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Standard-model bundles

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Abstract

We describe a family of genus one fibered Calabi-Yau threefolds with fundamental group $\mathbb{Z}/2$. On each Calabi-Yau Z in the family we exhibit a positive dimensional family of Mumford stable bundles whose symmetry group is the Standard Model group $SU(3) \times SU(2) \times U(1)$ and which have $c_3 = 6$. We also show that for each bundle V in our family, $c_2(Z) - c_2(V)$ is the class of an effective curve on Z. These conditions ensure that Z and V can be used for a phenomenologically relevant compactification of Heterotic M-theory.

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1 Introduction

In this paper we construct a particular class of bundles with constrained Chern classes on certain non-simply connected Calabi-Yau threefolds. These bundles are instrumental in deriving the Standard Model of particle physics in the context of the Heterotic M-theory [DOPWb]. Bundles of this type have been the subject of active research for quite some time [TY87], [Kac95], [PR99], [ACK], [DOPWc], [Tho]. In contrast with the classical constructions [TY87], [Kac95], [PR99], where the bundles obtained are associated with the tangent bundle of the Calabi-Yau manifold and tend to be rigid, our examples are independent of the geometry of the tangent bundle and vary in families. In particular we construct infinitely many positive dimensional families of bundles which are suitable for phenomenologically relevant compactifications of Heterotic M-theory. Our construction takes place entirely within the realm of algebraic geometry. The physical implications of our results are discussed in the companion paper [DOPWb], which also contains a summary of the construction written for physicists. In the remainder of this introduction we give a brief overview of the physical motivation for our work, followed by an outline of the actual geometric construction.

The search for exotic principal bundles on Calabi-Yau threefolds is motivated by string theory. To compactify the $E_8 \times E_8$-Heterotic string to four dimensions one prescribes:

- a Calabi-Yau 3-fold Z;
- a Ricci flat Kähler metric on Z with a Kähler form ω;
- an ω-instanton $\mathcal{E} \to Z$ with a structure group $E_8 \times E_8$.

The Hermit-Einstein connection on $\mathcal{E}$ is a vacuum of the Heterotic string theory. The moduli space of $\mathcal{E}$'s is a subspace of of the moduli space of vacua for the Heterotic string. In view of the Uhlenbeck-Yau theorem [UY86] every such $\mathcal{E}$ can be identified with an algebraic $E_8 \times E_8$-bundle on Z which is Mumford polystable with respect to the polarization ω. In view of this theorem one can use algebraic geometry to study the moduli space of Heterotic vacua.

The type of bundles $\mathcal{E}$ allowed in a Heterotic compactification is restricted in physics in three ways:

(Supersymmetry preservation) $\mathcal{E}$ has to be Mumford polystable.

(Anomaly cancellation) $c_2(\mathcal{E}) = c_2(Z)$.

(Gauge symmetries) If the compactification of the Heterotic string has a group of symmetries $G \subset E_8 \times E_8$, then the structure group of $\mathcal{E}$ can be reduced to the centralizer G' of G in $E_8 \times E_8$. Furthermore the corresponding G' bundle $\mathcal{E}_{G'} \to Z$ should also be supersymmetric and anomaly-free.

Using these three principles one can look for special compactifications of the Heterotic string that reproduce in their low energy limits well understood and experimentally confirmed quantum field theories. Of particular interest are compactifications that will lead to the Standard Model of particle physics. For such compactifications one imposes two additional requirements on the triple $(Z, \mathcal{E}, \omega)$:

(Standard Model gauge symmetries) The group G of symmetries of $\mathcal{E}$, i.e. the centralizer inside $E_8 \times E_8$ of a minimal subgroup $G' \subset E_8 \times E_8$ to which the structure group of $\mathcal{E}$ reduces, is $G = U(1) \times SU(2) \times SU(3)$.

(3-generations condition)

$$\chi(\mathrm{ad}(\mathcal{E})) = \frac{c_3(\mathrm{ad}(\mathcal{E}))}{2} = 3.$$

Triples $(Z, \omega, \mathcal{E})$ satisfying the five conditions above are hard to come by and tend to be rigid (see e.g. [Kac95]). However the recent advances in string theory prompted by the ground breaking work of Hořava-Witten [HW96b, HW96a, Wit96] on orbifold compactifications of M-theory, allow for a significant relaxation of the anomaly cancellation condition. This leads to two essential simplifications. First, it turns out that using the Hořava-Witten mechanisms one can suppress completely one copy of E_8 in the structure group of $\mathcal{E}$. Secondly it was argued in [DLOW99, DOW99, ACK] that one can use M-theory 5-branes to relax the equality in the anomaly cancellation condition to an inequality. This leads to the following purely mathematical problem.

Main Problem. Find a smooth Calabi-Yau 3-fold (Z, ω) and a reductive subgroup $G' \subset E_8$ so that

- ◊ the centralizer G of G' in E_8 is a group isogenous to $SU(3) \times SU(2) \times U(1)$;

◊ there exists an ω-stable $G'_{\mathbb{C}}$-bundle $\mathcal{V} \to Z$ so that

- $c_1(\mathcal{V}) = 0$,
- $c_2(Z) - c_2(\mathcal{V})$ is the class of an effective reduced curve on Z,
- $c_3(\mathcal{V}) = 6$.

Here the Chern classes of $\mathcal{V}$ are calculated in the adjoint representation of E_8 considered as a representation of G'. In fact for the physics applications it suffices for G to contain a group isogenous to $SU(3) \times SU(2) \times U(1)$ as a direct summand.

The groups $G' \subset E_8$ whose centralizer contains $SU(3) \times SU(2) \times U(1)$ as a direct summand can be classified. It turns out that there are no connected subgroups G' with $Z_{E_8}(G') = SU(3) \times SU(2) \times U(1)$. The stability assumption on $\mathcal{V}$ guarantees that the the structure group of $\mathcal{V}$ can not be reduced to a proper subgroup of $G'_{\mathbb{C}}$. Therefore the structure group of the associated $\pi_0(G'_{\mathbb{C}})$-bundle $\mathcal{V} \times_{G'_{\mathbb{C}}} \pi_0(G'_{\mathbb{C}})$ can not be reduced to a proper subgroup of $\pi_0(G'_{\mathbb{C}})$. Since $\mathcal{V} \times_{G'_{\mathbb{C}}} \pi_0(G'_{\mathbb{C}})$ is a Galois cover of Z with Galois group $\pi_0(G'_{\mathbb{C}})$, this just means that there should be a surjective homomorphism $\pi_1(Z) \twoheadrightarrow \pi_0(G'_{\mathbb{C}})$ and so we are forced to work with a non-simply connected Z.

Some possible choices for G' are: $SU(3) \times (\mathbb{Z}/6)$, $SU(4) \times (\mathbb{Z}/3)$ and $SU(5) \times (\mathbb{Z}/2)$. The corresponding centralizers are isogenous to $(SU(3) \times SU(2) \times U(1)) \times U(1) \times U(1)$, $(SU(3) \times SU(2) \times U(1)) \times U(1)$ and $SU(3) \times SU(2) \times U(1)$. When G'_0, the connected component of the identity in G', is a classical group, it turns out that the Chern classes of $\mathcal{V}$ in the fundamental representation of G'_0 coincide with the Chern classes of $\mathcal{V}$ in the adjoint representation of E_8.

In this paper we explain how to build a big family of solutions of the **Main Problem** above for $G' = SU(5) \times (\mathbb{Z}/2)$.

For concreteness we look for Z's with $\pi_1(Z) = \mathbb{Z}/2$. Let $\mathcal{V}$ be an $SL(5, \mathbb{C}) \times (\mathbb{Z}/2)$-bundle on such a Z. Then $\mathcal{V}$ splits as a product of a rank five vector bundle and the unique non-trivial local system on Z with monodromy $\mathbb{Z}/2$. Pulling back this vector bundle to the universal cover X of Z we get a rank five vector bundle on X which is invariant under the action of $\pi_1(Z)$ on X. Conversely every $\pi_1(Z)$-equivariant vector bundle $V \to X$ descends to a vector bundle on Z. Thus, in order to solve the **Main**

Problem, it suffices to construct a quadruple (X, τ_X, H, V) such that the following conditions hold:

($\mathbb{Z}/2$) X is a smooth Calabi-Yau 3-fold and $\tau_X : X \to X$ is a freely acting involution. H is a fixed ample line bundle (Kähler structure) on X.

(S) V is an H-stable vector bundle of rank five on X.

(I) V is τ_X-invariant.

(C1) $c_1(V) = 0$.

(C2) $c_2(X) - c_2(V)$ is effective.

(C3) $c_3(V) = 12$.

Since we need a mechanism for constructing bundles on X, we will choose X to be elliptically fibered and use the so called *spectral construction* [FMW97, FMW99, Don97, BJPS97] to produce bundles on X. Note that the spectral construction applies only to elliptic fibrations, i.e. genus one fibrations with a section. This is the reason we build an equivariant V on X rather than obtaining directly $\mathcal{V}$ on Z. In general, there are two ways in which the spectral construction can be modified to work on genus one fibrations such as Z. One is to work with a spectral cover in the Jacobian fibration of Z and an abelian gerbe on it. The other route (which is the one we chose) is to work with equivariant spectral data on the universal cover of Z. Note that there are higher algebraic structures involved in both approaches: the stackiness of the first approach is paralleled by complicated group actions on the derived category in the second.

More specifically we take X to be a Calabi-Yau of Schoen type [Sch88], i.e. a fiber product of two rational elliptic surfaces B and B' over $\mathbb{P}^1$, both in the four dimensional family described in [DOPWa]. The rank five bundle V is built as an extension of two vector bundles V_2 and V_3 of ranks two and three respectively. Each of these is manufactured by the spectral construction. Alternatively V may be viewed as a bundle corresponding to spectral data with a *reducible* spectral cover. Our preliminary research of this problem (some of which is recorded in [DOPWc]) showed that bundles corresponding to smooth spectral covers are unlikely to satisfy all of the above conditions. In fact, for the Calabi-Yau's we consider, one can show rigorously (see Remark 2.3) that V's coming from smooth spectral covers can never satisfy **(I)**, **(C1)** and **(C3)** at the same time.

The paper is organized as follows. Section 2 describes the construction of X and lists the geometric constraints on the spectral data which will ensure the validity of **(I)**. Section 3 deals with the actual construction. We describe V_2 and V_3 in terms of their spectral data. The data for each V_i involves a spectral curve C_i in the surface B, a line bundle $\mathcal{N}_i$ on C_i, another line bundle L_i on the surface B', and some optional parameters. The effect of taking these additional parameters to be non-zero is interpretted in section 3.2 as a series of Hecke transforms. The freedom to perform these Hecke transforms gives us at the end of the day infinitely many families of bundles. In section 4 we explain how the geometric information about the action of the spectral involution, obtained in [DOPWa, Theorem 7.1], takes care of condition **(I)**. A delicate point here is that we need two genericity assumptions on C_i. The first one is that C_i is finite over the base of the elliptic fibration on B. The second assumption is that $\mathrm{im}[\mathrm{Pic}(B) \to \mathrm{Pic}(C_i)]$ is Zariski dense in $\mathrm{Pic}^0(C_i)$. In sections 4.2 and 4.3 we check these two assumptions in the special case that is untimately utilized in the construction of V. In section 5 we translate the remaining conditions into a sequence of rather tight numerical inequalities. In Section 5.4 we show how the latter can be solved. In Section 6 we summarize the construction of (X, τ_X, H, V) and give an estimate on the dimension of the moduli space of such quadruples. Finally in Appendix A we have gathered some basic facts on Hecke transforms of vector bundles which are used in Section 3.

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Contents

2 Elliptic Calabi-Yau threefolds with free involutions

Our goal is to construct special $SU(5)$ bundles on smooth Calabi-Yau 3-folds with fundamental group $\mathbb{Z}/2$. We construct our Calabi-Yau 3-fold Z as the quotient of a smooth Calabi-Yau 3-fold X by a freely acting involution $\tau_X : X \to X$. Our X will be elliptic and the elliptic fibration will be preserved by τ_X, so that Z will still have a genus one fibration. This enables us to apply the spectral construction to produce bundles.

The manifold X is constructed as the fiber product $B \times_{\mathbb{P}^1} B'$ of two rational elliptic surfaces B and B' which live in the four dimensional family described in [DOPWa, Section 4]. For the first surface B we use the notation from [DOPWa]. In particular we have $\beta : B \to \mathbb{P}^1$, $e, \zeta : \mathbb{P}^1 \to B$ and the involutions $\alpha_B, \tau_B : B \to B$ and $\tau_{\mathbb{P}^1} : \mathbb{P}^1 \to \mathbb{P}^1$. We use the same symbols with primes for the corresponding objects on B'.

We choose an isomorphism of $\mathbb{P}^1$ with $\mathbb{P}^{1'}$ which identifies $\tau_{\mathbb{P}^1}$ with $\tau_{\mathbb{P}^{1'}}$ and sends $0 \in \mathbb{P}^1$ to $\infty' \in \mathbb{P}^{1'}$ and $\infty \in \mathbb{P}^1$ to $0' \in \mathbb{P}^{1'}$. With this convention we will make no distinction between $\mathbb{P}^1$ and $\mathbb{P}^{1'}$ from now on.

Define $X := B \times_{\mathbb{P}^1} B'$. For a generic choice of B and B' this X will be smooth. It is an elliptic 3-fold in two ways: via its projections $\pi : X \to B'$ and $\pi' : X \to B$. Since most of our analysis will involve the elliptic fibers we will work with the elliptic structure $\pi : X \to B'$ in order to avoid cumbersome notation. By construction the discriminant of π is in the linear system $\beta'^* \mathcal{O}_{\mathbb{P}^1}(12) = -12K_{B'}$ and so X is a Calabi-Yau 3-fold.

For the zero section of π we take the section $\sigma : B' \to X$ corresponding to $e : \mathbb{P}^1 \to B$. Let $\alpha_X := \alpha_B \times_{\mathbb{P}^1} \tau_{B'}$ and let $\tau_X := \tau_B \times_{\mathbb{P}^1} \tau_{B'}$. Since the fixed points of τ_B and $\tau_{B'}$ sit over ∞ and 0 respectively, we conclude that τ_X acts freely on X. In particular the quotient $Z := X/\tau_X$ is non-singular. We claim that Z is in fact a Calabi-Yau. This is equivalent to saying that τ_X preserves the holomorphic 3-form on X. Indeed, τ_X acts on $H^0(X, \Omega^3_X)$ as multiplication by a number $\lambda \in \mathbb{C}^\times$. Since the fiber $f_0 \times f_0'$ of $X \to \mathbb{P}^1$ is stable under τ_X it suffices to compute the action of τ_X on $H^0(f_0 \times f_0', \Omega^3_X)$. But

$$\Omega^3_{X|f_0 \times f_0'} = (K_{f_0} \boxtimes K_{f_0'}) \otimes T_0^* \mathbb{P}^1,$$

and τ_X acts as $\tau_{B|f_0}$ on K_{f_0}, $\tau_{B'|f_0'}$ on $K_{f_0'}$ and as $\tau_{\mathbb{P}^1}$ on $T_0^*\mathbb{P}^1$. Since $\tau_{B|f_0}$ is a translation on f_0, it acts on $H^0(f_0, K_{f_0})$ as $+1$. Since $\tau_{B'|f_0'}$ and $\tau_{\mathbb{P}^1}$ each have a fixed point, they act as -1 on $H^0(f_0', K_{f_0'})$ and $T_0^*\mathbb{P}^1$ respectively.

Hence $\lambda = 1 \cdot (-1) \cdot (-1) = 1$.

The vector bundles on Z can be interpreted as τ_X-invariant vector bundles on X. To construct vector bundles on X we will exploit the fact that X is an elliptically fibered 3-fold and so we can manufacture bundles by using a relative Fourier-Mukai transform.

Concretely, let $\mathcal{P}_X \to X \times_{B'} X$ be the Poincare sheaf corresponding to the section σ. That is $\mathcal{P}_X$ is the rank one torsion-free sheaf given by

$$\mathcal{P}_X = \mathcal{O}_{X\times_{B'}X}(\Delta - \sigma \times_{B'} X - X \times_{B'} \sigma - m^* c_1(B')) = p_{13}^* \mathcal{P}_B,$$

where $m : X \times_{B'} X \to B'$ and $p_{13} : X \times_{B'} X = B \times_{\mathbb{P}^1} B' \times_{\mathbb{P}^1} B \to B \times_{\mathbb{P}^1} B$ are the natural projections. As in [DOPWa, Section 6], one argues that $\mathcal{P}_X$ defines an autoequivalence (see [BM, Theorem 1.2]) of $D^b(X)$:

$$\begin{aligned} \boldsymbol{FM}_X : \qquad & D^b(X) \longrightarrow D^b(X) \\ & \mathcal{F} \longmapsto R^\bullet p_{1*}(p_2^* \mathcal{F} \overset{L}{\otimes} \mathcal{P}_X). \end{aligned}$$

If $V \to X$ is a vector bundle of rank r which is semistable and of degree zero on each fiber of $\pi : X \to B'$, then its Fourier-Mukai transform $\boldsymbol{FM}_X(V)[1]$ is a torsion sheaf of pure dimension two on X. The support of $\boldsymbol{FM}_X(V)[1]$ is a surface $i_\Sigma : \Sigma \hookrightarrow X$ which is finite of degree r over B'. Furthermore $\boldsymbol{FM}_X(V)$ is of rank one on Σ. In fact, if Σ is smooth, then $\boldsymbol{FM}_X(V)[1] = i_{\Sigma *}L$ is just the extension by zero of some line bundle $L \in \operatorname{Pic}(\Sigma)$. Conversely if $\mathcal{N} \to X$ is a sheaf of pure dimension two which is flat over B', then $\boldsymbol{FM}_X(\mathcal{N})$ is a vector bundle on X of rank equal to the degree of $\operatorname{supp}(\mathcal{N})$ over B' and whose first Chern class is vertical (for the projection $\pi : X \to B'$). This correspondence between vector bundles on X and sheaves on X supported on finite covers of B' is commonly known as the *spectral construction* and has been extensively studied in the context of Weierstrass elliptic fibrations [FMW97, FMW99, Don97, BJPS97]. The torsion sheaf $\mathcal{N}$ on X is called *spectral datum* and the surface $\Sigma = \operatorname{supp} \mathcal{N}$ is called a *spectral cover.*

Since our elliptic Calabi-Yau X is not Weierstrass we briefly describe how the spectral construction works (at least for generic spectral data) on X and how it interacts with the involution τ_X. First we need to understand the action of $\boldsymbol{FM}_X$ on line bundles on X. Note that since $X = B \times_{\mathbb{P}^1} B'$ is a fiber product we have $\operatorname{Pic}(X) = (\operatorname{Pic}(B) \times \operatorname{Pic}(B'))/\operatorname{Pic}(\mathbb{P}^1)$. In particular, every line bundle on X can be written as $L \boxtimes L' := \pi'^* L \otimes \pi^* L'$ for some $L \to B$ and $L' \to B'$.

Lemma 2.1. *For every line bundle $\mathcal{L} = L \boxtimes L'$ on X, the actions of the Fourier-Mukai transform and of the spectral involution are given by:*

(a) $\boldsymbol{FM}_X(\mathcal{L}) = \boldsymbol{FM}_X(L \boxtimes L') = \pi'^*\boldsymbol{FM}_B(L) \otimes \pi^* L' = \boldsymbol{FM}_B(L) \boxtimes L'$.

(b) $\boldsymbol{T}_X(\mathcal{L}) := (\boldsymbol{FM}_X^{-1} \circ \tau_X^* \circ \boldsymbol{FM}_X)(\mathcal{L}) = \pi'^*(\boldsymbol{T}_B(L)) \otimes \pi^*(\tau_{B'}^* L') = \boldsymbol{T}_B(L) \boxtimes \tau_{B'}^* L'$.

Proof. Part (b) is an obvious consequence of part (a). To prove part (a) we will use the identification $X \times_{B'} X = B \times_{\mathbb{P}^1} B' \times_{\mathbb{P}^1} B$. In terms of this identification we have:

$$\begin{aligned} \boldsymbol{FM}_X(\mathcal{L}) &= Rp_{23*}(p_{12}^*\mathcal{L} \otimes \mathcal{P}_X) \\ &= Rp_{23*}(p_{12}^*(L \boxtimes L') \otimes p_{13}^*\mathcal{P}_B) \\ &= Rp_{23*}(\pi_1^* L \otimes \pi_2^* L' \otimes p_{13}^*\mathcal{P}_B) \\ &= Rp_{23*}(p_{13}^*(p_1^* L \otimes \mathcal{P}_B) \otimes p_{23}^*(\pi^* L')) \\ &= Rp_{23*}(p_{13}^*(p_1^* L \otimes \mathcal{P}_B)) \otimes \pi^* L'. \end{aligned}$$

Here π_1, π_2 and π_3 are the natural projections of $B \times_{\mathbb{P}^1} B' \times_{\mathbb{P}^1} B$ onto B, B' and B respectively, $p_i : B \times_{\mathbb{P}^1} B \to B$ are the projections on the two factors, and in the last identity we have used the projection formula for p_{23}.

Now using the base change property for the fiber square

$$\begin{array}{ccc} X \times_{B'} X & \xrightarrow{p_{23}} & B' \times_{\mathbb{P}^1} B \\ {\scriptstyle p_{13}}\downarrow & & \downarrow{\scriptstyle \pi'} \\ B \times_{\mathbb{P}^1} B & \xrightarrow[p_2]{} & B \end{array}$$

we get $Rp_{23*}p_{13}^* = \pi'^* Rp_{2*}$ and so

$$\boldsymbol{FM}_X(\mathcal{L}) = \pi'^*(Rp_{2*}(p_1^* L \otimes \mathcal{P}_B)) \otimes \pi^* L' = \boldsymbol{FM}_B(L) \boxtimes L'.$$

The lemma is proven. □

Let now $i_\Sigma : \Sigma \hookrightarrow X$ be a surface which is finite and of degree r over B'. Then for any line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ the torsion sheaf $\mathcal{N} := i_{\Sigma*}i_\Sigma^*\mathcal{L}$ has a resolution by global line bundles. Namely

$$0 \to \mathcal{L}(-\Sigma) \to \mathcal{L} \to \mathcal{N} \to 0.$$

In particular $\mathcal{N}$ is quasi-isomorphic to the two-step complex of line bundles $[\mathcal{L}(-\Sigma) \to \mathcal{L}]$ on X and so the actions of $\boldsymbol{FM}_X$ and $\boldsymbol{T}_X$ on $\mathcal{N}$ can be computed via the formulas in Lemma 2.1. Specifically we have:

Lemma 2.2. *Let $\mathcal{L} = L \boxtimes L'$ be a global line bundle on X and let $i_\Sigma : \Sigma \hookrightarrow X$ be a surface finite over B'. Let $\mathcal{N} = i_{\Sigma*}i_\Sigma^*\mathcal{L}$ be such that $V = \boldsymbol{FM}_X(\mathcal{N})$ is a rank r vector bundle on X with $c_1(V) = 0$. Then $\tau_X^* V \cong V$ if and only if the following three conditions*

- $\alpha_X(\Sigma) = \Sigma$;
- $\tau_{B'}^* L' \cong L'$;
- $\boldsymbol{T}_B^* L \cong L$.

are satisfied.

Remark 2.3. *Notice that the τ_X invariance of V amounts to two separate conditions on the spectral data. The first is that the spectral surface Σ has to be invariant under the involution α_X. This condition is relatively easy to satisfy. It just means that Σ is pulled back from the quotient X/α_X. The second condition requires the $\tau_{B'}$ invariance of L' and the $\boldsymbol{T}_B$ invariance of L.*

In fact, the formulas in [DOPWa, Table 3] (written in terms of the basis of $H^2(B, \mathbb{Z})$ described in [DOPWa, Section 4.2]) show that $L \in \mathrm{Pic}(B) \otimes \mathbb{Q}$ will be $\boldsymbol{T}_B$-invariant if and only if L is in the affine subspace

$$-\frac{1}{2}e_1 + \mathrm{Span}_{\mathbb{Q}}(f, e_9, e_4 - e_5, e_4 - e_6, 3\ell - 2(e_4 + e_5 + e_6) - 3e_7, \ell - e_7 - 2e_8),$$

which does not intersect $\mathrm{Pic}(B) \subset \mathrm{Pic}(B) \otimes \mathbb{Q}$. This implies that V can not be τ_X-invariant if $\mathcal{N} = i_{\Sigma *} i_{\Sigma}^* \mathcal{L}$ for some global $\mathcal{L} \in \mathrm{Pic}(X)$. For Σ smooth and very ample the Lefschetz hyperplane section theorem asserts that every $\mathcal{N}$ comes from a global $\mathcal{L}$ and hence one is forced to work with singular or non-very ample surfaces Σ.

3 The construction

3.1 The basic construction

In this section we describe in detail our method of constructing τ_X-invariant vector bundles on X.

In order to circumvent the difficulty pointed out in Remark 2.3 we build our rank five bundle V on X not directly by the spectral construction but as an extension

$$0 \to V_2 \to V \to V_3 \to 0.$$

Here V_i, $i = 2, 3$ is a rank i bundle on X which is τ_X-invariant and satisfies some strong numerical conditions which will be discussed in the next section. In addition, we will see that the stability condition on V amounts to the extension being non split.

Each V_i is produced by an application of the spectral construction on X with a reducible spectral cover and a a line bundle on it which is *not* the restriction of a global line bundle on X. Define V_i from its spectral data as follows:

- Let C_i be a curve in the linear system $|\mathcal{O}_B(ie + k_i f)|$ where k_i is an integer. Let $\Sigma_i := C_i \times_{\mathbb{P}^1} B'$. Recall that $\beta' : B' \to \mathbb{P}^1$ has two I_2 fibers f'_1, f'_2. Let $F_j, j = 1, 2$ be the corresponding fibers of B; note that, because of the way we glued the $\mathbb{P}^1$ bases, these are *not* the reducible fibers f_j of B. Let
$$\{p_{ijk}\}_{k=1}^{i} := C_i \cap F_j.$$
Then $\Sigma_i \to C_i$ is an elliptic surface having $2i$ fibers of type I_2: $(n'_j \cup o'_j) \times \{p_{ijk}\}$ where $j = 1, 2$ and $k = 1, \dots, i$.
- Define V_i as
$$V_i = \boldsymbol{FM}_X\left(\left(\Sigma_i, (\pi'_{|\Sigma_i})^* \mathcal{N}_i \otimes \pi^* L_i \otimes \mathcal{O}_{\Sigma_i}\left(-\sum \{p_{ijk}\} \times (a_{ijk} n'_j + b_{ijk} o'_j)\right)\right)\right),$$
where L_i is a line bundle on B', $\mathcal{N}_i$ is a line bundle on the curve C_i and the optional parameters a_{ijk}, b_{ijk} are integers.

Note that there is a redundancy in our choices, because $n'_j + o'_j = f'_j$ is a pullback from $\mathbb{P}^1$, and so can be absorbed in L_i. In particular, we can always arrange for all the coefficients a_{ijk}, b_{ijk} to be non-negative. Also, without a loss of generality we may assume that for any given j we have $a_{ijk} \cdot b_{ijk} = 0$ for all i, k. With this convention, we have an alternative description of V_i: Put
$$\widetilde{W}_i := V_i \otimes \pi^* L_i^{-1}.$$
It turns out that the bundle $\widetilde{W}_i$ can be constructed directly. Consider the vector bundle W_i on B, built by the spectral construction as $W_i := \boldsymbol{FM}_B(C_i, \mathcal{N}_i)$. Then $\widetilde{W}_i$ is obtained from the vector bundle $\pi'^* W_i$ by a_{ijk} successive Hecke transforms along the divisors $\{p_{ijk}\} \times n'_j$ and b_{ijk} successive Hecke transforms along the divisors $\{p_{ijk}\} \times o'_j$. The center of each Hecke is

a line bundle on the surface $F_j \times n'_j$ or $F_j \times o'_j$. (For the definition and basic properties of Hecke transforms see appendix A).

In fact, V itself could be built by applying the spectral construction on X to the reducible spectral cover $\Sigma_2 \cup \Sigma_3$ and an appropriately chosen sheaf on it. However the construction with extensions is technically easier because it allows us to avoid dealing with sheaves on singular surfaces. This approach is a variation on the method employed by Richard Thomas in [Tho].

Remark 3.1. *Observe that in the definition of C_i we could have taken the linear system more generally to be of the form $|\mathcal{O}_B(ie + k_i f + \eta_i)|$ where k_i is an integer and $\eta_i \in \mathrm{Pic}(B)$ is a class perpendicular to e and f. If we impose the condition $c_1(W_i) = 0$, then the classes η_i are forced to be zero by the Riemann-Roch formula. However the introduction of the L_i's gives us the extra freedom of working with W_i's that have arbitrary vertical c_1. We will not exploit this extra freedom but we expect that many examples exist which are similar to ours but have $\eta_i \neq 0$.*

Since the Hecke interpretation of $\widetilde{W}_i$ will be important in determining the invariance properties of V and in implementing the numerical constraints, we proceed to spell it out explicitly.

3.2 Reinterpretation via Hecke transforms

Recall from section 2 that $X = B \times_{\mathbb{P}^1} B'$ fits into a commutative diagram of projections

(3.1)
$$\begin{tikzcd} & X \arrow[dl, "\pi'"'] \arrow[dr, "\pi"] & \\ B \arrow[dr, "\beta"'] & & B' \arrow[dl, "\beta'"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

Let $C \subset B$ be a smooth connected curve in the linear system $|\mathcal{O}_B(re + kf)|$. Let $\mathcal{N} \in \mathrm{Pic}^d(C)$ and let $W := \boldsymbol{FM}_B((C, \mathcal{N}))$ be the corresponding rank r vector bundle on B.

Consider $\Sigma = \pi'^{-1}(C) = C \times_{\mathbb{P}^1} B'$ and the line bundle $\mathcal{L} = (\pi'_{|\Sigma})^* \mathcal{N} = \mathcal{N} \boxtimes \mathcal{O}_{B'}$ on Σ.

Let $f'_j = n'_j \cup o'_j$, $F_j = \beta^{-1}(\beta'(f'_j))$, $j = 1, 2$ be as above. We assume that C is general enough so that the intersections $C \cap F_j$, $j = 1, 2$ are transversal.

Let $p \in C \cap F_j$ and let a be a non-negative integer. Define

$$W[a,p] := \boldsymbol{FM}_X((\Sigma, \mathcal{L}(-a(\{p\} \times n'_j)))).$$

Consider the divisor $D = F_j \times n'_j \subset X$ and the line bundles

$$\mathcal{O}_{F_j}(p-e) \boxtimes \mathcal{O}_{n'_j}(2a) \in \operatorname{Pic}(D),$$

where $a \in \mathbb{Z}$ and by a abuse of notation e denotes the point of intersection of the curves e, F_j in B. With this notation we have

Lemma 3.2. *Fix $a \geq 0$.*

(i) *There is a canonical surjective map $W[a,p]_{|D} \to \mathcal{O}_{F_j}(p-e) \boxtimes \mathcal{O}_{n'_j}(2a)$ which fits in a short exact sequence of vector bundles on D*

$$(\psi_{a+1}) \qquad 0 \to K_a \to W[a,p]_{|D} \to \mathcal{O}_{F_j}(p-e) \boxtimes \mathcal{O}_{n'_j}(2a) \to 0.$$

(ii) *For $a = 0$ we have $W[0,p] = \pi'^*W$ and for $a \geq 1$*

$$W[a,p] = \boldsymbol{Hecke}^-_{(\psi_a)} \circ \boldsymbol{Hecke}^-_{(\psi_{a-1})} \circ \ldots \circ \boldsymbol{Hecke}^-_{(\psi_1)}(\pi'^*W).$$

Proof. We will prove the lemma by induction in a. By definition we have $W[0,p] = \pi'^*W$ which takes care of the base of the induction. Assume that $W[i,p] = \boldsymbol{Hecke}^-_{(\psi_i)}(W[i-1,p])$ for all $0 < i < a$. Consider the short exact sequence of sheaves on Σ:

$$\begin{aligned} &(3.2) \quad 0 \to \mathcal{L}(-a(\{p\} \times n'_j)) &&\to \mathcal{L}(-(a-1)(\{p\} \times n'_j)) \\ &(3.3) &&\to \mathcal{L}(-(a-1)(\{p\} \times n'_j))_{|\{p\} \times n'_j} \to 0. \end{aligned}$$

We have $\mathcal{L}_{|\{p\} \times n'_j} = ((\pi'_{|\Sigma})^*\mathcal{N})_{|\{p\} \times n'_j} = \mathcal{O}_{\{p\} \times n'_j}$. Also $\{p\} \times n'_j$ is a component of an I_2 fiber of the elliptic surface $\pi'_{|\Sigma} : \Sigma \to C$ and so $\mathcal{O}_{\{p\} \times n'_j}(\{p\} \times n'_j) = \mathcal{O}_{\{p\} \times n'_j}(-2)$. Let $i_\Sigma : \Sigma \hookrightarrow X$ and $\iota : n'_j = \{p\} \times n'_j \hookrightarrow \Sigma \subset X$ denote the natural inclusions. Then if we extend each of the sheaves in the sequence (3.2) by zero we obtain a short exact sequence of sheaves on X:

(3.4)
$$0 \to i_{\Sigma *}\mathcal{L}(-a(\{p\} \times n'_j)) \to i_{\Sigma *}\mathcal{L}(-(a-1)(\{p\} \times n'_j)) \to \iota_*\mathcal{O}_{n'_j}(2(a-1)) \to 0.$$

Applying $\boldsymbol{FM}_X$ to (3.4) we get

$$(3.5) \qquad 0 \to W[a,p] \to W[a-1,p] \to \boldsymbol{FM}_X(\iota_*\mathcal{O}_{n'_j}(2(a-1))) \to 0.$$

By the definition of $\boldsymbol{FM}_X$ we have

$$\boldsymbol{FM}_X(\iota_*\mathcal{O}_{n'_j}(2(a-1))) = Rp^\bullet_{2*}(p_1^*(\iota_*\mathcal{O}_{n'_j}(2(a-1)) \overset{L}{\otimes} \mathcal{P}_X),$$

where $p_1, p_2 : X \times_{B'} X \to X$ are the natural projections.

If we use the identification $X \times_{B'} X \cong B \times_{\mathbb{P}^1} B \times_{\mathbb{P}^1} B'$, then the projection $p_i : X \times_{B'} X \to X$ gets identified with the projection $p_{i3} : B \times_{\mathbb{P}^1} B \times_{\mathbb{P}^1} B' \to B \times_{\mathbb{P}^1} B'$ and $\mathcal{P}_X = p_{12}^*\mathcal{P}_B$. In particular in terms of the identification $X \times_{B'} X \cong B \times_{\mathbb{P}^1} B \times_{\mathbb{P}^1} B'$ we see that $p_1^*(\iota_*\mathcal{O}_{n'_j}(2(a-1)))$ is supported on the surface $\{p\} \times F_j \times n'_j \subset B \times_{\mathbb{P}^1} B \times_{\mathbb{P}^1} B'$ is precisely

$$\begin{aligned} \mathrm{pr}^*_{n'_j}\, \mathcal{O}_{n'_j}(2(a-1)) \otimes \mathcal{P}_{X|\{p\}\times F_j \times n'_j} &= \mathrm{pr}^*_{n'_j}\, \mathcal{O}_{n'_j}(2(a-1)) \otimes \mathrm{pr}^*_{F_j}\, \mathcal{P}_{B|\{p\}\times F_j} \\ &= \mathrm{pr}^*_{n'_j}\, \mathcal{O}_{n'_j}(2(a-1)) \otimes \mathrm{pr}^*_{F_j}\, \mathcal{O}_{n'_j}(p-e). \end{aligned}$$

Also for the restricted map $p_{2|\{p\}\times F_j\times n'_j} : \{p\} \times F_j \times n'_j \to X$ we get

$$p_{2|\{p\}\times F_j\times n'_j} = p_{23|\{p\}\times F_j\times n'_j} = i_D,$$

and hence

$$\begin{aligned} \boldsymbol{FM}_X(\iota_*\mathcal{O}_{n'_j}(2(a-1))) &= R^\bullet_{p_{2*}}(p_1^*(\iota_*\mathcal{O}_{n'_j}(2(a-1)) \overset{L}{\otimes} \mathcal{P}_X) \\ &= i_{D*}(\mathcal{O}_{F_j}(p-e) \boxtimes \mathcal{O}_{n'_j}(2(a-1))). \end{aligned}$$

which combined with (3.5) concludes the proof of the lemma. □

If now $b \geq 0$ is another integer we may consider also the vector bundle

$$W\{b,p\} = \boldsymbol{FM}_X((\Sigma, \mathcal{L}(-b(\{p\} \times o'_j)))).$$

In exactly the same way we see that $W\{0\} = \pi'^*W$, that for every $b \geq 1$ there is a canonical exact sequence

$$(\phi_{b+1}) \qquad 0 \to M_b \to W\{b,p\}_{|\{F_j\}\times o'_j} \to \mathcal{O}_{F_j}(p-e) \boxtimes \mathcal{O}_{o'_j}(2b) \to 0,$$

and that

$$W\{b,p\} = \boldsymbol{Hecke}^-_{(\phi_a)} \circ \boldsymbol{Hecke}^-_{(\phi_{a-1})} \circ \ldots \circ \boldsymbol{Hecke}^-_{(\phi_1)}(\pi'^*W).$$

For future reference we record

Corollary 3.3. *The Chern classes of $W[a,p]$ and $W\{b,p\}$ are given by*

$$\begin{aligned} ch(W[a,p]) &= \pi'^*ch(W) - a\pi^*n'_j - a^2(f \times \mathrm{pt}); \\ ch(W\{b,p\}) &= \pi'^*ch(W) - b\pi^*o'_j - b^2(f \times \mathrm{pt}) \end{aligned}$$

Proof. Clearly it suffices to prove the corollary for $W[a,p]$. By Lemma 3.2 we have short exact sequences

$$0 \to W[n,p] \to W[n-1,p] \to i_{D*}\psi_n \to 0$$

for all $n \geq 1$. Here we have slightly abused the notation by writing ψ_a for the middle term of the short exact sequence (ψ_a). Hence $ch(W[n,p]) = ch(W[n-1,p]) - ch(i_{D*}\psi_n)$ and so

$$ch(W[a,p]) = \pi'^{*}ch(W) - \sum_{n=1}^{a} ch(i_{D*}\psi_n).$$

Using Grothendieck-Riemann-Roch we calculate

$$\begin{aligned} ch(i_{D*}\psi_n) &= i_{D*}(ch(\psi_n)td(D))td(X)^{-1} \\ &= i_{D*}\left(\left(1+\psi_n+\frac{\psi_n^2}{2}\right)(1+F_j \times \mathrm{pt})\right)(1-(f\times \mathrm{pt}+\mathrm{pt}\times f')) \\ &= i_{D*}((1+2(n-1)F_j \times \mathrm{pt})(1+F_j\times \mathrm{pt}))(1-(f\times \mathrm{pt}+\mathrm{pt}\times f')) \\ &= i_{D*}(1+(2n-1)F_j\times \mathrm{pt})(1-(f\times \mathrm{pt}+\mathrm{pt}\times f')) \\ &= D+(2n-1)(f\times \mathrm{pt}) = \pi^* n_j' + (2n-1)(f\times \mathrm{pt}). \end{aligned}$$

Consequently

$$\begin{aligned} ch(W[a,p]) &= \pi'^{*}ch(W) - \sum_{n=1}^{a}(\pi^* n_j' + (2n-1)(f\times \mathrm{pt}) \\ &= \pi'^{*}ch(W) - a\pi^* n_j' - a^2(f\times \mathrm{pt}). \end{aligned}$$

The corollary is proven. □

Finally, we are ready to give the Hecke interpretation of $\widetilde{W_i} = V_i \otimes \pi^* L_i^{-1}$. Recall that

$$\widetilde{W_i} = \boldsymbol{FM}_X\left(\left(\Sigma_i, \mathcal{L}_i\left(-\sum\{p_{ijk}\}\times(a_{ijk}n_j' + b_{ijk}o_j')\right)\right)\right)$$

where a_{ijk}, b_{ijk} are non-negative integers satisfying $a_{ijk}b_{ijk} = 0$. Since Hecke transforms whose centers have disjoint supports obviously commute, we see from the above discussion that

(3.6)
$$\widetilde{W_i} = W[a_{i11}, p_{i11}]\{b_{i11}, p_{i11}\}[a_{i12}, p_{i12}]\{b_{i12}, p_{i12}\}\dots[a_{i1i}, p_{i1i}]\{b_{i1i}, p_{i1i}\}.$$

4 Invariant spectral data

In this section we examine the conditions for V to be τ_X-invariant. It is easy to reduce this, first to invariance of the V_i, then to invariance of the W_i. Indeed, assume that the bundles V_i are τ_X-invariant, and choose liftings of the τ_X action to the V_i. The space $\mathrm{Ext}^1(V_3, V_2)$ parameterizing all extensions is a direct sum of its invariant and anti-invariant subspaces. So if $\mathrm{Ext}^1(V_3, V_2) \neq 0$ we also have an extension which is either invariant or anti-invariant. Finally, changing the lifted action of τ_X on one of the V_i interchanges invariants with anti-invariants, so we are done.

Since $V_i = \widetilde{W}_i \otimes \pi^* L_i$, we have

$$\tau_X^*(V_i) = \tau_X^*(\widetilde{W}_i) \otimes \pi^* \tau_{B'}^* L_i.$$

So it suffices to have a $\widetilde{W}_i$ which is τ_X-invariant and an L_i which is $\tau_{B'}$-invariant. From [DOPWa, Table 1] we know that there is a 6-dimensional lattice of $\tau_{B'}$-invariant classes on B', so we have lots of possibilities for the L_i. Now $\widetilde{W}_i$ is a Hecke transform of $\pi'^*(W_i)$, so we want W_i to be τ_B-invariant and the center of the Hecke transform to be τ_X-invariant. Above we took the support of this Hecke to be an arbitrary collection of components of the surfaces $F_j \times f'_j$ for $j = 1, 2$. It can be seen from [DOPWa, Table 1] together with the expression [DOPWa, Formula (4.2)] for the components of the I_2-fibers of B in terms of our basis, that the action of $\tau_{B'}$ interchanges o_1 with n_2 and o_2 with n_1. Therefore the condition for τ_X-invariance of the center of the Hecke transform becomes $a_{i1k} = b_{i2k}$ and $a_{i2k} = b_{i1k}$. Because of the redundancy in our choices we are free to take $a_{i2k} = b_{i1k} = 0$ and $a_{i1k} = b_{i2k} \geq 0$.

Finally we have to find the conditions that will ensure the τ_B-invariance of W_i.

4.1 The τ_B-invariance of W_i

Throughout this subsection we work with a spectral curve C_i in the linear system $|ie + k_i f|$, $i = 2$ or 3, which is *finite* over $\mathbb{P}^1$, and a line bundle $\mathcal{N}_i \in \mathrm{Pic}(C_i)$. The τ_B-invariance of $W_i = \boldsymbol{FM}_B((C_i, \mathcal{N}_i))$ is equivalent to the T_B invariance of $i_{C_i *}\mathcal{N}_i$.

In [DOPWa, Proposition 7.7] we saw that for any curve $C \subset B$ which is finite over $\mathbb{P}^1$ and for any line bundle $\mathcal{N}$ on B the image $\boldsymbol{T}_B(i_{C*}\mathcal{N})$ is again a sheaf of the form $i_{\alpha_B(C)*}(?)$ for some line bundle $? \in \mathrm{Pic}(\alpha_B(C))$. Therefore $\boldsymbol{T}_B$ induces a well defined map $\boldsymbol{T}_C : \mathrm{Pic}(C) \to \mathrm{Pic}(\alpha_B(C))$. Due to this, the τ_B-invariance of $\boldsymbol{FM}_B((C, \mathcal{N}))$ is equivalent to the following two conditions:

$$C = \alpha_B(C) \tag{4.1}$$

$$\mathcal{N} = \boldsymbol{T}_C(\mathcal{N}). \tag{4.2}$$

Lemma 4.1. *The linear system $|re + kf|$ contains smooth α_B-invariant curves if $r = 3$, $k \geq 3$ or if $r = 2$ and $k \geq 2$ is even.*

Proof. First of all, from the explicit equations of a spectral curve [FMW97] and Bertini's theorem, it is easy to see that the general curve C in the linear system $|re + kf|$ will be smooth as long as $k \geq r > 1$. The same kind of analysis allows one to understand the α_B-invariant members of these linear systems as well. Indeed, recall (see e.g. [FMW99]) that for every $a \geq 0$ we have an isomorphism $\beta_* \mathcal{O}_B(ae) = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \ldots \mathcal{O}_{\mathbb{P}^1}(-a)$. In particular, by the projection formula we get isomorphisms

$$\begin{aligned} H^0(B, \mathcal{O}_B(e)) &= H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \\ H^0(B, \mathcal{O}_B(2e + 2f)) &= H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \\ H^0(B, \mathcal{O}_B(3e + 3f)) &= H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}). \end{aligned}$$

Let $X \in H^0(B, \mathcal{O}_B(2e+2f))$, $Y \in H^0(B, \mathcal{O}_B(3e+3f))$ and $Z \in H^0(B, \mathcal{O}_B(e)$ be the preferred sections corresponding to the generator of the piece $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ under the above decompositions. Note that in terms of the sections $x \in H^0(P, \mathcal{O}_P(1) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(2)$, $y \in H^0(P, \mathcal{O}_P(1) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(3)$ and $z \in H^0(P, \mathcal{O}_P(1)$, which were used in [DOPWa, Section 3.2] to define the Weierstrass model of B, we have $x_{|W_\beta} = XZ$, $y_{|W_\beta} = Y$, $z_{|W_\beta} = Z^3$.

With this notation the isomorphism

$$H^0(\mathcal{O}_{\mathbb{P}^1}(k)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(k-2) \oplus \ldots \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(k-r) \to H^0(B, \mathcal{O}_B(re + kf))$$

is given explicitly by the formula

$$(a_k, a_{k-2}, \ldots, a_{k-r}) \mapsto (\beta^* a_k) Z^r + (\beta^* a_{k-2}) X Z^{r-2} + (\beta^* a_{k-3}) Y Z^{r-3} + \ldots .$$

In particular the curves C_2 and C_3 can be identified with the zero loci of

$$(\beta^* a_{k_2}) Z^2 + (\beta^* a_{k_2-2}) X \quad \text{and} \quad (\beta^* a_{k_3}) Z^3 + (\beta^* a_{k_3-2}) XZ + (\beta^* a_{k_3-3}) Y, \tag{4.3}$$

respectively.

Now recall, that in [DOPWa, Section 3.2] we identified α_B with the involution induced from $\tau_{P|W_\beta}$ and that τ_P acts trivially on the sections x, y, z. In view of the comparison formulas $x_{|W_\beta} = XZ$, $y_{|W_\beta} = Y$, $z_{|W_\beta} = Z^3$, this implies that $\alpha_B^*(X) = X$, $\alpha_B^*(Y) = Y$ and $\alpha_B^*(Z) = Z$. Here the lifting of the action of α_B to an action on line bundles of the form $\mathcal{O}_B(re + kf)$ is chosen in the way described in [DOPWa, Section 3.2]. In particular $\alpha_B^*(\beta^* s) = \beta^*(\tau_{\mathbb{P}^1}^* s)$ for any section $s \in \mathcal{O}_{\mathbb{P}^1}(k)$.

Since α_B^* acts linearly on the projective space $|re + kf|$ it follows that α_B will preserve a divisor $C \in |re + kf|$ if and only if

$$C \in \mathbb{P}(H^0(B, \mathcal{O}_B(re + kf))^+) \cup \mathbb{P}(H^0(B, \mathcal{O}_B(re + kf))^-) \subset |re + kf|,$$

where $H^0(B, \mathcal{O}_B(re + kf))^\pm$ denote the ± 1 eigenspaces of α_B^* acting on $H^0(B, \mathcal{O}_B(re + kf))$. Therefore we see that for each i there are two families of α_B-invariant C_i's, each parameterized by a projective space. In particular we will have $\alpha_B(C_i) = C_i$ if and only if all the coefficients in the polynomial expressions (4.3) are simultaneously $\tau_{\mathbb{P}^1}$-invariant or simulatneously $\tau_{\mathbb{P}^1}$-anti-invariant. Now the Bertini theorem immediately implies that we can find a smooth C_3, which is preserved by α_B as long as $k_3 \geq 3$ and we can find a smooth C_2, which is preserved by α_B as long as $k_2 \geq 2$ and k_2 is even. □

Remark 4.2. *Unfortunately, when k_2 is odd the linear systems $|2e + k_2 f|^\pm$ will each have a fixed component and so all the α_B-invariant curves C_2 will be reducible. In order to see this consider the homogeneous coordinates $(t_0 : t_1)$ on $\mathbb{P}^1$ which were used in [DOPWa, Section 3.2] to define the standard action of $\tau_{\mathbb{P}^1}$. In other words $(t_0 : t_1)$ are such that $\tau_{\mathbb{P}^1}^*(t_0) = t_0$, $\tau_{\mathbb{P}^1}^* t_1 = -t_1$ and $0 = (1 : 0)$ and $\infty = (0 : 1)$. Now it is clear that if a is a $\tau_{\mathbb{P}^1}$-invariant homogeneous polynomial in t_0 and t_1 of odd degree, then a is divisible by t_0. Similarly if a is a $\tau_{\mathbb{P}^1}$-anti-invariant polynomial of odd degree, then a is divisible by t_1. In particular, since k_2 and $k_2 - 2$ have the same parity we see that for k_2 odd, the fiber f_∞ is the fixed component of the linear system $|2e + k_2 f|^+$ and the fiber f_0 is the fixed component of the linear system $|2e + k_2 f|^-$.*

Lemma 4.3. *Let C be an α_B-invariant curve in $|re + kf|$ which is finite over $\mathbb{P}^1$, and assume that $i_C^* \operatorname{Pic}(B)$ is dense in $\operatorname{Pic}^0(C)$. Then for every $d \in \mathbb{Z}$ there exist line bundles $\mathcal{N} \in \operatorname{Pic}^d(C)$, s.t. $\boldsymbol{T}_C(\mathcal{N}) = \mathcal{N}$.*

Proof. The morphism $\boldsymbol{T}_C : \mathrm{Pic}(C) \to \mathrm{Pic}(C)$ is given explicitly by the formula:

$$\boldsymbol{T}_C(\mathcal{N}) = \alpha_C^*(\mathcal{N}) \otimes \mathcal{O}_C(e_9 - e_1 + f), \tag{4.4}$$

where $\alpha_C = \alpha_{B|C}$.

Indeed, by part (b) of [DOPWa, Proposition 7.7] this formula holds for all line bundles $\mathcal{N} \in \mathrm{Pic}(i_C^* \mathrm{Pic}(B))$. By the density assumption it holds for all $\mathcal{N} \in \mathrm{Pic}^0(C)$. But applying $\boldsymbol{T}_C$ to the short exact sequence

$$0 \to \mathcal{N}(-p) \to \mathcal{N} \to \mathcal{O}_p \to 0$$

we find $\boldsymbol{T}_C(\mathcal{N}(-p)) = \boldsymbol{T}_C(\mathcal{N})(-\alpha_C(p))$, so the formula extends to all components of $\mathrm{Pic}(C)$.

Thus a point $x \in \mathrm{Pic}^0(C)$ will be fixed under $\boldsymbol{T}_C$ if and only if

$$x - \alpha_C^*(x) = \mathcal{O}_C(e_9 - e_1 + f). \tag{4.5}$$

This equation is consistent exactly when

$$\mathcal{O}_C(e_9 - e_1 + f) \in \mathrm{im}[\ \mathrm{Pic}^0(C) \xrightarrow{\alpha_C^* - \mathrm{id}} \mathrm{Pic}^0(C)\ ].$$

Since α_C has fixed points on C, it follows [Mum74] that $\mathrm{im}(\alpha_C^* - \mathrm{id}) = \ker(\alpha_C^* + \mathrm{id})$. But from [DOPWa, Table 1 and Formula (4.2)] we see that

$$(\alpha_B^* + \mathrm{id})(e_9 - e_1 + f) = 2e_9 + 2f + e_7 - \ell = o_1 + o_2.$$

Since o_1 and o_2 do not intersect C, this implies that $\mathcal{O}_C(e_9 - e_1 + f)$ is α_C^*-anti-invariant. Hence there is a translate of of $\mathrm{Pic}^0(C/\alpha_C)$ in $\mathrm{Pic}^0(C)$ consisting of solutions of (4.5).

Shifting by an arbitrary multiple of an α_{C_i}-fixed point, we see that there are $\boldsymbol{T}_{C_i}$-fixed points in $\mathrm{Pic}^d(C_i)$ for every d. □

In view of this lemma it only remains to check the density of $i_{C_i}^* \mathrm{Pic}(B)$ in $\mathrm{Pic}^0(C_i)$. We do this only in the cases $(i = 2, k_2 = 3)$ and $(i = 3, k_3 = 6)$, which are the cases needed in section 5. Unfortunately the statement of Lemma 4.3 does not directly apply to the first of these cases (see Remark 4.2), so we will treat it separately next.

4.2 Invariance for $k_2 = 3$

By Remark 4.2, the general α_B-invariant curve C_2 in the linear system $|2e+3f|$ is of the form $C_2 = \overline{C}_2 + F$, where $\overline{C}_2$ is a smooth curve in the linear system $|2e+2f|^+$ and F denotes one of the elliptic curves f_0, f_∞. Assume that $\mathcal{N}_2 \to C_2$ is a line bundle and let $\overline{\mathcal{N}}_2 = \mathcal{N}_2 \otimes \mathcal{O}_{\overline{C}_2}$ be its restriction to $\overline{C}_2$. We know that $\overline{W}_2 := \boldsymbol{FM}_B(i_{\overline{C}_2*}\overline{\mathcal{N}}_2)$ is a vector bundle. We want $W_2 := \boldsymbol{FM}_B(i_{C_2*}\mathcal{N}_2)$ to be a vector bundle too.

Lemma 4.4. *W_2 is a vector bundle if and only if* $\deg(\mathcal{N}_{2|F}) = 1$.

Proof. We have a short exact sequence of torsion sheaves on B

$$0 \to i_{F*}(\mathcal{N}_{2|F}(-D)) \to i_{C_2*}\mathcal{N}_2 \to i_{\overline{C}_2*}\overline{\mathcal{N}}_2 \to 0,$$

where $D \subset F$ is the effective divisor $D = \overline{C}_2 \cap F$. Let $G := \mathcal{N}_{2|F}(-D)$. Since G is a line bundle on the fiber F we have $\boldsymbol{FM}_B(i_{F*}G) = i_{F*}(\boldsymbol{FM}_F(G))$, where $\boldsymbol{FM}_F : D^b(F) \to D^b(F)$ is the Fourier-Mukai transform with respect to the Poincare bundle $\mathcal{P}_{B|F\times F}$. If we apply $\boldsymbol{FM}_B$ to the above exact sequence, we will get the long exact sequence of cohomology sheaves

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H}^0(i_{F*}\boldsymbol{FM}_F(G)) \arrow[r] & \mathcal{H}^0\boldsymbol{FM}_B(i_{C_2*}\mathcal{N}_2) \arrow[r] & \overline{W}_2 \arrow[dll, rounded corners] \\ & \mathcal{H}^1(i_{F*}\boldsymbol{FM}_F(G)) \arrow[r] & \mathcal{H}^1\boldsymbol{FM}_B(i_{C_2*}\mathcal{N}_2) \arrow[r] & 0. \end{tikzcd}$$

Since we want the line bundle $\mathcal{N}_2 \to C_2$ to be chosen so that $\mathcal{H}^1\boldsymbol{FM}_B(i_{C_2*}\mathcal{N}_2) = 0$ and $W_2 = \mathcal{H}^0\boldsymbol{FM}_B(i_{C_2*}\mathcal{N}_2)$ is a rank two vector bundle on B, we must have $\mathcal{H}^0(i_{F*}\boldsymbol{FM}_F(G)) = 0$ and $\mathcal{H}^1(i_{F*}\boldsymbol{FM}_F(G))$ must be a line bundle on F such that there exists a surjection $\overline{W}_{2|f} \twoheadrightarrow \mathcal{H}^1(i_{F*}\boldsymbol{FM}_F(G))$. This can only happen if G has degree -1 on F. $\square$

We note that in the situation of the lemma W_2 fits in a short exact sequence

$$0 \to W_2 \to \overline{W}_2 \to i_{F*}(G^\vee) \to 0,$$

where G is defined in the proof of the lemma. Indeed, the proof of Lemma 4.4 gave us a short exact sequence

$$0 \to W_2 \to \overline{W}_2 \to i_{F*}(\boldsymbol{FM}_F(G)[1]) \to 0.$$

But every line bundle of degree -1 on an elliptic curve F is of the form $\mathcal{O}_F(-p)$ for some point $p \in F$. Applying $\boldsymbol{FM}_F$ to the short exact sequence

$$0 \to \mathcal{O}_F(-p) \to \mathcal{O}_F \to \mathcal{O}_p \to 0$$

we see that $\boldsymbol{FM}_F(G) = G^\vee[-1]$ for any line bundle of degree -1 on F.

Given $(\overline{C}_2, \overline{\mathcal{N}}_2)$ Lemma 4.4 produces a 2-parameter family of vector bundles W_2. Indeed, let $G \to F$ be any line bundle of degree -1. Consider the semi-stable bundle $\overline{W}_{2|F}$ on F. Since generically $\overline{C}_2$ intersects F at two distinct points we will have $\overline{W}_{2|F} = A \oplus A^\vee$, where A is a non-trivial line bundle of degree zero on F. Therefore $h^0(F, A^\vee \otimes G^\vee) = h^0(F, A \otimes G^\vee) = 1$ and so we have unique (up to scale) maps $A \to G^\vee$ and $A^\vee \to G^\vee$. Also since the degree one line bundles $A^\vee \otimes G^\vee$ and $A \otimes G^\vee$ are not isomorphic, it follows that their unique (up to a scale) sections vanish at different points on F. Hence we get a one parameter family of surjective maps of vector bundles $A \oplus A^\vee \to G^\vee$. The corresponding Hecke transform of $\overline{W}_2$:

$$W_2 = \ker[\overline{W}_2 \to i_{F*}(A \oplus A^\vee) \to i_{F*}(G^\vee)]$$

is a rank two vector bundle on B which is the Fourier-Mukai image of a line bundle $\mathcal{N}_2$ on $C_2 = \overline{C}_2 \cup F$. In particular W_2 is a deformation of a rank two vector bundle W corresponding to a spectral datum $(C, \mathcal{N})$ where C is a smooth (but non-invariant) curve in the linear system $|2e + 3f|$ and $\mathcal{N}$ is a line bundle on C. Even though this W can not be τ_B invariant, this shows that as far as Chern classes are concerned the bundle W_2 behaves as a bundle corresponding to a smooth spectral cover.

We are now ready to analyze the τ_B-invariance properties of W_2. First of all, since $\overline{C}_2$ is smooth Lemma 4.3 applies modulo the following density statement:

Lemma 4.5. *$i^*_{\overline{C}_2} \operatorname{Pic}(B)$ is Zariski dense in $\operatorname{Pic}^0(\overline{C}_2)$.*

Proof. The curves $\overline{C}_2$ have genus 2 and in the linear system $|2e + 2f|^+$ we have a degenerate curve consisting of the zero section e taken with multiplicity two and of two fibers of β which are exchanged by α_B. In particular the Jacobian of this degenerate curve is just the product of the two fibers. But the Mordell-Weil group of B has rank 6 and in particular we get elements of infinite order in the general fiber of β which are restrictions of global line bundles. By continuity this implies that for a general $\overline{C}_2 \in |2e + 2f|^+$ we can find both α_B-invariant and α_B-anti-invariant line bundles on B that restrict to elements of infinite order in $\operatorname{Pic}^0(\overline{C}_2)$. Finally α_B has two fixed points on a general $\overline{C}_2$ and so $g(\overline{C}_2/\alpha_{\overline{C}_2}) = 1$ and $\dim \operatorname{Prym}(\overline{C}_2, \alpha_{\overline{C}_2}) = 1$. Hence $i^*_{\overline{C}_2} \operatorname{Pic}(B)$ is dense in $\operatorname{Pic}^0(\overline{C}_2)$. □

We now reach the main point of this subsection:

Proposition 4.6. *For every integer d there exists a τ_B-invariant vector bundle W_2 whose spectral data $(C_2, \mathcal{N}_2 \in \mathrm{Pic}(C_2))$ deforms flatly to a smooth curve in B and a line bundle of degree d on it.*

Proof. We have a two parameter family of α_B-invariant curves $\overline{C}_2$. By Lemmas 4.3 and 4.5, there is a one parameter family of $\boldsymbol{T}_{\overline{C}_2}$-invariant line bundles $\overline{\mathcal{N}}_2$ on each. Althogether we get a three parameter family of τ_B-invariant bundles $\overline{W}_2$. We have seen above that each of these gives rise to a two parameter family of bundles W_2. We will check now that in each such two parameter family there is a finite number (in fact, four) of τ_B invariant W_2.

Indeed for every such $\overline{W}_2$ we must look for a τ_B-invariant Hecke transform W_2. For this we need to ensure that $G^\vee$ is preserved by $\tau_{B|F}$ and that the map $\overline{W}_2 \to i_{F*}(G^\vee)$ is τ_B-equivariant. We have two possibilities: $F = f_0$ or $F = f_\infty$. Recall that $\tau_{B|f_0} = t_{\zeta(0)}$ and $\tau_{B|f_\infty} = t_{\zeta(\infty)} \circ (-1)_{f_\infty}$ and that $\zeta(0) \in f_0$ and $\zeta(\infty) \in f_\infty$ are non-trivial points of order two. In particular $\tau_{B|f_0}$ does not fix any line bundle of degree one on f_0 and $\tau_{B|f_\infty}$ fixes precisely four such bundles, namely the four square roots of the degree two line bundle $\mathcal{O}_{f_\infty}(e(\infty) + \zeta(\infty))$.

Choose now $F = f_\infty$ and $G^\vee$ to be one of the four square roots of $\mathcal{O}_{f_\infty}(e(\infty) + \zeta(\infty))$. Choose a non-zero map $s : A \to G^\vee$. Then $\tau^*_{B|f_\infty} s : A^\vee \to G^\vee$ is also a non-zero map and so, as before, $s \oplus \tau^*_{B|f_\infty} s : A \oplus A^\vee \to G^\vee$ is surjective. Using this map as the center of a Hecke transform, we get a τ_B-invariant W_2. $\square$

4.3 Invariance for $k_3 = 6$

Let $W_3 = \boldsymbol{FM}_B(i_{C_3*}\mathcal{N}_3)$ for some curve in $|3e + 6f|$. As we saw above, in this case, we can chose C_3 to be smooth and preserved by α_B and so in order to find τ_B-invariant W_3's we only need to show that for a general $C_3 \in |3e + 6f|^\pm$ the image $i^*_{C_3} \mathrm{Pic}(B)$ will be Zariski dense in $\mathrm{Pic}^0(C_3)$.

The Zariski closure of the image $i^*_{C_3} \mathrm{Pic}(B)$ varies lower-semi-continuously with C_3, so it suffices to exhibit one good C_3. Our C_3 will be reducible, consiting of a generic α_B-invariant curve $\overline{C}_2$ in the linear system $|2e + 2f|$, plus the zero section e, plus two generic fibers ϕ_1 and ϕ_2, plus their images $\phi_3 := \alpha_B(\phi_1)$ and $\phi_4 := \alpha_B(\phi_2)$.

The arithmetic genus of C_3 is 13. The 13-dimensional $\mathrm{Pic}^0(C_3)$ is a $(\mathbb{C}^\times)^7$ extension of the six dimensional abelian variety $\mathcal{A} := \mathrm{Pic}^0(\overline{C}_2) \times \prod_{i=1}^4 \mathrm{Pic}^0(\phi_i)$. So our density statement follows from the from the following two lemmas.

Lemma 4.7. *For a generic choice of $\overline{C}_2$, ϕ_1, ϕ_2, the image of $i_{C_3*}\mathrm{Pic}^0(B)$ in $\mathcal{A}$ is Zariski dense.*

Lemma 4.8. *For a generic choice of $\overline{C}_2$, ϕ_1, ϕ_2, no proper subgroup of* $\mathrm{Pic}^0(C_3)$ *surjects onto $\mathcal{A}$.*

Proof of Lemma 4.7. Under the genericity assumption in the hypothesis of the lemma, it is clear that there are no isogenies among ϕ_1, ϕ_2, and the two elliptic curves $\mathrm{Pic}^0(\overline{C}_2/\alpha_{\overline{C}_2})$ and $\mathrm{Prym}(\overline{C}_2, \alpha_{\overline{C}_2})$. So it suffices to prove density separately in each of the two dimensional abelian varieties $\phi_1 \times \phi_3$, $\phi_2 \times \phi_4$ and $\mathrm{Pic}^0(\overline{C}_2)$. Density in $\mathrm{Pic}^0(\overline{C}_2)$ was already proved in Lemma 4.5 and density in say $\phi_1 \times \phi_3$ was established during the proof of that result. The lemma is proven. □

Proof of Lemma 4.8. Let $\mathcal{B}$ be a $\mathbb{C}^\times$-extension of an abelian variety $\mathcal{A}$. Such a $\mathcal{B}$ determines a line bundle $[\mathcal{B}] \in \mathrm{Pic}^0(\mathcal{A})$. A proper subgroup of $\mathcal{B}$ surjecting onto $\mathcal{A}$ will exist if and only if $[\mathcal{B}]$ is torsion.

Similarly, our $(\mathbb{C}^\times)^7$-extension $\mathrm{Pic}^0(C_3)$ will contain a proper subgroup surjecting onto $\mathcal{A}$ if and only there is a non-zero character $\chi : (\mathbb{C}^\times)^7 \to \mathbb{C}^\times$ such that the associated line bundle $L_\chi := \mathrm{Pic}^0(C_3) \times_\chi \mathbb{C}$ over $\mathcal{A}$ is torsion.

Therefore it suffices to find seven characters $\chi_1, \dots \chi_7$ of $(\mathbb{C}^\times)^7$ such that the associated line bundles are linearly independent over $\mathbb{Q}$. For this we will need an intrinsic description of the character lattice Λ of $(\mathbb{C}^\times)^7$ in terms of the geometry of the curve C_3. The singular set of C_3 is $S = \{s_{ij} | i = 1, 2, 3, 4 \text{ and } j = 1, 2, 3\}$, where $\phi_i \cap \overline{C}_2 = \{s_{ij}\}_{j=1}^2$, and $s_{i3} = \phi_i \cap e$. Here the singular points are labelled so that $\alpha_B(s_{1j}) = s_{3j}$ and $\alpha_B(s_{2j}) = s_{4j}$. Now the lattice Λ is explicitly described as:

$$\Lambda = \ker[\mathbb{Z}^S \to \pi_0(C_3 - S)].$$

Here the map $\mathbb{Z}^S \to \pi_0(C_3 - S)$ sends the characteristic function ϵ_{ij} of s_{ij} to the difference $\phi_i - \overline{C}_2$ for $j = 1, 2$ and to $\phi_i - e$ for $j = 3$. Our seven characters χ_k are the following seven elements in $\mathbb{Z}^S$:

$$
\begin{aligned}
\chi_1 &= \epsilon_{11} + \epsilon_{31} - \epsilon_{12} - \epsilon_{32}, \\
\chi_2 &= \epsilon_{21} + \epsilon_{41} - \epsilon_{22} - \epsilon_{42}, \\
\chi_3 &= \epsilon_{11} + \epsilon_{33} - \epsilon_{13} - \epsilon_{31}, \\
\chi_4 &= \epsilon_{21} + \epsilon_{43} - \epsilon_{23} - \epsilon_{41}, \\
\chi_5 &= \epsilon_{11} + \epsilon_{32} - \epsilon_{12} - \epsilon_{31}, \\
\chi_6 &= \epsilon_{21} + \epsilon_{42} - \epsilon_{22} - \epsilon_{41}, \\
\chi_7 &= \epsilon_{21} + \epsilon_{41} - \epsilon_{11} - \epsilon_{31} + \epsilon_{13} + \epsilon_{33} - \epsilon_{23} - \epsilon_{43}.
\end{aligned}
$$

To prove the independence of the L_{χ_k} over $\mathbb{Q}$ it suffices to prove the independence of the restrictions $\lambda_k := L_{\chi_k | \prod_{i=1}^4 \phi_i}$. We represent a degree zero line bundle λ on $\prod_{i=1}^4 \phi_i$ by a four-tuple $(\lambda^i \in \mathrm{Pic}^0(\phi_i))_{i=1}^4$. In this notation the λ_k's become:

$$
\begin{aligned}
\lambda_1 &= (a, 0, a, 0), \\
\lambda_2 &= (0, b, 0, b), \\
\lambda_3 &= (c, 0, -c, 0), \\
\lambda_4 &= (0, d, 0, -d), \\
\lambda_5 &= (a, 0, -a, 0), \\
\lambda_6 &= (0, b, 0, -b), \\
\lambda_7 &= (-c, d, -c, d),
\end{aligned}
$$

where

$$
\begin{aligned}
a &= \mathcal{O}_{\phi_1}(s_{11} - s_{12}) \in \mathrm{Pic}^0(\phi_1) \cong \mathrm{Pic}^0(\phi_3), \\
b &= \mathcal{O}_{\phi_2}(s_{21} - s_{22}) \in \mathrm{Pic}^0(\phi_2) \cong \mathrm{Pic}^0(\phi_4), \\
c &= \mathcal{O}_{\phi_1}(s_{11} - s_{13}) \in \mathrm{Pic}^0(\phi_1) \cong \mathrm{Pic}^0(\phi_3), \\
d &= \mathcal{O}_{\phi_2}(s_{21} - s_{23}) \in \mathrm{Pic}^0(\phi_2) \cong \mathrm{Pic}^0(\phi_4).
\end{aligned}
$$

So we only need to show the linear independence over $\mathbb{Q}$ of $a, c \in \mathrm{Pic}^0(\phi_1)$ and similarly for $b, d \in \mathrm{Pic}^0(\phi_2)$. This however is obvious from the fact that c, d are univalued as functions on the base $\mathbb{P}^1$, whereas a, b are two-valued. The lemma is proven. $\square$

5 Numerical conditions

Our goal is to construct a stable rank five holomorphic vector bundle on the Calabi-Yau manifold $Z := X/\tau_X$ which has a trivial determinant, three generations and an anomaly which can be absorbed into $M5$-branes. In terms of X this amounts to finding a rank five vector bundle $V \to X$ so that:

(S) V is a stable vector bundle.

(I) V is τ_X-invariant.

(C1) $c_1(V) = 0$.

(C2) $c_2(X) - c_2(V)$ is effective.

(C3) $c_3(V) = 12$.

We will construct a whole family of V's satisfying these conditions. As explained in section 3, each V will be constructed as an extension

$$0 \to V_2 \to V \to V_3 \to 0, \tag{5.1}$$

where the V_i's have special form. In fact, as we argued in section 3, in order to satisfy the condition **(I)** it is sufficient to take

$$V_i = \boldsymbol{FM}_X\left(\Sigma_i, (\pi'_{|\Sigma_i})^*\mathcal{N}_i \otimes \pi^* L_i \otimes \mathcal{O}_{\Sigma_i}\left(-\sum_{k=1}^{i} a_{ik}(\{p_{i1k}\}\times n'_1 + \{p_{i2k}\}\times o'_2)\right)\right)$$

with a_{ik} being positive integers, $\Sigma_i = \pi'^* C_i$ for some smooth curve $C_i \subset B$ satisfying (4.1) and $\mathcal{N}_i \in \mathrm{Pic}^{d_i}(C_i)$ satisfying (4.2). In fact, we do not need C_i to be smooth; it is sufficient for the pair $(C_i, \mathcal{N}_i)$ to be deformable to a pair $(C'_i, \mathcal{N}'_i)$ with C'_i being smooth but not necessarily α_B-invariant. Furthermore, we showed in section 3, that such $(C_i, \mathcal{N}_i)$ do exist and move in positive dimensional families, at least for specific values of k_i. From now on we will assume that C_i and $\mathcal{N}_i$ are chosen so that they are deformable to a smooth pair and that (4.1) and (4.2) hold.

Next we will rewrite the conditions **(S)** and **(C1-3)** as a sequence of numerical constraints on the numbers k_i, d_i and on the line bundles L_i.

5.1 The Chern classes of V

There are several ways of calculating the Chern classes of the bundles V_i. One possibility is to use the cohomological Fourier-Mukai transform on X. To avoid long and cumbersome calculations of $\boldsymbol{fm}_X$ we choose a slightly different approach which utilizes the details of the geometric structure of V.

Recall that in section 3 we gave an alternative description of the bundles V_i as

$$V_i = \widetilde{W_i} \otimes \pi^* L_i,$$

where $\widetilde{W_i}$ is the result of a_{ik}, $k = 1, \dots i$ Hecke transforms of the bundle $\pi'^* W_i$, where $W_i = \boldsymbol{FM}_B((C_i, \mathcal{N}_i))$. Due to Corollary 3.3 and the identity (3.6) we have

$$ch(\widetilde{W_i}) = \pi'^* ch(W_i) - \left(\sum_{k=1}^{i} a_{ik}\right) \pi'^*(n'_1 + o'_2) - 2\left(\sum_{k=1}^{i} a_{ik}^2\right)(f \times \mathrm{pt}).$$

Next we need the following

Lemma 5.1. *Let $C \subset B$ be a smooth curve in the linear system $|\mathcal{O}_B(re + mf)|$ and let $\mathcal{N} \in \mathrm{Pic}^d(C)$. Let $W = \boldsymbol{FM}_B((C, \mathcal{N}))$. Then*

$$ch(W) = r + \left(d + \binom{r+1}{2} - rm - r\right) f - m \cdot \mathrm{pt}\,.$$

Proof. The bundle W is defined as the Fourier-Mukai transform of the spectral datum $(C, \mathcal{N})$ on B. Explicitly this means that $W = \boldsymbol{FM}_B(i_{C*}\mathcal{N})$, where $i_C : C \hookrightarrow B$ is the inclusion map. In particular

$$ch(W) = \boldsymbol{fm}_B(ch(i_{C*}\mathcal{N}))$$

and so it suffices to calculate $ch(i_{C*}\mathcal{N})$.

By Groethendieck-Riemann-Roch theorem we have

$$\begin{aligned} ch(i_{C*}\mathcal{N}) &= (i_{C*}(ch(\mathcal{N}) td(C))) td(B)^{-1} \\ &= (i_{C*}(1 + (d + 1 - g) \cdot \mathrm{pt})) \cdot \left(1 + \frac{1}{2} f + \mathrm{pt}\right)^{-1} \\ &= ((re + mf) + (d + 1 - g) \cdot \mathrm{pt})) \cdot \left(1 - \frac{1}{2} f - \mathrm{pt}\right) \\ &= (re + mf) + \left(d + 1 - g - \frac{r}{2}\right) \cdot \mathrm{pt}\,. \end{aligned}$$

Also by adjunction we have $2g-2 = C\cdot(K_B+C) = 2rm - r^2 - r$. Hence

$$ch(i_{C*}\mathcal{N}) = (re+mf) + \left(d + \binom{r+1}{2} - rm - \frac{r}{2}\right)\cdot \mathrm{pt}\,.$$

Finally using [DOPWa, Table 2] we get

$$\begin{aligned} ch(W) &= \boldsymbol{fm}_B(ch(i_{C*}\mathcal{N})) \\ &= \boldsymbol{fm}_B\left((re+mf) + \left(d + \binom{r+1}{2} - rm - \frac{r}{2}\right)\cdot \mathrm{pt}\right) \\ &= r\left(1 - \frac{1}{2}f\right) + m\cdot(-\,\mathrm{pt}) + \left(d + \binom{r+1}{2} - rm - \frac{r}{2}\right)\cdot f \\ &= r + \left(d + \binom{r+1}{2} - rm - r\right) f - m\cdot \mathrm{pt}\,. \end{aligned}$$

The lemma is proven. □

Remark 5.2. *Since the Chern classes are topological invariants, the conclusion of the previous lemma still holds even if C is singular, as long as the sheaf $i_{C*}\mathcal{N}$ deforms flatly to a line bundle on some smooth spectral curve in the same linear system. In particular it will hold for the W_2 from section 4.2.*

Going back to the calculation of $ch(V)$ let us write S_i^α for the Newton sums

$$S_i^\alpha = \sum_{k=1}^{i} a_{ik}^\alpha.$$

In this notation we need to calculate the product

$$ch(V_i) = (\pi'^{*}ch(W_i) - S_i^1\pi^*(n_1' + o_2') - 2S_i^2(f\times \mathrm{pt}))\cdot ch(\pi^* L_i).$$

But

$$ch(\pi^*L_i) = \pi^* ch(L_i) = 1 + \pi^* L_i + \frac{L_i^2}{2}(f\times \mathrm{pt}),$$

and so

- $\pi^*(n_1'+o_2')\cdot ch(\pi^*L_i) = \pi^*(n_1'+o_2') + (L_i\cdot(n_1'+o_2'))\cdot(f\times \mathrm{pt})$;
- $(f\times \mathrm{pt})\cdot ch(\pi^*L_i) = f\times \mathrm{pt}$;

- Lemma 5.1 yields

$$
\begin{aligned}
&\pi'^{*} ch(W_i) \cdot ch(\pi^* L_i) = \\
&= \pi'^{*}\left(i + \left(d_i + \binom{i+1}{2} - k_i i - i\right)\cdot f - k_i \cdot \mathrm{pt}\right)\cdot\left(1 + \pi^* L_i + \frac{L_i^2}{2}(f \times \mathrm{pt})\right) \\
&= i + \pi^*\left(iL_i + \left(d_i + \binom{i+1}{2} - k_i i - i\right) f'\right) + \\
&+ \left[\left(\frac{i}{2}L_i^2 + \left(d_i + \binom{i+1}{2} - k_i i - i\right)\cdot (L_i \cdot f')\right)(f \times \mathrm{pt}) - k_i(\mathrm{pt} \times f')\right] \\
&- k_i(L_i \cdot f')\,\mathrm{pt}\,.
\end{aligned}
$$

Combining these formulas we get formulas for $ch(V_2)$ and $ch(V_3)$:

$$
\boxed{
\begin{aligned}
ch(V_2) = 2 &+ \pi^*(2L_2 + (d_2 - 2k_2 + 1)f' - S_2^1(n_1' + o_2')) \\
&+ (L_2^2 + (d_2 - 2k_2 + 1)(L_2 \cdot f') - S_2^1(L_2 \cdot (n_1' + o_2')) - 2S_2^2) \cdot (f \times \mathrm{pt}) \\
&\qquad - k_2(\mathrm{pt} \times f') \\
&- k_2(L_2 \cdot f')\,\mathrm{pt}\,.
\end{aligned}
}
$$

and similarly

$$
\boxed{
\begin{aligned}
ch(V_3) = 3 &+ \pi^*(3L_3 + (d_3 - 3k_2 + 3)f' - S_3^1(n_1' + o_2')) \\
&+ \left(\frac{3}{2}L_3^2 + (d_3 - 3k_3 + 3)(L_3 \cdot f') - S_3^1(L_3 \cdot (n_1' + o_2')) - 2S_3^2\right) \cdot (f \times \mathrm{pt}) \\
&\qquad - k_3(\mathrm{pt} \times f') \\
&- k_3(L_3 \cdot f')\,\mathrm{pt}\,.
\end{aligned}
}
$$

Together these formulas give

$$
\boxed{
\begin{aligned}
ch(V) &= 5 + \\
&+ \pi^*(2L_2 + 3L_3 + (d_2 + d_3 - 2k_2 - 3k_3 + 4)f' - (S_2^1 + S_3^1)(n_1' + o_2')) \\
&+ [(L_2^2 + (3/2)L_3^2 + (d_2 - 2k_2 + 1)(L_2 \cdot f') + (d_3 - 3k_3 + 3)(L_3 \cdot f') \\
&\qquad - (S_2^1 L_2 + S_3^1 L_3) \cdot (n_1' + o_2') - 2(S_2^2 + S_3^2))(f \times \mathrm{pt}) \\
&\qquad\qquad - (k_2 + k_3)(\mathrm{pt} \times f')] \\
&- ((k_2 L_2 + k_3 L_3) \cdot f')\,\mathrm{pt}\,.
\end{aligned}
}
$$

Therefore, taking into account that $c_2(X) = 12(f \times \mathrm{pt} + \mathrm{pt} \times f')$ we see that the conditions **(C1-3)** translate into the following numerical constraints:

(#**C1**) $2L_2 + 3L_3 = (S_2^1 + S_3^1)(n_1' + o_2') - (d_2 + d_3 - 2k_2 - 3k_3 + 4)f'$.

(**#C2**f) $k_2 + k_3 \leq 12$.

(**#C2**f') $L_2^2 + (3/2)L_3^2 + (d_2 - 2k_2 + 1)(L_2 \cdot f') + (d_3 - 3k_3 + 3)(L_3 \cdot f') - (S_2^1 L_2 + S_3^1 L_3)(n_1' + o_2') - 2(S_2^2 + S_3^2) \geq -12$.

(**#C3**) $k_2(L_2 \cdot f') + k_3(L_3 \cdot f') = -6$.

Our next task is to express the stability of V in a numerical form.

5.2 Stability of V

We need to make sure that the bundle V is Mumford stable with respect to some ample class $H \in H^2(X, \mathbb{Z})$. Recall (see e.g. [FMW99, Section 7]) that a polarization $H \in H^2(X, \mathbb{Z})$ is called *suitable* if up to an overall scale the components of the fibers of $\pi : X \to B'$ have sufficiently small volume. Starting with any polarization $H_0 \in H^2(X, \mathbb{Z})$ we can construct a suitable polarization by fixing some polarization $h' \in H^2(B', \mathbb{Z})$ and taking

$$H := H_0 + n \cdot \pi^* h',$$

with $n \gg 0$. As explained in [FMW99, Theorem 7.1] for a suitable H every vector bundle on X which comes from a spectral cover will be H-stable on X. From now on we will always assume that $H = H_0 + n \cdot \pi^* h'$ is chosen to be suitable. For a torsion free sheaf $\mathcal{F}$ on X denote by $\mu_H(\mathcal{F})$ the H-slope of $\mathcal{F}$, i.e. $\mu_H(\mathcal{F}) = (c_1(\mathcal{F}) \cdot H^2)/\operatorname{rk}(\mathcal{F})$. By repeating the argument in the proof of [FMW99, Theorem 7.1] one gets the following lemma whose proof is left to the reader.

Lemma 5.3. *The bundle V constructed in the previous section is H-stable if and only if*

(i) *The extension*

$$0 \to V_2 \to V \to V_3 \to 0,$$

is non-split.

(ii) $\mu_H(V_2) < \mu_H(V) = 0$.

Next we express both these conditions into a numerical form. Note that an extension

$$0 \to V_2 \to V \to V_3 \to 0$$

will be non-split if and only if it corresponds to a non-zero element in $\mathrm{Ext}^1(V_3, V_2) = H^1(X, V_3^\vee \otimes V_2)$. Thus we only need to ensure that $H^1(X, V_3^\vee \otimes V_2) \neq 0$. We have the following

Lemma 5.4. *For V_2 and V_3 as above one has $H^1(X, V_3^\vee \otimes V_2) \neq 0$ if $L_2 \cdot f' > L_3 \cdot f'$.*

Proof. We are assuming that the W_i's deform to vector bundles on B coming from smooth spectral covers. So by the upper-semi-continuity of $H^1(X, V_2^\vee \otimes V_3)$ it is enough to prove the lemma for W_i's arising from smooth spectral covers.

Let $L = L_3^{-1} \otimes L_2$. Then

$$V_3^\vee \otimes V_2 = \widetilde{W}_3^\vee \otimes \widetilde{W}_2 \otimes \pi^* L.$$

To calculate $H^1(X, \widetilde{W}_3^\vee \otimes \widetilde{W}_2 \otimes \pi^* L)$ we use the Leray spectral sequence for the projection $\pi : X \to B'$. It yields an exact sequence of vector spaces

$$0 \longrightarrow H^1(B', \pi_*(\widetilde{W}_3^\vee \otimes \widetilde{W}_2) \otimes L) \longrightarrow H^1(X, V_3^\vee \otimes V_2)$$

$$H^0(B', R^1\pi_*(\widetilde{W}_3^\vee \otimes \widetilde{W}_2) \otimes L) \longrightarrow \qquad \longrightarrow H^2(B', \pi_*(\widetilde{W}_3^\vee \otimes \widetilde{W}_2) \otimes L).$$

By construction $\widetilde{W}_2$ and $\widetilde{W}_3$ are vector bundles on X coming from the spectral construction applied to line bundles on *smooth* spectral covers, which are also finite over B'. In particular the restriction of $\widetilde{W}_3^\vee \otimes \widetilde{W}_2$ to the general fiber of π is regular and semistable of degree zero. Hence for general $\widetilde{W}_2$ and $\widetilde{W}_3$ we have

- $\pi_*(\widetilde{W}_3^\vee \otimes \widetilde{W}_2) = 0$;
- $R^1\pi_*(\widetilde{W}_3^\vee \otimes \widetilde{W}_2)$ is supported on a curve in B' and is a line bundle on its support.

Therefore

$$H^1(X, V_3^\vee \otimes V_2) = H^0(B', R^1\pi_*(\widetilde{W}_3^\vee \otimes \widetilde{W}_2) \otimes L).$$

To calculate the latter space notice that $R^1\pi_*(\widetilde{W}_3^\vee \otimes \widetilde{W}_2)$ is supported on the curve

$$\pi((-1)_X^* \Sigma_3 \cap \Sigma_2) \subset B'.$$

Since $\Sigma_i = \pi'^* C_i$, this implies

$$(-1)_X^* \Sigma_3 \cap \Sigma_2 = \coprod_{t \in (-1)_B^* C_3 \cap C_2} \{t\} \times f'.$$

Without a loss of generality we may assume that all $t \in (-1)^*_B C_3 \cap C_2$ project to distinct points in $\mathbb{P}^1$ under the map $\beta : B \to \mathbb{P}^1$. Consequently

$$R^1\pi_*(\widetilde{W}_3^\vee \otimes \widetilde{W}_2) = \oplus_{t\in(-1)^*_B C_3 \cap C_2} \Phi_t,$$

where Φ_t is a line bundle on the curve $\pi(\{t\} \times f') = f'_{\beta(t)}$. The line bundle Φ_t depends only on the the restriction of $\widetilde{W}_3^\vee \otimes \widetilde{W}_2$ to the surface $f_{\beta(t)} \times f'_{\beta(t)}$. Let $\mathrm{pr}_1 : f_{\beta(t)} \times f'_{\beta(t)} \to f_{\beta(t)}$ and $\mathrm{pr}_2 : f_{\beta(t)} \times f'_{\beta(t)} \to f'_{\beta(t)}$ denote the natural projections. Then we have

$$\begin{aligned}\Phi_t &= R^1\,\mathrm{pr}_{2*}((\widetilde{W}_3^\vee \otimes \widetilde{W}_2)_{|f_{\beta(t)}\times f'_{\beta(t)}}) = R^1\,\mathrm{pr}_{2*}(\mathrm{pr}_1^*(W_3^\vee \otimes W_2)) \\ &= R^1\,\mathrm{pr}_{2*}(\mathrm{pr}_1^*\mathcal{O}_{f_{\beta(t)}}) = H^1(f_{\beta(t)}, \mathcal{O}_{f_{\beta(t)}}) \otimes \mathcal{O}_{f'_{\beta(t)}}.\end{aligned}$$

In other words $H^1(X, V_3^\vee \otimes V_2) \neq 0$ if and only if $L_{|f'_{\beta(t)}}$ is effective for some $t \in (-1)^*_B C_3 \cap C_2$. For this it suffices to have $L \cdot f' > 0$, and it is necessary to have $L \cdot f' \geq 0$.

However we saw in the previous section that the condition **(C1)** implies $2L_2 \cdot f' + 3L_3 \cdot f' = 0$. If we assume that $L \cdot f' = 0$, then we will get $L_2 \cdot f' = L_3 \cdot f' = 0$ which contradicts **(C3)**. Thus $H^1(X, V_3^\vee \otimes V_2) \neq 0$ iff $L \cdot f' > 0$. The lemma is proven. □

Expressing the slope condition in numerical terms is completely straightforward. In the previous section we showed that

$$c_1(V_2) = \pi^*(2L_2 + (d_2 + 1 - 2k_2)f' - S_2^1(n_1' + o_2')).$$

Hence $\mu_H(V_2) < 0$ if and only if

$$\pi^*(2L_2 + (d_2 + 1 - 2k_2)f' - S_2^1(n_1' + o_2')) \cdot H^2 < 0.$$

It is more convenient to rewrite this as a condition on the surface B'. Recall that we take H to be of the form

$$H = H_0 + n \cdot \pi^* h'$$

where $h' \in H^2(B', \mathbb{Z})$ is some polarization and $n \gg 0$. Since $X = B \times_{\mathbb{P}^1} B'$ we see that any polarization H_0 on X can be written as

$$H_0 = \pi'^* h_0 + \pi^* h_0'$$

for some polarizations $h_0 \in H^2(B, \mathbb{Z})$ and $h_0' \in H^2(B', \mathbb{Z})$. In particular

$$\begin{aligned}H^2 &= (\pi'^* h_0 + \pi^* h_0')^2 + 2n(\pi'^* h_0 + \pi^* h_0')\pi^* h' + n^2 \cdot \pi^*(h'^2) \\ &= (h_0^2) \cdot (\{\mathrm{pt}\} \times f') + 2(\pi^* h_0' + n \cdot \pi^* h')\pi'^* h_0 + \pi^*((h_0' + n \cdot h')^2).\end{aligned}$$

By the projection formula we get

$$\begin{aligned}
\mu_H(V_2) &= \pi^*(2L_2 + (d_2+1-2k_2)f' - S_2^1(n_1' + o_2')) \cdot H^2 \\
&= (2L_2 + (d_2+1-2k_2)f' - S_2^1(n_1' + o_2')) \cdot \pi_*(H^2) \\
&= (2L_2 + (d_2+1-2k_2)f' - S_2^1(n_1' + o_2')) \\
&\quad \cdot ((h_0^2)f' + 2(h_0' + n \cdot h')\pi_*\pi'^* h_0) \\
&= (2L_2 + (d_2+1-2k_2)f' - S_2^1(n_1' + o_2')) \cdot ((h_0^2)f' + 2(h_0 \cdot f)h_0') \\
&\qquad + 2(h_0 \cdot f)n(2L_2 + (d_2+1-2k_2)f' - S_2^1(n_1' + o_2')) \cdot h'.
\end{aligned}$$

To derive the last identity we used (3.1) to write

$$\pi_*\pi'^* h_0 = \beta'^* \beta_* h_0 = (h_0 \cdot f) \cdot 1 + mf' \in H^\bullet(B', \mathbb{Z}),$$

with m being a positive integer. This implies that $\alpha \cdot \pi_*\pi'^* h_0 = (h_0 \cdot f)(\alpha \cdot 1) + m(\alpha \cdot f')$ for any cohomology class α on B'. In particular $\mathrm{pt} \cdot \pi_*\pi'^* h_0 = (h_0 \cdot f)\,\mathrm{pt}$ and hence the above formula.

In conclusion, we see that for $n \gg 0$ we have

$$\mu_H(V_2) < 2(h_0 \cdot f)n(2L_2 + (d_2+1-2k_2)f' - S_2^1(n_1' + o_2')) \cdot h',$$

and so $\mu_H(V_2) < 0$ provided that

$$(2L_2 + (d_2+1-2k_2)f' - S_2^1(n_1' + o_2')) \cdot h' < 0.$$

We are now ready to list all the conditions on V in a numerical form.

5.3 The list of constraints

In the previous two sections we translated the conditions **(S)**, **(I)**, **(C1-3)** into a set of numerical conditions. Together those read:

(#**S**e) $L_2 \cdot f' > L_3 \cdot f'$.

(#**S**s) $(2L_2 + (d_2+1-2k_2)f' - S_2^1(n_1' + o_2')) \cdot h' < 0$ for some ample class $h' \in \mathrm{Pic}(B')$.

(#**I**) $\tau_B^* L_i = L_i$, for $i = 2, 3$.

(#**C1**) $2L_2 + 3L_3 = (S_2^1 + S_3^1)(n_1' + o_2') - (d_2 + d_3 - 2k_2 - 3k_3 + 4)f'$.

(#**C2**f) $k_2 + k_3 \leq 12$.

(#**C2**f') $L_2^2 + (3/2)L_3^2 + (d_2 - 2k_2 + 1)(L_2 \cdot f') + (d_3 - 3k_3 + 3)(L_3 \cdot f') - (S_2^1 L_2 + S_3^1 L_3)(n_1' + o_2') - 2(S_2^2 + S_3^2) \geq -12.$

(#**C3**) $k_2(L_2 \cdot f') + k_3(L_3 \cdot f') = -6.$

Observe that these conditions already constrain severely the possible values of k_2, k_3, $L_2 \cdot f'$ and $L_3 \cdot f'$. Indeed, intersecting both sides of (#**C1**) with the curve $f' \subset B'$ we see that $2(L_2 \cdot f') + 3(L_3 \cdot f') = 0$. Recall also that we showed in section 4 that for the existence of smooth curves C_2 and C_3 one needs to take $k_2 \geq 2$ and $k_3 \geq 3$. Thus the integers k_2, k_3, $L_2 \cdot f'$ and $L_3 \cdot f'$ should satisfy:

- $L_2 \cdot f' > L_3 \cdot f'$
- $k_2 \geq 2$ and $k_3 \geq 3$;
- $k_2 + k_3 \leq 12$;
- $2(L_2 \cdot f') + 3(L_3 \cdot f') = 0$;
- $k_2(L_2 \cdot f') + k_3(L_3 \cdot f') = -6.$

Solving these, we find the following finite list of values for k_2, k_3, $L_2 \cdot f'$ and $L_3 \cdot f'$.

k_2	k_3	$L_2 \cdot f'$	$L_3 \cdot f'$
2	4	9	−6
2	6	3	−2
3	5	18	−12
3	6	6	−4
4	7	9	−6

Table 1: Possible values of k_2, k_3, $L_2 \cdot f'$, $L_3 \cdot f'$

5.4 Some solutions

In this section we show that the numerical constraints (#) can all be satisfied. In fact, we find infinitely many solutions of (#). These represent an infinite sequence of of moduli spaces (of arbitrarily large dimension) of all possible V's.

Fix k_2 and k_3 from the values in Table 1. For such a choice the corresponding numbers $L_2 \cdot f'$ and $L_3 \cdot f'$ are the solutions to the linear system

$$\begin{pmatrix} 2 & 3 \\ k_2 & k_3 \end{pmatrix} \cdot \begin{pmatrix} L_2 \cdot f' \\ L_3 \cdot f' \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \end{pmatrix}.$$

Thus in terms of k_2 and k_3 we have

$$L_2 \cdot f' = \frac{18}{2k_3 - 3k_2}, \quad L_3 \cdot f' = \frac{-12}{2k_3 - 3k_2}. \tag{5.2}$$

Put $k = 2k_3 - 3k_2$. Express L_i, $i = 2, 3$ in terms of the standard classes on B' as follows

$$L_2 = +\frac{9}{k}(e' + \zeta') + x_2 f' + y_2(n_1' + o_2') + 3M$$

$$L_3 = -\frac{6}{k}(e' + \zeta') + x_3 f' + y_3(n_1' + o_2') - 2M.$$

The most general way to make the L_i's satisfy (#**I**) together with (5.2) is to take M to be τ_B-invariant and perpendicular to $e' + \zeta'$, f' and $n_1' + o_2'$. This follows from the fact that the intersection form on $H^2(B', \mathbb{Z})$ is nondegenerate on the span of $e' + \zeta'$, f' and $n_1' + o_2'$. This is evident from Table 2

	$e' + \zeta'$	f'	$n_1' + o_2'$
$e' + \zeta'$	-2	2	2
f'	2	0	0
$n_1' + o_2'$	2	0	-4

Table 2: Intersection pairing on $e' + \zeta'$, f' and $n_1' + o_2'$

The condition (#**C1**) translates into

$$\begin{aligned} 2x_2 + 3x_3 &= -(d_2 + d_3 - 2k_2 - 3k_3 + 4) \\ 2y_2 + 3y_3 &= S_2^1 + S_3^1. \end{aligned} \tag{5.3}$$

Using Table 2 we compute

$$\begin{aligned} L_2^2 &= -\frac{162}{k^2} - 4y_2^2 + 9M^2 + \frac{36}{k}(x_2 + y_2) \\ L_3^2 &= -\frac{72}{k^2} - 4y_3^2 + 4M^2 - \frac{24}{k}(x_3 + y_3) \\ (n_1' + o_2') \cdot L_2 &= +\frac{18}{k} - 4y_2 \\ (n_1' + o_2') \cdot L_3 &= -\frac{12}{k} - 4y_3. \end{aligned}$$

So the condition ($\#\mathbf{C2}f'$) becomes

$$-\frac{270}{k^2}-4y_2^2-6y_3^2+15M^2+\frac{36}{k}(x_2+y_2-x_3-y_3)$$

$$+\frac{1}{k}(18d_2-12d_3-36k_2+36k_3-18)$$

$$-S_2^1\left(\frac{18}{k}-4y_2\right)-S_3^1\left(-\frac{12}{k}-4y_3\right)-2(S_2^2+S_3^2)\geq -12.$$

We eliminate d_3 using (5.3) and complete the squares involving y_2 and y_3, to find:

$$-\frac{135}{k^2}-4u_2^2-6u_3^2+15M^2+((S_2^1)^2-2S_2^2)+\left(\frac{2}{3}(S_3^1)^2-2S_3^2\right)$$

$$+\frac{30}{k}(2x_2+d_2-2k_2+1)\geq -12,$$

where

$$u_2=y_2-\frac{1}{2}\left(\frac{9}{k}+S_2^1\right)$$

$$u_3=y_3-\frac{1}{3}\left(-\frac{9}{k}+S_3^1\right).$$

Implementing the second condition in (5.3) we get $2u_2+3u_3=0$. Introduce new variables

$$\begin{aligned} u &:= 2u_2=-3u_3\\ x &:= 2x_2+d_2-2k_2+1. \end{aligned}$$

Substituting back into the expressions for L_2 and L_3 we get

$$(5.4)\qquad \begin{aligned} L_2 &= \frac{9}{k}\left(e'+\zeta'\right)+\frac{1}{2}\left(x-d_2+2k_2-1\right)f'\\ &\quad+\frac{1}{2}\left(u+\frac{9}{k}+S_2^1\right)\left(n_1'+o_2'\right)+3M\\ L_3 &= -\frac{6}{k}\left(e'+\zeta'\right)+\frac{1}{3}\left(-x-d_3+3k_3-3\right)f'\\ &\quad+\frac{1}{3}\left(-u-\frac{9}{k}+S_3^1\right)\left(n_1'+o_2'\right)-2M. \end{aligned}$$

Similarly for the conditions ($\#\mathbf{C2}f'$) and ($\#\mathbf{S}s$) we get

(5.5)

$$\frac{5}{3}u^2-15M^2\leq 12-\frac{135}{k^2}+\frac{30}{k}x+((S_2^1)^2-2S_2^2)+\left(\frac{2}{3}(S_3^1)^2-2S_3^2\right),$$

and

$$\left(\frac{18}{k}(e'+\zeta')+xf'+\left(u+\frac{9}{k}\right)(n_1'+o_2')+6M\right)\cdot h'<0. \tag{5.6}$$

respectively.

We will use the flexibility we have in choosing M to show that (5.5) and (5.6) have solutions that lead to *integral* coefficients in (5.4). The key observation here is that since $\operatorname{span}(e'+\zeta', f', n_1'+o_2') \subset H^2(B',\mathbb{Z})$ contains an ample class, the Hodge index theorem implies that $M^2 \leq 0$. Therefore, one expects that there will be non-effective admissible M's which will make (5.6) easier to satisfy.

Note that the *means inequality* implies that $(S_2^1)^2 - 2S_2^2 \leq 0$ with equality if and only if all the a_{2k}'s are equal to each other. Similarly $(2/3)(S_3^1)^2 - 2S_3^2 \leq 0$ with equality if and only if all the a_{3k}'s are equal to each other. In particular, for any choice of numbers u, x, k_2, k_3, a_{ik} which satisfies (5.5), the numbers u, x, k_2, k_3 will satisfy

$$\frac{5}{3}u^2 \leq 12 + 15M^2 - \frac{135}{k^2} + \frac{30}{k}x \tag{5.7}$$

as well.

But from Table 1 we see that $k = 2k_3 - 3k_2 > 0$ for all admissible values of k_2 and k_3. Combined with the fact that $e'+\zeta'$ is an effective curve this implies

$$\left(xf'+\left(u+\frac{9}{k}\right)(n_1'+o_2')+6M\right)\cdot h'<0.$$

On the other hand $f' = n_j' + o_j'$ for $j = 1, 2$ and so $f'\cdot h' > n_1'\cdot h'$ and $f'\cdot h' > o_1'\cdot h'$. Thus it suffices to check that

$$\left(\left(x+u+\frac{9}{k}\right)f'+6M\right)\cdot h'<0. \tag{5.8}$$

To make things more concrete recall that the only conditions that we need to impose on M are that M should be $\tau_{B'}^*$-invariant and that M should be perpendicular to $\operatorname{span}(e'+\zeta', f', n_1'+o_2')$. From [DOPWa, Table 1] we see that the classes $e_4'-e_5'$, $e_4'-e_6'$, $3\ell' - 2(e_4'+e_5'+e_6') - 3e_7'$ constitute a rational basis of the space of such M's. Let us choose for example M to be of the form

$$M = z(e_4' - e_5')$$

for some integer z. With this choice (5.7) and (5.8) become

$$\frac{5}{3}u^2 + 30z^2 - \frac{30}{k}x + \frac{135}{k^2} - 12 \leq 0, \tag{5.9}$$

and

$$\left(\left(x + u + \frac{9}{k} \right) f' + 6z \left(e'_4 - e'_5 \right) \right) \cdot h' < 0 \tag{5.10}$$

respectively.

Consider next the class $\gamma := (x+u+9/k)f'+6z(e'_4-e'_5)$. Since the Kähler cone is dual to the Mori cone we know that an ample class h' with $\gamma \cdot h' < 0$ will exist as long as γ is not effective. But γ satisfies $\gamma \cdot e'_4 = x+u+9/k-6z$ and so if $6z > x+u+9/k$ we will have $\gamma \cdot e'_4 < 0$. Under this assumption we have two alternatives: either γ is not effective or $\gamma - e'_4$ is effective. However we have $(\gamma - e'_4) \cdot f' = -1$ and f' moves, so $\gamma - e'_4$ and hence γ can not be effective.

In other words, as a first check for the consistency of the inequalities (5.5) and (5.6) it suffices to make sure that in the 3-space with coordinates (x, u, z) one can find points between the plane

$$6z = x + u + \frac{9}{k}$$

and the paraboloid

$$\frac{5}{3}u^2 + 30z^2 - \frac{30}{k}x + \frac{135}{k^2} - 12 = 0.$$

If we use the equation of the plane to eliminate x and substitute the result in (5.9) we obtain the quadratic inequality

$$\frac{5}{3}\left(u + \frac{9}{k} \right)^2 + 30 \left(z - \frac{3}{k} \right)^2 - 12 \leq 0, \tag{5.11}$$

which always has solutions regardless of the value of k.

To find an actual solution we will choose a particular value for k. By examining Table 1 we see that the possible values of $k = 2k_3 - 3k_2$ are $1, 2, 3$ and 6. Furthermore, since all the coefficients in (5.4) must be integers, k has to divide $\gcd(6,9) = 3$ i.e. we may have either $k = 1$ (which corresponds to $k_2 = 3$ and $k_3 = 5$) or $k = 3$ (which corresponds to $k_2 = 3$ and $k_3 = 6$. For concreteness we choose the second case, i.e.

$$k_2 = 3, \qquad k_3 = 6, \qquad k = 3.$$

Note that the geometry of this case has already been carefully analyzed in sections 4.2 and 4.3. In particular we showed that for these values of k_i there are spectral pairs $(C_i, \mathcal{N}_i)$ which are deformable to smooth pairs and which satisfy (4.1) and (4.2).

To minimize (5.11) we will take

$$u = -3, \qquad z = 1, \qquad x = 5,$$

where the value of x is chosen to satisfy (5.10).

We then have

(5.12)
$$L_2 = 3\left(e' + \zeta'\right) + \frac{1}{2}\left(4 - d_2\right) f' + \frac{1}{2}\left(6 + S_2^1\right)\left(n_1' + o_2'\right) + 3\left(e_4' - e_5'\right)$$
$$L_3 = -2\left(e' + \zeta'\right) + \frac{1}{3}\left(16 - d_3\right) f' + \frac{1}{3}\left(-6 + S_3^1\right)\left(n_1' + o_2'\right) - 2\left(e_4' - e_5'\right).$$

We see that all the coefficients in (5.12) will be integral as long as: d_2 is even, $d_3 \equiv 1 \pmod 3$, S_2^1 is even and S_3^1 is divisible by 3.

The inequality (5.5) now reads

$$-2 \le \left((S_2^1)^2 - 2S_2^2\right) + \left(\frac{2}{3}(S_2^1)^2 - 2S_2^2\right) \tag{5.13}$$

and the inequality (5.6) reads

$$\left(6\left(e' + \zeta'\right) + 5f' + 6\left(e_4' - e_5'\right)\right) \cdot h' < 0 \tag{5.14}$$

Note that (5.14) does not involve the numbers a_{ik} and so all the restrictions on the a_{ik}'s come from (5.13) and from the integrality of (5.12).

In view of the discussion about the means inequality above we see that (5.13) will be automatically satisfied if we take all a_{2k}'s to be equal to a fixed integer $a_2 \ge 0$ and all a_{3k}'s to be equal to another fixed integer $a_3 \ge 0$. Moreover with such a choice we clearly have $S_2^1 = 2a_2$ and $S_3^1 = 3a_3$ and so we have infinitely many possibilities for the numbers a_{ik}. Since d_2 and d_3 are unconstrained except for the conditions $d_2 \equiv 0 \pmod 2$ and $d_3 \equiv 1 \pmod 3$ we see that all the conditions **(S)**, **(I)** and **(C1-3)** will be satisfied if we can prove the following:

Claim 5.5. *There exists an ample class $h' \in \mathrm{Pic}(B')$ satisfying* (5.14).

Proof. As explained above the existence of h' is equivalent to showing that the class

$$6(e' + \zeta') + 5f' + 6(e_4' - e_5') = 6(e_1' + e_9') + 5f' + 6(e_4' - e_5')$$

is not in the Mori cone of B'. First consider the class $\xi' := e_4' - e_5' + e_9' + f' \in \mathrm{Pic}(B')$. We have $\xi'^2 = -1$ and $\xi \cdot f' = 1$. So ξ' is an exceptional class on B'. It is well known [DPT80] that on a *general* rational elliptic surface every exceptional class is effective and is a section. Since our B' is not generic we can't use this statement to conclude that ξ' is the class of a section. However we have

Lemma 5.6. *The divisor ξ' satisfies*

$$\mathcal{O}_{B'}(\xi') = c_1([e_4'] - [e_5']).$$

In particular ξ' is effective and is a section of $\beta' : B' \to \mathbb{P}^1$.

Proof. Let $\underline{\xi}'$ be the section of B' for which $[\underline{\xi}'] = [e_4'] - [e_5']$, that is $c_1([e_4'] - [e_5']) = \mathcal{O}_{B'}(\underline{\xi}')$. Since the group law on the general fiber f_t', $t \in \mathbb{P}^1$ is defined in terms of the Abel-Jacobi map and since we have taken $e_9'(t) \in f_t'$ to be the neutral element for the group law it follows that

$$\mathcal{O}_{B'}(\underline{\xi}' - e_9')_{|f_t'} = (c_1([e_4'] - [e_5']) \otimes \mathcal{O}_{B'}(-e_9'))_{|f_t'} = \mathcal{O}_{B'}(e_4' - e_5')_{|f_t'}$$

for the general $t \in \mathbb{P}^1$. Therefore the line bundle $\mathcal{O}_{B'}(\underline{\xi}' + e_5' - e_4' - e_9')$ must be a combination of vertical divisors on B', i.e. we can write

$$\underline{\xi}' = e_4' - e_5' + e_9' + a \cdot f' + b \cdot n_1 + c \cdot n_2 \tag{5.15}$$

for some integers a, b and c. By [DOPWa, Formula (4.2)] we have $e_4' \cdot n_i' = e_5' \cdot n_i' = 0$ and $e_9' \cdot n_i' = 1$ for $i = 1, 2$. Consider the I_2 fiber $n_1' \cup o_1'$ of B'. The smooth part $(n_1' \cup o_1')^\sharp := (n_1' \cup o_1') - (n_1' \cap o_1')$ of this fiber is an abelian group isomorphic to $\mathbb{Z}/2 \times \mathbb{C}^\times$ with $n_1' - (n_1' \cap o_1')$ being the connected component of the identity. By definition the section $\underline{\xi}'$ intersects $n_1' \cup o_1'$ at a point which is the difference of the points $e_4' \cdot (n_1' \cup o_1') = e_4' \cdot o_1'$ and $e_5' \cdot (n_1' \cup o_1') = e_5' \cdot o_1'$ in the group law of $(n_1' \cup o_1')^\sharp$. Since these two points belong to the same component of $(n_1' \cup o_1')^\sharp$ and the group of connected components of $(n_1' \cup o_1')^\sharp$ is $\mathbb{Z}/2$ it follows that $\underline{\xi}'$ intersects $n_1' \cup o_1'$ at a point in n_1', i.e. $\underline{\xi}' \cdot n_1' = 1$. Similarly $\delta' \cdot n_2' = 1$. Therefore, intersecting both sides of (5.15) with n_1' and n_2' we get $1 = 1 + b$ and $1 = 1 + c$ respectively. Thus $b = c = 0$. Finally from the fact that $\underline{\xi}'^2 = -1$ we compute that $a = 1$ and so $\underline{\xi}' = \xi'$. The lemma is proven. □

In view of the previous lemma we have a section ξ' of $\beta' : B' \to \mathbb{P}^1$ and we need to show that the class

$$\mu := 6e_1' + 6\xi' - f' \in \mathrm{Pic}(B')$$

is not in the Mori cone of B'.

Assume that μ is in the Mori cone. Note that by the definition of ξ' we have $e'_1 \cdot f' = \xi' \cdot f' = 1$, $e'^2_1 = \xi'^2 = -1$ and $e'_1 \cdot \xi' = 1$. Then $\mu \cdot \xi' = -1$ and so $\mu - \xi' = 6e'_1 + 5\xi' - f'$ will also have to be in the Mori cone. But now $(\mu - \xi') \cdot e'_1 = -2$ and so $\mu - 2e'_1 - \xi'$ will be in the Mori cone. Intersecting with ξ' again we get $(\mu - 2e'_1 - \xi') \cdot \xi' = -2$ and so continuing iteratively we conclude that $-\xi - f$ must be in the Mori cone which is obviously false. This shows that μ is not in the Mori cone of B' and so h' ought to exist.

For completeness we will identify an explicit ample class h' on B' with $\mu \cdot h' < 0$. We will look for h' of the form

$$h' = af' + be'_1 + c\xi'$$

and will try to adjust the coefficients a, b and c so that h' is ample and $\mu \cdot h' < 0$. First we have the following

Lemma 5.7. *The divisor class $h' = af' + be'_1 + c\xi'$ is ample provided that a, b and c are positive and $a > |b - c|$.*

Proof. Assume that a, b and c are positive and $a > |b - c|$. By the Nakai-Moishezon criterion for ampleness [Har77, Theorem 1.10], h' will be ample if $h'^2 > 0$ and if $h' \cdot C > 0$ for every irreducible curve $C \subset B'$.

Let $C \subset B'$ be an irreducible curve. Then we have two possibilities: either C is a component of a fiber of $\beta' : B' \to \mathbb{P}^1$, or $\beta' : C \to \mathbb{P}^1$ is a finite map. If $\beta' : C \to \mathbb{P}^1$ is finite and $C \neq \xi', e'_1$, then $C \cdot f' > 0$, $C \cdot e'_1 \geq 0$ and $C \cdot \xi' \geq 0$. In particular the fact that a, b and c are positive implies that $C \cdot h' > 0$. Hence h' will be ample if we can show that the intersections h'^2, $h' \cdot f'$, $h' \cdot n'_i$, $h' \cdot o'_i$, $h' \cdot e'_1$ and $h' \cdot \xi'$ are all positive. For this we calculate

$$\begin{aligned}
h' \cdot e'_1 &= a + c - b \\
h' \cdot \xi' &= a + b - c \\
h' \cdot f' &= b + c \\
h' \cdot n'_1 &= h' \cdot n'_2 = c \\
h' \cdot o'_1 &= h' \cdot o'_2 = b \\
h'^2 &= -b^2 - c^2 + 2ab + 2ac + 2bc = b(a + c - b) + c(a + b - c) + ab + ac,
\end{aligned}$$

which are manifestly positive provided that $a > |b - c|$. The lemma is proven. □

Now we see that the condition $\mu \cdot h' < 0$ translates into $12a < b + c$, i.e we may take $h' = 25f' + 144e'_1 + 168\xi'$. This finishes the proof of the claim. □

6 Summary of the construction

In this section we recapitulate the main points of the construction. Recall that we want to build a quadruple (X, H, τ_X, V) satisfying:

($\mathbb{Z}/2$) X is a smooth Calabi-Yau 3-fold and $\tau_X : X \to X$ is a freely acting involution. H is a fixed Kähler structure (ample line bundle) on X

(S) V is an H-stable vector bundle of rank five on X.

(I) V is τ_X-invariant.

(C1) $c_1(V) = 0$.

(C2) $c_2(X) - c_2(V)$ is effective.

(C3) $c_3(V) = 12$.

The construction is carried out in several steps.

6.1 The construction of (X, τ_X)

X is built as the fiber product of two rational elliptic surfaces of special type.

6.1.1. Building special rational elliptic surfaces. Let $\Gamma_1 \subset \mathbb{P}^2$ be a nodal cubic with a node A_8. Choose four generic points on Γ_1 and label them A_1, A_2, A_3, A_7. Let $\Gamma \subset \mathbb{P}^2$ be the unique smooth cubic which passes trough A_1, A_2, A_3, A_7, A_8 and is tangent to the line $\langle A_7 A_i \rangle$ for $i = 1, 2, 3$ and 8. Consider the pencil of cubics spanned by Γ_1 and Γ. All cubics in this pencil pass trough A_1, A_2, A_3, A_7, A_8 and are tangent to Γ at A_8. Let A_4, A_5, A_6 be the remaining three base points, and let B denote the blow-up of $\mathbb{P}^2$ at the points A_i, $i = 1, 2, \ldots, 8$ and the point A_9 which is infinitesimally near A_8 and corresponds to the line $\langle A_7 A_8 \rangle$.

The pencil becomes the anti-canonical map $\beta : B \to \mathbb{P}^1$ which is an elliptic fibration with a section. The map β has two reducible fibers $f_i = n_i \cup o_i$, $i = 1, 2$ of type I_2. We denote by e_i, $i = 1, \ldots, 7$ and e_9 the exceptional divisors corresponding to A_i, $i = 1, \ldots, 7$ and A_9, and by e_8 the reducible divisor $e_9 + n_1$. The divisors e_i together with the pullback ℓ of a class of a line from $\mathbb{P}^2$ form a standard basis in $H^2(B, \mathbb{Z})$.

The surface B has an involution α_B which is uniquely characterized by the properties: α_B commutes with β, α_B induces an involution on $\mathbb{P}^1$, and α_B fixes the proper transform of Γ pointwise.

Choosing e_9 as the zero section of β, we can interpret any other section ξ as an automorphism $t_\xi : B \to B$ which acts along the fibers of β. The automorphism $\tau_B = t_{e_1} \circ \alpha_B$ is again an involution of B which commutes with β, induces the same involution on $\mathbb{P}^1$ as α_B and has four isolated fixed points sitting on the same fiber of β.

The special rational elliptic surfaces form a four dimensional irreducible family. Their geometry was the subject of [DOPWa].

6.1.2. Building (X, τ_X)**.** Choose two special rational elliptic surfaces $\beta : B \to \mathbb{P}^1$ and $\beta' : B' \to \mathbb{P}^1$ so that the discriminant loci of β and β' in $\mathbb{P}^1$ are disjoint, α_B and $\alpha_{B'}$ induce the same involution on $\mathbb{P}^1$ and the fix loci of τ_B and $\tau_{B'}$ sit over different points in $\mathbb{P}^1$. The fiber product $X := B \times_{\mathbb{P}^1} B'$ is a smooth Calabi-Yau 3-fold which is elliptic and has a freely acting involution $\tau_B \times \tau_{B'}$ and another (non-free) involution $\alpha_X := \alpha_B \times \alpha_{B'}$. For concreteness we fix the elliptic fibration of X to be the projection $\pi : X \to B'$ to B'.

The Calabi-Yau's form a nine dimensional irreducible family.

6.1.3. Building H**.** Choose any ample divisor H_0 on X and take $H = H_0 + n \cdot \pi^* h'$ for some positive integer n. Then the divisor H will be ample as long as h' is ample on B' and $n \gg 0$.

Choose $h' = 25f' + 144e_1' + 168\xi'$ with ξ' being the unique section of $\beta' : \mathbb{P}^1 \to B'$ satisfying $[e_4'] - [e_5]' = [\xi']$. The divisor class $h' \in \mathrm{Pic}(B')$ is ample on B' by Lemma 5.7.

6.2 The construction of V

The bundle V is build as a non-split extension

$$0 \to V_2 \to V \to V_3 \to 0$$

of two τ_X-invariant stable vector bundles V_2 and V_3 of ranks 2 and 3 respectively.

Each V_i is constructed via the spectral cover construction on X.

6.2.1. Building V_2 **and** V_3**.** Choose curves $\overline{C}_2, C_3 \subset B$, so that

- $\overline{C}_2 \in |\mathcal{O}_B(2e_9+2f)|$ and $C_3 \in |\mathcal{O}_B(3e_9+6f)|$.
- $\overline{C}_2$, C_3 are α_B-invariant.
- $\overline{C}_2$ and C_3 are smooth and irreducible.

Set $C_2 := \overline{C}_2 + f_\infty$ where f_∞ is the smooth fiber of β containing the fixed points of τ_B.

The space of such $\overline{C}_2$'s is an open set (see section 4) in $\mathbb{P}(H^0(B, \mathcal{O}_B(2e_9+2f))^+)$ where $H^0(B, \mathcal{O}_B(2e_9+2f))^\pm$ denote the spaces of invariants/anti-invariants for the α_B action on $H^0(B, \mathcal{O}_B(2e_9+2f)$. Using the explicit equations (4.3) of the spectral curves we easily that all such C_2 form a 2 dimensional irreducible family. The space of permissible C_3's is an open subset in the disjoint union of the projective spaces $\mathbb{P}(H^0(B, \mathcal{O}_B(3e_9+6f))^\pm)$, which have dimensions 8 and 6.

Fix an even integer d_2 and an integer d_3 satisfying $d_3 \equiv 1 \pmod 3$. Choose line bundles $\mathcal{N}_2$ and $\mathcal{N}_3$ on C_2 and C_3 respectively, which satisfy

- $\mathcal{N}_i \in \operatorname{Pic}^{d_i}(C_i)$ for $i = 1, 2$,
- $\mathcal{N}_i = \boldsymbol{T}_{C_i}(\mathcal{N}_i) := \alpha^*_{B|C_i}\mathcal{N}_i \otimes \mathcal{O}_{C_i}(e_1 - e_9 + f)$ for $i = 1, 2$.

As explained in sections 4.2 and 4.3, such $\mathcal{N}_2$'s and $\mathcal{N}_3$'s are parameterized by abelian subvarieties of $\operatorname{Pic}^{d_2-1}(\overline{C}_2)$ and $\operatorname{Pic}^{d_3}(C_3)$ of dimensions equal to the genera of the quotient curves $\overline{C}_2/\alpha_{\overline{C}_2}$ and C_3/α_{C_3} respectively. Thus there is a one dimensional space of $\mathcal{N}_2$'s and a six dimensional space of $\mathcal{N}_3$'s.

Let $\Sigma_i = C_i \times_{\mathbb{P}^1} B'$ for $i = 1, 2$. Recall that $\beta' : B' \to \mathbb{P}^1$ has two I_2 fibers f_1' and f_2'. Let F_1, F_2 be the corresponding (smooth) fibers of $\beta : B \to \mathbb{P}^1$. Let $C_i \cap F_j = \{p_{ijk}\}_{k=1}^i$ for $i = 2, 3$, $j = 1, 2$. Then $\Sigma_i \to C_i$ is an elliptic surface having $2i$ fibers of type I_2: $\{p_{ijk}\} \times f_j'$. Also $\Sigma_i \subset X$ and the natural projection $\pi_{|\Sigma_i} : \Sigma_i \to B'$ is finite of degree i.

Fix non-negative integers a_2 and a_3. Define

$$V_i =$$
$$\boldsymbol{FM}_X\left(\left(\Sigma_i, (\pi_{|\Sigma_i})^*\mathcal{N}_i \otimes \mathcal{O}_{\Sigma_i}\left(-a_i \textstyle\sum_{k=1}^i (\{p_{i1k}\} \times n_1' + \{p_{i2k}\} \times o_2')\right)\right)\right) \otimes \pi^* L_i,$$

where L_2 and L_3 are the line bundles

$$\begin{aligned} L_2 &= 3(e_1' + e_4' - e_5' + e_9') + \frac{1}{4}(4 - d_2)f' + (3 - a_2)(n_1' + o_2') \\ L_3 &= -2(e_1' + e_4' - e_5' + e_9') + \frac{1}{3}(16 - d_3)f' + (-2 + a_3)(n_1' + o_2'), \end{aligned}$$

on B'.

6.2.2. Building V. Take V to be a non-split extension of V_2 by V_3 which is τ_X-invariant. As explained in section 5.2, the space of all such extensions of V_2 by V_3 is the union of projective spaces $\mathbb{P}(H^1(X, V_3^\vee \otimes V_2)^+) \cup \mathbb{P}(H^1(X, V_3^\vee \otimes V_2)^-)$ where $H^1(X, V_3^\vee \otimes V_2)^\pm$ denote the invariants/anti-invariants for the τ_X action on $H^1(X, V_3^\vee \otimes V_2)$.

Furthermore, it is shown in section 5.2 that

$$\dim H^1(X, V_3^\vee \otimes V_2) \geq (((-1)_B^* C_3) \cdot C_2) \cdot (L_2' \cdot f' - L_3' \cdot f') = 150,$$

and from the explicit description of $H^1(X, V_3^\vee \otimes V_2)$ we see that the $\pm$ decomposition breaks this as $150 = 70 + 80$, so the dimension of the admissible extensions of V_2 by V_3 is at least 79.

In other words, for a fixed (X, τ_X, H) as above we find infinitely many components of the moduli space of V's satisfying (**S**), (**I**) and (**C1-3**). Each component corresponds to a choice of the integers a_2, a_3, d_2 and d_3 and has dimension $2 + 8 + 1 + 6 + 79 = 96$.

Appendix A Hecke transforms

In this appendix we review the definition and some basic properties of the Hecke transforms (aka 'elementary modifications') of vector bundles along divisors. For more details the reader may wish to consult [Mar82, Mar87], [Fri98].

A.1 Definition and basic properties

Let X be a smooth complex projective variety. Let $\imath : D \hookrightarrow X$ be a divisor with normal crossings.

Let $E \to X$ be a vector bundle and let (ξ) be a short exact sequence of vector bundles on D of the form

$$(\xi) \quad : \quad 0 \longrightarrow F \longrightarrow E_{|D} \longrightarrow G \longrightarrow 0.$$

There are two Hecke transforms $\boldsymbol{Hecke}_{(\xi)}^{\pm}(E)$ attached to the pair $(E, (\xi))$.

Definition A.1.

(i) *The down-Hecke transform of E along (ξ) is the coherent sheaf $\boldsymbol{Hecke}^-_{(\xi)}(E)$ defined by the exact sequence*

$$0 \longrightarrow \boldsymbol{Hecke}^-_{(\xi)}(E) \longrightarrow E \longrightarrow \imath_* G \longrightarrow 0.$$

(ii) *The up-Hecke transform of E along (ξ) is the coherent sheaf*

$$\boldsymbol{Hecke}^+_{(\xi)}(E) = (\boldsymbol{Hecke}^-_{(\xi^\vee)}(E^\vee))^\vee$$

The first properties of the Hecke transforms are given by the following two lemmas.

Lemma A.2.

(a) *The sheaves $\boldsymbol{Hecke}^+_{(\xi)}(E)$ and $\boldsymbol{Hecke}^-_{(\xi)}(E)$ are locally free.*

(b) *The up-Hecke transform $\boldsymbol{Hecke}^+_{(\xi)}(E)$ of E along (ξ) fits in the exact sequence*

$$0 \longrightarrow E \longrightarrow \boldsymbol{Hecke}^+_{(\xi)}(E) \longrightarrow \imath_* F \otimes \mathcal{O}_X(D) \longrightarrow 0.$$

(c) *$\boldsymbol{Hecke}^+_{(\xi)}(E)_{|D}$ and $\boldsymbol{Hecke}^-_{(\xi)}(E)_{|D}$ are furnished with natural exact sequences*

$$(\xi^-) \quad : \quad 0 \longrightarrow G(-D) \longrightarrow \boldsymbol{Hecke}^-_{(\xi)}(E)_{|D} \longrightarrow F \longrightarrow 0,$$

$$(\xi^+) \quad : \quad 0 \longrightarrow G \longrightarrow \boldsymbol{Hecke}^+_{(\xi)}(E)_{|D} \longrightarrow F(D) \longrightarrow 0.$$

(d) *$\boldsymbol{Hecke}^+_{(\bullet)}(\bullet)$ and $\boldsymbol{Hecke}^-_{(\bullet)}(\bullet)$ are mutually inverse in the sense that*

$$\boldsymbol{Hecke}^+_{(\xi^-)}(\boldsymbol{Hecke}^-_{(\xi)}(E)) = E, \quad \boldsymbol{Hecke}^-_{(\xi^+)}(\boldsymbol{Hecke}^+_{(\xi)}(E)) = E.$$

(e) *$\boldsymbol{Hecke}^-_{(\xi)}(E) = \boldsymbol{Hecke}^+_{(\xi)}(E)(-D)$.*

Proof. The proof of **(a)** is straightforward. For the proof of **(b)** recall that by definition the dual bundle $\boldsymbol{Hecke}^{+}_{(\xi)}(E)^{\vee}$ fits in the short exact sequence of sheaves

$$0 \longrightarrow \boldsymbol{Hecke}^{+}_{(\xi)}(E)^{\vee} \longrightarrow E^{\vee} \longrightarrow \imath_{*}(F^{\vee}) \longrightarrow 0.$$

Application of $\mathcal{H}om_{\mathcal{O}_X}(\bullet, \mathcal{O}_X)$ combined with the fact that $\imath_*(F^{\vee})$ is torsion yields the long exact sequence

$$\begin{array}{cccccc} 0 & \to & \mathcal{H}om_{\mathcal{O}_X}(E^{\vee}, \mathcal{O}_X) & \to & \mathcal{H}om_{\mathcal{O}_X}(\boldsymbol{Hecke}^{+}_{(\xi)}(E)^{\vee}, \mathcal{O}_X) & \to \\ & \to & \mathcal{E}xt^1_{\mathcal{O}_X}(\imath_*(F^{\vee}), \mathcal{O}_X) & \to & \mathcal{E}xt^1_{\mathcal{O}_X}(E^{\vee}, \mathcal{O}_X) & \to \quad \ldots \end{array}$$

Furthermore $E^{\vee}$ and $\mathcal{O}_X$ are both locally free and hence every extension of $E^{\vee}$ by $\mathcal{O}_X$ splits locally yielding $\mathcal{E}xt^1_{\mathcal{O}_X}(E^{\vee}, \mathcal{O}_X) = 0$. Thus we obtain the exact sequence

$$0 \longrightarrow E \longrightarrow \boldsymbol{Hecke}^{+}_{(\xi)}(E) \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_X}(\imath_*(F^{\vee}), \mathcal{O}_X) \longrightarrow 0.$$

To calculate $\mathcal{E}xt^1_{\mathcal{O}_X}(\imath_*(F^{\vee}), \mathcal{O}_X)$ consider the ideal sequence of the divisor D:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_D(D) \longrightarrow 0.$$

After apllying $\mathcal{H}om_{\mathcal{O}_X}(\imath_*(F^{\vee}), \bullet)$ and taking into account that $\mathcal{O}_X$ and $\mathcal{O}_X(D)$ are locally free sheaves, we get the exact sequence

$$\begin{array}{cccccc} 0 & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(\imath_*(F^{\vee}), \mathcal{O}_D D) & \longrightarrow & \mathcal{E}xt^1_{\mathcal{O}_X}(\imath_*(F^{\vee}), \mathcal{O}_X) & \longrightarrow \\ & \longrightarrow & \mathcal{E}xt^1_{\mathcal{O}_X}(\imath_*(F^{\vee}), \mathcal{O}_X(D)) & \longrightarrow & \ldots & \end{array}$$

To understand the map

$$\mathcal{E}xt^1_{\mathcal{O}_X}(\imath_*(F^{\vee}), \mathcal{O}_X) \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_X}(\imath_*(F^{\vee}), \mathcal{O}_X(D)), \tag{A.1}$$

consider a point $p \in D \subset X$. Let $R := \mathcal{O}_{X,p}$ and let $t \in R$ be a local equation of D around p. Let M be the finitely generated R/tR module whose sheafification gives $F^{\vee} \to D$ in a neighborhood of p.

An element $(\alpha) \in \mathcal{E}xt^1_{\mathcal{O}_X}(\imath_*(F^{\vee}), \mathcal{O}_X)_p$ of the stalk of $\mathcal{E}xt^1_{\mathcal{O}_X}(\imath_*(F^{\vee}), \mathcal{O}_X)$ at p is an extension of R-modules of the form

$$(\alpha) \quad : \quad 0 \longrightarrow R \longrightarrow A \longrightarrow M \longrightarrow 0,$$

where M is given its R-module structure via $R \to R/tR$.

The image $(\beta) \in \mathcal{E}xt^1_{\mathcal{O}_X}(\imath_*(F^\vee), \mathcal{O}_X(D))_p$ of (α) under the map (A.1) is just the pushout of the extension (α) via the homomorphism

$$R \longrightarrow \frac{1}{t}R.$$

That is, there is a commutative diagram

$$\begin{tikzcd} (\alpha) & 0 \arrow[r] & R \arrow[r] \arrow[d] & A \arrow[r, "\pi"] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\ (\beta) & 0 \arrow[r] & \frac{1}{t}R \arrow[r] & B \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

and $B = (A \oplus \frac{1}{t}R)/R$.

On the other hand, since tR annihilates M we have $\pi(tx) = t\pi(x) = 0$, for all $x \in A$. In particular the map

$$\begin{array}{rccc} s: & A \oplus \frac{1}{t}R & \longrightarrow & \frac{1}{t}R \\ & x \oplus \frac{f}{t} & \longrightarrow & \frac{tx}{t} + \frac{f}{t}, \end{array}$$

is well defined and descends to B to a map splitting the exact sequence

$$0 \longrightarrow \frac{1}{t}R \longrightarrow B \longrightarrow M \longrightarrow 0.$$

Therefore the map

$$\mathcal{E}xt^1_{\mathcal{O}_X}(\imath_*(F^\vee), \mathcal{O}_X) \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_X}(\imath_*(F^\vee), \mathcal{O}_X(D))$$

is the zero map and we get an isomorphism

$$\begin{tikzcd} \mathcal{H}om_{\mathcal{O}_X}(\imath_*(F^\vee), \imath_*\mathcal{O}_D \otimes \mathcal{O}_X(D)) \arrow[r, "\simeq"] \arrow[d, equal] & \mathcal{E}xt^1_{\mathcal{O}_X}(\imath_*(F^\vee), \mathcal{O}_X) \\ \imath_*F \otimes \mathcal{O}_X(D) & \end{tikzcd}$$

which concludes the proof of the lemma. □

There is a natural symmetry between the up and down Hecke transforms. If X, D, E and (ξ) are as above, then we can form the dual exact sequence

$$(\xi^\vee) \quad : \quad 0 \longrightarrow G^\vee \longrightarrow E^\vee_{|D} \longrightarrow F^\vee \longrightarrow 0,$$

and the up and down Hecke transforms of $E^\vee$ along $(\xi^\vee)$. The relation with the Hecke transforms of E is given by the following lemma.

Lemma A.3.

$$\begin{aligned} \boldsymbol{Hecke}^{+}_{(\xi)}(E)^{\vee} &\simeq \boldsymbol{Hecke}^{-}_{(\xi^{\vee})}(E^{\vee}) \\ \boldsymbol{Hecke}^{-}_{(\xi)}(E)^{\vee} &\simeq \boldsymbol{Hecke}^{+}_{(\xi^{\vee})}(E^{\vee}) \end{aligned}$$

Proof. Clear. □

A.2 Geometric interpretation - flips

Let (E, ξ) be as in Section A.1 and let $\tau \to \mathbb{P}(E)$ be the relatively ample tautological line bundle. Denote by $Y := \mathrm{Bl}_{\mathbb{P}(F)}\mathbb{P}(E)$ the blow-up of $\mathbb{P}(E)$ along $\mathbb{P}(F)$.

Let $p : Y \to \mathbb{P}(E)$ be the blow-up morphism and let $\mathcal{E} \subset Y$ be the exceptional divisor. The image of Y under the full linear system $p^*\tau \otimes \mathcal{O}_Y(-\mathcal{E})$ is again a projective bundle $\mathbb{P}(E') \to X$. We have the following diagram

$$\begin{tikzcd} \tau \arrow[dr, dash] & & Y \arrow[dl, "p"'] \arrow[dr, "p'"] & & \tau' \arrow[dl, dash] \\ & \mathbb{P}(E) \arrow[dr, "f"'] & & \mathbb{P}(E') \arrow[dl, "f'"] & \\ & & X & & \end{tikzcd}$$

where $\tau \to \mathbb{P}(E)$ and $\tau' \to \mathbb{P}(E')$ are relatively ample line bundles having the properties

$$\begin{aligned} f_*\tau &= E^{\vee} \\ f'_*\tau' &= E'^{\vee} \\ p'^*\tau' &= p^*\tau \otimes \mathcal{O}_Y(-\mathcal{E}) \end{aligned}$$

To identify E' in terms of Hecke transforms consider the ideal sequence of $\mathcal{E}$:

$$0 \longrightarrow \mathcal{O}_Y(-\mathcal{E}) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{\mathcal{E}} \longrightarrow 0.$$

Tensoring by $p^*\tau$ we get

$$0 \longrightarrow {p'}^*\tau' \longrightarrow p^*\tau \longrightarrow p^*\tau \otimes \mathcal{O}_{\mathcal{E}} \longrightarrow 0. \tag{A.2}$$

Let $\pi : Y \to X$ be the composition $\pi = f \circ p = f' \circ p'$. Consider the π direct image of (A.2):

$$0 \longrightarrow \pi_* {p'}^*\tau' \longrightarrow \pi_* p^*\tau \longrightarrow \pi_*(p^*\tau \otimes \mathcal{O}_{\mathcal{E}}) \longrightarrow R^1\pi_* {p'}^*\tau' \longrightarrow \dots$$

Observe first that every fiber of π is either a projective space or has two irreducible components (meeting transversally) each of which is a projective space. Furthermore ${p'}^*\tau'$ restricted on a component P of the fiber is either $\mathcal{O}_P(1)$ or $\mathcal{O}_P$ and hence by Serre's vanishing theorem doesn't have higher cohomology. Thus by the base change and cohomology theorem $R^1\pi_* {p'}^*\tau' = 0$. Next

$$\pi_* {p'}^*\tau' = f'_* {p'}_* {p'}^*\tau' = f'_*\tau' = {E'}^\vee.$$

Here we used that $p' : Y \longrightarrow \mathbb{P}(E')$ has connected fibers.

Similarly $\pi_* p^*\tau = E^\vee$ and we get

$$0 \longrightarrow {E'}^\vee \longrightarrow E^\vee \longrightarrow \pi_*(p^*\tau \otimes \mathcal{O}_{\mathcal{E}}) \longrightarrow 0.$$

But $\pi_*(p^*\tau \otimes \mathcal{O}_{\mathcal{E}}) = \imath_* \bar{f}_*(\tau_{|\mathbb{P}(F)})$ where $\bar{f} : \mathbb{P}(F) \longrightarrow D$ is the natural projection. Hence we get the short exact sequence

$$0 \longrightarrow {E'}^\vee \longrightarrow E^\vee \longrightarrow \imath_* F^\vee \longrightarrow 0$$

and thus

$$E' = \boldsymbol{Hecke}^+_{(\xi)}(E).$$

A.3 An example

Let X and D be as before. One has the short exact sequence:

$$(D) \quad : \qquad \begin{tikzcd} 0 \arrow[r] & N^\vee_D X \arrow[r] \arrow[d, equal] & \Omega^1_{X|D} \arrow[r] & \imath_*\Omega^1_D \arrow[r] & 0 \\ & \mathcal{O}_D(-D) & & & \end{tikzcd}$$

We can form the up and down Hecke transforms of Ω^1_X along (D).

Lemma A.4. *Denote by $\Omega^1_X(\log D)$ the sheaf of one forms on X with logarithmic poles along D. Then the up and down Hecke transforms of Ω^1_X along (D) can be identified as follows*

$$\begin{aligned} \boldsymbol{Hecke}^+_{(D)}(\Omega^1_X) &= \Omega^1_X(\log D) \\ \boldsymbol{Hecke}^-_{(D)}(\Omega^1_X) &= \Omega^1_X(\log D) \otimes \mathcal{O}_X(-D). \end{aligned}$$

Proof. To prove the first equality observe that $\Omega^1_X(\log D)$ fits in the residue sequence

$$0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X(\log D) \longrightarrow \imath_*\mathcal{O}_D \longrightarrow 0,$$

where the map $\Omega^1_X(\log D) \to \imath_*\mathcal{O}_D$ is given by the residue along D. On the other hand, according to Lemma A.2 we have an exact sequence

$$0 \longrightarrow \Omega^1_X \longrightarrow \boldsymbol{Hecke}^+_{(D)}(\Omega^1_X) \longrightarrow \imath_*\mathcal{O}_D \longrightarrow 0$$

and it is easy to check that the two extension classes coincide. □

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