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Spectral involutions on rational elliptic surfaces

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Abstract

In this paper we describe a four dimensional family of special rational elliptic surfaces admitting an involution with isolated fixed points. For each surface in this family we calculate explicitly the action of a spectral version of the involution (namely of its Fourier-Mukai conjugate) on global line bundles and on spectral data. The calculation is carried out both on the level of cohomology and in the derived category. We find that the spectral involution behaves like a fairly simple affine transformation away from the union of those fiber components which do not intersect the zero section. These results are the key ingredient in the construction of Standard-Model bundles in [DOPWa].

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1 Introduction

Let $Z \to S$ be an elliptic fibration on a smooth variety Z, i.e. a flat morphism whose generic fiber is a curve of genus one, and which has a section $S \to Z$. The choice of such a section defines a Poincare sheaf $\mathcal{P}$ on $Z \times_S Z$. The corresponding Fourier-Mukai transform $\boldsymbol{FM} : D^b(Z) \to D^b(Z)$ is then an autoequivalence of the derived category $D^b(Z)$ of complexes of coherent sheaves on Z. It sets up an equivalence between $SL(r, \mathbb{C})$-bundles on Z and spectral data consisting of line bundles (and their degenerations) on spectral covers $C \subset Z$ which are of degree r over S. This equivalence has been used extensively to construct vector bundles on elliptic fibrations and to study their moduli [FMW97, Don97, BJPS97].

For many applications it is important to remove the requirement of the existence of a section, i.e. to allow genus one fibrations. This could be done in two ways.

The 'spectrum' of a degree zero semistable rank r bundle on a genus one curve E consists of r points in the Jacobian $\mathrm{Pic}^0(E)$, rather than in $E = \mathrm{Pic}^1(E)$ itself. So one approach is to consider spectral covers C contained in the relative Jacobian $\mathrm{Pic}^0(Z/S)$. But the spectral data in this case no longer involves a line bundle on C; instead, it lives in a certain non-trivial gerbe, or twisted form of $\mathrm{Pic}(C)$. So the essential problem becomes the analysis of this gerbe.

The second approach is to find an elliptic fibration $\pi : X \to B$ together with a group G acting compatibly on X and B (but not preserving the section of π) such that the action on X is fixed point free and the quotient is the original $Z \to S$. One can then use the Fourier-Mukai transform to construct vector bundles on X. The problem becomes the determination of conditions for such a bundle on X to be G-equivariant, hence to descend to Z. Equivalently we need to know the action of each $g \in G$ on spectral data. This is the restriction of the action on $D^b(X)$ of the Fourier-Mukai conjugate $\boldsymbol{FM}^{-1} \circ g^* \circ \boldsymbol{FM}$ of g^*. This will be referred to as the *spectral action* of g. Unfortunately, the spectral action can be quite complicated: both global vector bundles on X and sheaves supported on C can go to complexes on X of amplitude greater than one.

In this paper, we work out such a spectral action in one class of examples consisting of special rational elliptic surfaces. In the second part [DOPWa] of this paper we use this analysis to construct special bundles on certain non-simply connected smooth Calabi-Yau threefolds. These special bundles in turn are the main ingredient for the construction of Heterotic M-theory vacua having the Standard Model symmetry group $SU(3) \times SU(2) \times U(1)$ and three generations of quarks and leptons. The physical significance of such vacua is explained in [DOPWb] and was the original motivation of this work.

Here is an outline of the paper. We begin in section 2 with a review of the basic properties of rational elliptic surfaces. Within the eight dimensional moduli space of all rational elliptic surfaces we focus attention on a five dimensional family of rational elliptic surfaces admitting a particular involution τ, and then we restrict further to a four dimensional family of surfaces with reducible fibers. This seems to be the simplest family of surfaces for which one needs the full force of Theorem 7.1: for general surfaces in the five dimensional family, the spectral involution $\boldsymbol{T} := \boldsymbol{FM}^{-1} \circ \tau^* \circ \boldsymbol{FM}$ of τ takes line bundles to line bundles, while in the four dimensional subfamily it is possible for $\boldsymbol{T}$ to take a line bundle to a complex which can not be represented by any single sheaf. We study the five dimensional family in section 3 and the four dimensional subfamily in section 4. This section concludes, in subsection 4.3, with a synthetic construction of the surfaces in the four dimensional subfamily. This construction maybe less motivated than the original a priori analysis we use, but it is more concise and we hope it will make the exposition more accessible.

In the remainder of the paper we work out the actions of τ, $\boldsymbol{FM}$, $\boldsymbol{T}$, first at the level of cohomology in sections 5 and 6, and then on the derived category in section 7. The main result is Theorem 7.1, which says that $\boldsymbol{T}$ behaves like a fairly simple affine transformation *away from the union of those fiber components which do not intersect the zero section.* A corollary is that for spectral curves which do not intersect the extra vertical components, all the complications disappear. This fact together with the cohomological formulas from sections 5 and 6 will be used in [DOPWa] to build invariant vector bundles on a family of Calabi-Yau threefolds constructed from the rational elliptic surfaces in our four dimensional subfamily.

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2 Rational elliptic surfaces

A rational elliptic surface is a rational surface B which admits an elliptic fibration $\beta : B \to \mathbb{P}^1$. It can be described as the blow-up of the plane $\mathbb{P}^2$ at nine points $A_1, \ldots, A_9$ which are the base points of a pencil $\{f_t\}_{t\in\mathbb{P}^1}$ of cubics. The map β is recovered as the anticanonical map of B and the proper transform of f_t is $\beta^{-1}(t)$.

In particular the topological Euler characteristic of B is $\chi(B) = \chi(\mathbb{P}^2) + 9 = 12$. For a generic B the map β has twelve distinct singular fibers each of which has a single node. For future use we denote by $B^{\#} \subset B$ the open set of regular points of β and we set $\beta^{\#} := \beta_{|B^{\#}}$.

Under mild general position requirements [DPT80] each subset of eight of these points determines the pencil of cubics and hence the ninth point. In particular we see that the rational elliptic surfaces depend on $2 \cdot 8 - \dim \mathbb{P}GL(3, \mathbb{C}) = 8$ parameters.

Let $e_1, \ldots, e_9$ be the exceptional divisors in B corresponding to the A_i's. Let ℓ be the preimage of the class of a line in $\mathbb{P}^2$ and let $f := \beta^*\mathcal{O}_{\mathbb{P}^1}(1)$. Note that

$$f = -K_B = 3\ell - \sum_{i=1}^{9} e_i$$

and that $\ell, e_1, \ldots, e_9$ form a basis of $H^2(B, \mathbb{Z})$.

The curves $e_1, e_2, \ldots, e_9$ are sections of the map $\beta : B \to \mathbb{P}^1$. Choosing a section $e : \mathbb{P}^1 \to B$ determines a group law on the fibers of $\beta^{\#}$. The inversion for this group law is an involution on $B^{\#}$ which for a general B extends to a well defined involution $(-1)_{B,e} : B \to B$. When B or e are understood from the context we will just write $(-1)_B$ or (-1). The involution $(-1)_{B,e}$ fixes the section e as well as a tri-section of β which parameterizes the non-trivial points of order two. The quotient $W_\beta/(-1)_{B,e}$ is a smooth rational surface which is ruled over the base $\mathbb{P}^1$. For a general B this quotient is the Hirzebruch surface $\mathbb{F}_2$ and the image of e is the exceptional section of $\mathbb{F}_2$. This gives yet another realization of B as a branched double cover of $\mathbb{F}_2$.

A convenient way to describe the involution $(-1)_{B,e}$ is through the Weierstrass model $w : W_\beta \to \mathbb{P}^1$ of $B \underset{e}{\overset{\beta}{\rightleftarrows}} \mathbb{P}^1$.

The model W_β is described explicitly as follows. By relative duality $R^1\beta_*\mathcal{O}_B \cong \mathcal{O}_{\mathbb{P}^1}(-1)$. This implies that $\beta_*\mathcal{O}_B(3e) = (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus$

$\mathcal{O}_{\mathbb{P}^1}(3))^\vee$. Let

$$p : P := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3)) \to \mathbb{P}^1.$$

be the natural projection. The linear system $\mathcal{O}_B(3e)$ defines a map $\nu : B \to P$ compatible with the projections. The Weierstrass model W_β is defined to be the image of this map. It is given explicitly by an equation

$$y^2 z = x^3 + (p^* g_2) x z^2 + (p^* g_3) z^3$$

where $g_2 \in H^0(\mathcal{O}_{\mathbb{P}^1}(4))$ and $g_3 \in H^0(\mathcal{O}_{\mathbb{P}^1}(6))$ and x, y and z are the natural sections of $\mathcal{O}_P(1) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(2)$, $\mathcal{O}_P(1) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(3)$ and $\mathcal{O}_P(1)$ respectively.

In terms of W_β the section e is given by $x = z = 0$ and the involution $(-1)_{B,e}$ sends y to $-y$. The tri-section of fixed points of $(-1)_{B,e}$ is given by $y = 0$.

The Mordell-Weil group $\mathbb{MW} = \mathbb{MW}(B, e)$ is the group of sections of β. As a set $\mathbb{MW}$ is the collection of all sections of $\beta : B \to \mathbb{P}^1$ or equivalently all sections of $\beta^\# : B^\# \to \mathbb{P}^1$. The group law on $\mathbb{MW}$ is induced from the addition law on the group scheme $\beta^\# : B^\# \to \mathbb{P}^1$ and so e corresponds to the neutral element in $\mathbb{MW}(B, e)$. For a section $\xi \subset B$ we will put $[\xi]$ for the corresponding element of $\mathbb{MW}$. Note that the natural map

$$c_1 : \mathbb{MW}(B, e) \to \mathrm{Pic}(B), \qquad [\xi] \mapsto \mathcal{O}_B(\xi).$$

is not a group homomorphism. When written out in coordinates, it involves both a linear part and a quadratic term (see e.g. [Man64]). However, when B is smooth the map c_1 induces a linear map to a quotient of $\mathrm{Pic}(B)$ which describes $\mathbb{MW}(B, e)$ completely. Indeed, let B be smooth and let $\mathcal{T} \subset \mathrm{Pic}(B)$ be the sublattice generated by e and all the components of the fibers of β. Then c_1 induces a map

$$\bar{c}_1 : \mathbb{MW}(B, e) \to \mathrm{Pic}(B)/\mathcal{T}, \qquad [\xi] \mapsto (\mathcal{O}_B(\xi) \mod \mathcal{T})$$

which is a linear isomorphism [Shi90, Theorem 1.3]

There is a natural group homomorphism $t : \mathbb{MW} \to \mathrm{BirAut}(B)$ assigning to each section $\xi \in \mathbb{MW}$ the birational automorphism $t_\xi : B \dashrightarrow B$, which on the open set $B^\#$ is just translation by ξ with respect to the group law determined by e. When $\beta : B \to \mathbb{P}^1$ is relatively minimal the map t_ξ extends canonically to a biregular automorphism of B [Kod63, Theorem 2.9].

3 Special rational elliptic surfaces

In the second part of this paper [DOPWa] we will work with Calabi-Yau threefolds X which are elliptically fibered over a rational elliptic surface B. Any involution τ_X on an elliptic CY $\pi : X \to B$ commuting with π induces (either the identity or) an involution τ_B on the base B. In order for τ_X to act freely on X we need the fixed points of τ_B to be disjoint from the discriminant of π. If B is a rational elliptic surface, then the discriminant of π is a section in $K_B^{-12} = \mathcal{O}_B(12f)$ and so $(-1)_B$ will not do. We want to describe some special rational elliptic surfaces which admit additional involutions. Within the 8 dimensional family of rational elliptic surfaces we describe first a 5 dimensional family of surfaces which admit an involution α_B. The fixed locus of α_B has the right properties but it turns out that α_B does not lift to a free involution on X. However, one can easily show that each α_B can be corrected by a translation t_ζ (for a special type of section ζ) to obtain an additional involution τ_B which does the job. Unfortunately the general member of the 5 dimensional family leads to a Calabi-Yau manifold which does not admit any bundles satisfying all the constraints required by the Standard Model of particle physics (see [DOPWa]). We therefore specialize further to a 4 dimensional family of surfaces for which the extra involution τ_B can be constructed in an explicit geometric way. This provides some extra freedom which enables us to carry out the construction. The involution α_B fixes one fiber of β and four points in another fiber. The involution τ_B fixes only four points in one fiber. A special feature of the 4 dimensional family is that it consists of B's for which β has at least two I_2 fibers. This translates into a special position requirement on the nine points in $\mathbb{P}^2$. Another special feature of the 4 dimensional family is seen in the double cover realization of B where the quotient $B/(-1)$ becomes $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ instead of $\mathbb{F}_2$.

In the next several sections we will describe the structure of the rational elliptic surfaces that admit additional involutions. This rather extensive geometric analysis is ultimately distilled into a fairly simple synthetic construction of our surfaces which is explained in section 4.3. The impatient reader who is interested only in the end result of the construction and wants to avoid the tedious geometric details is advised to skip directly to section 4.3.

3.1 Types of involutions on a rational elliptic surfaces

Consider a smooth rational elliptic surface $B \underset{e}{\overset{\beta}{\rightleftarrows}} \mathbb{P}^1$ with a fixed section. For any automorphism τ_B of B we have $\tau_B^* K_B \cong K_B$. Since $K_B^{-1} = \beta^* \mathcal{O}_{\mathbb{P}^1}(1)$ this implies that τ_B induces an automorphism $\tau_{\mathbb{P}^1} : \mathbb{P}^1 \to \mathbb{P}^1$. If τ_B is an involution we have two possibilities: either $\tau_{\mathbb{P}^1} = \mathrm{id}_{\mathbb{P}^1}$ or $\tau_{\mathbb{P}^1}$ is an involution of $\mathbb{P}^1$.

Both of these cases occur and lead to Calabi-Yau manifolds with freely acting involutions. For concreteness here we only treat the case when $\tau_{\mathbb{P}^1}$ is an involution. The case $\tau_{\mathbb{P}^1} = \mathrm{id}_{\mathbb{P}^1}$ can be analyzed easily in a similar fashion.

If $\tau_{\mathbb{P}^1}$ is an involution, then $\tau_{\mathbb{P}^1}$ will have two fixed points on $\mathbb{P}^1$ which we will denote by $0, \infty \in \mathbb{P}^1$. Note that every involution on $\mathbb{P}^1$ is uniquely determined by its fixed points and so specifying $\tau_{\mathbb{P}^1}$ is equivalent to specifying the points $0, \infty \in \mathbb{P}^1$. Next we classify the types of involutions on B that lift a given involution $\tau_{\mathbb{P}^1}$.

Lemma 3.1. *Let $\beta : B \to \mathbb{P}^1$ be a rational elliptic surface and let $\tau_{\mathbb{P}^1} : \mathbb{P}^1 \to \mathbb{P}^1$ be a fixed involution. There is a canonical bijection*

$$\left\{ \begin{array}{l} \textit{Involutions } \tau_B : B \to B, \textit{ satis-} \\ \textit{fying } \tau_{\mathbb{P}^1} \circ \beta = \beta \circ \tau_B. \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \textit{Pairs } (\alpha_B, \zeta) \textit{ consisting of:} \\ \bullet \textit{ An involution } \alpha_B : B \to B, \textit{ satisfying } \tau_{\mathbb{P}^1} \circ \beta = \beta \circ \alpha_B \textit{ which leaves the zero section invariant, i.e. } \alpha_B(e) = e. \\ \bullet \textit{ A section } \zeta \textit{ of } \beta \textit{ satisfying } \alpha_B(\zeta) = (-1)_B(\zeta). \end{array} \right\}$$

Proof. Let $\tau_B : B \to B$ be such that $\tau_{\mathbb{P}^1} \circ \beta = \beta \circ \tau_B$. Put $\zeta = \tau_B(e)$ for the image of the zero section under τ_B and let $\alpha_B = t_{-\zeta} \circ \tau_B$.

Then α_B is an automorphism of B which induces $\tau_{\mathbb{P}^1}$ on $\mathbb{P}^1$ and preserves the zero section $e \subset B$. So $\alpha_B^2 : B \to B$ will be an automorphism of B which acts trivially on $\mathbb{P}^1$. But

$$t_{-\zeta} \circ \tau_B = \tau_B \circ t_{-\tau^{*-1}_{B/\mathbb{P}^1}(\zeta)}$$

where $\tau^*_{B/\mathbb{P}^1} : \mathrm{Pic}^0(B/\mathbb{P}^1) \to \mathrm{Pic}^0(B/\mathbb{P}^1)$ is the involution on the relative Picard scheme induced from τ_B. In particular we have that α_B^2 must be a translation by a section. Indeed we have

$$\alpha_B^2 = t_{-\zeta} \circ \tau_B \circ \tau_B \circ t_{-\tau^{*-1}_{B/\mathbb{P}^1}(\zeta)} = t_{-\zeta-\tau^{*-1}_{B/\mathbb{P}^1}(\zeta)}. \tag{3.1}$$

Combined with the fact that α_B^2 preserves e (3.1) implies that $\alpha_B^2 = \mathrm{id}_B$. On the other hand, if we use the zero section e to identify $\mathrm{Pic}^0(B/\mathbb{P}^1) \to \mathbb{P}^1$ with $\beta^\# : B^\# \to \mathbb{P}^1$, then $\tau^*_{B/\mathbb{P}^1} = \alpha_B$. Indeed, let $\xi \in \mathrm{Pic}^0(B/\mathbb{P}^1)$ and let $x \in \mathbb{P}^1$ be the projection of the point ξ. Let $f_x \subset B$ be the fiber of β over x. Denote by $m_\xi \in f_x$ the unique smooth point in f_x for which $\mathcal{O}_{f_x}(m_\xi) = \xi \otimes \mathcal{O}_{f_x}(e(x))$. Then by definition $\tau_B^*(\xi)$ is a line bundle of degree zero on f_x such that

$$\mathcal{O}_{f_x}(\tau_B(m_\xi)) = \tau_B\xi \otimes \mathcal{O}_{f_x}(\tau_B(e(x))) = \tau_B\xi \otimes \mathcal{O}_{f_x}(\zeta(x)).$$

In other words under the identification of $\mathrm{Pic}^0(f_x)$ with the smooth locus of f_x via $e(x)$ the line bundle $\tau_B^*\xi \to f_x$ corresponds to the unique point p_ξ of f_x such that

$$\mathcal{O}_{f_x}(p_\xi) = \mathcal{O}_{f_x}(\tau_B(m_\xi)) \otimes \mathcal{O}_{f_x}(e(x) - \zeta(x)).$$

But the right hand side of this identity equals $\mathcal{O}_{f_x}(\alpha_B(m_\xi))$ by definition and so $p_\xi = \alpha_B(m_\xi)$.

Combined with the identity (3.1) and the fact that $t : \mathbb{MW}(B) \to \mathrm{Aut}(B)$ is injective this yields

$$\alpha_B(\zeta) = (-1)_B(\zeta).$$

Conversely, given a pair (α_B, ζ) we set $\tau_B = t_\zeta \circ \alpha_B$. Clearly τ_B is an automorphism of B which induces $\tau_{\mathbb{P}^1}$ on $\mathbb{P}^1$. Furthermore we calculate $\tau_B^2 = t_\zeta \circ \alpha_B \circ t_\zeta \circ \alpha_B = t_\zeta \circ \alpha_B \circ \alpha_B \circ t_{-\zeta} = \mathrm{id}_B$. The lemma is proven. □

The above lemma implies that in order to understand all involutions τ_B it suffices to understand all pairs (α_B, ζ). Since the involutions α_B stabilize e it follows that α_B will have to necessarily act on the Weierstrass model of B. In the next section we analyze this action in more detail.

3.2 The Weierstrass model of B

Let as before $\tau_{\mathbb{P}^1} : \mathbb{P}^1 \to \mathbb{P}^1$ be an involution and let $(t_0 : t_1)$ be homogeneous coordinates on $\mathbb{P}^1$ such that $\tau_{\mathbb{P}^1}((t_0 : t_1)) = (t_0 : -t_1)$ and $0 = (1 : 0)$ and $\infty = (0 : 1)$. Since t_0 and t_1 are a basis of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ and since $\mathcal{O}_{\mathbb{P}^1}(1)$ is generated by global sections we can lift the action of $\tau_{\mathbb{P}^1}$ to $\mathcal{O}_{\mathbb{P}^1}(1)$. For concreteness choose the lift $t_0 \mapsto t_0$, $t_1 \mapsto -t_1$. Since $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)) =$

$S^k H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ we get a lift of the action of $\tau_{\mathbb{P}^1}$ to the line bundles $\mathcal{O}_{\mathbb{P}^1}(k)$ for all k. We will call this action *the standard action* of $\tau_{\mathbb{P}^1}$ on $\mathcal{O}_{\mathbb{P}^1}(k)$. Via the standard action the involution $\tau_{\mathbb{P}^1}$ acts also on the vector bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3)$ and hence we get a standard lift $\tau_P : P \to P$ of $\tau_{\mathbb{P}^1}$ satisfying $\tau_P^* \mathcal{O}_P(1) \cong \mathcal{O}_P(1)$.

Assume that we are given an involution $\alpha_B : B \to B$ which induces $\tau_{\mathbb{P}^1}$ on $\mathbb{P}^1$ and preserves the section e. We have the following

Lemma 3.2. (i) *There exists a unique involution* $\alpha_{W_\beta} : W_\beta \to W_\beta$ *such that the natural map* $\nu : B \to W_\beta$ *satisfies* $\alpha_{W_\beta} \circ \nu = \nu \circ \alpha_B$.

(ii) *Let* $W \subset P$ *be a Weierstrass rational elliptic surface. Then the involution* $\tau_{\mathbb{P}^1}$ *lifts to an involution on* W *which preserves the zero section if and only if* $\tau_P(W) = W$.

(iii) *If* $w : W_\beta \to \mathbb{P}^1$ *is not isotrivial, then* α_{W_β} *is either* $\tau_{P|W_\beta}$ *or* $\tau_{P|W_\beta} \circ (-1)_{W_\beta}$.

Proof. Since $\alpha_B^*(\mathcal{O}_B(e)) \cong \mathcal{O}_B(e)$, there exists an involution on the total space of the bundle $\mathcal{O}_B(e)$ which acts linearly on the fibers and induces the involution α_B on B. Indeed - the square $\gamma \circ \alpha_B^* \gamma$ of the isomorphism $\gamma : \alpha_B^*(\mathcal{O}_B(e)) \widetilde{\to} \mathcal{O}_B(e)$ is a bundle automorphism of $\mathcal{O}_B(e)$ (acting trivially on the base) and so is given by multiplication by some non-zero complex number $\lambda \in \mathbb{C}$. Rescaling the isomorphism γ by $\sqrt{\lambda^{-1}}$ then gives the desired lift.

In this way the involution α_B induces an involution on $\mathcal{O}_e(-e) = \mathcal{O}_{\mathbb{P}^1}(1)$ which lifts the action of $\tau_{\mathbb{P}^1}$. Let us normalize the lift of α_B to $\mathcal{O}_B(e)$ so that the induced action on $\mathcal{O}_e(-e) = \mathcal{O}_{\mathbb{P}^1}(1)$ coincides with the standard action of $\tau_{\mathbb{P}^1}$. Thus the Weierstrass model $W_\beta \subset P$ must be stable under the corresponding τ_P and the restriction of τ_P to W_β is an involution that preserves the zero section of w and induces $\tau_{\mathbb{P}^1}$ on the base. By construction $\tau_{P|W_\beta}$ coincides with the involution induced from α_B up to a composition with $(-1)_{W_\beta}$. This finishes the proof of the lemma. □

We are now ready to construct the Weierstrass models of all surfaces B that admit an involution α_B. Similarly to the proof of Lemma 3.2, the fact that $\tau_P^* \mathcal{O}_P(1) \cong \mathcal{O}_P(1)$ implies that the action of τ_P can be lifted to an action on $\mathcal{O}_P(1)$. Since there are two possible such lifts and they differ by multiplication by $\pm 1 \in \mathbb{C}^\times$ we can use the identification $\mathcal{O}_P(1)_{|B} = \mathcal{O}_B(3e)$ to choose the unique lift that will induce the standard action of $\tau_{\mathbb{P}^1}$ on $\mathcal{O}_{\mathbb{P}^1}(3) = \mathcal{O}_e(-3e)$. With these choices we define an action

$$\tau_P^* : H^0(P, \mathcal{O}_P(r) \otimes p^* \mathcal{O}_P(s)) \to H^0(P, \mathcal{O}_P(r) \otimes p^* \mathcal{O}_P(s))$$

of τ_P on the global sections of any line bundle on P. Note that by construction we have $\tau_P^* x = x$, $\tau_P^* y = y$ and $\tau_P^* z = z$.

Consider the general equation of the Weierstrass model W_β of B:

$$y^2 z = x^3 + (p^* g_2) x z^2 + (p^* g_3) z^3. \tag{3.2}$$

Here $g_2 \in H^0(\mathcal{O}_{\mathbb{P}^1}(4))$ and $g_3 \in H^0(\mathcal{O}_{\mathbb{P}^1}(6))$. The fact $W_\beta \subset P$ is stable under τ_P implies that the image of the Weierstrass equation (3.2) under τ_P^* must be a proportional Weierstrass equation. In particular we ought to have $\tau_{\mathbb{P}^1}^* g_2 = g_2$ and $\tau_{\mathbb{P}^1}^* g_3 = g_3$.

Conversely, for any $g_2 \in H^0(\mathcal{O}_{\mathbb{P}^1}(4))$ and $g_3 \in H^0(\mathcal{O}_{\mathbb{P}^1}(6))$ which are invariant for the standard action of $\tau_{\mathbb{P}^1}$ it follows that τ_P will preserve the Weierstrass surface W given by the equation (3.2). Note that for a generic choice of g_2 and g_3 the surface W will be smooth and so $B = W$, $\alpha_B = \tau_{P|W}$. When W is singular, the surface B is the minimal resolution of singularities of W and hence $\alpha_W = \tau_{P|W}$ determines uniquely α_B by the universal property of the minimal resolution.

Next we describe the fixed locus of α_B. Note that since α_B induces $\tau_{\mathbb{P}^1}$ on $\mathbb{P}^1$ the fixed points of α_B will necessarily sit over the two fixed points of $\tau_{\mathbb{P}^1}$. So in order to understand the fixed locus of α_B it suffices to understand the action of α_B on the two α_B-stable fibers of β - namely $f_0 = \beta^{-1}(0)$ and $f_\infty = \beta^{-1}(\infty)$.

Lemma 3.3. *Let α_B be the involution on B induced from $\tau_{P|W_\beta}$ (with the above normalizations). Then α_B fixes f_0 pointwise and has four isolated fixed points on f_∞, namely the points of order two.*

Proof. The curve f_0 is a smooth cubic in the projective plane

$$P_0 = \mathbb{P}(\mathcal{O}_0 \oplus \mathcal{O}(2)_0 \oplus \mathcal{O}(3)_0),$$

Where $\mathcal{O}(k)_0$ denotes the fiber of the line bundle $\mathcal{O}_{\mathbb{P}^1}(k)$ at the point $0 \in \mathbb{P}^1$. Note that 1, $t_0(0)^2$ and $t_0(0)^3$ span the lines $\mathcal{O}_0$, $\mathcal{O}(2)_0$ and $\mathcal{O}(3)_0$ respectively and so $\tau_{\mathbb{P}^1}$ acts trivially on those lines via its standard action. So if we identify those lines with $\mathbb{C}$ via the basis 1, $t_0(0)^2$ and $t_0(0)^3$, then $X_0 := x_{|P_0}$, $Y_0 := y_{|P_0}$ and $Z_0 := z_{|P_0}$ become identified with sections of the line bundle $\mathcal{O}_{P_0}(1)$ and can be used as homogeneous coordinates on P_0 in which $\tau_{P|P_0} : P_0 \to P_0$ is given by $(X_0 : Y_0 : Z_0) \mapsto (X_0 : Y_0 : Z_0)$. In other words $\tau_{P|P_0}$ acts as the identity on P_0 and hence α_B preserves pointwise the cubic

$$f_0 : \quad Y_0^2 Z_0 = X_0^3 + g_2(1:0) X_0 Z_0^2 + g_3(1:0) Z_0^3 \quad \subset B.$$

In a similar fashion f_∞ is a cubic in the projective plane

$$P_\infty = \mathbb{P}(\mathcal{O}_\infty \oplus \mathcal{O}(2)_\infty \oplus \mathcal{O}(3)_\infty).$$

In this case the lines $\mathcal{O}_\infty$, $\mathcal{O}(2)_\infty$ and $\mathcal{O}(3)_\infty$ have frames 1, t_1^2 and t_1^3 respectively and so $\tau_{\mathbb{P}^1}$ acts trivially on $\mathcal{O}_\infty$ and $\mathcal{O}(2)_\infty$ and by multiplication by -1 on $\mathcal{O}(3)_\infty$. This means that if we use these frames to identify $\mathcal{O}_\infty$, $\mathcal{O}(2)_\infty$ and $\mathcal{O}(3)_\infty$ with $\mathbb{C}$ we get projective coordinates $X_\infty := x_{|P_\infty}$, $Y_\infty := y_{|P_\infty}$ and $Z_\infty := z_{|P_\infty}$ in which $\tau_{P|P_\infty}$ acts as $(X_\infty : Y_\infty : Z_\infty) \mapsto (X_\infty : -Y_\infty : Z_\infty)$ and f_∞ has equation

$$Y_\infty^2 Z_\infty = X_\infty^3 + g_2(0:1)X_\infty Z_\infty^2 + g_3(0:1)Z_\infty^3.$$

In other words $\alpha_{B|f_\infty} = (-1)_{B|f_\infty}$ and so α_B has four isolated fixed points on f_∞ coinciding with the points of order two on f_∞. □

Note that if we consider the involution $\alpha_B \circ (-1)_B$ instead of α_B we will get the same distribution of fixed points with f_0 and f_∞ switched, i.e. we will get four isolated fixed points on f_0 and a trivial action on f_∞.

3.3 The quotient B/α_B.

Let $\beta : B \to \mathbb{P}^1$ be a rational elliptic surface whose Weierstrass model is given by (3.2), with $g_2 \in H^0(\mathcal{O}_{\mathbb{P}^1}(4))$ and $g_3 \in H^0(\mathcal{O}_{\mathbb{P}^1}(6))$ being invariant for the standard action of $\tau_{\mathbb{P}^1}$. For the time being we will assume that g_2 and g_3 are chosen generically so that $B = W$ is smooth and β has twelve I_1 fibers necessarily permuted by $\tau_{\mathbb{P}^1}$.

We have a commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & B/\alpha_B \\ {\scriptstyle\beta}\downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow[\text{sq}]{} & \mathbb{P}^1 \end{array}$$

where $\text{sq} : \mathbb{P}^1 \to \mathbb{P}^1$ is the squaring map $(t_0 : t_1) \mapsto (t_0^2 : t_1^2)$.

Now by the analysis of the fixed points of α_B above we have that $B/\alpha_B \to \mathbb{P}^1$ is a genus one fibration which has six I_1 fibers. Furthermore we saw that the only singularities of B/α_B are four singular points of type A_1 sitting on the fiber over $\infty = (0:1) \in \mathbb{P}^1$.

Lemma 3.4. *Assume that B is Weierstrass.*

(i) *The minimal resolution $\widehat{B/\alpha_B}$ of B/α_B is a rational elliptic surface with a $6I_1 + I_0^*$ configuration of singular fibers and $B/\alpha_B \to \mathbb{P}^1$ is its Weierstrass model.*

(ii) *The surface B is the unique double cover of B/α_B whose branch locus consists of the fiber of $B/\alpha_B \to \mathbb{P}^1$ over $0 = (1:0) \in \mathbb{P}^1$ and the four singular points of B/α_B.*

Proof. By construction $\widehat{B/\alpha_B} \to \mathbb{P}^1$ is a genus one fibered surface with seven singular fibers - six fibers of type I_1 (i.e. the images of the twelve I_1 fibers of β under the quotient map $B \to B/\alpha_B$) and one I_0^* fiber (i.e. the fiber of $\widehat{B/\alpha_B} \to \mathbb{P}^1$ over $\infty \in \mathbb{P}^1$). Moreover since the section $e : \mathbb{P}^1 \to B$ is stable under α_B we see that $e(\mathbb{P}^1)/\alpha_B \subset B/\alpha_B$ will again be a section of the genus one fibration that passes through one of the singular points. So the proper transform of $e(\mathbb{P}^1)/\alpha_B$ in $\widehat{B/\alpha_B}$ will be a section of $\widehat{B/\alpha_B} \to \mathbb{P}^1$ which intersects the I_0^* fiber at a point on one of the four non-multiple components. □

In fact the quotient $B \to B/\alpha_B$ can be constructed directly as a double cover of the quadric $Q \cong \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. In particular this gives a geometric construction of B as an iterated double cover of Q.

Lemma 3.5. *Every rational elliptic surface with $6I_1 + I_0^*$ configuration of singular fibers can be obtained as a minimal resolution of a double cover of the quadric Q branched along a curve $M \in \mathcal{O}_Q(2,4)$ which splits as a union of two curves of bidegrees $(1,4)$ and $(1,0)$ respectively.*

Proof. Indeed consider a curve $T \subset Q$ of bidegree $(1,4)$ and a ruling $r \subset Q$ of type $(1,0)$. Assume for simplicity that T is smooth and that T and r intersect transversally. The double cover W_M of Q branched along $M := T \cup r$ is singular at the ramification points sitting over the four points in $T \cap r$. The curve T is of genus zero and so for a general T the four sheeted covering map $p_{1|T} : T \to \mathbb{P}^1$ will have six simple ramification points. Thus

$$W_M \to Q \overset{p_1}{\to} \mathbb{P}^1$$

has six singular fibers of type I_1 and one fiber passing trough the four singularities of W_M.

Let $s \subset Q$ be any ruling of type $(0,1)$ that passes trough one of the points in $T \cap r$. Then s intersects M at one double point and so the preimage of s in W_M splits into two sections of the elliptic fibration $W_M \to \mathbb{P}^1$ that intersect at one of the singular points of W_M. This implies (as promised) that the minimal resolution $\widehat{W}_M$ of W_M is a rational elliptic surface of type $6I_1 + I_0^*$ and that W_M is its Weierstrass form.

Alternatively we can construct $\widehat{W}_M$ as follows. Label the four points in $T \cap r$ as $\{P_1, P_2, P_3, P_4\}$. Consider the blow-up $\phi : \widehat{Q} \to Q$ of Q at the points $\{P_1, P_2, P_3, P_4\}$ and let $\widehat{T}$ and $\hat{r}$ be the proper transforms of T and r under ϕ. We have

$$\mathcal{O}_{\widehat{Q}}(\widehat{T} + \hat{r}) = \phi^* \mathcal{O}_Q(T + r) \otimes \mathcal{O}_{\widehat{Q}} \left(-2 \sum_{i=1}^{4} E_i \right)$$

where $E_i \subset \widehat{Q}$ is the exceptional divisor corresponding to the point P_i. This shows that the line bundle $\mathcal{O}_{\widehat{Q}}(\widehat{T} + \hat{r})$ is uniquely divisible by two in $\mathrm{Pic}(\widehat{Q})$ and so we may consider the double cover of $\widehat{Q}$ branched along $\widehat{T} + \hat{r}$. Since each of the rational curves E_i intersects the branch divisor $\widehat{T} \cup \hat{r}$ at exactly two points it follows that the preimage D_i of E_i in the double cover of $\widehat{Q}$ is a smooth rational curve of self-intersection -2. But if we contract the curves D_i we will obtain a surface with four A_1 singularities which doubly covers Q with branching along $M = T \cup r$, i.e. we will get the surface W_M. In other words the double cover of $\widehat{Q}$ branched along $\widehat{T} + \hat{r}$ must be the surface $\widehat{W}_M$. Let $\psi : W_M \to Q$ and $\hat{\psi} : \widehat{W}_M \to \widehat{Q}$ denote the covering maps and let $\hat{\phi} : \widehat{W}_M \to W_M$ be the blow-up that resolves the singularities of W_M. Hence the elliptic fibrations on W_M and $\widehat{W}_M$ are given by the composition maps $\omega := p_1 \circ \psi : W_M \to \mathbb{P}^1$ and $\hat{\omega} := p_1 \circ \psi \circ \hat{\phi} : \widehat{W}_M \to \mathbb{P}^1$ respectively.

Finally to write W_M as a quotient $W_M = B/\alpha_B$ (respectively $\widehat{W}_M$ as a quotient $\widehat{W}_M = \widehat{B/\alpha_B}$ we proceed as follows. If there exists a Weierstrass rational elliptic surface $\beta : B \to \mathbb{P}^1$ so that $W_M = B/\alpha_B$, then $\kappa : B \to W_M$ will be the unique double cover of W_M branched along the fiber $(W_M)_0 := \omega^{-1}(0)$ and at the four singular points of W_M. In view of the universal property of the blow-up we may instead consider the unique double cover $\hat{\kappa} : \widehat{B} \to \widehat{W}_M$ which is branched along the divisor $(\widehat{W}_M)_0 + \sum_{i=1}^{4} D_i$. To see that such a cover exists observe that $\hat{\omega}^{-1}(\infty)$ is a Kodaira fiber of type I_0^* and we have $\hat{\omega}^{-1}(\infty) = 2V + \sum_{i=1}^{4} D_i$, where $2V = \hat{\psi}^*(\hat{r})$ is the double component of $\hat{\omega}^{-1}(\infty)$. This yields

$$\mathcal{O}_{\widehat{W}_M}\left((\widehat{W}_M)_0 + \sum_{i=1}^{4} D_i\right) = \hat{\omega}^*\mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{O}_{\widehat{W}_M}(-2V)$$

and so $\mathcal{O}_{\widehat{W}_M}((\widehat{W}_M)_0 + \sum_{i=1}^4 D_i)$ is divisible by two in $\mathrm{Pic}(\widehat{W}_M)$. But from the construction of $\widehat{W}_M$ it follows immediately that $\pi_1(\widehat{W}_M) = 0$ and so $\mathrm{Pic}(\widehat{W}_M)$ is torsion-free. Due to this there is a unique square root of the line bundle $\mathcal{O}_{\widehat{W}_M}((\widehat{W}_M)_0 + \sum_{i=1}^4 D_i)$ and we get a unique root cover $\hat{\kappa} : \widehat{B} \to \widehat{Q}$ as desired.

Let $\widehat{D}_i \subset \widehat{B}$ denote the component of the ramification divisor of $\hat{\kappa}$ which maps to D_i. Note that each $\widehat{D}_i$ is a smooth rational curve and that since $\hat{\kappa}^* D_i = 2\widehat{D}_i$ we have

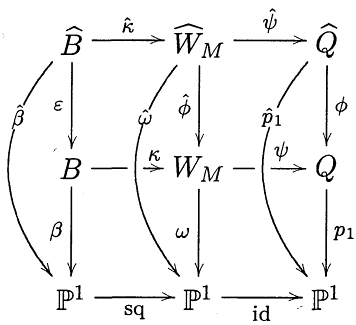
$$\widehat{D}_i \cdot \widehat{D}_i = \frac{1}{4}\hat{\kappa}^*(D_i^2) = \frac{1}{4} \cdot 2 \cdot D_i^2 = \frac{1}{4} \cdot 2 \cdot (-2) = -1.$$

Therefore we can contract the disjoint (-1) curves $\{\widehat{D}_i\}_{i=1}^4$ to obtain a smooth surface B which covers W_M two to one with branching exactly along $(W_M)_0$ and the the four singular points of W_M. If we now denote the covering involution of $\kappa : B \to W_M$ by α_B we have $W_M = B/\alpha_B$ and $\widehat{W}_M = \widehat{B/\alpha_B}$. This construction is clearly invertible, so the lemma a is proven. □

Corollary 3.6. *All rational elliptic surfaces $\beta : B \to \mathbb{P}^1$ which admit an involution α_B, which preserves the zero section e of β and induces an involution on $\mathbb{P}^1$, form a five dimensional irreducible family.*

Proof. According to lemma 3.5 every such surface B determines and is determined by the curve $M = T \cup r \subset Q$ and by the choice of a smooth fiber $(W_M)_0$ of W_M. The curve M depends on $\dim|\mathcal{O}_Q(1,4)| + \dim|\mathcal{O}_Q(1,0)| - \dim \mathrm{Aut}(Q) = 9 + 1 - 6 = 4$ parameters. Adding one more parameter for the choice of $(W_M)_0$ we obtain the statement of the corollary. □

It is convenient to assemble all the surfaces and maps described above in the following commutative diagram:



where the maps ϕ, $\hat{\phi}$ and ε are blow-ups. The maps ψ, $\hat{\psi}$, κ and $\hat{\kappa}$ are double covers and ω, $\hat{\omega}$, β and $\hat{\beta}$ are elliptic fibrations.

Now we are ready to look for the involutions τ_B.

Let B and α_B be as in the previous section. As explained in Section 3.1, in order to describe all possible involutions τ_B we need to describe all sections $\zeta : \mathbb{P}^1 \to B$ such that $\alpha_B^* \zeta = (-1)_B^* \zeta$.

Remark 3.7. *The existence of such a section ζ can be shown by solving an equation in the group* $\mathbb{MW}$. *For this, observe that since α_B preserves the fibers of β it must send a section to a section. Thus α_B induces a bijection $\alpha_{\mathbb{MW}} : \mathbb{MW} \to \mathbb{MW}$, which is uniquely characterized by the property*

$$c_1(\alpha_{\mathbb{MW}}([\xi])) = \mathcal{O}_B(\alpha_B(\xi)).$$

Also, by the definition of $(-1)_B$ we know that $c_1(-[\xi]) = (-1)_B(\xi)$ and hence we need to show the existence of a section ζ, such that $\alpha_{\mathbb{MW}}([\zeta]) = -[\zeta]$.

The first step is to observe that since the isomorphism $\tau_{\mathbb{P}^1}^* B \widetilde{\to} B$ preserves the group structure on the fibers, the induced bijection $\alpha_{\mathbb{MW}}$ on sections is actually a group automorphism.

Next note that for the general B in the five dimensional family from Corollary 3.6, the lattice $\mathcal{T}$ has rank two since the general such B has only singular fibers of type I_1 and so $\mathcal{T} = \mathbb{Z}e \oplus \mathbb{Z}f$. Moreover $\alpha_{B|\mathcal{T}} = \mathrm{id}_{\mathcal{T}}$, and so the space of anti-invariants of α_B^* acting on $\mathrm{Pic}(B) \otimes \mathbb{Q}$ *injects into the space of anti-invariants of $\alpha_{\mathbb{MW}}$. But in Section 3.3 we showed that B/α_B is again a rational elliptic surface which has four A_1 singularities. In particular* $\mathrm{rk}(\mathrm{Pic}(B/\alpha_B)) = 6$ *and so there is a 4-dimensional space of anti-invariants for the α_B^* action on* $\mathrm{Pic}(B) \otimes \mathbb{Q}$.

This implies that $\alpha_{\mathbb{MW}}$ has a 4 dimensional space of anti-invariants on $\mathbb{MW} \otimes \mathbb{Q}$ and hence we can find a section $\zeta \neq e$ with $\alpha_{\mathbb{MW}}([\zeta]) = -[\zeta]$. The involution τ_B corresponding to (α_B, ζ) will have only four isolated fixed points.

4 The four dimensional subfamily of special rational elliptic surfaces

From now on we will restrict our attention to a 4-dimensional subfamily of the 5-dimensional family of surfaces of Corollary 3.6. We do this for two reasons:

- Mathematically, this seems to be the simplest family where the full range of possible behavior of the spectral involution $\boldsymbol{T} = \boldsymbol{FM}^{-1} \circ \tau_B^* \circ \boldsymbol{FM}$ is present, see Proposition 7.1. Indeed, for a generic surface in the five dimensional family, $\boldsymbol{T}$ takes line bundles to line bundles, so everything can be rephrased without the use of the derived category.
- In terms of our motivation from the physics, this specialization is needed for the construction of the Standard Model bundles. By taking fiber products of surfaces from the five dimensional family one indeed gets a smooth Calabi-Yau with a freely acting involution. However, it turns out that for a generic such B, the cohomology of the resulting Calabi-Yau is not rich enough to lead to invariant vector bundles satisfying the Chern class constraints from [DOPWa].

4.1 The quotient B/τ_B

The starting point of the construction of the four dimensional family is the following simple observation: since ζ must satisfy $\alpha_B^*(\zeta) = (-1)_B^*(\zeta)$ it will help to work with rational elliptic surfaces B for which we know the geometric relationship between the two involutions α_B and $(-1)_B$. In the previous section we interpreted the involution α_B as the covering involution of the map κ. On the other hand the involution $(-1)_B$ was the group inversion along the fibers of β corresponding to a zero section $e : \mathbb{P}^1 \to B$ which was chosen to be one of the two components of the preimage in B of a ruling of type $(0, 1)$ in Q which passes trough one of the four points in $T \cap r$. Since in this setup the involutions α_B and $(-1)_B$ are generically unrelated it is natural to look for a special configuration of the curves T and r for which $(-1)_B$ can be related to the maps κ and ψ.

Lemma 4.1. *Consider the family of rational elliptic surfaces B obtained as an iterated double cover $B \to W_M \to Q$ for which the component T of the branch curve M is split further into a union $T = s \cup \mathfrak{T}$ where s is a ruling of Q of type $(0,1)$ and $\mathfrak{T}$ is a curve of type $(1,3)$. Let as before e be the section of B mapping to $s \subset Q$. Then we have:*

(i) *The involution $(-1)_{B,e}$ is a lift of the covering involution of the double cover $\psi : W_M \to \mathbb{P}^1$.*

(ii) *For a general pair (B, α_B) corresponding to a branch curve $M = s \cup \mathfrak{T} \cup r$ there exist three pairs of sections of β labeled by the non-trivial points of order two on f_0 and such that the two members of each pair are interchanged both by α_B and $(-1)_B$.*

Proof. If the curve $\mathfrak{T}$ is chosen to be general and smooth, then the branch curve M has five nodes $\{P, P_1, P_2, P_3, P_4\}$. Here as before $\{P_1, P_2, P_3, P_4\} = T \cap r$ and the extra point P is the intersection point of the curves $\mathfrak{T}$ and s.

Let $\{p, p_1, p_2, p_3, p_4\} \subset W_M$ denote the corresponding singularities of W_M. Observe that for a general choice of the curve $\mathfrak{T}$ and the point $0 \in \mathbb{P}^1$ the singularity $p \in W_M$ is not contained in the branch locus $(W_M)_0 \cup \{p_1, p_2, p_3, p_4\}$ of the map κ. In particular the double cover of W_M branched along $(W_M)_0 \cup \{p_1, p_2, p_3, p_4\}$ will have two A_1 singularities at the two preimages $\bar{p}_1$ and $\bar{p}_2$ of the point p. In order to get a smooth rational elliptic surface we have to to blow up this two points. Abusing slightly the notation we will denote by B the resulting smooth surface and by $\kappa : B \to W_M$ the composition of the blow-up map with the double cover of W_M branched along $(W_M)_0 \cup \{p_1, p_2, p_3, p_4\}$. Let $n_1, n_2 \subset B$ denote the exceptional curves corresponding to $\bar{p}_1$ and $\bar{p}_2$ and let o_1, o_2 denote proper transforms in B of the two preimages of the fiber $\omega^{-1}(\omega(p))$ in the double cover of W_M branched along $(W_M)_0 \cup \{p_1, p_2, p_3, p_4\}$. Here we have labeled o_1 and o_2 so that $\bar{p}_1 \in o_1$ and $\bar{p}_2 \in o_2$. From this picture it is clear that $\beta : B \to \mathbb{P}^1$ is a smooth rational elliptic surface with a $8I_1 + 2I_2$ configuration of singular fibers which is symmetric with respect to the involution $\tau_{\mathbb{P}^1}$.

Furthermore the two I_2 fibers of β are just the curves $o_1 \cup n_1$ and $o_2 \cup n_2$ and the two fixed points $\{0, \infty\}$ of $\tau_{\mathbb{P}^1}$ correspond to two smooth fibers f_0 and f_∞ of β. Note also that the proper transform of the section $s \subset Q$ via the generically finite map $\psi \circ \kappa : B \to Q$ is an irreducible rational curve $e \subset B$ which is a section of $\beta : B \to \mathbb{P}^1$. Moreover the inversion $(-1)_B$ with respect to e commutes with the covering involution α_B for the map κ and descends to an inversion $(-1)_{W_M}$ along the fibers of the elliptic fibration $\omega : W_M \to \mathbb{P}^1$ which fixes the image of e pointwise. But by construction the image of e in W_M is just the component of the ramification divisor of the cover $\psi : W_M \to Q$ sitting over $s \subset Q$. In particular $(-1)_{W_M}$ is just the covering involution for the map ψ.

We are now ready to construct a section $\zeta : \mathbb{P}^1 \to B$ of β satisfying $\alpha_B^*(\zeta) = (-1)_B(\zeta)$. Indeed, assume that such a section exists.

Due to the fact that $\alpha_{B|f_0} = \mathrm{id}_{f_0}$ we have $\zeta(0) = -\zeta(0)$ i.e. $\zeta(0)$ is a point of order two on f_0. Now from the Weierstrass equation (3.2) of B it is clear that the general B cannot have monodromy $\Gamma_0(2)$ and so without a loss of generality we may assume that $\zeta \neq -\zeta = \alpha_B^*\zeta$. Consider now the image $\kappa(\zeta) \subset W_M = B/\alpha_B$ of ζ in W_M. We have $\kappa^{-1}(\kappa(\zeta)) = \zeta \cup \alpha_B^*\zeta$. On the other hand the preimage of the general elliptic fiber of $\omega : W_M \to \mathbb{P}^1$ via κ splits as a disjoint union of two fibers of β and so $\alpha_{B|f_0} = \mathrm{id}_{f_0}$ we have $\zeta(0) = -\zeta(0)$ i.e. $\zeta(0)$ is a point of order two on f_0. Consider now the image $\kappa(\zeta) \subset W_M = B/\alpha_B$ of ζ in W_M. We have $\kappa^{-1}(\kappa(\zeta)) = \zeta \cup \alpha_B^*\zeta$. On the other hand the preimage of the general elliptic fiber of $\omega : W_M \to \mathbb{P}^1$ via κ splits as a disjoint union of two fibers of β and so

$$\kappa(\zeta) \cdot \omega^{-1}(\mathrm{pt}) = \frac{1}{2}\kappa^*(\kappa(\zeta) \cdot \omega^{-1}(\mathrm{pt})) = \frac{1}{2}(\zeta + \alpha^*\zeta) \cdot (2\beta^{-1}(\mathrm{pt})) = 2$$

i.e. the smooth rational curve $\kappa(\zeta)$ is a double section of ω. Moreover the condition $\alpha_B^*\zeta = -\zeta$ combined with the property $\alpha_{B|B_\infty} = (-1)_{B_\infty}$ implies that $(\alpha_B^*)\zeta(\infty) = \zeta(\infty)$ and so the double cover $\omega_{|\kappa(\zeta)} : \kappa(\zeta) \to \mathbb{P}^1$ is branched exactly over the points $0, \infty$. Furthermore since $\zeta(0)$ is a point of order two on f_0 it must lie on the preimage of T in B and so the two ramification points of the cover $\omega_{|\kappa(\zeta)} : \kappa(\zeta) \to \mathbb{P}^1$ must both lie on the ramification divisor of the double cover $\psi : W_M \to Q$ as depicted on Figure 1.

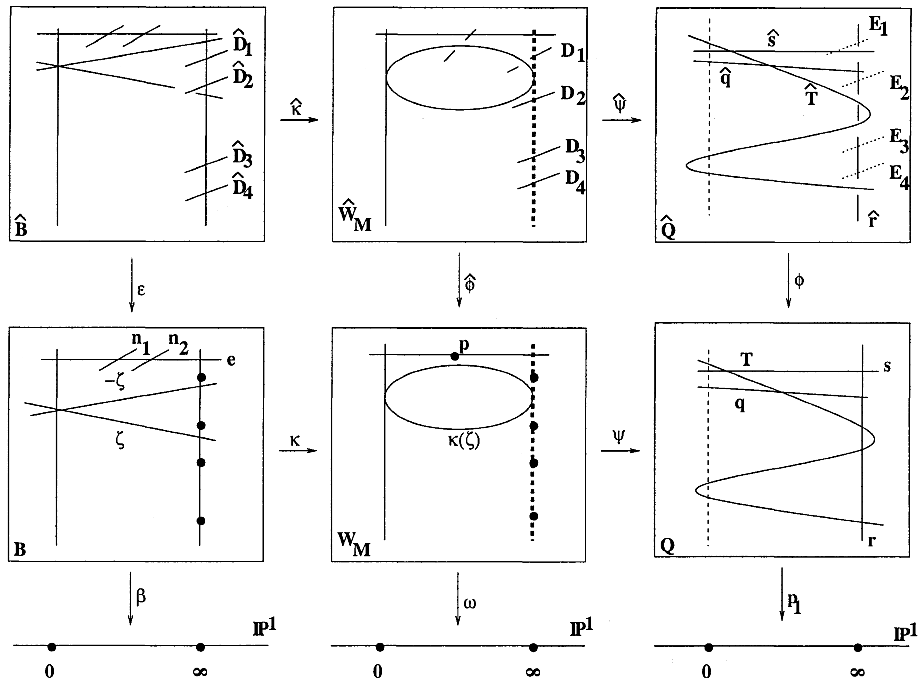


Figure 1: The section ζ

Also note that if we pullback to B the involution of W_M acting along the fibers of ψ we will get precisely $(-1)_B$. Combined with the fact that $\alpha_B^*\zeta = (-1)_B^*\zeta$ this shows that $\kappa(\zeta)$ is stable under the involution of W_M acting along the fibers of ψ and so $\psi^{-1}(\psi(\kappa(\zeta))) = \kappa(\zeta)$. Put $q := \psi(\kappa(\zeta))$. Then q is a smooth rational curve which intersects each of the curves T and r at a single point so that the double cover $\psi_{|\kappa(\zeta)} : \kappa(\zeta) \to q$ is branched exactly at $q \cap (T \cup r)$. So q is the unique ruling of type $(0,1)$ on Q which passes trough the point $\psi(\kappa(\zeta(0))) \in T \cap Q_0$.

Conversely if we start with any ruling q of type $(0,1)$ that passes trough one of the four points in $T \cap f_0$ we see that $\psi^{-1}(q)$ is a smooth rational curve which is a double cover of q with branch divisor $q \cap (T \cup r)$. Since the rulings of type $(1,0)$ pull back to a single fiber of ω via ψ we see that

$$\psi^{-1}(q) \cdot \omega^{-1}(\mathrm{pt}) = \psi^*(q \cdot p_1^{-1}(\mathrm{pt})) = 2q \cdot p_1^{-1}(\mathrm{pt}) = 2,$$

and so q is a double section of the elliptic fibration $\omega : W_M \to \mathbb{P}^1$ which is tangent to the fibers $(W_M)_0$ and $(W_M)_\infty$. Also it is clear that for T and r in general position the point $q \cap r$ is not one of the four points in $T \cap r$ and so the point of contact of $\psi^{-1}(q)$ and $(W_M)_\infty$ is not one of the four isolated branch points of the covering $\kappa : B \to W_M$. So $\psi^{-1}(q)$ intersects the branch locus of κ at a single point with multiplicity two - namely the point of contact of $(W_M)_0$ and $\psi^{-1}(q)$. This implies that the preimage of $\psi^{-1}(q)$ in B splits into two sections of β that intersect at a point on the fiber f_0 and are exchanged both by α_B and $(-1)_B$. The lemma is proven. □

Finally, let τ_B be the involution of B corresponding to the pair (α_B, ζ) constructed in the previous lemma. Then the quotient B/τ_B is again a genus one fibered rational surface which similarly to B/α_B has four A_1 singularities all sitting on fiber over $\infty \in \mathbb{P}^1$. However B/τ_B has also a smooth double fiber and so is only genus one fibered. The minimal resolution of B/τ_B in this case has a $4I_1 + I_2 + I_0^* +_2 I_0$ configuration of singular fibers.

4.2 The basis in $H^2(B, \mathbb{Z})$

In order to describe an integral basis of the cohomology of B we need to find a description of our B as a blow-up of $\mathbb{P}^2$ in the base points of a pencil of cubics.

To achieve this we will use a different fibration on B, namely the fibration

$$B \xrightarrow{\psi \circ \kappa} Q \xrightarrow{p_2} \mathbb{P}^1, \qquad \delta = p_2 \circ \psi \circ \kappa : B \to \mathbb{P}^1.$$

induced from the projection of the quadric Q onto its *second* factor. The fibers of δ can be studied directly in terms of the degree four map $\psi \circ \kappa : B \to Q$ but it is much more instructive to use instead an alternative description of B as a *double cover* of a quadric.

In section 4.1 we saw that the description of B as an iterated double cover

$$B \stackrel{\kappa}{\to} W_M \stackrel{\psi}{\to} Q$$

of the quadric Q yields two commuting involutions α_B and $(-1)_B$ on B. By construction the quotient B/α_B can be identified with the blow-up of the rational elliptic surface W_M at the A_1 singularity $p \in W_M$ sitting over the unique intersection point $\{P\} = s \cap \mathfrak{T}$. In particular if we consider the Stein factorization of the generically finite map $\kappa : B \to W_M$ we get

$$B \to W_\beta \to W_M.$$

Here W_β is the Weierstrass model of $\beta : B \to \mathbb{P}^1$ and $B \to W_\beta$ is the blow-up the two A_1 singularities of W_β and the map $W_\beta \to W_M$ is the double cover branched at $(W_M)_0 \cup \{p_1, p_2, p_3, p_4\}$.

Similarly we can describe the quotients $B/(-1)_B$ and $B/((-1)_B \circ \alpha_B)$ as blow-ups of appropriate double covers of Q. Indeed the curves on Q that play a special role in the description of B as an iterated double cover are: the $(1,3)$ curve $\mathfrak{T}$, the $(0,1)$ ruling s and the $(1,0)$ rulings $r = r_\infty = p_1^{-1}(\infty)$ and $r_0 = p_1^{-1}(0)$.

Consider the double cover $\omega' : W_{M'} \to Q$ branched along the curve $M' = T \cup r_0 = s \cup \mathfrak{T} \cup r_0$ and the double cover $\mathrm{Sq} : \widetilde{Q} \to Q$ branched along the union of rulings $r_0 \cup r_\infty$. Clearly $\widetilde{Q}$ is again a quadric which is just a the fiber product of $p_1 : Q \to \mathbb{P}^1$ with the squaring map $\mathrm{sq} : \mathbb{P}^1 \to \mathbb{P}^1$, i.e. we have a fiber-square

$$\begin{array}{ccc} \widetilde{Q} & \xrightarrow{\mathrm{Sq}} & Q \\ {\scriptstyle \tilde{p}_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ \mathbb{P}^1 & \xrightarrow[\mathrm{sq}]{} & \mathbb{P}^1 \end{array}$$

The preimage $\widetilde{\mathfrak{T}} := \mathrm{Sq}^{-1}(\mathfrak{T}) \subset \widetilde{Q}$ of $\mathfrak{T}$ in $\widetilde{Q}$ is a genus two curve doubly covering $\mathfrak{T}$ with branching at the six points $\mathfrak{T} \cap (r_0 \cup r_\infty)$. Also, the preimage $\tilde{s} = \mathrm{Sq}^{-1}(s)$ is a rational curve doubly covering the ruling s branched at the two points $s \cap (r_0 \cup r_\infty)$. In particular, $\tilde{s}$ is a ruling of type $(0,1)$ on $\widetilde{Q}$. Similarly, if we denote by $\tilde{r}_0$ and $\tilde{r}_\infty$ the two components of the ramification divisor of $\mathrm{Sq} : \widetilde{Q} \to Q$, then $\tilde{r}_0$ and $\tilde{r}_\infty$ are rulings of type $(1,0)$ on $\widetilde{Q}$.

Now it is clear that the Weierstrass model W_β of B can be described as either of the following

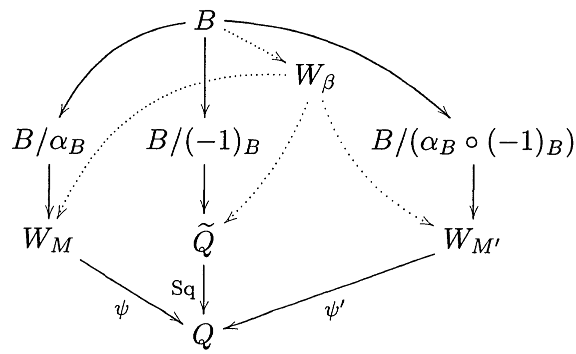
- $W_\beta \to W_M$ is the double cover branched at the fiber $(W_M)_0$ and the four points $\{p_1, p_2, p_3, p_4\}$ of order two of the fiber $(W_M)_\infty$.
- $W_\beta \to W_{M'}$ is the double cover branched at the fiber $(W_M)_\infty$ and the four points of order two of the fiber $(W_M)_0$.
- $W_\beta \to \widetilde{Q}$ is the double cover branched at the curve $\tilde{s} \cup \widetilde{\mathfrak{T}}$.

Furthermore

- The quotient $B/\alpha_B \to W_M$ is the blow-up of W_M at the A_1 singularity p sitting over the point $P \in Q$ of intersection of s and $\mathfrak{T}$. The map $B \to B/\alpha_B$ is the double cover of B/α_B branched at the fiber $(B/\alpha_B)_0$ and the four points of order two of $(B/\alpha_B)_\infty$.
- The quotient $B/(\alpha_B \circ (-1)_B) \to W_{M'}$ is the blow-up of $W_{M'}$ at the A_1 singularity sitting over the point of intersection of s and $\mathfrak{T}$. The map $B \to B/(\alpha_B \circ (-1)_B)$ is the double cover of $B/(\alpha_B \circ (-1)_B)$ branched at the fiber $(B/\alpha_B)_\infty$ and the four points of order two of $(B/\alpha_B)_0$.
- The quotient $B/(-1)_B$ is the blow-up of $\widetilde{Q}$ at the two intersection points of $\tilde{s}$ and $\widetilde{\mathfrak{T}}$. The map $B \to B/(-1)_B$ is the double cover branched at the strict transform of $\tilde{s} \cup \widetilde{\mathfrak{T}}$.

The action of the Klein group $\langle \alpha_B, (-1)_B \rangle$ on B and all of the above maps are most conveniently recorded in the commutative diagram

(4.1)



where the solid arrows in the first and third rows are all double covers, the solid arrows in the middle row are blow-ups and the dotted arrows are Stein factorization maps.

In order to visualize the system of maps (4.1) better it is instructive to label all the double cover maps appearing in (4.1) by a picture of their branch loci. This is recorded in the diagram in Figure 2.

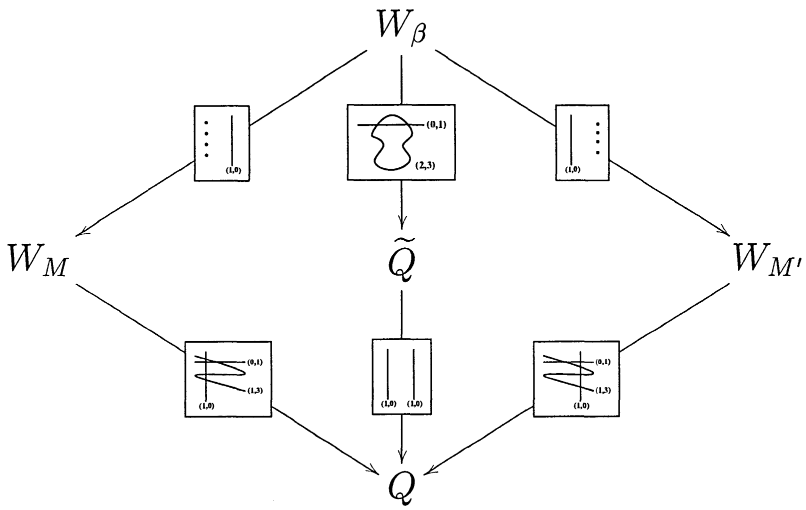


Figure 2: W_β as a double cover of a quadric

There is a definite advantage in interpreting geometric questions on B or W_β on all three surfaces W_M, $W_{M'}$ and $\widetilde{Q}$. For example, by viewing W_β as a double cover of the quadric $\widetilde{Q}$ we can easily describe the fibers of the rational curve fibration $\delta : B \to \mathbb{P}^1$ defined in the beginning of the section. Indeed, due to the commutativity of (4.1) the map $\delta = p_2 \circ \psi \circ \kappa$ decomposes also as

$$\underbrace{B \longrightarrow B/(-1)_B \longrightarrow \widetilde{Q} \xrightarrow{\tilde{p}_2} \mathbb{P}^1}_{\delta},$$

where $\tilde{p}_2 : \widetilde{Q} \to \mathbb{P}^1$ is the projection onto the ruling of type $(1,0)$. In particular we can view each fiber $\delta^{-1}(x)$ of the map $\delta : B \to \mathbb{P}^1$ as the double cover of the fiber $\tilde{p}_2^{-1}(x)$ of $\tilde{p}_2 : \widetilde{Q} \to \mathbb{P}^1$ branched along the degree two divisor $\widetilde{\mathfrak{T}} \cap \tilde{p}_2^{-1}(x) \subset \tilde{p}_2^{-1}(x)$. This shows that the singular fibers of δ are precisely the preimages under the map $B \to \widetilde{Q}$ of $\tilde{s}$ and of those $(0,1)$ rulings of $\widetilde{Q}$ which happen to be tangent to the curve $\widetilde{\mathfrak{T}}$.

Since the curve $\widetilde{\mathfrak{T}}$ is of type $(2,3)$ on $\widetilde{Q}$ we see by adjunction that $\widetilde{\mathfrak{T}}$ must have genus two and so by the Hurwitz formula the double cover map $\tilde{p}_2 : \widetilde{\mathfrak{T}} \to \mathbb{P}^1$ will have six ramification points. This means that there are six rulings of $\widetilde{Q}$ of type $(0,1)$ which are tangent to $\widetilde{\mathfrak{T}}$, i.e. generically δ will have seven singular fibers (see Figure 3). Six of those will be unions of two rational curves meeting at a point and the seventh one will have one rational component occurring with multiplicity two (the preimage in B of the strict transform of $\tilde{s}$ in $B/(-1)_B$) and two reduced rational components n_1 and n_2 (the exceptional divisors of the blow-up $B \to W_\beta$). Notice moreover that (4.1) implies that the preimage in B of the strict transform of $\tilde{s}$ in $B/(-1)_B$ is precisely the zero section e of the elliptic fibration $\beta : B \to \mathbb{P}^1$ and so the non-reduced fiber of δ is just the divisor $2e + n_1 + n_2$ on B.

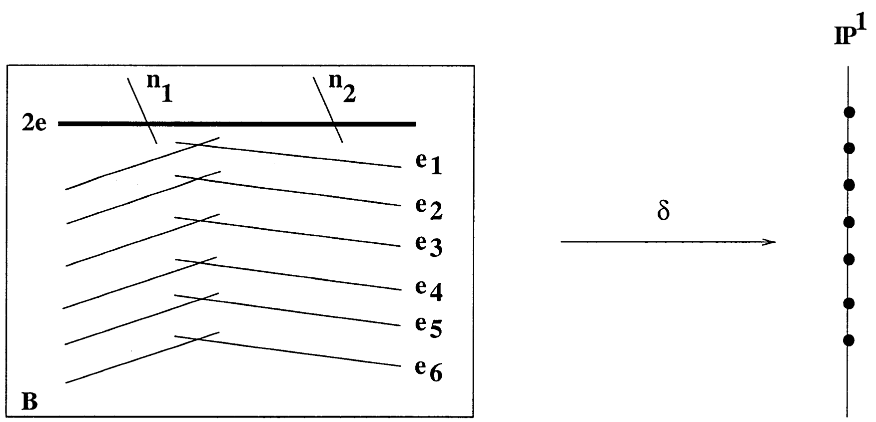


Figure 3: The singular fibers of δ

In fact, one can describe explicitly the $(0,1)$ rulings of $\widetilde{Q}$ that are tangent to the curve $\widetilde{\mathfrak{T}}$. Indeed let $\mathrm{pt} \in r_0 \cap \mathfrak{T}$ be one of the three intersection points of r_0 and $\mathfrak{T}$. Choose (analytic) local coordinates (x,y) on a neighborhood $\mathrm{pt} \in U \subset Q$ so that $\mathrm{pt} = (0,0)$, r_0 has equation $x = 0$ in U and the $(0,1)$ ruling through $\mathrm{pt} \in Q$ has equation $y = 0$ in U. Let $\widetilde{U} \subset \widetilde{Q}$ be the preimage of U in $\widetilde{Q}$. Then there are unique coordinates (u,v) on $\widetilde{U}$ such that the double cover $\widetilde{U} \to U$ is given by $(u,v) \mapsto (u^2, v) = (x,y)$. Due to our genericity assumption[1] the local equation of $\mathfrak{T}$ in U will be $x = ay +$ (higher order terms) for some number a. Thus the pullback of r_0 to $\widetilde{U}$ will be given by $u = 0$ and $\widetilde{\mathfrak{T}}$ will have equation $u^2 = av +$ (higher order terms). Since by construction $v = 0$ is the local equation of a $(0,1)$ ruling of $\widetilde{Q}$ it follows that $\widetilde{\mathfrak{T}}$ is tangent to the three $(0,1)$ rulings of $\widetilde{Q}$ passing through the three intersection points in $\widetilde{\mathfrak{T}} \cap \tilde{r}_0$. In the same way one sees that $\widetilde{\mathfrak{T}}$ is tangent to the three $(0,1)$ rulings of $\widetilde{Q}$ passing through the three intersection points in $\widetilde{\mathfrak{T}} \cap \tilde{r}_\infty$. This accounts for all six $(0,1)$ rulings of $\widetilde{Q}$ that are tangent to $\widetilde{\mathfrak{T}}$.

[1]We are assuming that $\mathfrak{T}$ meets r_0 and r_∞ transversally.

We are now ready to describe B as the blow-up of $\mathbb{P}^2$ at the base locus of a pencil of cubics. Each component of a reduced singular fiber of δ is a curve of self-intersection (-1) on B. For every such fiber choose one of the components and label it by e_i, $i = 1, 2, \dots, 6$ (see Figure 3). Now $e, e_1, e_2, \dots, e_6$ is a collection of seven disjoint (-1) curves on the rational elliptic surface B. The curves n_1 and n_2 are rational (-2) curves on B and so if we contract e each of them will become a (-1) curve. So if we contract $e, e_1, e_2, \dots, e_6$ and after that we contract n_1 we will end up with a Hirzebruch surface. Moreover numerically $e, e + n_1, e_1, e_2, \dots, e_6$ behave like eight disjoint (-1) curves on B and so the result of the contraction of $e, n_1, e_1, e_2, \dots, e_6$ should be $\mathbb{F}_1$. Contracting the infinity section of $\mathbb{F}_1$ we will finally obtain $\mathbb{P}^2$ as the blow down of nine (-1) divisors on B. Let e_7 denote the infinity section of $\mathbb{F}_1$. To make things explicit let us identify e_7 as a curve coming from $\widetilde{Q}$. Denote by $\mathfrak{e} \subset \widetilde{Q}$ the image of e_7 in $\widetilde{Q}$. Then $\mathfrak{e}$ is an irreducible curve which intersects the generic $(0,1)$ ruling at one point. This implies that $\mathfrak{e}$ is of type $(1,k)$ on $\widetilde{Q}$ and so $\mathfrak{e}$ must be a rational curve. In particular the map $e_7 \to \mathfrak{e}$ ought to be an isomorphism and $e_7 \cup (-1)^*_B(e_7)$ is the preimage in B of the strict transform of $\mathfrak{e}$ in $B/(-1)_B$. Equivalently $e_7 \cup (-1)^*_B(e_7)$ is the strict transform in B of the preimage of $\mathfrak{e}$ in W_β. This implies that the preimage of $\mathfrak{e}$ in W_β is reducible and so $\mathfrak{e}$ must have order of contact two with the branch divisor $\tilde{\mathfrak{s}} \cup \widetilde{\mathfrak{T}}$ of the covering $W_\beta \to \widetilde{Q}$ at each point where $\mathfrak{e}$ and $\tilde{\mathfrak{s}} \cup \widetilde{\mathfrak{T}}$ meet. Since $\mathfrak{e} \cdot \tilde{\mathfrak{s}} = (1,k) \cdot (0,1) = 1$ this implies that $\mathfrak{e}$ must pass through one of the two intersection points of $\tilde{\mathfrak{s}} \cap \widetilde{\mathfrak{T}}$ and be tangent to $\widetilde{\mathfrak{T}}$ at $(\mathfrak{e} \cdot \widetilde{\mathfrak{T}} - 1)/2$ points. But

$$\frac{\mathfrak{e} \cdot \widetilde{\mathfrak{T}} - 1}{2} = \frac{(1,k) \cdot (2,3) - 1}{2} = k + 1$$

and so $e_7 \cdot (-1)^*_B e_7 = k + 1$. From here we can calculate k. Indeed, on one hand we know that $e_7^2 = -1$ and so

$$(e_7 + (-1)^*_B e_7)^2 = -2 + 2 + 2k = 2k.$$

On the other hand $e_7 + (-1)^*_B e_7$ is the preimage in B of the strict transform of $\mathfrak{e}$ in $B/(-1)_B$. But $B/(-1)_B$ is simply the blow-up of $\widetilde{Q}$ at the two intersection points of $\tilde{\mathfrak{s}}$ and $\widetilde{\mathfrak{T}}$ and $\mathfrak{e}$ passes trough only one of those points and so the strict transform of $\mathfrak{e}$ in $B/(-1)_B$ has self-intersection $\mathfrak{e}^2 - 1$. In other words

$$(e_7 + (-1)^*_B e_7)^2 = 2(\mathfrak{e}^2 - 1) = 2(2k - 1) = 4k - 2,$$

and so $k = 1$.

Therefore, in order to reconstruct e_7 starting from $\widetilde{Q}$ we need to find a $(1,1)$ curve $\mathfrak{e}$ on $\widetilde{Q}$ which passes through one of the two points in $\tilde{s} \cap \widetilde{\mathfrak{T}}$ and tangent to $\widetilde{\mathfrak{T}}$ at two extra points. But curves like that always exist. Indeed, the linear system $|\mathcal{O}_{\widetilde{Q}}(1,1)|$ embeds $\widetilde{Q}$ in $\mathbb{P}^3$. Pick a point $J \in \tilde{s} \cap \widetilde{\mathfrak{T}}$ and let $j : \widetilde{Q} \dashrightarrow \mathbb{P}^2$ be the linear projection of $\widetilde{Q}$ from that point. Now the $(1,1)$-curves passing through J are precisely the preimages via j of all lines in $\mathbb{P}^2$ and so the curve $\mathfrak{e}$ will be just the preimage under j of a line in $\mathbb{P}^2$ which is bitangent to $j(\widetilde{\mathfrak{T}})$. To understand better the curve $j(\widetilde{\mathfrak{T}}) \subset \mathbb{P}^2$ note that it has degree $(1,1) \cdot (2,3) - 1 = 4$ and that the map $j : \widetilde{\mathfrak{T}} \to j(\widetilde{\mathfrak{T}})$ is a birational morphism. Furthermore any $(1,1)$-curve passing trough J and another point on the $(1,0)$ ruling through J will have to contain the whole $(1,0)$ ruling. Since the $(1,0)$ ruling trough J intersects $\widetilde{\mathfrak{T}}$ at J and two extra pointsJ' and J'', it follows that $j(J') = j(J'')$. Therefore $j(\widetilde{\mathfrak{T}})$ is a nodal quartic in $\mathbb{P}^2$ and the curve $\mathfrak{e} \subset \widetilde{Q}$ corresponds to a bitangent line of this nodal quartic. The normalization of this nodal quartic is just the genus two curve $\widetilde{\mathfrak{T}}$ and the lines in $\mathbb{P}^2$ correspond just to sections in the canonical class $\omega_{\widetilde{\mathfrak{T}}}$ that have poles at the two preimages of the node. But a linear system of degree 4 on a genus two curve is always two dimensional and so the space of lines in $\mathbb{P}^2$ is canonically isomorphic with $|\omega_{\widetilde{\mathfrak{T}}}(J' + J'')|$. In other words, finding the bitangent lines to $j(\widetilde{\mathfrak{T}})$ in $\mathbb{P}^2$ is equivalent to finding all divisors in $|\omega_{\widetilde{\mathfrak{T}}}(J' + J'')|$ of the form $2\mathcal{D}$ where $\mathcal{D}$ is an effective divisor of degree two on $\widetilde{\mathfrak{T}}$. Since every degree two line bundle on a genus two curve is effective we see that finding $\mathfrak{e}$ just amounts to choosing a non-trivial square root of the degree four line bundle $\omega_{\widetilde{\mathfrak{T}}}(J' + J'')$.

Going back to the description of B as the blow-up of $\mathbb{P}^2$ at the base points of a pencil of cubics assume for concreteness that J is the point in $\tilde{s} \cap \widetilde{\mathfrak{T}}$ corresponding to the exceptional curve $n_1 \subset B$. Let $\mathfrak{e} \subset \widetilde{Q}$ be a $(1,1)$ curve which passes trough J and is bitangent to $\widetilde{\mathfrak{T}}$ at two extra points. Let $e_7 \subset B$ be one of the components of the preimage in B of the strict transform of e_7 in $B/(-1)_B$. Label by $e_1, \ldots, e_6$ the components of the reduced singular fibers of $\delta : B \to \mathbb{P}^1$ which do not intersect e_7. Then $e_1, \ldots, e_6$ and e and e_7 are disjoint (-1) curves on B. After contracting these eight curves and the image of the curve n_1 we will get a $\mathbb{P}^2$.

Let $c : B \to \mathbb{P}^2$ denote this contraction map and let $\ell = c^* \mathcal{O}_{\mathbb{P}^2}(1)$ be the pullback of the class of a line via c. Thus $\mathrm{Pic}(B)$ is generated over $\mathbb{Z}$ by the classes of the curves ℓ, $e_1, \ldots, e_6$, e, e_7 and n_1. In particular, if we put

$$\begin{aligned} e_9 &:= e \\ e_8 &:= e + n_1 \end{aligned}$$

we see that

$$H^2(B, \mathbb{Z}) = \mathbb{Z}\ell \oplus (\oplus_{i=1}^9 \mathbb{Z}e_i),$$

with $\ell^2 = 1$, $\ell \cdot e_i = 0$ and $e_i \cdot e_j = -\delta_{ij}$.

Note that in this basis we have

$$\begin{aligned} n_1 &= e_8 - e_9 \\ o_1 &= f - e_8 + e_9 \\ n_2 &= \ell - e_7 - e_8 - e_9 \\ o_2 &= 2\ell - e_1 - e_2 - e_3 - e_4 - e_5 - e_6. \end{aligned} \tag{4.2}$$

4.3 A synthetic construction

Before we proceed with the calculation of the action of τ_B on $H^2(B, \mathbb{Z})$ it will be helpful to analyze how the surface B and the map $c : B \to \mathbb{P}^2$ can be reconstructed synthetically from geometric data on $\mathbb{P}^2$.

First we will need a general lemma describing a birational involution of $\mathbb{P}^2$ fixing some smooth cubic pointwise.

Lemma 4.2. *Let $\Gamma \subset \mathbb{P}^2$ be a smooth cubic and let $b \in \Gamma$. There exists a unique birational involution $\alpha : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ which preserves the general line through b and fixes the general point of Γ. Let $b_1, b_2, b_3, b_4 \in \Gamma$ be the four ramification points for the linear projection of Γ from b. Then*

(i) *α sends a general line to a cubic which is nodal at b and passes through the b_i's.*

(ii) *α sends the net of conics through b_1, b_2, b_3 to the net of cubics that are nodal at b_4 and pass through b, b_1, b_2, b_3.*

Proof. Let $\alpha : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational involution which fixes the general point of the cubic Γ and preserves the general line through $b \in \Gamma$. If $b \in L \subset \mathbb{P}^2$ is a general line, then $L \cap \Gamma$ consists of three distinct points $\{b, 0_L, \infty_L\}$. Since α preserves L it follows that $\alpha_{|L}$ is a birational involution of L which fixes the points 0_L and ∞_L. But any birational involution of $\mathbb{P}^1$ is biregular, has exactly two fixed points and is uniquely determined by its fixed points. Thus the restriction of α on the generic line through b is uniquely determined and so there can be at most one such α. Conversely we can use this uniqueness to show the existence of α. Indeed, choose coordinates $(x : y : z)$ in $\mathbb{P}^2$ so that $b = (0 : 0 : 1)$ and Γ is given by the equation $F(x, y, z) = 0$ with F a homogeneous cubic polynomial. Since $b \in \Gamma$ we can write $F = F_1 z^2 + F_2 z + F_3$ with F_d a homogeneous polynomial in (x, y) of degree d. Let $(x : y : z)$ be a point in $\mathbb{P}^2$ and let $L = \{(x : y : z + t)\}_{t \in \mathbb{P}^1}$ be the line through b and $(x : y : z)$. The involution $\alpha_{|L}$ will have to fix the two additional (besides b) intersection points of L and Γ. The values of t corresponding to these points are just the roots of the equation $F(x, y, z + t) = 0$, that is the solutions to

$$F_1(x, y)t^2 + F_z(x, y, z)t + F(x, y, z) = 0. \tag{4.3}$$

On the other hand since t is the affine coordinate on L the involution $\alpha_{|L} : \mathbb{P}^1 \to \mathbb{P}^1$ will be given by a fractional linear transformation

$$t \mapsto \frac{at + b}{ct + d}$$

for some complex numbers a, b, c and d. The condition that $\alpha_{|L} \neq \mathrm{id}_L$ but $\alpha^2_{|L} = \mathrm{id}_L$ is equivalent to $d = -a$.

In these terms the fixed points of $\alpha_{|L}$ correspond to the values of t for which

$$ct^2 - 2at - b = 0. \tag{4.4}$$

Comparing (4.3) with (4.4) we conclude that $a = -(1/2)F_z(x,y,z)$, $b = -F(x,y,z)$ and $c = F_1(x,y)$ and so

$$\alpha_{|L}((x:y:z+t)) = \left(x:y:z - \frac{F_z(x,y,z)t + 2F(x,y,z)}{2F_1(x,y)t + F_z(x,y,z)}\right).$$

In particular for $t = 0$ we must have

$$\alpha((x:y:z)) = \alpha_{|L}((x:y:z)) = \left(x:y:z - 2\frac{F(x,y,z)}{F_z(x,y,z)}\right). \tag{4.5}$$

Now the formula (4.5) clearly defines a birational automorphism α of $\mathbb{P}^2$ and it is straightforward to check that $\alpha^2 = \mathrm{id}_{\mathbb{P}^2}$. This shows the existence and uniqueness of α.

To prove the remaining statements note that the α that we have just defined lifts to a biregular involution $\hat{\alpha}$ on the blow-up $g : \widehat{\mathbb{P}^2} \to \mathbb{P}^2$ of $\mathbb{P}^2$ at the points b, b_1, b_2, b_3, b_4. Let $\Sigma, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4 \subset \widehat{\mathbb{P}^2}$ denote the corresponding exceptional divisors and let $\ell = g^*\mathcal{O}_{\mathbb{P}^2}(1)$ be the class of a line. By definition α preserves the general line through b and the cubic Γ. Hence $\hat{\alpha}$ will preserve the proper transforms of Γ and the general line through b, i.e.

$$\hat{\alpha}(\ell - \Sigma) = \ell - \Sigma$$
$$\hat{\alpha}\left(3\ell - \Sigma - \sum_{i=1}^{4}\Sigma_i\right) = 3\ell - \Sigma - \sum_{i=1}^{4}\Sigma_i.$$

Also it is clear (e.g. from (4.5)) that $\hat{\alpha}$ identifies the proper transform of the line through b and b_i with Σ_i and so

$$\hat{\alpha}(\Sigma_i) = \ell - \Sigma - \Sigma_i$$

for $i = 1, 2, 3, 4$. Therefore we get two equations for $\hat{\alpha}(\ell)$ and $\hat{\alpha}(\Sigma)$:

$$\hat{\alpha}(\ell) - \hat{\alpha}(\Sigma) = \ell - \Sigma$$
$$3\hat{\alpha}(\ell) - \hat{\alpha}(\Sigma) = 7\ell - 5\Sigma - 2\sum_{i=1}^{4}\Sigma_i,$$

which yield $\hat{\alpha}(\ell) = 3\ell - 2\Sigma - \sum_{i=1}^{4}\Sigma_i$ and $\hat{\alpha}(\Sigma) = 2\ell - \Sigma - \sum_{i=1}^{4}\Sigma_i$.

If now L is a line not passing through any of the points b, b_1, b_2, b_3, b_4 we see that the proper transform $\widehat{L}$ of L in $\widehat{\mathbb{P}^2}$ is an irreducible curve such that $\hat{\alpha}(\widehat{L})$ is in the linear system $|3\ell - 2\Sigma - \sum_{i=1}^{4}\Sigma_i|$. In particular $\hat{\alpha}(\widehat{L})$ intersects Σ at two points and intersects each Σ_i at a point. So $\alpha(L) = g(\hat{\alpha}(\widehat{L}))$ is a cubic which is nodal at b and passes through each of the b_i's. This proves part (i) of the lemma.

Similarly if C is a conic through b_1, b_2 and b_3, then $\widehat{C}$ is an irreducible curve in the linear system $|2\ell - \Sigma_1 - \Sigma_2 - \Sigma_3|$ on $\widehat{\mathbb{P}^2}$. Hence $\hat{\alpha}(\widehat{C})$ is an irreducible curve in the linear system $|3\ell - \Sigma - \Sigma_1 - \Sigma_2 - \Sigma_3 - 2\Sigma_4|$ and so $\alpha(C) = g(\hat{\alpha}(\widehat{C}))$ is a cubic passing through b, b_1, b_2, b_3 which is nodal at b_4. The lemma is proven. □

For our synthetic construction of B we will start with a nodal cubic $\Gamma_1 \subset \mathbb{P}^2$ and will denote its node by $A_8 \in \Gamma_1$. Pick four other points on Γ_1 and label them A_1, A_2, A_3, A_7. For generic such choices there is a unique smooth cubic Γ which passes through the points A_1, A_2, A_3, A_7, A_8 and is tangent to the line $\langle A_7 A_i \rangle$ at the point A_i for $i = 1, 2, 3$ and 8. Consider the pencil of cubics spanned by Γ and Γ_1. All cubics in this pencil pass through A_1, A_2, A_3, A_7, A_8 and are tangent to Γ at A_8. Let A_4, A_5, A_6 be the remaining three base points. Each cubic in the pencil intersects the line $N_2 := \langle A_7 A_8 \rangle$ in the same divisor $A_7 + 2A_8 \in \mathrm{Div}(N_2)$. Therefore there is a reducible cubic $\Gamma_2 = N_2 \cup O_2$ in the pencil. Generically O_2 will be a smooth conic as depicted on Figure 4.

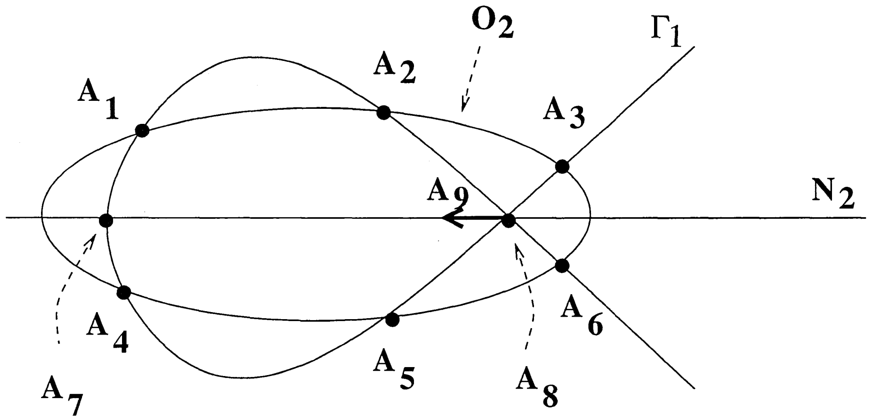


Figure 4: The pencil of cubics determining B

By Lemma 4.2 there is a birational involution α of $\mathbb{P}^2$ corresponding to Γ with $b = A_7$. Note that by construction $b_i = A_i$ for $i = 1, 2, 3$ and $b_4 = A_8$. By Lemma 4.2(ii) we know that $\alpha(O_2)$ is a nodal cubic with a node at A_8 which passes through A_1, A_2, A_3 and A_7. Since the involution α fixes $A_4, A_5, A_6 \in \Gamma$ it also follows that $\alpha(O_2)$ contains A_4, A_5, A_6. The intersection number $\alpha(O_2)$ with Γ_1 is therefore at least $6 + 2 \cdot 2 = 10$ and so $\alpha(O_2) = \Gamma_1$. Moreover α collapses N_2 to A_8. This shows that α preserves the pencil.

We define B to be the blow-up of $\mathbb{P}^2$ at the points A_i, $i = 1, \dots, 8$ and the point A_9 which is infinitesimally near to A_8 and corresponds to the tangent direction N_2. The pencil of cubics becomes the anticanonical map $\beta : B \to \mathbb{P}^1$. The reducible fibers are $f_i = n_i \cup o_i$, $i = 1, 2$ where n_2, o_2 are the proper transforms of N_2, O_2, o_1 is the proper transform of Γ_1 and n_1 is the proper transform of the exceptional divisor corresponding to A_8. In order to conform with the notation in Section 2 we denote by e_i for $i = 1, \dots, 7$ and 9 the exceptional divisors corresponding to A_i, $i = 1, \dots, 7$ and 9 and by e_8 the reducible divisor $e_9 + n_1$.

The involution $\alpha : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ lifts to a biregular involution $\alpha_B : B \to B$. The induced involution $\tau_{\mathbb{P}^1}$ of $\mathbb{P}^1$ has two fixed points $0, \infty \in \mathbb{P}^1$. One of them, say 0, will be the image $\beta(\Gamma)$. We will use e_9 as the zero section $e : \mathbb{P}^1 \to B$. Note that $(-1)_B^* e_i = \alpha_B^* e_i$ for $i = 1, 2, 3$ and so we can take $\zeta = e_1$.

5 Action on cohomology

First we describe the action of the automorphisms $(-1)_B$, α_B, t_ζ and τ_B on $H^\bullet(B,\mathbb{Z})$.

5.1 Action of $(-1)_B$

From the discussion in section 4.2 it is clear that $(-1)_B$ preserves the fibers of $\delta : B \to \mathbb{P}^1$ and exchanges the two components of the six singular fibers of δ which are unions of two rational curves meeting at a point. Furthermore from the description of B as a blow-up of $\mathbb{P}^2$ at nine points (see section 4.2) it follows that the class of the fiber of δ is $\ell - e_7$. Hence $(-1)_B(\ell - e_7) = \ell - e_7$ and $(-1)_B(e_i) + e_i = \ell - e_7$ for $i = 1, \ldots, 6$. Also, by the same analysis we see that $(-1)_B$ preserves n_1 and n_2 and since $(-1)_B$ preserves f by definition, it follows that $(-1)_B$ preserves o_1 and o_2 as well. Similarly $(-1)_B$ preserves e_9 by definition and so $(-1)_B^*(e_8) = (-1)_B^*(e_9 + n_1) = e_9 + n_1 = e_8$. Finally we can solve the equations $(-1)_B^*(\ell - e_7) = \ell - e_7$ and $(-1)_B^*(o_2) = o_2$ to get $(-1)_B(\ell) = f + \ell - 2e_7 + e_8 + e_9$ and $(-1)_B^*(e_7) = f - e_7 + e_8 + e_9$.

5.2 Action of α_B

Again from the analysis in section 4.2 and the geometric description of B/α_B and its Weierstrass model W_M we see that α_B preserves the classes of the fibers of the two fibrations $\beta : B \to \mathbb{P}^1$ and $\delta : B \to \mathbb{P}^1$. In particular we have $\alpha_B^*(f) = f$, $\alpha_B^*(\ell - e_7) = \ell - e_7$ and $\alpha_B^*(e_9) = e_9$. Also α_B interchanges o_1 and o_2 and hence interchanges n_1 and n_2. From the relationship between the ramification divisors defining W_M and $\widetilde{Q}$ we see that α_B will exchange the two components of the three singular fibers of δ corresponding to the three intersection points in $\mathfrak{T} \cap r_0$, i.e. $\alpha_B^*(e_j) + e_j = \ell - e_7$ for $j = 1, 2, 3$. Similarly α_B will preserve the two components of the singular fibers of δ corresponding to the three intersection points in $\mathfrak{T} \cap r_\infty$, that is $\alpha_B^*(e_i) = e_i$ for $i = 4, 5, 6$. Finally, solving the equations $\alpha_B^*(\ell - e_7) = \ell - e_7$ and $\alpha_B^*(o_1) = o_2$ we get $\alpha_B^*(\ell) = 3\ell - e_1 - e_2 - e_3 - 2e_7 - e_8$ and $\alpha_B^*(e_7) = 2\ell - e_1 - e_2 - e_3 - e_7 - e_8$.

5.3 Action of t_ζ^*

By definition we have $t_\zeta^*(f) = f$. In order to find the action of t_ζ on the classes e_i we will use the fact that t_ζ is defined in terms of the addition law on $\beta^\# : B^\# \to \mathbb{P}^1$.

Since t_ζ preserves each fiber of $\beta : B \to \mathbb{P}^1$, the curve $t_\zeta^*(n_1)$ will have to be either n_1 or o_1. But $\zeta = e_1$ and so $\zeta \cdot n_1 = 0$ and $\zeta \cdot o_1 = 1$, so since $n_1^\#$ is the identity component of the disconnected group $n_1^\# \cup o_1^\# = (n_1 \cup o_1) - (n_1 \cap o_1)$, we must have $t_\zeta^*(n_1) = o_1$. In the same way one can argue that $t_\zeta^*(n_2) = o_2$ and $t_\zeta^*(o_i) = n_i$ for $i = 1, 2$.

Next note that since t_ζ is compatible with the group scheme structure of $B^\#$ we must have $t_\zeta^*(\xi) = c_1([\xi] - [\zeta])$ for any section ξ of β. Using this relation we calculate:

$$\begin{aligned} t_\zeta^*(e_1) &= c_1([e_1] - [e_1]) = e_9, \\ t_\zeta^*(e_9) &= c_1([e_9] - [e_1]) = (-1)_B([e_1]) = \ell - e_1 - e_7, \end{aligned}$$

which in turn implies $t_\zeta^*(e_8) = t_\zeta^*(e_9 + n_1) = \ell - e_1 - e_7 + o_1 = f + \ell - e_1 - e_7 - e_8 + e_9$.

The previous formulas identify cohomology classes in $H^2(B, \mathbb{Z})$ or equivalently line bundles on B. However observe that the above formulas can also be viewed as equality of divisors, due to the fact that the line bundles in question correspond to sections of β, and so each of these is represented by a unique (rigid) effective divisor.

Also since the addition law on an elliptic curve is defined in terms of the Abel-Jacobi map we see that for a section ξ of β, the restriction of the line bundle $c_1([\xi] - [e_1]) \otimes \mathcal{O}_B(-e_9)$ to the generic fiber of β will be the same as the restriction of $\mathcal{O}_B(\xi - e_1)$. By the see-saw principle the difference of these two line bundles will have to be a combination of components of fibers of β, i.e.

$$t_\zeta^*(\xi) = c_1([\xi] - [e_1]) = \xi - e_1 + e_9 + a_1^\xi n_1 + a_2^\xi n_2 + a^\xi f.$$

Intersecting both sides with n_1 and taking into account that $(t_\zeta^{-1})^*(n_1) = o_1$ we get $o_1 \cdot \xi = \xi \cdot n_1 + 1 - 2a_1^\xi$. Similarly when we intersect with n_2 we get $o_2 \cdot \xi = \xi \cdot n_2 + 1 - 2a_2^\xi$. In particular since for $i = 2, \ldots, 6$ we have $e_i \cdot n_1 = e_i \cdot n_2 = 0$ and $e_i \cdot o_1 = e_i \cdot o_2 = 1$ we get $a_1^{e_i} = a_2^{e_i} = 0$ and so $t_\zeta^*(e_i) = e_i - e_1 + e_9 + a^{e_i} f$. Using the fact that $(t_\zeta^*(e_i))^2 = -1$ we find that $a^{e_i} = 1$ and thus

$$t_\zeta^*(e_i) = e_i - e_1 + e_9 + f$$

for $i = 2, \ldots, 6$.

Finally, for e_7 we have $e_7 \cdot n_1 = e_7 \cdot o_2 = 0$ and $e_7 \cdot n_2 = e_7 \cdot o_1 = 1$ and so $t_\zeta^*(e_7) = e_7 - e_1 + e_9 + n_2 + a^{e_7} f$. From $(t_\zeta^*(e_7))^2 = -1$ we find $a^{e_7} = 0$ and therefore $t_\zeta^*(e_7) = e_7 - e_1 + e_9 + n_2$.

This completes the calculation of the action of t_ζ^* on $H^2(B, \mathbb{Z})$. The action of τ_B^* is easily obtained since by definition we have $\tau_B^* = \alpha_B^* \circ t_\zeta^*$.

All these actions are summarized in Table 1 below.

	$(-1)_B^*$	t_ζ^*	α_B^*	τ_B^*
f	f	f	f	f
e_1	$\ell - e_1 - e_7$	e_9	$\ell - e_1 - e_7$	e_9
e_j, $j=2,3$	$\ell - e_j - e_7$	$f + e_j - e_1 + e_9$	$\ell - e_j - e_7$	$f - e_j + e_1 + e_9$
e_i, $i=4,5,6$	$\ell - e_i - e_7$	$f + e_i - e_1 + e_9$	e_i	$f - \ell + e_i + {} + e_1 + e_7 + e_9$
e_7	$f - e_7 + e_8 + e_9$	$\ell - e_1 - e_8$	$2\ell - (e_1 + e_2 + {} + e_3 + e_7 + e_8)$	$\ell - e_2 - e_3$
e_8	e_8	$f + \ell + e_9 - {} - e_1 - e_7 - e_8$	$\ell - e_7 - e_8$	$f - \ell + e_1 + {} + e_7 + e_8 + e_9$
e_9	e_9	$\ell - e_1 - e_7$	e_9	e_1
ℓ	$\ell + f - {} - 2e_7 + e_8 + e_9$	$2f + 2\ell - 3e_1 - {} - e_7 - e_8 + 2e_9$	$3\ell - (e_1 + e_2 + {} + e_3 + 2e_7 + e_8)$	$2f + 2(e_1 + e_9) - {} - (e_2 + e_3) + e_7$

Table 1: Action of $(-1)_B$, α_B, t_ζ and τ_B on $H^\bullet(B, \mathbb{Z})$

6 The cohomological Fourier-Mukai transform

For the purposes of the spectral construction we will need also the action of the relative Fourier-Mukai transform for $\beta : B \to \mathbb{P}^1$ on the cohomology of B. By definition the Fourier-Mukai transform is the exact functor on the bounded derived category $D^b(B)$ of B given by the formula

$$\boldsymbol{FM}_B : \quad D^b(B) \longrightarrow D^b(B)$$
$$\mathcal{F} \longmapsto R^\bullet p_{1*}(p_2^* \mathcal{F} \overset{L}{\otimes} \mathcal{P}_B).$$

Here p_1, p_2 are the projections of $B \times_{\mathbb{P}^1} B$ to its two factors, and $\mathcal{P}_B$ is the Poincare sheaf:

$$\mathcal{P}_B := \mathcal{O}_B(\Delta - e \times_{\mathbb{P}^1} B - B \times_{\mathbb{P}^1} e - q^* \mathcal{O}_{\mathbb{P}^1}(1)),$$

with $q = \beta \circ p_1 = \beta \circ p_2$. Using the zero section $e : \mathbb{P}^1 \to B$ we can identify B with the relative moduli space $\mathcal{M}(B/\mathbb{P}^1)$ of semistable (w.r.t. to a suitable polarization), rank one, degree zero torsion free sheaves along the fibers of $\beta : B \to \mathbb{P}^1$. Under this identification, the sheaf $\mathcal{P}_B \to B \times_{\mathbb{P}^1} B = B \times_{\mathbb{P}^1} \mathcal{M}(B/\mathbb{P}^1)$ becomes the universal sheaf. This puts us in the setting of [BM, Theorem 1.2] and implies that $\boldsymbol{FM}_B$ is an *autoequivalence* of $D^b(B)$. In particular we can view any vector bundle $V \to B$ in two different ways - as V and as the object $\boldsymbol{FM}_B(V) \in D^b(B)$.

The cohomological Fourier-Mukai transform is defined as the unique linear map

$$\boldsymbol{fm}_B : H^\bullet(B, \mathbb{Q}) \to H^\bullet(B, \mathbb{Q})$$

satisfying:

$$\boldsymbol{fm}_B \circ ch = ch \circ \boldsymbol{FM}_B. \tag{6.1}$$

Explicitly,

$$\boldsymbol{fm}_B(x) = \mathrm{pr}_{2*}(\mathrm{pr}_1^*(x) \cdot ch(j_*\mathcal{P}) \cdot td(B \times B)) \cdot td(B)^{-1},$$

where pr_i are the projections of $B \times B$ to its factors and $j : B \times_{\mathbb{P}^1} B \hookrightarrow B \times B$ is the natural inclusion.

We will need an explicit description of the cohomological spectral involution

$$\boldsymbol{t}_B := \boldsymbol{fm}_B^{-1} \circ \tau_B^* \circ \boldsymbol{fm}_B.$$

For this we proceed to calculate the action of $\boldsymbol{fm}_B$ and $\boldsymbol{fm}_B^{-1}$ in the obvious basis in cohomology.

Let $\mathrm{pt} \in H^4(B, \mathbb{Z})$ denote the class Poincare dual to the homology class of a point in B and let $1 \in H^0(B, \mathbb{Z})$ be the class which is Poincare dual to the fundamental class of B. The classes 1, f, $e_1, \ldots, e_9$, pt constitute a basis of $H^\bullet(B, \mathbb{Q})$.

To calculate $\boldsymbol{fm}_B$ we will use the identity (6.1) together with a calculation of the action of $\boldsymbol{FM}_B$ on certain basic sheaves, which is carried out in Lemma 6.1 below.

The first observation is that there are two ways to lift a sheaf G on $\mathbb{P}^1$ to a sheaf on B. First we may consider the pullback $\beta^*(G)$. Second, for any section $\xi : \mathbb{P}^1 \to B$ of β we may form the push-forward ξ_*G. These two lifts behave quite differently. For example, if G is a line bundle, then β^*G is a line bundle on B, whereas ξ_*G is a torsion sheaf on B supported on ξ. The action of $\boldsymbol{FM}_B$ interchanges these two types of sheaves (up to a shift):

Lemma 6.1. *For any sheaf G on $\mathbb{P}^1$ and any section ξ of β we have:*

$$\begin{aligned}\boldsymbol{FM}_B(\beta^*G) &= e_*(G \otimes \mathcal{O}_{\mathbb{P}^1}(-1))[-1]\\ \boldsymbol{FM}_B(\xi_*G) &= \beta^*G \otimes \mathcal{O}_B(\xi - e) \otimes \beta^*\mathcal{O}_{\mathbb{P}^1}(-e\cdot\xi - 1),\end{aligned}$$

where as usual for a complex $K^\bullet = (K^i, d_K^i)$ and an integer $n \in \mathbb{Z}$ we put $K^\bullet[n]$ for the complex having $(K[n])^i = K^{n+i}$ and $d_{K[n]} = (-1)^n d_K$.

Proof. By definition we have $\boldsymbol{FM}_B(\beta^*G) = Rp_{2*}(p_1^*\beta^*G \otimes \mathbb{P}_B)$. But $\beta \circ p_1 = \beta \circ p_2$ and so by the projection formula we get $\boldsymbol{FM}_B(\beta^*G) = Rp_{2*}(p_2^*\beta^*G \otimes \mathbb{P}_B) = \beta^*G \otimes Rp_{2*}\mathcal{P}_B$. In order to calculate $Rp_{2*}\mathcal{P}_B$, note first that $Rp_{2*}\mathcal{P}_B$ is a complex concentrated in degrees zero and one since p_2 is a morphism of relative dimension one. Next observe that $R^0p_{2*}\mathcal{P}_B = 0$. Indeed, by definition $\mathcal{P}_B$ is a rank one torsion free sheaf on $B \times_{\mathbb{P}^1} B$, and so $R^0p_{2*}\mathcal{P}_B$ must be a torsion free sheaf on B. On the other hand, from the definition of $\mathcal{P}_B$ we see that both $R^0p_{2*}\mathcal{P}_B$ and $R^1p_{2*}\mathcal{P}_B$ are torsion sheaves on B whose reduced support is precisely $e \subset B$. Therefore $R^0p_{2*}\mathcal{P}_B$ is torsion and torsion free at the same time and so $R^0p_{2*}\mathcal{P}_B = 0$. This implies that $Rp_{2*}\mathcal{P}_B = R^1p_{2*}\mathcal{P}_B[-1]$. Now, since $R^2p_{2*}\mathcal{P}_B = 0$ we can apply the cohomology and base change theorem [Har77, Theorem 12.11] to conclude that $R^1p_{2*}\mathcal{P}_B$ has the base change property for arbitrary (i.e. not necessarily flat) morphisms. In particular considering the base change diagram

$$\begin{array}{ccc} B = B \times_{\mathbb{P}^1} e & \hookrightarrow & B \times_{\mathbb{P}^1} B \\ \downarrow \beta & & \downarrow p_2 \\ \mathbb{P}^1 & \overset{e}{\hookrightarrow} & B \end{array}$$

we have that

$$\begin{aligned} e^*R^1p_{2*}\mathcal{P}_B &= R^1\beta_*(\mathcal{P}_{B|B\times_{\mathbb{P}^1}e}) = R^1\beta_*\mathcal{O}_B = (\beta_*\omega_{B/\mathbb{P}^1})^\vee \\ &= (\beta_*(\mathcal{O}_B(-f) \otimes \beta^*\mathcal{O}(2)))^\vee = \mathcal{O}_{\mathbb{P}^1}(-1). \end{aligned}$$

Since $e \subset B$ is the reduced support of $R^1p_{2*}\mathcal{P}_B$ and $(R^1p_{2*}\mathcal{P}_B)_{|e}$ is a line bundle, it follows that $e \subset B$ is actually the scheme theoretic support of $R^1p_{2*}\mathcal{P}_B$ and so $R^1p_{2*}\mathcal{P}_B = e_*\mathcal{O}_{\mathbb{P}^1}(-1)$, which finishes the proof of the first part of the lemma.

Let now $\xi : \mathbb{P}^1 \to B$ be a section of β. Then $\boldsymbol{FM}_B(\xi_*G) = Rp_{2*}(p_1^*\xi_*G \otimes \mathcal{P}_B)$. But $p_1^*\xi_*G$ is a sheaf on $B \times_{\mathbb{P}^1} B$ supported on $\xi \times_{\mathbb{P}^1} B \subset B \times_{\mathbb{P}^1} B$ and is in fact the extension by zero of the sheaf β^*G on $B = \xi \times_{\mathbb{P}^1} B$. Moreover by definition we have $\mathbb{P}_{B|\xi\times_{\mathbb{P}^1}B} = \mathcal{O}_B(\xi - e - (e\cdot\xi + 1)f)$. Taking into account that $p_2 : \xi \times_{\mathbb{P}^1} B \to B$ is an isomorphism, we get the second statement of the lemma. □

With all of this said we are now ready to derive the explicit formulas for $\boldsymbol{fm}_B$. First, observe that $ch(\mathcal{O}_B) = 1$ and so by (6.1) and Lemma 6.1 we have

$$\begin{aligned}\boldsymbol{fm}_B(1) &= ch(\boldsymbol{FM}_B(\mathcal{O}_B)) = ch(\boldsymbol{FM}_B(\beta^*\mathcal{O}_{\mathbb{P}^1})) \\ &= ch(e_*(\mathcal{O}_{\mathbb{P}^1}(-1))[-1]) = -ch(e_*(\mathcal{O}_{\mathbb{P}^1}(-1)).\end{aligned}$$

But from the short exact sequence of sheaves on B

$$0 \to \mathcal{O}_B(-e-f) \to \mathcal{O}_B(-f) \to e_*\mathcal{O}_{\mathbb{P}^1}(-1) \to 0$$

we calculate

$$\begin{aligned}ch(e_*(\mathcal{O}_{\mathbb{P}^1}(-1)) &= ch(\mathcal{O}_B(-f)) - ch(\mathcal{O}_B(-e-f)) \\ &= (1 - f + 0\cdot \mathrm{pt})\cdot\left(1 + (e-f) + \frac{1}{2}\,\mathrm{pt}\right) \\ &= e - \frac{1}{2}\,\mathrm{pt}\,.\end{aligned}$$

In other words $\boldsymbol{fm}_B(1) = -e + (1/2)\,\mathrm{pt} = -e_9 + (1/2)\,\mathrm{pt}$.

Next we calculate $\boldsymbol{fm}_B(\mathrm{pt})$. Let $t \in \mathbb{P}^1$ be a fixed point. Then $\mathrm{pt} = ch(\mathcal{O}_{e(t)}) = ch(e_*\mathcal{O}_t)$ and so

$$\begin{aligned}\boldsymbol{fm}_B(\mathrm{pt}) &= ch(\boldsymbol{FM}_B(e_*\mathcal{O}_t))\\ &= ch(\mathcal{O}_f) = ch(\mathcal{O}_B) - ch(\mathcal{O}_B(-f))\\ &= 1 - (1 - f + 0\cdot \mathrm{pt}) = f.\end{aligned}$$

To calculate $\boldsymbol{fm}_B(f)$ note that $ch(\mathcal{O}_B(f)) = 1 + f$ and so

$$\begin{aligned}\boldsymbol{fm}_B(f) &= ch(\boldsymbol{FM}_B(\mathcal{O}_B(f))) - \boldsymbol{fm}_B(1)\\ &= ch(\boldsymbol{FM}_B(\beta^*\mathcal{O}_{\mathbb{P}^1}(1))) - \boldsymbol{fm}_B(1)\\ &= ch(e_*\mathcal{O}_{\mathbb{P}^1}[-1]) - \left(-e + \frac{1}{2}\,\mathrm{pt}\right)\\ &= -[ch(\mathcal{O}_B) - ch(\mathcal{O}_B(-e))] + e - \frac{1}{2}\,\mathrm{pt}\\ &= -\left[1 - \left(1 - e - \frac{1}{2}\,\mathrm{pt}\right)\right] + e - \frac{1}{2}\,\mathrm{pt}\\ &= -\,\mathrm{pt}\,.\end{aligned}$$

Finally we calculate $\boldsymbol{fm}_B(e_i)$. If $i = 1, \ldots, 7$, the class e_i is a class of a section $e_i : \mathbb{P}^1 \to B$ of β and so we can apply Lemma 6.1 to $\mathcal{O}_{e_i}$. We have $ch(\mathcal{O}_{e_i}) = e_i + (1/2)\,\mathrm{pt}$ and hence

$$\begin{aligned}\boldsymbol{fm}_B(e_i) &= ch(\boldsymbol{FM}_B(\mathcal{O}_{e_i})) - \frac{1}{2}\boldsymbol{fm}_B(\mathrm{pt})\\ &= ch(\boldsymbol{FM}_B(e_{i*}\mathcal{O}_{\mathbb{P}^1})) - \frac{1}{2}\boldsymbol{fm}_B(\mathrm{pt})\\ &= ch(\mathcal{O}_B(e_i - e_9 - f)) - \frac{1}{2}f\\ &= 1 + (e_i - e_9 - f) - \mathrm{pt} - \frac{1}{2}f\\ &= 1 + (e_i - e_9 - \frac{3}{2}f) - \mathrm{pt}\,.\end{aligned}$$

For e_9 we get in the same way

$$\boldsymbol{fm}_B(e_9) = ch(\mathcal{O}_B) - \frac{1}{2}f = 1 - \frac{1}{2}f,$$

and so it only remains to calculate $\boldsymbol{fm}_B(e_8)$.

Unfortunately we can not use the same method for calculating $\boldsymbol{fm}_B(e_8)$ since e_8 is only a numerical section of β and splits as a union of two irreducible curves $e_8 = e_9 + n_1$. However, recall that the automorphism $\alpha_B : B \to B$ moves a section to a section. Consequently $\alpha_B(e_7)$ will be another section of β. Let $a : \mathbb{P}^1 \to B$ denote the map corresponding to $\alpha_B(e_7)$. Then

$$ch(\mathcal{O}_{\alpha_B(e_7)}) = ch(\mathcal{O}_B) - ch(\mathcal{O}_B(-\alpha_B(e_7)) = \alpha_B(e_7) + \frac{1}{2}\,\mathrm{pt}\,.$$

Thus

$$\begin{aligned}\boldsymbol{fm}_B(\alpha_B(e_7)) &= ch(\boldsymbol{FM}_B(a_*\mathcal{O}_{\mathbb{P}^1})) - \frac{1}{2}f\\ &= ch(\mathcal{O}_B(\alpha_B(e_7) - e_9 - (e_9\cdot\alpha_B(e_7)+1)f) - \frac{1}{2}f.\end{aligned}$$

But according to Table 1 we have $e_9\cdot\alpha_B(e_7) = e_9\cdot(2\ell - e_1 - e_2 - e_3 - e_7 - e_8) = 0$ and so

$$\boldsymbol{fm}_B(\alpha_B(e_7)) = 1 + \alpha_B(e_7) - e_9 - \frac{3}{2}f - \mathrm{pt}\,.$$

In terms of e_8 this reads

$$\begin{aligned}2\boldsymbol{fm}_B(\ell) - \boldsymbol{fm}_B(e_8) &= 1 + 2\ell - \sum_{i=1}^{3} e_i - e_7 - e_8 - e_9 - \frac{3}{2}f - \mathrm{pt}\\ &\quad + \boldsymbol{fm}_B(\sum_{i=1}^{3} e_i + e_7)\\ &= 1 + 2\ell - \sum_{i=1}^{3} e_i - e_7 - e_8 - e_9 - \frac{3}{2}f - \mathrm{pt} +\\ &\quad + \left(4 + \sum_{i=1}^{3} e_i + e_7 - 4e_9 - 6f - 4\,\mathrm{pt}\right)\\ &= 5 + (2\ell - \frac{15}{2}f - e_8 - 5e_9) - 5\,\mathrm{pt}\,.\end{aligned}$$

Also from $\boldsymbol{fm}_B(f) = -\,\mathrm{pt}$ we get

$$3\boldsymbol{fm}_B(\ell) - \boldsymbol{fm}_B(e_8) = 8 + (3\ell - 12f - e_8 - 8e_9) - 8\,\mathrm{pt}\,.$$

Solving these two equations for $\boldsymbol{fm}_B(e_8)$ results in

$$\boldsymbol{fm}_B(e_8) = 1 + (e_8 - e_9 - \frac{3}{2}f) - \mathrm{pt},$$

which completes the calculation of $\boldsymbol{fm}_B$.

	$\boldsymbol{fm}_B$	$\boldsymbol{fm}_B^{-1}$
1	$-e_9 + \frac{1}{2}\,\mathrm{pt}$	$e_9 + \frac{1}{2}\,\mathrm{pt}$
pt	f	$-f$
f	$-\,\mathrm{pt}$	pt
e_i, $i \neq 9$	$1 + e_i - e_9 - \frac{3}{2}f - \mathrm{pt}$	$-1 + e_i - e_9 - \frac{3}{2}f + \mathrm{pt}$
e_9	$1 - \frac{1}{2}f$	$-1 - \frac{1}{2}f$

Table 2: Action of the cohomological Fourier-Mukai transform

	$\boldsymbol{t}_B$
1	1
pt	pt
f	f
e_j $j = 1, 2, 3$	$2f + 2e_9 - e_j - 2\,\mathrm{pt}$
e_i, $i = 4, 5, 6$	$2f - \ell + 2e_9 + e_7 + e_i - \mathrm{pt}$
e_7	$f + \ell - e_1 - e_2 - e_3 + e_9 - \mathrm{pt}$
e_8	$2f - \ell + 2e_9 + e_7 + e_8 - \mathrm{pt}$
e_9	e_9
ℓ	$5f - e_1 - e_2 - e_3 + e_7 + 5e_9 - 3\,\mathrm{pt}$

Table 3: Action of $\boldsymbol{fm}_B^{-1} \circ \tau_B^* \circ \boldsymbol{fm}_B$ on cohomology

In summary, the action of $\boldsymbol{t}$ and the auxiliary actions of $\boldsymbol{fm}_B$ and $\boldsymbol{fm}_B^{-1}$ are recorded in tables 3 and 2 respectively.

7 Action on bundles

In this section we show how the cohomological computations in the previous section lift to actions of the Fourier-Mukai transform $\boldsymbol{FM}_B$ and the spectral involution $\boldsymbol{T}_B := \boldsymbol{FM}_B^{-1} \circ \tau_B^* \circ \boldsymbol{FM}_B$ on (complexes of) sheaves on B. Recall that the Chern character intertwines $\boldsymbol{FM}_B$ and $\boldsymbol{fm}_B$: $\boldsymbol{fm}_B \circ ch = ch \circ \boldsymbol{FM}_B$. Similarly, it intertwines $\boldsymbol{T}_B$ and $\boldsymbol{t}_B$: $\boldsymbol{t}_B \circ ch = ch \circ \boldsymbol{T}_B$.

Note that the Fourier-Mukai transform of a general sheaf $\mathcal{F}$ on B is a complex of sheaves, not a single sheaf. Nevertheless, all the sheaves we are interested in are taken by $\boldsymbol{T}_B$ again to sheaves. To explain what is going on exactly we will need to introduce some notation first. Put $c_1 : D^b(B) \to \mathrm{Pic}(B)$ for the first Chern class map in Chow cohomology. In combination with $\boldsymbol{T}_B$, the map c_1 induces a well defined map

$$\mathcal{P}\mathrm{ic}(B) \to \mathrm{Coh}(B) \subset D^b(B) \overset{\boldsymbol{T}_B}{\to} D^b(B) \overset{c_1}{\to} \mathrm{Pic}(B), \tag{7.1}$$

where $\mathcal{P}\mathrm{ic}(B)$ denotes the Picard category whose objects are all line bundles on B and whose morphisms are the isomorphisms of line bundles. Since $\boldsymbol{T}_B$ is an autoequivalence, the map (7.1) descends to a well defined map of sets

$$\widetilde{\boldsymbol{T}}_B : \mathrm{Pic}(B) = \pi_0(\mathcal{P}\mathrm{ic}(B)) \to \mathrm{Pic}(B).$$

If we identify $\mathrm{Pic}(B)$ and $H^2(B, \mathbb{Z})$ via the first Chern class map, we can describe $\widetilde{\boldsymbol{T}}_B$ alternatively as $\widetilde{\boldsymbol{T}}_B(-) = [\boldsymbol{t}_B(\exp(c_1(-)))]_2 \in H^2(B, \mathbb{Z})$.

Denote by $\mathrm{Pic}^W(B) \subset \mathrm{Pic}(B)$ the subgroup generated by f and the classes of all sections of β that meet the neutral component of each fiber. A straightforward calculation shows that $\mathrm{Pic}^W(B) = \mathrm{Span}(f, e_9, \{f + e_i - e_1 + e_9\}_{i=2}^6, 2e_7 - e_8 + 2f)$ (note that $f + e_i - e_1 + e_9$ is the class of the section $[e_i] - [e_1]$ and $2e_7 - e_8 + 2f$ is the class of the section $2[e_7]$) and that $\mathrm{Span}(o_1, o_2)^{\perp} = \mathrm{Span}(e_9, \{e_i - e_1\}_{i=2}^6, \ell - e_7 - 2e_1, 2\ell - e_8 - 4e_1)$. In particular $\mathrm{Pic}^W(B)$ is a sublattice of index 3 in $\mathrm{Span}(o_1, o_2)^{\perp}$. With this notation we have:

Theorem 7.1. *Let L be a line bundle on B. Then*

(i) *The complex $\boldsymbol{T}_B(L) \in D^{[0,1]}(B)$ becomes a line bundle when restricted on the open set $B - (o_1 \cup o_2)$. More precisely, the zeroth cohomology sheaf $\mathcal{H}^0(\boldsymbol{T}_B(L))$ is a line bundle on B and the first cohomology sheaf $\mathcal{H}^1(\boldsymbol{T}_B(L))$ is supported on the divisor $o_1 + o_2$.*

(ii) *The map $\widetilde{\boldsymbol{T}}_B$ satisfies*

$$\widetilde{\boldsymbol{T}}_B(L) = \tau_B^*(L) \otimes \mathcal{O}_B((c_1(L) \cdot (e - \zeta))f + (c_1(L) \cdot f + 1)(e - \zeta + f)).$$

(iii) *For every $L \in \mathrm{Pic}^W(B)$ the image $\boldsymbol{T}_B(L)$ is a line bundle on B and so*

$$\boldsymbol{T}_B(L) = \tau_B^*(L) \otimes \mathcal{O}_B((c_1(L) \cdot (e - \zeta))f + (c_1(L) \cdot f + 1)(e - \zeta + f)).$$

In particular $\boldsymbol{T}_B : \mathrm{Pic}^W(B) \to (\mathrm{Pic}^W(B) + (e - \zeta + f)) \subset \mathrm{Pic}(B)$ is an affine isomorphism.

Proof. The proof of this proposition is rather technical and involves some elementary but long calculations in the derived category $D^b(B)$.

Since $\boldsymbol{T}_B = \boldsymbol{FM}_B^{-1} \circ \tau_B^* \circ \boldsymbol{FM}_B$ we need to understand $\boldsymbol{FM}_B^{-1}$. The following lemma is standard.

Lemma 7.2. *The inverse* $\boldsymbol{FM}_B^{-1}$ *of the Fourier-Mukai functor* $\boldsymbol{FM}_B$ *is isomorphic to the functor*

$$\boldsymbol{D}_B \circ \boldsymbol{FM}_B \circ \boldsymbol{D}_B : D^b(B) \to D^b(B),$$

where $\boldsymbol{D}_B$ *is the (naive) Serre duality functor* $\boldsymbol{D}_B(F) := R^\bullet\mathcal{H}om(F, \omega_B)$ *with* ω_B *being the canonical line bundle on* B.

Proof. It is well known (see e.g. [Orl97, Section 2]) that $\boldsymbol{FM}_B$ has left and right adjoint functors $\boldsymbol{FM}^*_B$ and $\boldsymbol{FM}^!_B$ which are both isomorphic to $\boldsymbol{FM}_B^{-1}$. Furthermore, these adjoint functors can be defined by explicit formulas, see [Orl97, Section 2], e.g. the right adjoint is given by:

$$\boldsymbol{FM}^!_B(F) = R\,\mathrm{pr}_{1*}(\mathrm{pr}_2^* F \overset{L}{\otimes} \mathcal{P}^\vee))\otimes\omega_B[2].$$

Here $\mathrm{pr}_i : B \times B \to B$ are the projections onto the two factors, $\mathcal{P} \to B \times B$ is the extension by zero of $\mathcal{P}_B$ and $K^\vee := R^\bullet\mathcal{H}om(K, \mathcal{O}_{B\times B})$. Using e.g. the formula for the right adjoint functor, the relative duality formula [Har66] and the fact that ω_B is a line bundle, one calculates

$$\begin{aligned}
\boldsymbol{FM}^!_B(F) &= R\,\mathrm{pr}_{1*}(\mathrm{pr}_2^* F \overset{L}{\otimes} \mathcal{P}^\vee)\otimes\omega_B[2] \\
&= R\,\mathrm{pr}_{1*}((\mathrm{pr}_2^* F \overset{L}{\otimes} \mathcal{P}^\vee) \otimes \mathrm{pr}_2^*\omega_B[2] \otimes \mathrm{pr}_2^*\omega_B^{-1}) \otimes \omega_B \\
&= R\,\mathrm{pr}_{1*}(\mathrm{pr}_2^*(F \otimes \omega_B^{-1}) \overset{L}{\otimes} \mathcal{P}^\vee \otimes \mathrm{pr}_2^*\omega_B[2]) \otimes \omega_B \\
&= (R\,\mathrm{pr}_{1*}(\mathrm{pr}_2^*(F^\vee \otimes \omega_B) \overset{L}{\otimes} \mathcal{P})^\vee \otimes \omega_B \\
&= (\boldsymbol{FM}_B(\boldsymbol{D}_B(F)))^\vee \otimes \omega_B \\
&= \boldsymbol{D}_B \circ \boldsymbol{FM}_B \circ \boldsymbol{D}_B(F).
\end{aligned}$$

which proves the lemma. □

Next observe that $\mathrm{Pic}(B)$ is generated by all sections of β. Indeed $\mathrm{Pic}(B)$ is generated by ℓ and $e_1, e_2, \ldots, e_9$. The divisor classes $e_1, \ldots, e_7$ and e_9 are already sections of β. Also $\alpha_B(e_1) = \ell - e_1 - e_9$ is a section and so ℓ is contained in the group generated by all sections. Furthermore, $\alpha_B(e_7) = 2\ell - e_1 - e_2 - e_3 - e_7 - e_8$ is a section and so e_8 is contained in the group generated by all sections.

In view of this it suffices to prove parts (i) and (ii) of the theorem for line bundles of the form $L = \mathcal{O}_B(\sum a_i\xi_i)$ where $a_i \in \mathbb{Z}$ and ξ_i are sections of β.

Put $\mathcal{V}_0 := e_*\mathcal{O}_{\mathbb{P}^1}(-1)$. Consider the group $\mathrm{Ext}^1(\mathcal{V}_0, \mathcal{O}_B)$ of extensions of $\mathcal{V}_0$ by $\mathcal{O}_B$.

Since $e^2 = -1$ we have $\mathcal{V}_0 = e_* e^* \mathcal{O}_B(e)$ and so $\mathcal{V}_0$ fits in a short exact sequence

$$0 \to \mathcal{O}_B \to \mathcal{O}_B(e) \to \mathcal{V}_0 \to 0. \tag{7.2}$$

In particular we have a quasi-isomorphism $[\mathcal{O}_B \to \mathcal{O}_B(e)] \widetilde{\to} \mathcal{V}_0$ where in the complex

$$[\mathcal{O}_B \to \mathcal{O}_B(e)],$$

the sheaf $\mathcal{O}_B$ is placed in degree -1 and $\mathcal{O}_B(e)$ is placed in degree 0. Thus we have

$$\begin{aligned} \operatorname{Ext}^1(\mathcal{V}_0, \mathcal{O}_B) &= \operatorname{Hom}_{D^b(B)}(\mathcal{V}_0, \mathcal{O}_B[1]) = \operatorname{Hom}_{D^b(B)}([\mathcal{O}_B \to \mathcal{O}_B(e)], \mathcal{O}_B[1]) \\ &= \mathbb{H}^0(B, [\mathcal{O}_B \to \mathcal{O}_B(e)]^\vee[1]) = \mathbb{H}^0(B, [\mathcal{O}_B(-e) \to \mathcal{O}_B]), \end{aligned}$$

where in the complex $[\mathcal{O}_B(-e) \to \mathcal{O}_B]$ the sheaf $\mathcal{O}_B$ is placed in degree zero. In particular we have a quasi-isomorphism $[\mathcal{O}_B(-e) \to \mathcal{O}_B] \widetilde{\to} e_* \mathcal{O}_{\mathbb{P}^1}$ and hence $\operatorname{Ext}^1(\mathcal{V}_0, \mathcal{O}_B) = H^0(B, e_* \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$. This shows that there is a unique (up to isomorphism) sheaf $\mathcal{V}_1$ which is a non-split extension of $\mathcal{V}_0$ by $\mathcal{O}_B$. But from (7.2) we see that the line bundle $\mathcal{O}_B(e)$ is one such extension, i.e. we must have $\mathcal{V}_1 \cong \mathcal{O}_B(e)$.

Next consider the group of extensions $\mathrm{Ext}^1(\mathcal{V}_1, \mathcal{O}_B(f)) = H^1(B, \mathcal{V}_1^\vee \otimes \mathcal{O}(f))$. By the Leray spectral sequence we have a short exact sequence

$$\begin{aligned} 0 \to H^1(\mathbb{P}^1, (\beta_*\mathcal{V}_1^\vee) \otimes \mathcal{O}(1)) &\to H^1(B, \mathcal{V}_1^\vee \otimes \mathcal{O}(f)) \\ &\to H^0(\mathbb{P}^1, (R^1\beta_*\mathcal{V}_1^\vee) \otimes \mathcal{O}(1)) \to 0. \end{aligned}$$

But $\beta_*(\mathcal{V}_1^\vee) = \beta_*\mathcal{O}(-e) = 0$ and $R^1\beta_*(\mathcal{V}_1^\vee) = R^1\beta_*\mathcal{O}(-e) = \mathcal{O}(-1)$. Thus $\mathrm{Ext}^1(\mathcal{V}_1, \mathcal{O}_B(f)) = H^0(\mathbb{P}^1, \mathcal{O}) = \mathbb{C}$ and so there is a unique (up to isomorphism) non-split extension

$$0 \to \mathcal{O}_B(f) \to \mathcal{V}_2 \to \mathcal{V}_1 \to 0.$$

Arguing by induction we see that for every $a \geq 1$ there is a unique up to isomorphism vector bundle $\mathcal{V}_a \to B$ of rank a on B satisfying $\beta_*(\mathcal{V}_a^\vee) = 0$, $R^1\beta_*(\mathcal{V}_a^\vee) = \mathcal{O}(-a)$ and $\mathrm{Ext}^1(\mathcal{V}_a, \mathcal{O}_B(af)) = \mathbb{C}$ is generated by the non-split short exact sequence

$$0 \to \mathcal{O}_B(af) \to \mathcal{V}_{a+1} \to \mathcal{V}_a \to 0.$$

Alternatively, for each positive integer we can consider the vector bundle Ψ_a of rank a which is defined recursively as follows:

- $\Psi_1 := \mathcal{O}_B$, and
- Ψ_{a+1} is the unique non-split extension

$$0 \to \mathcal{O}_B(af) \to \Psi_{a+1} \to \Psi_a \to 0.$$

The fact that the Ψ_a's are correctly defined can be checked exactly as above. Moreover for each $a \geq 1$ $\mathcal{V}_a$ can be identified with the unique non-split extension

$$0 \to \Psi_a \to \mathcal{V}_a \to e_*\mathcal{O}_{\mathbb{P}^1}(-1) \to 0.$$

Let now $\xi : \mathbb{P}^1 \to B$ be a section of β. The first step in calculating $\boldsymbol{T}_B$ is given in the following lemma.

Lemma 7.3. *For any integer a we have*

$$\boldsymbol{FM}(\mathcal{O}_B(a\xi)) = \begin{cases} \mathcal{V}_{-a} \otimes \mathcal{O}_B(\xi - e - (\xi \cdot e + 1)f)[-1], \\ \qquad \textit{for } a \leq 0 \\ \mathcal{V}_a^{\vee} \otimes \mathcal{O}_B(-f) \otimes \mathcal{O}_B(\xi - e - (\xi \cdot e + 1)f), \\ \qquad \textit{for } a > 0 \end{cases}$$

Proof. By Lemma 6.1 we know that $\boldsymbol{FM}_B(\mathcal{O}_B) = e_*\mathcal{O}(-1)[-1]$ which gives the statement of the lemma for $a = 0$. To prove the statement for $a = -1$ consider the short exact sequence

$$0 \to \mathcal{O}_B(-\xi) \to \mathcal{O}_B \to \xi_*\mathcal{O}_{\mathbb{P}^1} \to 0 \tag{7.3}$$

of sheaves on B. For an object $K \in D^b(B)$ let $\boldsymbol{FM}^i_B(K)$ denote the i-th cohomology sheaf of the complex $\boldsymbol{FM}_B(K)$. Since $\boldsymbol{FM}_B$ is an exact functor on $D^b(B)$ it sends any short exact sequence to a long exact sequence of cohomology sheaves. Applying $\boldsymbol{FM}_B$ to (7.3) and using Lemma 6.1 we get

$$0 \longrightarrow \boldsymbol{FM}^0_B(\mathcal{O}_B(-\xi)) \longrightarrow 0 \longrightarrow \mathcal{O}_B(\xi - e - (1 + \xi \cdot e)f) \longrightarrow \boldsymbol{FM}^1_B(\mathcal{O}_B(-\xi)) \longrightarrow e_*\mathcal{O}(-1) \longrightarrow 0.$$

Thus $\boldsymbol{FM}^0_B(\mathcal{O}_B(-\xi)) = 0$ and $\boldsymbol{FM}^1_B(\mathcal{O}_B(-\xi))$ fits in a short exact sequence

(7.4)
$$0 \to \mathcal{O}_B(-e) \to \boldsymbol{FM}^1_B(\mathcal{O}_B(-\xi)) \otimes \mathcal{O}(-\xi + (1+\xi\cdot e)f) \to e_*\mathcal{O}_{\mathbb{P}^1} \to 0.$$

Since (7.3) is non-split and $\boldsymbol{FM}_B$ is an additive functor, it follows that (7.4) will not split. But $\mathrm{Ext}^1(e_*\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_B(-e)) = \mathrm{Ext}^1(e_*\mathcal{O}_{\mathbb{P}^1}(e), \mathcal{O}_B) = \mathbb{C}$ as we saw above and therefore we must have

$$\begin{aligned} \boldsymbol{FM}_B(\mathcal{O}_B(-\xi)) &\cong \mathcal{O}_B(\xi - (1+\xi\cdot e)f)[-1] \\ &= \mathcal{V}_1 \otimes \mathcal{O}_B(\xi - e - (1+\xi\cdot e)f)[-1]. \end{aligned}$$

Assume that the Lemma is proven for $\mathcal{O}_B(-a\xi)$ for some positive a. Then we have a short exact sequence of sheaves on B

(7.5) $$0 \to \mathcal{O}_B(-(a+1)\xi) \to \mathcal{O}_B(-a\xi) \to \xi_*\mathcal{O}_{\mathbb{P}^1}(a) \to 0.$$

Applying $\boldsymbol{FM}_B$ to (7.5) and using Lemma 6.1 we get

$$0 \to \boldsymbol{FM}^0_B(\mathcal{O}(-(a+1)\xi)) \to 0 \to \mathcal{O}(\xi - e + (a-1-\xi\cdot e)f) \to \boldsymbol{FM}^1_B(\mathcal{O}(-(a+1)\xi)) \to \boldsymbol{FM}^1_B(\mathcal{O}(-a\xi)) \to 0.$$

and so again $\boldsymbol{FM}^0_B(\mathcal{O}_B(-(a+1)\xi)) = 0$. Furthermore, by the inductive hypothesis we have $\boldsymbol{FM}^1_B(\mathcal{O}(-a\xi)) = \mathcal{V}_a \otimes \mathcal{O}_B(\xi - e - (1+\xi\cdot e)f)$ and so by the same reasoning as above the short exact sequence

$$0 \to \mathcal{O}_B(af) \to \boldsymbol{FM}^1_B(\mathcal{O}_B(-(a+1)\xi)) \otimes \mathcal{O}(e - \xi + (1+\xi\cdot e)f) \to \mathcal{V}_a \to 0$$

must be non-split. Since $\mathcal{V}_{a+1}$ is the only such non-split extension, we must have

$$\boldsymbol{FM}_B(\mathcal{O}_B(-(a+1)\xi)) = \mathcal{V}_{a+1} \otimes \mathcal{O}_B(\xi - e - (1+\xi\cdot e)f)[-1].$$

This completes the proof of the lemma for all $a \leq 0$. The argument for $a > 0$ is exactly the same and is left as an exercise. □

The next step is to calculate the action of $\boldsymbol{T}_B$ on line bundles of the form $\mathcal{O}_B(a\xi)$.

Due to Lemma 7.2 we have $\boldsymbol{T}_B = \boldsymbol{D}_B \circ \boldsymbol{FM}_B \circ \boldsymbol{D}_B \circ \tau_B^* \circ \boldsymbol{FM}_B$. Since τ_B is an automorphism of B we have $\boldsymbol{D}_B \circ \tau_B^* = \tau_B^* \circ \boldsymbol{D}_B$ and so

$$\boldsymbol{T}_B = (\boldsymbol{D}_B \circ \boldsymbol{FM}_B) \circ \tau_B^* \circ (\boldsymbol{D}_B \circ \boldsymbol{FM}_B). \tag{7.6}$$

To calculate $\boldsymbol{D}_B(\boldsymbol{FM}_B(\mathcal{O}_B(a\xi))$ we need to distinguish two cases: $a = 0$ and $a \neq 0$. When $a = 0$, we have $\boldsymbol{D}_B((\boldsymbol{FM}_B(\mathcal{O}_B)) = \boldsymbol{D}_B(e_*\mathcal{O}(-1)[-1])$. But as we saw above the short exact sequence (7.2) induces a quasi-isomorphism

$$\begin{bmatrix} \mathcal{O}_B \\ \downarrow \\ \mathcal{O}_B(e) \end{bmatrix}_{1}^{0} \xrightarrow{\text{q.i.}} e_*\mathcal{O}(-1)[-1].$$

Applying duality one gets

$$\begin{aligned} \boldsymbol{D}_B(e_*\mathcal{O}(-1)[-1]) &= \begin{bmatrix} \mathcal{O}_B(-e) \\ \downarrow \\ \mathcal{O}_B \end{bmatrix}_{0}^{-1} \otimes \mathcal{O}_B(-f) \\ &= \begin{bmatrix} \mathcal{O}_B(-e-f) \\ \downarrow \\ \mathcal{O}_B(-f) \end{bmatrix}_{0}^{-1} = e_*\mathcal{O}_{\mathbb{P}^1}(-1). \end{aligned}$$

But for $a \neq 0$ the sheaves $\boldsymbol{FM}_B(\mathcal{O}_B(a\xi))$ are locally free and so we get

$$\boldsymbol{D}_B \circ \boldsymbol{FM}_B(\mathcal{O}_B(a\xi)) = \begin{cases} \mathcal{V}_{-a}^{\vee} \otimes \mathcal{O}_B(e - \xi + (\xi \cdot e)f)[1], & \text{for } a < 0 \\ \mathcal{V}_0, & \text{for } a = 0 \\ \mathcal{V}_a \otimes \mathcal{O}_B(e - \xi + (1 + \xi \cdot e)f), & \text{for } a > 0. \end{cases}$$

To apply τ_B^* next we need to calculate $\tau_B^*\mathcal{V}_a$. For this recall that $\mathcal{V}_a$ is isomorphic to the unique non-split extension

$$0 \to \Psi_a \to \mathcal{V}_a \to e_*\mathcal{O}_{\mathbb{P}^1}(-1) \to 0.$$

Since $\tau_B(f) = f$ and Ψ_a is built by successive extensions of multiples of f, it follows that $\tau_B^*\Psi_a \cong \Psi_a$ for every a. So $\mathcal{W}_a := \tau_B^*\mathcal{V}_a$ is the unique non-split extension

$$0 \to \Psi_a \to \mathcal{W}_a \to \zeta_*\mathcal{O}_{\mathbb{P}^1}(-1) \to 0,$$

where as before $\zeta = \tau_B^*(e)$. With this notation we have

$$\tau_B^* \circ \boldsymbol{D}_B \circ \boldsymbol{FM}_B(\mathcal{O}_B(a\xi)) = \begin{cases} \mathcal{W}_{-a}^{\vee} \otimes \mathcal{O}_B(\zeta - \tau_B^*(\xi) + (\xi \cdot e)f)[1], & \text{for } a < 0 \\ \zeta_*\mathcal{O}_{\mathbb{P}^1}(-1), & \text{for } a = 0 \\ \mathcal{W}_a \otimes \mathcal{O}_B(\zeta - \tau_B^*(\xi) + (1 + \xi \cdot e)f), & \text{for } a > 0. \end{cases}$$

Now to finish the calculation of $\boldsymbol{T}_B(\mathcal{O}_B(a\xi))$ we have to work out $\boldsymbol{FM}_B(\mathcal{W}_a \otimes \mathcal{O}_B(\zeta - \phi))$ and $\boldsymbol{FM}_B(\mathcal{W}_a^\vee \otimes \mathcal{O}_B(\zeta - \phi))$ for all $a > 0$ and all sections $\phi : \mathbb{P}^1 \to B$ of β. Again we proceed by induction in a.

Let $a = 1$. By definition $\mathcal{W}_1$ is the unique non-split extension

$$0 \to \mathcal{O}_B \to \mathcal{W}_1 \to \zeta_* \mathcal{O}(-1) \to 0,$$

and hence $\mathcal{W}_1 = \mathcal{O}_B(\zeta)$ and $\mathcal{W}_1^\vee = \mathcal{O}_B(-\zeta)$. In particular $\mathcal{W}_1^\vee \otimes \mathcal{O}_B(\zeta - \phi) = \mathcal{O}_B(-\phi)$. Consequently by Lemma 7.3 we get

$$\begin{aligned} \boldsymbol{FM}_B(\mathcal{W}_1^{vee} \otimes \mathcal{O}_B(\zeta - \phi)) &= \mathcal{V}_1 \otimes \mathcal{O}_B(\phi - e - (1 + \phi \cdot e)f)[-1] \\ &= \mathcal{O}_B(\phi - (1 + \phi \cdot e)f)[-1]. \end{aligned}$$

Substituting $\phi = \tau_B^*(\xi)$ we get

$$\begin{aligned} \boldsymbol{FM} \circ \tau_B^* \circ \boldsymbol{D}_B \circ \boldsymbol{FM}_B(\mathcal{O}_B(-\xi)) &= \mathcal{O}_B(\tau_B^*(\xi) - (1 + \tau_B^*(\xi) \cdot e - \xi \cdot e)f) \\ &= \mathcal{O}_B(\tau_B^*(\xi) - (1 + \xi \cdot \zeta - \xi \cdot e)f). \end{aligned}$$

Let now $a = 2$. We have a short exact sequence

$$0 \to \mathcal{O}_B(f) \to \mathcal{W}_2 \to \mathcal{O}_B(\zeta) \to 0$$

and so

$$0 \to \mathcal{O}_B(-\phi) \to \mathcal{W}_2^\vee \otimes \mathcal{O}_B(\zeta - \phi) \to \mathcal{O}_B(\zeta - \phi - f) \to 0. \tag{7.7}$$

In particular we need to calculate $\boldsymbol{FM}_B(\mathcal{O}_B(\zeta - \phi))$. For this note that since $\mathcal{O}_B(\zeta - \phi)$ is a line bundle which has degree zero on the fibers of β, the sheaf $\boldsymbol{FM}_B^0(\mathcal{O}_B(\zeta - \phi))$ will have to be torsion free and torsion at the same time and so $\boldsymbol{FM}_B^0(\mathcal{O}_B(\zeta - \phi)) = 0$ (see the argument on p. 536). Consequently if we apply $\boldsymbol{FM}_B$ to the exact sequence

$$0 \to \mathcal{O}_B(\zeta - \phi) \to \mathcal{O}_B(\zeta) \to \phi_* \mathcal{O}_{\mathbb{P}^1}(\zeta \cdot \phi) \to 0,$$

we will get a short exact sequence of sheaves

$$\begin{aligned} 0 \to \mathcal{O}_B(\zeta - 2e - 2f) &\to \mathcal{O}_B(\phi - e - (1 + \phi \cdot e - \phi \cdot \zeta)f) \\ &\to \boldsymbol{FM}_B^1(\mathcal{O}_B(\zeta - \phi)) \to 0. \end{aligned}$$

In other words $\boldsymbol{FM}_B^1(\mathcal{O}_B(\zeta - \phi)) \otimes \mathcal{O}_B(e - \phi + (1 + \phi \cdot e - \phi \cdot \zeta)f) = \mathcal{O}_D$, where D is an effective divisor in the linear system $|\mathcal{O}_B(\phi - \zeta + e + (1 - \phi \cdot e + \phi \cdot \zeta)f)|$.

To understand this linear system better consider the section $\mu : \mathbb{P}^1 \to B$ for which $[\mu] = [\phi] - [\zeta]$ in $\mathbb{MW}(B, e)$. Then as in section 5 we can write

$$\mathcal{O}_B(\phi - \zeta) = \mathcal{O}_B(\mu - e + af + bn_1 + cn_2).$$

Taking into account that $\mu \cdot n_i = 1 - \phi \cdot n_i$ and that $\mu^2 = -1$ we can solve for a, b and c to get

$$a = -1 + \phi \cdot e - \phi \cdot \zeta + \phi \cdot n_1 + \phi \cdot n_2, \qquad b = -\phi \cdot n_1, \qquad c = -\phi \cdot n_2,$$

which yields

$$\begin{aligned} \mathcal{O}_B(\phi - \zeta + e + (1 - \phi \cdot e + \phi \cdot \zeta)f) &= \mathcal{O}_B(\mu + (\phi \cdot n_1)o_1 + (\phi \cdot n_2)o_2) \\ &= \mathcal{O}_B(\mu + (\mu \cdot o_1)o_1 + (\mu \cdot o_2)o_2). \end{aligned}$$

Therefore, the numerical section $\mu + (\phi \cdot n_1)o_1 + (\phi \cdot n_2)o_2$ is the only effective divisor in the linear system $|\mathcal{O}_B(\phi - \zeta + e + (1 - \phi \cdot e + \phi \cdot \zeta)f)|$ and so $D = \mu + (\phi \cdot n_1)o_1 + (\phi \cdot n_2)o_2$ as divisors. Note that the fact that ϕ is a section implies that $\phi \cdot n_i$ is either zero or one, and so D is always reduced.

This implies $\boldsymbol{FM}_B(\mathcal{O}_B(\zeta - \phi)) = i_{D*}\mathcal{O}_D \otimes \mathcal{O}_B(\phi - e - (1 + \phi \cdot e - \phi \cdot \zeta)f)[-1]$, where $i_D : D \hookrightarrow B$ is the natural inclusion. Next note that by definition of $\boldsymbol{FM}_B$ we have $\boldsymbol{FM}_B(K \otimes \beta^* M) = \boldsymbol{FM}_B(K) \otimes \beta^* M$ for any locally free sheaf $M \to \mathbb{P}^1$. Thus

$$\boldsymbol{FM}_B(\mathcal{O}_B(\zeta - \phi - f)) = i_{D*}\mathcal{O}_D \otimes \mathcal{O}_B(\phi - e - (2 + \phi \cdot e - \phi \cdot \zeta)f)[-1].$$

We are now ready to apply $\boldsymbol{FM}_B$ to (7.7). The result is

$$\begin{array}{l} 0 \longrightarrow 0 \longrightarrow \mathcal{S}^0 \longrightarrow 0 \\ \hookrightarrow \mathcal{O}_B(\phi - (1 + \phi \cdot e)f) \longrightarrow \mathcal{S}^1 \longrightarrow i_{D*}i_D^*\mathcal{O}_B(\phi - e - (2 + \phi \cdot e - \phi \cdot \zeta)f) \longrightarrow 0, \end{array}$$

where $\mathcal{S}^i := \boldsymbol{FM}^i_B(\mathcal{W}_2^\vee \otimes \mathcal{O}_B(\zeta - \phi))$.

Writing $\mathcal{L} := \mathcal{O}_B(-e-(1-\phi\cdot\zeta)f)$ and $\mathcal{F} := \mathcal{S}^1 \otimes \mathcal{O}_B(-\phi+(1+\phi\cdot e)f)$, we find a non-split short exact sequence

$$0 \to \mathcal{O}_B \to \mathcal{F} \to i_{D*}i_D^*\mathcal{L} \to 0. \tag{7.8}$$

Next we analyze the space of such extensions. We want to calculate

$$\mathrm{Ext}^1(i_{D*}i_D^*\mathcal{L}, \mathcal{O}_B) = Hom_{D^b(B)}(i_{D*}i_D^*\mathcal{L}, \mathcal{O}_B[1]) = \mathbb{H}^0(B, (i_{D*}i_D^*\mathcal{L})^\vee[1]).$$

As before, after tensoring the short exact sequence

$$0 \to \mathcal{O}_B(-D) \to \mathcal{O}_B \to i_{D*}\mathcal{O}_D \to 0$$

of the effective divisor D by $\mathcal{L}$ we get a quasi-isomorphism

$$\begin{bmatrix} \mathcal{L}(-D) \\ \downarrow \\ \mathcal{L} \end{bmatrix}^{-1}_{0} \xrightarrow{\text{q.i.}} i_{D*}i_D^*\mathcal{L},$$

and so

$$(i_{D*}i_D^*\mathcal{L})^\vee[1] = \begin{bmatrix} \mathcal{L}^\vee \\ \downarrow \\ \mathcal{L}^\vee(D) \end{bmatrix}^{-1}_{0} = i_{D*}i_D^*(\mathcal{L}^\vee(D)).$$

In particular $\mathrm{Ext}^1(i_{D*}i_D^*\mathcal{L}, \mathcal{O}_B) = H^0(B, i_{D*}i_D^*(\mathcal{L}^\vee(D))) = H^0(D, i_D^*(\mathcal{L}^\vee(D)))$. Since D is a tree of smooth rational curves, the dimension of the space of global sections of the line bundle $i_D^*(\mathcal{L}^\vee(D))$ will depend only on the degree of $\mathcal{L}^\vee(D)$ on each component of D. But $D = \mu + (\phi\cdot n_1)o_1 + (\phi\cdot n_2)o_2 = \mu + (\mu\cdot o_1)o_1 + (\mu\cdot o_2)o_2$ and since μ is a section of β we know that $\mu\cdot o_i$ is either 0 or 1. We can distinguish three cases:

(a) $\mu\cdot o_1 = \mu\cdot o_2 = 0$, i.e. $\mu \in \mathrm{Pic}^W(B)$ and $D = \mu$;

(b) μ intersects only one of the o_i's, i.e. D is the union of μ and that o_i;

(c) $\mu\cdot o_1 = \mu\cdot o_2 = 1$ and so $D = \mu + o_1 + o_2$.

Also since D is linearly equivalent to $\phi - \zeta + e + (1 - \phi\cdot e + \phi\cdot\zeta)f$ we find

$$\mathcal{L}\cdot\mu = -1, \qquad \mathcal{L}\cdot o_1 = \mathcal{L}\cdot o_2 = 0.$$

This gives the following answers for $\mathrm{Ext}^1(i_{D*}i_D^*\mathcal{L}, \mathcal{O}_B)$:

in case (a): Since $D = \mu$ we have $(\mathcal{L}^\vee(D))_{|D} = (\mathcal{L}^\vee(\mu))_{|\mu} = \mathcal{O}_\mu(1) \otimes \mathcal{O}_\mu(-1) = \mathcal{O}_\mu$ and so $\mathrm{Ext}^1(i_{D*}i_D^*\mathcal{L}, \mathcal{O}_B) = H^0(\mu, \mathcal{O}_\mu) = \mathbb{C}$.

in case (b): Say for concreteness $\mu \cdot o_1 = 1$ and $\mu \cdot o_2 = 0$. Then $D = \mu + o_1$ is a normal crossing divisor with a single singular point $\{x\} = \mu \cap o_1$. Then $(\mathcal{L}^\vee(D))_{|\mu} = \mathcal{O}_\mu(1) \otimes \mathcal{O}_\mu = \mathcal{O}_\mu(1)$ and $(\mathcal{L}^\vee(D))_{|o_1} = \mathcal{O}_{o_1} \otimes \mathcal{O}_{o_1}(-1) = \mathcal{O}_{o_1}(-1)$. Hence $(\mathcal{L}^\vee(D))_{|D}$ is the line bundle on D obtained by identifying the fiber $(\mathcal{O}_\mu(1))_x$ with the fiber $(\mathcal{O}_{o_1}(-1))_x$. Since $H^0(\mathcal{O}_{o_1}(-1)) = 0$ it follows that $\mathrm{Ext}^1(i_{D*}i_D^*\mathcal{L}, \mathcal{O}_B) = H^0(D, (\mathcal{L}^\vee(D))_{|D})$ can be identified with the space of all sections of $\mathcal{O}_\mu(1)$ that vanish at $x \in \mu$, i.e. we again have $\mathrm{Ext}^1(i_{D*}i_D^*\mathcal{L}, \mathcal{O}_B) = \mathbb{C}$.

in case (c): The divisor $D = \mu + o_1 + o_2$ is again a normal crossings divisor but has now two singular points x_1 and x_2, where $\{x_i\} = \mu \cap o_i$ for $i = 1, 2$. In this case we have $(\mathcal{L}^\vee(D))_{|\mu} = \mathcal{O}_\mu(2)$ and $(\mathcal{L}^\vee(D))_{|o_i} = \mathcal{O}_{o_i}(-1)$. Hence $\mathrm{Ext}^1(i_{D*}i_D^*\mathcal{L}, \mathcal{O}_B)$ gets identified with the space of all sections in $\mathcal{O}_\mu(2)$ vanishing at the points x_1 and x_2 and is therefore one dimensional.

In other words we always have a unique (up to isomorphism) choice for the sheaf $\mathcal{F}$. In fact, it is not hard to identify the middle term of the non-split extension (7.8). Indeed, let $o := D - \mu$ be the union of the vertical components of D. We have a short exact sequence:

$$0 \to \mathcal{O}_o(-\mu) \to H^0(o, \mathcal{O}_o(\mu)) \otimes \mathcal{O}_o \to \mathcal{O}_o(\mu) \to 0.$$

When we pull it back via

$$\mathcal{O}_B(\mu) \to \mathcal{O}_o(\mu)$$

we get a non-split sequence

$$0 \to \mathcal{O}_o(-\mu) \to \mathcal{F}' \to \mathcal{O}_B(\mu) \to 0.$$

Since we have already seen that such an extension is unique, we conclude that $\mathcal{F}' = \mathcal{F}$.

We have shown that $\boldsymbol{FM}_B(\mathcal{W}_2^\vee \otimes \mathcal{O}_B(\zeta - \phi))[1]$ is a rank one sheaf on B such that:

- The torsion in $\boldsymbol{FM}_B(\mathcal{W}_2^\vee \otimes \mathcal{O}_B(\zeta - \phi))[1]$ is $\mathcal{O}_o(-\mu)$.
- $\boldsymbol{FM}_B(\mathcal{W}_2^\vee \otimes \mathcal{O}_B(\zeta - \phi))/(\text{torsion}) = \mathcal{O}_B(2\phi - \zeta + e + (\phi\cdot\zeta - 2\phi\cdot e)f - o)$.
- The sheaf $\boldsymbol{FM}_B(\mathcal{W}_2^\vee \otimes \mathcal{O}_B(\zeta - \phi))[1]$ is the unique non-split extension of the line bundle $\mathcal{O}_B(2\phi - \zeta + e + (\phi\cdot\zeta - 2\phi\cdot e)f - o)$ by the torsion sheaf $\mathcal{O}_o(-\mu)$.

Let $a = 3$. Then the short exact sequence

$$0 \to \mathcal{O}_B(2f) \to \mathcal{W}_3 \to \mathcal{W}_2 \to 0$$

induces a short exact sequence

$$0 \to \mathcal{W}_2^\vee \otimes \mathcal{O}_B(\zeta - \phi) \to \mathcal{W}_3^\vee \otimes \mathcal{O}_B(\zeta - \phi) \to \mathcal{O}_B(\zeta - \phi - 2f) \to 0.$$

Applying $\boldsymbol{FM}_B$ one gets again that $\boldsymbol{FM}_B^0(\mathcal{W}_3^\vee \otimes \mathcal{O}_B(\zeta - \phi)) = 0$ and $\boldsymbol{FM}_B^1(\mathcal{W}_3^\vee \otimes \mathcal{O}_B(\zeta - \phi))$ fits in the non-split short exact sequence

$$\begin{aligned} 0 \to \boldsymbol{FM}_B^1(\mathcal{W}_2^\vee(\zeta - \phi)) &\to \boldsymbol{FM}_B^1(\mathcal{W}_3^\vee(\zeta - \phi)) \\ &\to \boldsymbol{FM}_B^1(\mathcal{O}(\zeta - \phi)) \otimes \mathcal{O}(-2f) \to 0. \end{aligned}$$

Now recall that

$$\boldsymbol{FM}_B(\mathcal{O}_B(\zeta - \phi)) = i_{D*}i_D^*\mathcal{O}_B(\phi - e - (1 + \phi\cdot e - \phi\cdot\zeta)f),$$

where $D = \mu + (\mu\cdot o_1)o_1 + (\mu\cdot o_2)o_2$ is the unique effective divisor in the linear system $|\mathcal{O}_B(\phi - \zeta + e + (1 - \phi\cdot e + \phi\cdot\zeta)f)|$.

In particular we have

$$\begin{aligned} \mu\cdot e &= \phi\cdot e - 1 + 1 - \phi\cdot e + \phi\cdot\zeta = \phi\cdot\zeta \\ \mu\cdot\zeta &= \phi\cdot\zeta + 1 + 1 - \phi\cdot e + \phi\cdot\zeta - \mu\cdot o_1 - \mu\cdot o_2 \\ &= 2 - \phi\cdot e + 2\phi\cdot\zeta - \mu\cdot o_1 - \mu\cdot o_2 \\ \mu\cdot\phi &= -1 - \phi\cdot\zeta + \phi\cdot e + 1 - \phi\cdot e + \phi\cdot\zeta - (\mu\cdot o_1)(\phi\cdot o_1) - (\mu\cdot o_2)(\phi\cdot o_2) \\ &= 0, \end{aligned}$$

and hence

$$i_\mu^*\mathcal{O}_B(\phi - e - (1 + \phi\cdot e - \phi\cdot\zeta)f) = \mathcal{O}_\mu(-1 - \phi\cdot e).$$

We are now ready to calculate $\boldsymbol{FM}_B(\mathcal{O}_B(\zeta - \phi)) \otimes \mathcal{O}(-2f)$ for the three possible shapes of the divisor D.

Case (a) $D = \mu$ and so $\boldsymbol{FM}_B(\mathcal{O}_B(\zeta - \phi)) \otimes \mathcal{O}(-2f) = \mathcal{O}_\mu(-3 - \phi \cdot e)$. Furthermore we showed that in this case we have $\boldsymbol{FM}_B^1(\mathcal{W}_2^\vee(\zeta - \phi)) = \mathcal{O}_B(2\phi - \zeta + e + (\phi \cdot \zeta - 2\phi \cdot e)f)$ and so after twisting (7.9) by $\mathcal{O}_B(2\phi - \zeta + e + (\phi \cdot \zeta - 2\phi \cdot e)f)^{-1}$ we get a non-split short exact sequence

$$0 \to \mathcal{O}_B \to ? \to \mathcal{O}_\mu(a) \to 0,$$

where

$$? = \boldsymbol{FM}_B^1(\mathcal{W}_3^\vee(\zeta - \phi)) \otimes \mathcal{O}_B(2\phi - \zeta + e + (\phi \cdot \zeta - 2\phi \cdot e)f)^{-1},$$

and

$$a = -3 - \phi \cdot e + \mu \cdot (-2\phi + \zeta - e - (\phi \cdot \zeta - 2\phi \cdot e)f) = -1.$$

Therefore we must have $? = \mathcal{O}_B(\mu) = \mathcal{O}_B(\phi - \zeta + e + (1 - \phi \cdot e + \phi \cdot \zeta)f)$ and so

$$\boldsymbol{FM}_B^1(\mathcal{W}_3^\vee(\zeta - \phi)) = \mathcal{O}_B(3\phi - 2\zeta + 2e + (1 - 3\phi \cdot e + 2\phi \cdot \zeta)f).$$

Case (b) In this case μ intersects exactly one of the o_i, say o_1. Then $D = \mu + o_1$ and so $\boldsymbol{FM}_B^1(\mathcal{O}_B(\zeta - \phi)) \otimes \mathcal{O}_B(-2f) = \mathcal{O}_\mu(-3 - \phi \cdot e) \cup_x \mathcal{O}_{o_1}$ Moreover the torsion in $\boldsymbol{FM}_B^1(\mathcal{W}_2^\vee(\zeta - \phi))$ is $\mathcal{O}_{o_1}(-1)$ and $\boldsymbol{FM}_B^1(\mathcal{W}_2^\vee(\zeta - \phi))/(\text{torsion}) = \mathcal{O}_B(2\phi - \zeta + e + (\phi \cdot \zeta - 2\phi \cdot e)f - o_1)$. Tensoring (7.9) with $\mathcal{O}_{o_1}$ and taking into account the fact that $\boldsymbol{FM}_B^1(\mathcal{W}_2^\vee(\zeta - \phi))_{|o_1} = \mathbb{C}^2 \otimes \mathcal{O}_{o_1}$ we get a long exact sequence of $\mathcal{T}or$ sheaves

$$\cdots \longrightarrow \mathcal{T}or_1^{\mathcal{O}_B}(\mathcal{O}_\mu(-3-\phi\cdot e)\cup_x \mathcal{O}_{o_1}, \mathcal{O}_{o_1}) \hookrightarrow \mathbb{C}^2 \otimes \mathcal{O}_{o_1} \longrightarrow \boldsymbol{FM}_B^1(\mathcal{W}_3^\vee(\zeta - \phi))_{|o_1} i \longrightarrow \mathcal{O}_{o_1} \longrightarrow 0.$$

Next we calculate $\mathcal{T}or_1^{\mathcal{O}_B}(\mathcal{O}_\mu(-3 - \phi \cdot e) \cup_x \mathcal{O}_{o_1}, \mathcal{O}_{o_1})$.

Lemma 7.4. $\mathcal{T}or_1^{\mathcal{O}_B}(\mathcal{O}_\mu(-3 - \phi \cdot e) \cup_x \mathcal{O}_{o_1}, \mathcal{O}_{o_1}) = 0$

Proof. Recall that for any integer a we have the following short exact sequence of sheaves on B:

$$0 \to \mathcal{O}_{o_1}(-1) \to \mathcal{O}_\mu(a) \cup_x \mathcal{O}_{o_1} \to \mathcal{O}_\mu(a) \to 0.$$

Tensoring this sequence with $\mathcal{O}_{o_1}$ we obtain a long exact sequence of $\mathcal{T}or$ sheaves:

$$\mathcal{T}or_1^{\mathcal{O}_B}(\mathcal{O}_{o_1}(-1),\mathcal{O}_{o_1}) \to \mathcal{T}or_1^{\mathcal{O}_B}(\mathcal{O}_\mu(a)\cup_x \mathcal{O}_{o_1},\mathcal{O}_{o_1}) \to \mathcal{T}or_1^{\mathcal{O}_B}(\mathcal{O}_\mu(a),\mathcal{O}_{o_1}) \to \mathcal{O}_{o_1}(-1) \longrightarrow \mathcal{O}_{o_1} \longrightarrow \mathcal{O}_x \longrightarrow 0.$$

In order to calculate the sheaves $\mathcal{T}or_1^{\mathcal{O}_B}(\mathcal{O}_{o_1}(-1),\mathcal{O}_{o_1})$ and $\mathcal{T}or_1^{\mathcal{O}_B}(\mathcal{O}_\mu(a),\mathcal{O}_{o_1})$ recall that we have $\mathcal{T}or_i^{\mathcal{O}_B}(K,M) = \mathcal{H}^{-i}(K \overset{L}{\otimes}_{\mathcal{O}_B} M)$ for any two objects $K, M \in D^b(B)$. Now note that $\mathcal{O}_{o_1}(-1) = \mathcal{O}_{o_1} \otimes \mathcal{O}_B(-\mu)$ and that

$$\mathcal{O}_{o_1} \overset{\text{q.i.}}{=} \left[\begin{array}{c} \mathcal{O}_B(-o_1) \\ \downarrow \\ \mathcal{O}_B \end{array}\right]\begin{array}{l}-1\\ \\ 0\end{array}, \qquad \mathcal{O}_\mu(a) \overset{\text{q.i.}}{=} \left[\begin{array}{c} \mathcal{O}_B(af-\mu) \\ \downarrow \\ \mathcal{O}_B(af) \end{array}\right]\begin{array}{l}-1\\ \\ 0\end{array},$$

and so

$$\mathcal{O}_\mu(a) \overset{L}{\otimes} \mathcal{O}_{o_1} \overset{\text{q.i.}}{=} \left[\begin{array}{c} \mathcal{O}_B(-\mu-o_1) \\ \downarrow \\ \mathcal{O}_B(-\mu)\oplus\mathcal{O}_B(-o_1) \\ \downarrow \\ \mathcal{O}_B \end{array}\right]\begin{array}{l}-2\\ \\ -1\\ \\ 0\end{array} \otimes \mathcal{O}_B(af).$$

Similarly

$$\mathcal{O}_{o_1}(-1) \overset{L}{\otimes} \mathcal{O}_{o_1} \overset{\text{q.i.}}{=} \left[\begin{array}{c} \mathcal{O}_B(-2o_1) \\ \downarrow \\ \mathcal{O}_B(-o_1)\oplus\mathcal{O}_B(-o_1) \\ \downarrow \\ \mathcal{O}_B \end{array}\right]\begin{array}{l}-2\\ \\ -1\\ \\ 0\end{array} \otimes \mathcal{O}_B(-\mu).$$

Consequently $\mathcal{T}or_i^{\mathcal{O}_B}(\mathcal{O}_{o_1}(-1),\mathcal{O}_{o_1}) = \mathcal{T}or_i^{\mathcal{O}_B}(\mathcal{O}_\mu(a),\mathcal{O}_{o_1}) = 0$ for all $i \neq 0$. This proves the lemma. □

The previous lemma implies that $\boldsymbol{FM}_B^1(\mathcal{W}_3^\vee(\zeta-\phi))_{|o_1} = \mathbb{C}^3 \otimes \mathcal{O}_{o_1}$ and that $\boldsymbol{FM}_B^1(\mathcal{W}_3^\vee(\zeta-\phi))$ fits in the commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_B(2\phi-\zeta+e+(\phi\zeta-2\phi e)f-2o_1) & \to & ? & \to & \mathcal{O}_\mu(-4-\phi e) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \boldsymbol{FM}^1_B(\mathcal{W}_2^\vee(\zeta-\phi)) & \to & \boldsymbol{FM}^1_B(\mathcal{W}_3^\vee(\zeta-\phi)) & \to & \mathcal{O}_\mu(-3-\phi e)\cup_x \mathcal{O}_{o_1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{C}^2\otimes\mathcal{O}_{o_1} & \to & \mathbb{C}^3\otimes\mathcal{O}_{o_1} & \to & \mathcal{O}_{o_1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where ? is a non-split extension of $\mathcal{O}_\mu(-4-\phi e)$ by $\mathcal{O}_B(2\phi-\zeta+e+(\phi\zeta-2\phi e)f-2o_1)$. This implies that $? = \mathcal{O}_B(2\phi-\zeta+e+(\phi\zeta-2\phi e)f-3o_1)$ and that $\boldsymbol{FM}^1_B(\mathcal{W}_3^\vee(\zeta-\phi))$ fits in a short exact sequence

$$
\begin{aligned}
0 &\to \mathcal{O}_B(3\phi-2\zeta+2e+(1+2\phi\zeta-3\phi e)f-3o_1) \\
&\to \boldsymbol{FM}^1_B(\mathcal{W}_3^\vee(\zeta-\phi)) \to \mathbb{C}^3\otimes\mathcal{O}_{o_1}\to 0.
\end{aligned}
$$

In particular we see that the torsion in $\boldsymbol{FM}^1_B(\mathcal{W}_3^\vee(\zeta-\phi))$ is supported on o_1.

The same reasoning applied to the restriction of (7.9) to μ instead of o_1 implies that $\boldsymbol{FM}^1_B(\mathcal{W}_3^\vee(\zeta-\phi))/(\text{torsion})$ is isomorphic to the line bundle $\mathcal{O}_B(3\phi-2\zeta+2e+(1+2\phi\zeta-3\phi e)f-2o_1)$. Since $\mathcal{O}_B(3\phi-2\zeta+2e+(1+2\phi\zeta-3\phi e)f-2o_1)_{|o_1} = \mathcal{O}_{o_1}(2)$ we conclude that the torsion in $\boldsymbol{FM}^1_B(\mathcal{W}_3^\vee(\zeta-\phi))$ is isomorphic to the kernel of the natural map $\mathbb{C}^3\otimes\mathcal{O}_{o_1}\cong H^0(o_1,\mathcal{O}_{o_1}(2x))\otimes\mathcal{O}_{o_1}\to\mathcal{O}_{o_1}(2x)\cong\mathcal{O}_{o_1}(2)$. In particular we see that the torsion in $\boldsymbol{FM}^1_B(\mathcal{W}_3^\vee(\zeta-\phi))$ is a rank two vector bundle on o_1, which has no sections and is of degree -2, i.e. is isomorphic to $\mathcal{O}_{o_1}(-1)\oplus\mathcal{O}_{o_1}(-1)$.

<u>Case (c)</u> In this case $D=\mu+o_1+o_2$. An analysis, analogous to the one used in case (b), now shows that the torsion in $\boldsymbol{FM}^1_B(\mathcal{W}_3^\vee(\zeta-\phi))$ is isomorphic to $\mathcal{O}_{o_1}(-1)^{\oplus 2}\oplus\mathcal{O}_{o_2}(-1)^{\oplus 2}$ and that $\boldsymbol{FM}^1_B(\mathcal{W}_3^\vee(\zeta-\phi))/(\text{torsion})$ is isomorphic to the line bundle $\mathcal{O}_B(3\phi-2\zeta+2e+(1+2\phi\zeta-3\phi e)f-2o_1-2o_2)$.

Continuing inductively we get that for every $a \geq 1$ the object $\boldsymbol{FM}_B(\mathcal{W}_a^\vee \otimes \mathcal{O}_B(\zeta - \phi))[1]$ is a rank one sheaf on B such that

- The torsion in $\boldsymbol{FM}_B(\mathcal{W}_a^\vee \otimes \mathcal{O}_B(\zeta - \phi))[1]$ is isomorphic to
$$\mathcal{O}_{o_1}^{\oplus(a-1)(\phi\cdot n_1)}(-1) \oplus \mathcal{O}_{o_2}^{\oplus(a-1)(\phi\cdot n_2)}(-1).$$
(In this formula it is tacitly understood that the direct sum of zero copies of a sheaf is the zero sheaf.)
- The sheaf $\boldsymbol{FM}_B(\mathcal{W}_a^\vee \otimes \mathcal{O}_B(\zeta - \phi))[1]/(\text{torsion})$ is isomorphic to the line bundle
$$\begin{aligned}\mathcal{O}_B(a\phi - (a-1)\zeta + (a-1)e + ((a-2) + (a-1)\phi\cdot\zeta - a\phi\cdot e)f \\ - (a-1)(\phi\cdot n_1)o_1 - (a-1)(\phi\cdot n_2)o_2).\end{aligned}$$

Now by substituting $\phi = \tau_B^*(\xi)$ in the above formula and by noticing that $D(\mathcal{O}_{o_i}(-1)) = \mathcal{O}_{o_i}(-1)[-1]$ we obtain

$$\begin{aligned}\mathcal{H}^0\boldsymbol{T}_B(\mathcal{O}_B(-a\xi)) &= \\ &= \mathcal{O}_B(\tau_B^*(-a\xi) + ((-a\xi)\cdot(e-\zeta))f + (1-a)(e-\zeta+f) \\ &\qquad + (a-1)(\xi\cdot o_1)o_1 + (a-1)(\xi\cdot o_2)o_2), \\ \mathcal{H}^1\boldsymbol{T}_B(\mathcal{O}_B(-a\xi)) &= \mathcal{O}_{o_1}^{\oplus(a-1)(\xi\cdot o_1)}(-1) \oplus \mathcal{O}_{o_2}^{\oplus(a-1)(\xi\cdot o_2)}(-1),\end{aligned}$$

for all $a \geq 1$. We have already analyzed the case $a = 0$ above and so this proves the theorem for $L = \mathcal{O}_B(-a\xi)$ and $a \geq 0$. The cases $L = \mathcal{O}_B(a\xi)$ with $a > 0$ or $L = \mathcal{O}_B(\sum a_i\xi_i)$ with different ξ_i's are analyzed in exactly the same way. □

Remark 7.5. **(i)** *The calculation of $\boldsymbol{T}_B(L)$ in the proof of Theorem 7.1 works equally well on a rational elliptic surface in the five dimensional family from Corollary 3.6 (with the choice of ζ as in Remark 3.7). Since in this case* $\text{Pic}^W(B) = \text{Pic}(B)$, *we see that for a general B in the five dimensional family we have $\boldsymbol{T}_{B|\text{Pic}(B)} = \widetilde{T}_B$. In particular $\boldsymbol{T}_B$ sends all line bundles to line bundles and induces an affine automorphism on* $\text{Pic}(B)$.

(ii) *In the proof of Theorem 7.1 we also showed that the statement of Theorem 7.1(iii) admits a partial inverse. Namely, we showed that if L is a multiple of a section, then $\boldsymbol{T}_B(L)$ is a line bundle if* and only if $L \in \text{Pic}^W(B)$.

The previous theorem shows that the $\boldsymbol{T}_B$ action on $\mathrm{Pic}(B)$ is somewhat complicated. If we work modulo the exceptional curves o_1, o_2, the formulas simplify considerably. (Working modulo o_1, o_2 amounts to contracting these two curves.)

Corollary 7.6. *The action of $\widetilde{\boldsymbol{T}}_B$ induces an affine automorphism of $\mathrm{Pic}(B)/(\mathbb{Z}o_1 \oplus \mathbb{Z}o_2)$, namely:*

$$\widetilde{\boldsymbol{T}}_B(L) = \alpha_B^*(L) \otimes \mathcal{O}_B(e - \zeta + f) \qquad \mathrm{mod}\ (o_1, o_2).$$

Proof. Apply Theorem 7.1 together with (4.2). □

Using these two results we can now describe the action of $\boldsymbol{T}_B$ on sheaves supported on curves in B. Let $C \subset B$ be a curve which is finite over $\mathbb{P}^1$. Denote by $i_C : C \hookrightarrow B$ the inclusion map. For the purposes of the spectral construction we will need to calculate the action of the spectral involution $\boldsymbol{T}_B$ on sheaves of the form $i_{C*}i_C^*L$ for some $L \in \mathrm{Pic}(B)$:

Proposition 7.7. *Let $C \subset B$ be a curve which is finite over $\mathbb{P}^1$ and such that $\mathcal{O}_B(C) \in \mathrm{Pic}^W(B)$ (for example we may take C in the linear system $|re + kf|$ for some integers r, k). Let $L \in \mathrm{Pic}(B)$. Put $D := \alpha_B(C)$. Then*

(a) $\boldsymbol{T}_B(i_{C*}i_C^*L) = i_{D*}i_D^*(\boldsymbol{T}_B(L))$.

(b) $\boldsymbol{T}_B(i_{C*}i_C^*L) = i_{D*}i_D^*(\alpha_B^*(L) \otimes \mathcal{O}_B(e - \zeta + f))$.

Proof. Since C is assumed to be finite over $\mathbb{P}^1$ it follows that i_C^*L will be flat over $\mathbb{P}^1$ and so $V = \boldsymbol{FM}_B(L)$ will be a vector bundle on B of rank $r = C \cdot f$, which is semistable and of degree zero on every fiber of β. But then τ_B^*V will be again a vector bundle of this type. Moreover if f_t is a general fiber of β then we can write $V_{|f_t} \cong a_1 \oplus \ldots \oplus a_r$, where a_i are line bundles of degree zero on f_t. In fact if we put $\{p_1, \ldots, p_r\} = C \cap f_t$ for the intersection points of C and f_t we have $a_i = \mathcal{O}_{f_t}(p_i - e(t))$. Now τ_B induces an isomorphism $\tau_B : f_{\tau_{\mathbb{P}^1}(t)} \to f_t$ and

$$(\tau_B^*V)_{|f_{\tau_{\mathbb{P}^1}(t)}} = \tau_B^*a_1 \oplus \ldots \oplus \tau_B^*a_r.$$

By definition $\tau_B = t_\zeta \circ \alpha_B$. Since every translation on an elliptic curve induces the identity on Pic^0 it follows that $\tau_B^*a_i = \alpha_B^*a_i = \mathcal{O}_{f_{\tau_{\mathbb{P}^1}(t)}}(\alpha_B(p_i) - e(f_{\tau_{\mathbb{P}^1}(t)}))$. This shows that $\boldsymbol{FM}_B^{-1}(\tau_B^*V)$ will be a line bundle supported on $D = \alpha_B(C)$ and so to prove (a) we only need to identify this line bundle explicitly.

Consider the short exact sequence

$$0 \to L(-C) \to L \to i_{C*}i_C^*L \to 0.$$

Applying the exact functor $\boldsymbol{T}_B$ we get a long exact sequence of sheaves

$$\begin{array}{l} 0 \longrightarrow \mathcal{H}^0\boldsymbol{T}_B(L(-C)) \longrightarrow \mathcal{H}^0\boldsymbol{T}_B(L) \longrightarrow \boldsymbol{T}_B(i_{C*}i_C^*L) \\ \quad \longrightarrow \mathcal{H}^1\boldsymbol{T}_B(L(-C)) \longrightarrow \mathcal{H}^1\boldsymbol{T}_B(L) \longrightarrow 0. \end{array}$$

However, by parts (i) and (iii) of Theorem 7.1 we have

$$\mathcal{H}^1\boldsymbol{T}_B(L(-C)) = \mathcal{H}^1\boldsymbol{T}_B(L)$$

and so $\boldsymbol{T}_B(i_{C*}i_C^*L)$ fits in a short exact sequence

$$0 \to \mathcal{H}^0\boldsymbol{T}_B(L(-C)) \to \mathcal{H}^0\boldsymbol{T}_B(L) \to \boldsymbol{T}_B(i_{C*}i_C^*L) \to 0.$$

But in the proof of Theorem 7.1 we showed that for any line bundle $K \in \mathrm{Pic}(B)$ one has

$$\mathcal{H}^0\boldsymbol{T}_B(K) = \widetilde{T}_B(K) \otimes \mathcal{O}_B((c_1(K)\cdot o_1)o_1 + (c_1(K)\cdot o_2)o_2).$$

Taking into account that $\mathcal{O}(o_i)_{|C} = \mathcal{O}_C$ we can twist the above exact sequence by

$$\mathcal{O}_B(-(c_1(K)\cdot o_1)o_1 - (c_1(K)\cdot o_2)o_2)$$

to obtain

$$0 \to \widetilde{\boldsymbol{T}}_B(L(-C)) \to \widetilde{\boldsymbol{T}}_B(L) \to \boldsymbol{T}_B(i_{C*}i_C^*L) \to 0.$$

To calculate $\widetilde{\boldsymbol{T}}_B(L(-C))$ let $\Omega : \mathrm{Pic}(B) \to \mathrm{Pic}(B)$ denote the linear part of the affine map $\widetilde{\boldsymbol{T}}_B$. In other words $\Omega(L) = \tau_B^*(L) + (c_1(L)\cdot(e-\zeta))f + (c_1(L)\cdot f)(e-\zeta+f)$ and $\widetilde{\boldsymbol{T}}_B(L) = \omega(L) + (e-\zeta+f)$. Then $\widetilde{\boldsymbol{T}}_B(L(-C)) = \widetilde{\boldsymbol{T}}_B(L) \otimes \mathcal{O}_B(-\Omega(C))$.

Using the formula describing Ω one checks immediately that Ω is a linear involution of $\mathrm{Pic}(B)$ which preserves the intersection pairing. Also we have $\Omega(o_1) = -o_2$ and $\Omega(o_2) = -o_2$ and so Ω preserves $\mathrm{Span}(o_1, o_2)^{\perp}$. But according to Corollary 7.6 the restriction of Ω to $\mathrm{Span}(o_1, o_2)^{\perp} \supset \mathrm{Pic}^W(B)$ coincides with the restriction of α_B^*, which yields

$$\begin{aligned}\widetilde{\boldsymbol{T}}_B(L(-C)) &= \widetilde{\boldsymbol{T}}_B(L) \otimes \mathcal{O}_B(-\Omega(C)) \\ &= \widetilde{\boldsymbol{T}}_B(L) \otimes \mathcal{O}_B(-\alpha_B^*(C)) = \widetilde{\boldsymbol{T}}_B(L) \otimes \mathcal{O}_B(-D).\end{aligned}$$

Consequently $\boldsymbol{T}_B(i_{C*}i_C^*L)$ fits in the exact sequence

$$0 \to \widetilde{\boldsymbol{T}}_B(L) \otimes \mathcal{O}_B(-D) \to \widetilde{\boldsymbol{T}}_B(L) \to \boldsymbol{T}_B(i_{C*}i_C^*L) \to 0.$$

But as we saw above $\boldsymbol{T}_B(i_{C*}i_C^*L)$ is the extension by zero of some line bundle on D and so we must have $\boldsymbol{T}_B(i_{C*}i_C^*L) = i_{D*}i_D^*\widetilde{\boldsymbol{T}}_B(L)$. Finally note that α_B^* preserves $\mathrm{Pic}^W(B)$ since $\alpha_B^*(o_1) = o_2$. Therefore D is disjoint from o_1 and o_2 and so the restriction of $\widetilde{\boldsymbol{T}}_B(L)$ to D will be the same as the restriction of the projection of $\widetilde{\boldsymbol{T}}_B(L)$ onto $\mathrm{Span}(o_1, o_2)^{\perp}$. Applying again Corollary 7.6 we get that $i_{D*}i_D^*\widetilde{\boldsymbol{T}}_B(L) = i_{D*}i_D^*\mathcal{O}_B(\alpha_B^*L + (e - \zeta + f))$. The Proposition is proven. □

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