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Duality and Confinement in $\mathcal{N} = 1$ Supersymmetric Theories from Geometric Transitions

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Abstract

We study large N dualities for a general class of $\mathcal{N} = 1$ theories realized on type IIB D5 branes wrapping 2-cycles of local Calabi-Yau threefolds or as effective field theories on D4 branes in type IIA brane configurations. We completely solve the issue of the classical moduli space for $\mathcal{N} = 2, \prod_{i=1}^{n} U(N_i)$ theories deformed by a general superpotential for the adjoint and bifundamental fields. The $\mathcal{N} = 1$ geometries in type IIB and its T-dual brane configurations are presented and they agree with the field theory analysis. We investigate the geometric transitions in the ten dimensional theories as well as in M-theory. Strong coupling effects in field theory are analyzed in the deformed geometry with fluxes. Gluino condensations are identified the normalizable deformation parameters while the vacuum expectation values of the bifundamental fields are with the non-normalizable ones. By lifting to

1 Introduction

Duality is one of the most fascinating aspects in string theory. Recently, geometric transitions became an important tool in understanding large N dualities between open string theory and closed string theory. In geometric transitions, one begins with D-branes wrapped around cycles of a Calabi-Yau threefold and the field theory is described in the small 't Hooft parameter region. After a geometric transition of the Calabi-Yau threefold to another Calabi-Yau threefold, the D-branes disappear and they are replaced by RR and NS fluxes through the dual cycles, the field theory being described in the large 't Hooft parameter region.

The large N duality between Chern-Simons theory on the $\mathbf{S^3}$ cycle of the deformed conifold and topological closed strings on the resolved conifold was observed in [1] and the result was embedded in type IIA strings by Vafa [2]. The topological transition becomes a geometric transition between D6 branes on the $\mathbf{S^3}$ of the deformed conifold and type IIA strings on the resolved conifold, with fluxes or between D5 branes wrapped on the $\mathbf{P}^1$ cycle of the resolved conifold and type IIB on the deformed conifold with fluxes. The type IIB formulation has been extended to a large class of geometric transition dualities, for geometries which are more complicated than the conifold [3, 4, 5, 6, 7, 8, 9, 10, 11].

In the present work we explore the large N dualities from type IIB, type IIA and M-theory perspectives for large classes of the $\mathcal{N} = 1$ supersymmetric gauge theories by wrapping D5 branes on blown-up $\mathbf{P}^1$ cycles in Calabi-Yau threefolds which are obtained by deforming resolved ALE spaces of the ADE singularities. We begin with the $\mathcal{N} = 2$ quiver gauge associated with the ADE Dynkin diagrams, which can be geometrically engineered on the resolved ALE spaces. Then the $\mathcal{N} = 1$ theory is obtained by adding a superpotential for the adjoint fields, $W_i(\Phi_i)$, thus the full tree-level superpotential is

$$W = \sum_{i=1}^{n} W_i - \mathrm{Tr} \sum_{i=1}^{n} \sum_{j=1}^{n} s_{i,j} Q_{i,j} \Phi_j Q_{j,i}, \quad \text{where } W_i = \mathrm{Tr} \sum_{j=1}^{d_i+1} \frac{g_{i,j-1}}{j} \Phi_i^j, \quad (1)$$

with adjoints Φ and bifundamentals $Q_{i,j}$, which fixes the moduli space of the $\mathcal{N} = 1$ theory. This case has not been explicitly discussed in the field theory literature and we provide full and detailed explanation on how the moduli space of the $\mathcal{N} = 1$ theory appears as a solution of the F-term and D-term equations from $\mathcal{N} = 2$ field theory deformed by the superpotential

(1). In the vacuum, the gauge group is broken as:

$$\prod_{i=1}^{n} U(N_i) \to \prod_{j=1}^{n} \prod_{k=j}^{n} \prod_{l=1}^{d_{j,k}} U(M_{j,k,l}) \tag{2}$$

where $d_{j,k}$ is the maximum of the degrees W_i' for $j \leq i \leq k$.

Having identified the classical moduli, we interpret the expectation values of the adjoints and adjoints in terms of complex and topological structure changes in Calabi-Yau threefolds in type IIB picture, and correspondingly curving of NS branes and fixing of D4 branes in type IIA picture. Moreover, we identify the number of the Higgsed gauge groups in the $\mathcal{N} = 1$ theory with the number of the $\mathbf{P}^1$ cycles in type IIB and the number of intersection points of NS branes in type IIA. These interpretations enable us to write down the $\mathcal{N} = 1$ geometry explicitly, which is a small resolution of the singular threefold given by

$$xy - u \prod_{p=1}^{n} \left(u - \sum_{i=1}^{p} W_i'(v) \right) = 0. \tag{3}$$

The deformations of three dimensional ADE-type of singularities can be divided into two classes in the sense of [45]. In the $\mathcal{N} = 1$ theory, the non-normalizable deformations are fixed by the expectation values of the bifundamentals which result in topological changes of the Calabi-Yau spaces by creating $\mathbf{S}^3$ cycles in type IIB, and in turn non-hyperelliptic curves in M-theory. After the non-normalizable deformations, the gauge groups are decoupled and the geometry has only conifold singularities. Now in the strong coupling regime, each gauge group has a gluino condensation. Via a geometric transition which is obtained by shrinking $\mathbf{P}^1$ cycles and making a normalizable deformation, the $\mathbf{P}^1$ cycles vanish and are replaced by the $\mathbf{S}^3$ cycles with RR, NS fluxes. The gluino condensations are mapped into the sizes of $\mathbf{S}^3$ cycles arising from the normalizable deformation. The NS fluxes, which are related to the couplings of the remaining Abelian gauge fields after gluino condensations, come from the Kähler structure changes of the deformed geometry after the geometric transition while the RR fluxes are originated from the vanished D5 branes.

By taking T-duality along the $U(1)$ direction of a natural $\mathbf{C}^*$ action on the $\mathcal{N} = 1$ geometry (3) given by

$$\lambda \cdot (x, y, u, v) \to (\lambda x, \lambda^{-1} y, u, v) \quad \text{for } \lambda \in \mathbf{C}^*, \tag{4}$$

one can obtain type IIA pictures which have been developed in a series of papers [8, 9, 30]. This T-duality gives the dictionary between the geometric engineering construction and the Hanany-Witten type brane construction. D5 branes on the $\mathbf{P}^1$ cycles become D4 branes on intervals as $U(1)$ acts along the angular direction of the $\mathbf{P}^1$'s and NS branes appear when the $U(1)$ orbits degenerate. When the NS branes are projected onto the $u-v$ plane, they will be a collection of the holomorphic curves given by

$$u\prod_{p=1}^{n}\left(u-\sum_{i=1}^{p}W_i'(v)\right)=0, \tag{5}$$

and their intersection points are in one-to-one correspondence with the Higgsed gauge groups.

By lifting to eleven dimensional M theory, the brane configuration of D4 branes and NS branes in type IIA becomes a single M5 brane represented by a Riemann surface Σ in a complex 3 dimensional space. As the transition occurs in the limit where the sizes of $\mathbf{P}^1$ are very small, the information pertaining to the D4 branes is lost and we obtain an M-theory plane M-theory curve which describes the remaining Abelian theory after confinement. We then go down to 10 dimensions and we get a brane configurations with NS branes which can then be mapped to a deformed geometry with $\mathbf{S}^3$ cycles and fluxes after a T-duality.

Another interesting issue is on how to derive the remaining abelian gauge theories which remain after confinement from the $\mathcal{N}=2$ Seiberg-Witten curves. The point is that it is unknown how to reduce the Seiberg-Witten curve from $\mathcal{N}=2$ to $\mathcal{N}=1$ in the presence of bifundamental fields. Nevertheless, we show how to obtain the $\mathcal{N}=1$ parts and identify with the M-theory curve after the transition.

Finally, we discuss orientifold theories which are obtained by complex conjugation on the geometry, Seiberg dualities by introducing matter fields and birational flops and the relation to G_2 holonomy manifolds via mirror symmetry.

2 Vafa's Large N dualities

We will briefly review some of features of Vafa's $\mathcal{N}=1$ large N dualities which will be used later. Consider type IIB theory on a non-compact Calabi-Yau threefold $O(-1)+O(-1)$ of $\mathbf{P}^1$ which is a small resolution of the conifold :

$$xy-uv=0 \tag{6}$$

by wrapping N D5 branes on $\mathbf{P}^1$. This gives a four dimensional $\mathcal{N} = 1$ $U(N)$ pure Yang-Mills theory described by open strings ending on the D5 branes, in the small 't Hooft parameter regime. Vafa's duality states that in the large N limit (large 't Hooft parameter regime), this is equivalent to type IIB on the deformed conifold:

$$f = xy - uv - \mu = 0. \tag{7}$$

In the deformed conifold, the $\mathbf{P}^1$ cycle is shrunken and replaced by $\mathbf{S}^3$ of size μ which is identified with the condensate of the $SU(N)$ glueball superfield $S = -\frac{1}{32\pi^2}\mathrm{Tr}W_\alpha W^\alpha$. The description is now in terms of closed strings. Rather than the N original D5 branes, there are now N units of RR flux through $\mathbf{S}^3$, and also some NS flux through the non-compact cycle dual to $\mathbf{S}^3$. The glueball S is identified with the flux of the holomorphic 3-form on the compact 3-cycle of the deformed conifold

$$S = \int_A \Omega \tag{8}$$

and the integral of holomorphic 3-form on the noncompact 3-cycle is made with introducing a cut-off Λ_0:

$$\Pi = \int_B^{\Lambda_0} \Omega = \frac{1}{2\pi i}(-3S\log\Lambda_0 - S + S\log S) \tag{9}$$

The effective superpotential is written as

$$W_{\text{eff}} = \int (H_{RR} + \tau H_{NS}) \wedge \Omega \tag{10}$$

where H_{RR} is the RR flux on the A-cycle and is due to the N D5 branes and H_{NS} is the NS flux on the noncompact B-cycle. By using the usual IR/UV identification in the AdS/CFT conjecture, we identify the large distance Λ_0 (small IR scale) in supergravity with the small distance (large UV scale) in the field theory such that the coupling constant in field theory is constant and finite in UV. After doing so, the form of the effective superpotential is

$$W_{\text{eff}} = S\log[\Lambda^{3N}/S^N] + NS \tag{11}$$

The condition of supersymmetry implies that the derivative of W_{eff} with respect to S is zero which implies that S gets N discrete values, separated by a phase. This is the gluino condensation in the field theory and signals the breaking of the chiral symmetry $\mathbf{Z_{2N}} \to \mathbf{Z_2}$. The gluons of $SU(N)$ get a mass so the $SU(N)$ gets a mass gap and confines. What remains is the $U(1)$ part of $U(N)$ whose coupling constant is equal to the coupling constant of the $U(N)$ theory divided by N.

We consider the lift of the transition to M theory by using MQCD [8, 9, 30]. A T-duality of the geometrical constructions takes the N D5 branes wrapped on $\mathbf{P}^1$ to a brane configuration with two orthogonal NS branes on the directions x and y (together with four directions corresponding to the Minkowski space) and N D4 branes in the the direction x_n. The lift to M theory involves a single M5 brane which has the worldvolume $R^{1,3} \times \Sigma$ where Σ is a 2-dim. manifold holomorphically embedded in (x, y, t) where $t = exp(\frac{x_n}{R_{10}} + i\ x_{10})$. When the $\mathbf{P}^1$ cycle shrinks, the direction x_n goes to zero and eventually the coordinate t becomes the coordinate of a circle. Because we cannot embed holomorphically into a circle, it results that the coordinate t of the M5 brane become constant and Σ is embedded inside $x\ y\ =$ const. where the constant is related to the scale of the $U(N)$ theory. After reducing to ten dimensions, $x\ y\ =$ const. becomes the equations for a 2-dimensional surface where an NS brane is wrapped, which is T-dual to the deformed conifold and the constant is related to the size of the $\mathbf{S}^3$ cycle. Therefore we could explicitly see the relation between the scale of the $U(N)$ theory and the size of the $\mathbf{S}^3$ cycle.

The transition has been generalized to more complicated geometries in [3, 4, 5, 6, 7], where the blown-up geometry involves more $\mathbf{P}^1$ cycles and the deformed geometry involves more $\mathbf{S}^3$ cycles. The effective superpotential involves integrals of the holomorphic 3-form on the several $\mathbf{S}^3$ cycles and several noncompact cycles. Because the UV field theory involves several $U(N_i)$ groups, the running of different gauge group couplings gives rise to different bare couplings at the cut-offs on the noncompact cycles which corresponds to different UV scales for the $U(N_i)$ groups. For different $U(N_i)$ groups in the $\mathcal{N} = 2$ theory we have different β-functions and the $\mathcal{N} = 2$ exact β-function for the coupling $\tau_i \equiv \frac{\theta_i}{2\pi} + 4\pi i g_i^{-2}$ of $U(N_i)$ is

$$\beta_i \equiv -2\pi i \beta(\tau_i) = \sum_j C_{ij} N_j, \tag{12}$$

with $C_{ij} = 2\delta_{ij} - |s_{ij}| = \vec{e}_i \cdot \vec{e}_j$ the Cartan matrix of the A-D-E diagram.

3 $\mathcal{N} = 2$ A-D-E Quiver Gauge Theories

One can associate $\mathcal{N} = 2$ supersymmetric gauge theories with gauge group $\prod_{i=1}^n U(N_i)$ to each Dynkin diagram of the simple complex Lie algebras of type $\mathbf{A}_n, \mathbf{D}_n, \mathbf{E}_n$ (Figure 1). Each factor $U(N_i)$ corresponds to a vertex v_i in the Dynkin diagram and bifundamental hypermultiplet $Q_{i,j}$ correspond to an edge from v_i to v_j. The bifundamental $Q_{i,j}$ is in the $(\mathbf{N}_i, \bar{\mathbf{N}}_j)$ representation of $U(N_i) \times U(N_j)$.

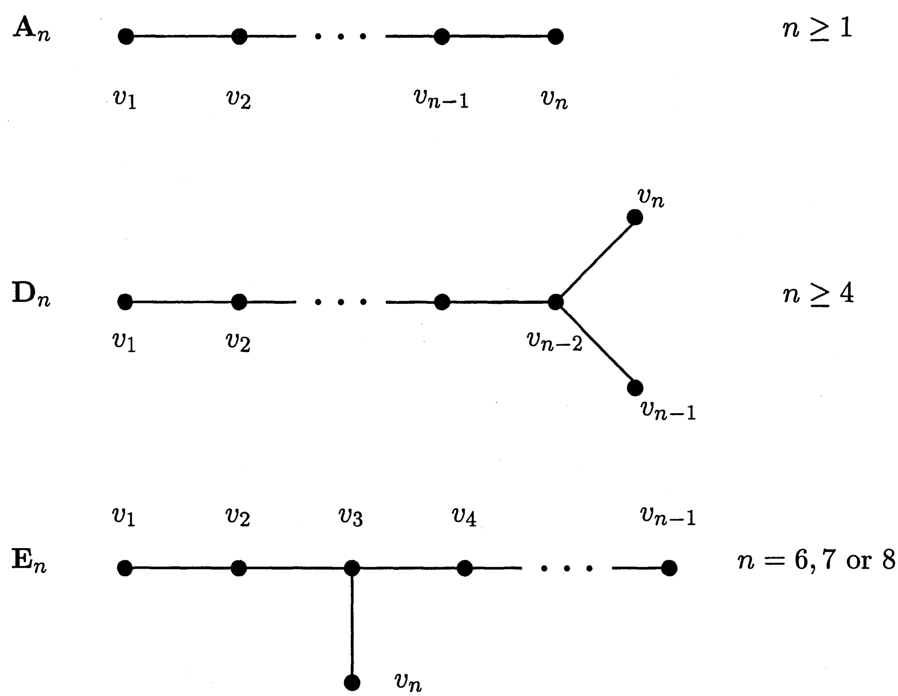

Figure 1: A-D-E Dynkin Diagrams

The A-D-E Dynkin diagram also arise as a resolution graph of two dimensional quotient singularity $(\mathbf{C}^2/G, 0)$ by a finite subgroup $G \subset SU(2)$, which is called A-D-E singularity. The A-D-E singularity can be blown-up to a smooth ALE space where the singular point is replaced by a configuration of rational curves $\mathbf{P}^1$. The configuration can be explained in terms of the resolution graph which consists of n vertices corresponding to the rational curves $\mathbf{P}^1$ and edges between the vertices when the corresponding $\mathbf{P}^1$'s intersect. The intersection matrix of the resolution graph is the negative of the Cartan matrix. The A-D-E singularities can be embedded as hypersufaces $f(x, y, u) = 0$ in $\mathbf{C}^3$:

$$\begin{array}{ll} \mathbf{A}_n: & f = xy + u^{n+1} \\ \mathbf{D}_n: & f = x^2 + y^2 u + u^{n-1} \\ \mathbf{E}_6: & f = x^2 + y^4 + u^3 \\ \mathbf{E}_7: & f = x^2 + uy^3 + u^3 \\ \mathbf{E}_8: & f = x^2 + y^5 + u^3 \end{array} \tag{13}$$

Furthermore, we can realize the $\mathcal{N} = 2$ A-D-E quiver gauge theories on these resolved ALE spaces. Consider type IIB string theory compactified on the product of the resolved ALE space and the flat complex plane $\mathbf{C}^1$. This product space can be view as the normal bundle $\mathcal{N}$ of the exceptional locus whose restriction to each $\mathbf{P}^1$ is $\mathcal{O}(-2) \oplus \mathcal{O}$ of $\mathbf{P}^1$. If we wrap N_i D5 branes around each $\mathbf{P}^1$, then we obtain the corresponding $\mathcal{N}=2$ quiver gauge theories on the uncompatified world-volume of the D5 brane which have been studied [51]. The sections of the $\mathcal{O}$ factor of $\mathcal{N}$ on each $\mathbf{P}^1$, which are given by the eigenvalues of the adjoints Φ_i of $U(N_i)$ are identified with the Coulomb branch of the moduli.

We will now explicitly construct the resolved ALE space for the A-D-E singularities and the brane configuration description via T-duality. The resolutions of the A-D-E singularities can be obtained by 'plumbing' n $\mathcal{O}(-2)$ over $\mathbf{P}^1$. To be precise, we introduce two $\mathbf{C}^2$ for each $\mathcal{O}(-2)$ of $\mathbf{P}^1$, denoted by $\mathbf{C}^2_{i,0}$ and $\mathbf{C}^2_{i,\infty}$, whose coordinates are Z_i, Y_i for $\mathbf{C}^2_{i,0}$ and Z_i', Y_i' for $\mathbf{C}^2_{i,\infty}$. Then the total space of the normal bundle $\mathcal{O}(-2)$ over the i-th $\mathbf{P}^1_i$ is given by gluing $\mathbf{C}^2_{i,0}$ and $\mathbf{C}^2_{i,\infty}$ with the identification:

$$Z_i' = 1/Z_i, \quad Y_i' = Y_i Z_i^2, \tag{14}$$

and $\mathbf{P}^1_i$ sits in as a zero section of the bundle. We denote the total space of $\mathcal{O}(-2)$ of $\mathbf{P}^1_i$ by $\mathrm{Tot}(\mathcal{O}(-2))$ and choose a real four dimensional small tubular neighborhood $\mathcal{T}_i$ in $\mathrm{Tot}(\mathcal{O}(-2))$ of the zero section $\mathbf{P}^1_i$ i.e.

$$\mathcal{T}_i = \{p \in \mathrm{Tot}(\mathcal{O}(-2)) | \mathrm{dist}(p, (Z_i(p), Y_i(p) = 0)) < \epsilon\}, \quad \text{for a small } \epsilon \tag{15}$$

where $(Z_i(p), Y_i(p) = 0)$ is the point on the zero section $\mathbf{P}^1_i$ obtained by the projection along the fiber in $\mathrm{Tot}(\mathcal{O}(-2))$. We glue together $\mathcal{T}_1, \cdots, \mathcal{T}_n$ by plumbing a neighborhood of the north pole ($Z_i' = X_i' = Y_i' = 0$) of $\mathbf{P}^1_i$ in $\mathcal{T}_i$ and a neighborhood of the south pole ($Z_{i+1} = X_{i+1} = Y_{i+1} = 0$) of $\mathbf{P}^1_{i+1}$ in $\mathcal{T}_{i+1}$ by exchanging the fiber (resp. base) coordinate Y_i' (resp. Z_i') of $\mathcal{O}(-2)$ over the i-th $\mathbf{P}^1$ with the base (resp. fiber) coordinate Z_{i+1} (resp. Y_{i+1}) of $\mathcal{O}(-2)$ over the $(i+1)$-th $\mathbf{P}^1$ (Figure 2).

In other words, the plumbing is an isomorphism between portions of $\mathcal{T}_i$ and $\mathcal{T}_{i+1}$ induced by the map

$$Y_i' \to Z_{i+1}, \quad Z_i' \to Y_{i+1}. \tag{16}$$

Note that the north pole of the i-th $\mathbf{P}^1$ will meet the south pole of the $(i+1)$-th $\mathbf{P}^1$ after the plumbing. Then the minimal resolution of the $\mathbf{A}_n$ singularity is isomorphic to a union of the tubular neighborhoods

$$\mathcal{T} = \mathcal{T}_1 \cup \ldots \cup \mathcal{T}_n \tag{17}$$

with the plumbing. For other A-D-E resolutions, we glue $\mathcal{T}_i$ following the Dynkin diagram.

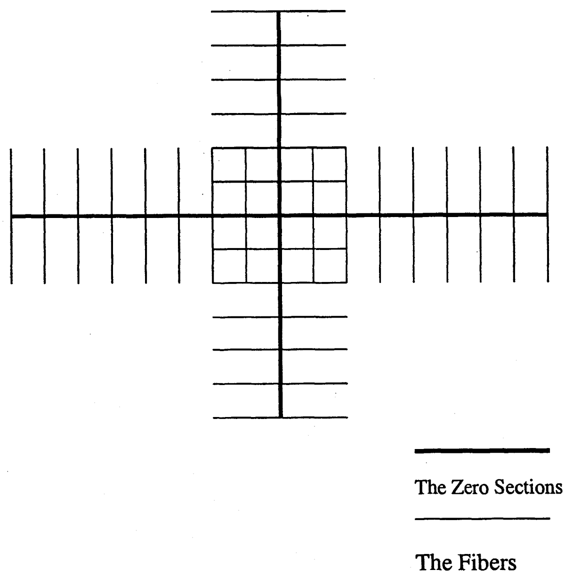

Figure 2: The fibers of $\mathcal{O}(-2)$ over $\mathbf{P}^1_i$ are identified with the sections of $\mathcal{O}(-2)$ over $\mathbf{P}^1_{i+1}$ and vice versa.

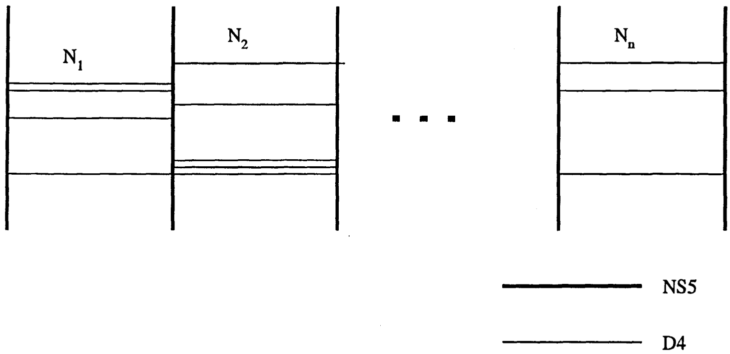

Figure 3: $\mathbf{A}_n$ brane configuration: D5 branes wrapping $\mathbf{P}^1$ cycles are T-dualized to D4 branes between NS branes.

Consider a circle action S_2 on $\mathcal{T}_i$:

$$\begin{aligned}(e^{i\theta}, Z_i) = e^{i\theta} Z_i, \quad (e^{i\theta}, Y_i) = e^{-i\theta} Y_i \\ (e^{i\theta}, Z_i') = e^{-i\theta} Z_i', \quad (e^{i\theta}, Y_i') = e^{i\theta} Y_i'.\end{aligned} \tag{18}$$

Then this action is compatible with the plumbing because the plumbing exchanges Z_i' with Y_{i+1} and Y_i with Z_{i+1}, and thus globalizes to the action on the $\mathcal{T}$. Since the orbits of the action degenerate along $Z_i = Y_i = 0$ and $Z_i' = Y_i' = 0$, we have two NS branes along the transversal direction to $\mathcal{T}$ at $Z_i = Y_i = 0$ and $Z' = Y' = 0$ on each open set $\mathcal{T}_i$ after T duality. We will then have $(n+1)$ NS branes labelled from 0 to n, which are parallel because the transversal direction to $\mathcal{T}$ is chosen to be flat. Thus the T-dual of N_i D5 branes wrapping $\mathbf{P}_i^1$ of the resolution of $\mathbf{A}_n$ singularity will be a brane configuration of N_i D4 branes between the $(i-1)$-th and i-th NS branes as in Figure 3. The method works for other A-D-E singularities.

4 Field Theory Analysis-Classical Vacua

We now consider deformations of the $\mathcal{N} = 2$ gauge quiver theories to $\mathcal{N} = 1$ supersymmetric theory by adding a tree-level superpotential, which is in general of the form

$$W = \sum_{i=1}^{n} W_i - \mathrm{Tr} \sum_{i=1}^{n} \sum_{j=1}^{n} s_{i,j} Q_{i,j} \Phi_j Q_{j,i}, \quad \text{where } W_i = \mathrm{Tr} \sum_{j=1}^{d_i+1} \frac{g_{i,j-1}}{j} \Phi_i^j, \tag{19}$$

and $s_{i,i} = 0$ and $s_{i,j} = -s_{j,i} = 1$ if there is an edge between v_i, v_j in the Dynkin diagram which means that the corresponding $\mathbf{P}^1$ cycles intersect. [1] The matrix $C_{ij} = 2\delta_{ij} - |s_{ij}|$ is the Cartan matrix of the A-D-E diagram. As pointed out in [7, 43], for $m > 1$ the non-renormalizable interaction terms seem to be irrelevant for the long distance behavior of the theory, but these terms have in general strong effects on the infrared dynamics. These are examples of operators known as 'dangerously irrelevant'.

In this section, we find the supersymmetric classical vacua by solving D-term and F-term equations. The D-flat condition implies that:

$$D_i^a = \mathrm{Tr}\left(\Phi_i^\dagger t_a^{(\mathrm{N_i})}\Phi_i + \Phi_i t_a^{(\bar{\mathrm{N}}_\mathrm{i})}\Phi_i^\dagger + \sum_j s_{ij}\left(Q_{i,j}^\dagger t_a^{(\mathrm{N_i})} Q_{i,j} - Q_{j,i}^\dagger t_a^{(\bar{\mathrm{N}}_\mathrm{i})} Q_{j,i}\right)\right) = 0 \quad (20)$$

where $t_a^{(\mathrm{N_i})}$ are the generators in the fundamental of $U(N_i)$ and $t_a^{(\bar{\mathrm{N}}_\mathrm{i})}$ are those in the anti-fundamental of $U(N_i)$. This can be rewritten as:

$$[\Phi_i, \Phi_i^\dagger] = 0, \quad (21)$$

$$s_{ij}\left(Q_{i,j}Q_{i,j}^\dagger - Q_{j,i}^\dagger Q_{j,i}\right) = -\eta_i \mathrm{Id}_{N_i}. \quad (22)$$

As noticed in [34, 35], the reason for this is that imposing D-terms equations together with the gauge equivalences is the same as taking the quotient under the complexified gauge groups which can used to diagonalize Φ_i, thus automatically satisfying (21). In this context, the D-terms equations are the moment map in symplectic quotient. (22) can be proved inductively on i for the A-D-E quiver theory. In the case of the $\mathbf{A}_n$ theory, as shown in the appendix of [42] that, after color rotations, the $Q_{i,j}$ can be simultaneously diagonalized, so that they are of the form

$$Q_{i,j} = \mathrm{diag}\{q_{i,j}^{(a)}\}, \quad Q_{j,i} = \mathrm{diag}\{q_{j,i}^{(a)}\} \quad (23)$$

with

$$|q_{j,i}^{(a)}| = |q_{i,j}^{(a)}|, \quad a = 1, \dots, N_{i,i+1} := \min\,(N_i, N_{i+1}). \quad (24)$$

For the $\mathbf{D}_n$ or $\mathbf{E}_n$ theories, this does not hold in general. [2] In the following discussions, the diagonalization of $Q_{i,j}$ is not used in an essential way. Sometimes we will write $q_i^{(a)}$ for $q_{i,j}^{(a)}$ and $\tilde{q}_i^{(a)}$ for $q_{j,i}^{(a)}$ when $j = i+1$ if there is no confusion.

[1] The choice of the sign for $s_{i,j}$ does not make any differences and we will assume $s_{i,j} = 1$ if $i < j$ in the case of $\mathbf{A}_n$ or $\mathbf{D}_n$.

[2] We thank the referee for pointing out this.

We now look at the F-term equations $dW = 0$. The equations are

$$\begin{aligned} s_{i,j}(Q_{i,j}\Phi_j - \Phi_i Q_{i,j}) = 0, \\ W_i'(\Phi_i) = \textstyle\sum_j s_{ij} Q_{i,j} Q_{j,i} \end{aligned} \tag{25}$$

The first equation also implies that

$$s_{i,j}(Q_{i,j}Q_{j,i}\Phi_i - \Phi_i Q_{i,j}Q_{j,i}) = 0, \tag{26}$$

From this and the second equation, it is clear that the unbroken gauge symmetry of Φ_i will also preserve $Q_{i,j}Q_{j,i}$. Therefore, we can solve the F-term equations in terms of the unbroken gauge symmetries of $Q_{i,j}Q_{j,i}$, which are determined by the eigenvalues of $Q_{i,j}Q_{j,i}$. For each factor group $U(N_i)$, we restrict the F-term equations on the gauge invariant subspace for which the eigenvalues of the mesons $Q_{i,j}Q_{j,i}$ is fixed and we can divide into two cases. [3]

• **Case 1.** On the gauge invariant subspace for which the eigenvalues of $Q_{i,j}Q_{j,i}$ are zero for all j.

Let $U(M_{i,i})$ be the maximal gauge symmetry subgroup of $U(N_i)$ for which the eigenvalues of $Q_{i,j}$ are zero. Then (25) implies that

$$W_i'(\Phi_i) = 0, \tag{27}$$

on the subspace invariant under $U(M_{i,i})$. Hence the eigenvalues v_i of Φ_i is given by

$$W_i'(v_i) = \sum_{l=1}^{d_i} g_{i,l} v_i^l := g_{i,d_i} \prod_{l=1}^{d_i} (v_i - b_{i,l}) = 0, \tag{28}$$

and the gauge group will break

$$U(M_{i,i}) \to \prod_{l=1}^{d_i} U(M_{i,i,l}). \tag{29}$$

• **Case 2.** On the gauge invariant subspace for which the eigenvalues of $Q_{i,j}Q_{j,i}$ are non-zero for some j. Then for a given non-zero eigenvalue of $Q_{i,j}Q_{j,i}$, there exist $j \le i < k$ such that

$$\prod_{l=j}^{k-1} Q_{l,l+1} \prod_{l=1}^{k-j} Q_{k-l+1,k-l} \tag{30}$$

[3] We will restrict to the $\mathbf{A}_n$ quiver theory, the method works also for the $\mathbf{D}_n$ and $\mathbf{E}_n$ quiver theories.

have non-zero eigenvalues and

$$\prod_{l=j-1}^{k-1} Q_{l,l+1} \prod_{l=1}^{k-j+1} Q_{k-l+1,k-l} = 0, \tag{31}$$

$$\prod_{l=j}^{k} Q_{l,l+1} \prod_{l=0}^{k-j} Q_{k-l+1,k-l} = 0. \tag{32}$$

Let $U(M_{j,k})$ be the maximal gauge symmetry subgroup for the given non-zero eigenvalue of $Q_{i,j}Q_{j,i}$. We first note that $U(M_{j,k})$ embeds diagonally into the factor $\prod_{i=j}^{k} U(N_i)$ of $\prod_{i=1}^{n} U(N_i)$ as all bifundamentals $Q_{l,l+1}, Q_{l+1,i}$ for $l = j, \ldots, k-1$ have the unique non-zero expectation value. The first equation of (25) implies that the expectation values of Φ_m for $m = j, \ldots, k$ are the same, and hence from the second F-term equation (25), we see that the eigenvalue values $v_{j,k}$ of Φ_m must satisfy

$$\sum_{l=j}^{k} W_l'(v_{j,k}) = 0. \tag{33}$$

Generically, the number of different solutions is given by the maximum degree of W_l':

$$d_{j,k} = \max\{d_j, d_{j+1}, \ldots, d_k\}. \tag{34}$$

By considering the solutions $v_{j,k,l}$, $l = 1, \ldots, d_{j,k}$ of (33), the expectation values $q_{m,m+1,l}q_{m+1,m,l}$ of the mesons $Q_{m,m+1}Q_{m+1,m}$ (i.e. of the bifundamentals) are determined uniquely from the second F-term equation (25):

$$q_{m,m+1,l}q_{m+1,m,l} = \sum_{i=j}^{m} W_i'(v_{j,k,l}). \tag{35}$$

Then the gauge group is generically Higgsed to

$$U(M_{j,k}) \to \prod_{l=1}^{d_{j,k}} U(M_{j,k,l}), \tag{36}$$

and $U(M_{j,k,l})$'s are diagonally embedded into $U(N_j) \times U(N_{j+1}) \times \cdots U(N_k)$. There are $n(n-1)/2$ possibilities for $U(M_{j,k})$. Combining with **Case 1**, the number of gauge group breakings is

$$d := \sum_{j=1}^{n} \sum_{k=j}^{n} d_{j,k} \tag{37}$$

where $d_{ii} = d_i$. If all degrees of the adjoint superpotential term W_i are equal to the same value p, then there are $p(n+1)n/2$ breakings which agrees with [6]. Therefore in the case of $\mathbf{A}_n$ quiver theory, there are finitely many discrete branches of the vacua parameterized by

$$M_{j,k,l}, \quad j=1,\dots,n, \quad k=j,\dots,n, \quad l=1,\dots,d_{j,k} \tag{38}$$

for which the gauge group is Higgsed to:

$$\prod_{i=1}^{n} U(N_i) \to \prod_{j=1}^{n}\prod_{k=j}^{n}\prod_{l=1}^{d_{j,k}} U(M_{j,k,l}), \tag{39}$$

where $d_{j,k}$ are defined in (34), provided that a set of equations

$$W'_j(v) + W'_{j+1}(v) + \dots + W'_k(v) = 0 \tag{40}$$

have exactly $d_{j,k}$ different solutions. This particularly means that any two curves

$$u - \sum_{i=1}^{j-1} W'_i(v) = 0, \quad u - \sum_{i=1}^{k} W'_k(v) = 0 \tag{41}$$

meet transversally at exactly $d_{j,k}$ different points. The equation (40) is equivalent to the condition that the system

$$\begin{aligned} \frac{d^2 \sum_{i=j}^{k} W_i(v)}{dv^2} &= 0, \\ \frac{d \sum_{i=j}^{k} W_i(v)}{dv} &= 0. \end{aligned} \tag{42}$$

does not have any common solution, and $\sum_{i=j}^{k} g_{i,d_i} \neq 0$.

We will now consider two particular cases of the superpotential (19) in great details, which will be called quadratic and degenerate superpotential, respectively:

$$W_{\mathrm{q}} = \mathrm{Tr}\left(\sum_{i=1}^{n}(\frac{g_i}{2}\Phi_i^2 + h_i\Phi_i) - \sum_{i=1}^{n}\sum_{j=1}^{n} s_{i,j}Q_{i,j}\Phi_j Q_{j,i}\right) \tag{43}$$

$$W_{\mathrm{deg}} = \mathrm{Tr}\left(\sum_{i=1}^{n}\frac{g_i}{m+1}\Phi_i^{m+1} - \sum_{i=1}^{n}\sum_{j=1}^{n} s_{i,j}Q_{i,j}\Phi_j Q_{j,i}\right) \tag{44}$$

The $\mathcal{N}=1$ theory deformed by (44) or (43) contains certain universal IR aspects of theories with more general superpotential (19), while they are amenable to a more concrete and detailed analysis due to their simplicity and they will be also a basis for our analysis of the general case. We assume that $g_{n-1} = g_n$ for $\mathbf{D}_n$ theory which means that we are giving the same mass to the last two adjoints Φ_{n-1}, Φ_n. We do this in order to preserve the $\mathbf{Z}_2$ symmetry of exchanging the last two vertices in the $\mathbf{D}_n$ Dynkin diagram.

• The case of the $\mathbf{A}_n$ quiver theory with quadratic superpotential W_{q} (43):

We may rewrite the F-term equation (25) as

$$\begin{aligned} Q_{i,i+1}\Phi_{i+1} - \Phi_i Q_{i,i+1} = 0, \quad i = 1, \dots, n-1, \\ g_i \Phi_i + h_i = Q_{i,i+1}Q_{i+1,i} - Q_{i,i-1}Q_{i-1,i}, \quad i = 1, \dots, n \end{aligned} \tag{45}$$

where $Q_{1,0}, Q_{0,1}, Q_{n,n+1}, Q_{n+1,n}$ are defined to be zero. By substituting the adjoints Φ_i in the first equation in terms of the mesons as given by the second equation, we conclude that either $q_i^{(a)} = 0$ or

$$g_{i+1} q_{i-1}^{(a)} \tilde{q}_{i-1}^{(a)} - (g_{i+1} + g_i)\, q_i^{(a)} \tilde{q}_i^{(a)} + g_i q_{i+1}^{(a)} \tilde{q}_{i+1}^{(a)} = h_{i+1} g_i - h_i g_{i+1}, \tag{46}$$

for $i = 1, \dots, n$ and $a = 1, \dots, N_{i,i+1}$. Here the indexed quantity is assumed to be zero if the lower index is not between 1 and n. For the $\mathbf{D}_n$ theories, the equations (46) for $i = n-1, n$ should be replaced by

$$\begin{aligned} g_{n-1} q_{n-3}^{(a)} \tilde{q}_{n-3}^{(a)} - (g_{n-1} + g_{n-2})\, q_{n-2}^{(a)} \tilde{q}_{n-2}^{(a)} - g_{n-1} q_{n-2,n}^{(a)} q_{n,n-2}^{(a)} \\ = h_{n-1} g_{n-2} - h_{n-2} g_{n-1}, \\ g_n q_{n-3}^{(a)} \tilde{q}_{n-3}^{(a)} - (g_n + g_{n-2})\, q_{n-2,n}^{(a)} q_{n,n-2}^{(a)} - g_n q_{n-2}^{(a)} \tilde{q}_{n-2}^{(a)} = h_n g_{n-2} - h_{n-2} g_n. \end{aligned} \tag{47}$$

When $q_j^{(a)} q_{j+1}^{(a)} \cdots q_{k-1}^{(a)} \neq 0$ for $j < k$ and $q_{j-1}^{(a)} = q_k^{(a)} = 0$, the equations (46) become a system of $(k-j)$ linear equations in $(k-j)$ unknowns $q_j^{(a)} \tilde{q}_j^{(a)}, \dots, q_{k-1}^{(a)} \tilde{q}_{k-1}^{(a)}$:

$$\begin{aligned} (g_j + g_{j+1})\, q_j^{(a)} \tilde{q}_j^{(a)} - g_j q_{j+1}^{(a)} \tilde{q}_{j+1}^{(a)} = h_j g_{j+1} - h_{j+1} g_j. \\ -g_{j+2} q_j^{(a)} \tilde{q}_j^{(a)} + (g_{j+1} + g_{j+2})\, q_{j+1}^{(a)} \tilde{q}_{j+1}^{(a)} - g_{j+1} q_{j+2}^{(a)} \tilde{q}_{j+2}^{(a)} = h_{j+1} g_{j+2} - h_{j+2} g_{j+1}. \\ \vdots \\ -g_k q_{k-2}^{(a)} \tilde{q}_{k-2}^{(a)} + (g_{k-1} + g_k)\, q_{k-1}^{(a)} \tilde{q}_{k-1}^{(a)} - g_{k-1} q_k^{(a)} \tilde{q}_k^{(a)} = h_{k-1} g_k - h_k g_{k-1}. \end{aligned} \tag{48}$$

Hence it has a unique solution if the system is linearly independent which is equivalent to having a non-zero determinant of the following $(k-j+1) \times$

$(k-j+1)$ matrix:

$$\begin{vmatrix} -g_{j+1} & g_j & 0 & 0 & \dots & 0 & 0 \\ 0 & -g_{j+2} & g_{j+1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -g_k & g_{k-1} \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 \end{vmatrix}. \tag{49}$$

Here only two entries in each row are non-zero except in the last row where all entries are 1's. Thus either q_i's are zero or if $q_j^{(a)} q_{j+1}^{(a)} \cdots q_{k-1}^{(a)} \neq 0$ for $j<k$ and $q_{j-1}^{(a)} = q_k^{(a)} = 0$, there is a unique non-zero solution. As the components of $Q_{j,j+1}, \dots, Q_{k-1,k}$ have non-zero expectation values, the gauge group is Higgsed to

$$\prod_{i=1}^{n} U(N_i) \to \prod_{i=1}^{j-1} U(N_i) \times U(M_{j,k}) \times \prod_{i=j}^{k} U(N_i - M_{j,k}) \times \prod_{i=k+1}^{n} U(N_i), \tag{50}$$

where $M_{j,k} \leq \min\{N_i, N_{i+1}, \dots, N_k\}$. The subgroup $U(M_{j,k})$ is diagonally embedded in $U(N_j) \times \cdots \times U(N_k)$ such that when $Q_{l,l+1}Q_{l+1,l}$ is restricted to the fundamental and anti-fundamental representations of $U(M_{j,k})$ factor of $U(N_l)$, it is a diagonal matrix with the same diagonal entries:

$$Q_{l.l+1}Q_{l+1,l} = \mathrm{diag}\{q_l \tilde{q}_l\}, \quad \text{for } l = j, \dots, k-1 \tag{51}$$

where $q_l\tilde{q}_l$ is the solution of (48).

- Moduli space of $\mathbf{A}_3$ quiver theory:

We first consider the $\mathbf{A}_3$ quiver theory. After possible changes of bases, we may assume that

$$\begin{aligned} Q_{1,2}Q_{2,1} &= \mathrm{diag}\{p_1^{(1)}, \dots, p_1^{(M_{1,3})}, q_1^{(1)}, \dots, q_1^{(M_{1,2})}, 0, \dots, 0\} && (52) \\ Q_{2,3}Q_{3,2} &= \mathrm{diag}\{p_2^{(1)}, \dots, p_2^{(M_{1,3})}, \underbrace{0, \dots, 0}_{M_{1,2}}, q_2^{(1)}, \dots, q_2^{(M_{2,3})}, 0, \dots, 0\} && (53) \end{aligned}$$

where

$$\begin{aligned} p_1^{(1)} = \dots = p_1^{(M_{1,3})} &= \frac{h_1g_2 - g_1h_2 + h_1g_3 - g_1h_3}{g_1+g_2+g_3}, \\ q_1^{(1)} = \dots = q_1^{(M_{1,2})} &= \frac{g_2h_1 - g_1h_2}{g_1+g_2}, \\ p_2^{(1)} = \dots = p_2^{(M_{1,3})} &= \frac{h_2g_3 - g_2h_3 + h_1g_3 - g_1h_3}{g_1+g_2+g_3}, \\ q_2^{(1)} = \dots = q_2^{(M_{2,3})} &= \frac{g_3h_2 - g_2h_3}{g_2+g_3}. \end{aligned} \tag{54}$$

This will imply that

$$\begin{aligned}\Phi_1 &= \frac{1}{g_1}(Q_{12}Q_{21} - h_1\mathrm{Id}) \\ \Phi_2 &= \frac{1}{g_2}(Q_{23}Q_{32} - Q_{21}Q_{12} - h_2\mathrm{Id}) \\ \Phi_3 &= \frac{1}{g_3}(-Q_{32}Q_{23} - h_3\mathrm{Id}) \end{aligned} \tag{55}$$

With the above choice for the Higgsing, the gauge group is broken to:

$$\begin{aligned}&U(N_1) \times U(N_2) \times U(N_3) \\ &\quad \to U(N_1 - M_{1,2} - M_{1,3}) \times U(N_2 - M_{1,2} - M_{2,3} - M_{1,3}) \times U(N_3 - M_{2,3} - M_{1,3}) \\ &\quad \times U(M_{1,2}) \times U(M_{2,3}) \times U(M_{1,3}).\end{aligned} \tag{56}$$

where $U(M_{1,2})$ is diagonally embedded into the product $U(N_1) \times U(N_2)$, $U(M_{2,3})$ into $U(N_2) \times U(N_3)$, and $U(M_{1,3})$ into $U(N_1) \times U(N_2) \times U(N_3)$ respectively.

Generally, in the case of $\mathbf{A}_n$ quiver theory, there are finitely many discrete branches of the vacua parameterized by $M_{j,k}, j = 1, \ldots, n, k = j+1, \ldots, n$ in which the gauge group is broken as follows:

$$\prod_{i=1}^{n} U(N_i) \to \prod_{j=1}^{n}\prod_{k=j}^{n} U(M_{j,k}), \tag{57}$$

$$\text{where } M_{i,i} = N_i - \sum_{\substack{j \leq i \leq k \\ j \neq k}} M_{j,k}.$$

Here $U(M_{j,k})$ is the largest group which can be diagonally embedded into $U(N_j) \times U(N_{j+1}) \times \cdots \times U(N_k)$ of $\prod_{i=1}^{n} U(N_i)$, but cannot be embedded into a larger group $\prod_{i=1}^{n} U(N_i)$. The group $U(M_{j,k})$ appears when the blocks of $Q_{j,j+1}, \cdots, Q_{k-1,k}$ have simultaneously non-zero expectation values. Therefore the product of n gauge groups is broken into the product of $\frac{n(n+1)}{2}$ gauge groups in a generic branch in the $\mathbf{A}_n$ quiver theory. Notice that the vacua is completely determined by the following set of the complex lines in the $u - v$ space up to discrete moduli $M_{j,k}$:

$$u - \sum_{j=1}^{i}(g_i v + h_i) = 0, \quad , i = 0, 1, \ldots, n, \tag{58}$$

where we set $g_0 = h_0 = 0$. Conversely, these lines are determined by the vacua. There are only $2n$ free continuous parameters in this correspondence and the vacua cannot be arbitrary, we will interpret these lines as NS branes and the intersection points of these lines as the location of D-branes in type IIA picture in the next section (Figure 4).

• The case of the $\mathbf{D}_n$ quiver theory with quadratic superpotential W_{q} (43):

In the $\mathbf{D}_n$ quiver theory studied in [36], the F-term equations are the same as in the $\mathbf{A}_n$ quiver theory except the last two, corresponding to the roots v_{n-1}, v_n and there is no difference in the procedure of finding the vacua. Because of our mass assumption $g_{n-1} = g_n$, we will have one less gauge group breaking in the generic Higgs branch. So the vacua are parameterized by $n(n+1)/2 - 1$ natural numbers. In the branch parameterized by $M_{j,k}$, the gauge group is Higgsed to:

$$\prod_{i=1}^{n} U(N_i) \to \prod_{j=1}^{n-1}\prod_{k=j}^{n-1} U(M_{jk}) \times \prod_{j=1}^{n-2} U(M_{j,n}), \tag{59}$$
$$\text{where } M_{i,i} = N_i - \sum_{\substack{j\leq i\leq k \\ j\neq k}} M_{j,k}.$$

In terms of the line configurations in the $u - v$ plane, one missing gauge group in the Higgsing is due to the fact that two lines corresponding to the last two vertices of the $\mathbf{D}_n$ diagram are parallel.

• The case of the $\mathbf{A}_n$ quiver theory with degenerate superpotential W_{deg} (44):

The F-term equation (25) can be rewritten as follows:

$$\begin{gathered} Q_{i,i+1}\Phi_{i+1} - \Phi_i Q_{i,i+1} = 0, \\ g_i\Phi_i^m = Q_{i,i+1}Q_{i+1,i} - Q_{i,i-1}Q_{i-1,i}, \quad i = 1, \dots, n \end{gathered} \tag{60}$$

where $Q_{1,0}, Q_{0,1}, Q_{n,n+1}, Q_{n+1,n}$ are defined to be zero. The first equation also implies that

$$Q_{i,j}\Phi_j^m - \Phi_i^m Q_{i,j} = 0. \tag{61}$$

So, by eliminating the Φ_i^m, we conclude that the bifundamentals $Q_{i,i+1}, Q_{i+1,i}$ are zero and hence the solutions are

$$Q_{i,i+1} = 0, \quad Q_{i+1,i} = 0, \quad \Phi_i^m = 0, \tag{62}$$

provided that the condition (49) for g_i's are satisfied. Classically this means that $\Phi_i = 0$, but at quantum level Φ_i is allowed to be a nilpotent i.e. $\Phi^m = 0$.

The nilpotent solutions $\Phi_i^k = 0, \Phi_i^{k-1} \neq 0, k = 2, \ldots, m$ are the infinitesimal deviations from the vacuum at $\Phi_i = 0$, which means that the $\mathbf{P}^1$ cycles on which D5 branes wrap on can be deformed in an infinitesimal neighborhood. In terms of the brane configuration, there will be m NS branes infinitesimally deviated from each other which bound the infinitesimally deviated D4 branes. We will revisit these issues later.

5 $\mathcal{N} = 1$ theory - Brane Interpretation

We give a brane configuration interpretation of the results of the previous section for the $\mathbf{A}_n$ quiver theory. Before we begin our discussion with the arbitrary superpotential, we illustrate the idea for the case of the $\mathbf{A}_2$ quiver theory with the quadratic superpotential W_{q} (43). We will also consider the $\mathbf{D}_n$ theory and deal with the degenerate superpotential in the next section.

Recall that the $\mathcal{N} = 2$ brane configuration, where D4 branes with three NS branes, denoted as zeroth, first and second from the left to the right, can move freely along the direction of NS branes which was denoted as v-direction. Therefore we can identify the NS branes with the moduli of the positions of the D4 branes. When the superpotential (43) is introduced, the expectation values of $\Phi_i, \quad i = 1, 2$ are determined by

$$W_i' = g_i(v + \frac{h_i}{g_i}) = 0 \tag{63}$$

when the expectation value of the meson $Q_{1,2} = Q_{2,1}$ is zero. We denoted the maximal unbroken gauge symmetry groups by $U(M_{i,i})$ at the vacua with these expectation values. Hence the M_{ii} of N_i D4 branes stretched between $(i-1)$-th and i-th NS branes will be fixed at the position $v = -h_i/g_i$ in the vacua. When the expectation values of the meson $Q_{1,2}Q_{2,1}$ is non-zero, there is a common expectation value of the adjoints Φ_1 and Φ_2 given by

$$W_1'(v) + W_2'(v) = (g_1 + g_2)(v + \frac{h_1 + h_2}{g_1 + g_2}) = 0. \tag{64}$$

The maximal unbroken gauge group is denoted by $U(M_{1,2})$ which is diagonally embedded into $U(N_1) \times U(N_2)$. It means that the $M_{1,2}$ D4 branes connecting the 0-th and the first NS and $M_{1,2}$ D4 branes from connecting the first and the second are merged together at $v = -\frac{h_1+h_2}{g_1+g_2}$ so that they form a long D4 brane stretched from the zeroth to the second NS branes. Also, by giving the masses to the adjoints Φ_i, the NS branes are rotated. It seems impossible to do so because D4 branes between adjacent NS branes are split into two stacks rather than one stack so the supersymmetry would not be preserved. But if we make a translation of the first and the second NS branes into the u direction, one can rotate the NS branes. This translation in u direction is due to the non-zero expectation value of the meson $Q_{1,2}Q_{2,1}$. Thus we identify the v-direction with the freedom of the adjoints moving along the Coulomb branch and u-direction with that of the mesons. In describing the positions of NS branes, we need to fix the reference NS brane defined by $u = 0$ in the $u - v$ plane. We choose this to be the zeroth NS brane, and then we consider the relation of other NS branes w.r.t. the zeroth one. It is most natural to consider the location of the first NS brane in the $\mathcal{N} = 1$ as the moduli of the right-end position of the D4 branes which were stretched between the zeroth and the first NS brane in the $\mathcal{N} = 2$ picture, and the location of the second NS brane in the $\mathcal{N} = 1$ as that of the long D4 branes which were stretched between the zeroth through the second NS brane. Hence the first NS brane is given by

$$u = W_1'(v) \tag{65}$$

and the second NS brane is given by

$$u = W_1'(v) + W_2'(v), \tag{66}$$

and the intersection of the first and the second NS branes is given by $W_2'(v) = 0$ and so its v-coordinate is exactly the expectation value of Φ_2 when it is restricted to the invariant subspace of $U(M_{22})$. Therefore, the D4 branes are located exactly at the intersection points of the curves describing the NS branes (See Figure 4). Extending these arguments to the $\mathbf{A}_n$ quiver theory with the quadratic superpotential (43), we obtain $(n+1)$ lines representing the NS branes, and thus there are $\frac{(n+1)n}{2}$ intersection points corresponding to the stacks of D4 branes. This is exactly the same as the number of the Higgsed gauge groups (57). For the $\mathbf{D}_n$ case, the analysis is the same except the fact the last two lines are parallel (Figure 5).

The general superpotential (19) case is a straightforward extension. The l-th NS brane is described as the moduli of the right-end position of the D4 brane connecting the zeroth and the l-th NS brane in the $\mathcal{N} = 2$ theory and it is given by

$$u = \sum_{i=1}^{l} W_i'(v), \tag{67}$$

and the v-coordinates of its intersection with the first NS brane $u = 0$ are the common expectation values $\Phi_1, \ldots, \Phi_l$ when they are restricted to the invariant subspace under $U(M_{1,l})$. More generally, the v-coordinates of the intersection of the $(l-1)$-th and the m-th NS branes are given by

$$W_l'(v) + W_{l+2}'(v) + \ldots + W_m'(v) = 0, \tag{68}$$

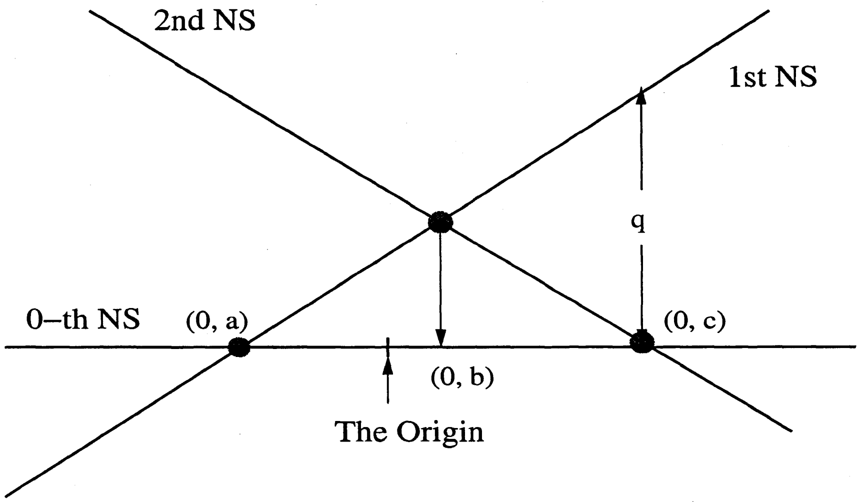

Figure 4: A brane configuration for the $\mathcal{N}=1$ $\mathbf{A}_2$ theory with the quadratic superpotential. Here a (resp. b) is the expectation value of Φ_1 (resp. Φ_2) on $U(M_{1,1})$ (resp. $U(M_{2,2})$), c is the common expectation value of Φ_1 and Φ_2 on $U(M_{1,2})$ and q is the expectation value of the meson $Q_{1,2}Q_{2,1}$. $\bullet$'s are the location of D4 branes.

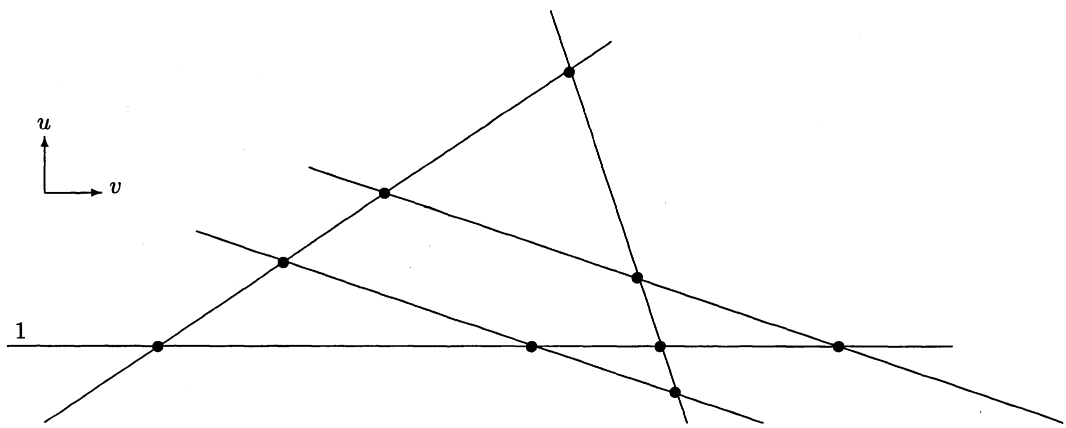

Figure 5: The brane configuration in $u-v$ space for $\mathbf{D}_n$. Note that the last two NS branes are parallel.

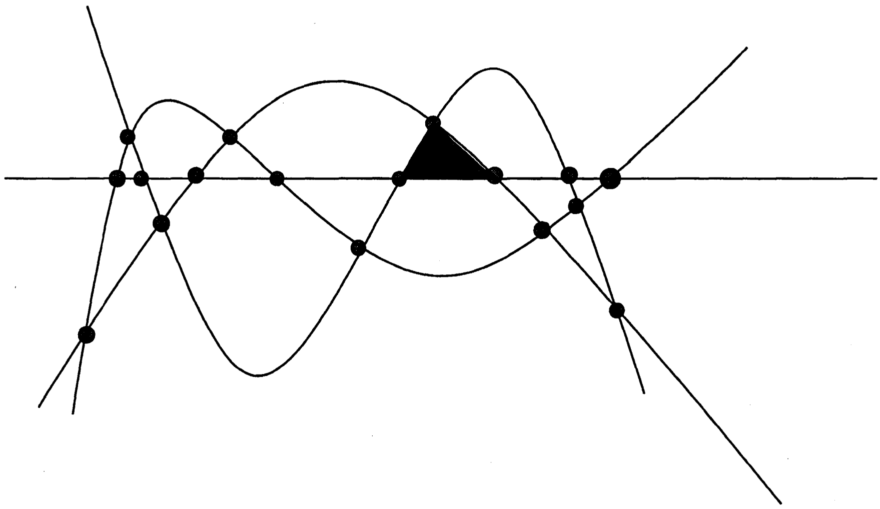

Figure 6: A brane configuration for the $\mathcal{N} = 1$ $\mathbf{A}_3$ theory with $d_1 = 2,\ \ d_{1,2} = 3$ and $d_{1,3} = 3$.

and hence they are the common expectation values of $\Phi_l, \ldots, \Phi_m$ on $U(M_{l,m})$. The total number of intersection points is $d_{l,m}$ which agrees with the number of the Higgsed gauge groups. One can give interpretation of the u-coordinates of the intersection points in terms of the expectation values of the mesons. For $j = 1$, (33) means that the v-coordinates of the intersection of the first and the k-th NS brane is given by $v_{1,k,l}$ and (35) implies that the u-coordinates of the m-th curve

$$u = \sum_{i=1}^{m} W_i'(v), \tag{69}$$

evaluated at $v = v_{1,k,l}$, will give the expectation values of the corresponding mesons $Q_{m,m+1}Q_{m+1,m}$. Figure 6 shows a brane configuration for the $\mathcal{N} = 1$ $\mathbf{A}_3$ theory with 17 gauge groups Higgsed by a superpotential with $d_1 = 2,\ \ d_{1,2} = 3$ and $d_{1,3} = 3$.

6 Geometric Engineering for $\mathcal{N} = 1$ A-D-E Quiver Theory

In this section, we describe Calabi-Yau threefolds which are T-dual of the brane configurations of the $\mathcal{N} = 1$ theory. We first present the Calabi-Yau threefolds and then we show that the T-dual picture is exactly the same as the brane configuration for the vacua.

The $\mathcal{N}=1$ geometry for the $\mathbf{A}_n$ quiver theory with general superpotential W (19) is the minimal resolution of a Calabi-Yau threefold defined in $\mathbf{C}^4$ by

$$X:\, xy-\prod_{p=0}^{n}\left(u-\sum_{i=0}^{p}W_i'(v)\right)=0 \tag{70}$$

where W_0' is defined to be zero. The singularities are isolated and located at $x=y=0$ and the intersection of any two curves in the $u-v$ plane defined by

$$u=\sum_{i=1}^{j-1}W_i'(v),\quad u=\sum_{i=1}^{k}W_i'(v). \tag{71}$$

The singularities can be resolved by successive blow-ups which replace each singular point by a $\mathbf{P}^1$ cycle. Therefore we see that the number of $\mathbf{P}^1$ cycles match the number of the Higgsed gauge groups. The resolved space is covered by $(n+1)$ three dimensional complex spaces $U_p, p=0,\dots n$ with coordinates

$$u_p=\frac{\prod_{j=0}^{p}\left(u-\sum_{i=0}^{j}W_i'(v)\right)}{x},\quad x_p=\frac{x}{\prod_{j=0}^{p-1}\left(u-\sum_{i=0}^{j}W_i'(v)\right)},\quad v_p=v, \tag{72}$$

where $x_0=x$. They blow down to the singular threefold (70) by

$$\sigma:\quad \tilde{X}:=U_0\sqcup U_2\sqcup\ldots\sqcup U_n\to X, \tag{73}$$

$$U_p\ni(u_p,x_p,v_p)\mapsto\begin{cases} x=\begin{cases} x_0 \text{ if } p=0,\\ x_p\prod_{j=0}^{p-1}\left(x_pu_p+\sum_{i=j+1}^{p}W_i'(v_p)\right) \text{ otherwise}\end{cases}\\ y=u_p\prod_{j=p+1}^{n}\left(x_pu_p-\sum_{i=p+1}^{j}W_i'(v_p)\right)\\ u=x_pu_p+\sum_{i=0}^{p}W_i'(v_p)\\ v=v_p\end{cases} \tag{74}$$

where $U_p\sqcup U_{p+1}$ means that the three spaces U_p, U_{p+1} are glued together by

$$x_{p+1}=u_p^{-1},\ v_{p+1}=v_p,\ u_{p+1}=x_pu_p^2-W_{p+1}'(v_p)u_p. \tag{75}$$

Thus the complex lines $\mathbf{C}^1$ defined by

$$W_{p+1}'(v_p)=0,\quad x_p=0 \tag{76}$$

in U_p together with the complex lines $\mathbf{C}^1$ in U_{p+1} defined by

$$W'_{p+1}(v_{p+1}) = 0, \quad u_{p+1} = 0 \tag{77}$$

form the $\mathbf{P}^1$ cycles and there are no other $\mathbf{P}^1$ cycles in $U_p \sqcup U_{p+1}$. This is a generalization of the $\mathbf{A}_1$ quiver theory considered in [4]. While the $\mathbf{P}^1$ in the resolution (17) of the $\mathbf{A}_n$ singularity can move freely in the v-direction, the above $\mathbf{P}^1$ cycles are frozen at

$$W'_{p+1}(v) = 0. \tag{78}$$

Hence the supersymmetry is broken from $\mathcal{N} = 2$ to $\mathcal{N} = 1$.

To study this phenomenon in details, consider the extreme case where

$$W_{p+1}(v) = \frac{g_{m+1}}{m+1} v^{m+1}, \quad m > 1. \tag{79}$$

Then the condition (75) becomes

$$x_{p+1} = u_p^{-1}, \; v_{p+1} = v_p, \; u_{p+1} = x_p u_p^2 - v_p^m u_p. \tag{80}$$

Locally in the neighborhood $U_p \sqcup U_{p+1}$, this can be considered as the limit case of the $\mathbf{A}_1$ quiver theory considered in [4]. In [4], the $\mathcal{N} = 2$ theory has been deformed by a tree-level superpotential

$$W_{\mathrm{CIV}} = \sum_{p=1}^{m+1} \frac{g_p}{p} \mathrm{Tr}\Phi^p. \tag{81}$$

Then the classical vacua are located at

$$W'_{\mathrm{CIV}}(v) = \sum_{p=0}^{m} g_{p+1} v^p = g_{m+1} \prod_{p=1}^{d_k} (v - a_p). \tag{82}$$

While the D5 branes wrapping $\mathbf{P}^1$ in the $\mathcal{N} = 2$ geometry can move freely along the v-direction, the D5 branes wrapped on $\mathbf{P}^1$ cycles in the $\mathcal{N} = 1$ geometry will be fixed at the vacua i.e. $v = a_p$ and the $\mathcal{N} = 1$ geometry can be described as a union of two $\mathbf{C}^3$'s with the patching condition:

$$x' = u^{-1}, \quad v' = v, \quad u' = xu^2 + g_{m+1} \prod_{p=1}^{m} (v - a_p) u, \tag{83}$$

where (x', v', u'), (u, v, x) are coordinates systems for two $\mathbf{C}^3$'s respectively. As we will see, the T-dual picture is a brane configuration with p stacks of D4 branes between a straight NS and a curved NS brane of degree m as in

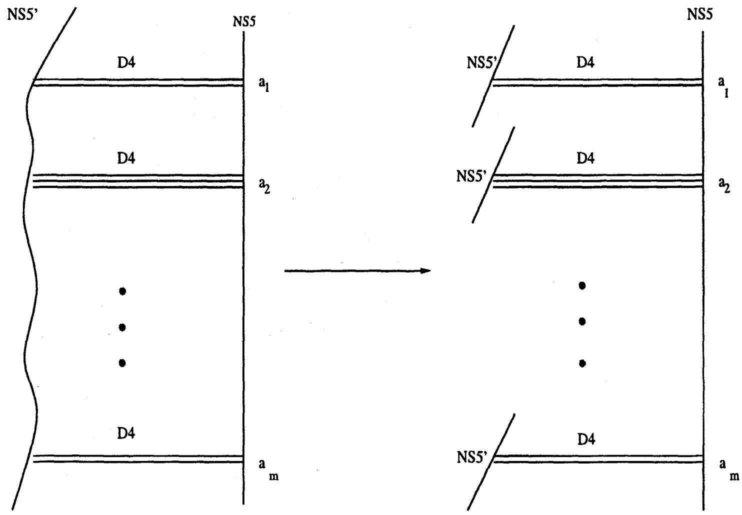

Figure 7: The $g_{m+1} \to \infty$ limit of $\mathbf{A}_1$ brane configuration with a non-degenerate superpotential.

the far-left of Figure 8 or 7.

We may consider two limit of this configuration. If we take the limit where $g_{m+1} \to \infty$ while keeping the ratio g_i/g_{m+1} finite for $i = 1, \dots, m$, then the curved NS brane will break into m lines and all of them are separated from each other and completely rotated so that the corresponding adjoint masses are infinity in this limit. The configuration is shown in the right of Figure 7.

The $\mathcal{N} = 1$ geometry will be given by the resolution of

$$xy + u \prod_{p=1}^{m} (v - a_p) = 0, \tag{84}$$

because the curve $u - g_{m+1} \prod_{p=1}^{m} (v - a_p) = 0$ will approach $\prod_{p=1}^{m} (v - a_p) = 0$ in the limit.

On the other hand, if we take the limit where $g_i \to 0$ for $i = 1, \dots, m$, then all a_i become zero and the NS brane is curved so that it intersect with the straight NS branes in the $u - v$ plane with high multiplicities which is shown in the center of Figure 8. In the limit $g_{m+1} \to \infty$, the curved NS brane breaks into m lines and these lines will be on top of each other as in the far-right of Figure 8. Then the gluing data (83) becomes

$$x' = u^{-1}, \quad v' = v, \quad u' = xu^2, \quad (\mathrm{mod}\, \mathcal{I}^2), \tag{85}$$

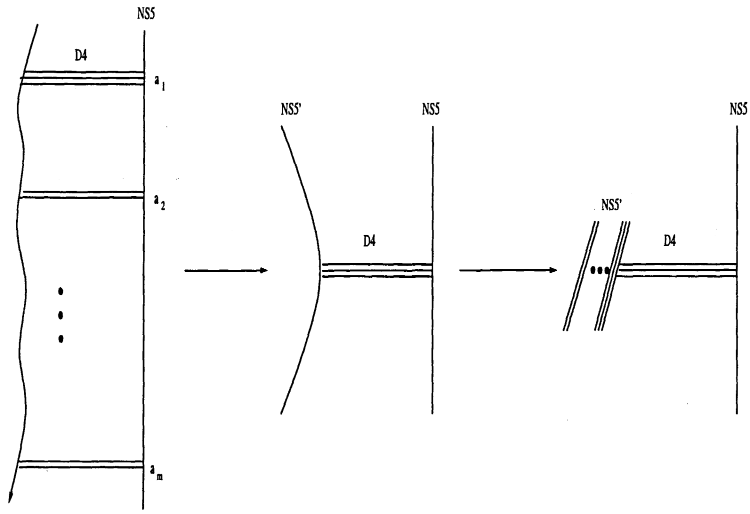

Figure 8: The transition $\mathbf{A}_1$ brane configuration with a non-degenerate superpotential to a degenerate superpotential and the $g_{m+1} \to \infty$ limit.

where $\mathcal{I}$ is the ideal sheaf defining $\mathbf{P}^1$, the normal bundle $\mathcal{N} = \mathcal{I}/\mathcal{I}^2$ of $\mathbf{P}^1$ is $\mathcal{O}(-2) + \mathcal{O}$ so that the first order deformation space $H^0(\mathbf{P}^1, \mathcal{N})$ is one dimensional, but there is an obstruction to deform this curve in the m-th order because the relation (85) breaks down in the m-th neighborhood $\mathcal{I}^{m+1}$ [37]. As we observe previously, this is due to the superpotential term $W_{CIV}(\Phi) = g_{m+1}\Phi^{m+1}$ [49, 38]. This superpotential can be also generated by the holomorphic Chern-Simons actions [38, 40, 41].

More generally, recall that the $\mathcal{N} = 1$ geometry for the degenerate superpotential is given by:

$$x_{p+1} = u_p^{-1}, \ v_{p+1} = v_p, \ u_{p+1} = x_p u_p^2 - v_p^m u_p. \tag{86}$$

Let $E = \cup_{i=1}^n E_i$ be the exceptional locus of the resolution $\sigma : \tilde{X} \to X$ where E_i is isomorphic to $\mathbf{P}^1$. Let $\mathcal{I}_E$ be the ideal sheaf of E. The relation (86) is reduced to

$$x_{p+1} = u_p^{-1}, \ v_{p+1} = v_p, \ u_{p+1} = x_p u_p^2, \qquad (\mathrm{mod}\ \mathcal{I}_E^2), \tag{87}$$

which can be identified with a tubular neighborhood of the total space of the normal bundle $\mathcal{N}_E = \mathcal{I}_E/\mathcal{I}_E^2$ and E can be identified with the zero section. The space (86) (more generally (75)) can be obtained as a modification of the complex structure from (87). The modification of complex structure is realized by perturbing the Cauchy-Riemann operator $\bar{\partial}_{\bar{j}}$ on (86) by

$$\bar{D} = \bar{\partial}_{\bar{j}} + A^i_{\bar{j}}\partial_i \tag{88}$$

where $A^{\bar{i}}_{\bar{j}}$ is an anti-holomorphic one form taking values in the tangent bundle of (86). We assume that the zero section E of the normal bundle $\mathcal{N}_E$ remains holomorphic. The space of C^∞ deformations of E is identified with the space (s_1, s_2) of C^∞ sections of the normal bundle $\mathcal{N}_E$. The relevant holomorphic Chern-Simon action is

$$\int_E (s_1 \bar{D} s_2 - s_2 \bar{D} s_1). \tag{89}$$

More generally, we may consider the normal bundle $\mathcal{N}_{E_{j,k}}$ of

$$E_{j,k} := E_j \cup E_{j+1} \cup \cdots \cup E_k \tag{90}$$

and restrict the Chern-Simon action. Then the variations with respect to the section (s_1, s_2) of $\mathcal{N}_{E_{j,k}}$ give the conditions that the corresponding curve is holomorphic.

So far we have counted the $\mathbf{P}^1$ cycles corresponding to the Higgsed branch whose gauge group is of the form $U(M_{i,i})$ i.e. ones with the zero meson expectation values. There should be more $\mathbf{P}^1$ cycles corresponding to the non-zero meson expectation values. Where are they? They are not confined in a union of two open sets as $\mathbf{P}^1$ cycles consider above, and rather spread out into several open sets. Consider the inverse images of the singular points $p_{j,k,l}$ of X, which is one of the intersection points given by (71), under the blowing-down map σ (73). Then the inverse image $\sigma^{-1}(p_{j,k,l})$ in each open set U_p, $p = j, \ldots, k$ will be a non-compact curve isomorphic to $\mathbf{C}^1$ and that they will glue together to form a $\mathbf{P}^1$.

To see the exceptional $\mathbf{P}^1$'s corresponding to the non-zero meson expectation values more manifestly, we introduce another resolution picture of (70) which is essentially the same as the one considered above. Consider a subvariety

$$\tilde{Y} \subset \underbrace{\mathbf{P}^1 \times \cdots \times \mathbf{P}^1}_{n} \times X \tag{91}$$

defined by for $p = 1, \ldots, n$

$$\begin{aligned} s_p x = t_p \textstyle\prod_{j=0}^{p-1} \left(u - \sum_{i=0}^{j} W_i'(v)\right), \quad s_p \prod_{j=p}^{n} \left(u - \sum_{i=0}^{j} W_i'(v)\right) = t_p y, \\ t_{p-1} s_p = s_{p-1} t_p \left(u - \textstyle\sum_{i=0}^{p-1} W_i'(v)\right), \end{aligned} \tag{92}$$

where (s_p, t_p) is the homogeneous coordinates of the p-th $\mathbf{P}^1$ and (s_0, t_0) is set to be $(0, 0)$. Then the natural projection

$$\tau : \tilde{Y} \to X \tag{93}$$

will be the resolution of X. It is easy to check that $\tilde{Y}$ is smooth. Consider a singular point given by $x=y=0$ and the intersection of two curves (71) with $j<k$. The fiber of τ (93) over such a point is a $\mathbf{P}^1$ given by

$$\begin{aligned} &t_p = 0 && \text{for } p<j,\\ &t_{p-1}s_p = s_{p-1}t_p \textstyle\sum_{i=l}^{p-1} W_i'(v) && \text{for } j<p\le k,\\ &s_p = 0 && \text{for } p>k. \end{aligned} \tag{94}$$

Notice that $\sum_{i=l}^{p-1} W_i'(v)$ is non-zero for $j<p\le k$. This shows that $\mathbf{P}^1$ is diagonally embedded into the product of $k-j+1$ $\mathbf{P}^1$'s beginning at the j-th and ending in k-th $\mathbf{P}^1$ in (91) which corresponds to the fact that the gauge groups $U(M_{j,k,l})$, $l=1,\dots,d_{j,k}$ are diagonally embedded in $U(N_j)\times\cdots\times U(N_k)$.

To see the T-dual picture, note that the circle action (18) can be extended to X and lifted to $\tilde{X}$ as follows:

$$\begin{aligned} S_X:\quad & X\times\mathbf{S}^1\to X, && (95)\\ & (e^{i\theta},x)=e^{i\theta}x,\quad (e^{i\theta},y)=e^{-i\theta}y,\quad (e^{i\theta},u)=u,\quad (e^{i\theta},v)=v,\\ S_{\tilde{X}}:\quad & \tilde{X}\times\mathbf{S}^1\to\tilde{X}, && (96)\\ & (e^{i\theta},u_p)=e^{i\theta}u_p,\quad (e^{i\theta},x_p)=e^{-i\theta}x_p,\quad (e^{i\theta},v_p)=v_p. \end{aligned}$$

Under this action, the blowing-up map σ (73) and the gluing map (75) are equivariant. In particular, the $\mathbf{P}^1$ cycles defined by (76) and (77) are invariant under the action. So if we take T-dual along the orbits of the circle action $S_{\tilde{X}}$, then D5 branes wrapping $\mathbf{P}^1$'s will become D4 branes on the interval and NS branes will appear where the orbits degenerate [44]. So there will be one NS brane for each open set U_p located at $u_p=x_p=0$ and stretched along v_p. Under the blow-down map, it will map to

$$x=y=0,\quad u=\sum_{i=0}^{p} W_i'(v). \tag{97}$$

which shows NS brane is curved into the u-direction. So we will have NS branes wrapping $(n+1)$ holomorphic curves in the $u-v$ space and the stacks of D4 branes between them. This proves that the T-dual picture of D5 branes wrapping $\mathbf{P}^1$ cycles in $\tilde{X}$ is exactly the same as the brane configuration constructed for the $\mathcal{N}=1$ $\mathbf{A}_n$ quiver theory for the arbitrary superpotential W (19).

The geometry for the $\mathcal{N}=1$ $\mathbf{D}_n$ quiver theory is similar. It will be given by the resolution of a singular threefold defined by the same equation as in (70) except the fact that the mass of the last two adjoints are assumed to be the same i.e. $g_{n-1}=g_n$. So the analysis is similar.

7 Large N Duality Proposal and Normalizable Deformations

In the IR limit, the $\mathcal{N}=1$ A-D-E quiver theory is equivalent to a pure $\mathcal{N}=1$ gauge theory. So we expect to have gaugino condensation and mass gap as noticed in [6]. In the large N description, the theory lives on a geometry where the $\mathbf{P}^1$ cycles have shrunk and $\mathbf{S}^3$ cycles have grown and RR fluxes through them and NS fluxes through their dual cycles have been created. We will mainly consider the case when W_i's are of the same degree.

We first consider the case of the degenerate superpotential (44) whose $\mathcal{N}=1$ geometry is :

$$X_{\deg}: F(x,y,u,v) := xy - \prod_{k=0}^{n}\left(u - \sum_{i=0}^{k} g_i v^m\right) = 0, \tag{98}$$

where g_0 is defined to be zero.

The singularity $X_{\deg}$ admits the $U(1)$ symmetry group

$$x \to e^{i\theta/2}x, \quad y \to e^{i\theta/2}y, \quad u \to e^{i\theta/(n+1)}u, \quad v \to e^{i\theta/m(n+1)}v. \tag{99}$$

If F is viewed as the superpotential of a Landau-Ginzburg theory [45], it would flow to a superconformal theory with central charge given by

$$\hat{c} = (1-2Q(x)) + (1-2Q(y)) + (1-2Q(u)) + (1-2Q(v)) = \frac{2mn-2}{m(n+1)}. \tag{100}$$

The miniversal deformation space [39] of the singularity, which describes the most general complex deformations, is given by the chiral (or Milnor) ring

$$\mathcal{R} := \frac{\mathbf{C}\{x,y,u,v\}}{(\partial F/\partial x, \partial F/\partial y, \partial F/\partial u, \partial F/\partial v)}, \tag{101}$$

which is generated by the monomials $m_{i,j} = u^i v^j$ with the charges

$$Q_{i,j} = \frac{i}{n+1} + \frac{j}{m(n+1)}. \tag{102}$$

The Poincaré series is given by

$$P_{\mathcal{R}}(t) := \sum_{m_{i,j}\in\mathcal{R}} t^{Q_{i,j}} = \frac{(1-t^{1-Q(x)})(1-t^{1-Q(y)})(1-t^{1-Q(u)})(1-t^{1-Q(v)})}{(1-t^{Q(x)})(1-t^{Q(y)})(1-t^{Q(u)})(1-t^{Q(v)})}, \tag{103}$$

and so the dimension of the ring $\mathcal{R}$ is

$$\dim_{\mathbf{C}}\mathcal{R} = P_{\mathcal{R}}(1) = n(nm+m-1). \tag{104}$$

The Milnor ring characterizes the geometry of the generic deformation of $F = 0$. Namely the Milnor fiber

$$\{(x, y, u, v) \in \mathbf{C}^4 | F(x, y, u, v) = \epsilon \ll 1\} \tag{105}$$

is homotopic to a bouquet of $n(nm+m-1)$ $\mathbf{S}^3$'s. Moreover the most general deformation of $F(x, y, u, v)$ is given by

$$X_\epsilon : F_\epsilon(x, y, u, v) := F(x, y, u, v) + \sum_{m_{i,j} \in \mathcal{R}} \epsilon_{i,j} m_{i,j} = 0. \tag{106}$$

As shown in [45], in order for the deformation to corresponds to the dynamical parts of the theory at the singularity, the cohomology classes created by the deformation should be supported on the singularity i.e. vanishing cohomologies. The Poincaré duals to the vanishing cohomology classes are the vanishing homologies which arises from quadratic singularities (i.e. conifold singularities). These classes correspond to the normalizable (including log-normalizable) three forms which satisfies:

$$\lim_{\delta \to 0} \int_{X_\epsilon \cap \mathbf{B}^6_\delta} \left| \frac{\partial \Omega_{X_\epsilon}}{\partial \epsilon_{i,j}} \right|^2 \to \infty, \tag{107}$$

where Ω_{X_ϵ} is the holomorphic three form on X_ϵ and $\mathbf{B}^6_\delta$ is the six dimensional ball with radius δ located at the singularity. This normalizability is necessary in order for the deformation to describe the large N dual of the original theory because the original theory should be valid near the singularity, and also insures that the geometric transitions will be the conifold transitions locally.

The $U(1)$ symmetry of the singularity $F = 0$ can be extended to the deformed space $F_\epsilon = 0$ by giving the charges to the deformation parameters $\epsilon_{i,j}$ and further extended to $\wedge^3 T^*(F_\epsilon = 0)$. The holomorphic three form

$$\Omega_\epsilon = \frac{dy \wedge du \wedge dv}{\partial F_\epsilon / \partial x} \tag{108}$$

has charge

$$Q(\Omega_\epsilon) = Q(x) + Q(y) + Q(u) + Q(v) - 1 = 1 - \frac{\hat{c}}{2}. \tag{109}$$

The integral (107) can be written as

$$\frac{\partial^2}{\partial \epsilon_{i,j} \overline{\partial} \epsilon_{i,j}} \int_{X_\epsilon \cap \mathbf{B}^6_\delta} \Omega_{X_\epsilon} \wedge \overline{\Omega}_{X_\epsilon}, \tag{110}$$

and it diverges when

$$Q(\Omega_{X_\epsilon}) + Q(u^i v^j) - 1 \leq 0. \tag{111}$$

This holds when

$$Q(u^i v^j) = \frac{i}{n+1} + \frac{j}{m(n+1)} \leq \frac{\hat{c}}{2} = \frac{mn-1}{m(n+1)}. \tag{112}$$

Therefore, the normalizable deformations are generated by the monomials $u^i v^j$ with $mi + j \leq mn - 1$ and there are $m(n+1)n/2$ of them which give the dynamical parts of the dual theory (gluino condensation). Since there are $n(nm + m - 1)$ deformation parameters in total, there are

$$\frac{m(n+1)n}{2} - n \tag{113}$$

non-normalizable deformations which must be used to specify the theory externally (i.e. fixing the parameters of the tree-level superpotential). So in the Milnor fiber (105), there are two kinds of $\mathbf{S}^3$, ones corresponding to the normalizable deformations and the others corresponding to the non-normalizable deformations.

We now further deform the theory by adding lower order terms in the superpotential i.e. we consider the general superpotential (19) with $d_i = m$. Then the $\mathcal{N} = 1$ geometry is given by the deformation of :

$$X^m : F_m(x, y, u, v) := xy - \prod_{i=0}^{n} \left(u - \sum_{i=0}^{k} W_i'(v) \right) = 0, \tag{114}$$

which is the same as (70) with constraint $d_i = \deg(W_i) = m$ (by abusing the notation, we write X for X^m if there is no confusion). Now X has only conifold singularities (this is due to our assumption that all the curves $u = \sum_{i=0}^{k} W_i'(v)$ are smooth and they meet transversally each other.) and all non-normalizable deformations are absorbed in the lower order terms of the superpotential and the remaining deformations are all normalizable. Hence the deformation of X is given by

$$X_{\text{def}} : G_m(x, y, u, v) := F_m(x, y, u, v) + \sum_{mi+j \leq mn-1} \epsilon_{i,j} u^i v^j = 0. \tag{115}$$

Therefore the geometric transition is the transition from the resolution $\tilde{X}$ of X to the deformation X_{def} where the exceptional $\mathbf{P}^1$ cycles have been replaced by the Lagrangean three cycles $\mathbf{S}^3$'s.

Before we discuss the geometric transition in full details, we would like to investigate the deformation from the $\mathcal{N} = 1$ geometry X_{deg} with the degenerate superpotential W_{deg} to the $\mathcal{N}=1$ geometry X with the general superpotential W. In deforming from X_{deg} to X^m, there will be $m(n+1)n/2 - n$ $\mathbf{S}^3$'s appearing in the deformed geometry X^m corresponding to the non-normalizable deformations. The way the $\mathbf{S}^3$ cycles appear can be seen as follows. There are $m(n+1)n/2 - n$ non-contractible 1-cycles in the NS brane configuration in the $u - v$ plane defined by (67). Figure 6 has 14 non-contractible 1-cycles. In general, one can prove that there are

$$d - n = \sum_{j=1}^{n}\sum_{k=j}^{n} d_{j,k} - n = \sum_{k=1}^{n}\sum_{j=1}^{k} d_{j,k} - n \tag{116}$$

non-contractible 1-cycles by induction on n. For $n = 1$, the first NS brane $u = W_1'(v)$ meets the v-axis at $d_{1,1}$ points so that there are $d_{1,1} - 1$ enclosed 1-cycles since any two consecutive intersection points determine 1-cycle. The k-th NS brane meets the $(j-1)$-th NS brane at $d_{j,k}$ points and so there are $\sum_{j=1}^{k} d_{j,k}$ new intersection points comparing with the case $n = j - 1$. This produces $\sum_{j=1}^{k} d_{j,k} - 1$ new non-contractible 1-cycles. Hence this proves (116).

Now note that each 1-cycle γ bounds a compact real two dimensional domain Δ_γ in the $u - v$ plane. For example, one of them in the $\mathbf{A}_3$ quiver theory is shown in Figure 6 as a shaded region. We claim that the inverse image of the compact domain bounded by the 1-cycle under the projection map

$$\pi : X^m \to \mathbf{C}^2 : \ (x, y, u, v) \to (u, v). \tag{117}$$

contains a $\mathbf{S}^3$. One can see this heuristically as follows: the inverse image of the interior point of the domain is a hyperboloid i.e. $\mathbf{C}^*$ and at the boundary point of the domain, the waists will shrink to zero i.e. the inverse image of a union of two complex lines meeting transversely. In Figure 7, the inverse image, up to homotopy, of the line joining the boundary points of the polyhedron formed by the NS branes in the case of the quadratic superpotential is shown, which is homotopic to $\mathbf{S}^2$. More rigorously, we may assume, after coordinate changes, that the domain Δ_γ lies in the real part of the $u - v$ plane and X^m is defined by an equation

$$x^2 + y^2 + \prod_{i=0}^{n}\left(u - \sum_{i=0}^{k} W_i'(v)\right) = 0. \tag{118}$$

and the function

$$f(u, v) = \prod_{i=0}^{n}\left(u - \sum_{i=0}^{k} W_i'(v)\right) \tag{119}$$

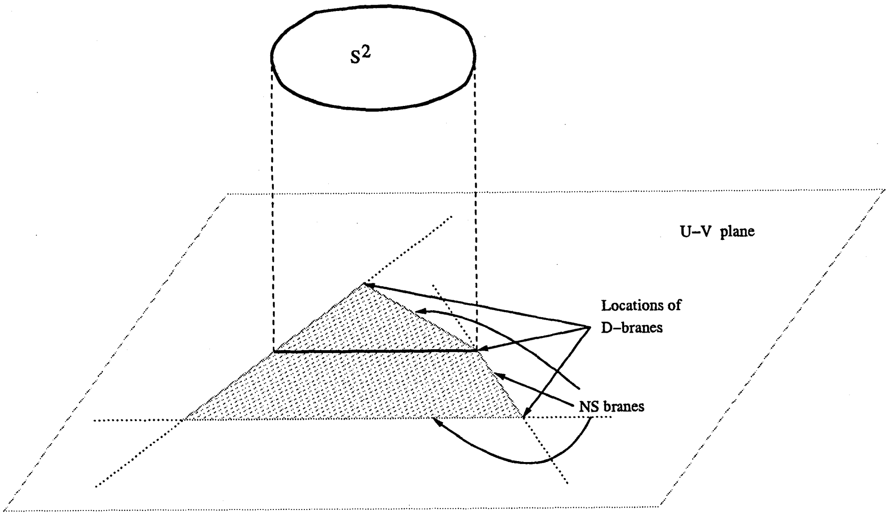

Figure 9: The threefold geometry over $u-v$ plane: An $\mathbf{S}^2$ lies over the thick line and an $\mathbf{S}^3$ lies over the shaded quadrilateral.

as a function of the real variables is not positive. Note that $f(u,v)$ is zero on the boundary of each domain. Moreover, one can show that the function $f(u,v)$ has a unique extremum in the interior of each domain bounded by 1-cycles by induction on n. Then one can construct a homeomorphism h from a unit disk D in the real $u-v$ plane to a domain bounded by 1-cycle sending each concentric circles to the isothermal curves of $f(u,v)$. By pulling back the fiberation

$$\pi_{\Delta_\gamma} : \pi^{-1}(\Delta_\gamma) \to \Delta_\gamma \tag{120}$$

to the disk D, we obtain a fiberation $h^*(\pi_{\Delta_\gamma})$ over a unit disk D which can be written as

$$h^*(\pi) : \{(x,y,u,v) \in \mathbf{R}^4 | x^2+y^2+u^2+v^2 = 1\} \to D, \quad (x,y,u,v) \to (u,v) \tag{121}$$

and this shows that $\pi^{-1}(\Delta_\gamma)$ is homeomorphic to S^3.

So there are $mn(n+1)/2 - n$ (in general $d-n$) $\mathbf{S}^3$ cycles in the blown-down geometry X^m (in general X) which are non-dynamical parameters of the theory. Of course, the appearing $\mathbf{S}^3$'s are completely fixed by the superpotential so by the expectation values of the adjoints and bifundamentals. Conversely, the superpotential is fixed by the $\mathbf{S}^3$'s. Having all non-dynamical deformations fixed by expectation values of the adjoints and bifundamentals (in geometry, there are only conifold singularities in the blown-down geometry), the $\mathcal{N}=1$ geometry (73) can go through the geometric transition where the rigid $\mathbf{P}^1$'s will disappear and will be replaced by the finite size of $\mathbf{S}^3$. The equation of the new geometry Y_ϵ is given by

$$Y_\epsilon:\ G_m(x,y,u,v) = xy - \prod_{i=0}^{n}\left(u - \sum_{i=0}^{k} W_i'(v)\right) + \sum_{mi+j\leq mn-1} \epsilon_{i,j} u^i v^j = 0. \quad (122)$$

Under this deformation, each singular point of the blown-down geometry X which corresponds to the intersection points of the NS branes (Figure 4) split into two points on the smooth curve in the $u-v$ plane and replaced by $\mathbf{S}^3$ cycles. The holomorphic three form Ω on Y_ϵ is given by

$$\Omega = \frac{dxdydudv}{dG_\epsilon} = \frac{dydudv}{y}. \quad (123)$$

Since the NS branes meet transversally at each singular point $p_{j,k,l}$, we can locally write the equation of X as $x^2+y^2+u^2+v^2=0$ and, thus the deformed geometry Y_ϵ can be written as

$$x^2+y^2+u^2+v^2 = \mu_{j,k,l} \quad (124)$$

with holomorphic three form

$$\Omega = \frac{dx\, dy\, du}{\sqrt{\mu_{j,k,l}-x^2-y^2-u^2}} \quad (125)$$

Therefore, the period of the holomorphic three-form Ω_ϵ over the 3-cycle $A_{j,k,l}$ of (122), which is a compact 3-sphere, is given by

$$S_{j,k,l} = \int_{A_{j,k,l}} \Omega_\epsilon \sim \frac{\mu_{j,k,l}}{4} \quad (126)$$

and the period over the dual $B_{j,k,l}$ cycle is

$$\Pi_{j,k,l} = \int_{B_{j,k,l}}^{\Lambda_{j,k,l}} \Omega_\epsilon \sim \frac{1}{2\pi i}\left(-3S_{j,k,l}\log\Lambda_{j,k,l} - S + S\log S\right) + \ldots. \quad (127)$$

As in the other geometric transitions [4, 6], $S_{j,k,l}$ is identified with the glueball field $S_{j,k,l} = -\frac{1}{32\pi^2}\mathrm{Tr}_{SU(M_{j,k,l})} W_\alpha W^\alpha$ of the non-Abelian factor $SU(M_{j,k,l})$ of $U(M_{j,k,l})$ in (57) in the dual theory. The $S_{j,k}$ will be massive and obtain particular expectation values due to the superpotential $W_{\rm eff}$. The dual superpotential $W_{\rm eff}$ arises from the non-zero fluxes left after the transition. The deformed geometry will have $M_{j,k,l}$ units of H_R flux through the $A_{j,k,l} \cong \mathbf{S}^3$ cycle due to $M_{j,k,l}$ D5 branes wrapped on the $\mathbf{P}^1$ cycle before the transition, and there is also an H_{NS} flux $\alpha_{j,k}$ through each of the dual non-compact cycle $B_{j,k,l}$ with $2\pi i\alpha_{j,k} = 8\pi^2/g_0^2$ given in terms of the bare coupling constant g_0 of the $U(M_{j,k,l})$ theory.

The appearance of an NS field in the deformed geometry can be understood as follows. In the mirror resolve geometry it is an NS 4-form related to the change in complex structure [2] (see also [28]). This should be mapped in the the resolved geometry to H_{NS} which is due to a change in Kähler structure of Y_ϵ whose origin comes from the Kähler structure change of $\tilde{X}$ due to the superpotential term

$$\sum_{i=j}^{k} W_i \tag{128}$$

whose leading coefficient gives the size of $\mathbf{P}^1$ arising by blowing up the singular point $p_{j,k,l}$. So H_{NS} is of the form

$$H_{NS} = \frac{1}{2}(\partial + \bar{\partial})J \tag{129}$$

where J is the $(1,1)$ Kähler form representing the Kähler structure change of Y_ϵ.

Thus the effective superpotential is

$$-\frac{1}{2\pi i}W_{\text{eff}} = \sum_{j=1}^{n}\sum_{k=j}^{n}\sum_{l=1}^{d_{j,k}}(M_{j,k,l}\Pi_{j,k,l} + \alpha_{j,k}S_{j,k,l}). \tag{130}$$

After identifying $\Lambda_{j,k,l}$ with the UV-cutoff, we obtain the usual lower energy superpotential associated with the $SU(M_{j,k,l})$ glueballs:

$$W_{\text{eff}} = \sum_{j=1}^{n}\sum_{k=j}^{n} S_{j,k,l}\left(\log\frac{\Lambda_{j,k,l}^{3M_{j,k,l}}}{S_{j,k,l}^{M_{j,k,l}}} + M_{j,k,l}\right). \tag{131}$$

Integrating out the massive $S_{j,k,l}$ by solving

$$\frac{\partial W_{\text{eff}}}{\partial S_{j,k,l}} = 0 \tag{132}$$

leads to $M_{j,k,l}$ supersymmetric vacua of $SU(M_{j,k,l})$ $\mathcal{N}=1$ supersymmetric Yang-Mills:

$$\langle S_{j,k,l}\rangle = \exp(2\pi i m/M_{j,k,l})\Lambda_{j,k,l}^3, \quad m = 1, \dots, M_{j,k,l}. \tag{133}$$

The dual theory obtained after the transition is an $\mathcal{N}=2$

$$\prod_{j=1}^{n}\prod_{k=j}^{n}\prod_{l=1}^{d_{j,k}} U(1) \equiv U(1)^d \tag{134}$$

gauge theory broken to $\mathcal{N} = 1$ $U(1)^d$ by the superpotential W_{eff} (130) [52]. The $S_{j,k,l}$, which are the same $\mathcal{N} = 2$ multiplet as the the $U(1)^d$, get masses and frozen to particular $\langle S_i \rangle$ by W_{eff}. On the other hand, the $\mathcal{N} = 1$ $U(1)^d$ gauge fields remain massless. The couplings $\tau_{j,k,l,j',k',l'}$ of these $U(1)$'s can be determined by $\Pi_{j,k,l}$ or the $\mathcal{N} = 2$ prepotential $\mathcal{F}(S_{j,k,l})$, with $\Pi_{j,k,l} = \partial \mathcal{F} / \partial S_{j,k,l}$:

$$\tau_{i,j,k,i',j',k'} = \frac{\partial \Pi_{i,j,k}}{\partial S_{j',k',l'}} = \frac{\partial^2 \mathcal{F}(S_{j,k,l})}{\partial S_{j,k,l} \partial S_{j',k',l'}}. \tag{135}$$

The couplings should be evaluated at the vacua $\langle S_{j,k,l} \rangle$ obtained in (133). As in [4], the coupling constants of the $U(1)$ factors are related to the period matrix of the curve

$$\prod_{i=0}^{n} \left(u - \sum_{i=0}^{k} W_i'(v) \right) + \sum_{mi+j \leq mn-1} \epsilon_{i,j} u^i v^j = 0, \tag{136}$$

which is an $(n+1)$-fold covering of the v-space. By integrating the period integral of Y_ϵ first along the fibers of the projection from Y_ϵ to the v-space, one might be able to compute the coupling constants $\tau_{j,k,l,j'k'l'}$ in terms of the periods integral of the curve (136). Instead of doing this, we will take type IIA picture by T-duality and then lift to M-theory. From M-theory perspective, it will be clear that the curve (136) is indeed a Seiberg-Witten curve for the $\mathcal{N} = 1$ $U(1)^d$ theory.

To consider type IIA picture, note that the circle action considered for the various geometries so far can also be extended to the geometry after the transition because it acts only on the $x - y$ space. The orbits degenerate along $x = y = 0$. So in the T-dual picture, we have an NS brane wrapping the curve in the $u - v$ plane defined by

$$\prod_{i=0}^{n} \left(u - \sum_{i=0}^{k} W_i'(v) \right) + \sum_{mi+j \leq mn-1} \epsilon_{i,j} u^i v^j = 0, \tag{137}$$

which is exactly (136).

8 M Theory and the M5 brane Transition

We now lift the type IIA brane configurations for the $\mathcal{N} = 1$ $\mathbf{A}_n$ quiver theory, deformed by the superpotential (43), to M theory and investigate the large N limit via Witten's MQCD formalism. For simplicity, we will often restrict to the case with the degree of W_i are the same, denoted by $m+1$. We denote the finite direction of D4 branes by x^7 and the angular coordinate of the circle $\mathbf{S}^1$ in the 11-th dimension by x^{10}. Thus the NS branes are separated along the x^7 direction. We combine them into a complex coordinate

$$t = \exp(-R^{-1}x^7 - ix^{10}) \tag{138}$$

where R is the radius of the circle $\mathbf{S}^1$ in the 11-th dimension. In MQCD [50], the classical type IIA brane configuration turns into a single fivebrane whose world-volume is a product of the Minkowski space $\mathbf{R}^{1,3}$ and the M-theory curve Σ in a flat Calabi-Yau manifold

$$M = \mathbf{C}^2 \times \mathbf{C}^*, \tag{139}$$

whose coordinates are u, v, t.

Classically , the u, v-coordinates of the M-theory curve Σ describes the location of NS branes in the type IIA picture. Hence it is given by Figure 6:

$$\prod_{i=0}^{n} \left(u - \sum_{i=0}^{k} W_i'(v) \right) = 0. \tag{140}$$

By lifting this to the M-theory, the location of the D4 branes on the NS brane will be smeared out as D4 brane acquire 11-th direction. So, at quantum level, the configuration (140) will be deformed and the u, v coordinates of Σ will satisfy:

$$\prod_{i=0}^{n} \left(u - \sum_{i=0}^{k} W_i'(v) \right) + \sum_{mi+j \leq mn-1} \epsilon_{i,j} u^i v^j = 0. \tag{141}$$

Note that we have added the only deformations which will modify the intersection of NS5 and D4 branes in type IIA picture and there are $m(n+1)n/2$ complex parameters corresponding to the number of the stacks of D4 branes. Figure 10 shows the quantum deformation of the NS brane configuration for the $\mathbf{A}_2$ theory with quadratic superpotential (Figure 4) where the union of three lines is deformed into a curve of genus one.

When Σ is projected onto the $u - v$ plane, it will be denoted by Σ_{NS} and becomes a non-hyperelliptic curve for $n > 1$, unlike most cases in the literature. In fact, Σ_{NS} is a $(n+1)$-fold branched covering of a complex plane and can be compactified to a projective curve $\overline{\Sigma_{NS}}$ with singularities at infinity for $m > 1$ with geometric genus (i.e. the genus after resolving singularities) is

$$g(\overline{\Sigma_{NS}}) = \frac{n(mn+m-2)}{2} \tag{142}$$

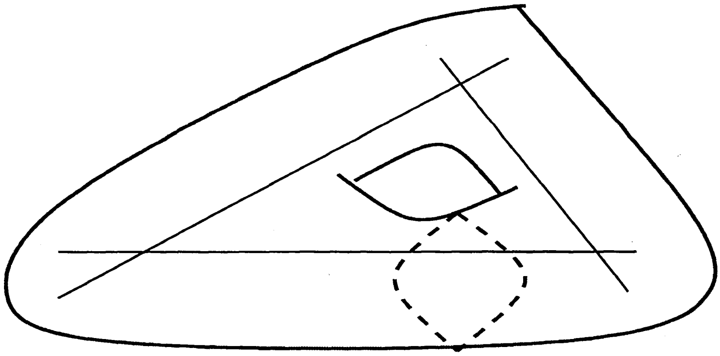

Figure 10: The line configuration has deformed to a smooth curve by quantum effect in M-theory.

which is the same as the rank of the first homology of the NS brane configurations in the $u - v$ plane. Heuristically, one can see from Figure 10 that Σ_{NS} is a small deformation of the NS brane configuration Figure 4, so the number of holes (i.e. genus) is the same as the number of non-contractible 1-cycles in the NS brane configuration. In general, the genus of $\overline{\Sigma_{NS}}$ is

$$g(\overline{\Sigma_{NS}}) = d - n \tag{143}$$

where d is the number of the Higgsed groups defined in (37). For example, Σ_{NS} corresponding to Figure 6 will be a curve of genus 14. As explained in [8], the coupling of over-all $U(1)$ of each $U(N_i)$ is described by the non-compact part of the Jacobian $J(\Sigma_{NS})$ which fits into an exact sequence of algebraic groups:

$$1 \longrightarrow (\mathbf{C}^*)^n \longrightarrow J(\Sigma_{NS}) \longrightarrow J(\overline{\Sigma_{NS}}) \longrightarrow 0. \tag{144}$$

Hence this produces the exact number of $U(1)$ gauge factors in the $\mathcal{N} = 1$ transition!

Because Σ is not rational, one cannot parameterize the t-coordinates of Σ in terms of u or v. Since there are $M_{j,k,l}$ D4 branes at each intersection point $p_{j,k,l}$, $l = 1, \ldots, m$ of the NS branes

$$u = \sum_{i=0}^{j-1} W_i'(v), \quad u = \sum_{i=0}^{k} W_i'(v) \tag{145}$$

in the $u - v$ plane corresponding to the gauge group $U(M_{jk})$ classically, Σ is a deformation of a finite covering of the NS brane configuration ramified over the intersection points $p_{j,k,l}$ and it is possible to give a parametric description locally which will be used later.

As we go through the transition, the sizes of $\mathbf{P}^1$'s get smaller and there are gaugino condensations. In M-theory the t-coordinates which describe the $SU(M_{j,k,l}$ parts of $U(M_{j,k,l})$ will be fixed and then the remaining $U(1)$ parts of the theory will be given by a plane curve in the $u-v$ plane given by

$$\prod_{i=0}^{n}\left(u-\sum_{i=0}^{k}W_i'(v)\right)+\sum_{mi+j\leq mn-1}\zeta_{i,j}u^iv^j=0. \tag{146}$$

where $\zeta_{i,j}$ are fixed by the gaugino condensation which breaks the chiral symmetries. So this is the quantum moduli of the remaining $U(1)^d$ theory where d is the total number of the Higgsed gauge groups given in (37) since the $U(1)$ parts of $U(N)$ theory is parameterized by the center of mass coordinates of D4 branes. Hence from the M-theory, the geometric transition is nothing but a transition from the space curve Σ to the plane curve.

Let us consider this dual theory from $\mathcal{N}=2$ Seiberg-Witten theory. To motivate our method, we consider the $\mathbf{A}_1$ case. The $\mathcal{N}=2$ theory deformed by $W_{\text{tree}}=\sum_{i=1}^{m+1}g_ru_r$ has unbroken supersymmetry only on submanifolds of the Coulomb branch, where there are additional massless fields besides the u_r. The additional massless fields are the magnetic monopoles or dyons, which become massless on some particular submanifolds $\langle u_p\rangle$ [48]. Near a point with l massless monopoles, the superpotential is

$$W=\sum_{k=1}^{l}M_k(u_r)q_k\tilde{q_k}+\sum_{p=1}^{m+1}g_pu_p, \tag{147}$$

and the supersymmetric vacua are $\langle u_p\rangle$ satisfying

$$M_k(\langle u_p\rangle)=0,\quad \sum_{k=1}^{l}\frac{\partial M_k(\langle u_p\rangle)}{\partial u_p}\langle q_k\tilde{q_k}\rangle+g_p=0, \tag{148}$$

where the first equations are for all $k=1,ldots,l$ and the second for all $r=1,\dots,N$ (with $g_p=0$ for $p>m+1$). The Seiberg-Witten curve of the $U(N)$ theory is

$$u^2+P(v,u_r)u+\Lambda^{2N}=0,\quad P(v,u_r)\equiv\det(v-\Phi)=\prod_{i=1}^{N}(v-e_i). \tag{149}$$

After the coordinate change $2(u+P(v,u_r)/2)$ by y, the curve becomes

$$y^2=P(x,u_r)^2-4\Lambda^{2N}. \tag{150}$$

This is a hyperelliptic curve which has N branch cuts centered about the eigenvalues e_i with endpoints e_i^- and e_i^+. The condition for having $N-n$ mutually local massless magnetic monopole is that

$$P(x, \langle u_p \rangle) - 4\Lambda^{2N} = (H_{N-n}(v))^2 F_{2n}(v), \tag{151}$$

where H_{N-n} is polynomial in v of degree $N-n$ and F_{2n} is a polynomial in x of degree $2n$. Out of N branch cuts, $N-n$ disappear and the curve becomes singular at those points. The remaining n massless photons is described by the reduced curve

$$y^2 = F_{2n}(x, \langle u_r \rangle) = F_{2n}(x, g_p, \Lambda). \tag{152}$$

Classically, we have

$$P(v, u_r) = \prod_{i=1}^{n} (v - v_i)^{N_i} \tag{153}$$

where v_i are the eigenvalues of Φ obtained by solving the tree-level superpotential. Since $\langle u_r \rangle$ are obtained from the classical values after quantum corrections, we have

$$F_{2n} = \prod_{i=1}^{n} (v - v_i^+)(v - v_i^-) \tag{154}$$

where $v_i^{\pm}$ are quantum corrections of v_i. After possible change of the scale Λ, we can consider

$$y^2 = \prod_{i=1}^{n} (v - v_i)^2 - 4\Lambda^{2n} \tag{155}$$

or

$$u^2 + \prod_{i=1}^{n} (v - v_i) u + \Lambda^{2n} = 0 \tag{156}$$

as a reduced curve describing the remaining massless photons.

More generally we begin with the $\mathbf{A}_n$ brane configuration for the $\mathcal{N} = 2$ theory with the gauge group $\prod_{i=1}^{n} U(N_i)$ and hypermultiplets in the representation $\sum_{i=1}^{n-1} (\mathbf{N}_i, \bar{\mathbf{N}}_{i+1})$. The corresponding Seiberg-Witten curve will be a curve defined by a polynomial $F(t, v)$ [47, 29]:

$$\begin{aligned} u^{n+1} - P_1(v) u^n \quad & + \textstyle\sum_{j=2}^{n} (-1)^j (\prod_{i=1}^{j-1} \Lambda_i^{(j-i)\beta_i}) P_j(v) u^{n+1-j} \\ & + (-1)^{n+1} (\prod_{i=1}^{n} \Lambda_i^{(n+1-i)\beta_i}) = 0, \end{aligned} \tag{157}$$

where $\beta_i = 2N_i - N_{i-1} - N_{i+1}$ is the β-function coefficient for $U(N_i)$. Here Λ_i is a $\mathcal{N}=2$ QCD scale for $U(N_i)$ theory and $F(u,v)$ is of degree $(n+1)$ in u so that for each v there are $(n+1)$ roots corresponding to $(n+1)$ NS branes and P_i is of degree N_i whose zeroes are the positions of the N_i D4 branes stretched between the $(i-1)$-th and i-th NS brane. Hence we have

$$P_i(v) = \det(v - \Phi_i) = \prod_{k=1}^{N_i} (v - e_k^{(i)}), \tag{158}$$

where $e_k^{(i)}$ are eigenvalues of Φ_i. The curve (157) describes an $(n+1)$-fold branched covering of the complex plane with $(i-1)$ and i-th sheets connected by N_i branch cuts centered about the eigenvalues $e_k^{(i)}$ with endpoints $e_k^{(i)-}$ and $e_k^{(i)+}$. According to Seiberg-Witten theory [48], the renormalized order parameters, their duals and the prepotential $\mathcal{F}$ are given by

$$a_k^{(i)} = \frac{1}{2\pi}\oint_{A_k^{(i)}} \lambda, \quad a_{D,k}^{(i)} = \frac{1}{2\pi}\oint_{B_k^{(i)}} \lambda, \quad a_{D,k}^{(i)} = \frac{\partial \mathcal{F}}{\partial a_k^{(i)}}, \tag{159}$$

where λ is the Seiberg-Witten differential, and $A_k^{(i)}$ and $B_k^{(i)}$ are a set of canonical homology cycles for the curve (157). The cycle $A_k^{(i)}$ is chosen to be simple contour on sheet i enclosing the branch cut centered about $e_k^{(i)}$ and $B_k^{(i)}$ is the dual cycle. The gauge symmetry is generically broken to

$$\prod_{i=1}^{n} U(1)^{N_i} \tag{160}$$

including the one corresponding to the trace of $U(N_i)$.

At the Higgsed branch with the gauge group

$$\prod_{j=1}^{n}\prod_{k=j}^{n}\prod_{l=1}^{d_{j,k}} U(M_{j,k,l}) \tag{161}$$

the adjoints get the expectation values:

$$P_i(v) = \det(v - \Phi_i) = \prod_{j=1}^{i}\prod_{k=i}^{n}\prod_{l=1}^{d_{j,k}} (v - e_{j,k,l})^{M_{j,k,l}}. \tag{162}$$

In terms of $U(1)^{N_i}$ theory which comes from the factor $U(N_i)$, there are $N_i - \sum_{j=1}^{i}\sum_{k=i}^{n} d_{j,k}$ mutually local massless magnetic monopoles when $N_i -$

$\sum_{j=1}^{i}\sum_{k=i}^{n} d_{j,k}$ corresponding one cycles collapse. Thus the massless photons can be described by the reduced curve

$$\begin{aligned} u^{n+1} - \tilde{P}_1(v)u^n \quad &+ \textstyle\sum_{j=2}^{n}(-1)^j(\prod_{i=1}^{j-1}\Lambda_i^{(j-i)\beta_i'})\tilde{P}_j(v)u^{n+1-j} \\ &+(-1)^{n+1}(\textstyle\prod_{i=1}^{n}\Lambda_i^{(n+1-i)\beta_i'}) = 0, \end{aligned} \tag{163}$$

where

$$\begin{aligned} \tilde{P}_i(v) &= \textstyle\prod_{j=1}^{i}\prod_{k=i}^{n}\prod_{l=1}^{d_{j,k}}(v - e_{j,k,l}), \\ \beta_i' &= 2\textstyle\sum_{j=1}^{i}\sum_{k=i}^{n} d_{j,k} - \sum_{j=1}^{i-1}\sum_{k=i-1}^{n} d_{j,k} - \sum_{j=1}^{i+1}\sum_{k=i+1}^{n} d_{j,k}. \end{aligned} \tag{164}$$

The reduced curve is an $(n+1)$-fold branched covering of the complex plane whose branch cuts are centered around the eigenvalues $e_{j,k,l}$ of Φ. The eigenvalues of two different adjoints can be the same. The number of different eigenvalues of all the adjoints is exactly d which is given by (37). After the NS branes become curved in the u, v-space due to the expectation values of the adjoints and the bifundamentals, every different eigenvalue $e_{j,k,l}$ will split into two points with ramification index two. Hence the Euler characteristic χ of the reduced curve after curving have the following relation by Riemann-Hurwitz formula:

$$\chi := 2 - 2g = 2(n+1) - 2d \tag{165}$$

so that the genus g is equal to $d - n$ which is the same as (142). When $\deg\,\Phi_i = m$ for all i, then the genus is

$$g = \frac{m(n+1)n}{2} - n, \tag{166}$$

which is equal to (143). Moreover, the reduced curve after curving and translation is the same as (141) as they describe the same field theory.

In fact, we can show more precisely how the parameters $\zeta_{i,j}$ of (146) are related to the parameters and the $\mathcal{N} = 2$ QCD scales. For simplicity, we consider the $\mathbf{A}_2$ theory where the gauge group breaks as

$$U(N_1) \times U(N_2) \to U(M_1) \times U(M_2) \times U(M_{12}) \tag{167}$$

The $\mathcal{N} = 2$ Seiberg-Witten curve is written as:

$$\begin{aligned} &\Lambda_1^{2N_1-N_2}t^3 - (v-a_1)^{M_1}(v-a_{12})^{M_{12}}t^2 \\ &\qquad + (v-a_2)^{M_2}(v-a_{12})^{M_{12}}t - \Lambda_2^{2N_2-N_1} = 0, \end{aligned} \tag{168}$$

when the D4 between the first and second NS branes are split into two stacks located at $v = a_1, a_{12}$ and D4 between the second and the third at

$v = a_2, a_{12}$. From (46), a_{12} is necessarily the center of the mass coordinates i.e. $a_{12} = (a_1 g_1 + a_2 g_2)/(g_1 + g_2)$. Then in the limit, we have

- $v \to a_1$, $\quad (v - a_1)^{M_1} \sim \Lambda_1^{2N_1 - N_2} t (a_1 - a_{12})^{-M_{12}} + (a_1 - a_2)^{M_2} t^{-1} + \Lambda_2^{2N_2 - N_1} t^{-2} (a_1 - a_{12})^{-M_{12}}$,
- $v \to a_2$, $\quad (v - a_2)^{M_2} \sim -\Lambda_1^{2N_1 - N_2} t^2 (a_2 - a_{12})^{-M_{12}} + (a_2 - a_1)^{M_1} t (a_2 - a_{12})^{-M_{12}} + \Lambda_2^{2N_2 - N_1} t^{-1} (a_2 - a_{12})^{-M_{12}}$,
- $v \to a_{12}$, $\quad (v - a_{12})^{M_{12}} \sim \dfrac{\Lambda_1^{2N_1 - N_2} t^3 + \Lambda_2^{2N_2 - N_1}}{-(a_{12} - a_1)^{M_1} t^2 + (a_{12} - a_2)^{M_2} t}$. $\quad$ (169)

From the projection (146) of $\mathcal{N} = 1$ M-theory curve onto $u - v$ plane , we have

$$u(v - a_i) \sim (-1)^{i-1} \frac{\zeta(0, a_i)}{g_1 g_2 (a_1 - a_2)} := \mu_{ii} \tag{170}$$

around $u = 0, v = a_i, i = 1, 2$. In order to match the first two parameters μ_{11} and μ_{22}, you restrict $\mathcal{N} = 1$ curve over the line $u = 0$. Then we have

$$t \sim (v - a_1)^{-M_1} (v - a_2)^{M_2}. \tag{171}$$

So we have

$$\begin{aligned} t &\sim (v - a_1)^{-M_1} (a_1 - a_2)^{M_2}, \ \lim(u, v) = (0, a_1) \\ t &\sim (a_2 - a_1)^{-M_1} (v - a_2)^{M_2}, \ \lim(u, v) = (0, a_2). \end{aligned} \tag{172}$$

The mass g_i of Φ_i causes the i-th NS brane in $\mathcal{N} = 1$, on which M_i of D4 brane end on the left, rotate to $u = g_i(v - a_i)$. This together with (169) implies that

$$\begin{aligned} g_1^{-M_1} u^{M_1} = (v - a_1)^{M_1} &\sim \Lambda_1^{2N_1 - N_2} t (a_1 - a_{12})^{-M_{12}} + (a_1 - a_2)^{M_2} t^{-1} \\ &\quad + \Lambda_2^{2N_2 - N_1} t^{-2} (a_1 - a_{12})^{-M_{12}} \\ g_2^{-M_2} u^{M_2} = (v - a_2)^{M_2} &\sim -\Lambda_1^{2N_1 - N_2} t^2 (a_2 - a_{12})^{-M_{12}} \\ &\quad + (a_2 - a_1)^{M_1} t (a_2 - a_{12})^{-M_{12}} \\ &\quad - \Lambda_2^{2N_2 - N_1} t^{-1} (a_2 - a_{12})^{-M_{12}} \end{aligned} \tag{173}$$

On the other hand, we obtain from (170) and (172) that

$$\begin{aligned} \mu_{11}^{-M_1} u^{M_1} &\sim t (a_1 - a_2)^{-M_2} \\ \mu_{22}^{-M_2} u^{M_2} &\sim t^{-1} (a_2 - a_1)^{-M_1}. \end{aligned} \tag{174}$$

By comparing (173) and (174) for the large (resp. small) t for μ_{11} (resp. μ_{22}), we obtain

$$\mu_{11}^{M_1} = g_1^{M_1} (a_1 - a_{12})^{-M_{12}} (a_1 - a_2)^{M_2} \Lambda_1^{2N_1 - N_2} \tag{175}$$

$$\mu_{22}^{M_2} = g_2^{M_2} (a_1 - a_{12})^{-M_{12}} (a_2 - a_1)^{M_2} \Lambda_2^{2N_2 - N_1} \tag{176}$$

To compare the scales for the remaining gauge group $U(M_{1,2})$, we take the following form of the $\mathcal{N}=2$ curve

$$t^3-(v-a_1)^{M_1}(v-a_{12})^{M_{12}}t^2+\Lambda_1^{\beta_1}(v-a_2)^{M_2}(v-a_{12})^{M_{12}}t-\Lambda_1^{2\beta_1}\Lambda_2^{\beta_2}=0, \quad (177)$$

which gives

- $v \to a_{12}, \quad (v-a_{12})^{M_{12}} \sim \dfrac{t^2-\Lambda_1^{2\beta_1}\Lambda_2^{\beta_2}t^{-1}}{-(a_{12}-a_1)^{M_1}t+\Lambda^{\beta_1}(a_{12}-a_2)^{M_2}}$. (178)

In order to match the deformation parameter corresponding to $U(M_{12})$, we consider $\mathcal{N}=1$ curve along the line $u=g_1(v-a_1)$. First we will make coordinate changes so that the lines $u-g_1(v-a_1)$ and $u+g_2(v-a_2)$ move to the lines $u=0$ and $v-a_{12}$ and the points $(0,a_1)$ and $(g_1(a_{12}-a_1),a_{12})$ to $(0,a_1)$ and $(0,a_{12})$ in the new coordinate system. This can be done by the change $u-g_1(v-a_1)\to u$ and $(a_1-a_{12})(u+g_2(v-a_2))/(g_2(a_1-a_2))+a_{12}\to v$. Under the new coordinate system, the special direction of $\mathcal{N}=1$ curve is given by

$$t \sim (v-a_1)^{-M_1}(v-a_{12})^{M_{12}} \quad (179)$$

Thus $t\sim(a_{12}-a_1)^{-M_1}(v-a_{12})^{M_{12}}$ around $(0,a_{12})$. We may approximate (141) by

$$u(v-a_{12}) \sim \frac{g_2(a_1-a_2)\zeta(a_{12},g_1(a_{12}-a_1))}{a_{12}(a_1-a_{12})} := \mu_{12}. \quad (180)$$

Since there will be mass contributions from both of the adjoins Φ_1,Φ_2, the curve will rotate to $u=(g_1+g_2)(v-a_{12})$ by adding superpotential. By comparison for small t, we obtain

$$\mu_{12}=g_{12}^{M_{12}}(a_{12}-a_1)^{-M_1}(a_{12}-a_2)^{-M_2}\Lambda^{\beta_1}\Lambda^{\beta_2} \quad (181)$$

where $g_{12}=g_1+g_2$ and β_i is the beta-function for $U(N_i)$.

As in [7], the scales $\Lambda_{j,k}$ of the low energy $U(M_{j,k})$ theory can be determined by naive threshold matching relations at the scales of all massive $U(M_{j,k})$ matter and W-boson fields:

$$\Lambda_{j,k}{}^{3M_{j,k}}=g_{j,k}{}^{M_{j,k}}\prod_{(l,m)\in I^+}(a_{j,k}-a_{l,m})^{M_{l,m}}\prod_{(l,m)\in I^-}(a_{j,k}-a_{l,m})^{-M_{l,m}}\prod_{i=j}^{k}\Lambda_i{}^{\beta_i} \quad (182)$$

where $I^+=\{(l,m)|m=j-1 \text{ or } l=k+1\}$ and $I^-=\{(l,m)|l=j \text{ or } m=k\}-\{(j,k)\}$. Recall that we always assume that the first index is less than equal to the second index in the double indices. For the case under consideration, (182) can be written as

$$\begin{aligned}\Lambda_{1,1}^{3M_1} &= g_1^{M_1}(a_1-a_{1,2})^{-M_{1,2}}(a_1-a_2)^{M_2}\Lambda_1^{2M_1-M_2+M_{1,2}},\\ \Lambda_{2,2}^{3M_2} &= g_2^{M_2}(a_2-a_{1,2})^{-M_{1,2}}(a_2-a_1)^{M_1}\Lambda_2^{2M_2-M_1+M_{1,2}} \quad (183)\\ \Lambda_{1,2}^{3M_{1,2}} &= g_{1,2}^{M_{1,2}}(a_{1,2}-a_1)^{-M_1}(a_3-a_2)^{-M_2}\Lambda_1^{2M_1-M_2+M_{1,2}}\Lambda_2^{2M_2-M_1+M_{1,2}}.\end{aligned}$$

Therefore, we can identify the geometric deformation parameter μ_{ij} with $\mathcal{N}=1$ QCD scale $\Lambda^3_{M_{i,j}}$ of $U(M_i)$.

9 Orientifold Theories

The whole discussions can be extended to the SO/Sp gauge theories as in [5, 9] in the presence of an orientifold O6 plane. The A-D-E singularities defined by (13) are invariant under the complex conjugation on the ambient space $\mathbf{C}^3$

$$(x,y,u)\to(\bar{x},\bar{y},\bar{u}). \quad (184)$$

Moreover, this action can be extended to the resolved ALE space where the exceptional $\mathbf{P}^1$ becomes $\mathbf{RP}^2$. By wrapping N_i D5 branes on each $\mathbf{P}^1$ we obtain the $\mathcal{N}=2$ supersymmetric gauge theories with gauge group

$$\prod_{i=1}^{n} O(N_i) \quad (185)$$

from type IIB theory with orientifolding.

Now consider the $\mathcal{N}=1$ supersymmetric gauge theory deformed by a tree-level superpotential of the form:

$$W_{SO}=\sum_{i=1}^{n}W_i-\mathrm{Tr}\sum_{i=1}^{n}\sum_{j=1}^{n}s_{i,j}Q_{i,j}\Phi_jQ_{j,i}, \text{where } W_i=\mathrm{Tr}\sum_{j=1}^{d_i+1}\frac{g_{i,2j-1}}{2j}\Phi_i^{2j} \quad (186)$$

We can obtain the classical vacua using the method used in Section 4. The eigenvalues of the adjoints are the roots of

$$W_j'(v)+W_{j+1}'(v)+\ldots+W_k'(v)=g_{2d_{j,k}}v\prod_{l=1}^{d_{j,k}}(v^2+b_{j,k,l}^2),\quad b_{j,k,l}>0, \quad (187)$$

where $g_{2d_{j,k}}$ is the sum of the highest coefficient of W_i whose degree is maximal among $W_j, \ldots, W_k$. Following the brane interpretation considered before, we conclude that the $\mathcal{N} = 1$ geometry is given by the resolution of

$$xy - u \prod_{p=1}^{n} \left(u - \sum_{i=1}^{p} W_i'(v) \right) = 0. \tag{188}$$

Here the singularities are arranged so that the geometry is invariant under the orientifold action

$$(x, y, u, v) \to (\bar{x}, \bar{y}, \bar{u}, \bar{v}). \tag{189}$$

Because of the presence of the orientifold plane located at $u = v = 0$, the $\mathbf{P}^1$ cycle at $u = v = 0$ becomes an $\mathbf{RP}^2$ cycle stuck on the orientifold plane and the gauge theory on the D5 brane wrapped on it is $O(2M_0)$. The D5 branes on $\mathbf{P}^1$ located at $v = ib_{j,k,l}$ are identified with the D5 brane on $\mathbf{P}^1$ located at $v = -ib_{j,k,l}$ and the gauge theory on the corresponding D5 brane is $U(M_{j,k,l})$. Therefore the gauge group will break:

$$\prod_{i=1}^{n} O(N_i) \to \prod_{i=1}^{n} O(M_i) \prod_{j=1}^{n} \prod_{k=j}^{n} \prod_{l=1}^{d_{j,k}} U(M_{j,k,l}), \tag{190}$$

where $d_{j,k}$ are defined in (34). Under geometric transition, the $\mathbf{P}^1$ cycles shrink and is replaced by $\mathbf{S}^3$. The $\mathbf{S}^3$ located at $u = v = 0$ is invariant under the orientifold action, and the $\mathbf{S}^3$ located at $v = ib_{j,k,l}$ maps to one located at $v = -b_{j,k,l}$ and vice-versa.

Since the circle actions are compatible with the orientifolding, we can take T-dual pictures and lift to M-theory. By M-theory transition, we obtain a Seiberg-Witten curve in the $u - v$ plane for $\mathcal{N} = 1$ $U(1)^d$ theory with $\mathbf{Z}_2^n$ global symmetries generalizing the results of [5, 9].

One can also use orientifolds O4 planes which would imply that we start with an $\mathcal{N} = 2$ theory with product group

$$O(N_1) \times Sp(N_2) \times O(N_3) \times Sp(N_4) \times \cdots \tag{191}$$

This can be again deformed to an $\mathcal{N} = 2$ supersymmetric theory and the result is similar to the ones obtained before and are a generalization of the ones of [53].

10 Matter Fields and Seiberg Dualities

We can also discuss the Seiberg duality [54] which takes place in the presence of quark chiral superfields in the fundamental representation of

$$\prod_{j=1}^{n}\prod_{k=j}^{n}\prod_{l=1}^{d_{j,k}} U(M_{j,k,l}) \tag{192}$$

in $\mathcal{N}=1$ theory, by generalizing the results of [9] (for related discussions concerning connections between Seiberg duality and toric dualities see [55, 7]). Here we consider the case of massive fundamental fields, which will be integrated out in the infrared which changes the scale of the $\mathcal{N}=1$ theory so changes the flux on the $\mathbf{S}^3$ cycles.

As in [4, 9], the massive matter fields can be obtained by wrapping D5 branes on non-compact holomorphic curves at distance from the $\mathbf{P}^1$ cycles and the distances are identified with mass of the hypermultiplet. Recall that the $\mathcal{N}=1$ geometry is covered by open sets $U_p, \quad p=0,\dots,n$ whose coordinates are given by u_p, x_p, v_p. The holomorphic curve defined by

$$v_p = m, \quad x_p = 0 \tag{193}$$

in U_p maps to $x=0, u=\sum_{i=0}^{p} W_i'(m), v=m$ under the blown-down map σ (73). Hence if $v=m$ does not pass through one of the intersection points of the NS branes in the $u-v$ plane, then the holomorphic curve (193) cannot be compactified in $\tilde{X}$ and D5 branes wrapped on the curve become semi-infinite D4 branes bounded by a NS brane on one side.

The $\mathcal{N}=1$ Seiberg dualities for these models are generalizations for the results of [56] to the case of product gauge group. In order to describe the $\mathcal{N}=1$ Seiberg dualities one starts with the $\mathcal{N}=2$ Seiberg dualities discussed in [35]. The $\mathcal{N}=1$ theories with fundamental matter will have a classical moduli space given by the solution of the D-term and F-term equations which for the case of [35] imply different branches, the Coulomb branch (when the vacuum expectation values of the fundamental fields are zero) and the Higgs branch (when the vacuum expectation values of the adjoint field is zero), the latter being split into a non-baryonic branch and a baryonic branch. By adding a superpotential for the adjoint fields, the results of [35] imply that the baryonic branch and some isolated points on the non-baryonic branch are non lifted and they are points where $\mathcal{N}=1$ supersymmetry is preserved. It has been observed that, in the $\mathcal{N}=2$ theories, points in the baryonic branch have a double description, either as classical solutions for an $\mathcal{N}=2$ $U(N)$ theory with F fundamental flavors or for an $\mathcal{N}=2$ $U(F-N)$ theory with F fundamental flavors and these two theories are Seiberg dual to each other. We can break the supersymmetry in both theories to $\mathcal{N} = 1$ theory by adding a superpotential for the adjoint fields. If we denote the adjoint field for the $U(N)$ theory by Φ and the one for the $U(F-N)$ theory by $\tilde{\Phi}$, the corresponding $\mathcal{N} = 1$ theories are obtained by adding the superpotentials

$$W(\Phi) = \sum_{k>0} \mathrm{Tr}\Phi^k \tag{194}$$

or

$$W(\tilde{\Phi}) = \sum_{k>0} \mathrm{Tr}\tilde{\Phi}^k \tag{195}$$

respectively. Moreover, the two corresponding $\mathcal{N} = 1$ theories are Seiberg dual to each other. To give a geometrical picture, we start in the brane side of the type IIB picture where the matter field is added as D5 branes wrapping holomorphic 2-cycles separated by a distance m from the exceptional $\mathbf{P}^1$.

As discussed in [9], the Seiberg duality is seen in geometry as a birational flop transition where the exceptional $\mathbf{P}^1$ cycles are replaced by another $\mathbf{P}^1$ cycles with negative volume. From the above discussion, we conclude that all the exceptional $\mathbf{P}^1$ cycles in the $\mathcal{N} = 1$ theory are replaced together in a single flop because all of them originate from one $\mathbf{P}^1$ cycle in $\mathcal{N} = 2$ theory after the superpotential deformation. In type IIA picture, this implies that one have to move one NS brane through another NS brane completely and one cannot move just some part of NS brane, while keeping other part fixed, through another NS brane as all of D4 branes stretched between them change together. In terms of geometry, one possible way of achieving the Seiberg-duality by changing the coordinates of U_k (73):

$$u_p = \frac{\prod_{j=0}^{p}\left(u - \sum_{i=0}^{j} W_i'(v)\right)}{x}, x_p = \frac{x}{\prod_{j=0}^{p-1}\left(u - \sum_{i=0}^{j} W_i'(v)\right)}, v_p = v, \tag{196}$$

to

$$\tilde{u_p} = \frac{\prod_{j=p}^{n}\left(u - \sum_{i=0}^{j} W_i'(v)\right)}{x}, \tilde{x_p} = \frac{x}{\prod_{j=p+1}^{n}\left(u - \sum_{i=0}^{j} W_i'(v)\right)}, v_p = v \tag{197}$$

Hence we conclude that we cannot randomly close $\mathbf{P}^1$ cycles but we need to close all the $\mathbf{P}^1$ cycles corresponding to the intersection points of two curved NS branes. This is related to the fact that the deformations of the complex structure for the dual theories are related.

To clarify the above claim, we consider the example of the $\mathbf{A_3}$ theory as in Figure 6. We have 17 gauge groups in the $\mathcal{N} = 1$ theory for which we could discuss the Seiberg duality. But the electric and magnetic theory should come from Seiberg dual $\mathcal{N} = 2$ theories with the corresponding superpotential deformations. As discussed before, this means that we have to close groups of $\mathbf{P}^1$ cycles which correspond to all the intersection points between two NS curved branes. In the previous section notations, this means that we need to close the $\mathbf{P}^1$ cycles corresponding to groups of points (p_{011}, p_{012}) (which correspond to the intersections between the 0-th NS brane and the 1-st NS brane), $(p_{021}, p_{022}, p_{023})$ (which correspond to the intersections between the 0-th NS brane and the 2-nd NS brane), $(p_{031}, p_{032}, p_{033})$ (intersections 0-th and 3-rd), $(p_{121}, p_{122}, p_{123})$ (intersections 1-st and 2-nd), $(p_{131}, p_{132}, p_{133})$ (intersections 1-st and 3-rd), $(p_{231}, p_{232}, p_{233})$ (intersections 2-nd and 3-rd). In terms of the gauge groups, this means that we have to consider the Seiberg dual for the product group $U(M_{121}) \times U(M_{122}) \times U(M_{123})$ and other product groups read from the previous intersection points.

This is an interesting and somehow unexpected feature of the Seiberg dualities for the product groups with bifundamentals. In order to understand it better, we need to know the moduli space for these theories in the presence of fundamental matter, which involves a discussion similar to the one in the previous sections. One thing appears in the type IIA brane configuration related to the positions of the semi-infinite D4 branes corresponding to massive flavors. In the case of non-curved NS branes, the positions of the semi-infinite D4 branes are related to the masses of the fundamental flavors but in the case of curved NS branes this is more difficult to see. The semi-infinite D4 branes will be distributed around the intersection points and so we have a flavor symmetry breaking in the $\mathcal{N} = 1$ theory. This might be related to the above described phenomenon of $\mathbf{P}^1$ cycles closing and opening in groups. We hope to return to this issue in the future.

11 Discussion

One important question is whether we can compare our type IIA brane configuration construction with other type IIA constructions as the ones in [19, 26]. This would relate our MQCD with M-theory curves to M-theory with G_2 manifolds. Moreover, one may be able to construct the various G_2 holonomy manifolds which have been used in M-theory lifts of the geometrical type IIA transitions [12] (see other important developments in [13]-[28]).

The type IIA and type IIB pictures are related by mirror symmetry which was interpreted as three T-dualities in [57]. In toric geometry, the natural T-dualites come from $U(1)^3$ of the torus embedding. In fact, the case for the conifold and its Abelian quotients have been studied in [60]. Recently, geometric transitions in the toric situation has been studied in [26] utilizing the mirror symmetries of [58, 59].

Our type IIA picture has been obtained by taking one T-duality from type IIB picture. Hence we need to take two more T-dualities to obtain the mirror type IIA. It would be interesting to work out in details for the toric cases as in [26] to see how two type IIA picture appear and how their M-theory transitions are related. More generally, the geometry is not toric and there are no obvious three T-dualities one can take. In the case of degenerate superpotential with the same degree, the geometry (98) become quasi-homogeneous and there is extra $U(1)$ action. Brane configurations have been extensively studied during the last years and in the present paper we extend them to more general cases. Therefore, after two extra T-dualities we could get a rich class of G_2 manifolds.

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