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dS/CFT and spacetime topology

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1 Introduction

In the general Euclidean formulation of the AdS/CFT correspondence put forth in [25], one considers Riemannian manifolds of the form $M^{n+1} \times Y$, where M^{n+1} is conformally compactifiable with conformal boundary-at-infinity N^n. It is then a problem of fundamental interest to determine, for a given compact manifold N with a given conformal structure, the complete Einstein manifolds of negative Ricci curvature having N as conformal boundary. An early result of this type, predating AdS/CFT, was obtained by Graham and Lee [16], who showed that for a conformal structure sufficiently close to the standard one on S^n, there is a unique Einstein metric with a prescribed curvature close to the standard hyperbolic metric on the $n+1$ ball which induces the conformal boundary N.

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More recently, motivated by certain issues in AdS/CFT, Witten and Yau [28] obtained some general topological restrictions on M. They proved in this context that if M is an Einstein manifold with negative Ricci curvature, such that the conformal class of N admits a metric of positive scalar curvature then the co-dimension one homology of M vanishes, $H_n(M, \mathbb{Z}) = 0$. This implies, in particular, that N is connected. As discussed in [28, 26], these results resolve certain "puzzles" concerning the AdS/CFT correspondence. These, and related results, were extended by Cai and Galloway [7] to the case of zero scalar curvature; see also [24] for further developments. This, in a sense, covers all cases relevant to the AdS/CFT correspondence, since, as argued in [28], CFT's defined on conformal boundaries of negative scalar curvature are unstable. Results of a related nature in the Lorentzian context for asymptotically AdS spacetimes follow from results on topological censorship [11, 12, 26].

In the present paper we study similar issues for spacetimes of de Sitter type, i.e., spacetimes satisfying the Einstein equation with cosmological constant $\Lambda > 0$, which admit a regular conformal (Penrose) compactification. More specifically, we study the influence of the curvature and topology of the conformal boundary (at past or future infinity) on the bulk spacetime, for spacetimes of de Sitter type. Our motivation for this study comes from recent proposals for a de Sitter analogue of the AdS/CFT correspondence (see e.g., [22, 23]), and also from current developments in cosmology, in particular the supernova observations which have led cosmologists to include a positive Λ among the cosmological parameters of the standard model of the universe, see [5].

Consider a spacetime M of de Sitter type which admits a conformal completion to the past and future, such that the past conformal boundary $\mathcal{J}^-$ and future conformal boundary $\mathcal{J}^+$ are spacelike and compact. When M is globally hyperbolic, this implies that $\mathcal{J}^-$ and $\mathcal{J}^+$ are each connected, and homeomorphic, irrespective of any field equations. A spacetime M with these properties is nonsingular, in the sense of being timelike and null geodesically complete. Although conformal infinity still consists of two components, $\mathcal{J}^\pm$, there are reasons to view this situation as analogous to a conformally compactifiable Riemannian manifold M with *connected* conformal boundary N, as considered above. For example, in the dS/CFT proposal recently put forward by Strominger [22], it is argued that $\mathcal{J}^+$ and $\mathcal{J}^-$ become effectively identified, and give rise to a single conformal field theory; see also [27]. At the purely classical level, a recent result of Anderson [2] shows that the boundary map from Riemannian AdS metrics on B^4, restricted to the connected component containing the hyperbolic metric, to the component of the space $\mathcal{C}^0$ of conformal classes on S^3 with positive scalar curvature, containing the round sphere, has degree one, and is hence surjective [2, Theorem C]. In the Lorentzian case, results by Friedrich [9, 10], suggest a similar relation between $\mathcal{C}^0 \times \mathcal{C}^0$ and the space of asymptotically de Sitter spacetimes on $S^3 \times [0,1]$. Roughly stated, there is an analogy between "filling in S^3" in the Riemannian AdS case, and "filling in two copies of S^3" in the Lorentzian de Sitter case.

Thus, for globally hyperbolic spacetimes of de Sitter type with compact conformal boundaries $\mathcal{I}^{\pm}$, the notion of "connectedness of the boundary" is in some sense built in. Our main results then imply that for such spacetimes, which obey suitable energy conditions, the curvature and topology of $\mathcal{I}^+$ and $\mathcal{I}^-$ are quite restricted: Each must have finite fundamental group and the associated conformal class of each must contain a metric of positive scalar curvature. Thus, in analogy with the results of Witten and Yau [28] pertaining to the AdS/CFT correspondence, we establish here, for asymptotically de Sitter spacetimes, some connections between the bulk spacetime (e.g., its being nonsingular) and the topology of the conformal boundary. Further discussion of the role of topology in the dS/CFT correspondence may be found in [19].

In the following subsection we give a somewhat more detailed description of our main results. From a rather different point of view, our results can be interpreted as statements about the topology and completeness of inflationary cosmological models; see the comment at the end of the next subsection.

1.1 Overview of the paper

We consider spacetimes which are asymptotically de Sitter either to the future or the past. To fix the time orientation for the present discussion, let M be a globally hyperbolic spacetime of de Sitter type with regular *past* conformal boundary $\mathcal{I}^-$, see Section 2 for definitions, and assume $\mathcal{I}^-$ is compact. Then the Cauchy surfaces of M are compact, and in fact homeomorphic to $\mathcal{I}^-$, see Proposition 2.1. Subject to appropriate energy conditions, our results show that, due to the development of singularities, or other irregularities, M cannot be asymptotically de Sitter to the future; i.e., cannot have a regular future conformal infinity $\mathcal{I}^+$, unless the curvature and topology of $\mathcal{I}^-$ is suitably restricted. We briefly discuss here the various curvature and topology restrictions obtained.

The Riemannian metric $\tilde{h}_{\alpha\beta}$ induced by $\tilde{g}_{\alpha\beta}$ on $\mathcal{I}^-$ changes by a conformal factor with a change in the *defining function* Ω, and thus $\mathcal{I}^-$ is endowed with a natural conformal structure $[\tilde{h}_{\alpha\beta}]$. By the Yamabe theorem, the conformal class $[\tilde{h}_{\alpha\beta}]$ contains a metric of constant scalar curvature -1, 0, or $+1$, exclusively, in which case we will simply say that $\mathcal{J}^-$ has negative, zero, or positive scalar curvature, respectively.

In Section 3, we show that, with the setting as above, if $\mathcal{J}^-$ has negative scalar curvature then all the timelike geodesics of M are future incomplete. We further show that if $\mathcal{J}^-$ has zero scalar curvature, M can contain a future complete timelike geodesic only under special circumstances: M must split as a warped product. As discussed in Section 3, these results can be expressed in terms of the *Yamabe type* of $\mathcal{J}^-$, and hence the Yamabe type of the Cauchy surfaces of M. The upshot is, in order for M to be timelike geodesically complete, $\mathcal{J}^-$ must be of positive Yamabe type. Hence, in $3+1$ dimensions, $\mathcal{J}^-$ cannot have any $K(\pi, 1)$ factors in its prime decomposition. Thus, modulo the Poincaré conjecture, $\mathcal{J}^-$ must be covered by a 3-sphere, be diffeomorphic to $S^1 \times S^2$, or be a connected sum of such manifolds.

Some results of a related nature are obtained in Section 4. Corollary 4.2 shows that with the setting as above, M cannot admit a regular future conformal boundary $\mathcal{J}^+$, compact or otherwise, unless $\mathcal{J}^-$ has finite fundamental group. Perhaps somewhat surprisingly, this rules out, in particular, a scenario in which a black hole forms from a regular past (with compact $\mathcal{J}^-$), such that M, with Cauchy surface topology $S^1 \times S^2$, is future asymptotically similar to Schwarzschild-de Sitter spacetime. In a somewhat related vein, it is shown in Theorem 4.3 that M^{n+1}, $n \leq 7$, must be future null geodesically incomplete, unless $H_{n-1}(\mathcal{J}^-, \mathbb{Z})$ vanishes, or, equivalently, by Poincaré duality, etc., unless $H_1(\mathcal{J}^-, \mathbb{Z})$ is pure torsion and finite. In $3+1$ dimensions, Corollary 4.2 implies that in order for M to have a conformal structure similar to that of de Sitter space, $\mathcal{J}^-$ must be covered by a homotopy 3-sphere; see also Theorem 4.1. With regard to energy conditions (see Section 2), the results of Section 4 only require the null energy condition.

Finally, we remark that Theorems 3.1, 3.2, and 4.3, discussed here in a time dual manner, can be interpreted as singularity results, which establish, as a consequence of certain curvature or topology assumptions, the occurence of *past* singularities in inflationary cosmological models.

2 Preliminaries

Let $(M, g_{\alpha\beta})$ be an $n+1$ dimensional space–time, $n \geq 2$, with covariant derivative D_α, Ricci tensor $R_{\alpha\beta}$ and scalar curvature R. The Einstein equation with cosmological constant Λ is

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = S_{\alpha\beta}, \tag{2.1}$$

where $S_{\alpha\beta}$ is the stress energy tensor. Let $S = g^{\alpha\beta} S_{\alpha\beta}$. The stress energy tensor satisfies the weak, dominant and strong energy conditions respectively, if for any causal vector field V^α, it holds that

$$\text{(W.E.C.)} \qquad S_{\alpha\beta} V^\alpha V^\beta \geq 0 \tag{2.2a}$$

$$\text{(D.E.C.)} \qquad S_{\alpha\beta} V^\alpha V^\beta \geq 0, \quad \text{and} \quad S^{\alpha\beta} V_\beta \text{ is causal} \tag{2.2b}$$

$$\text{(S.E.C.)} \quad (S_{\alpha\beta} - \frac{1}{n-1} S g_{\alpha\beta}) V^\alpha V^\beta \geq 0 \tag{2.2c}$$

We will consider the case $\Lambda > 0$. After a rescaling, we may assume that $\Lambda = n(n-1)/2$. With this normalization, the strong energy condition (2.2c) is equivalent to

$$R_{\alpha\beta} V^\alpha V^\beta \geq n g_{\alpha\beta} V^\alpha V^\beta \tag{2.3}$$

for any causal vector field V^α. Both the weak and the strong energy conditions imply the null energy condition

$$R_{\alpha\beta} X^\alpha X^\beta \geq 0, \tag{2.4}$$

for any null vector X^α.

Our proofs occasionally make use of notions from causal theory. We briefly recall here some basic notation, terminology and results; for further details, see e.g. [17, 20]. For a subset A of a spacetime M, the timelike future of A, denoted $I^+(A)$, consists of all points in M that can be reached from A by future directed timelike curves. (Sometimes this is written as $I^+(A, M)$ to emphasize the particular spacetime involved.) Similarly the causal future of A, denoted $J^+(A)$, consists of the points of A together with the points in M that can reached from A by future directed causal curves. Sets of the form $\partial I^+(A)$ are called achronal boundaries, and, when nonempty, are achronal (meaning that no two points can be joined by a timelike curve) C^0 hypersurfaces. For a closed achronal set $S \subset M$, the future domain of

Greek indices $\alpha, \beta, \ldots$ run over $0, \ldots, n$ while lower case latin indices $a, b, c, \ldots$ run over $1, 2, \ldots n$.

dependence of S, denoted $D^+(S)$, consists of all points $p \in M$ such that each past inextendible causal curve from p meets S. The future Cauchy horizon of S, denoted $H^+(S)$, is the future boundary of $D^+(S)$; one has $\partial D^+(S) = H^+(S) \cup S$. The sets $I^-(A)$, $J^-(A)$, $D^-(S)$, $H^-(S)$ are defined in a time dual manner. S is a Cauchy surface if and only if $D(S) := D^+(S) \cup D^-(S) = M$, or equivalently, $H(S) := H^+(S) \cup H^-(S) = \emptyset$. If M is globally hyperbolic, i.e., if M has a Cauchy surface S then M has topology $\mathbb{R} \times S$.

2.1 Spacetimes of de Sitter type

We use Penrose's notion of conformal infinity to to make precise what it means for a spacetime to be asymptotically de Sitter. We will say that M has a *regular future conformal completion* provided there is a spacetime-with-boundary $\tilde{M}$ with C^2 metric $\tilde{g}_{\alpha\beta}$ such that,

1. M is the interior of $\tilde{M}$, and hence $\tilde{M} = M \cup \mathcal{J}$, where $\mathcal{J} = \partial\tilde{M}$,

2. $\mathcal{J}$ is spacelike, and $\mathcal{J} \subset I^+(M, \tilde{M})$, i.e., $\mathcal{J}$ is the future conformal boundary of M, and

3. $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$ are related by, $\tilde{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$, where $\Omega \in C^2(\tilde{M})$ satisfies: (i) $\Omega > 0$ on M and (ii) $\Omega = 0$ and $d\Omega \neq 0$ along $\mathcal{J}$. (Then, since $\mathcal{J}$ is spacelike $\tilde{D}^\gamma\Omega$ must be timelike along $\mathcal{J}$, $\tilde{D}_\gamma\Omega\tilde{D}^\gamma\Omega\big|_{\mathcal{J}} < 0$.)

Similarly, we say that a spacetime M has a *regular past and future conformal completion* if the above definition holds, but with condition 2 modified as follows:

2′. $\mathcal{J}$ is spacelike, and decomposes into disjoint nonempty sets, $\mathcal{J} = \mathcal{J}^+ \cup \mathcal{J}^-$, where $\mathcal{J}^+ \subset I^+(M, \tilde{M})$ and $\mathcal{J}^- \subset I^-(M, \tilde{M})$, i.e., $\mathcal{J}^+$ and $\mathcal{J}^-$ are, respectively, the future and past conformal boundaries of M.

A spacetime admitting a regular future, or regular past and future, conformal completion, as described above, will be said to be of *de Sitter type.* In general, as a matter of notation, geometric quantities associated with $\tilde{g}_{\alpha\beta}$ will be decorated with a tilde $\sim$, for example the covariant derivative $\tilde{D}_\alpha$ and Ricci tensor $\tilde{R}_{\alpha\beta}$.

2.2 Asymptotically simple spacetimes of de Sitter type

Let $(M, g_{\alpha\beta})$ be a spacetime of de Sitter type with regular future conformal infinity, $\mathcal{I}^+$. M is said to be *future asymptotically simple* provided every future inextendible null geodesic in M has a future end point on $\mathcal{I}^+$. Future asymptotic simplicity is a global assumption which rules out the presence of singularities, black holes, etc. Past asymptotic simplicity is defined time-dually.

The following proposition relates asymptotic simplicity to the causal structure of M.

Proposition 2.1. *Let $(M, g_{\alpha\beta})$ be a spacetime of de Sitter type with regular future conformal infinity $\mathcal{I}^+$.*

1. *If M is globally hyperbolic and $\mathcal{I}^+$ is compact then M is future asymptotically simple.*

2. *If M is future asymptotically simple then M is globally hyperbolic.*

In either case, the Cauchy surfaces of M are homeomorphic to $\mathcal{I}^+$.

Proof. By extending $M \cup \mathcal{I}^+$ a little beyond $\mathcal{I}^+$, one can obtain a spacetime without boundary Q such that $\mathcal{I}^+$ separates Q, and $Q = M \cup D^+(\mathcal{I}^+, Q)$. Suppose M is globally hyperbolic and $\mathcal{I}^+$ is compact. Since any Cauchy surface for M is clearly a Cauchy surface for Q, Q is globally hyperbolic. Since $\mathcal{I}^+$ is compact, it is necessarily a Cauchy surface for Q ([6, 14]). It follows that the Cauchy surfaces for M are homeomorphic to $\mathcal{I}^+$. Let γ be a null geodesic in M. Since $\mathcal{I}^+$ is a Cauchy surface for Q, the extension of γ to the future in Q must meet $\mathcal{I}^+$. It follows that M is future asymptotically simple. This proves point 1.

Now assume M is future asymptotically simple. We claim that $\mathcal{I}^+$ is a Cauchy surface for Q. Since, by construction, $H^+(\mathcal{I}^+, Q) = \emptyset$, we need only show $H^-(\mathcal{I}^+, Q) = \emptyset$. If $H^-(\mathcal{I}^+, Q) \neq \emptyset$, consider a future directed null geodesic generator γ of $H^-(\mathcal{I}^+, Q)$. By future asymptotic simplicity, γ meets $\mathcal{I}^+$. But a generator of $H^-(\mathcal{I}^+, Q)$ can meet $\mathcal{I}^+$ only at an edge point of $\mathcal{I}^+$, yet $\mathcal{I}^+$, being closed in Q, is edgeless. So we must have $H^-(\mathcal{I}^+, Q) = \emptyset$.

Thus, $\mathcal{I}^+$ is a Cauchy surface for Q, and Q is globally hyperbolic. It follows that M can be foliated by Cauchy surfaces for Q. These Cauchy surfaces for Q are also Cauchy surfaces for M, and we conclude that M is globally hyperbolic, with Cauchy surfaces homeomorphic to $\mathcal{I}^+$. This proves point 2. $\square$

3 Past incomplete spacetimes of de Sitter type

This section is devoted to the proof of the following two theorems, which deal with spacetimes of de Sitter type with compact future conformal infinity of nonpositive scalar curvature. The first theorem deals with the case when $\mathcal{J}^+$ has negative scalar curvature.

Theorem 3.1. *Let $(M, g_{\alpha\beta})$ be a globally hyperbolic spacetime satisfying the Einstein equations (2.1) with cosmological constant $\Lambda = n(n-1)/2$. Assume*

1. *M is of de Sitter type with future conformal completion $(\tilde{M}, \tilde{g}_{\alpha\beta}, \tilde{D}_\alpha, \Omega)$, with future conformal boundary $\mathcal{J}^+$, which is compact.*
2. *The stress energy tensor $S_{\alpha\beta}$ of $(M, g_{\alpha\beta})$ satisfies the strong energy condition (2.2c) and the fall-off condition,*

$$\lim_{p \to \mathcal{J}} [S_{\alpha\beta} \tilde{D}^\alpha \Omega \tilde{D}^\beta \Omega](p) = 0. \tag{3.1}$$

Then if $\mathcal{J}^+$ has negative scalar curvature, every timelike geodesic in $(M, g_{\alpha\beta})$ is past incomplete.

Theorem 3.1 may be viewed as a Lorentzian analogue of the main result of Witten-Yau [28]. The next theorem deals with the case when $\mathcal{J}^+$ has zero scalar curvature. This case is more subtle than the negative scalar curvature case. Here we need to assume that both the strong and dominant energy conditions hold, and in addition we require that M is maximal.

Theorem 3.2. *Let $(M, g_{\alpha\beta})$ be a globally hyperbolic spacetime satisfying the Einstein equations (2.1) with cosmological constant $\Lambda = n(n-1)/2$. Assume that in addition to conditions 1 and 2 of Theorem 3.1, the following holds.*

3. *$(M, g_{\alpha\beta})$ satisfies the dominant energy condition.*
4. *$(M, g_{\alpha\beta})$ is maximal among all spacetimes satisfying conditions 1–3.*

Then, if $\mathcal{J}^+$ has zero scalar curvature, either $(M, g_{\alpha\beta})$ contains no past complete timelike geodesic, or else $(M, g_{\alpha\beta})$ is isometric to the warped product with line element

$$ds^2 = -d\tau^2 + e^{2\tau} h_{ab} dx^a dx^b \tag{3.2}$$

with h_{ab} a Ricci flat metric on $\mathcal{J}^+$. In particular, $(M, g_{\alpha\beta})$ satisfies the vacuum Einstein equations with cosmological constant $\Lambda = n(n-1)/2$.

Remark 3.1. The condition (3.1) may be weakened to

$$\lim_{p\to\mathcal{J}} [\Omega^2 S_{\alpha\beta}\tilde{D}^\alpha\Omega\tilde{D}^\beta\Omega](p) = 0$$
$$\liminf_{p\to\mathcal{J}} [S_{\alpha\beta}\tilde{D}^\alpha\Omega\tilde{D}^\beta\Omega](p) \geq 0$$

Remark 3.2. It follows from Proposition 2.1 that the Cauchy surfaces of M are homeomorphic to $\mathcal{J}^+$, and in particular are compact.

Remark 3.3. Let N^n be a smooth compact manifold of dimension $n \geq 3$. By definition, N is of Yamabe type -1, if N admits a metric of constant negative scalar curvature, but not of zero or constant positive curvature; N is of Yamabe type 0 if N admits a metric of zero scalar curvature, but not constant positive scalar curvature; N is of Yamabe type $+1$ if N admits a metric of constant positive curvature. The definition of Yamabe type partitions the class of all compact n-manifolds into these three sub-classes. Thus, according to Theorem 3.1, if $\mathcal{J}^+$ is of Yamabe type -1 then all timelike geodesics in M are past incomplete. Or, to change the viewpoint slightly, if M contains a past complete timelike geodesic, then the Yamabe type of $\mathcal{J}^+$ must be 0 or $+1$, and is 0, only if M splits as a warped product, as described. □

Theorems 3.1 and 3.2 shall be obtained as consequences of the following basic singularity theorem, and a rigid version of it, for spacetimes obeying the energy condition (2.2c).

Proposition 3.3. *Let M^{n+1} be a spacetime satisfying the energy condition,*

$$\mathrm{Ric}\,(V, V) = R_{\alpha\beta}V^\alpha V^\beta \geq -n$$

for all unit timelike vectors V^α. Suppose that M has a smooth compact Cauchy surface N with mean curvature H satisfying $H > n$. Then every timelike geodesic in M is past incomplete.

Remark 3.4. By our sign conventions, $H = \mathrm{div}_N T = D_a T^a$, where T is the future pointing unit normal along N. Proposition 3.3 is an extension of an old singularity theorem of Hawking to the case of negative Ricci curvature.

Proof. Fix $\delta > 0$ so that the mean curvature of N satisfies $H \geq n(1+\delta)$. Let $\rho : I^-(N) \to \mathbb{R}$ be the Lorentzian distance function to N,

$$\rho(x) = d(x, N) = \sup_{y\in N} d(x, y)\,; \tag{3.3}$$

ρ is continuous and smooth outside the past focal cut locus of N. We will show that ρ is bounded from above,

$$\rho(x) \leq \coth^{-1}(1+\delta) \qquad \text{for all } x \in I^-(N)\,. \tag{3.4}$$

This implies that every past inextendible timelike curve with future end point on N has length $\leq \coth^{-1}(1+\delta)$.

Suppose to the contrary, there is a point $q \in I^-(N)$ such that $d(q,N) = \ell > \coth^{-1}(1+\delta)$. Let $\gamma : [0,\ell] \to M$, $t \to \gamma(t)$, be a past directed unit speed timelike geodesic from $p \in N$ to q that realizes the distance from q to N. γ meets N orthogonally, and, because it maximizes distance to N, ρ is smooth on an open set U containing $\gamma \setminus \{q\}$. For $0 \leq t < \ell$, the slice $\rho = t$ is smooth near $\gamma(t)$; let $H(t)$ be the mean curvature, with respect to the future pointing normal $D^\alpha \rho$, at $\gamma(t)$ of the slice $\rho = t$.

$H = H(t)$ obeys the traced Riccati (Raychaudhuri's) equation,

$$H' = \mathrm{Ric}(\gamma', \gamma') + |K|^2 , \tag{3.5}$$

where $' = d/dt$ and $|K|^2 = K_{ab}K^{ab}$ is the square of the second fundametal form K_{ab} of N. Equation (3.5), together with the inequalities $|K|^2 \geq (\mathrm{tr}K)^2/n = H^2/n$, $\mathrm{Ric}(\gamma', \gamma') \geq -n$ and $H(0) \geq n(1+\delta)$, implies that $\mathcal{H}(t) := H(t)/n$ satisfies,

$$\mathcal{H}' \geq \mathcal{H}^2 - 1, \qquad \mathcal{H}(0) \geq 1 + \delta . \tag{3.6}$$

By an elementary comparison with the unique solution to: $h' = h^2 - 1$, $h(0) = 1 + \delta$, we obtain $\mathcal{H}(t) \geq \coth(a - t)$, where $a = \coth^{-1}(1+\delta) < \ell$, which implies that $\mathcal{H} = \mathcal{H}(t)$ is unbounded on $[0, a)$, contradicting the fact that $\mathcal{H}$ is smooth on $[0, \ell)$. □

Proposition 3.3 admits the following rigid generalization.

Proposition 3.4. *Let M^{n+1} be a spacetime satisfying the energy condition,*

$$\mathrm{Ric}\,(V, V) = R_{\alpha\beta}V^\alpha V^\beta \geq -n \tag{3.7}$$

for all unit timelike vectors V^α, and suppose M has a smooth compact Cauchy surface N with mean curvature H satisfying $H \geq n$. If there exists at least one past complete timelike geodesic in M, then a neighborhood of N in $J^-(N)$ is isometric to $(-\epsilon, 0] \times N$, with warped product metric $ds^2 = -d\tau^2 + e^{2\tau}h_{ab}dx^a dx^b$, where h_{ab} is the induced metric on N. If the timelike geodesics orthogonal to N are all past complete, then this warped product splitting extends to all of $J^-(N)$.

Proof. The proof method we employ is standard. Let h_{ab} be the induced metric on N, and let K_{ab} be the second fundamental form of N, $K_{ab} = -D_a T_b$, where T is the past pointing unit normal along N.

Let $t \to N_t$ be a variation of $N_0 = N$, with variation vector field ϕT, where ϕ is a smooth function on N. Let $H = H_t$ be the mean curvature function of N_t. A standard computation gives

$$\frac{\partial H}{\partial t}\Big|_{t=0} = -\Delta\phi + (R_{TT} + \frac{H_0^2}{n} + \sigma_{ab}\sigma^{ab})\phi\,, \tag{3.8}$$

where $R_{TT} = \mathrm{Ric}(T,T)$, and σ_{ab} is the trace free part of K_{ab}, $\sigma_{ab} = K_{ab} - \frac{H_0}{n}h_{ab}$. In view of the energy condition and the fact that $H_0 \geq n$, the quantity $R_{TT} + \frac{H_0^2}{n} + \sigma_{ab}\sigma^{ab}$ is nonnegative. If it were positive at some point then, by standard results, there would exist a function ϕ for which the right hand side of (3.8) were strictly positive. Since $H_0 \geq n$, this would imply that for small $t > 0$, $H_t > n$. Proposition 3.3 would then imply that all timelike geodesics are past incomplete, contrary to assumption. Thus, σ_{ab} must vanish along N, and hence N is totally umbilic, $K_{ab} = h_{ab}$ and $H = n$.

Now introduce Gaussian normal coordinates in a neighborhood U of N in $J^-(N)$,

$$U = [0,\epsilon) \times N, \quad ds^2 = -du^2 + h_{ab}(u)dx^a dx^b\,. \tag{3.9}$$

Let $K_{ab} = K_{ab}(u)$ and $H = H_u$ be the second fundamental form and mean curvature, respectively, of the u-slice N_u. $H = H_u$ obeys the traced Riccati equation,

$$\frac{\partial H}{\partial u} = R_{uu} + |K|^2\,, \tag{3.10}$$

where R_{uu} is the Ricci tensor contracted with the coordinate vector ∂_u. Since $R_{uu} \geq -n$, $|K|^2 \geq H^2/n$, and $H_0 = n$, it follows that $\mathcal{H} := H/n$ satisfies,

$$\frac{\partial \mathcal{H}}{\partial u} \geq \mathcal{H}^2 - 1, \quad \mathcal{H}(0) = 1\,, \tag{3.11}$$

which by an elementary comparison, implies $\mathcal{H} \geq 1$ on U. Hence, $H|_{N_u} \geq n$ for all $u \in [0,\epsilon)$. But the argument above then implies that each N_u is totally umbilic, $K_{ab}(u) = h_{ab}(u)$ for each u. Since $K_{ab} = -\frac{1}{2}\frac{\partial h_{ab}}{\partial u}$, we obtain the warped product splitting of U,

$$ds^2 = -du^2 + e^{-2u}h_{ab}(0)dx^a dx^b\,, \tag{3.12}$$

which, upon the substitution $\tau = -u$, yields the local warped product splitting asserted in the proposition. If the normal geodesics to N are all past complete then this splitting can be extended indefinitely to the past. □

We now proceed to the proofs of Theorems 3.1 and 3.2.

Proof of Theorem 3.1. We will begin by proving that (3.1) implies $\tilde{D}_\gamma\Omega\tilde{D}^\gamma\Omega\big|_{\mathcal{I}} = -1$. To see this, note that $\tilde{D}^\alpha\Omega\big|_{\mathcal{I}}$ is perpendicular to $\mathcal{I}^+$, and is past oriented. Introduce a coordinate system $(x^\alpha) = (s, x^a)$ near $\mathcal{I}^+$ so that $\partial_s\big|_{\mathcal{I}}$ agrees with $\tilde{D}^\alpha\Omega\partial_{x^\alpha}\big|_{\mathcal{I}}$. Then $Y^\alpha = \Omega(\partial_s)^\alpha = \Omega\tilde{D}^\alpha\Omega + O(\Omega^2)$. The Ricci tensor $R_{\alpha\beta}$ of $(M, g_{\alpha\beta})$ and the Ricci tensor $\tilde{R}_{\alpha\beta}$ of $(M, \tilde{g}_{\alpha\beta})$ are related by

$$R_{\alpha\beta} = \tilde{R}_{\alpha\beta} + \Omega^{-1}[(n-1)\tilde{D}_\alpha\tilde{D}_\beta\Omega + \tilde{D}_\gamma\tilde{D}^\gamma\Omega\tilde{g}_{\alpha\beta}] - n\Omega^{-2}\tilde{D}_\gamma\Omega\tilde{D}^\gamma\Omega\tilde{g}_{\alpha\beta} \tag{3.13}$$

A computation shows that

$$S_{\alpha\beta}Y^\alpha Y^\beta = \left[\frac{n(n-1)}{2}\tilde{D}_\gamma\Omega\tilde{D}^\gamma\Omega + \Lambda\right] g_{\alpha\beta}Y^\alpha Y^\beta + O(\Omega)$$

and hence (3.1) implies

$$\tilde{D}_\gamma\tilde{D}^\gamma\Omega\big|_{\mathcal{I}} = -1. \tag{3.14}$$

Let $\tilde{h}^0_{ab}$ be the metric on $\mathcal{I}^+$ induced from $\tilde{g}_{\alpha\beta}$. By the Yamabe theorem, there is a positive function θ on $\mathcal{I}^+$ such that the scalar curvature $\bar{r}^0$ of $\theta^{-2}\tilde{h}^0_{ab}$ equals -1, 0, or 1 on $\mathcal{I}^+$. Further, there is a neighborhood U of $\mathcal{I}^+$, and a conformal gauge transformation Θ with $\Theta\big|_{\mathcal{I}} = \theta$, such that after replacing Ω by $\bar{\Omega} = \Omega\Theta^{-1}$, and $\tilde{g}_{\alpha\beta}$ by $\bar{g} = \Theta^{-2}\tilde{g}_{\alpha\beta}$, we have

$$\Omega(p) = d_{\bar{g}}(p, \mathcal{I}^+)$$

on U where $d_{\bar{g}}$ denotes the Lorentz distance to $\mathcal{I}^+$. This is achieved, following [4, §5] by solving the equation

$$-1 = \bar{g}^{\alpha\beta}D_\alpha\bar{\Omega}D_\beta\bar{\Omega} \tag{3.15}$$

By (3.14) the function $a = \Omega^{-1}(1 + \tilde{g}^{\alpha\beta}D_\alpha\Omega D_\beta\Omega)$ is in $C^1(\tilde{M})$. A computation shows that (3.15) is equivalent to the system

$$2\Theta\tilde{g}^{\alpha\beta}D_\alpha\Theta D_\beta\Omega - \Omega\tilde{g}^{\alpha\beta}D_\alpha\Theta D_\beta\Theta = \Theta^2 a$$

This equation with initial data $\Theta = \theta$ on $\mathcal{I}^+$, has a unique solution in a neighborhood of $\mathcal{I}^+$ [21, pp. 39-40]. In a sufficiently small neighborhood U of $\mathcal{I}^+$, the solution is positive, and we continue this to a positive function Θ on all of M for which $\bar{g}^{\alpha\beta}D_\alpha\bar{\Omega}D_\beta\bar{\Omega} = -1$ on U. This implies that the gradient curves of $\bar{\Omega}$ on U are unit-speed timelike geodesics with respect to $\bar{g}$ and since $\bar{\Omega} = 0$ on $\mathcal{I}^+$, we have $\bar{\Omega}(p) = d_{\bar{g}}(p, \mathcal{I}^+)$. Finally, we rename $\bar{g}_{\alpha\beta}, \bar{\Omega}$ to $\tilde{g}_{\alpha\beta}, \Omega$. By construction, the metric $\tilde{h}^0_{ab}$ induced on $\mathcal{I}^+$ by $\tilde{g}_{\alpha\beta}$ has scalar curvature $\tilde{r}^0 = \bar{r}^0$.

Letting $t = \Omega$, so that t *increases to the past* near $\mathcal{I}^+$, the foliation of level sets N_t of t is the Gaussian foliation with respect to $\mathcal{I}^+$ on U. Let h_{ab}, r, K_{ab}, H be the induced metric on N_t, its scalar curvature function, the second fundamental form of N_t and the mean curvature $H = h^{ab}K_{ab}$, respectively. Here $K_{ab} = -D_a T_b$ is the second fundamental form of N_t defined with respect to the past directed timelike normal T to N_t, so that $K_{ab} = -\frac{1}{2}\mathcal{L}_T g_{ab}$. Similarly, $\tilde{h}_{ab}, \tilde{r}, \tilde{K}_{ab}, \tilde{H}$ are the metric, scalar curvature, second fundamental form, and mean curvature of N_t, defined with respect to the conformally rescaled metric $\tilde{g}_{\alpha\beta}$.

By the above, we may without loss of generality assume that on U, $g_{\alpha\beta}$ is of the form

$$g_{\alpha\beta}dx^\alpha dx^\beta = \frac{1}{t^2}(-dt^2 + \tilde{h}_{ab}dx^a dx^b) \tag{3.16}$$

where $\tilde{h}_{ab} = \tilde{h}_{ab}(t,x)$, $\tilde{h}_{ab}(0,x) = \tilde{h}^0_{ab}(x)$ and the scalar curvature $\tilde{r}^0$ of $\tilde{h}^0_{ab}$ is constant $= -1, 0, +1$. With this form for $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$, we have $T = t\partial_t$. We will use an index T to denote contraction with T, for example $u_T = u_\alpha T^\alpha$, and an index 0 for contraction with ∂_t, for example $u_0 = u_\alpha(\partial_t)^\alpha$.

To prove Theorem 3.1, it is sufficient to show, assuming $\tilde{r}_0 = -1$, that $H|_{N_t} > n$ for some $t > 0$, for then Proposition 3.3 applies. The mean curvature functions H and $\tilde{H}$ are related by

$$H = t\tilde{H} + n\,. \tag{3.17}$$

In particular, $H\big|_{N_t} > 0$ for t sufficiently small. The Gauss equation (in the physical metric $g_{\alpha\beta}$) applied to each N_t, together with the Einstein equation, yields the constraint,

$$\begin{aligned} H^2 &= 2S_{TT} + 2\Lambda + |K|^2 - r \\ &= 2t^2 S_{00} + n(n-1) + |K|^2 - t^2\tilde{r}\,, \end{aligned} \tag{3.18}$$

which, since $|K|^2 \geq H^2/n$, implies,

$$H^2 \geq n^2 + \frac{n}{n-1}t^2(S_{00} - \tilde{r})\,. \tag{3.19}$$

Using $\tilde{r}_0 = -1$ and the energy condition (3.1), the above inequality implies $H|_{N_t} > n$ for all $t > 0$ sufficiently small. □

Finally, we give the proof of Theorem 3.2.

Proof of Theorem 3.2. Let the notation be as in the proof of Theorem 3.1. We will show, by adapting an argument in [1] to the Lorentzian setting, that

$H|_{N_t} \geq n$ for each $t > 0$, so that Propsition 3.4 may be applied. To this end, we first show that the quantity $t^{-1}\tilde{H}$ is nondecreasing along the flow lines of ∂_t.

The conformal transformation rule for the Ricci tensor, shows that on U,

$$R_{00} = \tilde{R}_{00} - t^{-1}\tilde{H} - t^{-2}n \tag{3.20a}$$

$$R = n(n+1) + 2nt\tilde{H} + t^2\tilde{R} \tag{3.20b}$$

The traced Riccati (Raychaudhuri) equation in the conformally rescaled metric $\tilde{g}_{\alpha\beta}$ is

$$\partial_t\tilde{H} = |\tilde{K}|^2 + \tilde{R}_{00}. \tag{3.21}$$

Using (3.20a), (3.21) gives,

$$\begin{aligned} \partial_t(t^{-1}\tilde{H}) &= t^{-1}[|\tilde{K}|^2 + t^{-2}(R_{TT} + n)] && (3.22)\\ &\geq 0, && (3.23)\end{aligned}$$

since, by the energy condition (2.3), $R_{TT} + n \geq 0$. Integrating from ϵ to t along each flow line of ∂_t, and letting $\epsilon \to 0$, gives,

$$t^{-1}\tilde{H}(t) \geq \liminf_{\epsilon\to 0} \epsilon^{-1}\tilde{H}(\epsilon). \tag{3.24}$$

The contracted Gauss equation (in the unphysical metric $\tilde{g}_{\alpha\beta}$) states

$$\tilde{r} + \tilde{H}^2 - |\tilde{K}|^2 = 2\tilde{R}_{00} + \tilde{R}.$$

Using (3.20) we find after some manipulations, that on U,

$$2\tilde{R}_{00} + \tilde{R} = 2t^{-2}S_{TT} - 2(n-1)t^{-1}\tilde{H}$$

The previous two equations imply,

$$2(n-1)t^{-1}\tilde{H}\left(1 + \frac{t\tilde{H}}{2(n-1)}\right) = |\tilde{K}|^2 - \tilde{r} + 2t^{-2}S_{TT}.$$

Setting $t = \epsilon$ in the above, and letting $\epsilon \to 0$, while making use of the asymptotic form of the weak energy condition (3.1), and the fact that $\tilde{H}$ is bounded, shows that the right hand side of Equation (3.24) is greater than or equal to $-\tilde{r}_0/2(n-1)$. Since we are now considering the case $\tilde{r}_0 = 0$, we conclude that there exists $c > 0$ sufficiently small so that $\tilde{H}|_{N_t} \geq 0$ for $t \in (0, c]$. Equation (3.17) then implies $H|_{N_t} \geq n$ for each $t \in (0, c]$. Let $M_c = \{p \in M, t(p) \in (0, c]\}$.

By applying Proposition 3.4 to $N_t \subset M$ for $t \in (0, c]$, we conclude that the line element on M_c is of the form (3.2) and hence is conformal to $-dt^2 + \tilde{h}_{ab}dx^a dx^b$. Taking into account the assumption that $\mathcal{I}^+$ is a regular conformal boundary, it follows that $\tilde{h}_{ab} = \tilde{h}^0_{ab}$, the metric on $\mathcal{I}^+$. Hence, for $t \in (0, c]$, N_t with induced metric h_{ab} has zero scalar curvature and the second fundamental form of N_t satisfies $K_{ab} = h_{ab}$. The Hamiltonian constraint

$$r + (\mathrm{tr}K)^2 - |K|^2 = 2\Lambda + S_{TT}$$

then implies

$$S_{TT} = 0$$

By assumption the dominant energy condition holds, and hence by the conservation theorem [17, §4.3], $S_{\alpha\beta} = 0$ in the domain of dependence of N_c. By construction, N_c is a Cauchy surface in M, so we find that $S_{\alpha\beta} = 0$ in M. A calculation shows that a warped product line element of the form (3.2) satisfies the vacuum Einstein equations with cosmological constant $\Lambda = n(n-1)/2$ only if $\mathrm{Ric}[\tilde{h}^0_{ab}] = 0$.

Summarizing our conclusions so far, we have that $(M, g_{\alpha\beta})$ satisfies the vacuum Einstein equations with cosmological constant Λ and we have a Cauchy surface N_c in M with induced data equivalent to that of a hypersurface in the warped product spacetime with line element given by (3.2). By assumption $(M, g_{\alpha\beta})$ is maximal. The global splitting asserted in Theorem 3.2 is a consequence of uniqueness of the maximal Cauchy development for the Einstein equations, see [8] for a discussion of the $\Lambda = 0$ case. □

4 Spacetimes of de Sitter type and the null energy condition

In this section we obtain restrictions on the topology of spacetimes of de Sitter type which obey the null energy condition.

Theorem 4.1. *Let $(M, g_{\alpha\beta})$ be a spacetime of de Sitter type with regular past and future conformal boundaries $\mathcal{I}^\pm$. Assume that $(M, g_{\alpha\beta})$ is future or past asymptotically simple and satisfies the null energy condition (2.4). Then $(M, g_{\alpha\beta})$ is globally hyperbolic, and the Cauchy surfaces of M are compact with finite fundamental group.*

Remark 4.1. Put another way, given a globally hyperbolic spacetime M of de Sitter type with regular $\mathcal{I}^\pm$, which obeys the null energy condition, if the fundamental group of the Cauchy surfaces of M is infinite then M can be neither past nor future asymptotically simple. This is illustrated by the spatially closed version of Schwarzschild-de Sitter spacetime, whose Cauchy surfaces have topology $S^1 \times S^2$. Asymptotic simplicity fails in this model due to the presence of a black hole and a white hole.

Proof. For definiteness, assume M is future asymptotically simple. By Proposition 2.1, M is globally hyperbolic, with Cauchy surfaces homeomorphic to $\mathcal{J}^+$. Then $\tilde{M}$ can be extended a little beyond $\mathcal{J}^\pm$ to obtain a spacetime (without boundary) P, with $\tilde{M} \subset P$, such that the Cauchy surfaces for M are also Cauchy surfaces for P, so that P is globally hyperbolic.

Consider the achronal boundary $\partial I^+(p,P)$, which is an achronal C^0 hypersurface in P. We claim that $\partial I^+(p,P)$ is compact. Suppose not. By the global hyperbolicity of P, $\partial I^+(p,P) = J^+(p,P) \setminus I^+(p,P)$, and hence the null geodesic generators of $\partial I^+(p,P)$ extend back to the point p. Using the noncompactness of $\partial I^+(p,P)$, one easily constructs a future directed null geodesic $\gamma \subset \partial I^+(p,P)$ starting at p, which is future inextendible in P. In particular, γ meets $\mathcal{J}^+$ at a point q, say, and enters the interior of $D^+(\mathcal{J}^+,P)$.

Let η be the portion of γ from p to q, excluding these end points. Then η is a *null line*, i.e., a complete achronal null geodesic in $(M, g_{\alpha\beta})$. Observe that $I^+(\eta,M) = I^+(p,P) \cap M$, from which it follows that $\partial I^+(\eta,M) = \partial I^+(p,P) \cap M$ (where $\partial I^+(A,X)$ refers to the boundary in X). It follows that the generators of the achronal boundary $\partial I^+(\eta,M)$ extend back to p and hence are past complete in $(M,g_{\alpha\beta})$. By the time-dual of these arguments, we have that $\partial I^-(\eta,M) = \partial I^-(q,P) \cap M$, and that the null generators of $\partial I^-(\eta,M)$ are future complete in $(M,g_{\alpha\beta})$. Then, since the null energy condition holds, we may apply the null splitting theorem [15] to conclude that $\partial I^+(\eta,M)$ and $\partial I^-(\eta,M)$ agree, and, in fact, form a smooth achronal edgeless totally geodesic null hypersurface in M. Hence, $\partial I^+(p,P) \cap M = \partial I^-(q,P) \cap M$, from which it follows that the null generators of $\partial I^+(p,P)$ reconverge, and, by the achronality of $\partial I^+(p,P)$, terminate at q. But this contradicts the fact that the generator γ enters the interior of $D^+(\mathcal{J}^+,P)$.

Thus, $\partial I^+(p,P)$ is compact, and, by standard results [6, 14], is a Cauchy surface for P. So the Cauchy surfaces of P, and hence, the Cauchy surfaces of M are compact. Now let M^* denote the universal covering spacetime of M. M^* will be globally hyperbolic; in fact if S is a Cauchy surface for M, so that $M \approx \mathbb{R} \times S$, then $M^* \approx \mathbb{R} \times S^*$, where S^* is the universal cover of S and each slice $\{t\} \times S^*$ is a Cauchy surface for M^*. It is easily seen that the assumptions on M in the theorem lift to M^*. Then from the above, we conclude that the Cauchy surfaces of M^*, and hence S^*, are compact. It follows that S^* is a finite cover of S, and hence S has a finite fundamental group. □

The following corollary to Theorem 4.1, is an immediate consequence of Theorem 4.1, and Proposition 2.1. It replaces the assumption of asymptotic simplicity with other natural assumptions.

Corollary 4.2. *Let $(M, g_{\alpha\beta})$ be a globally hyperbolic spacetime of de Sitter type with regular past and future conformal boundaries $\mathcal{I}^{\pm}$. Assume that $(M, g_{\alpha\beta})$ obeys the null energy condition, and that $\mathcal{I}^{+}$ (or $\mathcal{I}^{-}$) is compact. Then the Cauchy surfaces of M, which by Proposition 2.1 are homeomorphic to $\mathcal{I}^{+}$ (or $\mathcal{I}^{-}$), have finite fundamental group.*

We conclude with the following theorem.

Theorem 4.3. *Let $(M^{n+1}, g_{\alpha\beta})$, $n \leq 7$, be a globally hyperbolic spacetime of de Sitter type with regular future conformal boundary $\mathcal{I}^{+}$. Assume that $(M^{n+1}, g_{\alpha\beta})$ obeys the null energy condition, and that $\mathcal{I}^{+}$ is compact and orientable. If M is past null geodesically complete then the Cauchy surfaces of M, which by Proposition 2.1, are homeomorphic to $\mathcal{I}^{+}$, have vanishing co-dimension one homology, i.e., $H_{n-1}(N, \mathbb{Z}) = 0$, N a Cauchy surface for M. In particular, there can be no worm holes in N.*

Proof. The proof is an application of the Penrose singularity theorem [17, 20] applied to a suitable covering spacetime of M.

As in the proof of Theorem 3.1, introduce coordinates so that the physical metric $g_{\alpha\beta}$ takes the form of Equation (3.16). The second fundamental forms K_{ab} and $\tilde{K}_{ab}$, of the t-slices N_t (notation as in the proof of Theorem 3.1) are related by

$$K_{ab} = t^{-1}\tilde{K}_{ab} + g_{ab}\,. \tag{4.1}$$

Let $\tilde{X}$ be a $\tilde{g}$-unit vector field defined in a neighborhood U of a point $p \in \mathcal{I}^{+}$, which is everywhere orthogonal to ∂_t. Then $X = t^2\tilde{X}$ is a g-unit vector field defined on $U \setminus \mathcal{I}^{+}$, everywhere orthogonal to ∂_t. ¿From (4.1), we have

$$K_{ab}X^aX^b = t\,\tilde{K}_{ab}\tilde{X}^a\tilde{X}^b + 1\,, \tag{4.2}$$

which is positive for t sufficiently small. Hence, for t sufficiently small, $K_{ab} = -D_aT_b$ is positive definite along N_t.

Thus, by fixing t_0 sufficiently small, there exists a compact Cauchy surface $N = N_{t_0}$ for M which is strictly convex to the past, i.e. for which D_aT_b is negative definite along N, where T is the past pointing unit normal along N. Suppose $H_{n-1}(N, \mathbb{Z}) \neq 0$. By well known results of geometric measure theory (see [18, p. 51], for discussion), every nontrivial class in $H_{n-1}(N, \mathbb{Z})$

has a least area representative which can be expressed as a linear combination of smooth, orientable, connected, compact, embedded minimal (mean curvature zero) hypersurfaces in N. Let Σ be such a hypersurface; we may assume Σ represents a nontrivial element of $H_{n-1}(N,\mathbb{Z})$. Note Σ is spacelike and has co-dimension two in M. As described in [13], since Σ is minimal in N, and N is strictly convex to the past in M, Σ must be a past trapped surface in M, i.e., the two families of past directed null geodesics issuing orthogonally from Σ are converging in the mean along Σ.

The next step is to construct a certain covering spacetime M^*. Since N is a Cauchy surface for M, so that $M \approx \mathbb{R} \times N$, each covering space N^* of N gives rise, in an essentially unique way, to a covering spacetime M^* of M, such that $M^* \approx \mathbb{R} \times N^*$, where the slices $\{t\} \times N^*$ are Cauchy surfaces for M^*. Since Σ is two-sided, loops in N have a well-defined oriented intersection number with respect to Σ. The intersection number is a homotopy invariant, and so gives rise to a well-defined subgroup G of $\Pi_1(N)$, corresponding to the loops in N having zero intersection number with respect to Σ. N^* is defined to be the covering space of N associated with the subgroup G, i.e., satisfying $\pi_*(\Pi_1(N^*)) = G$, where $\pi : N^* \to N$ is the covering map. N^* has a simple description in terms of cut-and-paste operations. Σ does not separate N, for otherwise it would bound in N. By making a cut along Σ, we obtain a compact manifold N' with two boundary components, each isometric to Σ. Taking $\mathbb{Z}$ copies of N', and gluing these copies end-to-end we obtain the covering space N^* of N. The inverse image $\pi^{-1}(\Sigma)$ consists of $\mathbb{Z}$ copies of Σ, each one separating N^*. Let $\Sigma_0 \subset N^*$ denote one of these copies.

As per the discussion above, there exists a covering spacetime $M^* \approx \mathbb{R}\times N^*$, with Cauchy surfaces homeomorphic to N^*. Since the covering map is a local isometry, the assumptions that M obeys the null energy condition and is past null geodesically complete lift to M^*. Moreover, Σ_0 will be a past trapped surface in M^*. Then, according to the Penrose singularity theorem (cf., [17, Theorem 1] or [20, Theorem 61]), the achronal boundary $\partial I^-(\Sigma_0)$ is a *compact* Cauchy surface for M^*. This implies that N^* is compact, and hence, a finite covering of N, which is a contradiction. We conclude that $H_{n-1}(N,\mathbb{Z}) = 0$. □

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