the topological stability condition $g \geq 0$, $2 - 2g - N_0(T) < 0$ holds. Then, we can associate with each polygonal 2-cell $\{\rho^2(i)\}_{i=1}^{N_0(T)}$ of the conical polytope $|P_{T_a}| \to M$ dual to a $|T_{l=a}| \to M \in \mathcal{DT}[\{q(k)\}_{k=1}^{N_0}]$ the quadratic differential

$$\rho^2(k) \longmapsto \psi(k) \doteq -\frac{\left(\frac{\sqrt{3}}{3}aq(k)\right)^2}{4\pi^2\zeta^2(k)} d\zeta(k) \otimes d\zeta(k), \tag{67}$$

where $\zeta(k)$ is a locally uniformizing complex coordinate in the unit disk. The set of such $\{\psi(k), \zeta(k)\}_{k=1}^{N_0(T)}$ naturally restricts to the edges $\{\rho^1(j)\}_{j=1}^{N_1(T)}$ of $|P_{T_a}| \to M$ and characterizes uniquely a complex structure $\mathcal{C}$ and a meromorphic quadratic differential $\psi d\zeta \otimes d\zeta$ which define a decorated punctured

Riemann surface $(((M;N_0),\mathcal{C}),\psi)$ associated with $|T_{l=a}|\rightarrow M$. It follows that there is an injective mapping

$$f_{T_a}:\mathcal{DT}[\{q(k)\}_{k=1}^{N_0}]\longrightarrow\mathfrak{M}_{g,N_0}\times\left(\frac{\sqrt{3}}{3}a\right)N_+^{N_0} \tag{68}$$
$$\left(|T_{l=a}|\rightarrow M;\{q(k)\}_{k=1}^{N_0}\right)\longmapsto[((M;N_0),\mathcal{C}),\psi],$$

defining the generalized dynamical triangulations $|T_{l=a}|\rightarrow M\in\mathcal{DT}[\{q(k)\}]$ as distinguished labellings (i.e., providing a reference complex structure $\mathcal{C}_{T_a}$) in the top-dimensional orbicells

$$h^{-1}\left(\frac{R_+^{|e(\Gamma)|}}{Aut_\partial(P_L)}\right)\subset\mathfrak{M}_{g,N_0}\times R_+^N.$$

Proof. In the light of the above remarks, very little remains at issue here. The first part of the proposition is a trivial consequence of the explicit construction of the mapping (45) $h^{-1}:RGB_{g,N_0}^{met}\rightarrow\mathfrak{M}_{g,N}\times R_+^N$ for the trivalent ribbon graphs associated with $|P_{T_L}|\rightarrow M$. In order to prove the second part let us denote by

$$\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})\doteq\frac{R_+^{|e(\Gamma)|}}{Aut_\partial(P_{T_a})} \tag{69}$$

the rational cell associated with the dynamical triangulation $|T_{l=a}|\rightarrow M\in\mathcal{DT}[\{q(k)\}_{k=1}^{N_0}]$. Note that the automorphism group $Aut_\partial(P_{T_a})$ is isomorphic to the isotropy subgroup $\subset\mathfrak{Map}(M;N_0)$ of the generic Riemann surface in $h^{-1}(\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0}))$. The orbicell (69) contains the ribbon graph associated with the conical polytope $|P_{T_a}|\rightarrow M$ dual to $|T_{l=a}|\rightarrow M$, and all (trivalent) metric ribbon graphs $|P_{T_L}|\rightarrow M$ with the same combinatorial structure of $|P_{T_a}|\rightarrow M$ but with all possible length assignments $\{L(\rho^1(h))\}_1^{N_1(T)}$ associated with the corresponding set of edges $\{\rho^1(h)\}_1^{N_1(T)}$. Up to the action of $Aut_\partial(P_{T_a})$, (69) is naturally identified with the convex polytope (of dimension $(2N_0(T)+6g-6)$) in $\mathbb{R}_+^{N_1(T)}$ defined by

$$\Delta_{T_a}(\{q(k)\}_{k=1}^{N_0})=$$

$$=\left\{\{L(\rho^1(j))\}\in\mathbb{R}_+^{N_1(T)}:\sum_{j=1}^{q(k)}A_{(k)}^j(T_a)L(\rho^1(j))=\frac{\sqrt{3}}{3}aq(k)\Bigg|_{k=1}^{N_0(T)}\right\},$$

where $A^j_{(k)}(T_a)$ is a $(0,1)$ indicator matrix, depending on the given dynamical triangulation $|T_{l=a}| \to M$, with $A^j_{(k)}(T_a) = 1$ if the edge $\rho^1(j)$ belongs to $\partial(\rho^2(k))$, and 0 otherwise. Note that $|P_{T_a}| \to M$ appears as the barycenter of such a polytope.

Since the cell decomposition (57) of the space of trivalent metric ribbon graphs RGP^{met}_{g,N_0} depends only on the combinatorial type of the ribbon graph, we can use the equilateral polytopes $|P_{T_a}| \to M$, dual to dynamical triangulations in $\mathcal{DT}[\{q(k)\}_{k=1}^{N_0}]$, as the set over which the disjoint union in (57) runs. Thus we can write

$$RGP^{met}_{g,N_0} = \bigsqcup_{\mathcal{DT}[\{q(k)\}_{k=1}^{N_0}]} \Omega_{T_a}(\{q(k)\}_{k=1}^{N_0}). \tag{70}$$

Note that the decoration associated with the quadratic differential ψ is provided by

$$\oint_{\partial(\rho^2(k))} \sqrt{\psi(k)} = \sum_{h=1}^{q(k)} L(\rho^1(h)) \underset{DT}{=} \left(\frac{\sqrt{3}}{3}a\right) q(k). \tag{71}$$

Remark 8. *In the case of dynamical triangulations the above integral can be actually exploited for providing the conical angles $\theta(k) = \frac{\pi}{3}q(k)$ associated with the dynamical triangulation $|T_{l=a}| \to M$. In such a sense, the decoration defined by the quadratic differentials $\{\psi(k)\}$ allows for a reconstruction of the original triangulation $|T_{l=a}| \to M$ characterizing the orbicell $\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})$. Thus, the quadratic differential $\psi(k)$ can be interpreted as a field on the Riemann surface $((M;N_0),\mathcal{C})$ associated with $|T_{l=a}| \to M$ and carrying curvature. It couples with the punctures by dressing them with a gravitational charge provided by (71).*

Roughly speaking the above remarks imply that we can view dynamical triangulations as punctured Riemann surfaces dressed by a discretized curvature operator. The distinguished role that dynamical triangulations play in the correspondence (65) may appear a little bit mysterious and its worthwhile discussing its origin. The starting point is the observation that there is a particular simplicial refinement of a given Regge triangulation $|T_l| \to M$ which, at the expenses of adding new regular (*i.e.*, with $\theta(k) = 2\pi$) vertices generates from $|T_l| \to M$ an almost-equilateral triangulations $|T_{l\approx a}| \to M$

characterized by the same (essential) divisor $Div(T)$ of the original triangulation $|T_l| \to M$. The refinement in question is the so called *hex refinement* familiar in the theory of circle packing, [22], [23], [24]. The hex refinement $HX\{|T_l| \to M\}$ of $|T_l| \to M$ is the triangulation formed by adding a vertex to each edge of $|T_l| \to M$ and adding an edge between new vertices lying on the same triangular face. In this way each face of $|T_l| \to M$ will be subdivided into four new faces, and any new interior vertex added to $|T_l| \to M$ is a flat regular vertex (of degree $q(k) = 6$ and $\theta(k) = 2\pi$). The original conical angles $\{\theta(k)\}$ of $|T_l| \to M$ can be, at each step $(HX\{|T_l| \to M\})^n$, changed by retaining the original incidence numbers $\{q(k)\}$, and eventually $(n \to \infty)$ fine-tuned to $\theta(k) \rightsquigarrow q(k)\frac{\pi}{3}$. As a matter of fact, by iterating such a procedure it is possible to prove [25] that the resulting triangulation $(HX\{|T_l| \to M\})^n$ converges to the triangulated surface formed by glueing together equilateral triangles in the pattern given by the original $|T_l| \to M$. The Riemann surface associated with $(HX\{|T_l| \to M\})^n$ correspondingly converges, in the Teichmüller metric (see the appendix, (113)), to the surface associated with the equilateral triangulation.

This property of dynamically triangulated surfaces is strictly related to a result, due to V. A. Voevodskii and G. B. Shabat [26], which establishes a remarkable bijection between dynamical triangulations and curves over algebraic number fields. The proof in [26] exploits the characterization of the collection of algebraic curves defined over the algebraic closure $\overline{\mathbb{Q}}$ of the field of rational numbers, (*i.e.*, over the set of complex numbers which are roots of non-zero polynomials with rational coefficients), provided by Belyĭ's theorem (see e.g., [20]). According to such a result, a nonsingular Riemann surface M has the structure of an algebraic curve defined over $\overline{\mathbb{Q}}$ if and only if there is a holomorphic map (a branched covering of M over the sphere)

$$f : M \to \mathbb{CP}^1 \tag{72}$$

that is ramified only at $0, 1$ and ∞, (such maps are known as Belyĭ maps). In other words, the Riemann surface associated with a dynamical triangulation (with edge-lengths normalized to $a = 1$) is, in a canonical way, a ramified covering of the Riemann sphere $\mathbb{CP}^1$ with ramification locus contained in $\{0, 1, \infty\}$. The triangulation (actually its barycentrically dual ribbon graph decorated with the quadratic differential ψ) is the preimage of the interval $[0, 1]$, in particular the set of vertices appears as the preimage of 0 and the set of half-cylinders over the cells $\{\rho^2(k)\}$ as the preimage of ∞. The mid points of the edges $\{\rho^1(h)\}$, (identified with the barycenters of the edges of $|T_{a=1}| \to M$), correspond to the preimage of 1. Moreover, every branched covering of $\mathbb{CP}^1$ defining a Riemann surface over $\overline{\mathbb{Q}}$ arises in this fashion. It is worthwhile remarking that the inverse image of the line segment $[0, 1] \subset \mathbb{CP}^1$

under a Belyĭ map is a Grothendieck's *dessin d'enfant*, thus dynamically triangulated surfaces are eventually connected with the theory of the Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ action on the branched coverings $f : M \to \mathbb{CP}^1$. The correspondence between Belyĭ maps, *dessin d'enfant* and JS quadratic differentials has been recently analyzed in depth by M. Mulase and M. Penkava [20], an equally inspiring paper is [27] by M. Bauer and C. Itzykson.

Remark 9. *Since Belyĭ surfaces are dense in the moduli space of Riemann surfaces, such density carries over to the set of dynamically triangulated surfaces.*

In this connection, we are actually interested in understanding to what extent the map f_{T_a} (68) and more generally $RGP^{met}_{g,N_0} \to \mathfrak{M}_{g,N_0} \times R^{N_0}_{+}$ is an approximation to the conformal parametrization of the space of Riemannian structures on a surface M, used in the standard field-theoretic approach to $2D$ quantum gravity. To begin with, let us remark that the above analysis provides a formal dictionary between the smooth and the combinatorial description of a Riemannian surface (M, ds^2) according to

$$\begin{array}{ccc}
\text{Riemannian Surfaces} & & \text{Triangulations} \\
\frac{Riem(M)}{Diff(M)} & \rightsquigarrow\rightsquigarrow & RGP^{met}_{g,N_0} \\
\mathfrak{M}_g & \hookrightarrow\longrightarrow & \mathfrak{M}_{g,N_0} \\
 & \nearrow & \\
W(M) & \longrightarrow\longrightarrow & \mathbb{R}^{N_0}_{+}
\end{array} \tag{73}$$

Where $\frac{Riem(M)}{Diff(M)} \rightsquigarrow RGP^{met}_{g,N_0}$ stands for the approximation of a two dimensional Riemannian structure defined by a dynamical triangulation and

$$\begin{array}{c}
\mathfrak{M}_g \hookrightarrow \mathfrak{M}_{g,N_0} \\
W(M) \xrightarrow{\nearrow} \mathbb{R}^{N_0}_{+}
\end{array} \tag{74}$$

represents a correspondence between $\mathfrak{M}_{g,N_0}$, the moduli space of Riemann surfaces of genus g, $\mathfrak{M}_g$, and the discretization of the conformal degrees of freedom $W(M)$. The nature of such a correspondence is rather non-trivial and its analysis will occupy us in the remaining part of this paper. First of all, it cannot be explained if we simply regard dynamical triangulations as providing a sort of approximating net in the space of Riemannian structures $\frac{Riem(M)}{Diff(M)}$. Actually, the rationale we are seeking lies at a deeper level and is a direct consequence of the interpretation of dynamical triangulation as punctured Riemann surfaces dressed by a discretized curvature, (see remark 8). If we approximate a Riemannian structure with a (dynamical) triangulation, $(M, ds^2) \rightsquigarrow (|T_l| \to M)$, then the metric information ds^2_T of the approximating $|T_l| \to M$ is encoded into the lengths of the $N_1(T)$ edges of the triangulation. This translates into $6g - 6 + 3N_0(T)$ parameters, as we have seen in section 2, where we wrote $ds^2_T = e^{2v}|dt|^2$ with the conformal factor locally provided by $v \doteq u - \sum_{k=1}^{N_0(T)} \left(\frac{\varepsilon(k)}{2\pi}\right) \ln|t - t_k|$. Thus, the conformal mode is approximated by $3N_0(T)$ parameters. However, out of these $3N_0(T)$ quantities, the $N_0(T)$ deficit angles $\{\varepsilon(k)\}$ (*i.e.* the curvatures) are basically free variables being subjected only to the linear topological constraint $\sum_{k=1}^{N_0(T)} \left(-\frac{\varepsilon(k)}{2\pi}\right) = 2g - 2$, (this follows directly from the Picard-Troyanov theorem). More delicate is the situation of the remaining $2N_0(T)$ degrees of freedom associated with the (complex) coordinates t_k of the conical singularities (vertices), and described by $\ln|t - t_k|$. These latter conformal degrees of freedom basically describe the conical points where the curvature of $|T_l| \to M$ couples with the underlying complex structure of (M, ds^2_T) and cannot be easily handled (see [7]). However, if we go to the barycentrically dual complex associated with $(|T_l| \to M)$, then the $2N_0(T)$ conformal degrees of freedom associated with the (complex) coordinates t_k get naturally *traded* into the modular degrees of freedom of $\mathfrak{M}_{g,N_0}$ representing punctures of the underlying Riemann surface $((M; N_0), \mathcal{C})$. These punctures (which vary holomorphically in $((M; N_0), \mathcal{C})$) are the points where we have the coupling between $((M; N_0), \mathcal{C})$ and the $N_0(T)$ conformal degrees of freedom associated with the deficit angles $\{\varepsilon(k)\}$ (the curvature charges). As we shall see in the next section, such a factorization of the conformal degrees of freedom into the positional degrees of freedom of the interaction vertices and the degrees of freedom associated with the corresponding curvature charges has important consequences in understanding the dynamics of simplicial quantum gravity as compared to its continuous field-theoretic counterpart.

4 Trading conformal modes for moduli: The case of pure gravity

In order to formalize a procedure for describing at the level of measures the trading a (discretized) conformal degree of freedom into a moduli parameter (a puncture), let us define the forgetful mapping

$$\phi : \mathfrak{M}_{g,N_0} \times R_+^{N_0} \to \mathfrak{M}_{g,N_0} \tag{75}$$

as the application which forgets the decorations on $\mathfrak{M}_{g,N_0} \times R_+^{N_0}$ provided by the perimeters $\{L(k)\} = \left(\frac{\sqrt{3}}{3}a\right)\{q(k)\}_{i=1}^{N_0(T)}$, (*i.e.*, ϕ associates to $(\widetilde{\psi})$ the underlying Riemann surface $((M;N_0),\mathcal{C})$ defined by (48)). Then the map

$$\begin{aligned} f_{T_a}:\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0}) &\xrightarrow{h^{-1}} \mathfrak{M}_{g,N_0}\times R_+^{N_0} \xrightarrow{\phi} \mathfrak{M}_{g,N_0} \\ (|P_{T_L}|\to M) &\longmapsto ((M;N_0),\mathcal{C},\psi)\longmapsto ((M;N_0),\mathcal{C}) \end{aligned} \tag{76}$$

defines an open cell in the moduli space $\mathfrak{M}_{g,N_0}$, labelled by the given (generalized) dynamical triangulation $|T_{l=a}|\to M\in\mathcal{DT}[\{q(k)\}_{k=1}^{N_0}]$, (recall that $\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})$ is the rational cell associated with the dynamical triangulation $|T_{l=a}|\to M$; see (69)). As $|T_{l=a}|\to M$ varies in the discrete set $\mathcal{DT}[\{q(k)\}_{k=1}^{N_0}]$, we get in this way a cell decomposition of $\mathfrak{M}_{g,N_0}$ which is parametrized by $\mathcal{DT}[\{q(k)\}_{k=1}^{N_0}]$, *i.e.*,

$$\mathfrak{M}_{g,N_0}\left(\{q(k)\}_{k=1}^{N_0}\right)=\bigsqcup_{T_a\in\mathcal{DT}[\{q(k)\}_{k=1}^{N_0}]} f_{T_a}\left(\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})\right). \tag{77}$$

Note that, as the notation suggests, such a cell decomposition depends on the set of curvature assignments considered $\{q(k)\}_{k=1}^{N_0}$. Distinct curvature assignments $\{\widehat{q(k)}\}_{k=1}^{N_0}$ gives rise to possibly distinct cell decompositions of $\mathfrak{M}_{g,N_0}$ labelled by the corresponding $\mathcal{DT}[\{\widehat{q(k)}\}_{k=1}^{N_0}]$.

It is worthwhile to note that the cell decomposition (77) is the combinatorial counterpart of the slice theorem (1). There, we identified a local slice for the action of $W(M)\ltimes Diff(M)$ with a deformation of a Riemann surface (*i.e.*, a point of $\mathfrak{M}_g$) generated by a (holomorphic) quadratic differential, together with a family of corresponding conformal factors on the fibers. Here, such a local slice is combinatorially described by the set of triangulations $\mathcal{DT}[\{q(k)\}_{k=1}^{N_0}]$ which through the corresponding meromorphic quadratic differentials $((M;N_0),\mathcal{C},\psi)$ also generate deformations of Riemann surfaces, (now represented by points of $\mathfrak{M}_{g,N_0}$), and a family of (discretized) conformal degrees of freedom on the fibers provided by the curvature charges $\oint_{\partial(\rho^2(k))}\sqrt{\psi(k)}$ associated with ψ.

If the cell decomposition (77) is the combinatorial counterpart of the slice theorem (1) then, the combinatorial aspects of the anomalous scaling of the Weyl measure $D_h[v]$, (see also (4)), should be related to integration over the moduli space $\overline{\mathfrak{M}}_{g,N_0}$. In order to prove this latter statement, let us denote by ω_{WP} the Weil-Petersson 2-form on $\overline{\mathfrak{M}}_{g,N_0}$ (see *e.g.* (124) in the appendix for a definition). According to a recent result [11] by P. Zograf, one can explicitly represent ω_{WP} in terms of the Witten-Kontsevich intersection numbers. Since these latter can be expressed in terms of the form (64) defined on $\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})$, it follows that under the map f_{T_a} defined by (76), we can pull back ω_{WP} to a form $f^*_{T_a}(\omega_{WP})$ defined on the cellular orbicells $\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})$. Explicitly, let us consider, for each partition $d_1 + d_2 + ... + d_{N_0} = 3g - 3 + N_0(T)$, of $\dim_C \overline{\mathfrak{M}}_{g,N_0}$ the form

$$\tau_0^{l_0}\tau_1^{l_1}...\tau_{3g-3+N_0}^{l_{3g-3+N_0}} \doteq \omega_1^{d_1} \wedge \omega_2^{d_2} \wedge ... \wedge \omega_{N_0}^{d_{N_0}}, \tag{78}$$

where each ω_k is the (combinatorial) Chern form (64) associated with the generic Regge polytope $(|P_{T_L}| \to M)$ in $\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})$, and where the notation $\tau_0^{l_0}\tau_1^{l_1}...\tau_{3g-3+N_0}^{l_{3g-3+N_0}}$ keeps track of the integers $\{l_k \geq 0\}$ enumerating how many of the summands d_i are equal to k. The integral of $\tau_0^{l_0}\tau_1^{l_1}...\tau_{3g-3+N_0}^{l_{3g-3+N_0}}$ over the cellular decomposition of $\overline{\mathfrak{M}}_{g,N_0}$ provided by (77) defines the (combinatorial) Witten-Kontsevich intersection numbers $\left\langle \tau_0^{l_0}\tau_1^{l_1}...\tau_{3g-3+N_0}^{l_{3g-3+N_0}} \right\rangle$ of $\overline{\mathfrak{M}}_{g,N_0}$, [8], [9],[21]. In terms of the family of forms (78) the Weil-Petersson volume form associated with the cellular decomposition (77) can be written as [11]

$$\frac{f_{T_a}^*(\omega_{WP})^{3g-3+N_0(T)}}{(3g-3+N_0(T))!} = \tag{79}$$

$$= \sum_{|l|=3g-3+N_0(T)} \frac{(-1)^{g-1+N_0(T)+\|l\|}}{\prod_{i=2}^{3g-2+N_0(T)} l_i!\,((i-1)!)^{l_i}} \tau_0^{N_0(T)}\tau_2^{l_2}...\tau_{3g-2+N_0}^{l_{3g-2+N_0}},$$

where $l \doteq (l_2, l_3, ..., l_{3g-2+N_0(T)})$ is a multi-index and where

$$\begin{aligned} |l| &\doteq \textstyle\sum_{k=2}^{3g-2+N_0(T)} (k-1) l_k \\ \|l\| &\doteq \textstyle\sum_{k=2}^{3g-2+N_0(T)} l_k \end{aligned} \quad . \tag{80}$$

With these preliminary remarks along the way, let us consider the Weil-Petersson volume $VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right)$ of the (compactified) moduli space $\overline{\mathfrak{M}}_{g,N_0}$,

$$VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right) = \frac{1}{N_0!}\int_{\overline{\mathfrak{M}}_{g,N_0}} \frac{{\omega_{WP}}^{3g-3+N_0(T)}}{(3g-3+N_0(T))!} = \tag{81}$$

$$= \frac{1}{N_0!}\sum_{T\in\mathcal{DT}[\{q(i)\}_{i=1}^{N_0}]} \int_{f_{T_a}\left(\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})\right)} \frac{{\omega_{WP}}^{3g-3+N_0(T)}}{(3g-3+N_0(T))!}$$

where we have exploited the cell decomposition (77) of $\overline{\mathfrak{M}}_{g,N_0}$, and where we have divided by $N_0(T)!$ in order to factor out the labelling of the $N_0(T)$ punctures.

By pulling ω_{WP} back to the orbicells $\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})$ by means of the map f_{T_a} and by taking into account that each $\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})$ is a (smooth) orbifold acted upon by the automorphism group $Aut_{\partial}(P_{T_a})$, we can explicitly write (81) as an orbifold integration,(the orbifold integration over moduli space and consistent with the orientation of $\overline{\mathfrak{M}}_{g,N_0}$ is defined in [28], Th. 3.2.1), according to

$$VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right) = \frac{1}{N_0!}\int_{\overline{\mathfrak{M}}_{g,N_0}} \frac{{\omega_{WP}}^{3g-3+N_0(T)}}{(3g-3+N_0(T))!} = \tag{82}$$

$$= \frac{1}{N_0!}\sum_{T\in\mathcal{DT}[\{q(i)\}_{i=1}^{N_0}]} \frac{1}{|Aut_{\partial}(P_{T_a})|}\int_{\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})} \frac{f^*_{T_a}(\omega_{WP})^{3g-3+N_0(T)}}{(3g-3+N_0(T))!},$$

where the integration measure is defined by (79), and where the summation is over all distinct dynamical triangulations with given unlabeled curvature assignments weighted by the order $|Aut_{\partial}(P_T)|$ of the automorphisms group of the corresponding dual polytope. We need the following technical

Lemma 10. *For any orbicell $\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})$ we have*

$$\lim_{N_0\to\infty}\int_{\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})} \frac{f^*_{T_a}(\omega_{WP})^{3g-3+N_0(T)}}{(3g-3+N_0(T))!} \longrightarrow \delta_{\widetilde{T}_a}, \tag{83}$$

where the limit is in the sense of distributions, and where $\delta_{\widetilde{T}_a}$ denotes the Dirac measure concentrated on a dynamical triangulation $|T_{l=a}| \to M \in \mathcal{DT}[\{q(i)\}_{i=1}^{\infty}]$ with the essential incidence of the given $T_a \in \mathcal{DT}[\{q(k)\}_{k=1}^{N_0}]$, i.e., *with*

$$\{q(i)\}_{i=1}^{\infty} = \left\{\{q(k)\}_{k=1}^{N_0}, \{q(h)\}_{N_0+1}^{\infty} = 6\right\}. \tag{84}$$

In order to prove this statement, let us recall that according to a result of C.McMullen [29] the Weil-Petersson form ω_{WP} can be written as $d\Theta$ for some bounded 1-form Θ. For instance, one can use the representation

$$\Theta = -i\beta_M,\ \ d\Theta = \omega_{WP}, \tag{85}$$

where β_M is the Bers embedding

$$\beta_M : \mathfrak{T}_{g,N_0}(M) \to Q_{N_0}(M) \simeq T^*_M\mathfrak{T}_{g,N_0}(M), \tag{86}$$

and where $Q_{N_0}(M)$ denotes the space of (holomorphic) quadratic differentials (see the appendix for a characterization of β_M). This implies that for any compact complex submanifold $V^{2m} \subset \mathfrak{T}_{g,N_0}(M)$, one has

$$Vol_{WP}(V^{2m}) = \int_{V^{2m}} \omega_{WP}^m = \int_{\partial V} \Theta \wedge \omega_{WP}^{m-1}, \tag{87}$$

where the $2m-1$ form $\Theta \wedge \omega_{WP}^{m-1}$ is bounded. The cellular decomposition f_{T_a} defined by (76) may be thought of as representing the combinatorial counterpart of the Bers embedding, and, if V^{2m} is the image under the map (76) of the closure $\overline{\Omega}_{T_a}$ of $\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})$, we can write

$$\int_{\overline{\Omega}_{T_a}} \frac{f_{T_a}^*(\omega_{WP})^{3g-3+N_0(T)}}{(3g-3+N_0(T))!} = \int_{\partial[\overline{\Omega}_{T_a}]} \frac{f_{T_a}^*\Theta \wedge f_{T_a}^*(\omega_{WP})^{3g-4+N_0(T)}}{(3g-3+N_0(T))!} = \tag{88}$$
$$= \delta_{\partial\overline{\Omega}_{T_a}},$$

where we have considered

$$f_{T_a}^*(\omega_{WP})^{3g-4+N_0(T)} \tag{89}$$

as a Leray form based on the hypersurface $\partial\overline{\Omega}_{T_a}$ and have interpreted the surface integral appearing in (88) as the definition (with respect to the Weil-Petersson volume) of the Dirac measure $\delta_{\partial\overline{\Omega}_{T_a}}$ based on the hypersurface $\partial\overline{\Omega}_{T_a}$. With such a characterization of the surface Dirac measure associate with $f_{T_a}^*(\omega_{WP})$, the lemma follows from (88) if we can show that as $N_0 \to \infty$, $\partial\overline{\Omega}_{T_a}$ converges suitably to a dynamical triangulation $|\widetilde{T}_{l=a}| \to M \in \mathcal{DT}[\{q(i)\}_{i=1}^{\infty}]$. To this end, let us fix some N large enough and consider for $N_0 << N$ the convex subset $\Delta_{T_a}(\{q(k)\}_{k=1}^{N_0})$ of $\mathbb{R}^{2N+6g-6}$ associated with the orbicell $\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})$. Let $|T_l| \to M$ a triangulation in $\Delta_{T_a}(\{q(k)\}_{k=1}^{N_0})$ distinct from the given dynamical triangulation $|T_{l=a}| \to M$. By iterating an hex refinement on $|T_l| \to M$ we can always assume that for $N \to \infty$, the triangulation $|T_l| \to M$ converges (in the standard topology of $\mathbb{R}^{2N+6g-6}$) to a dynamical triangulation $|\widetilde{T}_{l=a}| \to M$ with curvature assignments given by (84). Under the same sequence of hex refinement also $|T_{l=a}| \to M$ converges to $|\widetilde{T}_{l=a}| \to M$. Since $|T_l| \to M$ is a generic triangulation of $\Delta_{T_a}(\{q(k)\}_{k=1}^{N_0})$, it follows that $\Delta_{T_a}(\{q(k)\}_{k=1}^{N_0})$ shrinks, as $N \to \infty$, to $|\widetilde{T}_{l=a}| \to M$. All triangulations of $\Delta_{T_a}(\{q(k)\}_{k=1}^{N_0})$ have the same automorphism group $Aut_\partial(P_{T_a})$ of $|T_{l=a}| \to M$, and since the hex refinement does not alter $Aut_\partial(P_{T_a})$, we get that in $\mathbb{R}^{2N+6g-6}$

$$\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0}) \longrightarrow \left(|\widetilde{T}_{l=a}| \to M\right), \tag{90}$$

as $N \to \infty$. Going to the closure $\partial\overline{\Omega}_{T_a} \longrightarrow (|\widetilde{T}_{l=a}| \to M\ )$ and $\delta_{\partial\overline{\Omega}_{T_a}}$ converges (weakly) to $\delta_{\widetilde{T}_a}$ as stated.

By introducing a smooth, compactly supported approximation to the characteristic function of $\overline{\Omega}_{T_a}$ we get from the above lemma an immediate but important consequence

Proposition 11. *In the large N_0 limit we have the asymptotic relation*

$$VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right) \approx \frac{1}{N_0!} \sum_{T \in \mathcal{DT}[\{q(i)\}_{i=1}^{N_0}]} \frac{1}{|Aut_{\partial}(P_{T_a})|}. \tag{91}$$

Note that

$$Card\left[\mathcal{DT}[\{q(i)\}_{i=1}^{N_0}]\right] \doteq \frac{1}{N_0!} \sum_{T \in \mathcal{DT}[\{q(i)\}_{i=1}^{N_0}]} \frac{1}{|Aut_{\partial}(P_{T_a})|} \tag{92}$$

provides the number of distinct dynamical triangulations with given (unlabeled) curvature assignments $\{q(i)\}_{i=1}^{N_0}$ over the $N_0(T)$ vertices $[\{\sigma^0(i)\}_{i=1}^{N_0}]$. Such a counting function is related to the enumeration of all distinct (generalized) triangulations with a given number N_0 of unlabeled vertices by

$$Card\left[\mathcal{DT}\left(N_0\right)\right] = \sum_{\{q(i)\}_{i=1}^{N_0}} Card\left[\mathcal{DT}[\{q(i)\}_{i=1}^{N_0}]\right] \tag{93}$$

where the summation $\sum_{\{q(i)\}_{i=1}^{N_0}}$ is over all possible curvature assignments on the N_0 (unlabeled) vertices. In simplicial quantum gravity $Card\left[\mathcal{DT}\left(N_0\right)\right]$ is a basic quantity providing the canonical entropy function associated with the set of triangulations considered [5]. Since $VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right)$ does not depend on the particular cellular decomposition defined by the given choice of the curvature assignments $\{q(i)\}_{i=1}^{N_0}$, according to (91) and (93) we get the asymptotic relation

$$Card\left[\mathcal{DT}\left(N_0\right)\right] \approx VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right) Card\left[\{q(i)\}_{i=1}^{N_0}\right], \tag{94}$$

where $Card\left[\{q(i)\}_{i=1}^{N_0}\right]$ denotes the number of possible curvature assignments over the N_0 (unlabeled) vertices $[\{\sigma^0(i)\}_{i=1}^{N_0}]$. (For caveats concerning the meaning of this formula as compared to some of our previous work, see Appendix II).

At this stage, it is important to remark that we have two independent asymptotic evaluations of $Card\left[\mathcal{DT}\left(N_0\right)\right]$ and of $VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right)$. Thus, the relations (91) and (94) provide a non trivial geometrical interpretation of the canonical entropy function, $Card\left[\mathcal{DT}\left(N_0\right)\right]$ which can be profitably confronted with the field-theoretic measure. The large N_0 asymptotics of $Vol_{W-P}(\overline{\mathfrak{M}}_{g,N_0})$ has been discussed by Manin and Zograf [10], [11]. They obtained

$$Vol_{W-P}(\overline{\mathfrak{M}}_{g,N_0}) = \pi^{2(3g-3+N_0)} \times \tag{95}$$
$$\times (N_0+1)^{\frac{5g-7}{2}} C^{-N_0} \left(B_g + \sum_{k=1}^{\infty} \frac{B_{g,k}}{(N_0+1)^k} \right),$$

where $C = -\frac{1}{2} j_0 \frac{d}{dz} J_0(z)|_{z=j_0}$, ($J_0(z)$ the Bessel function, j_0 its first positive zero); (note that $C \simeq 0.625....$). The genus dependent parameters B_g are explicitly given [10] by

$$\begin{cases} B_0 = \frac{1}{A^{1/2}\Gamma(-\frac{1}{2})C^{1/2}}, & B_1 = \frac{1}{48}, \\ B_g = \frac{A^{\frac{g-1}{2}}}{2^{2g-2}(3g-3)!\Gamma(\frac{5g-5}{2})C^{\frac{5g-5}{2}}} \left\langle \tau_2^{3g-3} \right\rangle, & g \geq 2 \end{cases} \tag{96}$$

where $A \doteq -j_0^{-1} J_0'(j_0)$, and $\left\langle \tau_2^{3g-3} \right\rangle$ is a Kontsevich-Witten intersection number, (the coefficients $B_{g,k}$ can be computed similarly-see [10] for details). Thus, from (91) we get that in the large N_0 limit

$$Card\left[\mathcal{DT}[\{q(i)\}_{i=1}^{N_0}]\right] \approx B_g \pi^{2(3g-3+N_0)} N_0^{\frac{5g-7}{2}} C^{-N_0} \left(1 + O(\frac{1}{N_0})\right), \tag{97}$$

which does not depend on the actual distribution of the curvature assignments $\{q(i)\}_{i=1}^{N_0}$, (this is not surprising since in the large N_0 limit, the average number of triangles incident on a generic vertex is $q(i) = 6$, see (11)). It is important to stress that $Card\left[\mathcal{DT}[\{q(i)\}_{i=1}^{N_0}]\right]$ is the combinatorial counterpart of the local slice in $Riem(M)/Diff(M)$ defined by the metrics with constant curvature. In this connection, note that (97) shows that it is precisely this part of the full triangulation counting (94) that contributes the genus-g pure gravity critical exponent

$$\gamma_g = \frac{5g-1}{2}. \tag{98}$$

The relations (95) and (97) also show that γ_g has a modular origin, arising from the cardinality of the cell decomposition of $\overline{\mathfrak{M}}_{g,N_0}$. To go deeper into such an issue, let us recall that the Weil-Petersson volume of the moduli space $\overline{\mathfrak{M}}_{g,N_0}$ for any fixed value of N_0 is such that

$$C_1^g(2g)! \leq Vol_{W-P}(\overline{\mathfrak{M}}_{g,N_0}) \leq C_2^g(2g)!, \tag{99}$$

where the constants $0 < C_1 < C_2$ are independent of N_0 (see [30], [31]). Thus the modular volume $Vol_{W-P}(\overline{\mathfrak{M}}_g)$, (assuming $g \geq 2$, or inserting at least 3 stabilizing punctures), does not contribute to the anomalous scaling term $N_0(T)^{\frac{5g-7}{2}}$ in the asymptotics (97) for $Card\left[\mathcal{DT}[\{q(i)\}_{i=1}^{N_0}]\right]$. Such a scaling is generated only by the modular parameters associated with the location of vertices, (this does not surprise, since γ_g enters directly topological triangulations counting regardless of any metric structure the triangulation can carry). Recall that there is an explicit connection (53) between the variables $\{\zeta(k)\}$ uniformizing the polygonal cells of $|P_{T_a}| \to M$ and the variables $\{t_k\}$ uniformizing the corresponding vertex stars in $|T_a| \to M$. We have also an explicit expression $v \doteq u - \sum_{k=1}^{N_0(T)}\left(\frac{\varepsilon(k)}{2\pi}\right)\ln|t-t_k|$ (see (25)) for the conformal factor defining the metric of the triangulation $|T_a| \to M$ in terms of the underlying conformally flat structure (see (24)). In such a framework we are fully entitled to interpret the above remark, which sees the scaling as associated with the modular parameters describing the punctures, as the counterpart in DT theory of the anomalous scaling of the formal Weyl measure $D_h[v]$.

As mentioned, the large $N_0(T)$ asymptotics of the full triangulation counting $Card[\mathcal{DT}[N_0]]$ can be obtained from purely combinatorial (and matrix theory) arguments [5], [32] to the effect that

$$Card[\mathcal{DT}[N_0]] \sim \frac{16c_g}{3\sqrt{2\pi}} \cdot e^{\mu_0 N_0(T)} N_0(T)^{\frac{5g-7}{2}}\left(1+O(\frac{1}{N_0})\right), \tag{100}$$

where c_g is a numerical constant depending only on the genus g, and $e^{\mu_0} = (108\sqrt{3})$ is a (non-universal) parameter depending on the set of triangulations considered (here the generalized triangulations, barycentrically dual to trivalent graphs; in the case of regular triangulations in place of $108\sqrt{3}$ we would get $e^{\mu_0} = (\frac{4^4}{3^3})$). As already stressed, such a parameter does not play any relevant role in 2D quantum gravity. Through a comparison of the asymptotics (97) and (100), the relation (94) immediately yields that

$$Card\left[\{q(i)\}_{i=1}^{N_0}\right] \sim \frac{c_g}{\pi^{6g-6}B_g}\left[\frac{Ce^{\mu_0}}{\pi^2}\right]^{N_0(T)}, \tag{101}$$

where corresponding to the given values of the parameters C and e^{μ_0} (see (95) and (100)) we have $(Ce^{\mu_0}/\pi^2) \sim 11.846$. Thus, as $N_0 \to \infty$, the cardinality $Card\left[\{q(i)\}_{i=1}^{N_0}\right]$ of the set of possible curvature assignments grows exponentially, and only provides a non-universal contribution to the whole triangulation counting. Summarizing the results obtained so far we can state the following

Proposition 12. *The anomalous scaling behavior of the canonical entropy function* $Card\,[\mathcal{DT}[N_0]]$ *comes only from* $Card\left[\mathcal{DT}[\{q(i)\}_{i=1}^{N_0}]\right]$. *In particular, up to an overall normalization, the combinatorial counterpart of the anomalous scaling of the field theoretic measure*

$$Z^{\pm}_{(n)}(h)D_h[v]\prod_{\beta=1}^{3g-3}dm_\beta d\overline{m}_\beta, \tag{102}$$

is provided, in the case of pure gravity, by the large N_0 *asymptotics of*

$$Vol_{W-P}(\overline{\mathfrak{M}}_{g,N_0})\exp\left[-vN_0(T)\right] \tag{103}$$

where $\nu \doteq 2\ln\frac{\pi}{C}$.

The overall normalization referred to in the above statement simply reflects the presence of the leading exponential contributions to triangulations counting associated both to $Card\left[\{q(i)\}_{i=1}^{N_0}\right]$ and to the term $\pi^{2N_0}C^{-N_0}$ in (97). Such terms affect the normalization $Z_g[1]$ in the continuum expression (7) for the scaling, with the surface area, of the partition function of 2D gravity. The exponential term $\exp\left[-vN_0(T)\right]$ takes care of such normalization and isolates the polynomial subleading asymptotics of $Vol_{W-P}(\overline{\mathfrak{M}}_{g,N_0})$ which is responsible, according to (91), of the anomalous scaling in dynamical triangulation theory.

5 Concluding remarks: matter fields

From a moduli theory point of view, the above analysis explains the origin of the critical exponents in the canonical entropy function (100) for pure gravity. According to (94) and to the Manin-Zograf asymptotics, the critical exponents are related to the subleading polynomial asymptotics of the volume of the moduli space $\overline{\mathfrak{M}}_{g,N_0}$. In order to extend the factorization (94) to the case of gravity coupled to matter, we need to discuss how the discretization of the matter fields interacts with such factorization. If we parametrize the set of possible conformal fields $\{\Phi\}$ over M in terms of a smooth projective manifold (actually a variety) of dimension r, then a basic ingredient of the quantum theory describing the interaction of $\{\Phi\}$ with 2D gravity is the path integral over the space of maps $(M, \mathcal{C}) \to X$, between a marked Riemann surface and the target X. Such path integrals tend to localize around the space of holomorphic maps which thus provides a sort of discretization of the matter functional integral, (see *e.g.*, [12]). In a simplicial quantum gravity setting a distribution of matter fields $\{\Phi\}$ over a dynamical triangulation $|T_a| \to M$ can be described by a map

$$\begin{aligned} \overline{\mathfrak{M}}_{g,N_0} &\to RGB^{met}_{g,N_0} \times X \\ ((M;N_0),\mathcal{C}) &\longmapsto (\Gamma, \Phi_{(k)}) \end{aligned} \tag{104}$$

where $((M;N_0),\mathcal{C})$ is the punctured Riemann surface associated with $|T_a| \to M$, Γ is the corresponding ribbon graph in RGB^{met}_{g,N_0}, and $\Phi_{(k)}$ is a locally constant map describing conformal fields attached to the polygonal faces $\{\rho^2(k)\}$ of the corresponding dual polytope $|P_{Ta}| \to M$. According to the bijection (65), we can write (104) equivalently as

$$\overline{\mathfrak{M}}_{g,N_0} \to \left(\overline{\mathfrak{M}}_{g,N_0} \times X\right) \times \mathbb{R}^N_+ . \tag{105}$$

Since $\overline{\mathfrak{M}}_{g,N_0} \times X$ is isomorphic to the space $\overline{\mathfrak{M}}_{g,N_0}(X,0)$ of stable constant maps from $((M;N_0),\mathcal{C})$ to X , we have a natural candidate which should play in the matter setting the same role of $\overline{\mathfrak{M}}_{g,N_0}$ in the pure gravity case. More generally, it is technically convenient to introduce the space $\overline{\mathfrak{M}}_{g,N_0}(X,\beta)$ of stable maps from $((M;N_0),\mathcal{C})$ to X representing the homology class $\beta \in H_2(X,\mathbb{Z})$, (for instance, in the case of the Polyakov action, the map can be thought of as providing the embedding of $((M;N_0),\mathcal{C})$ in the target (Euclidean) space). According to these remarks it is rather natural to conjecture that, for dynamical triangulations, the measure describing the statistical distribution of conformal matter interacting with 2D gravity is provided by a (generalized) Weil-Petersson volume of $\overline{\mathfrak{M}}_{g,N_0}(X,\beta)$. This observation calls into play the (descendent) Gromov-Witten invariants of the manifold X (in order to provide the analogous of (79)) and to a large extent can be discussed along the same lines of the pure gravity case. As already mentioned in the introduction, such a connection is in line with the fact that many relevant properties of $\overline{\mathfrak{M}}_{g,N_0}(X,\beta)$ are governed, via its intersection theory, by matrix models. This is a consequence of the localization of path integrals over maps (conformal matter) which computes such path integrals as sums, labelled by ribbon graphs, of finite dimensional integrals over space of stable holomorphic maps. We thus expect that the techniques related to the study of the properties of $\overline{\mathfrak{M}}_{g,N_0}(X,\beta)$, on which there has been much progress recently [12], will led to a deeper understanding of the anomalous scaling of conformal matter interacting with 2D gravity (*i.e.*, non-critical strings).

6 Appendix I : Teichmüller theory of pointed Riemann surfaces

In this section we summarize, for the reader's convenience, basic facts about Teichmüller space theory. For details and proofs we refer to [33]. If $S_2(M)$ denotes the space of symmetric bilinear forms on M, let us consider the set of all Riemannian metrics on $(M; N_0)$, *i.e.*

$$Riem(M; N_0) \doteq \{g \in S_2(M) |\ g(x)(u,u) > 0\ if\ u \neq 0\}, \tag{106}$$

and let

$$Riem_{-1}(M; N_0) \hookrightarrow Riem(M; N_0) \tag{107}$$

be the set of metrics of constant curvature -1, describing the hyperbolic structures on $(M; N_0)$. If $Diff_+(M)$ is the group of all orientation preserving diffeomorphisms then

$$Diff_+(M; N_0) =$$

$$= \left\{ \psi \in Diff_+(M) : \psi\ preserves\ setwise\ \{\sigma^0(i)\}_{i=1}^{N_0(T)} \right\} \tag{108}$$

acts by pull-back on the metrics in $Riem_{-1}(M; N_0)$. Let $Diff_0(M; N_0)$ be the subgroup consisting of diffeomorphisms which when restricted to $(M; N_0)$ are isotopic to the identity, then the Teichmüller space $\mathfrak{T}_{g,N_0}(M)$ associated with the genus g surface with $N_0(T)$ punctures M is defined by

$$\mathfrak{T}_{g,N_0}(M) = \frac{Riem_{-1}(M; N_0)}{Diff_0(M; N_0)}. \tag{109}$$

Recall that a Riemann surface is a complex analytic structure on M consisting of an atlas of charts $\{U_\alpha, \zeta_\alpha\}$ where $\{U_\alpha\}$ is a covering of M by open sets, $\zeta_\alpha : U_\alpha \to \mathbb{C}$ is a homeomorphism and if $U_\alpha \cap U_\beta \neq \emptyset$, then $\zeta_\alpha \circ \zeta_\beta^{-1} : \zeta_\beta(U_\alpha \cap U_\beta) \to \mathbb{C}$ is complex analytic. The local maps $\zeta_\alpha : U_\alpha \to \mathbb{C}$ are the uniformizing parameters of the surface. From such a complex function theory perspective, $\mathfrak{T}_{g,N_0}(M)$ is defined by fixing a reference complex structure $\mathcal{C}_0$ on $(M; N_0)$ (a marking) and considering the set of equivalence classes of complex structures $(\mathcal{C}, f)$ where $f : \mathcal{C}_0 \to \mathcal{C}$ is an orientation preserving quasi-conformal map, and where any two pairs of complex structures $(\mathcal{C}_1, f_1)$ and $(\mathcal{C}_2, f_2)$ are considered equivalent if $h \circ f_1$ is homotopic to f_2 via a conformal map $h : \mathcal{C}_1 \to \mathcal{C}_2$. Let us assume that the reference complex structure $\mathcal{C}_0$ admits an antiholomorophic reflection $j : \mathcal{C}_0 \to \mathcal{C}_0$. Since any orientation reversing diffeomorphism $\widetilde{\varphi}$ can be written as $\varphi \circ j$ for some orientation preserving diffeomorphism φ, the (extended) mapping class group

$\mathfrak{Map}(M;N_0) \doteq Diff(M;N_0)/Diff_0(M;N_0)$ acts naturally on $\mathfrak{T}_{g,N_0}(M)$ according to

$$\mathfrak{Map}(M;N_0) \times \mathfrak{T}_{g,N_0}(M) \longrightarrow \mathfrak{T}_{g,N_0}(M) \tag{110}$$
$$\begin{cases} (\varphi,(\mathcal{C},f)) \longmapsto (\mathcal{C}, f\circ\varphi^{-1}), \\ (\widetilde{\varphi},(\mathcal{C},f)) \longmapsto (\mathcal{C}^*, f\circ j\circ\varphi^{-1}) \end{cases},$$

where

$$\begin{cases} \varphi \in Diff_+(M;N_0) \\ \widetilde{\varphi} \in Diff(M;N_0) - Diff_+(M;N_0) \end{cases},$$

and where the conjugate surface $\mathcal{C}^*$ is the Riemann surface locally described by the complex conjugate coordinate charts associated with $\mathcal{C}$. It follows that the Teichmüller space $\mathfrak{T}_{g,N_0}(M)$ can be also seen as the universal cover of the moduli space $\mathfrak{M}_{g,N_0}$ of genus g Riemann surfaces with $N_0(T)$ punctures defined by

$$\mathfrak{M}_{g,N_0} = \frac{\mathfrak{T}_{g,N_0}(M)}{\mathfrak{Map}(M;N_0)} \tag{111}$$

It is well-known that $\mathfrak{M}_{g,N_0}$ is a connected orbifold space of complex dimension $3g-3+N_0(T)$ and that, although in general non complete, it admits a stable curve compactification (Deligne-Mumford) $\overline{\mathfrak{M}}_{g,N_0}$. The orbifold $\overline{\mathfrak{M}}_{g,N_0}$ is endowed with $N_0(T)$ natural line bundles $\mathcal{L}_i$ (the cotangent space to M at the i-th marked point), and with a natural rank g vector bundle $\mathbb{E}$, the Hodge bundle whose fibers are top wedge powers of Abelian differentials on the surface.

For a genus g Riemann surfaces with $N_0(T) > 3$ punctures the complex vector space $Q_{N_0}(M)$ of (holomorphic) quadratic differentials is defined by tensor fields ψ described, in a locally uniformizing complex coordinate chart (U,ζ), by a holomorphic function $\mu : U \to \mathbb{C}$ such that $\psi = \mu(\zeta)d\zeta \otimes d\zeta$. Away from the discrete set of the zeros of ψ, we can locally choose a canonical conformal coordinate $z_{(\psi)}$ (unique up to $z_{(\psi)} \mapsto \pm z_{(\psi)} + const$) by integrating the holomorphic 1-form $\sqrt{\psi}$, *i.e.*,

$$z_{(\psi)} = \int^{\zeta} \sqrt{\mu(\zeta')d\zeta' \otimes d\zeta'}, \tag{112}$$

so that $\psi = dz_{(\psi)} \otimes dz_{(\psi)}$.
If we endow $Q_{N_0}(M)$ with the L^1-(Teichmüller) norm

$$||\psi|| \doteq \int_M |\psi|, \tag{113}$$

then the Banach space of integrable quadratic differentials on M, $Q_{N_0}(M) \doteq \{\psi|,\ ||\psi|| < +\infty\}$, is non-empty and consists of meromorphic quadratic differentials whose only singularities are (at worst) simple poles at the N_0 distinguished points of (M, N_0). $Q_{N_0}(M)$ is finite dimensional and, according to the Riemann-Roch theorem, it has complex dimension $\dim_{\mathbb{C}} Q_{N_0}(M) = 3g - 3 + N_0(T)$.

From the viewpoint of Riemannian geometry, a quadratic differential is basically a transverse-traceless two tensor deforming a Riemannian structure to a nearby inequivalent Riemannian structure. Thus a quadratic differential $\psi = \mu(\zeta) d\zeta \otimes d\zeta$ also encodes information on possible deformations of the given complex structure. Explicitly, by performing an affine transformation with constant dilatation $K > 1$, one defines a new uniformizing variable $z'_{(\psi)}$ associated with $\psi = \mu(\zeta) d\zeta \otimes d\zeta$ by deforming the variable $z_{(\psi)}$ defined by (112) according to

$$z'_{(\psi)} = K \operatorname{Re}(z_{(\psi)}) + \sqrt{-1} \operatorname{Im}(z_{(\psi)}). \tag{114}$$

The new metric associated with such a deformation is provided by

$$\left| dz'_{(\psi)} \right|^2 = \frac{(K+1)^2}{4} \left| dz_{(\psi)} + \frac{K-1}{K+1} d\bar{z}_{(\psi)} \right|^2. \tag{115}$$

Since $dz^2_{(\psi)}$ is the given quadratic differential $\psi = \mu(\zeta) d\zeta \otimes d\zeta$, we can equivalently write $\left| dz'_{(\psi)} \right|^2$ as

$$\left| dz'_{(\psi)} \right|^2 = \frac{(K+1)^2}{4} |\mu| \left| d\zeta + \frac{K-1}{K+1} \left(\frac{\overline{\mu}}{|\mu|^{1/2}} \right) d\overline{\zeta} \right|^2, \tag{116}$$

where $(\overline{\mu}/|\mu|)\, d\overline{\zeta} \otimes d\zeta^{-1}$ is the (Teichmüller-) Beltrami form associated with the quadratic differential ψ. If we consider quadratic differentials $\psi = \mu(\zeta) d\zeta \otimes d\zeta$ in the open unit ball $Q^{(1)}_{N_0}(M) \doteq \{\psi|,\ ||\mu(\zeta)|| < 1\}$ in the Teichmüller norm (113), then there is a natural choice for the constant K provided by

$$K = \frac{1 + ||\mu(\zeta)||}{1 - ||\mu(\zeta)||}. \tag{117}$$

In this latter case we get

$$\left| dz'_{(\psi)} \right|^2 = \frac{||\mu(\zeta)||^2}{(||\mu(\zeta)|| - 1)^2} |\mu| \left| d\zeta + \frac{1}{||\mu(\zeta)||} \left(\frac{\overline{\mu}}{|\mu|^{1/2}} \right) d\overline{\zeta} \right|^2. \tag{118}$$

According to Teichmüller's existence theorem any complex structure on can be parametrized by the metrics (118) as $\psi = \mu(\zeta) d\zeta \otimes d\zeta$, varies in $Q^{(1)}_{N_0}(M)$.

This is equivalent to saying that for any given (M, g), (with $(M, N_0; [g])$ defining a reference complex structure $\mathcal{C}_0$ on (M, N_0)), and any diffeomorphism $f \in Diff_0(M; N_0)$ mapping (M, g) into (M, g_1), with $(M, N_0; [g_1])$ a complex structure distinct from $(M, N_0; [g])$, there is a quadratic differential $\psi \in Q^{(1)}_{N_0}(M)$ and a biholomorphic map $F \in Diff_0(M; N_0)$, homotopic to f such that $[F^* g_1]$ is given by the conformal class associated with (118). This is the familiar point of view which allows to identify Teichmüller space with the open unit ball $Q^{(1)}_{N_0}(M)$ in the space of quadratic differentials $Q_{N_0}(M)$. It is also worthwhile noticing that (118) allows us to consider the open unit ball $Q^{(1)}_{N_0}(M)$ in the space of quadratic differential as providing a slice for the combined action of $Diff_0(M; N_0)$ and of the conformal group $W^s(M; N_0)$ on the space of Riemannian metrics $Riem(M; N_0)$, *i.e.*

$$Q^{(1)}_{N_0}(M) \hookrightarrow \frac{Riem(M; N_0)}{Diff_0(M; N_0)} \simeq W^s(M; N_0) \times \mathfrak{T}_{g,N_0}(M) \tag{119}$$

$$\left[|dz'_\psi|^2 \right] \longmapsto \frac{||\mu||^2}{(||\mu|| - 1)^2} |\mu| \cdot \left| d\zeta + \frac{1}{||\mu||} \left(\frac{\overline{\mu}}{|\mu|^{1/2}} \right) d\overline{\zeta} \right|^2 ,$$

where

$$W^s(M; N_0) \doteq \{ f : M \to \mathbb{R}^+ | \, f \in H^s(M, \mathbb{R}) \} \tag{120}$$

denotes the (Weyl) space of conformal factors defined by of all positive (real valued) functions on M whose derivatives up to the order s exist in the sense of distributions and are represented by square integrable functions.

Together with $Q_{N_0}(M)$ one introduces also the space $B_{N_0}(M)$ of (L^∞ measurable) Beltrami differentials, *i.e.* of tensor fields $\omega = \nu(\zeta) d\overline{\zeta} \otimes d\zeta^{-1}$, sections of $k^{-1} \otimes \overline{k}$, ($k$ being the holomorphic cotangent bundle to M), with $\sup_M |\nu(\zeta)| < \infty$. The space of Beltrami differentials is naturally identified with the tangent space to $\mathfrak{T}_{g,N_0}(M)$, *i.e.*, $\omega = \nu(\zeta) d\overline{\zeta} \otimes d\zeta^{-1} \in T_C \mathfrak{T}_{g,N_0}(M)$, (with C a complex structure in $\mathfrak{T}_{g,N_0}(M)$).

The two spaces $Q_{N_0}(M)$ and $B_{N_0}(M)$ can be naturally paired according to

$$\langle \psi | \omega \rangle = \int_M \mu(\zeta) \nu(\zeta) d\zeta d\overline{\zeta}. \tag{121}$$

In such a sense $Q_{N_0}(M)$ is $\mathbb{C}$-anti-linear isomorphic to $T_C \mathfrak{T}_{g,N_0}(M)$, and can be canonically identified with the cotangent space $T^*_C \mathfrak{T}_{g,N_0}(M)$ to $\mathfrak{T}_{g,N_0}(M)$. On the cotangent bundle $T^*_C \mathfrak{T}_{g,N_0}(M)$ we can define the Weil-Petersson metric as the inner product between quadratic differentials corresponding to the

L^2- norm provided by

$$\|\psi\|^2_{WP} \doteq \int_M h^{-2}(\zeta)\,|\psi(\zeta)|^2\,|d\zeta|^2, \tag{122}$$

where $\psi \in Q_{N_0}(M)$ and $h(\zeta)\,|d\zeta|^2$ is the hyperbolic metric on M disk-wise described by (54). Note that $\frac{\psi}{h}$ is a Beltrami differential on M, thus if we introduce a basis $\{\mu_\alpha\}_{\alpha=1}^{3g-3+N_0}$ of the vector space of harmonic Beltrami differentials on (M, N_0), we can write

$$G_{\alpha\overline{\beta}} = \int_M \mu_\alpha \overline{\mu_\beta} h(\zeta)\,|d\zeta|^2 \tag{123}$$

for the components of the Weil-Petersson metric on the tangent space to $\mathfrak{T}_{g,N_0}(M)$. We can introduce the corresponding Weil-Petersson Kähler form according to

$$\omega_{WP} = \sqrt{-1} G_{\alpha\overline{\beta}} dZ^\alpha \wedge d\overline{Z^\beta}. \tag{124}$$

Such Kähler potential can be made invariant under the mapping class group $\mathfrak{Map}(M; N_0)$ to the effect that the Weil-Petersson volume 2-form ω_{WP} on $\mathfrak{T}_{g,N_0}(M)$ descends on $\mathfrak{M}_{g,N_0}$, and it has a (differentiable) extension, in the sense of orbifold, to $\overline{\mathfrak{M}}_{g,N_0}$.

7 Appendix II : relations with curvature assignments enumeration techniques

To the attentive reader it will not have escaped the fact that the statement of proposition 12 seems partly in contrast with some of the results discussed in our monograph [34], where $\gamma_g = \frac{5g-1}{2}$ is to some extent related to a quantity resembling $Card\left[\{q(i)\}_{i=1}^{N_0}\right]$. The origin of such a contrast is formula (94) which is similar to formula 5.33 (p. 103) of [34]. There, one introduces a quantity p_λ^{curv}, (where $\lambda = N_0$ in the present notation), which plays the role of $Card\left[\{q(i)\}_{i=1}^{N_0}\right]$ and which is related to the number of distinct (unlabelled) curvature assignments over the N_0 vertices. p_λ^{curv} is actually combinatorially estimated by a rather delicate procedure which calls into play a rooting (at a curvature blob). Note in particular that strictly speaking while p_λ^{curv} is obtained by an unlabelled rooted counting (using partition of integers techniques), $Card\left[\{q(i)\}_{i=1}^{N_0}\right]$ is actually obtained by a labelled enumeration with the labellings formally factorized out by dividing by $N_0!$, (already in such a sense it is difficult to compare the subleading asymptotics of the two quantities, since, even if it is true that by dividing by $N_0!$ one removes the labels, there are necessary conditions on the large N_0 limit of $Card\left[\{q(i)\}_{i=1}^{N_0}\right]$ in order to carry a subdominant asymptotic comparison with p_λ^{curv}. Typical example in such a direction are provided by labelled and unlabelled counting of graphs, see *e.g.* [35] section 9.3). At any rate, this is not the real source of ambiguity in comparing our results with those described in [34]. In 5.33 it further appears the factor $< Card\{T_a^{(i)}\}_{curv} >$ which in its role it is analogous to $VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right)$, and which must be normalized by a modular part (see chapter 4 of [34]), in order to provide the final triangulation counting formula 6.22 of [34]. As a matter of fact, the estimation of $< Card\{T_a^{(i)}\}_{curv} >$ in [34] requires that one should specify also a rooting fixing the modular degrees of freedom of the possible deformations of the dynamical triangulations. By specifying the dimension to $n = 2$ and by considering the case of the sphere, where modular degrees of freedom in the sense of [34] are absent, 6.22 shows that the corresponding $\gamma_{g=2} = -\frac{1}{2}$ originates from p_λ^{curv} which seems at variance with the results of the present analysis. However, in the present case, also for the sphere, triangulations in $\mathcal{DT}[\{q(i)\}_{i=1}^{N_0}]$ do possess modular degrees of freedom (since the counting procedure involving the use of $VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right)$ maps a triangulated sphere into a punctured sphere). Thus, the correct dictionary between (94) and 5.33 and 6.22 of [34], in the case of the sphere, is that one should compare p_λ^{curv} with $Card\left[\{q(i)\}_{i=1}^{N_0}\right] VOL\left(\overline{\mathfrak{M}}_{0,N_0}\right)$, a comparison which indeed provides the correct interplay between these two rather different formalisms. Further geometrical aspects clarifying the role of the Weil-Petersson volume in simplicial quantum gravity are discussed in [36].

Acknowledgements

The authors wish to express their gratitude to C. Dappiaggi for many invaluable comments. This work has been supported in part by the Ministero dell'Universita' e della Ricerca Scientifica under the PRIN project *The geometry of integrable systems*.

References

[1] E. D'Hoker, *Lectures on Strings*, IASSNS-HEP-97/72.

[2] V.G. Knizhnik, A. M. Polyakov, A. B. Zamolodchikov, *Fractal structure of 2d quantum gravity*, Mod. Phys. Lett. **A3** (1988), 819-826.

[3] J. Distler, H. Kaway, *Conformal field theory and 2d quantum gravity*, Nucl. Phys. **B321** (1989), 509-527.

[4] F. David, *Conformal field theories coupled to 2D gravity in the conformal gauge*, Mod. Phys. Lett. **A3** (1988), 1651-1656.

[5] J. Ambjørn, B. Durhuus, T. Jonsson, *Quantum Geometry*, Cambridge Monograph on Mathematical Physics, Cambridge Univ. Press (1997).

[6] S. Catterall and E. Mottola, *The conformal mode in 2D simplicial gravity,* Phys. Lett. B **467** (1999) 29 [arXiv:hep-lat/9906032].

[7] P. Menotti, P. Peirano, *Functional integration on two-dimensional Regge geometries*, Nucl. Phys. **B473** (1996) 426 [arXiv:hep-th/9602002], see also Phys. Lett. B **353**, (1995) 444 [arXiv:hep-th/9503181].

[8] M. Kontsevich, *Intersection theory on moduli space of curves*, Commun. Math. Phys. **147**, (1992) 1.

[9] E. Witten, *Two dimensional gravity and intersection theory on moduli space*, Surveys in Diff. Geom. **1.** (1991) 243.

[10] Y. I. Manin, P. Zograf, *Invertible cohomological filed theories and Weil-Petersson volumes*, Annales de l' Institute Fourier, **Vol. 50**, (2000), 519-535 [arXiv:math-ag/9902051].

[11] P. Zograf, *Weil-Petersson volumes of moduli spaces of curves and curves and the genus expansion in two dimensional* [arXiv:math.AG/9811026]

[12] A. Okounkov, R. Pandharipande, *Gromov-Witten theory, Hurwitz numbers, and Matrix models*, I, [arXiv:math.AG/0101147].

[13] T. Eguchi, K. Hori, C. S. Xiong, *Quantum cohomology and Virasoro algebra*, Phys. Lett. B **402** (1997), 71-80 [hep-th/9703086].

[14] M. Troyanov, *Prescribing curvature on compact surfaces with conical singularities*, Trans. Amer. Math. Soc. **324**, (1991) 793; see also M. Troyanov, *Les surfaces euclidiennes a' singularites coniques*, L'Enseignment Mathematique, **32** (1986) 79.

[15] W. P. Thurston, *Shapes of polyhedra and triangulations of the sphere*, Geom. and Topology Monog. **Vol.1** (1998) 511.

[16] E. Picard, *De l'integration de l'equation* $\Delta u = e^u$ *sur une surface de Riemann fermee'*, Crelle's Journ. **130** (1905) 243.

[17] W. M. Goldman, *Moduli spaces*, Lectures given at the workshop at Korea Nat. Univ. of Education, Ch'onju, Korea (1994).

[18] P. Deligne, G. D. Mostow, *Monodromy of hypergeometric functions and non-lattice integral monodromy*, Publ. Math. I.H.E.S. **63** (1986), 5-106.

[19] K. Strebel, *Quadratic differentials*, Springer Verlag, (1984).

[20] M. Mulase, M. Penkava, *Ribbon graphs, quadratic differentials on Riemann surfaces, and algebraic curves defined over* $\overline{\mathbb{Q}}$, The Asian Journal of Mathematics **2**, 875-920 (1998) [math-ph/9811024 v2].

[21] E. Looijenga, *Intersection theory on Deligne-Mumford compactifications*, Séminaire Bourbaki, (1992-93), 768.

[22] B. Rodin, D. Sullivan, *The convergence of circle packings to the Riemann mappings*, J. Differential Geometry **26**, (1987), 349-360.

[23] P.L.Bowers, K. Stephenson, *Circle packings in surfaces of finite type: An in situ approach with application to moduli*, Topology **32** (1993), 157-183.

[24] R. W. Barnard, G.B. Williams, *Combinatorial excursions in moduli space*, Preprint, URL: http://www.math.ttu.edu/~williams.

[25] P. L. Bowers, K. Stephenson, *Uniformizing dessins and Belyĭ maps via circle packing*, preprint (1997) URL: http://web.math.fsu.edu/ bowers/Papers/recentPapers.html.

[26] V. A. Voevodskii, G.B. Shabat, *Equilateral triangulations of Riemann surfaces, and curves over algebraic number fields*, Soviet Math. Dokl. **39** (1989) 38.

[27] M. Bauer, C. Itzykson, *Triangulations*, in: *The Grothendieck Theory of Dessins d'Enfants*, ed. L. Schneps, Lond. Math. Soc. Lect. Notes Series, **Vol. 200**, Cambridge Univ. Press (1994) 179.

[28] R. C. Penner, *Weil-Petersson volumes*, J. Diff. Geom. **35** (1992) 559-608.

[29] C. T. McMullen, *The moduli space of Riemann surfaces is Kähler hyperbolic*, Annals of Math., **151** (2000), 327-357 [arXiv:math.CV/0010022].

[30] G. Schumacher, S. Trapani, *Estimates of Weil-Petersson volumes via effective divisors*, Commun. Math. Phys. **222**, (2001) 1-7 [arXiv:math.AG/0005094 v2].

[31] S. Grushevsky, *Explicit upper bound for the Weil-Petersson volumes*, Math. Ann. **321**, (2001) 1-13 [arXiv:math.AG/0003217].

[32] E. Brézin, C. Itzykson, G. Parisi, J.B. Zuber, *Planar Diagrams*, Commun. Math. Phys. **59**, 25-51 (1978).

[33] S. Nag, *The complex analytic theory of Teichmüller spaces*, Canad. Math. Soc. monographs, J. Wiley &Sons (1988).

[34] J. Ambjørn, M. Carfora, A. Marzuoli, *The geometry of dynamical triangulations*, Lecture Notes in Physics **m50**, Springer Verlag (1997).

[35] F. Harary, E. M. Palmer, *Graphical enumeration*, Academic Press (1973).

[36] M. Carfora, C. Dappiaggi, A. Marzuoli, *The modular geometry of random Regge triangulations*, [arXiv:gr-qc/0206077].