

Geometry of three manifolds and existence of Black Hole due to boundary effect

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1 Introduction

In this paper, we observe that the brane functional studied in [5] can be used to generalize some of the works that Schoen and I [4] did many years ago. The key idea is that if a three dimensional manifold M has a boundary with strongly positive mean curvature, the effect of this mean curvature can influence the internal geometry of M . For example, if the scalar curvature of M is greater than certain constant related to this boundary effect, no incompressible surface of higher genus can exist.

A remarkable statement in general relativity is that if the mean curvature of ∂M is strictly greater than the trace of p_{ij} (the second fundamental form of M in space time), the value of this difference can provide the existence of apparent horizon in M . In fact, matter density can even be allowed to be negative if this boundary effect is very strong. Theorem 5.2 is the major result of this paper.

2 Existence of stable incompressible surfaces with constant mean curvature

We shall generalize some of the results of Schoen-Yau [1] and Meeks-Simon-Yau [3].

Let M be a compact three dimensional manifold whose boundary ∂M has mean curvature (with respect to the outward normal) not less than $c > 0$. Assume the volume form of M can be written as $d\Lambda$ where Λ is a smooth two form.

Let $f : \Sigma \rightarrow M$ be a smooth map from a surface Σ into M which is one to one on $\Pi_1(\Sigma)$. We are interesting in minimizing the energy

$$E_c(f) = \frac{1}{2} \int_{\Sigma} |\nabla f|^2 - c \int_{\Sigma} f^* \Lambda.$$

There are two different hypothesis we shall make for the existence of surfaces which minimizes E_c .

Theorem 2.1. *Assume the existence of embedded Σ with $\Pi_1(\Sigma) \rightarrow \Pi_1(M)$ to be one to one. Assume that for any ball B in M , the volume of the ball B is not greater than $\frac{1}{c} \text{Area}(\partial B)$. Then we can find a surface isotopic to Σ which minimizethe functional $\text{Area}(\Sigma) - c \int_{\Sigma} \Lambda$.*

Proof. This follows from the argument of Meeks-Simon-Yau [3]. The hypothesis is method to deal with the cut and paste argument. \square

Theorem 2.2. *Assume that the supremum norm of Λ is not greater than c^{-1} . Then for any $c' < c$, we can find a conformal map from some conformal*

structure on Σ to M which induces the same map is $f_* = \Pi_1(\Sigma) \rightarrow \Pi_1(M)$ and minimize the energy $\frac{1}{2} \int_{\Sigma} |\nabla f|^2 - c' \int_{\Sigma} f^* \Lambda$.

Proof. Since $c' < c$, the energy is greater than a positive multiple of the standard energy. Hence the argument of Schoen-Yau [1] works. \square

Remark. It should be possible to choose $c' = c$ in this last theorem.

3 Second variational formula

Let Σ be the stable surface established in section 2. Then the variational formula shows that the mean curvature of Σ is equal to c . The second variational formula shows that for all φ defined on Σ ,

$$\int_{\Sigma} |\nabla \varphi|^2 - \int_{\Sigma} (\text{Ric}_M(\nu, \nu) + \Sigma h_{ij}^2) \varphi^2 \geq 0 \tag{3.1}$$

where $\text{Ric}_M(\nu, \nu)$ is the Ricci curvature of M along the normal of Σ and h_{ij} is the second fundamental form.

The Gauss equation shows that

$$\text{Ric}_M(\nu, \nu) = \frac{1}{2} R_M - K_{\Sigma} + \frac{1}{2} (H^2 - \Sigma h_{ij}^2) \tag{3.2}$$

where R_M is the scalar curvature, K_{Σ} is the Gauss curvature of Σ and $H = \text{trace of } h_{ij}$ is the mean curvature.

Hence

$$\int_{\Sigma} |\nabla \varphi|^2 \geq \frac{1}{2} \int_{\Sigma} (R_M + \Sigma h_{ij}^2 + H^2) \varphi^2 - \int_{\Sigma} K_{\Sigma} \varphi^2. \tag{3.3}$$

Since $\Sigma h_{ij}^2 \geq \frac{1}{2} H^2$, we conclude that

$$\int_{\Sigma} |\nabla \varphi|^2 \geq \frac{1}{2} \int_M \left(R_M + \frac{3}{2} H^2 \right) \varphi^2 - \int_{\Sigma} K_{\Sigma} \varphi^2. \tag{3.4}$$

If $\chi(\Sigma) \leq 0$, we conclude by choosing $\varphi = 1$, that

$$\int_{\Sigma} \left(R_M + \frac{3}{2} c^2 \right) \leq 0. \tag{3.5}$$

Theorem 3.1. *If $R_M + \frac{3}{2}c^2 \geq 0$, any stable orientable surface Σ with $\chi(\Sigma) \leq 0$ must have $\chi(\Sigma) = 0$ and $R_M + \frac{3}{2}c^2 = 0$ along Σ . Furthermore Σ must be umbilical.*

Let us now see whether stable orientable Σ with $\chi(\Sigma) = 0$ can exist or not. If $R_M + \frac{3}{2}c^2 > 0$ at some point of M and $R_M + \frac{3}{2}c^2 \geq 0$ everywhere. We can deform the metric conformally so that $R_M + \frac{3}{2}c^2 > 0$ everywhere while keeping mean curvature of ∂M not less than c . (This can be done by arguments of Yamabe problem.) In this case, incompressible torus does not exist.

Hence we may assume $R_M + \frac{3}{2}c^2 = 0$ everywhere. In this case, we deform the metric to $g_{ij} - t(R_{ij} - \frac{R}{3}g_{ij})$. By computation, one sees that unless $R_{ij} - \frac{R}{3}g_{ij}$ everywhere, the (new) scalar curvature will be increased.

Let Σ be the stable surface with constant mean curvature with respect to the metric g_{ij} . We can deform the surface Σ along the normal by multiplying the normal with a function f . For this surface Σ_f , we look at the equation $H_t(\Sigma_f) = c$ where H_t is the mean curvature with respect to the new metric at time t . As a function of t and f , $H_t(\Sigma_f)$ define a mapping into the Hilbert space of functions on Σ . The linearized operator with respect to the second (function) variable is $-\Delta - (\text{Ric}(\nu, \nu) + \Sigma h_{ij}^2)$. This operator is self-adjoint and if there is no kernel, we can solve the equation $H_t(\Sigma_f) = c$ for t small.

We conclude that if $-\Delta - (\text{Ric}(\nu, \nu) + \Sigma h_{ij}^2)$ has no kernel and if the metric is not Einstein, we can keep mean curvature constant and scalar curvature greater than $-\frac{3c^2}{2}$. On the other hand, if the metric is Einstein, we can use argument in [5] to prove that M is the warped product of the flat torus with R .

If $-\Delta - (\text{Ric}(\nu, \nu) + \Sigma h_{ij}^2)$ has kernel, it must be a positive function f defined on Σ . (This comes from the fact that it must be the first eigenfunction of the operator.) Hence

$$\begin{aligned} \Delta(\log f) + |\nabla \log f|^2 &= -(\text{Ric}(\nu, \nu) + \Sigma h_{ij}^2) \\ &\leq -\frac{1}{2} \left(R_M + \frac{3}{2}H^2 \right) + K_\Sigma. \end{aligned}$$

Since

$$\int_{\Sigma} \left(R_M + \frac{3}{2} H^2 \right) \geq 0$$

and

$$\int_{\Sigma} K_{\Sigma} = 0,$$

f must be a constant and

$$K_{\Sigma} \geq -\frac{1}{2} \left(R_M + \frac{3}{2} H^2 \right) = 0.$$

Hence $K_{\Sigma} = 0$ and $R_M = -\frac{3}{2} H^2$ is constant along Σ . Also $h_{ij} = \frac{H}{2} g_{ij}$ and $\text{Ric}(\nu, \nu) + \Sigma h_{ij}^2 = 0$ along Σ .

If we compute the first order deformation of the mean curvature of Σ along the normal, it is trivial as $h_{ij} = \frac{H}{2} g_{ij}$, $R = -\frac{3}{2} H^2$ and $R(\nu, \nu) = -\Sigma h_{ij}$.

In conclusion, the mean curvature is equal to H up to first order in t while we can increase the scalar curvature of M up to first order (unless $R_{ij} = \frac{R}{3} g_{ij}$ everywhere). We can therefore prove the following

Theorem 3.2. *Let M be a three dimensional complete manifold with scalar curvature not less than $-\frac{3}{2} c^2$ and one of the component of ∂M is an orientable incompressible surface with nonpositive Euler number and mean curvature $\geq c$. Suppose that for any ball B in M , the area of ∂B is not less than $c \text{Vol}(B)$. Then M is isometric to the warped product of the flat torus with a half line.*

4 Geometry of manifolds with lower bound on scalar curvature.

In this section, we generalize the results of Schoen-Yau [4].

Given a region Ω and a Jordan curve $\Gamma \subset \partial\Omega$ which bounds an embedded disk in Ω and a subdomain in $\partial\Omega$ we define R_{Γ} to be the supremum of $r > 0$ so that Γ does not bound a disk inside the tube of Γ with radius $r < R_{\Gamma}$. We define $\text{Rad}(\Omega)$ to be the supremum of all such R_{Γ} . (Note that this concept can be generalized to higher homology or homotopic groups. The

Radius that is defined in this manner will be sensitive to geometry of higher homology or higher homotopic groups.)

Let R be the scalar curvature of M and h be a function defined on M and k be a function defined on ∂M so that for any smooth function φ

$$\int_M |\nabla\varphi|^2 + \frac{1}{2} \int_M R\varphi^2 + \int_{\partial M} k\varphi^2 \geq \int_M h\varphi^2. \quad (4.1)$$

Let f be the positive first eigenfunction of the operator $-\Delta + \frac{1}{2}R - h$ so that

$$\begin{cases} -\Delta f + \frac{1}{2}Rf - hf = \lambda f \\ \frac{\partial f}{\partial\nu} + kf = 0 \end{cases} \quad \text{on } \partial M. \quad (4.2)$$

Let Γ be a Jordan curve on $\partial\Omega$ which defines $\text{Rad}(\Omega)$ up to a small constant. Let Σ be a disk in Ω with boundary Γ such that Σ together with a region on $\partial\Omega$ bounds a region Ω_Σ .

Assume that $\partial\Omega$ has mean curvature H so that $f(H - k)$ is greater than cf . Then we define a functional

$$L(\Sigma) = \int_\Sigma f - c \int_{\Omega_\Sigma} f. \quad (4.3)$$

Let us now demonstrate that $\partial\Omega$ forms a "barrier" for the existence of minimum of $L(\Sigma)$.

Let r be the distance function to $\partial\Omega$. Let us assume that Γ is in the interior of Ω . If Σ touches $\partial\Omega$, we look at the domain $\Omega_\Sigma \cap \{0 < r < \varepsilon\} = \Omega_{\Sigma,\varepsilon}$. Then

$$\int_{\partial(\Omega_{\Sigma,\varepsilon})} f \frac{\partial r}{\partial\nu} = \int_{\Omega_{\Sigma,\varepsilon}} f \Delta r + \int_{\Omega_{\Sigma,\varepsilon}} \nabla f \cdot \nabla r. \quad (4.4)$$

When ε is small, $f\Delta r + \nabla f \cdot \nabla r$ is close to the boundary value $-\frac{\partial f}{\partial\nu} - Hf$ on $\partial\Omega$. Hence

$$\int_{\partial(\Omega_{\Sigma,\varepsilon})} f \frac{\partial r}{\partial\nu} < - \int_{\Omega_{\Sigma,\varepsilon}} cf. \quad (4.5)$$

Since $|\frac{\partial r}{\partial \nu}| \leq 1$ and $\frac{\partial r}{\partial \nu} = 1$ along $r = \varepsilon$, we conclude that if we replace Σ by $(\Sigma \setminus \partial\Omega_{\Sigma,\varepsilon}) \cup (\partial\Omega_{\Sigma,\varepsilon} \cap \{r = \varepsilon\})$, then the new surface will have strictly less energy than $L_f(\Sigma)$. Hence when we minimize L , $\partial\Omega$ forms a barrier.

By standard geometric measure theory, we can find a surface Σ which minimize the functional L_f . (We start to minimize the functional $\int_{\Sigma} f - tc \int_{\Omega_{\Sigma}} f$ when t is small.)

For this surface, we can compute both the first variational and second variational formula and obtain from the first variational formula

$$\frac{\partial f}{\partial \nu} + Hf = cf. \tag{4.6}$$

The second variational formula has contributions from two terms. The second term gives rise to

$$\int c \left(\frac{\partial f}{\partial \nu} + Hf \right) = c^2 \int_{\Sigma} f. \tag{4.7}$$

Using (4.6) the first term of the second variational formula gives

$$\begin{aligned} 0 &\leq \int_{\Sigma} |\nabla\varphi|^2 f && (4.8) \\ &\quad - \int_{\Sigma} \left(\frac{1}{2}R_M - K_{\Sigma} \right) \varphi^2 f - \int \det(h_{ij})\varphi^2 f \\ &\quad + \int_{\Sigma} \left(\Delta_M f - \Delta_{\Sigma} f - H \frac{\partial f}{\partial \nu} \right) \varphi^2 - \int (\Sigma h_{ij}^2 - H^2)\varphi \\ &\quad + 2 \int_{\Sigma} \frac{\partial f}{\partial \nu} H \varphi^2 - \int_{\Sigma} c^2 f \varphi^2 \\ &\leq \int_{\Sigma} |\nabla\varphi|^2 + \int_{\Sigma} \left(\Delta_M f - \frac{1}{2}R_M f \right) \varphi^2 \\ &\quad - \int_{\Sigma} (\Delta_{\Sigma} f - K_{\Sigma} f) \varphi^2 + \frac{1}{4} \int_{\Sigma} H^2 f \varphi^2 + \int_{\Sigma} \frac{\partial f}{\partial \nu} H \varphi^2 - c^2 \int_{\Sigma} f \varphi^2 \end{aligned}$$

where Δ_M and Δ_{Σ} are the Laplacian of M and Σ respectively and φ is any function vanishing on $\partial\Sigma$.

We conclude from $\frac{\partial f}{\partial \nu} + Hf = cf$ that

$$\begin{aligned} & \frac{1}{4} \int_{\Sigma} H^2 f \varphi^2 + \int_{\Sigma} \frac{\partial f}{\partial \nu} H \varphi^2 - c^2 \int_{\Sigma} f \varphi^2 \\ & \leq -\frac{3}{4} \int_{\Sigma} H^2 f \varphi^2 + c \int_{\Sigma} H f \varphi^2 - c^2 \int_{\Sigma} f \varphi^2 \\ & \leq -\frac{2}{3} c^2 \int_{\Sigma} f \varphi^2, \end{aligned} \quad (4.9)$$

$$\int_{\Sigma} |\nabla \varphi|^2 f - \int_{\Sigma} (\Delta_{\Sigma} f - K_{\Sigma} f) \varphi^2 - \int_{\Sigma} \left(h + \lambda + \frac{2c^2}{3} \right) f \varphi^2 \geq 0 \quad (4.10)$$

where λ is the first eigenvalue of the operator $-\Delta + \frac{R}{2} - h$ with boundary value given by $\frac{\partial f}{\partial \nu} + kf = 0$.

By the argument of [4], we see that for any point $p \in \Sigma$, there exists a curve σ from p to $\partial \Sigma$ with length l such that

$$\int_0^l \left(h + \lambda + \frac{2c^2}{3} \right) \varphi^2 \leq \frac{3}{2} \int_0^l (\varphi')^2 \quad (4.11)$$

where l is the length of the curve σ and φ vanishes at 0 and l .

Theorem 4.1. *Let M be a three dimensional manifold so that (4.1) holds. Let λ be the first eigenvalue of the operator (4.2). Suppose that the mean curvature of ∂M minus k is greater than a constant $c > 0$. Then for any closed curve $\Gamma \subset M$, there is a surface Σ that Γ bounds in M so that for any point $p \in \Sigma$, there is a curve σ from p to $\partial \Sigma$, inequality (4.11) holds.*

5 Existence of Black Holes

For a general initial data set for the Einstein equation, we have two tensors g_{ij} and p_{ij} . The local energy density and linear momentum are given by

$$\begin{aligned} \mu &= \frac{1}{2} [R - \Sigma p^{ij} p_{ij} + (\Sigma p_i^i)] \\ J^i &= \sum_j D_j [p^{ij} - (\Sigma p_k^k) g^{ij}] \end{aligned} \tag{5.1}$$

In [2], Schoen and I studied extensively the following equation initiated by Jung

$$\sum_{i,j} \left(g^{ij} - \frac{f^i f^j}{1 + |\nabla f|^2} \right) \left(\frac{D_i D_j f}{(1 + |\nabla f|^2)^{1/2}} - p_{ij} \right) = 0. \tag{5.2}$$

For the metric

$$\tilde{g}_{ij} = g_{ij} + \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j},$$

one has the following inequality

$$\begin{aligned} 2(\mu - |J|) &\leq \bar{R} - \sum_{i,j} (h_{ij} - p_{ij})^2 \\ &\quad - 2 \sum (h_{i4} - p_{i4})^2 + 2 \sum D_i (h_{i4} - p_i). \end{aligned} \tag{5.3}$$

Hence for any function φ

$$\begin{aligned} 2 \int_M (\mu - |J|) \varphi^2 &\leq \int_M \bar{R} \varphi^2 - 2 \int_M \sum (h_{i4} - p_{i4})^2 \\ &\quad - \int_M 4\varphi (\nabla_i \varphi) (h_{i4} - p_{i4}) + 2 \int_{\partial M} (h_{\nu 4} - p_{\nu 4}) \varphi^2 \\ &\leq \int_M \bar{R} \varphi^2 + 2 \int_M |\nabla \varphi|^2 + 2 \int_{\partial M} (h_{\nu 4} - p_{\nu 4}) \varphi^2. \end{aligned} \tag{5.4}$$

Hence in (4.1) we can take

$$h = (\mu - |J|), \quad k = h_{\nu 4} - p_{\nu 4}. \tag{5.5}$$

Let e_1, e_2, e_3 and e_4 be orthonormal frame of the graph so that e_1, e_2 is tangential to ∂M (assuming $f = 0$ on ∂M) and e_3 is tangential to the graph but normal to ∂M . By assumption,

$$h_{\nu 4} = h_{34} = \langle \nabla_{e_4} e_4, e_3 \rangle. \quad (5.6)$$

Let w be the outward normal vector of ∂M in the horizontal space where $f = 0$. Hence

$$\begin{aligned} h_{34} &= \langle e_4, w \rangle \langle \nabla_w e_4, e_3 \rangle \\ &= -\langle e_4, w \rangle \langle e_4, \nabla_w e_3 \rangle \\ &= \frac{-\langle e_4, w \rangle}{\langle e_3, w \rangle} \langle e_4, \nabla_{e_3} e_3 \rangle. \end{aligned} \quad (5.7)$$

Since $-\sum_{i=1}^3 \langle e_4, \nabla_{e_i} e_i \rangle$ is the mean curvature of the graph of f which is $\text{tr } p$, we conclude that

$$\begin{aligned} h_{34} &= \frac{\langle e_4, w \rangle}{\langle e_3, w \rangle} \left(\text{tr } p + \sum_{i=1}^2 \langle e_4, \nabla_{e_i} e_i \rangle \right) \\ &= \frac{\langle e_4, w \rangle}{\langle e_3, w \rangle} \left(\text{tr } p + \langle e_4, w \rangle \sum_{i=1}^2 \langle w, \nabla_{e_i} e_i \rangle \right). \end{aligned} \quad (5.8)$$

The mean curvature of $\partial\Omega$ with respect to the metric $g_{ij} + \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}$ is given by

$$-\sum_{i=1}^2 \langle \nabla_{e_i} e_i, e_3 \rangle = -\sum_{i=1}^2 \langle \nabla_{e_i} e_i, w \rangle \langle w, e_3 \rangle.$$

Hence the difference between mean curvature and k is given by

$$-\frac{\langle e_4, w \rangle}{\langle e_3, w \rangle} \text{tr } p + p_{34} + \left(\langle w, e_3 \rangle + \frac{\langle w, e_4 \rangle^2}{\langle w, e_3 \rangle} \right) H_{\partial\Omega} \quad (5.9)$$

where $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$ with respect to the metric g_{ij} .

Since

$$\begin{aligned} p_{34} &= \langle e_4, w \rangle p(e_3, w) \\ &= \frac{\langle e_4, w \rangle}{\langle e_3, w \rangle} p(e_3, e_3). \end{aligned}$$

We conclude that the expression (5.9) is given by

$$\begin{aligned}
 & -\frac{\langle e_4, w \rangle}{\langle e_3, w \rangle} (\text{tr } \partial\Omega p) + \frac{1}{\langle e_3, w \rangle} H_{\partial\Omega} \\
 & \geq (H_{\partial\Omega} - |\text{tr } \partial\Omega p|) \langle e_3, w \rangle^{-1}.
 \end{aligned}
 \tag{5.10}$$

We shall assume $H_{\partial\Omega} > |\text{tr } \partial\Omega p|$ and we can choose c to be lower bound of $H_{\partial\Omega} - |\text{tr } \partial\Omega p|$.

We need to solve the Dirichlet problem for f with $f = 0$ on $\partial\Omega$. While most of the estimates were made in [2], we need to construct a barrier for the boundary valued problem.

Let φ be an increasing function defined on the interval $[0, \varepsilon]$ so that $\varphi'(\varepsilon) = \infty$. Let d be the distance function from $\partial\Omega$ measured with respect to g_{ij} .

Then $\varphi(d)$ can be put in (5.2) and when ε is small, we obtain the expression

$$\frac{\varphi'}{\sqrt{1 + (\varphi')^2}} (-H_{\partial\Omega}) - \text{tr}_{\partial\Omega} p + \frac{\varphi''}{(1 + \varphi'^2)^{3/2}} - \frac{p_{\nu\nu}}{1 + \varphi'^2}.
 \tag{5.11}$$

To construct a supersolution, we need this expression to be nonpositive. When ε is small, and φ' is very large, the condition is simply $H_{\partial\Omega} > \text{tr}_{\partial\Omega} p$. Similarly, we can construct a subsolution using $-\varphi(d)$. The conclusion is that we can solve (5.2) if $H_{\partial\Omega} > |\text{tr}_{\partial\Omega} p|$. We have therefore arrived at the following conclusion

Theorem 5.1. *Let M be a space like hypersurface in a four dimensional spacetime. Let g_{ij} be the induced metric and p_{ij} be the second fundamental form. Let μ and J be the energy density and local linear momentum of M . Suppose the mean curvature H of ∂M is greater than $\text{tr}_{\partial M}(p)$. Assume that $H - |\text{tr}_{\partial M} p| \geq c \geq 0$. Let Γ be a Jordan curve in ∂M that bounds a domain in ∂M . If M admits no apparent horizon, then there exists a surface Σ in M bounds by Γ so that for any point $p \in \Sigma$, there is a curve σ with length l from p to Γ and*

$$\int_0^l \left((\mu - |J|) + \frac{3}{2}c^2 \right) \varphi^2 ds \leq \int_0^l |\nabla\varphi|^2 ds
 \tag{5.12}$$

where φ is any function vanishing at 0 and l .

Theorem 5.2. *Let M be a space like hypersurface in a spaetime. Let g_{ij} be its induced metric and p_{ij} be its second fundamental form. Assume that the mean curvature H of ∂M is strictly greater than $|\text{tr}_{\partial m}(p)|$. Let $c = \min(H - |\text{tr}_{\partial m}(p)|)$ if $\text{Rad}(M) \geq \sqrt{\frac{3}{2}} \frac{\Pi}{\sqrt{\Lambda}}$ where $\Lambda = \frac{2}{3}c^2 + \mu - |J|$, then M must admit apparent bonizons in its interior.*

An important point here is that the curvature $H - |\text{tr}_{\partial m}(p)|$ of the boundary itself can give rise to Black Hole.

The inequality actually shows that as long as $\mu - |J| \geq 0$ everywhere, $\frac{2c^2}{3} + \mu - |J|$ to be large in a reasonable ringed region and $\text{Rad}(\Omega)$ is large, an apparent horizon will form in M .

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