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Ground state degeneracy of the Pauli-Fierz Hamiltonian with spin

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Abstract

We consider an electron, spin 1/2, minimally coupled to the quantized radiation field in the nonrelativistic approximation, a situation defined by the Pauli-Fierz Hamiltonian H. There is no external potential and H fibers as $\int^{\oplus} H_p dp$ according to the total momentum p. We prove that the ground state subspace of H_p is two-fold degenerate provided the charge e and the total momentum p are sufficiently small. We also establish that the total angular momentum of the ground state subspace is $\pm 1/2$ and study the case of a confining external potential.

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1 Introduction and main results

Nonrelativistic quantum electrodynamics predicts the anomalous magnetic moment (g-factor) of the electron with an error of less than 1%. One definition of the g-factor comes from the motion of the electron in a weak uniform external magnetic field. $g/2$ is then the ratio of the spin precession relative to the orbital precession. To provide a theoretical analysis of such an experiment one has to show that the ground state band of the Hamiltonian is adiabatically decoupled from the spectrum of excitations and has to determine the effective Hamiltonian governing the motion in the ground state band. The details are given elsewhere [7]. They result in a nonperturbative microscopic definition of the g-factor, which, when computed to order e^2, yields $g/2 = 1.0031$ as compared to the accurate relativistic value of $g/2 = 1.0012$. One crucial input of the analysis is the microscopic Hamiltonian at fixed total momentum to have an exactly two-fold degenerate ground state. In our paper we will, under suitable conditions, establish such a spectral property.

Let us first introduce the Pauli-Fierz Hamiltonian for a single electron coupled to the quantized radiation field. By translation invariance the total momentum is conserved and the operator of interest will be the Pauli-Fierz Hamiltonian fibered with respect to the total momentum. The Hilbert space for the coupled system is

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \otimes \mathcal{F}.$$

$L^2(\mathbb{R}^3)$ is the Hilbert space for the translational degrees of freedom of the electron, multiplication by x stands for the position, $-i\nabla_x$ for the momentum of the electron. $\mathbb{C}^2$ is the Hilbert space for the spin, σ, of the electron, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the spin 1/2 Pauli spin matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$\mathcal{F}$ is the symmetric Fock space for the photons given by

$$\mathcal{F} = \oplus_{n=0}^{\infty} \left(L^2(\mathbb{R}^3) \otimes \mathbb{C}^2\right)^n_{\text{sym}},$$

where $(\cdots)^n_{\text{sym}}$ denotes the n-fold symmetric tensor product of $(\cdots)$. $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ will be identified with $L^2(\mathbb{R}^3 \times \{1,2\})$. The Fock vacuum is denoted by Ω and a vector $\psi \in \mathcal{F}$ by $\psi = (\psi^{(0)}, \psi^{(1)}, ...)$, $\psi^{(0)} = c\Omega$, $c \in \mathbb{C}$. The photons live in $\mathbb{R}^3$ and have helicity ± 1. The photon field is thus represented by the two-component Bose field $a(k,j)$ on $\mathcal{F}$ with commutation relations

$$[a(k,j), a^*(k',j')] = \delta_{jj'}\delta(k-k'),$$

$$[a(k,j), a(k',j')] = 0, \quad [a^*(k,j), a^*(k',j')] = 0,$$

$k, k' \in \mathbb{R}^3$, $j, j' = 1, 2$. The energy of the photons is given by

$$H_{\mathrm{f}} = \sum_{j=1,2} \int \omega(k) a^*(k,j) a(k,j) dk, \tag{1}$$

i.e. H_{f} restricted to $(L^2(\mathbb{R}^3)\otimes\mathbb{C}^2)^n_{\mathrm{sym}}$ is multiplication by $\sum_{j=1}^n \omega(k_j)$. Throughout units are such that $\hbar = 1$, $c = 1$. Physically $\omega(k) = |k|$. This case is somewhat singular. To regularize, the photons are assumed to have a small mass as

$$\omega(k) = \sqrt{|k|^2 + m_{\mathrm{ph}}^2}, \quad m_{\mathrm{ph}} > 0. \tag{2}$$

In fact we can allow for a more general class of dispersion relations. We assume $\omega : \mathbb{R}^3 \to \mathbb{R}$ to be continuous with the properties

(A.1) $\inf_{k\in\mathbb{R}^3} \omega(k) \geq \omega_0 > 0$,

(A.2) $\omega(k_1) + \omega(k_2) \geq \omega(k_1 + k_2)$,

(A.3) $\omega(k) = \omega(Rk)$ for an arbitrary rotation R.

The quantized transverse vector potential is defined through

$$A_\varphi(x) = \sum_{j=1,2} \int \frac{\widehat{\varphi}(k)}{\sqrt{2\omega(k)}} e_j(k) \left(e^{-ik\cdot x} a^*(k,j) + e^{ik\cdot x} a(k,j)\right) dk.$$

Here e_1 and e_2 are polarization vectors which together with $\widehat{k} = k/|k|$ form a standard basis in $\mathbb{R}^3$. $\varphi : \mathbb{R}^3 \to \mathbb{R}$ is the form factor which ensures an ultraviolet cutoff. It is assumed to be rotational invariant, $\varphi(Rx) = \varphi(x)$ for an arbitrary rotation R, continuous, bounded with some decay at infinity, and normalized as $\int \varphi(x)dx = 1$. We will mostly work with the Fourier transform $\widehat{\varphi}(k) = (2\pi)^{-3/2} \int \varphi(x) e^{-ik\cdot x} dx$. It satisfies (1) $\widehat{\varphi}(Rk) = \widehat{\varphi}(k)$ for an arbitrary rotation R, (2) $\overline{\widehat{\varphi}} = \widehat{\varphi}$ for notational simplicity, (3) the normalization $\widehat{\varphi}(0) = (2\pi)^{-3/2}$, and (4) the decay

$$\int \left(\omega(k)^{-2} + \omega(k)^{-1} + 1 + \omega(k)\right) |\widehat{\varphi}(k)|^2 dk < \infty.$$

With these preparations the Pauli-Fierz Hamiltonian, including spin, is defined by

$$H = \frac{1}{2m} \left\{\sigma \cdot (-i\nabla_x - eA_\varphi(x))\right\}^2 + H_{\mathrm{f}}, \tag{3}$$

where m is the bare mass and e the charge of the electron.

Translations for the electron are generated by $-i\nabla_x$ and translations for the photon field by the field momentum

$$P_{\mathrm{f}} = \sum_{j=1,2} \int k a^*(k,j) a(k,j) dk. \tag{4}$$

By translation invariance of H the total momentum

$$p = -i\nabla_x + P_{\mathrm{f}}$$

is thus conserved,

$$[H, p] = 0,$$

as can be checked also directly. We unitarily transform H such that the fibering with respect to p becomes transparent. In momentum representation, momentum multiplication by k, position $i\nabla_k$, an element $\psi \in \mathcal{H}$ is written as $\psi^{(n)}(k, k_1, j_1, ..., k_n, j_n)$ with values in $\mathbb{C}^2$. For each n let

$$U\psi^{(n)}(k, k_1, j_1, ..., k_n, j_n) = \psi^{(n)}(k - \sum_{j=1}^{n} k_j, k_1, j_1, ..., k_n, j_n) \tag{5}$$

with inverse

$$U^{-1}\psi^{(n)}(k, k_1, j_1, ..., k_n, j_n) = \psi^{(n)}(k + \sum_{j=1}^{n} k_j, k_1, j_1, ..., k_n, j_n). \tag{6}$$

Clearly U is unitary. We set

$$A_\varphi = A_\varphi(0)$$

as an operator on $\mathcal{F}$. Similarly for the quantized magnetic field,

$$B_\varphi(x) = i \sum_{j=1,2} \int \frac{\widehat{\varphi}(k)}{\sqrt{2\omega(k)}} (k \wedge e_j(k)) \left(e^{-ik\cdot x} a^*(k, j) - e^{ik\cdot x} a(k, j)\right) dk,$$

and we denote

$$B_\varphi = B_\varphi(0)$$

as an operator on $\mathcal{F}$. Then, first working out the square in (3), one obtains

$$UHU^{-1} = \frac{1}{2m}(p - P_\mathrm{f} - eA_\varphi)^2 - \frac{e}{2m}\sigma \cdot B_\varphi + H_\mathrm{f}. \tag{7}$$

In (7) p is multiplication by k in the representation from (5) and (6). Thus the fibering is

$$\mathcal{H} = \int_{\mathbb{R}^3}^{\oplus} \mathcal{H}_p dp$$

with $\mathcal{H}_p$ isomorphic to $\mathbb{C}^2 \otimes \mathcal{F}$ and

$$UHU^{-1} = \int_{\mathbb{R}^3}^{\oplus} H_p dp.$$

In the following we will regard $p \in \mathbb{R}^3$ simply as a parameter. We also choose units such that $m = 1$. Then the operator under study is

$$H_p = \frac{1}{2}\left(p - P_\mathrm{f} - eA_\varphi\right)^2 - \frac{e}{2}\sigma \cdot B_\varphi + H_\mathrm{f} \tag{8}$$

acting on

$$\mathcal{H}_p = \mathbb{C}^2 \otimes \mathcal{F}.$$

For simplicity the index p of $\mathcal{H}_p$ will be omitted.

For $e = 0$ (8) reduces to the noninteracting Hamiltonian

$$H_{p0} = \frac{1}{2}(p - P_\mathrm{f})^2 + H_\mathrm{f}. \tag{9}$$

Clearly, with the definitions (1) and (4), H_{p0} is self-adjoint with the domain $D(H_{\mathrm{f}} + P_{\mathrm{f}}{}^2) = D(H_{\mathrm{f}}) \cap D(P_{\mathrm{f}}{}^2)$. The rest of (8) is regarded as the interaction part of the Hamiltonian,

$$H_{p\mathrm{I}} = H_p - H_{p0} = -e(p - P_{\mathrm{f}}) \cdot A_\varphi + \frac{e^2}{2} A_\varphi^2 - \frac{e}{2}\sigma \cdot B_\varphi. \tag{10}$$

For sufficiently small e, $|e| < e^*$, $H_{p\mathrm{I}}$ is bounded relative to H_{p0} with a bound less than 1, which by a theorem of Kato and Rellich implies that H_p is a self-adjoint operator for every $p \in \mathbb{R}^3$. In addition H_p is bounded from below. To see this let $a^\sharp(f) = \sum_{j=1,2} \int f(k,j) a^\sharp(k,j) dk$. Using the inequalities

$$\|a^\sharp(f)\psi\| \le c_1 \left\{ \sum_{j=1,2} \int \left(\omega(k)^{-1} + 1\right) |f(k,j)|^2 dk \right\}^{1/2} \|(H_{\mathrm{f}} + 1)^{1/2}\psi\|,$$

$$\|a^\sharp(f) a^\sharp(f)\psi\| \le c_2 \sum_{j=1,2} \int \left(\omega(k)^{-1} + \omega(k)\right) |f(k,j)|^2 dk \|(H_{\mathrm{f}} + 1)\psi\|,$$

with some constants c_1 and c_2, one has

$$\|H_{p\mathrm{I}}\psi\| \le c_0(e) \|(H_{p0} + 1)\psi\|. \tag{11}$$

Here

$$c_0(e) = c_0 \left\{ |e| \left\{ \int \left(\omega(k)^{-2} + \omega(k)\right) |\hat{\varphi}(k)|^2 dk \right\}^{1/2} + |e|^2 \int \left(\omega(k)^{-2} + 1\right) |\hat{\varphi}(k)|^2 dk \right\}$$

with some numerical constant c_0 of order one. Thus $|e| < e^*$ with a suitable $e^* > 0$ implies $c_0(e) < 1$.

The goal of our paper is to study some ground state properties of H_p. The ground state energy of H_p is

$$E(p) = \inf \mathrm{Spec}(H_p) = \inf_{\psi \in D(H_p), \|\psi\| = 1} (\psi, H_p \psi).$$

It is easily seen that resolvent $(H_p - z)^{-1}$ with $z \notin \mathrm{Spec}(H_p)$ is continuous in both p and e in the operator norm. Thus $E(p)$ is continuous in both p and e. If $E(p)$ is an eigenvalue, the corresponding spectral projection is denoted by P_{g}. $\mathrm{Tr} P_{\mathrm{g}}$ is the degeneracy of the ground state. The bottom of the continuous spectrum is denoted by $E_{\mathrm{c}}(p)$. Under the assumptions (A.1)–(A.3) one knows that

$$E_{\mathrm{c}}(p) = \inf_{k \in \mathbb{R}^3} \left\{ E(p-k) + \omega(k) \right\},$$

see [3, 8]. Thus it is natural to set

$$\Delta(p) = E_{\mathrm{c}}(p) - E(p) = \inf_{k \in \mathbb{R}^3} \left\{ E(p-k) + \omega(k) - E(p) \right\}.$$

Let $e = 0$. Then $\Delta(p) > 0$ for $|p| < p_0$ with some p_0 by (A.1). Thus, by the continuity of $E(p)$ mentioned above, $\{(p,e) \in \mathbb{R}^3 \times \mathbb{R} | \Delta(p) > 0\} \ne \emptyset$. As our main result we state

Theorem 1.1 *Suppose* $|e| < e_0$ *and* $\Delta(p) > 0$. *Then* $\mathrm{Tr} P_{\mathrm{g}} = 2$.

We briefly comment on our assumptions. $\Delta(p) > 0$ means that we assume, rather than prove, a spectral gap. If instead one would merely impose the implicit but rather natural condition that

$$E(p) \geq E(0), \tag{12}$$

then, following the arguments in [3], one concludes in case of the dispersion relation (2) that $\Delta(p) > 0$ for $|p| < \sqrt{3} - 1$ and arbitrary $e \in \mathbb{R}$. For a general dispersion relation satisfying (A.1) to (A.3) a corresponding bound can be established. For the spinless Pauli-Fierz Hamiltonian (12) is proven through a suitable variant of a diamagnetic inequality. We did not succeed to establish (12) for the Hamiltonian (8). Note that on physical grounds one expects $\Delta(p) > 0$ for $|p| < p_{\mathrm{c}}$ and $\Delta(p) = 0$ for $|p| > p_{\mathrm{c}}$ with a simultaneous loss of the ground state. For the Hamiltonian (8) with ω given by (2) one has $p_{\mathrm{c}} < \infty$ at $e = 0$ and expects $p_{\mathrm{c}} < \infty$ to persist for all $e \neq 0$ provided the spatial dimension $d \geq 3$.

The constant e_0 does *not* depend on m_{ph}. Thus, together with (12), the domain of validity of Theorem 1.1 is independent of m_{ph} and it may seem that one could take the limit $m_{\mathrm{ph}} \to 0$. Of course, thereby $\Delta(p) \to 0$. Unfortunately, this will not work, since the Pauli-Fierz Hamiltonian is infrared divergent and the number of photons increases without bound as $m_{\mathrm{ph}} \to 0$ [2]. The physical ground states are no longer in Fock space. The only exception is $p = 0$, and one might hope to apply our method directly to the case $p = 0$ and $m_{\mathrm{ph}} = 0$. This problem will be taken up in Section 3.

Let us indicate the strategy for the proof of Theorem 1.1. From a pull-through formula one estimates the overlap of a ground state with the subspace $P_0 \mathcal{H} = \mathbb{C}^2 \otimes \{\mathbb{C}\Omega\}$. If $|e|$ is sufficiently small, the overlap is large which implies $\mathrm{Tr} P_{\mathrm{g}} < 3$. For a lower bound we will derive the algebraic relation $P_0 P_{\mathrm{g}} P_0 = a P_0$, $a > 0$, which implies $\mathrm{Tr} P_{\mathrm{g}} \geq 2$.

The Hamiltonian (8) is invariant under rotations with axis $\widehat{p} = p/|p|$. To understand the implications let us define the field angular momentum relative to the origin by

$$J_{\mathrm{f}} = -i \sum_{j=1,2} \int a^*(k,j)(k \wedge \nabla_k) a(k,j) dk$$

and the helicity by

$$S_{\mathrm{f}} = i \int \widehat{k} \left\{ a^*(k,2)a(k,1) - a^*(k,1)a(k,2) \right\} dk.$$

Let $\vec{n} \in \mathbb{R}^3$ be an arbitrary unit vector, $\theta \in \mathbb{R}$, and let $R = R(\vec{n}, \theta)$ be the rotation around $\vec{n}$ through the angle θ. We note that

$$e^{i\theta\vec{n}\cdot(J_{\mathrm{f}}+S_{\mathrm{f}})} H_{\mathrm{f}} e^{-i\theta\vec{n}\cdot(J_{\mathrm{f}}+S_{\mathrm{f}})} = H_{\mathrm{f}},$$

$$e^{i\theta\vec{n}\cdot(J_{\mathrm{f}}+S_{\mathrm{f}})} P_{\mathrm{f}} e^{-i\theta\vec{n}\cdot(J_{\mathrm{f}}+S_{\mathrm{f}})} = R P_{\mathrm{f}},$$

$$e^{i\theta\vec{n}\cdot(J_f+S_f)}A_\varphi e^{-i\theta\vec{n}\cdot(J_f+S_f)} = RA_\varphi.$$

Define the total angular momentum through

$$J = J_f + S_f + \frac{1}{2}\sigma.$$

Then it follows that

$$e^{i\theta\vec{n}\cdot J}H_p e^{-i\theta\vec{n}\cdot J} = H_{R^{-1}p}.$$

In particular, the ground state energy is rotation invariant,

$$E(p) = E(Rp). \tag{13}$$

If one sets $\vec{n} = \hat{p}$, then

$$e^{i\theta\hat{p}J}H_p e^{-i\theta\hat{p}J} = H_p, \tag{14}$$

which expresses the rotation invariance relative to $\hat{p}$.

Since $\mathrm{Spec}(\hat{p}\cdot(J_f+S_f)) = \mathbb{Z}$ and $\mathrm{Spec}(\hat{p}\cdot\sigma) = \{-1,+1\}$, it follows that

$$\mathrm{Spec}(\hat{p}\cdot J) = \mathbb{Z}_{1/2},$$

where $\mathbb{Z}_{1/2}$ is the set of half integers $\{\pm 1/2, \pm 3/2, \pm 5/2, ...\}$. By virtue of (14), $\mathcal{H}$ and H_p are decomposable as

$$\mathcal{H} = \bigoplus_{z\in\mathbb{Z}_{1/2}} \mathcal{H}(z),$$

$$H_p = \bigoplus_{z\in\mathbb{Z}_{1/2}} H_p(z).$$

Therefore, if $\mathrm{Tr}P_g = 2$, the ground states $\psi_{g\pm} \in P_g\mathcal{H}$ can be chosen such that $\psi_{g+} \in \mathcal{H}(z)$ and $\psi_{g-} \in \mathcal{H}(z')$ for some $z, z' \in \mathbb{Z}_{1/2}$.

Theorem 1.2 *Suppose $|e| < e_0$ and $\Delta(p) > 0$. Then H_p has two orthogonal ground states, $\psi_{g\pm}$, with the property $\psi_{g\pm} \in \mathcal{H}(\pm 1/2)$.*

2 Spectral properties

2.1 Upper bound

Let us denote the number operator by

$$N_f = \sum_{j=1,2}\int a^*(k,j)a(k,j)dk.$$

In what follows ψ_g denotes an *arbitrary* normalized ground state of H_p. We note that $a(k,j)\psi$, $\psi \in D(N_f{}^{1/2})$, is well defined and

$$(a(k,j)\psi)^{(n)}(k_1,j_1,...,k_n,j_n) = \sqrt{n+1}\psi^{(n+1)}(k,j,k_1,j_1,...,k_n,j_n).$$

Moreover it follows that $\|a(k,j)\psi\| \le \|N_{\rm f}{}^{1/2}\psi\|$ and

$$(\psi, N_{\rm f}\phi) = \sum_{j=1,2} \int (a(k,j)\psi, a(k,j)\phi) dk.$$

Lemma 2.1 *Suppose* $\Delta(p) > 0$. *Then*

$$(\psi_{\rm g}, N_{\rm f}\psi_{\rm g}) \le 2e^2 \int \frac{(|k|^2/4) + 6E(p)}{(E(p-k) + \omega(k) - E(p))^2} \frac{|\widehat{\varphi}(k)|^2}{\omega(k)} dk.$$

Proof: By the pull-through formula

$$a(k,j)H_p\psi_{\rm g} = (H_{p-k} + \omega(k))\, a(k,j)\psi_{\rm g}$$

$$-e\frac{\widehat{\varphi}(k)}{\sqrt{2\omega(k)}} \left\{ (p - P_{\rm f} - eA_\varphi) \cdot e_j(k) + \frac{1}{2}\sigma \cdot (ik \wedge e_j(k)) \right\} \psi_{\rm g}.$$

Hence we have

$$a(k,j)\psi_{\rm g} = e\frac{\widehat{\varphi}(k)}{\sqrt{2\omega(k)}} (H_{p-k} + \omega(k) - E(p))^{-1}$$

$$\times \left\{ (p - P_{\rm f} - eA_\varphi) \cdot e_j(k) + \frac{1}{2}\sigma \cdot (ik \wedge e_j(k)) \right\} \psi_{\rm g}.$$

Thus

$$(\psi_{\rm g}, N_{\rm f}\psi_{\rm g}) = \sum_{j=1,2} \int \|a(k,j)\psi_{\rm g}\|^2 dk$$

$$\le \sum_{j=1,2} \int \left| \frac{\widehat{\varphi}(k)}{\sqrt{2\omega(k)}} \frac{\|(p - P_{\rm f} - eA_\varphi) \cdot e_j(k)\psi_{\rm g} + (1/2)\sigma \cdot (ik \wedge e_j(k))\psi_{\rm g}\|}{E(p-k) + \omega(k) - E(p)} \right|^2 dk.$$

We note that

$$\|(p - P_{\rm f} - eA_\varphi) \cdot e_j(k)\psi_{\rm g}\|^2 = \| \sum_{\mu=1,2,3} \sigma_\mu (p - P_{\rm f} - eA_\varphi)_\mu \sigma_\mu e_j^\mu(k)\psi_{\rm g}\|^2$$

$$\le 3 \sum_{\mu=1,2,3} \|\sigma_\mu (p - P_{\rm f} - eA_\varphi)_\mu \psi_{\rm g}\|^2 \le 6E(p),$$

and

$$\|\sigma \cdot (ik \wedge e_j(k))\psi_{\rm g}\|^2 \le |k|^2 \sum_{\mu=1,2,3} \|(i\widehat{k} \wedge e_j(k))_\mu \psi_{\rm g}\|^2 \le |k|^2.$$

Thus

$$\|(p - P_{\rm f} - eA_\varphi) \cdot e_j(k)\psi_{\rm g} + (1/2)\sigma \cdot (ik \wedge e_j(k))\psi_{\rm g}\|^2 \le 2\left\{(|k|^2/4) + 6E(p)\right\},$$

which leads to

$$(\psi_{\rm g}, N_{\rm f}\psi_{\rm g}) \le 2e^2 \int \frac{(|k|^2/4) + 6E(p)}{(E(p-k) + \omega(k) - E(p))^2} \frac{|\widehat{\varphi}(k)|^2}{\omega(k)} dk$$

and the lemma follows. *QED*

We set

$$\theta(p)=\theta(p,e)=2\int\frac{(|k|^2/4)+6E(p)}{(E(p-k)+\omega(k)-E(p))^2}\frac{|\widehat{\varphi}(k)|^2}{\omega(k)}dk.$$

Note that $\theta(p)$ is rotation invariant, i.e. $\theta(Rp)=\theta(p)$ for an arbitrary rotation R. Let $P_0=1\otimes P_\Omega$ be the projection onto $\mathbb{C}^2\otimes\{\mathbb{C}\Omega\}$.

Lemma 2.2 *Let* $\Delta(p)>0$. *Suppose* $|e|<1/\sqrt{3\theta(p)}$. *Then* $\mathrm{Tr}P_{\mathrm{g}}\leq 2$.

Proof: By Lemma 2.1 we have $\mathrm{Tr}(P_{\mathrm{g}}N_{\mathrm{f}})\leq e^2\theta(p)\mathrm{Tr}P_{\mathrm{g}}$. Therefore

$$\mathrm{Tr}P_{\mathrm{g}}-\mathrm{Tr}(P_{\mathrm{g}}P_0)=\mathrm{Tr}P_{\mathrm{g}}(I-P_0)\leq\mathrm{Tr}(P_{\mathrm{g}}N_{\mathrm{f}})\leq e^2\theta(p)\mathrm{Tr}P_{\mathrm{g}},$$

and

$$(1-e^2\theta(p))\mathrm{Tr}P_{\mathrm{g}}\leq\mathrm{Tr}(P_{\mathrm{g}}P_0)\leq 2,$$

which implies

$$\mathrm{Tr}P_{\mathrm{g}}\leq\frac{2}{1-e^2\theta(p)}<3.$$

Thus the lemma follows. *QED*

2.2 Lower bound

We say that $\psi\in\mathcal{F}$ is *real*, if for all $n\geq 0$, $\psi^{(n)}$ is a real-valued function on $L^2(\mathbb{R}^{3n}\times\{1,2\})$. The set of real ψ is denoted by $\mathcal{F}_{\mathrm{real}}$. We define the set of reality-preserving operators $\mathcal{O}_{\mathrm{real}}$ as follows:

$$\mathcal{O}_{\mathrm{real}}=\{A|A:\mathcal{F}_{\mathrm{real}}\cap D(A)\longrightarrow\mathcal{F}_{\mathrm{real}}\}.$$

It is seen that H_{f} and P_{f} are in $\mathcal{O}_{\mathrm{real}}$. Since for all $k\in\mathbb{R}$ and $z\in\mathbb{R}$ with $z\notin\mathrm{Spec}(H_{p0})$,

$$((H_{p0}-z)^k\psi)^{(n)}(k_1,j_1,...,k_n,j_n)$$
$$=\left(\frac{1}{2}\left(p-\sum_{i=1}^n k_i\right)^2+\sum_{i=1}^n\omega(k_i)+z\right)^k\psi^{(n)}(k_1,j_1,...,k_n,j_n),$$

$(H_{p0}-z)^k$ is also in $\mathcal{O}_{\mathrm{real}}$. Moreover $a(f)$ and $a^*(f)$ are in $\mathcal{O}_{\mathrm{real}}$ for real f's. In particular A_φ and iB_φ are in $\mathcal{O}_{\mathrm{real}}$. Note that, if $\psi\in D(B_\varphi)\cap\mathcal{F}_{\mathrm{real}}$, then $(\psi,B_\varphi\psi)=0$. Let $\mathcal{F}_{\mathrm{fin}}=\bigcup_{N=0}^\infty\oplus_{n=0}^N\left(L^2(\mathbb{R}^3\times\{1,2\})\right)^n_{\mathrm{sym}}$ denote the finite particle subspace of $\mathcal{F}$. Note that A_φ, B_φ, $(z-H_{p0})^{-1/2}$ and $(z-H_{p0})^{-1}$ leave $\mathcal{F}_{\mathrm{fin}}$ invariant.

Lemma 2.3 *Suppose $|e| < e^*$. Let $x \in \mathbb{C}^2$. Then there exists $a(t) \in \mathbb{R}$ independent of x such that for $t \geq 0$*

$$(x \otimes \Omega, e^{-t(H_p - E(p))} x \otimes \Omega)_{\mathcal{H}} = a(t)(x, x)_{\mathbb{C}^2}. \tag{15}$$

Proof: Note that $\|H_{p\mathrm{I}}(1 + H_{p0})^{-1}\| < 1$ for $|e| < e^*$ by (11). Then, by spectral theory, one has

$$e^{-t(H_p - E(p))} = \lim_{n\to\infty} \left(1 + \frac{t}{n}(H_p - E(p))\right)^{-n}$$

$$= \lim_{n\to\infty} \lim_{m\to\infty} \left\{ \sum_{k=0}^{m} \left(1 + \frac{t}{n} H_{p0}\right)^{-1} \left\{ \left(-\frac{t}{n}(H_{p\mathrm{I}} - E(p))\right) \left(1 + \frac{t}{n} H_{p0}\right)^{-1} \right\}^k \right\}^n$$

$$= \lim_{n\to\infty} \lim_{m\to\infty} \left\{ \left(1 + \frac{t}{n} H_{p0}\right)^{-1/2} \left(\sum_{k=0}^{m} \left(-\frac{t}{n} \widetilde{H_{\mathrm{I}}}\right)^k \right) \left(1 + \frac{t}{n} H_{p0}\right)^{-1/2} \right\}^n.$$

Here

$$\widetilde{H_{\mathrm{I}}} = \widetilde{H}_{p\mathrm{II}} + i\sigma \cdot \widetilde{B}_{\varphi},$$

$$\widetilde{H}_{p\mathrm{II}} = \left(1 + \frac{t}{n} H_{p0}\right)^{-1/2} (H_{p\mathrm{II}} - E(p)) \left(1 + \frac{t}{n} H_{p0}\right)^{-1/2},$$

$$\widetilde{B}_{\varphi} = \left(1 + \frac{t}{n} H_{p0}\right)^{-1/2} (iB_{\varphi}) \left(1 + \frac{t}{n} H_{p0}\right)^{-1/2},$$

$$H_{p\mathrm{II}} = -e(p - P_{\mathrm{f}}) \cdot A_{\varphi} + \frac{e^2}{2} A_{\varphi}^2.$$

It is seen that

$$\widetilde{H_{\mathrm{I}}}^2 = \widetilde{H}_{p\mathrm{II}} \widetilde{H}_{p\mathrm{II}} - \widetilde{B}_{\varphi} \cdot \widetilde{B}_{\varphi} + i\sigma \cdot (\widetilde{H}_{p\mathrm{II}} \widetilde{B}_{\varphi} + \widetilde{B}_{\varphi} \widetilde{H}_{p\mathrm{II}} - \widetilde{B}_{\varphi} \wedge \widetilde{B}_{\varphi}) = M + i\sigma \cdot L.$$

Here both of $M = \widetilde{H}_{p\mathrm{II}} \widetilde{H}_{p\mathrm{II}} - \widetilde{B}_{\varphi} \cdot \widetilde{B}_{\varphi}$ and $L = \widetilde{H}_{p\mathrm{II}} \widetilde{B}_{\varphi} + \widetilde{B}_{\varphi} \widetilde{H}_{p\mathrm{II}} - \widetilde{B}_{\varphi} \wedge \widetilde{B}_{\varphi}$ are in $\mathcal{O}_{\mathrm{real}}$. Moreover

$$\widetilde{H_{\mathrm{I}}}^3 = \widetilde{H}_{p\mathrm{II}} M - \widetilde{B}_{\varphi} L + i\sigma \cdot (\widetilde{B}_{\varphi} M + \widetilde{H}_{p\mathrm{II}} L - \widetilde{B}_{\varphi} \wedge L),$$

where both of $\widetilde{H}_{p\mathrm{II}} M - \widetilde{B}_{\varphi} L$ and $\widetilde{B}_{\varphi} M + \widetilde{H}_{p\mathrm{II}} L - \widetilde{B}_{\varphi} \wedge L$ are also in $\mathcal{O}_{\mathrm{real}}$. Thus, repeating above procedure, one obtains

$$\sum_{k=0}^{m} \left(-\frac{t}{n} \widetilde{H_{\mathrm{I}}}\right)^k = a_m + i\sigma \cdot b_m,$$

where a_m and b_m are in $\mathcal{O}_{\mathrm{real}}$. Hence there exist $a_{nm} \in \mathcal{O}_{\mathrm{real}}$ and $b_{nm} \in \mathcal{O}_{\mathrm{real}}$ such that

$$\left\{ \left(1 + \frac{t}{n} H_{p0}\right)^{-1/2} \left(\sum_{k=0}^{m} \left(-\frac{t}{n} \widetilde{H_{\mathrm{I}}}\right)^k \right) \left(1 + \frac{t}{n} H_{p0}\right)^{-1/2} \right\}^n = a_{nm} + i\sigma \cdot b_{nm}.$$

Finally

$$\begin{aligned}&(x\otimes\Omega, e^{-t(H_p-E(p))}x\otimes\Omega)\\ &=\lim_{n\to\infty}\lim_{m\to\infty}(x,x)(\Omega,a_{nm}\Omega)+i\lim_{n\to\infty}\lim_{m\to\infty}(x,\sigma x)(\Omega,b_{nm}\Omega).\end{aligned}$$

Since the left-hand side is real, the second term of the right-hand side vanishes and $a(t)=\lim_{n\to\infty}\lim_{m\to\infty}(\Omega,a_{nm}\Omega)$ exists, which establishes the desired result. *QED*

Lemma 2.4 *Suppose that* $\Delta(p)>0$ *and* $|e|<1/\sqrt{\theta(p)}$. *Then* $(\psi_{\rm g},P_0\psi_{\rm g})\neq 0$.

Proof: Since $P_\Omega+N_{\rm f}\geq 1$, we have from Lemma 2.1

$$(\psi_{\rm g},P_0\psi_{\rm g})\geq\|\psi_{\rm g}\|^2-\|(1\otimes N_{\rm f}{}^{1/2})\psi_{\rm g}\|^2>1-e^2\theta(p)>0.$$

Thus the lemma follows. *QED*

Lemma 2.5 *Suppose* $|e|<e^*$ *and* $|e|<1/\sqrt{\theta(p)}$. *Then there exists* $a>0$ *such that*

$$P_0P_{\rm g}P_0=aP_0. \tag{16}$$

Proof: Note that

$$P_{\rm g}=s-\lim_{t\to\infty}e^{-t(H_p-E(p))}.$$

Thus by Lemma 2.3,

$$(x\otimes\Omega,P_{\rm g}x\otimes\Omega)=\lim_{t\to\infty}(x\otimes\Omega,e^{-t(H_p-E(p))}x\otimes\Omega)=\lim_{t\to\infty}a(t)(x,x)$$

for all $x\in\mathbb{C}^2$. Since by Lemma 2.4, $(x\otimes\Omega,P_{\rm g}x\otimes\Omega)\neq 0$ for some $x\in\mathbb{C}^2$, $\lim_{t\to\infty}a(t)$ exists and it does not vanish. For arbitrary $\phi_1,\phi_2\in\mathcal{H}$, the polarization identity leads to $(\phi_1,P_0P_{\rm g}P_0\phi_2)=a(\phi_1,P_0\phi_2)$. The lemma follows. *QED*

Lemma 2.6 *Suppose* $|e|<e^*$ *and* $|e|<1/\sqrt{\theta(p)}$. *Then* ${\rm Tr}P_{\rm g}\geq 2$.

Proof: Suppose ${\rm Tr}P_{\rm g}=1$. Then $P_0P_{\rm g}P_0/{\rm Tr}P_0P_{\rm g}P_0$ is a one-dimensional projection which contradicts (16). Thus the lemma follows. *QED*

2.3 Proofs of Theorems 1.1 and 1.2

We define

$$e_0=\inf\left\{|e|\,\Big||e|<1/\sqrt{3\theta(p)},|e|<e^*\right\}.$$

Proof of Theorem 1.1:
Since $\Delta(p) > 0$ and $|e| < e_0$, we conclude $\mathrm{Tr}P_{\mathrm{g}} \geq 2$ by Lemma 2.6 and $\mathrm{Tr}P_{\mathrm{g}} \leq 2$ by Lemma 2.2. Hence the theorem follows. *QED*

Proof of Theorem 1.2:
Without restriction in generality we may suppose $\hat{p} = (0,0,1)$. Let $\psi_\pm$ be ground states of H_p such that $\psi_+ \in \mathcal{H}(z)$ and $\psi_- \in \mathcal{H}(z')$ with some $z, z' \in \mathbb{Z}_{1/2}$. $\Omega_+ = \begin{pmatrix} \Omega \\ 0 \end{pmatrix}$ and $\Omega_- = \begin{pmatrix} 0 \\ \Omega \end{pmatrix}$ are ground states of H_{p0} and $\Omega_\pm \in \mathcal{H}(\pm 1/2)$. Let $P_{\mathrm{g}}\Omega_+ = c_1\psi_+ + c_2\psi_-$ and $P_{\mathrm{g}}\Omega_- = c_3\psi_+ + c_4\psi_-$ with some $c_j \in \mathbb{C}$ and $Q_{\pm 1/2}$ be the projection of $\mathcal{H}$ onto $\mathcal{H}(\pm 1/2)$. Since $P_0 P_{\mathrm{g}} P_0 = aP_0$ we conclude that

$$(\Omega_+, P_{\mathrm{g}}\Omega_+) = a > 0, \tag{17}$$

$$(\Omega_-, P_{\mathrm{g}}\Omega_-) = a > 0. \tag{18}$$

Then $Q_{1/2}P_{\mathrm{g}}\Omega_+ \neq 0$ and $Q_{-1/2}P_{\mathrm{g}}\Omega_- \neq 0$. The alternative $Q_{1/2}\psi_+ \neq 0$ or $Q_{1/2}\psi_- \neq 0$ holds by (17), the alternative $Q_{-1/2}\psi_+ \neq 0$ or $Q_{-1/2}\psi_- \neq 0$ by (18). We may set $Q_{1/2}\psi_+ \neq 0$. Then $\psi_+ \in \mathcal{H}(1/2)$ and $\psi_- \in \mathcal{H}(-1/2)$. *QED*

3 Zero total momentum

3.1 Spinless Hamiltonian

In the spinless case H_p simplifies to

$$H_p = \frac{1}{2}(p - P_{\mathrm{f}} - eA_\varphi)^2 + H_{\mathrm{f}}$$

acting on $\mathcal{F}$. The bound (12) is available and H_p has at least one ground state for $|p| < p_{\mathrm{c}}$ with some $p_{\mathrm{c}} > 0$ and arbitrary e. To show the uniqueness of the ground state only the pull-through argument seems to be available. The details of Section 2.1 remain unchanged and one concludes that if

$$|e|^2 \leq \frac{1}{2}\left\{\int \frac{E(p)}{(E(p-k) + \omega(k) - E(p))^2} \frac{|\hat{\varphi}(k)|^2}{\omega(k)} dk\right\}^{-1},$$

then $\mathrm{Tr}P_{\mathrm{g}} \leq 1$, which implies that the ground state of H_p is unique.

3.2 Confining potentials

For $p = 0$ no infrared divergence is expected and for the remainder of this section we set

$$\omega(k) = |k|$$

as the physical dispersion relation. We did not succeed to apply the methods of Section 2 to this case. Therefore rather than considering $p = 0$ directly we add to the Hamiltonian (3) a confining potential, $V : \mathbb{R}^3 \to \mathbb{R}$, which in spirit amounts to the same physical situation. The Hamiltonian under study is

$$H_V = \frac{1}{2}\{\sigma \cdot (-i\nabla_x - eA_\varphi(x))\}^2 + V(x) + H_{\mathrm{f}}. \tag{19}$$

Let

$$H_{\mathrm{el}} = -\frac{1}{2}\Delta + V$$

on $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$,

$$\Sigma_{\mathrm{el}} = \inf \mathrm{Spec}_{\mathrm{ess}}(H_{\mathrm{el}}),$$

$$E_{\mathrm{el}} = \inf \mathrm{Spec}(H_{\mathrm{el}})$$

and

$$E = \inf \mathrm{Spec}(H_V).$$

Let V be relatively bounded with respect to $-\Delta$ with a bound less than 1. Then H_V is self-adjoint on $D(\Delta) \cap D(H_{\mathrm{f}})$ as established in [6]. For arbitrary e the existence of ground states has been proven by Griesemer, Lieb, and Loss [4] under the condition that H_{el} has a ground state separated by a gap from the continuous spectrum, i.e.

$$\Sigma_{\mathrm{el}} - E_{\mathrm{el}} > 0. \tag{20}$$

We also refer to [1] for prior results, where in particular it is proven that the charge density of an arbitrary ground state ψ_{g} is localized, i.e.

$$\|e^{c|x|}\psi_{\mathrm{g}}\| \le c_1\|\psi_{\mathrm{g}}\| \tag{21}$$

with some constant c. In the spinless case the uniqueness of the ground state, $\mathrm{Tr}P_{\mathrm{g}} = 1$, would follow from a positivity argument [5]. Let P_{el} be the projection of the subspace spanned by ground states of H_{el}. Suppose (20). Take e such that

$$\Sigma_{\mathrm{el}} - E > 0, \tag{22}$$

which can be satisfied by the continuity of E in e. Then a pull-through argument and (21) yield that

$$(\psi_{\mathrm{g}}, (1 \otimes N_{\mathrm{f}} + P_{\mathrm{el}}^{\perp} \otimes P_0)\psi_{\mathrm{g}}) \le \eta(e), \tag{23}$$

where $\lim_{\epsilon\to 0}\eta(e) = 0$. Hence in the similar way as Lemma 2.2 with P_0 and N_{f} replaced by $P_{\mathrm{el}} \otimes P_0$ and $1 \otimes N_{\mathrm{f}} + P_{\mathrm{el}}^{\perp} \otimes P_0$, respectively, we see that if, in addition to (22), e satisfies $\eta(e) < 1/3$, then

$$\mathrm{Tr}P_{\mathrm{g}} < 3. \tag{24}$$

The realness argument of Section 2.2 requires some extra conditions on V.

Theorem 3.1 *Suppose $V(x) = V(-x)$, $\Sigma_{\mathrm{el}} - E_{\mathrm{el}} > 0$, and $\mathrm{Tr}P_{\mathrm{el}} = 2$. Then there exists a positive constant e_{00} such that, if $|e| < e_{00}$, then H_V has a two-fold degenerate ground state.*

Proof: Suppose that e satisfies (22) and $\eta(e) < 1$. (23) yields

$$(\psi_{\rm g}, P_{\rm el} \otimes P_0 \psi_{\rm g}) \geq \|\psi_{\rm g}\|^2 - (\psi_{\rm g}, (1 \otimes N_{\rm f} + P_{\rm el}^{\perp} \otimes P_0)\psi_{\rm g}) \geq 1 - \eta(e) > 0. \quad (25)$$

Let F denote the Fourier transform of $L^2(\mathbb{R}^3)$ and define the unitary operator of $\mathcal{H}$ by $T = Fe^{ix\cdot P_{\rm f}}$. Then we have

$$TH_V T^{-1} = \frac{1}{2}\{\sigma \cdot (x - P_{\rm f} - eA_\varphi(0))\}^2 + FVF^{-1} + H_{\rm f}.$$

The assumption $V(x) = V(-x)$ implies that FVF^{-1} is a reality preserving operator on $L^2(\mathbb{R}^3)$. Let $\varphi_{\rm el}$ be the ground state of $H_{\rm el}$. In the similar way as Lemmas 2.3 and 2.5 with Ω and P_0 replaced by $\varphi_{\rm el}\otimes\Omega$ and $P_{\rm el}\otimes P_0$, respectively, it is established that by (25) there exists a positive constant c^* such that if $|e| < c^*$, then (22) and $\eta(e) < 1$ hold, and ${\rm Tr}P_{\rm g} \geq 2$. Take $e_{00} = \sup\{|e| | \eta(e) < 1/3, |e| < c^*\}$. Then the theorem follows from (24). *QED*

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