Adv. Theor. Math. Phys. **6** (2002) 107–139

Classification of Asymptotic Profiles for Nonlinear Schrödinger Equations with Small Initial Data

Tai-Peng Tsai
Institute for Advanced Study, Princeton, NJ 08540
ttsai@ias.edu

Horng-Tzer Yau
Courant Institute, New York University, New York, NY 10012
yau@cims.nyu.edu

Abstract

We consider a nonlinear Schrödinger equation with a bounded local potential in $\mathbb{R}^3$. The linear Hamiltonian is assumed to have two bound states with the eigenvalues satisfying some resonance condition. Suppose that the initial data are localized and small in H^1. We prove that exactly three local-in-space behaviors can occur as the time tends to infinity: 1. The solutions vanish; 2. The solutions converge to nonlinear ground states; 3. The solutions converge to nonlinear excited states. We also obtain upper bounds for the relaxation in all three cases. In addition, a matching lower bound for the relaxation to nonlinear ground states was given for a large set of initial data which is believed to be generic. Our proof is based on outgoing estimates of the dispersive waves which measure the relevant time-direction dependent information of the dispersive wave. These estimates, introduced in [16], provides the first general notion to measure the out-going tendency of waves in the setting of nonlinear Schrödinger equations.

e-print archive: http://xxx.lanl.gov/abs/math-ph/0205015

1 Introduction

Consider the nonlinear Schrödinger equation

$$i\partial_t\psi = (-\Delta + V)\psi + \lambda|\psi|^2\psi, \qquad \psi(t=0) = \psi_0, \tag{1.1}$$

where V is a smooth localized real potential, $\lambda = \pm 1$ and $\psi = \psi(t,x)$: $\mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{C}$ is a wave function. For any solution $\psi(t) \in H^1(\mathbb{R}^3)$ the L^2-norm and the Hamiltonian

$$\mathcal{H}[\psi] = \int \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}V|\psi|^2 + \frac{1}{4}\lambda|\psi|^4\, dx \tag{1.2}$$

are constants for all t. The global well-posedness for small solutions in $H^1(\mathbb{R}^3)$ can be proved using these conserved quantities and a continuity argument. We assume that the linear Hamiltonian $H_0 := -\Delta + V$ has two simple eigenvalues $e_0 < e_1 < 0$ with normalized eigen-functions ϕ_0, ϕ_1. The nonlinear bound states to the Schrödinger equation (1.1) are solutions to the equation

$$(-\Delta + V)Q + \lambda|Q|^2Q = EQ. \tag{1.3}$$

They are critical points to the Hamiltonian $\mathcal{H}[\psi]$ defined in (1.2) subject to the constraint that the L^2-norm of ψ is fixed. For any nonlinear bound state $Q = Q_E$, $\psi(t) = Qe^{-iEt}$ is a solution to the nonlinear Schrödinger equation.

We may obtain two families of such bound states by standard bifurcation theory, corresponding to the two eigenvalues of the linear Hamiltonian. For any E sufficiently close to e_0 so that $E - e_0$ and λ have the same sign, there is a unique positive solution $Q = Q_E$ to (1.3) which decays exponentially as $x \to \infty$. See Lemma 2.1 of [16]. We call this family the *nonlinear ground states* and we refer to it as $\{Q_E\}_E$. Similarly, there is a *nonlinear excited state* family $\{Q_{1,E_1}\}_{E_1}$ for E_1 near e_1. We will abbreviate them as Q and Q_1. From the same Lemma 2.1 of [16], these solutions are small, localized and $\|Q_E\| \sim |E - e_0|^{1/2}$ and $\|Q_{1,E_1}\| \sim |E_1 - e_1|^{1/2}$.

Our goal is to classify the asymptotic dynamics for small initial data. We have proved [17] that there exists a family of "finite co-dimensional manifolds" in the space of initial data so that the dynamics asymptotically converge to some excited states. Outside a small neighborhood of these manifolds, the asymptotic profiles are given by some ground states [16]. In this article, we shall extend the result of [16] and prove that the possible asymptotic profiles are either vacuum (i.e., vanishing in L^∞ norm), the ground states or the excited states. Furthermore, we obtain the rates of the convergence for all cases.

We first state the assumptions on the potential V, which is the same as in [15]. Denote by L^2_r the weighted L^2 spaces (r may be positive or negative)

$$L^2_r(\mathbb{R}^3) \equiv \{\phi \in L^2(\mathbb{R}^3) : \langle x \rangle^r \phi \in L^2(\mathbb{R}^3)\}. \tag{1.4}$$

The space for initial data in [15] is

$$Y \equiv H^1(\mathbb{R}^3) \cap L^2_{r_0}(\mathbb{R}^3), \qquad r_0 > 3. \tag{1.5}$$

We shall use L^2_{loc} to denote $L^2_{-r_0}$. The parameter $r_0 > 3$ is fixed and we can choose, say, $r_0 = 4$ for the rest of this paper.

Assumption A0: $-\Delta + V$ acting on $L^2(\mathbb{R}^3)$ has 2 simple eigenvalues $e_0 < e_1 < 0$, with normalized eigenvectors ϕ_0 and ϕ_1.

Assumption A1: Resonance condition. Let $e_{01} = e_1 - e_0$ be the spectral gap of the ground state. We assume that $2e_{01} > |e_0|$, i.e., $e_0 < 2e_1$. Let

$$\gamma_0 := \lim_{\sigma \to 0+} \text{Im}\left(\phi_0\phi_1^2, \frac{1}{H_0 + e_0 - 2e_1 - \sigma i} \mathbf{P}_\text{c}{}^{H_0} \phi_0\phi_1^2\right). \tag{1.6}$$

Since the expression is quadratic, we have $\gamma_0 \geq 0$. We assume, for some $s_0 > 0$,

$$\inf_{|s|<s_0} \lim_{\sigma \to 0+} \text{Im}\left(\phi_0\phi_1^2, \frac{1}{H_0 + e_0 - 2e_1 + s - \sigma i} \mathbf{P}_\text{c}{}^{H_0} \phi_0\phi_1^2\right) \geq \frac{3}{4}\gamma_0 > 0. \tag{1.7}$$

We shall use $0i$ to replace σi and the limit $\lim_{\sigma \to 0+}$ later on.

Assumption A2: For λQ_E^2 sufficiently small, the bottom of the continuous spectrum to $-\Delta + V + \lambda Q_E^2$, 0, is not a generalized eigenvalue, i.e., not a resonance. Also, we assume that V satisfies the assumption in [18] so that the $W^{k,p}$ estimates $k \leq 2$ for the wave operator $W_{H_0} = \lim_{t\to\infty} e^{itH_0} e^{it\Delta}$ hold for $k \leq 2$, i.e., there is a small $\sigma > 0$ such that,

$$|\nabla^\beta V(x)| \leq C \langle x \rangle^{-5-\sigma}, \qquad \text{for } |\beta| \leq 2.$$

Also, the functions $(x \cdot \nabla)^k V$, for $k = 0, 1, 2, 3$, are $-\Delta$ bounded with a $-\Delta$-bound < 1:

$$\left\| (x \cdot \nabla)^k V \phi \right\|_2 \leq \sigma_0 \|-\Delta \phi\|_2 + C \|\phi\|_2, \qquad \sigma_0 < 1, \quad k = 0, 1, 2, 3.$$

The main assumption in A0-A2 is the condition $2e_{01} > |e_0|$ in A1. It guarantees that twice the excited state energy of $H_0 - e_0$ becomes a resonance in the continuum spectrum (of $H_0 - e_0$). This resonance produces the main relaxation mechanism. If this condition fails, the resonance occurs in higher order terms and a proof of relaxation will be much more complicated. Also, the rate of decay will be different.

Theorem 1.1. *There is a small number $n_0 > 0$ such that if $\|\psi_0\|_Y := \alpha \le n_0$ then exactly three possible long time dynamics may occur as $t \to \infty$:*

I. $\|\psi(t)\|_{L^2_{\mathrm{loc}}} \le C_{\psi_0} t^{-6/11}$;

II. $\left\|\psi(t) - Q_E e^{-iEt + i\omega(t)}\right\|_{L^2_{\mathrm{loc}}} \le C_{\psi_0} t^{-1/2}$ *for some nonlinear ground state $Q_E \neq 0$ and real function $\omega(t) = O(\log t)$;*

III. $\left\|\psi(t) - Q_{1,E_1} e^{-iE_1 t + i\omega(t)}\right\|_{L^2_{\mathrm{loc}}} \le C_{\psi_0} t^{-1/2}$ *for some nonlinear excited state $Q_{1,E_1} \neq 0$ and real function $\omega(t) = O(\sqrt{t})$.*

Sufficient conditions guaranteeing the convergence to the vacuum (type I), the ground states (type II), or the excited states (type III) are provided in [16, 17]. The type I or III solutions constructed in [17] are finite co-dimensional subset of all small solutions, i.e., the initial data for these solutions form a finite co-dimensional subset of $\{\psi_0 : \|\psi_0\|_Y \le n_0\}$, the set of all small initial data. We believe that the type I or III solutions in general can be constructed in this way and thus have measure zero. The decay rates of the type I or III solutions constructed in [17] are of order $t^{-3/2}$; the corresponding upper bounds provided in Theorem 1.1 are $t^{-6/11}$ or $t^{-1/2}$. Since we believe that all type I or III solutions originate from the construction in [17], these bounds are far from optimal. They result from technical considerations of our classification scheme (which will be explained in the following).

The upper bound $t^{-1/2}$ obtained for the type II solutions in Theorem 1.1 are optimal for initial data considered in [16], where an lower bound of the same order was provided. However, there exists a measure zero set [15] such that the decay rate is at least of order $t^{-3/2}$. We believe that all solutions decaying to the ground states faster than $t^{-1/2}$ have measure zero. Summarizing, we believe that the type II solutions with decay rate exactly of order $t^{-1/2}$ are generic; all other behavior are of measure zero.

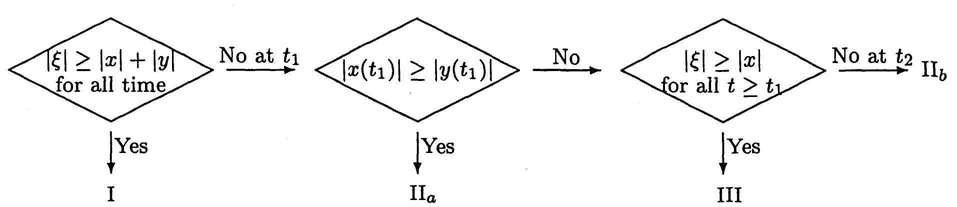
There is a vast literature concerning the classification of asymptotic dynamics for nonlinear Schrödinger equations with small initial data. We shall only be able to mention a few: the one bound state case [1, 4, 9, 12], the one dimension with two bound states [2, 3], the three dimension with two bound states [5] and [14] where results similar to Theorem 1.1 were considered. Earlier works concerning the related linear analysis were obtained in [6, 7, 10, 11].

To explain the idea for the proof, we decompose the wave function using the eigenspaces of the Hamiltonian H_0 as

$$\psi = \underline{x}\phi_0 + \underline{y}\phi_1 + \underline{\xi}, \quad \underline{\xi} = \mathbf{P}_c{}^{H_0}\,\psi. \tag{1.8}$$

This decomposition is not suitable for estimation and will be replaced by the decomposition (2.3) emphasizing the role of the excited states in the next section. It is useful for the following heuristic explanation.

The key ingredients for proving Theorem 1.1 were originated from the previous work [15, 16]. Apart from the standard arguments based on the normal form and resonance decay, the main new idea introduced in [16] was the concept of outgoing estimates. This concept allows us to capture the time-direction dependent information of waves. Therefore, even though the L^2 norm of the dispersive wave may not change much in the time evolution, its "size" will decay in time when measured in terms of "outgoing estimates" (see Propositions 3.1, 3.2, 3.4). Thus after certain initial time, the wave function will fall essentially into the region considered in [16] provided that it does not converge to some excited state or the vacuum. Hence we set up the following flow chart:



I: Dispersion dominated region. Convergence to the vacuum.

II_a, II_b: Nonlinear ground states dominated region. Convergence to nonlinear ground states.

III: Nonlinear excited states dominated region. Convergence to nonlinear excited states.

We first ask the question whether the dispersive part ξ dominates for all time. If it is, the dynamics will converge to the vacuum, the case I. If the dynamic fail this test at t_1, we then ask the second question whether

$|x(t_1)| \geq |y(t_1)|$. If yes, the ground state component dominates and we are in the region II_a, which was considered in [15] (formulated in a stronger form in [16]). Otherwise, the excited component dominates. We then test again whether the dispersive wave dominates the ground state component for all time $t \geq t_1$. If yes, this produces the excited state dominated region III. Otherwise, we reach the region II_b at the time t_2. At this point both the ground state component and the L^∞ norm of the dispersive wave can be arbitrarily small compared with the excited state component. Furthermore, the L^2 norm of the dispersive wave can be much larger than even the excited state component. In other words, we may have

$$\|\xi\|_{L^2} \gg |y| \gg |x| \gg \|\xi\|_{L^\infty}$$

Notice that the occurrence of this scenario is due to the existence of the stable and unstable manifolds, i.e., the dynamics may follow the stable manifold (or the unstable manifold backward in time) for almost infinite time.

In order to understand the dispersive wave ξ at the time t_2, we first notice that the radiation generated by the changes of the masses of the bound states contribute to the dispersive wave. We shall call it the local part of the dispersive wave. This local part, responsible for the relaxation of the excited states, will always be of the same order as the main decay term and will not be small. Our key observation is that the rest of $\xi(t_2)$, call the global part, is negligible when measured by an outgoing estimate. To control the local part, we apply an initial layer argument in the interval $[t_2, t_2 + \Delta t_2]$ until it becomes small at the time $t_2 + \Delta t_2$. Thus up to minor changes, we can now apply the argument of [16] from this time and the dynamics will converge to some ground state. The essence of this approach is that it *extracts the local relevant part of the dispersive wave while treating the global part as an error term by measuring it with an outgoing estimate.*

The scheme we just described is for heuristic explanation. Its precise form will be given in section 3. For nonlinear Schrödinger equations with general potentials, the analysis will be more complicated. In the two bound states case, if the condition $e_0 < 2e_1$ fails, the decay will be much slower than $1/\sqrt{t}$ and thus all errors have to be controlled much more accurately. The picture is even more complicated for multiple-bound states. The resonance decay may be extremely slow (such as $t^{-\varepsilon}$); the excited states may decay to other lower energy excited states before finally decay to a ground state. So far there has been no rigorous work in this direction. However, the notion of outgoing estimates and the initial layer argument seem to provide the right general notion for estimating the dispersive wave.

2 Preliminaries

Denote $\langle t\rangle = 1+|t|$ and $\langle x\rangle = 1+|x|$. We define L^2_{loc} and L^1_{loc}-norms by

$$\|f\|_{L^2_{\text{loc}}} = \left\|\langle x\rangle^{-r_1} f\right\|_{L^2}, \qquad \|f\|_{L^1_{\text{loc}}} = \left\|\langle x\rangle^{-2r_1} f\right\|_{L^1},$$

where $r_1 > 3$ is a constant to be determined by (2.2).

2.1 Nonlinear bound states and linear decay estimates

We recall some results for nonlinear bound states and linear estimates from [15, 16].

Lemma 2.1. *Suppose that $-\Delta + V$ satisfies the assumptions A0 and A2. There is a small constant $n_0 > 0$ such that the following hold. For any E between e_0 and $e_0 + \lambda n_0^2$ there is a nonlinear ground state Q_E solving* (1.3). *The nonlinear ground state Q_E is real, local, smooth, $\lambda^{-1}(E-e_0) > 0$, and*

$$Q_E = n\,\phi_0 + h, \qquad h \perp \phi_0, \qquad h = O(n^3),$$

where $n = [(E-e_0)/(\lambda \int \phi_0^4\,dx)]^{1/2}$. Moreover, we have $R_E \equiv \partial_E Q_E = Cn^{-2}\,Q_E + O(n) = O(n^{-1})$ and $\partial_E^2 Q_E = O(n^{-3})$. If we define $c_1 \equiv (Q,R)^{-1}$, then $c_1 = O(1)$ and $\lambda c_1 > 0$.

There is also a family of nonlinear excited states $\{Q_{E_1}\}_{E_1}$ for E_1 between e_1 and $e_1 + \lambda n_0^2$ satisfying similar properties: $Q_{E_1} = m\phi_1 + O(m^3)$ solves (1.3) *with $m \sim C[\lambda^{-1}(E_1 - e_1)]^{1/2}$, etc.*

This lemma can be proven using standard perturbation argument, see [15]. For the purpose of this paper, we prefer to use the value $m = (\phi_1, Q_1)$ as the parameter and refer to the family of excited states as $Q_1(m)$.

Lemma 2.2 (decay estimates for e^{-itH_0}). *Suppose that $H_0 = -\Delta + V$ satisfies the Assumptions A0–A2. For $q \in [2,\infty]$ and $q' = q/(q-1)$,*

$$\left\|e^{-itH_0}\,\mathbf{P}_c^{H_0}\phi\right\|_{L^q} \le C\,|t|^{-3\left(\frac12-\frac1q\right)}\,\|\phi\|_{L^{q'}}. \tag{2.1}$$

For sufficiently large r_1, we have

$$\lim_{\sigma\to 0+}\left\|\langle x\rangle^{-r_1}\,e^{-itH_0}\,\frac{1}{H_0+e_0-2e_1-\sigma i}\,\mathbf{P}_c^{H_0}\,\langle x\rangle^{-r_1}\,\phi\right\|_{L^2} \le C\,\langle t\rangle^{-3/2}\,\|\phi\|_{L^2}. \tag{2.2}$$

The decay estimate (2.1) is contained in [8] and [18]; the estimate (2.2) is taken from [13] and [15]. The estimate (2.2) holds only if we take $\sigma \to 0+$, not $\sigma \to 0-$.

2.2 Equations and decompositions

For initial data near excited states, the decomposition (1.8) contains an error of order y^3 and it is difficult to read from (1.8) whether the wave function is exactly an excited state. Thus we shall use the decomposition

$$\psi = x\phi_0 + Q_1(y) + \xi, \tag{2.3}$$

where

$$y = \underline{y}, \quad x = \underline{x} - (\phi_0, Q_1(y)), \quad \xi = \underline{\xi} - \mathbf{P}_{\mathrm{c}} Q_1(y). \tag{2.4}$$

Here we have used the convention that

$$Q_1(y) := Q_1(m)e^{i\Theta}, \qquad m = |y|, \;\; me^{i\Theta} = y.$$

For ψ with sufficiently small L^2 norm, such a decomposition exists and is unique [16]. Thus we shall write

$$\psi(t) = x(t)\phi_0 + Q_1(m(t))e^{i\Theta(t)} + \xi(t), \qquad \xi(t) \in \mathbf{H}_{\mathrm{c}}(H_0). \tag{2.5}$$

If we write $\Theta(t) = \theta(t) - \int_0^t E_1(m(s))\, ds$, we can write $y(t)$ as

$$y = me^{i\Theta} = m \exp\left\{ i\theta(t) - i\int_0^t E_1(m(s))\, ds \right\}. \tag{2.6}$$

Denote the part orthogonal to ϕ_1 by $h = x\phi_0 + \xi$. From the Schrödinger equation (1.1), h satisfies the equation

$$\begin{aligned} i\partial_t h &= H_0 h + G + \Lambda, \\ G &= \lambda|\psi|^2\psi - \lambda Q_1^3 e^{i\Theta} \\ &= \lambda Q_1^2(e^{i2\Theta}\bar{h} + 2h) + \lambda Q_1(e^{i\Theta}2h\bar{h} + e^{-i\Theta}h^2) + \lambda|h|^2 h, \qquad (2.7) \\ \Lambda &= \left(\dot{\theta}Q_1 - i\dot{m}Q_1'\right)e^{i\Theta}, \qquad \left(Q_1'(m) := \tfrac{d}{dm}Q_1(m)\right). \qquad (2.8) \end{aligned}$$

Since $m(t)$ and $\theta(t)$ are chosen so that (2.5) holds, we have $0 = (\phi_1, i\partial_t h(t)) = \left(\phi_1,\, G + (\dot{\theta}Q_1 - i\dot{m}Q_1')e^{i\Theta}\right)$. Hence $m(t)$ and $\theta(t)$ satisfy

$$\dot{m} = \left(\phi_1, \operatorname{Im} Ge^{-i\Theta}\right), \qquad \dot{\theta} = -\frac{1}{m}\left(\phi_1, \operatorname{Re} Ge^{-i\Theta}\right). \tag{2.9}$$

We also have the equation for y:

$$i\dot{y} = i\dot{m}e^{i\Theta} - (\dot{\theta} - E_1(m))me^{i\Theta} = E_1(m)y + e^{i\Theta}(i\dot{m} - m\dot{\theta}) = E_1(m)y + (\phi_1, G).$$

Here we have used (2.9). Denote $\Lambda_\pi = \pi\Lambda$ where π is the orthogonal projection $\pi\psi = \psi - (\phi_1, \psi)\phi_1$. We can decompose the equation for h into equations for x and ξ. Thus the original Schrödinger equation is equivalent to

$$\begin{cases} i\dot{x} = e_0\, x + (\phi_0,\, G + \Lambda_\pi)\,, \\ i\dot{y} = E_1(m)y + (\phi_1, G), \\ i\partial_t \xi = H_0\, \xi + \mathbf{P}_c\, (G + \Lambda_\pi)\,. \end{cases} \tag{2.10}$$

Clearly, x has an oscillation factor $e^{-ie_0 t}$, and y has a factor $e^{-ie_1 t}$ since $E_1(m) \sim e_1$. Hence we define

$$x(t) = e^{-ie_0 t}u(t), \qquad y(t) = e^{-ie_1 t}v(t). \tag{2.11}$$

Together with the integral form of the equation for ξ, we have

$$\dot{u} = -ie^{ie_0 t}\,(\phi_0,\, G + \Lambda_\pi)\,, \tag{2.12}$$

$$\dot{v} = -ie^{ie_1 t}\,[(E_1(m) - e_1)y + (\phi_1, G)]\,, \tag{2.13}$$

$$\xi(t) = e^{-iH_0 t}\xi_0 + \int_0^t e^{-iH_0(t-s)}\, \mathbf{P}_c^{H_0}\, G_\xi(s)\, ds, \quad G_\xi = i^{-1}(G + \Lambda_\pi). \tag{2.14}$$

This is the system we shall study.

We denote by G_3 the leading terms of G, which consists of cubic monomials in x and y:

$$G_3 = \lambda(y^2\bar{x} + 2|y|^2 x)\phi_0\phi_1^2 + \lambda(2|x|^2 y + x^2\bar{y})\phi_0^2\phi_1 + \lambda|x|^2 x\phi_0^3. \tag{2.15}$$

We can expand $E_1(m)$ in m as

$$E_1(m) = e_1 + E_{1,2}m^2 + E_{1,4}m^4 + E_1^{(6)}(m), \quad E_1^{(6)}(m) = O(m^6). \tag{2.16}$$

We think of x and y as order n, and ξ as order n^3. Since, by (2.8)–(2.9), Λ_π is local and

$$\|\Lambda_\pi\| \le |\dot{\theta}|\, \|\pi Q_1\| + |\dot{m}|\, \|\pi Q_1'\| \le C|y|^2\, \|G\|_{\text{loc}}\,, \tag{2.17}$$

the main terms in $G_\xi = i^{-1}(G + \Lambda_\pi)$ is $i^{-1}G_3$. These terms are explicit and can be integrated. We integrate the first term $\lambda y^2 \bar{x} \phi_0 \phi_1^2$ in G_3 as an example:

$$\begin{aligned}
&- i\lambda \int_0^t e^{-iH_0(t-s)}\, \mathbf{P}_c y^2 \bar{x} \phi_0 \phi_1^2 \, ds \\
&= -i\lambda e^{-iH_0 t} \int_0^t e^{i(H_0 - 0i)s}\, e^{i(e_0 - 2e_1)s}\, v^2 \bar{u}\, \mathbf{P}_c \phi_0 \phi_1^2 \, ds \\
&= y^2 \bar{x} \Phi_1 - e^{-iH_0 t} y^2 \bar{x}(0) \Phi_1 - \int_0^t e^{-iH_0(t-s)}\, e^{i(e_0 - 2e_1)s} \frac{d}{ds}\left(v^2 \bar{u}\right) \Phi_1 \, ds,
\end{aligned}$$

where

$$\Phi_1 = \frac{-\lambda}{H_0 - 0i + e_0 - 2e_1}\, \mathbf{P}_c \phi_0 \phi_1^2. \tag{2.18}$$

This term, with the phase factor $e_0 - 2e_1$, is the only one in G_3 having a negative phase factor. Since $-(e_0 - 2e_1)$ is in the continuous spectrum of H_0, $H_0 + e_0 - 2e_1$ is not invertible, and needs a regularization $-0i$. We choose $-0i$, not $+0i$, so that the term $e^{-iH_0 t} y^2 \bar{x}(0) \Phi_1$ decays as $t \to \infty$, see Lemma 2.2.

We can integrate all terms in G_3 and obtain the main terms of $\xi(t)$ as

$$\xi^{(2)}(t) = y^2 \bar{x} \Phi_1 + |y|^2 x \Phi_2 + |x|^2 y \Phi_3 + x^2 \bar{y} \Phi_4 + |x|^2 x \Phi_5, \tag{2.19}$$

where

$$\begin{aligned}
\Phi_2 &= \frac{-2\lambda}{H_0 - e_0}\, \mathbf{P}_c \phi_0 \phi_1^2, & \Phi_3 &= \frac{-2\lambda}{H_0 - e_1}\, \mathbf{P}_c \phi_0^2 \phi_1, \\
\Phi_4 &= \frac{-\lambda}{H_0 - 2e_0 + e_1}\, \mathbf{P}_c \phi_0^2 \phi_1, & \Phi_5 &= \frac{-\lambda}{H_0 - e_0}\, \mathbf{P}_c \phi_0^3.
\end{aligned} \tag{2.20}$$

The rest of $\xi(t)$ is

$$\begin{aligned}
\xi^{(3)}(t) &= e^{-iH_0 t}\xi_0 - e^{-iH_0 t}\xi^{(2)}(0) - \int_0^t e^{-iH_0(t-s)}\, \mathbf{P}_c\, G_4 \, ds \\
&\quad + \int_0^t e^{-iH_0(t-s)}\, \mathbf{P}_c \left(G_\xi - i^{-1}G_3 - i^{-1}\lambda |\xi|^2 \xi\right) ds \\
&\quad + \int_0^t e^{-iH_0(t-s)}\, \mathbf{P}_c \left(i^{-1}\lambda |\xi|^2 \xi\right) ds \\
&\equiv \xi_1^{(3)}(t) + \xi_2^{(3)}(t) + \xi_3^{(3)}(t) + \xi_4^{(3)}(t) + \xi_5^{(3)}(t).
\end{aligned} \tag{2.21}$$

The integrand G_4 in $\xi_3^{(3)}(t)$ consists of the remainders from the integration by parts:

$$G_4 = e^{i(e_0-2e_1)s}\frac{d}{ds}\left(v^2\bar{u}\right)\Phi_1 + e^{i(-e_0)s}\frac{d}{ds}\left(|v|^2u\right)\Phi_2 + e^{i(-e_1)s}\frac{d}{ds}\left(|u|^2v\right)\Phi_3 + e^{i(-2e_0+e_1)s}\frac{d}{ds}\left(u^2\bar{v}\right)\Phi_4 + e^{i(-e_0)s}\frac{d}{ds}\left(u^2\bar{u}\right)\Phi_5. \tag{2.22}$$

The integrands of $\xi_4^{(3)}(t)$ and $\xi_5^{(3)}(t)$ are higher order terms of G_ξ which we did not integrate. We single out $\xi_5^{(3)}(t)$ since $|\xi|^2\xi$ is a non-local term. Thus we have the following decomposition for ξ:

$$\xi(t) = \xi^{(2)}(t) + \xi^{(3)}(t) = \xi^{(2)} + \left(\xi_1^{(3)} + \cdots + \xi_5^{(3)}\right). \tag{2.23}$$

Denote $\xi_{1-2}^{(3)} = \xi_1^{(3)} + \xi_2^{(3)}$ and $\xi_{3-5}^{(3)} = \xi_3^{(3)} + \xi_4^{(3)} + \xi_5^{(3)}$. We have

$$\begin{aligned} \xi_{1-2}^{(3)}(t) &= e^{-itH_0}[\xi_0 - \xi^{(2)}(0)], \\ \xi_{3-5}^{(3)}(t) &= \int_0^t e^{-i(t-s)H_0}\,\mathbf{P}_c(G_\xi - i^{-1}G_3 - G_4)(s)ds. \end{aligned} \tag{2.24}$$

We now derive a bound for $\left\|\xi_{3-5}^{(3)}(t)\right\|_{L^2_{\text{loc}}}$. Using Lemma 2.2 to estimate the integrand of $\xi_3^{(3)}$ and bounding the L^2_{loc}-norm of the integrand of $\xi_4^{(3)} + \xi_5^{(3)}$ by either its L^∞ or L^4-norm, we have, assuming (2.27) below,

$$\left\|\xi_{3-5}^{(3)}(t)\right\|_{L^2_{\text{loc}}} \le \int_0^t \min\left\{|t-s|^{-3/2}, |t-s|^{-3/4}\right\} g_{\xi,3-5}(s)ds, \tag{2.25}$$

where

$$g_{\xi,3-5}(t) \equiv C\left\|G_\xi - i^{-1}G_3\right\|_{L^1\cap L^{4/3}} + Cn^2|\dot{u}| + Cn|u\dot{v}|. \tag{2.26}$$

Lemma 2.3. *Suppose*

$$|x|, |y| \le n \le \alpha \ll 1, \quad \|\xi\|_{L^2\cap L^4} \le \alpha. \tag{2.27}$$

Denote $X = n\alpha\|\xi\|_{L^2_{\text{loc}}} + \alpha\|\xi\|^2_{L^4}$. *We have*

$$\|G\|_{L^1_{\text{loc}}} + \|G_\xi(t)\|_{L^1\cap L^{4/3}} \lesssim n^2x + X, \tag{2.28}$$

$$\|G - G_3\|_{L^1_{\text{loc}}} + g_{\xi,3-5}(t) \lesssim n^4x + X. \tag{2.29}$$

PROOF: From the definitions of G, G_3 and by Hölder inequality, we have

$$\|G - G_3\|_{L^1 \cap L^{4/3}} \lesssim (*),$$

where

$$(*) = n^4 x + n^2 \|\xi\|_{L^2_{\rm loc}} + n \|\xi\|_{L^2 \cap L^4} \|\xi\|_{L^2_{\rm loc}} + \|\xi\|_{L^2 \cap L^4} \|\xi\|^2_{L^4}.$$

We have $\|G - G_3\|_{L^1_{\rm loc}} \lesssim \|G - G_3\|_{L^1 \cap L^{4/3}} \lesssim (*)$ and $\|G\|_{L^1_{\rm loc}} \lesssim \|G_3\|_{L^1_{\rm loc}} + \|G - G_3\|_{L^1_{\rm loc}} \lesssim n^2 x + (*)$. Since $G_\xi = i^{-1}(G + \Lambda_\pi)$ with $\|\Lambda_\pi\| \le n^2 \|G\|_{L^1_{\rm loc}}$ by (2.17), we have

$$\left\|G_\xi - i^{-1} G_3\right\|_{L^1 \cap L^{4/3}} \lesssim (*), \qquad \|G_\xi\|_{L^1 \cap L^{4/3}} \lesssim n^2 x + (*).$$

By (2.12)–(2.13), also using (2.17),

$$|\dot{u}| \lesssim \|G\|_{L^1_{\rm loc}} + \|\Lambda_\pi\|_{L^1_{\rm loc}} \lesssim \|G\|_{L^1_{\rm loc}}, \qquad |\dot{v}| \lesssim \|G\|_{L^1_{\rm loc}} + n^3. \tag{2.30}$$

From the definition (2.26) of $g_{\xi,3-5}(t)$, (2.30) and $\|G\|_{L^1_{\rm loc}} \lesssim n^2 x + (*)$, we have

$$g_{\xi,3-5}(t) \lesssim (*) + n^2 \|G\|_{L^1_{\rm loc}} + n|u|n^3 \lesssim (*).$$

From the assumption (2.27), $(*) \lesssim n^4 x + n\alpha \|\xi\|_{L^2_{\rm loc}} + \alpha \|\xi\|^2_{L^4}$. Thus we have proved the Lemma. **Q.E.D.**

2.3 Normal form for equations of bound states

Recall that we write $x(t) = e^{-ie_0 t}u(t)$ and $y(t) = e^{-ie_1 t}v(t)$. We have the following normal form for the equations of $\dot{u}$ and $\dot{v}$.

Lemma 2.4 (Normal form). *Suppose*

$$|x(t)|, |y(t)| \le n \ll 1, \qquad \|\xi(t)\|_{L^2 \cap L^4} \ll 1. \tag{2.31}$$

There are perturbations μ of u and ν of v satisfying

$$|u(t) - \mu(t)| + |v(t) - \nu(t)| \le C_1 n^2 |x(t)|, \tag{2.32}$$

such that

$$\begin{aligned} \dot{\mu} &= (c_1|\mu|^2 + c_2|\nu|^2)\mu + (c_3|\mu|^4 + c_4|\mu|^2|\nu|^2 + c_5|\nu|^4)\mu + g_u, \\ \dot{\nu} &= (c_6|\mu|^2 + c_7|\nu|^2)\nu + (c_8|\mu|^4 + c_9|\mu|^2|\nu|^2 + c_{10}|\nu|^4)\nu + g_v. \end{aligned} \tag{2.33}$$

Here g_u and g_v are error terms. All coefficients $c_1, \cdots, c_{10}$ are of order one and, except c_5 and c_9, purely imaginary. We have

$$\operatorname{Re} c_5 = \gamma_0, \qquad \operatorname{Re} c_9 = -2\gamma_0, \tag{2.34}$$

where $\gamma_0 > 0$ is defined in (1.6). *Moreover, we can write g_v as*

$$g_v = -iE^{(6)}(|y|)\nu + \widetilde{g}_v, \tag{2.35}$$

where $E^{(6)}(|y|) = O(|y|^6)$ is defined in (2.16), *and*

$$\begin{aligned} &|g_u(t)| + |\widetilde{g}_v(t)| \\ &\le C_1 \left\{ \alpha n^5 |x| + n^2 \left\| \xi^{(3)} \right\|_{L^2_{\text{loc}}} + n \|\xi\|^2_{L^2_{\text{loc}}} + \left(\|\xi\|_{L^2_{\text{loc}}} + \alpha n^2 \right) \|\xi\|^2_{L^4} \right\}, \end{aligned} \tag{2.36}$$

for some explicit constant C_1.

Proof: This is Lemma 3.4 of [16]. The definitions of μ, ν, g_u, g_v are exactly the same. The only difference is the error estimates (2.36) since our assumption (2.31) is different from that in [16]. Since μ is of the form $u + n^2 u + n^4 u$ and ν of the form $v + n^2 u + n^4 u$, their estimates remain the same. We only need to prove (2.36).

In [16] g_u and g_v are defined as

$$g_u = g_{u,4} + g_{u,5} + g_{u,3} + R_{u,7} - ie^{ie_0 t}(\phi_0,\, G_{5,3}),$$
$$g_v = g_{v,4} + g_{v,5} + g_{v,3} + R_{v,7} - ie^{ie_1 t}(\phi_1,\, G_{5,3}).$$

See [16] for their exact definitions. Recall from [16] that $g_{u,3}$ consists of higher order terms of $g_{u,1}$ and $g_{u,2}$. Note $g_{u,1}$ consists of terms of the form $n^2\dot{u} + nu\dot{v}$, and $g_{u,2}$ consists of terms of the form $n^2(u-\mu) + nu(v-\nu)$. Together with (2.12), (2.13) and (2.17), we can bound $g_{u,3}$ by

$$|g_{u,3}| \lesssim n^2 \left\|G - G_3\right\|_{L^1_{\rm loc}} + n^4 \left\|G\right\|_{L^1_{\rm loc}} + n^6|x|.$$

The other terms in g_u are of the form:

$$\begin{aligned} g_{u,4} &= n^4\dot{u} + n^3u\dot{v}, \\ g_{u,5} &= n^4(u-\mu) + n^3u(v-\nu), \\ R_{u,7} &= (\phi_0,\, n^6x + n^4\xi + n\xi^2 + \xi^3 + n^2 \left\|G - G_3\right\|_{L^1_{\rm loc}} + n^4 \left\|G\right\|_{L^1_{\rm loc}}), \\ G_{5,3} &= n^2\xi^{(3)}. \end{aligned}$$

We can bound $\dot{u}$, $\dot{v}$, $\|G\|_{L^1_{\rm loc}}$ and $\|G - G_3\|_{L^1_{\rm loc}}$ by (2.30), (2.28) and (2.29). Summing the estimates, we have

$$\begin{aligned} |g_u| &\lesssim n^2 \left\|G - G_3\right\|_{L^1_{\rm loc}} + n^4 \left\|G\right\|_{L^1_{\rm loc}} + n^6x + n^4\xi + n\xi^2 + \xi^3 + n^2\xi^{(3)} \\ &\lesssim n^6x + n^2\xi^{(3)} + n^4\xi + n\xi^2 + \xi^3 + n^2X, \end{aligned}$$

where $X = n\alpha \left\|\xi\right\|_{L^2_{\rm loc}} + \alpha \left\|\xi\right\|_{L^4}^2$ and all terms with ξ are measured in $L^1_{\rm loc}$. Therefore

$$|g_u(t)| \lesssim n^6|x| + n^2 \left\|\xi^{(3)}\right\|_{L^2_{\rm loc}} + n^3\alpha \left\|\xi\right\|_{L^2_{\rm loc}} + n \left\|\xi\right\|_{L^2_{\rm loc}}^2 + \left(\left\|\xi\right\|_{L^2_{\rm loc}} + n^2\alpha\right) \left\|\xi\right\|_{L^4}^2.$$

Note $\|\xi\|_{L^2_{\rm loc}} \lesssim n^2x + \left\|\xi^{(3)}\right\|_{L^2_{\rm loc}}$. Hence we obtain the estimate of g_u in (2.36). The estimate of $\widetilde{g}_v$ is proved in the same way. **Q.E.D.**

As a result of the lemma, we have

$$\frac{d}{dt}|\mu| = \frac{1}{2|\mu|}\frac{d}{dt}|\mu|^2 = |\mu|^{-1}\,\mathrm{Re}\,\bar{\mu}\dot{\mu} = \gamma_0|\nu|^4|\mu| + \mathrm{Re}\,g_\mu\bar{\mu}/|\mu|. \tag{2.37}$$

Similarly, using $\mathrm{Re}\,g_v\bar{\nu} = \mathrm{Re}\,\widetilde{g}_v\bar{\nu}$,

$$\frac{d}{dt}|\nu| = -2\gamma_0|\mu|^2|\nu|^3 + \mathrm{Re}\,\widetilde{g}_\nu\bar{\nu}/|\nu|. \tag{2.38}$$

2.4 Relaxation to Ground States

We shall need Theorem 4.3 of [16] which provides a relaxation estimates to ground states from initial data near some ground state. It is a strengthened form of Theorem 1.3 in [15]. For the purpose of the application in this paper, we start the dynamics at $t = t_4$.

Theorem 2.5 ([16]). *There are small constants $n_0, \varepsilon_0 > 0$ such that the following hold. Suppose $\psi(t_4) = x(t_4)\phi_0 + Q_1(y(t_4)) + \xi(t_4)$ with*

$$|x(t_4)| = n \ll n_0, \quad |y(t_4)| \le \varepsilon_0 n,$$

and that $\xi(t_4)$ satisfies, for all $s \ge 0$,

$$\begin{aligned} &\|\xi(t_4)\|_{H^1} \ll 1, \\ &\left\|e^{-isH_0}\xi(t_4)\right\|_{L^4} \le Cn^3 \Delta t(\Delta t + s)^{-3/4}, \\ &\left\|e^{-isH_0}\xi(t_4)\right\|_{L^2_{\rm loc}} \le Cn^3 \frac{\Delta t}{\Delta t + s} (1+s)^{-1/2}, \end{aligned} \tag{2.39}$$

for some $\Delta t \in [1, n^{-4-1/4}]$. Then there is a frequency E_∞ and a function $\Theta(t)$ such that $\|Q_{E_\infty}\|_Y \sim n$, $\Theta(t) = -E_\infty t + O(\log t)$ and, for some constant C_2,

$$\left\|\psi(t) - Q_{E_\infty} e^{i\Theta(t)}\right\|_{L^2_{\rm loc}} \le C_2 \left((\varepsilon n)^{-2} + \gamma_0 n^2 (t - t_4)\right)^{-1/2}.$$

2.5 Inequalities

For convenience of reference, we collect some integral inequalities here.

Lemma 2.6. *(1) Suppose $t \ge T$, $\Delta t \ge 1$.*

$$\int_{T-\Delta t}^{T} |t-s|^{-3/4}\, ds \le C\Delta t(\Delta t + t - T)^{-3/4}. \tag{2.40}$$

$$\int_{T-\Delta t}^{T} \min\left\{(t-s)^{-3/2},\ (t-s)^{-3/4}\right\} ds \le C\frac{\Delta t}{\Delta t + t - T} \langle t - T\rangle^{-1/2}. \tag{2.41}$$

(2) For $t \ge T \ge 1$,

$$\int_T^t (t-s)^{-3/4} s^{-3/2}\, ds \le CT^{-1/2} t^{-3/4}. \tag{2.42}$$

$$\int_T^t \min\left\{(t-s)^{-3/2},\ (t-s)^{-3/4}\right\} s^{-3/2}\, ds \le Ct^{-3/2}. \tag{2.43}$$

PROOF: (1) Let $t = T + \tau$. If $\tau > \Delta t$, $(\Delta t + \tau) \sim \tau$ and the integral in (2.40) is bounded by $\int_{T-\Delta t}^{T} \tau^{-3/4}\, ds = C\tau^{-3/4}\Delta t \sim C\Delta t(\Delta t + \tau)^{-3/4}$. If $\tau < \Delta t$, $(\Delta t + \tau) \sim \Delta t$ and the integral is bounded by $\int_{T-\Delta t}^{T} |T - s|^{-3/4}\, ds = C(\Delta t)^{1/4} \sim C\Delta t(\Delta t + \tau)^{-3/4}$.

For (2.41), if $t \le T + 1$, we have LHS $\le$ constant $\le$ RHS. Hence we assume $t \ge T + 1$. By a translation, (2.41) is equivalent to $\int_0^{\Delta t} (t-s)^{-3/2} ds \le C(\Delta t)t^{-1} \langle t - \Delta t\rangle^{-1/2}$. The integral is bounded by

$$\begin{aligned}\int_0^{\Delta t} (t-s)^{-3/2} ds &= 2(t - \Delta t)^{-1/2} - 2t^{-1/2} \\ &= 2[(t-\Delta t)^{-1/2} + t^{-1/2}]^{-1} \left[(t-\Delta t)^{-1} - t^{-1}\right] \\ &\le 2(t-\Delta t)^{1/2} \left[(t-\Delta t)^{-1} t^{-1} \Delta t\right].\end{aligned}$$

(2) Note, by rescaling,

$$\int_{t/2}^{t} (t-s)^{-3/4} s^{-3/2}\, ds = Ct^{-5/4} \le CT^{-1/2} t^{-3/4}.$$

If $t < 2T$, the integral on the left of (2.42) is bounded by the above integral. If $t \ge 2T$, it is bounded by the sum of the above integral and

$$\int_T^{t/2} (t-s)^{-3/4} s^{-3/2}\, ds \le Ct^{-3/4} \int_T^{t/2} s^{-3/2}\, ds \le Ct^{-3/4} T^{-1/2}.$$

Hence (2.42) is proven. For (2.43), note the left side is bounded by

$$\int_0^t \langle t-s\rangle^{-3/2} \langle s\rangle^{-3/2}\, ds + \int_{t-1/2}^{t} (t-s)^{-3/4} t^{-3/2} ds,$$

and both integrals are bounded by $Ct^{-3/2}$. **Q.E.D.**

3 Estimates

We have assumed that the initial data is small in $\|\cdot\|_Y$ in Theorem 1.1. We shall however use only the following properties: Let $\psi_0 = x_0\phi_0 + Q_1(y_0) + \xi_0$ with $\xi_0 \in \mathbf{H}_c$. Then we have for all $t \geq 0$,

$$\begin{aligned} &|x_0| + |y_0| + \|\xi_0\|_{L^2} \leq \alpha, \\ &\left\|e^{-itH_0}\xi_0\right\|_{L^4} \leq \alpha \langle t \rangle^{-3/4}, \\ &\left\|e^{-itH_0}\xi_0\right\|_{L^2_{\rm loc}} \leq \alpha \langle t \rangle^{-3/2}. \end{aligned} \tag{3.1}$$

From now on, we shall use these three conditions as our assumption for Theorem 1.1.

Recall the orthogonal decomposition (1.8) that $\psi(t) = \underline{x}\phi_0 + \underline{y}\phi_1 + \underline{\xi}$. We have $|\underline{x}(t)|^2 + |\underline{y}(t)|^2 + \left\|\underline{\xi}(t)\right\|_{L^2}^2 = \|\psi(t)\|_{L^2}^2 \leq \alpha^2$. If we decompose $\psi(t)$ via (2.3), i.e.,

$$\psi(t) = x\phi_0 + Q_1(y) + \xi, \tag{3.2}$$

we have $y = \underline{y}$, $x = \underline{x} + O(y^3)$ and $\xi = \underline{\xi} + O(y^3)$. Thus

$$|x(t)|,\ |y(t)|,\ \|\xi(t)\|_{L^2} \leq \tfrac{5}{4}\alpha. \tag{3.3}$$

Choose ι and δ so that

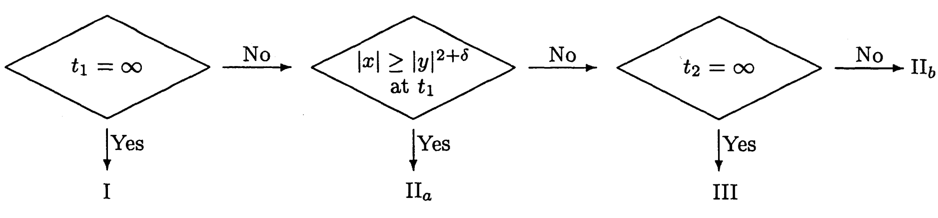
$$0 < \iota < 0.2, \quad 0.6 < \delta < 1, \quad \delta + \iota < 1. \tag{3.4}$$

(We set $\delta = 3/4$ in the statement of case I in Theorem 1.1.)

Let

$$t_1 \equiv \sup\left\{t \geq 0 :\ (\max\{|x(s)|, |y(s)|\})^{2+\delta} \leq \alpha \langle s \rangle^{-3/2},\ \forall s \in [0,t]\right\}. \tag{3.5}$$

t_1 may be ∞; we may assume $t_1 \geq 1$ by enlarging α. Our guiding principle is the following chart. The time t_2 is defined in Proposition 3.2.



I. $\psi(t)$ vanishes locally.

II_a. $\psi(t)$ relaxes to a ground state and stays away from nonlinear excited states for all time.

II_b. $\psi(t)$ approaches some nonlinear excited state but then relaxes to a ground state.

III. $\psi(t)$ converges to a nonlinear excited state.

The analysis of case II_b is very subtle since the time scale that $\psi(t)$ stays near an excited state may be infinite compared to its local size. We have the following time line picture for this case:

0	t_1	t_2	t_3	t_4
	$y=n$	$y=n$	$y=n$	$x=n$
	$x \le n^{2+\delta} = \alpha t_1^{-3/2}$	$n^{2+\iota}x = \alpha t_2^{-3/2}$	$x = \varepsilon n$	$y = \varepsilon n$

We first establish an estimate in the interval $[0, t_1)$.

Proposition 3.1. *For $t \in [0, t_1)$, we have*

$$\begin{aligned} |x(t)|, |y(t)| &\le [\alpha \langle t\rangle^{-3/2}]^{1/(2+\delta)}, \\ \|\xi(t)\|_{L^4} &\le (1+\iota)\ \alpha \langle t\rangle^{-3/4}, \\ \|\xi(t)\|_{L^2_{\text{loc}}} &\le (1+\iota)\ \alpha \langle t\rangle^{-3/2}, \\ \|G_\xi(t)\|_{L^1\cap L^{4/3}} &\le C\alpha^{3/(2+\delta)} \langle t\rangle^{-3/2}. \end{aligned} \tag{3.6}$$

Suppose $t_1 = \infty$. Then

$$|x(t)| + |y(t)| + \|\xi(t)\|_{L^2_{\text{loc}}} \le Ct^{-3/(4+2\delta)}, \qquad \text{as } t \to \infty. \tag{3.7}$$

Suppose $t_1 < \infty$. Let

$$n \equiv \max\{|x(t_1)|, |y(t_1)|\}. \tag{3.8}$$

We have $0 < n < 2\alpha$ and

$$n^{2+\delta} = \alpha \langle t_1\rangle^{-3/2}. \tag{3.9}$$

Moreover, for all $t \ge t_1$, we have the following outgoing estimates on the dispersive wave ξ:

$$\begin{aligned} \left\| e^{-i(t-t_1)H_0}\xi(t_1)\right\|_{L^4} &\le (1+\iota)\ \alpha t^{-3/4}, \\ \left\| e^{-i(t-t_1)H_0}\xi(t_1)\right\|_{L^2_{\text{loc}}} &\le (1+\iota)\ \alpha t^{-3/2}. \end{aligned} \tag{3.10}$$

PROOF: The estimate for $|x(t)|, |y(t)|$ in $(3.6)_1$ is by the definition of t_1. We will prove the rest of (3.6) by a continuity argument and assume that

$$\|\xi(t)\|_{L^4} \le 2\alpha \langle t\rangle^{-3/4}, \qquad \|\xi(t)\|_{L^2_{\text{loc}}} \le 2\alpha \langle t\rangle^{-3/2}. \tag{3.11}$$

We explain the idea of continuity argument: Suppose the estimates in (3.6) is true only up to $t \le T$ with $T < t_1$. Since the estimates (3.11) are weaker than those in (3.6), they remain true for $t \in [0, T+\tau]$ for some $\tau > 0$, $T + \tau \le t_1$, by continuity. Our proof then implies (3.6) for $t \in [0, T+\tau]$. This is a contradiction to the choice of T. Hence (3.6) holds for all $t \in [0, t_1]$. We will use similar continuity arguments to prove Propositions 3.2–3.4.

Recall

$$\xi(t) = e^{-itH_0}\xi_0 + \int_0^t e^{-i(t-s)H_0}\, \mathbf{P}_c G_\xi(s)\, ds, \tag{3.12}$$

and $G_\xi = i^{-1}(G + \Lambda_\pi)$. Since

$$\left\|\xi^2\bar{\xi}\right\|_{L^{4/3}} \le \|\xi\|_{L^4}^3, \qquad \left\|\xi^2\bar{\xi}\right\|_{L^1} \le \|\xi\|_{L^2}\, \|\xi\|_{L^4}^2, \tag{3.13}$$

and $\|\xi\|_{L^4\cap L^2} \le 2\alpha$, assuming (3.11) we have

$$\begin{aligned}
\|G_\xi(s)\|_{L^1\cap L^{4/3}} &\le C(|x(s)| + |y(s)|)^{3/(2+\delta)} + \|\xi(s)\|_{L^4\cap L^2}\, \|\xi(s)\|_{L^4}^2 \\
&\le C(\alpha(1+s)^{-3/2})^{3/(2+\delta)} + C\alpha^3(1+s)^{-3/2} \\
&\le C\alpha^{3/(2+\delta)}(1+s)^{-3/2} = o(1)\ \alpha(1+s)^{-3/2}.
\end{aligned}$$

Here we have used $\delta < 1$, see (3.4). Using (3.1) and (2.42), $\xi(t)$ is bounded in L^4 by

$$\|\xi(t)\|_{L^4} \le \alpha \langle t\rangle^{-3/4} + \int_0^t C(t-s)^{-3/4}\, \|G_\xi(s)\|_{L^{4/3}}\, ds \le \frac{(1+\iota)\alpha}{(1+t)^{3/4}}.$$

To bound $\|\xi(t)\|_{L^2_{\text{loc}}}$, we bound the integrand in (3.12) in L^∞ for s small and in L^4 for s large. Hence, using (3.1) and (2.43),

$$\begin{aligned}
\|\xi(t)\|_{L^2_{\text{loc}}} &\le \alpha \langle t\rangle^{-3/2} + \int_0^t C \min\left\{(t-s)^{-3/2}, (t-s)^{-3/4}\right\} \|G_\xi(s)\|_{L^1\cap L^{4/3}}\, ds \\
&\le (1+\iota)\alpha \langle t\rangle^{-3/2}.
\end{aligned}$$

Hence we have shown all estimates in (3.6), by a continuity argument.

Suppose $t_1 = \infty$. It follows from (3.6) that everything vanishes and we have (3.7). Suppose $t_1 < \infty$. That $n \le 2\alpha$ is by (3.3). Eq. (3.9) is by the definition of t_1. We want to show (3.10). For $t \ge t_1$ we have

$$e^{-i(t-t_1)H_0}\xi(t_1) = e^{-itH_0}\xi_0 + \int_0^{t_1} e^{-i(t-\tau)H_0}\, \mathbf{P}_c G_\xi(\tau)\, d\tau.$$

Hence, using $\|G_\xi(\tau)\|_{L^1\cap L^{4/3}} \le o(1)\alpha(1+s)^{-3/2}$ and (2.42)–(2.43), we have

$$\left\|e^{-i(t-t_1)H_0}\xi(t_1)\right\|_{L^4} \le \alpha t^{-3/4} + \int_0^{t_1} (t-\tau)^{-3/4} \|G_\xi(\tau)\|_{L^{4/3}}\, d\tau$$
$$\le \alpha t^{-3/4} + o(1)\alpha t^{-3/4} \le (1+\iota)\alpha t^{-3/4},$$

$$\left\|e^{-i(t-t_1)H_0}\xi(t_1)\right\|_{L^2_{\mathrm{loc}}}$$
$$\le \alpha t^{-3/2} + \int_0^{t_1} \min\left\{(t-\tau)^{-3/2}, (t-\tau)^{-3/4}\right\} \|G_\xi(\tau)\|_{L^1\cap L^{4/3}}\, d\tau$$
$$\le (1+\iota)\alpha t^{-3/2}.$$

This proves (3.10) and we conclude the proof of Proposition 3.1. **Q.E.D.**

The significance of t_1 is that it is a time when the dispersion loses its dominance over the bound states. If $t_1 = \infty$, the dispersion dominates for all the time and everything vanishes locally by (3.7). This gives us case I of Theorem 1.1. Suppose now $t_1 < \infty$. There are two possibilities:

1. $|x(t_1)| \ge |y(t_1)|^{2+\delta}$,
2. $|x(t_1)| < |y(t_1)|^{2+\delta}$.

We will focus on the second case since it is more subtle. We will come back to the first case, which corresponds to case II_a, at the end.

Proposition 3.2. *Suppose $t_1 < \infty$ and*

$$|y(t_1)| = n, \quad |x(t_1)| \le n^{2+\delta}, \quad \alpha t_1^{-3/2} = n^{2+\delta}. \tag{3.14}$$

Define

$$t_2 \equiv \sup\left\{t \ge t_1 : \ n^{2+\iota}|x(s)| \le \alpha \langle s\rangle^{-3/2},\ \forall s \in [t_1, t]\right\}. \tag{3.15}$$

For $t \in [t_1, t_2)$, we have

$$\begin{aligned}
&|y(t)/y(t_1)| \in [\tfrac{7}{8},\ \tfrac{9}{8}],\\
&|x(t)| \le \min\left\{2n^{2+\delta}, n^{-2-\iota}\alpha\langle t\rangle^{-3/2}\right\},\\
&\|\xi(t)\|_{L^4} \le (1+2\iota)\alpha\langle t\rangle^{-3/4},\\
&\|\xi(t)\|_{L^2_{\mathrm{loc}}} \le Cn^{-\iota}\alpha\langle t\rangle^{-3/2}.
\end{aligned} \tag{3.16}$$

Suppose $t_2 = \infty$. Then there is a $y_\infty \sim n$ such that

$$\big||y(t)| - y_\infty\big| + |x(t)| + \|\xi(t)\|_{L^2_{\text{loc}}} \le Ct^{-1/2}, \qquad \textit{as } t \to \infty. \tag{3.17}$$

Moreover, $\Theta(t) = -E_1(y_\infty)t + O(t^{1/2})$ as $t \to \infty$, where $\Theta(t)$ is the phase of $y(t)$, defined in (2.6).

Suppose $t_2 < \infty$. We have $n^{2+\iota}|x(t_2)| = \alpha \langle t_2\rangle^{-3/2}$ and, for all $t \ge t_2$, the following outgoing estimates on the dispersive wave ξ:

$$\begin{aligned} \left\| e^{-i(t-t_2)H_0}\xi(t_2)\right\|_{L^4} &\le (1+2\iota)\alpha t^{-3/4}, \\ \left\| e^{-i(t-t_2)H_0}\xi(t_2)\right\|_{L^2_{\text{loc}}} &\le (1+2\iota)\alpha t^{-3/2} + \frac{Cn^2|x(t_2)|t_1}{t_1+t-t_2}\langle t-t_2\rangle^{-1/2}. \end{aligned} \tag{3.18}$$

PROOF: We first consider $t \in [t_1, t_2)$. By definition of t_2, we have

$$|x(t)| \le n^{-2-\iota}\alpha t^{-3/2}, \qquad (t_1 \le t < t_2). \tag{3.19}$$

Using a continuity argument we may assume

$$\begin{aligned} &|y(t)/y(t_1)| \in [\tfrac{1}{2}, \tfrac{3}{2}], && |x(t)| \le 3n^{2+\delta}, \\ &\|\xi(t)\|_{L^4} \le 2\alpha\langle t\rangle^{-3/4}, && \|\xi(t)\|_{L^2_{\text{loc}}} \le 2Cn^{-\iota}\alpha\langle t\rangle^{-3/2}. \end{aligned} \tag{3.20}$$

We first estimate $\xi(t)$. By Lemma 2.3, $\|G_\xi(s)\|_{L^1\cap L^{4/3}} \lesssim n^2x + X$ with

$$X(s) = n\alpha\|\xi\|_{L^2_{\text{loc}}} + \alpha\|\xi\|^2_{L^4} \le C(n^{1-\iota}\alpha^2 + \alpha^3)s^{-3/2},$$

where we have used (3.20) in the last inequality. Using (3.19), we thus have $\|G_\xi(s)\|_{L^1\cap L^{4/3}} \le Cn^{-\iota}\alpha s^{-3/2}$. For $\xi(t)$ with $t \in [t_1, t_2)$ we have

$$\xi(t) = e^{-i(t-t_1)H_0}\xi(t_1) + J(t), \qquad J(t) \equiv \int_{t_1}^t e^{-i(t-s)H_0}\,\mathbf{P}_c^{H_0}\,G_\xi(s)\,ds.$$

The estimate for $e^{-i(t-t_1)H_0}\xi(t_1)$ is provided by (3.10) of Proposition 3.1. Hence it suffices to estimate the integral $J(t)$. We have

$$\begin{aligned} \|J(t)\|_{L^4} &\le C\int_{t_1}^t |t-s|^{-3/4}\|G_\xi(s)\|_{L^{4/3}}\,ds \le C\int_{t_1}^t |t-s|^{-3/4}n^{-\iota}\alpha s^{-3/2}\,ds \\ &\le Cn^{-\iota}\alpha t_1^{-1/2}t^{-3/4} \le C\alpha^{2/3}n^{(2+\delta)/3-\iota}t^{-3/4} \ll \alpha t^{-3/4}, \end{aligned}$$

where we have used the inequality (2.42) to bound the last integral, and also $\alpha^{1/3}t_1^{-1/2} = n^{(2+\delta)/3}$ by (3.14). Similarly,

$$\begin{aligned}\|J(t)\|_{L^2_{\text{loc}}} &\leq C\int_{t_1}^{t} \min\left\{|t-s|^{-3/2},\ |t-s|^{-3/4}\right\} \|G_\xi(s)\|_{L^1\cap L^{4/3}}\, ds \\ &\leq C\int_{t_1}^{t} \min\left\{|t-s|^{-3/2},\ |t-s|^{-3/4}\right\} n^{-\iota}\alpha s^{-3/2}\, ds \leq Cn^{-\iota}\alpha t^{-3/2}\end{aligned}$$

by (2.43). We have proven the estimates of $\xi(t)$ in (3.16).

We will estimate x and y using the normal form in Lemma 2.4 with the initial time $t = t_1$. Recall that $x(t) = e^{-ie_0 t}u(t)$, $y(t) = e^{-ie_0 t}v(t)$ and the perturbations μ of u and ν of v satisfy (2.37) and (2.38). We first estimate the error terms g_u and $\widetilde{g}_v$ in (2.37)–(2.38), for which we need a bound on $\left\|\xi^{(3)}(t)\right\|_{L^2_{\text{loc}}}$.

Recall $\xi^{(3)} = \xi_1^{(3)} + \xi_2^{(3)} + \xi_{3-5}^{(3)}$ is defined in (2.21). We set the initial time to t_1 and replace ξ_0 by $\xi(t_1)$ in (2.21). The estimate of $\xi_1^{(3)}$ is given by (3.10). $\left\|\xi_2^{(3)}(t)\right\|_{L^2_{\text{loc}}}$ is bounded by $Cn^2|x(t_1)|\,\langle t-t_1\rangle^{-3/2}$ by Lemma 2.2 and the definition (2.19) of $\xi^{(2)}(t_1)$. For $\xi_{3-5}^{(3)}$ we have the integral estimate (2.25) and we can use Lemma 2.3 to bound the integrand, $g_{\xi,3-5}(s) \lesssim n^4 x + X \leq C(n^{2-\iota}\alpha + n^{1-\iota}\alpha^2 + \alpha^3)s^{-3/2} = o(1)\alpha s^{-3/2}$. Summing all the estimates, we can bound $\xi^{(3)}(t)$ by

$$\begin{aligned}\left\|\xi^{(3)}(t)\right\|_{L^2_{\text{loc}}} &\leq (1+\iota)\alpha t^{-3/2} + Cn^2 n^{2+\delta}\,\langle t-t_1\rangle^{-3/2} \\ &\quad + \int_{t_1}^{t} \min\left\{|t-s|^{-3/2}, |t-s|^{-3/4}\right\} o(1)\alpha s^{-3/2} ds \\ &\leq (1+\iota)\alpha t^{-3/2} + Cn^2 t^{-3/2} + o(1)\alpha t^{-3/2} \leq 2\alpha t^{-3/2}.\end{aligned}$$

Here we have used $n^{2+\delta}\,\langle t-t_1\rangle^{-3/2} \leq Ct^{-3/2}$ for $t \geq t_1$. Using (2.36), (3.19), (3.20) and (3.9), we can bound the error terms g_u and $\widetilde{g}_v$ by

$$\begin{aligned}|g_u|, |\widetilde{g}_v| &\lesssim \alpha n^5|x| + n^2\left\|\xi^{(3)}\right\|_{L^2_{\text{loc}}} + n\,\|\xi\|^2_{L^2_{\text{loc}}} + (\|\xi\|_{L^2_{\text{loc}}} + \alpha n^2)\,\|\xi\|^2_{L^4} \\ &\lesssim n^2\alpha t^{-3/2}.\end{aligned}$$

We now estimate $x(t)$. If $t > n^{-10/3}$, using $\delta + \iota < 1$ we have $|x(t)| \leq n^{-2-\iota}\alpha t^{-3/2} \leq n^{-2-\iota}\alpha(n^{-10/3})^{-3/2} \leq n^{2+\delta}$.

If $t \in [t_1, n^{-10/3}]$, using (2.37) we have,

$$\begin{aligned} \big||\mu(t)| - |\mu(t_1)|\big| &\le \int_{t_1}^t \left|\frac{d}{ds}|\mu(s)|\right| ds \le C \int_{t_1}^t n^4 |x(s)| + |g_u(s)|\, ds \\ &\le C \int_{t_1}^t n^4 n^{2+\delta} + n^2 \alpha s^{-3/2}\, ds \\ &\le C n^{6+\delta} n^{-10/3} + C n^2 \alpha \langle t_1 \rangle^{-1/2} \ll n^{2+\delta}. \end{aligned} \tag{3.21}$$

Here we have used $\alpha^{1/3} t_1^{-1/2} = n^{(2+\delta)/3}$ and $\delta < 1$. Since $\mu = u + O(n^2 u)$, together with (3.19) we have proved the estimate for $x(t)$ in (3.16).

We now estimate $y(t)$. Using (2.38) and (3.19), for all $t \in [t_1, t_2)$ we have

$$\begin{aligned} \big||\nu(t)| - |\nu(t_1)|\big| &\le \int_{t_1}^t \left|\frac{d}{ds}|\nu(s)|\right| ds \le C \int_{t_1}^t n^4 |x(s)| + |\widetilde{g}_\nu(s)|\, ds \\ &\le C \int_{t_1}^t n^{2-\iota} \alpha s^{-3/2}\, ds \le C n^{2-\iota} \alpha \langle t_1 \rangle^{-1/2} \ll n. \end{aligned} \tag{3.22}$$

Since $\nu = v + O(n^2 u)$, we have proved the estimate for $y(t)$ in (3.16). The proof of (3.16) is complete.

Suppose $t_2 = \infty$. The bounds of $x(t)$ and $\|\xi(t)\|_{L^2_{\text{loc}}}$ in (3.17) are given by (3.16). By the same argument as in (3.22), we have for all $t > \tau \ge t_1$,

$$\big||\nu(t)| - |\nu(\tau)|\big| \le C \int_\tau^t n^{2-\iota} \alpha s^{-3/2}\, ds \le C n^{2-\iota} \alpha \langle \tau \rangle^{-1/2},$$

which converges to zero as $t, \tau \to \infty$. Hence $|\nu(t)|$ and $|y(t)|$ have a limit y_∞. Moreover, $\big||y_\infty| - |\nu(\tau)|\big| \le C\tau^{-1/2}$ as $\tau \to \infty$ and $y_\infty - |y(t_1)|$ is bounded by $C n^{2-\iota} \alpha \langle t_1 \rangle^{-1/2} \ll n$. Hence $y_\infty \sim n$. The phase $\Theta(t)$ of $y(t)$ is given in (2.6), $\Theta(t) = \theta(t) - \int_0^t E_1(|y(s)|) ds$. Let $E_\infty = E_1(y_\infty)$. Using (2.9) we have

$$\begin{aligned} |\Theta(t) + E_\infty t| &\le |\theta(0)| + \int_0^t |\dot{\theta}| + |E_1(y(s)) - E_\infty|\, ds \\ &\le C + \int_0^t C n^{-1} \|G(s)\|_{L^1_{\text{loc}}} + C n^2 \big||y(s)| - y_\infty\big|\, ds \\ &\le C + \int_0^t C(1+s)^{-1/2} ds \le C(1+t)^{1/2}. \end{aligned}$$

We have completed the proof of (3.17).

Suppose now $t_2 < \infty$. For $t \ge t_2$ we have

$$e^{-i(t-t_2)H_0} \xi(t_2) = e^{-i(t-t_1)H_0} \xi(t_1) + J_2(t),$$

where

$$J_2(t) = \int_{t_1}^{t_2} e^{-i(t-s)H_0}\, \mathbf{P}_c^{H_0}\, G_\xi(s)\, ds.$$

The estimate for $e^{-i(t-t_1)H_0}\xi(t_1)$ is provided by (3.10) of Proposition 3.1. Hence we only need to estimate $J_2(t)$. Recall $\|G_\xi(s)\|_{L^{4/3}\cap L^1} \le Cn^{-\iota}\alpha \langle s\rangle^{-3/2}$ for $s \in [t_1, t_2]$. Hence, by (2.42) and (3.14),

$$\begin{aligned}\|J_2(t)\|_{L^4} &\le C\int_{t_1}^{t_2} |t-s|^{-3/4}\,\|G_\xi(s)\|_{L^{4/3}}\, ds \le C\int_{t_1}^{t} |t-s|^{-3/4} n^{-\iota}\alpha s^{-3/2} ds \\ &\le Cn^{-\iota}\alpha t_1^{-1/2}\langle t\rangle^{-3/4} \le C\alpha^{2/3} n^{(2+\delta)/3-\iota} t^{-3/4} \ll \alpha t^{-3/4}.\end{aligned}$$

This proves the first bound in (3.18).

For the $L^2_{\rm loc}$ norm we have

$$\|J_2(t)\|_{L^2_{\rm loc}} \le \int_{t_1}^{t_2} \Omega(s)\, ds \le \int_{\ell}^{t-\ell} \Omega(s) ds + \int_{t_2-\ell}^{t_2} \Omega(s) ds,$$

where $\ell = t_1/2$ and $\Omega(s) := \min\{|t-s|^{-3/2}, |t-s|^{-3/4}\}\, Cn^{-\iota}\alpha s^{-3/2}$. We have

$$\int_{\ell}^{t-\ell} \Omega(s) ds \le Cn^{-\iota}\alpha \int_{\ell}^{t-\ell} |t-s|^{-3/2} s^{-3/2} ds = 2Cn^{-\iota}\alpha \int_{\ell}^{t/2} |t-s|^{-3/2} s^{-3/2} ds.$$

The last integral is bounded by

$$\int_{\ell}^{t/2} t^{-3/2} Cn^{-\iota}\alpha s^{-3/2} ds \le Cn^{-\iota}\alpha \ell^{-1/2} t^{-3/2} \ll \alpha t^{-3/2}.$$

Recall $\alpha t_2^{-3/2} = n^{2+\iota}|x(t_2)|$. Since $s \sim t_2$ for $s \in [t_2-\ell, t_2]$, by (2.41) we can bound the second integral by

$$\begin{aligned}\int_{t_2-\ell}^{t_2} \Omega(s) ds &\le \int_{t_2-\ell}^{t_2} \min\left\{|t-s|^{-3/2}, |t-s|^{-3/4}\right\} Cn^{-\iota}\alpha t_2^{-3/2} ds \\ &\le Cn^2|x(t_2)| \frac{\ell}{\ell+t-t_2} \langle t-t_2\rangle^{-1/2}.\end{aligned}$$

Combining these two bounds, we have proved the second bound in (3.18). **Q.E.D.**

The significance of t_2 is that it is a time when the dispersion loses its dominance over the ground state. If $t_2 = \infty$, by (3.17) the solution $\psi(t)$ converges locally to an excited state $Q_1(y_\infty)$. This gives us case III of Theorem 1.1. We shall consider the other case $t_2 < \infty$, which corresponds to case II_b, in Propositions 3.3–3.4.

Proposition 3.3. *Suppose that the assumptions of Proposition 3.2 hold and* $t_2 < \infty$. *Let*

$$t_3 \equiv \inf\{t \geq t_2 : \ |x(s)| < 0.001n, \ \forall s \in [t_2, t]\}.$$

We have

$$t_2 + n^{-4} \leq t_3 < \infty, \qquad |x(t_3)| = 0.001n,$$

and the following estimates for all s, t *with* $t_2 \leq s \leq t \leq t_3$:

$$\begin{aligned} |y(t)/y(t_2)| &\in [\tfrac{24}{25}, \ \tfrac{26}{25}], \\ |x(t)/x(s)| &\in [\tfrac{3}{4} \ e^{\frac{3}{4}\gamma_0 n^4 (t-s)}, \ \tfrac{5}{4} \ e^{\frac{5}{4}\gamma_0 n^4 (t-s)}], \\ \|\xi(t)\|_{L^4} &\leq C_3 \, n|x(t)| + (1+3\iota)\alpha t^{-3/4}, \\ \|\xi(t)\|_{L^2_{\text{loc}}} &\leq C_3 \, n^2|x(t)| + (1+3\iota)\alpha t^{-3/2}, \\ \left\|\xi^{(3)}(t)\right\|_{L^2_{\text{loc}}} &\leq C_3 \alpha n^3 |x(t)| + (1+3\iota)\alpha t^{-3/2} \\ &\quad + C_3 n^2 |x(t_2)| \frac{t_1}{t_1 + t - t_2} \langle t - t_2 \rangle^{-1/2}, \end{aligned} \tag{3.23}$$

for some explicit constant $C_3 > 0$.

PROOF: Using a continuity argument we may assume

$$\begin{aligned} |y(t)/y(t_2)| &\in [\tfrac{19}{20}, \ \tfrac{21}{20}] \\ |x(t)/x(s)| &\in [\tfrac{1}{2} \ e^{\frac{1}{2}\gamma_0 n^4 (t-s)}, \ \tfrac{3}{2} \ e^{\frac{3}{2}\gamma_0 n^4 (t-s)}] \\ \|\xi(t)\|_{L^4} &\leq 2C_3 \, n|x(t)| + 2\alpha t^{-3/4}, \\ \|\xi(t)\|_{L^2_{\text{loc}}} &\leq 2C_3 \, n^2|x(t)| + 2\alpha t^{-3/2}, \\ \left\|\xi^{(3)}(t)\right\|_{L^2_{\text{loc}}} &\leq 2C_3 \alpha n^3 |x(t)| + 2\alpha t^{-3/2} + \frac{2C_3 n^2 |x(t_2)| t_1}{t_1 + t - t_2} \langle t - t_2 \rangle^{-1/2}. \end{aligned} \tag{3.24}$$

We now apply Lemma 2.4 with *the initial time set to* $t = t_2$ to obtain the normal form for x and y and the decomposition of ξ.

We first estimate G_ξ and $g_{\xi,3-5}$. By Lemma 2.3, $\|G_\xi(s)\|_{L^1 \cap L^{4/3}} \lesssim n^2 x + X$ with

$$X(s) = n\alpha \, \|\xi\|_{L^2_{\text{loc}}} + \alpha \, \|\xi\|_{L^4}^2 \leq C\alpha n^3 |x(s)| + C\alpha^3 s^{-3/2}.$$

Hence $\|G_\xi(s)\|_{L^1 \cap L^{4/3}} \leq Cn^2|x(s)| + \alpha^3 s^{-3/2}$. By the same Lemma $g_{\xi,3-5}(s) \lesssim n^4|x(s)| + X \leq C\alpha n^3|x(s)| + \alpha^3 s^{-3/2}$. Note we have $|x(s)| \leq 2|x(t)| e^{-\frac{1}{2}\gamma_0 n^4 (t-s)}$ for $t_2 \leq s \leq t \leq t_3$ by (3.24).

We now estimate $\xi(t)$. For $t \in [t_2, t_3]$ we have

$$\xi(t) = e^{-i(t-t_2)H_0}\xi(t_2) + J(t), \qquad J(t) \equiv \int_{t_2}^{t} e^{-i(t-s)H_0}\, \mathbf{P}_c^{H_0}\, G_\xi(s)\, ds.$$

The estimate for $e^{-i(t-t_2)H_0}\xi(t_2)$ is by (3.18) of Proposition 3.2. Hence it suffices to estimate the integral. By the above estimate of G_ξ we have, using (2.42),

$$\begin{aligned}\|J(t)\|_{L^4} &\le C\int_{t_2}^{t} |t-s|^{-3/4}\, \|G_\xi(s)\|_{L^{4/3}}\, ds\\ &\le C\int_{t_2}^{t} |t-s|^{-3/4}\left(n^2|x(t)|e^{-\frac{1}{2}\gamma_0 n^4(t-s)} + \alpha^3 s^{-3/2}\right) ds\\ &\le Cn^2|x(t)|(n^{-4})^{1/4} + C\alpha^3 t_2^{-1/2}t^{-3/4} \le Cn|x(t)| + o(1)\,\alpha t^{-3/4}.\end{aligned}$$

For the $L^2_{\rm loc}$ norm, since $\xi = \xi^{(2)} + \xi^{(3)}$ and $\left\|\xi^{(2)}(t)\right\|_{L^2_{\rm loc}} \le Cn^2|x(t)|$ by its explicit form, it suffices to estimate $\xi^{(3)} = \xi^{(3)}_1 + \xi^{(3)}_2 + \xi^{(3)}_{3-5}$. By the above estimate of $g_{\xi,3-5}$ and (2.25),

$$\begin{aligned}\left\|\xi^{(3)}_{3-5}(t)\right\|_{L^2_{\rm loc}} &\le \int_{t_2}^{t} \min\left\{|t-s|^{-3/2}, |t-s|^{-3/4}\right\} g_{\xi,3-5}(s)\, ds\\ &\le C\int_{t_2}^{t} \min\left\{|t-s|^{-3/2}, |t-s|^{-3/4}\right\}\\ &\qquad\cdot\left(\alpha n^3|x(t)|e^{-\frac{1}{2}\gamma_0 n^4(t-s)} + \alpha^3 s^{-3/2}\right) ds\\ &\le C\alpha n^3|x(t)| + C\alpha^3 t^{-3/2}.\end{aligned}$$

The estimate of $\xi^{(3)}_1(t)$ is given in (3.18). We also have

$$\left\|\xi^{(3)}_2(t)\right\|_{L^2_{\rm loc}} \le Cn^2|x(t_2)|\,\langle t-t_2\rangle^{-3/2} \le Cn^2|x(t_2)|\frac{t_1}{t_1+t-t_2}\,\langle t-t_2\rangle^{-1/2}$$

by its explicit form and Lemma 2.2. Hence the $L^2_{\rm loc}$-bounds of ξ and $\xi^{(3)}$ are proved.

We next estimate g_u and $\widetilde{g}_v$. By (2.36) of Lemma 2.4, (3.24), and $\alpha t^{-3/2} \le n^{2+\delta}$,

$$\begin{aligned}|g_u|, |\widetilde{g}_v| &\lesssim \alpha n^5|x| + n^2\left\|\xi^{(3)}\right\|_{L^2_{\rm loc}} + n\,\|\xi\|^2_{L^2_{\rm loc}} + (\|\xi\|_{L^2_{\rm loc}} + \alpha n^2)\,\|\xi\|^2_{L^4}\\ &\lesssim \alpha n^5|x(t)| + \alpha n^2 t^{-3/2} + n^4|x(t_2)|\frac{t_1}{t_1+t-t_2}\,\langle t-t_2\rangle^{-1/2}.\end{aligned}$$

We now estimate $x(t)$ and $y(t)$ using the normal form (2.33) in Lemma 2.4 for the perturbation $\mu(t)$ of $u(t) = e^{ie_0 t}x(t)$ and $\nu(t)$ of $v(t) = e^{ie_1 t}y(t)$. Recall that the initial time for normal form and the decomposition of ξ is reset at $t = t_2$. We first consider $t \in [t_2, t_2 + n^{-1}]$. Using the estimate of g_u and $\alpha t_2^{-3/2} = n^{2+\iota}|x(t_2)|$,

$$\left|\frac{d}{dt}|\mu|\right| \lesssim n^4|x| + |g_u| \lesssim n^4|x| + \alpha n^2 t^{-3/2} \lesssim n^4|x(t_2)|.$$

Therefore $\big||\mu(t)| - |\mu(t_2)|\big| \le Cn^4|x(t_2)|(t - t_2) \ll |x(t_2)|$, and hence $|x(t)| \sim |x(t_2)|$. Similarly $|y(t)| \sim |y(t_2)|$.

We next consider $t \in [t_2 + n^{-1}, t_3]$. The previous estimate of g_u and $\widetilde{g}_v$ becomes

$$|g_u|, |\widetilde{g}_v| \lesssim \alpha n^5|x(t)| + \alpha n^2 t^{-3/2} + n^{4+1/2}|x(t_2)| \le Cn^{4+\iota}|x(t)|. \tag{3.25}$$

Here we have used $\alpha t^{-3/2} \le n^{2+\iota}|x(t_2)|$ and $|x(t_2)| \le |x(t)|$.

By the estimate of g_u and (2.37), we have

$$|\mu|^{-1}\frac{d}{dt}|\mu| \in [\tfrac{3}{4}\gamma_0 n^4,\ \tfrac{5}{4}\gamma_0 n^4].$$

Hence, for all s, t with $t_2 \le s \le t \le t_3$,

$$|\mu(t)/\mu(s)| \in [e^{\frac{3}{4}\gamma_0 n^4(t-s)},\ e^{\frac{5}{4}\gamma_0 n^4(t-s)}].$$

Since $\big||x| - |\mu|\big| \le Cn^2|x|$, we have proven $(3.23)_2$ for $|x(t)/x(s)|$. Since $|x(t)| \le 0.001n$ for all $t < t_3$, we must have $t_3 < \infty$. Moreover,

$$(\tfrac{5}{4}\gamma_0 n^4)^{-1}\ \log\frac{4|x(t_3)|}{5|x(t_2)|} \le t_3 - t_2 \le (\tfrac{3}{4}\gamma_0 n^4)^{-1}\ \log\frac{4|x(t_3)|}{3|x(t_2)|}. \tag{3.26}$$

We now estimate $|y(t)/y(s)|$. By the estimate of $\widetilde{g}_v$ and (2.38), we have

$$\left|\frac{d}{dt}|\nu|\right| = \left|-2\gamma_0|\mu|^2|\nu|^3 + \mathrm{Re}\,\widetilde{g}_v\bar{\nu}/|\nu|\right| \le 3\gamma_0 n^4|x(s)|. \tag{3.27}$$

By (3.24)

$$\begin{aligned}\big||\nu(t)| - |\nu(t_2)|\big| &\le \int_{t_2}^{t} 3\gamma_0 n^4|x(s)|\,ds \le \int_{t_2}^{t} 3\gamma_0 n^4 2|x(t)|e^{-\frac{1}{2}\gamma_0 n^4(t-s)}\,ds \\ &\le 12|x(t)| \le 0.012n.\end{aligned}$$

Since $\big||y| - |\nu|\big| \le Cn^2|x|$, we have proven $(3.23)_1$ for $|y(t)/y(s)|$. **Q.E.D.**

Proposition 3.4. *Assume the same assumptions of Proposition 3.2. Let*

$$t_4 \equiv \sup\{t \geq t_3 : \ |y(s)| \geq \varepsilon n, \quad \forall s \in [t_3, t)\}, \tag{3.28}$$

where $\varepsilon = \varepsilon_0/4$ and $\varepsilon_0 > 0$ is the small constant in Theorem 2.5. We have

$$t_3 \leq t_4 \leq t_3 + C(\gamma_0 \varepsilon^2 n^4)^{-1}. \tag{3.29}$$

We also have the following estimates for $t_3 \leq t \leq t_4$:

$$\begin{aligned} &\tfrac{1}{1200} n \leq |x(t)| \leq 2n, \qquad |y(t)| \leq 2n, \\ &|x(t_4)| \geq \tfrac{1}{2} n, \qquad |y(t_4)| = \varepsilon n. \end{aligned} \tag{3.30}$$

$$\begin{aligned} \|\xi(t)\|_{L^4} &\leq C_4\, n|x(t)| + (1+4\iota)\alpha t^{-3/4}, \\ \|\xi(t)\|_{L^2_{\rm loc}} &\leq C_4\, n^2|x(t)| + (1+4\iota)\alpha t^{-3/2}, \\ \left\|\xi^{(3)}(t)\right\|_{L^2_{\rm loc}} &\leq C_4 n^{2+\iota}|x(t)|, \end{aligned} \tag{3.31}$$

for some explicit constant $C_4 > 0$. Moreover, for $t \geq t_4$ and $\Delta t = \varepsilon^{-2} n^{-4}$, we have the following outgoing estimates on the dispersive wave ξ:

$$\begin{aligned} \left\|e^{-i(t-t_4)H_0}\xi(t_4)\right\|_{L^4} &\leq C n^3 \Delta t (\Delta t + t - t_4)^{-3/4}, \\ \left\|e^{-i(t-t_4)H_0}\xi(t_4)\right\|_{L^2_{\rm loc}} &\leq C n^3 \frac{\Delta t}{\Delta t + t - t_4} (1 + t - t_4)^{-1/2}. \end{aligned} \tag{3.32}$$

Hence the conditions of Theorem 2.5 are satisfied at $t = t_4$ and the solution $\psi(t)$ converges locally to a nonlinear ground state.

PROOF: By a continuity argument and (3.3), we may assume

$$\begin{aligned} \tfrac{1}{1400} n \leq |x(t)| &\leq 2n, \qquad |y(t)| \leq 2n, \\ \|\xi(t)\|_{L^4} &\leq 2C_4\, n|x(t)| + 2\alpha t^{-3/4}, \\ \|\xi(t)\|_{L^2_{\rm loc}} &\leq 2C_4\, n^2|x(t)| + 2\alpha t^{-3/2}, \\ \left\|\xi^{(3)}(t)\right\|_{L^2_{\rm loc}} &\leq 2C_4 n^{2+\iota}|x(t)|. \end{aligned} \tag{3.33}$$

The estimates for G_ξ, ξ, $g_{\xi,3-5}$, $\xi^{(3)}$, g_u and $\widetilde{g}_v$ can be proved in the same way as those in Proposition 3.3. The only difference is on the estimates of the bound states x and y, which we now focus on.

For any $t \leq t_4$, we have $|y(t)| \geq \varepsilon n$. By (2.37), (3.25) and (3.33), we have

$$\frac{d}{dt}|\mu| = \gamma_0 |\nu|^4 |\mu| + \operatorname{Re} g_\mu \bar{\mu}/|\mu| \geq \gamma_0 \varepsilon^4 n^4 |\mu| - C n^{4+\iota}|\mu| \geq \tfrac{7}{8}\gamma_0 \varepsilon^4 n^4 |\mu|.$$

Hence $|x(t)|, |\mu(t)| \geq |x(t_3)|(1-Cn^2) \geq \frac{1}{1200}n$. By (2.38), (3.25), (3.33), and $\nu = v + O(n^2 x)$,

$$\frac{d}{dt}|\nu| = -2\gamma_0|\mu|^2|\nu|^3 + \operatorname{Re}\widetilde{g}_v\bar{\nu}/|\nu|$$
$$\leq -2\gamma_0(\tfrac{n}{1400})^2|\nu|^3 + Cn^{5+\iota} < -10^{-6}\gamma_0 n^2|\nu|^3.$$

Let $\rho(t) = \left\{|\nu(t_3)|^{-2} + 2(10^{-6})\gamma_0 n^2(t-t_3)\right\}^{-1/2}$. We have $\rho(t_3) = |\nu(t_3)|$ and $\frac{d}{dt}\rho = -10^{-6}\gamma_0 n^2\rho^3$. By comparison principle,

$$|\nu(t)| \leq \rho(t) = \left\{|\nu(t_3)|^{-2} + 2(10^{-6})\gamma_0 n^2(t-t_3)\right\}^{-1/2}.$$

Since $|\nu(t)| \geq \varepsilon n/2$ for $t \in [t_3, t_4]$, we have $t_4 < \infty$ and $t_4 - t_3 \leq C\gamma_0^{-1}\varepsilon^{-2}n^{-4}$. Similarly,

$$\frac{d}{dt}|\nu| \geq -3\gamma_0 n^2|\nu|^3.$$

We can compare it with $\rho_-(t) = \left\{|\nu(t_3)|^{-2} + 6\gamma_0 n^2(t-t_3)\right\}^{-1/2}$ and obtain $\nu(t) \geq \rho_-(t)$ by comparison principle. This gives a lower bound $t_4 - t_3 \geq C\Delta t$ if $|\nu(t_3)| \geq 2\varepsilon n$.

We finally prove the outgoing estimates (3.32) for $\xi(t_4)$. For $t \geq t_4$,

$$e^{-i(t-t_4)H_0}\xi(t_4) = e^{-i(t-t_2)H_0}\xi(t_2) + J_4(t),$$

where $J_4(t)$ denotes the integral

$$J_4(t) = \int_{t_2}^{t_4} e^{-i(t-s)H_0}\,\mathbf{P}_c^{H_0}\,G_\xi(s)\,ds.$$

The estimate of $e^{-i(t-t_2)H_0}\xi(t_2)$ is by (3.18) of Proposition 3.2. Hence we only need to estimate $J_4(t)$. We have shown in the proof of Proposition 3.3 that, for $s \in [t_2, t_3]$,

$$\|G_\xi(s)\|_{L^1\cap L^{4/3}} \leq n^2|x(t_3)|e^{-\frac{1}{2}\gamma_0 n^4(t_3-s)} + \alpha^3 s^{-3/2}, \qquad (t_2 \leq s \leq t_3).$$

For $s \in [t_3, t_4]$, using $t_3 \geq n^{-4}$,

$$\|G_\xi(s)\|_{L^1\cap L^{4/3}} \leq Cn^3, \qquad (t_3 \leq s \leq t_4).$$

By the exponential decay, (2.43), (2.42) and $t_4 - t_3 \le C\Delta t$, $\Delta t = \varepsilon^{-2} n^{-4}$,

$$\begin{aligned}
\|J_4(t)\|_{L^4} &\le C \int_{t_2}^{t_4} |t-s|^{-3/4} \|G_\xi(s)\|_{L^{4/3}}\, ds \\
&\le \int_{t_2}^{t_3} |t-s|^{-3/4} \left(n^2 |x(t_3)| e^{-\frac{1}{2}\gamma_0 n^4 (t_3 - s)} + \alpha^3 s^{-3/2} \right) ds \\
&\quad + \int_{t_3}^{t_4} |t-s|^{-3/4} C n^3 ds \\
&\le C \int_{t_3 - n^{-4}}^{t_3} |t-s|^{-3/4} n^3\, ds + C\alpha^3 t_2^{-1/2} t^{-3/4} + C \int_{t_3}^{t_4} |t-s|^{-3/4} n^3\, ds \\
&\le C n^3 \Delta t (\Delta t + t - t_4)^{-3/4} + C\alpha^3 t_2^{-1/2} t^{-3/4}.
\end{aligned}$$

Since $\alpha t^{-3/4} \le C n^3 \Delta t (\Delta t + t - t_4)^{-3/4}$ for $t \ge t_4$, we have the L^4 estimate in (3.32). For L^2_{loc}-norm, we have

$$\begin{aligned}
\|J_4(t)\|_{L^2_{\mathrm{loc}}} &\le C \int_{t_2}^{t_4} \min\left\{ |t-s|^{-3/2}, |t-s|^{-3/4} \right\} \|G_\xi(s)\|_{L^1 \cap L^{4/3}}\, ds \\
&\le \int_{t_2}^{t_3} |t-s|^{-3/2} \left(n^2 |x(t_3)| e^{-\frac{1}{2}\gamma_0 n^4 (t_3 - s)} + \alpha^3 s^{-3/2} \right) ds \\
&\quad + \int_{t_3}^{t_4} \min\left\{ |t-s|^{-3/2}, |t-s|^{-3/4} \right\} C n^3 ds \\
&\le C \int_{t_3 - n^{-4}}^{t_3} |t-s|^{-3/2} n^3\, ds + C\alpha^3 t^{-3/2} \\
&\quad + C n^3 \frac{\Delta t}{\Delta t + t - t_4} (1 + t - t_4)^{-1/2} \\
&\le C n^3 \frac{\Delta t}{\Delta t + t - t_4} (1 + t - t_4)^{-1/2}.
\end{aligned}$$

Here we have used the exponential decay, (2.41), (2.40), and that $\alpha^3 t^{-3/2}$ is less than the last quantity for $t \ge t_4 \ge n^{-4}$. The proof is complete. **Q.E.D.**

We now come back to case II$_a$ where $t = t_1$ and we have $|x(t_1)| \ge |y(t_1)|^{2+\delta}$. We further divide it to three subcases:

1. $|y(t_1)| = n$, $n^{2+\delta} \le |x(t_1)| \le 0.001n$.

2. $\max(|x(t_1)|, |y(t_1)|) = n$, $0.001n \le |x(t_1)| \le n$, $\varepsilon_0 n \le |y(t_1)| \le n$.

3. $|x(t_1)| = n$, $|y(t_1)| \le \varepsilon_0 n$.

In case 1, we can set $t_2 = t_1$ and our analysis in Propositions 3.3–3.4 and

Theorem 2.5 for $t \in [t_2, \infty)$ goes through. In case 2, we set $t_3 = t_2 = t_1$ and apply Proposition 3.4 and Theorem 2.5. In case 3, we can set $t_4 = t_1$ and apply Theorem 2.5 directly. The proof of Theorem 1.1 is complete.

Acknowledgments

Tsai was partially supported by NSF grant DMS-9729992. Yau was partially supported by NSF grant DMS-0072098.

References

[1] V.S. Buslaev and G.S. Perel'man, Scattering for the nonlinear Schrödinger equations: states close to a soliton, St. Petersburg Math J. **4** (1993) 1111–1142.

[2] V.S. Buslaev and G.S. Perel'man, On the stability of solitary waves for nonlinear Schrödinger equations. Nonlinear evolution equations, 75–98, Amer. Math. Soc. Transl. Ser. 2, 164, Amer. Math. Soc., Providence, RI, 1995.

[3] V.S. Buslaev and C. Sulem, On the asymptotic stability of solitary waves of nonlinear Schrödinger equations, preprint.

[4] S. Cuccagna, Stabilization of solutions to nonlinear Schrödinger equations. *Comm. Pure Appl. Math.* **54** (2001), no. 9, 1110–1145.

[5] S. Cuccagna, On asymptotic stability of ground states of NLS, preprint.

[6] M. Grillakis: Linearized instability for nonlinear Schrödinger and Klein-Gordon equations. *Comm. Pure Appl. Math.* **41** (1988), no. 6, 747–774.

[7] M. Grillakis: Analysis of the linearization around a critical point of an infinite dimensional Hamiltonian system, *Comm. Pure Appl. Math.* **43** (1990), 299–333.

[8] J.-L. Journe, A. Soffer and C.D. Sogge, Decay estimates for Schrödinger operators. *Comm. Pure Appl. Math.* **44** (1991), no. 5, 573–604.

[9] C. A. Pillet and C. E. Wayne: Invariant manifolds for a class of dispersive, Hamiltonian, partial differential equations. *J. Differential Equations* **141**, no. 2, (1997), 310–326.

[10] J. Shatah and W. Strauss: Instability of nonlinear bound states. *Comm. Math. Phys.* **100** (1985), no. 2, 173–190.

[11] J. Shatah and W. Strauss: Spectral condition for instability. Nonlinear PDE's, dynamics and continuum physics (South Hadley, MA, 1998), 189–198, Contemp. Math., 255, Amer. Math. Soc., Providence, RI, 2000.

[12] A. Soffer and M.I. Weinstein, Multichannel nonlinear scattering theory for nonintegrable equations I, II, *Comm. Math. Phys.* **133** (1990), 119–146; *J. Diff. Eqns.* **98**, (1992), 376–390.

[13] A. Soffer and M.I. Weinstein, Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations, *Invent. math.* **136**, (1999), 9–74.

[14] A. Soffer and M.I. Weinstein, Selection of the ground state for nonlinear Schrödinger equations, preprint.

[15] T.-P. Tsai and H.-T. Yau: Asymptotic dynamics of nonlinear Schrödinger equations: resonance dominated and dispersion dominated solutions, *Comm. Pure Appl. Math.* **55** (2002) 153–216.

[16] T.-P. Tsai and H.-T. Yau: Relaxation of excited states in nonlinear Schrödinger equations, *IMRN*, to appear.

[17] T.-P. Tsai and H.-T. Yau: Stable directions for excited states of nonlinear Schrödinger equations, *Comm. P.D.E.*, to appear.

[18] K. Yajima, The $W^{k,p}$ continuity of wave operators for Schrödinger operators, *J. Math. Soc. Japan* **47**, no. 3, (1995), 551–581.