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Holomorphic Vector Bundles, Knots and the Rozansky-Witten Invariants

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Abstract

Link invariants, for 3-manifolds, are defined in the context of the Rozansky-Witten theory. To each knot in the link one associates a holomorphic bundle over a holomorphic symplectic manifold X. The invariants are evaluated for $b_1(M) \geq 1$ and X Hyper-Kähler. To obtain invariants of Hyper-Kähler X one finds that the holomorphic vector bundles must be hyper-holomorphic. This condition is derived and explained. Some results for X not Hyper-Kähler are presented.

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1 Introduction

This paper is concerned with the definition and evaluation of invariants that can be associated with knots and links in the context of the Rozansky-Witten model [RW]. This theory has as its basic data a 3-manifold M and a holomorphic symplectic manifold X. The path integral for this theory is a supersymmetric theory based on maps from M to X. I will give a quick review of this in the next section. For more details of the construction one should consult the references.

Rozansky and Witten observed that their theory is a kind of Grassmann odd version of Chern-Simons theory where, amongst other things, the structure constants, $f^a_{\ bc}$ of the Lie algebra in Chern-Simons theory go over to $R^I_{\ JK\bar{L}}\eta_0^{\bar{L}}$ in the Rozansky-Witten model. The comparisons that are to be made are between the n-th order terms in a $1/\sqrt{k}$ expansion in Chern-Simons theory and the Rozansky-Witten invariant evaluated for some $\dim_{\mathbb{C}} X = 2n$ Hyper-Kähler manifold. More precisely, for a $\mathbb{Q}\mathrm{H}S$ (rational homology sphere), the n-th order term in the Chern-Simons theory for group G can be written as

$$Z_n^{CS}[M] = \sum_{\Gamma_n} b_{\Gamma_n}(G) \sum_a I_{\Gamma_n,a}(M) \tag{1}$$

while for the $dim_{\mathbb{C}} X = 2n$ Hyper-Kähler manifold the Rozansky-Witten invariant reads as

$$Z_X^{RW}[M] = \sum_{\Gamma_n} b_{\Gamma_n}(X) \sum_a I_{\Gamma_n,a}(M). \tag{2}$$

The notation is as follows. The Γ_n represent all the possible Feynman graphs of the theory and the sum over the label a is that of all possible ways of assigning Feynman diagrams to the same graph. The Feynman diagrams and graphs are the same in the Chern-Simons and Rozansky-Witten theories. The $I_{\Gamma_n,a}(M)$ are the integrals over M of products of Greens functions that appear in both theories.

The interesting part corresponds to the weights b_{Γ_n} as this is 'all' that distinguishes the two theories. Different weight systems will yield topological field theories providing the b_{Γ_n} obey the IHX relations [LMO]. Indeed both the $b_{\Gamma_n}(G)$ of Chern-Simons theory and the $b_{\Gamma_n}(X)$ of the Rozansky-Witten theory satisfy the IHX relations.

While one class of knot observables was defined in [RW] (and an algorithm given for the associated weights) they played no essential role there.

However, one can show that the expectation values of Wilson loop observables in the Chern-Simons theory, and of the knot observables of Rozansky and Witten take a form analogous to (1) and (2) respectively. Once more the differences lie in the weights. To obtain a topological theory of knots (and links) the weights associated with the knot observables need to satisfy the STU relation. This is clearly satisfied by the Wilson loop observables in Chern-Simons theory and, as I will show below, also satisfied by the weights of the knot invariants in the Rozansky-Witten model.

Why write this paper? While they are of interest in themselves, I believe that, amongst other things, we also need to have a better understanding of these observables in order to get at the surgery formulae for the Rozansky-Witten invariants $Z_X^{RW}[M]$ when the Hyper-Kähler manifold X has $dim_{\mathbb{C}} X \geq 4$. Of course one of the things one would like to know about these invariants is if they are part of the "universal" knot invariants that arise in the LMO construction. Another reason for studying these is that recently, Hitchin and Sawon [HS] have found that the Rozansky-Witten theory provides information about Hyper-Kähler manifolds. Hopefully one will have a bigger set of invariants for Hyper-Kähler manifolds by allowing for knot observables.

Before going on it is appropriate to ask why should this topological field theory give us invariants of Hyper-Kähler manifolds? In order to answer this question let us recall some other topological field theories. We know that certain supersymmetric quantum mechanics models yield information about a manifold X (the models depend on how much extra structure we are willing to place on X). So for example such supersymmetric models provide simple "proofs" of the index theorems[1]. These theories involve maps $S^1 \to X$. There are also topological field theories which yield the Gromov-Witten invariants of a complex manifold X. These models are based on holomorphic maps from a Riemann surface to a compact closed complex X, $\Sigma \to X$.

From this perspective one would expect that a theory based on maps from a 3-manifold into a Hyper-Kähler X, $M \to X$, would indeed give rise to invariants for X and that the correct question is, instead, why do they give invariants of 3-manifolds?

The crux of the matter is that what we learn depends by and large on what we know. A topological field theory from one point of view is a theory

[1] In the present setting there are topological field theories that yield the index formulae for the Euler characteristic of X (Gauss-Bonnet), the signature of X (Hirzebruch), if X is a spin manifold for the $\hat{A}$ genus (Atiyah-Singer), while if X is a complex manifold one can also obtain the Riemann-Roch formula for the T*odd* genus.

defined on the space of sections of a certain bundle. To define the theory it may be necessary to make certain additional choices, for example, to fix on a preferred Riemannian metric on the total space of the bundle. In the best cases this theory will give invariants that do not depend on any of the particular choices made. What information there is to extract will depend crucially on the bundle in question. In principle, however, the topological field theory will provide invariants for the total space of the bundle. If either the base or the fibre is well understood then the invariants are really invariants for the fibre or the base respectively.

In the case of supersymmetric quantum mechanics we know all there is to know about S^1 and so the topological field theories will yield information about X. The same is true for the Gromov-Witten theory, since Riemann surfaces are completely classified. The case of the Rozansky-Witten theory is very different. We do not know much about 3-manifolds nor about Hyper-Kähler manifolds. By fixing on ones favourite Hyper-Kähler manifold and varying the 3-manifold we get invariants for the 3-manifolds. On the other hand on picking a particular 3-manifold or by using other knowledge about the 3-manifold invariants and varying X we learn about X.

This paper is organized as follows. In the next section there is a brief summary of the Rozansky-Witten theory. In section 3 knot and link observables are introduced. The expectation values of the link observables are the link invariants. The concept of a hyper-holomorphic bundle is seen to arise naturally from the requirement that the observables will correspond to invariants for Hyper-Kähler X. That the observables do correspond to link invariants for the 3-manifold M requires that they satisfy the STU relation. This relation is derived in section 4. Section 5 is devoted to stating some explicit results that I have derived, the derivation being postponed till section 7. Section 6 is by way of a digression on the properties of the theory when $X = T^{4n}$ while in section 7 an outline of the proofs is given, the bulk of the work being defered to the references. Finally, in the appendix, a slightly more general class of observables is introduced.

All calculations are done using path integrals. The normalization that I have taken is so that the Rozansky-Witten for a 3-torus, T^3 is the Euler characteristic of the Hyper-Kähler manifold X (or more generally the integral of the Euler density of X if X is non-compact) [T].

Some of the results presented here have also been obtained by J. Sawon [S].

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Hitchin kindly explained to me the relevance of the Hyper-Kähler quotient construction as a means for finding hyper-holomorphic bundles. Boris Pioline brought my attention to the paper [MS] which in turn led me to Nigel Hitchin. I have benefited from conversations with M. Blau and L. Göttsche. I am grateful to the referee who suggested numerous improvements. I am especially indebted to M.S. Narasimhan for his interest and his kind advice at all stages of this work. This work was supported in part by the EC under TMR contract ERBF MRX-CT 96-0090.

2 The Rozansky-Witten Theory

The construction of the Rozansky-Witten model for holomorphic symplectic X is described in the appendix of [RW]. I will not need that level of generality here, though to prove some of the results that I present below one does need to have the full theory at ones disposal.

The action, for Hyper-Kähler X can be written down without picking a preferred complex structure from the S^2 of available complex structures on X. In this way one establishes that the theory yields invariants of X as a Hyper-Kähler manifold. Since the knot observables, in any case, require us to make such a choice fix on the complex structure, I, on X so that the ϕ^I are local holomorphic coordinates with respect to this complex structure. The action is, in the preferred complex structure,

$$S = \int_M L_1 \sqrt{h}\, d^3x + \int_M L_2 \tag{3}$$

where

$$\begin{aligned} L_1 &= \frac{1}{2} g_{ij} \partial_\mu \phi^i \partial^\mu \phi^j + g_{I\overline{J}} \chi^I_\mu D^\mu \eta^{\overline{J}} && (4)\\ L_2 &= \frac{1}{2}\left(\epsilon_{IJ} \chi^I D \chi^J - \frac{1}{3} \epsilon_{IJ} R^J_{\ KL\overline{M}} \chi^I \chi^K \chi^L \eta^{\overline{M}} \right). && (5) \end{aligned}$$

The covariant derivative is

$$D^{\overline{I}}_{\mu\,\overline{J}} = \partial_\mu \delta^{\overline{I}}_{\overline{J}} + (\partial_\mu \phi^i) \Gamma^{\overline{I}}_{i\overline{J}}. \tag{6}$$

The tensor ϵ_{IJ} is the holomorphic symplectic 2-form that is available on a Hyper-Kähler manifold. It is a closed, covariantly constant non-degenerate holomorphic 2-form. Non-degeneracy means that there exists a holomorphic tensor ϵ^{IJ} such that

$$\epsilon^{IJ} \epsilon_{JK} = \delta^I_K. \tag{7}$$

Both of the Lagrangians L_1 and L_2 are invariant under two independent BRST supersymmetries. In fact L_1 is BRST exact. These supersymmetries are also defined without the need of picking a prefered complex structure on X. But, since a prefered complex structure has already been chosen in writing the theory, it is easiest to exhibit the BRST operators in this complex structure. $\overline{Q}$ acts by

$$\begin{array}{ll} \overline{Q}\phi^I = 0, & \overline{Q}\phi^{\overline{I}} = \eta^{\overline{I}}, \\ \overline{Q}\eta^I = 0, & \overline{Q}\chi^I = -d\phi^I, \end{array} \tag{8}$$

while Q acts by

$$\begin{array}{ll} Q\phi^I = T^I_{\overline{J}}\eta^{\overline{J}}, & Q\phi^{\overline{I}} = 0, \\ Q\eta^{\overline{I}} = 0, & Q\chi^I = -T^I_{\overline{J}}d\phi^{\overline{J}} - \Gamma^I_{JK}T^I_{\overline{J}}\eta^{\overline{J}}\chi^K, \end{array} \tag{9}$$

where $T^I_{\overline{J}} = \epsilon^{IK}g_{K\overline{J}}$ and represents an isomorphism between $TX^{(1,0)}$ and $TX^{(0,1)}$. The BRST charges satisfy the algebra,

$$\overline{Q}^2 = 0, \quad \{Q, \overline{Q}\} = 0, \quad Q^2 = 0. \tag{10}$$

Set $\eta^I = T^I_{\overline{J}}\eta^{\overline{J}}$ in order to make contact with the notation of the bulk of [RW] and that used in [T] and [HT].

3 The Knot Observables

I will define knot and link invariants by associating holomorphic bundles over a holomorphic symplectic manifold X to the knot or link. However, special issues which arise when X is Hyper-Kähler are addressed in some detail.

3.1 Associating the Holomorphic Tangent Bundle

The observables associated to a knot, $\mathcal{K}$, that were suggested in [RW] are

$$\mathcal{O}_\alpha(K) = \mathrm{Tr}_\alpha \, \mathrm{P}\, e^{\oint_K A}, \tag{11}$$

where the $sp(n)$ connection is

$$A^I{}_J = d\phi^L \Gamma^I_{LJ} - \epsilon^{IM}\Omega_{MJKL}\chi^K\eta^L, \tag{12}$$

and α designates a representation Υ_α of $sp(n)$. Some properties of the connection are:

1. The connection is $\overline{Q}$ exact

$$A^I{}_J = -\overline{Q}\left(\chi^L \Gamma^I_{LJ}\right). \tag{13}$$

This does not mean that it is a "trivial" observable since it is the BRST variation of a connection.

2. On a general holomorphic symplectic manifold X, the connection is

$$\begin{aligned} A^I{}_J &= -\overline{Q}\left(\chi^L \Gamma^I_{LJ}\right) \\ &= d\phi^L \Gamma^I_{LJ} + R^I{}_{JK\overline{L}} \chi^K \eta^{\overline{L}} \end{aligned} \tag{14}$$

where Γ^I_{LJ} is some symmetric connection on the holomorphic tangent bundle and

$$R^I{}_{JK\overline{L}} = \overline{\partial}_{\overline{L}} \Gamma^I{}_{JK}, \tag{15}$$

is the Atiyah class of X. The Atiyah class is the obstruction to the connection being holomorphic [A].

3. If X is Hyper-Kähler then we might also want to be sure the observable does not depend on the particular choice of the S^2 of complex structures. This is indeed the case since,

$$QA^I{}_J = d_A \Lambda^I{}_J = d\Lambda^I{}_J + [A, \Lambda]^I{}_J, \tag{16}$$

where $\Lambda^I{}_J = \eta^L \Gamma^I_{LJ}$. A Q transformation is, therefore, equivalent to a gauge transformation and we are assured that the Wilson loop is Q invariant since it is gauge invariant. (The situation will be made clearer below)

3.2 Associating Holomorphic Vector Bundles

Let $E \to X$ be a holomorphic vector bundle over a holomorphic symplectic X with fibre V. The reason for choosing E to be a holomorphic bundle is that we want the STU relations to be satisfied, see section 4. In trying to mimic the construction of the observables (11) we will find some more stringent conditions on E. Let ω be a connection on E whose, on fixing the complex structure of X, $(0,1)$ component, in a holomorphic frame, vanishes that is

$$\begin{aligned} \omega &= \omega^{(1,0)}, \\ &= \omega_I \, dz^I. \end{aligned} \tag{17}$$

Since $\Phi : M \to X$ one can pull E back to the 3-manifold M and consider the connection

$$\begin{aligned} A &= -\overline{Q}\left(\chi^I \omega_I\right) \\ &= d\phi^I \omega_I + \overline{\partial}_{\overline{J}} \omega_I \, \chi^I \eta^{\overline{J}}, \end{aligned} \tag{18}$$

which, due to the presence of the fermion terms, is not just the usual pull back $\Phi^*(\omega)$. Associate to a knot the observable

$$\mathcal{O}_E(K) = \mathrm{Tr}_V \; \mathrm{P} \; \exp\left(\oint_K A\right). \tag{19}$$

Next we list the relevant properties of this connection and specify extra requirements on E so that we obtain a good observable:

1. The connection A is $\overline{Q}$ exact but non-trivial.

2. If X is Hyper-Kähler and one wants (19) to also be invariant under Q (and so not to depend on the choice of complex structure on X) then the connection must satisfy

$$F_\omega^{(2,0)} = 0, \tag{20}$$

as well as

$$T_K^{\overline{J}} \, \overline{\partial}_{\overline{J}} \, \omega_I = T_I^{\overline{J}} \, \overline{\partial}_{\overline{J}} \, \omega_K. \tag{21}$$

The condition (20) can be satisfied by choosing a Hermitian metric on the bundle and then taking ω to be the unique hermitian connection. Holomorphic bundles E that also satisfy (21) are said to be hyper-holomorphic: a term coined by Verbitsky [V]. It is an immediate consequence that such bundles are **stable**, since contracting (21) with ϵ^{IK} yields

$$g^{I\overline{J}} F_{I\overline{J}} = 0, \tag{22}$$

which was conjectured to be equivalent to the condition of stability by Hitchin and Kobayashi and proven to be so by Donaldson and Uhlenbeck and Yau. One easy result, by counting equations, is that (21) and (22) are equivalent when X is a Hyper-Kähler surface. There is an important converse due to Verbitsky [V]. Let E be a stable holomorphic bundle over a Hyper-Kähler manifold for a given complex structure I then, if $c_1(E)$ and $c_2(E)$ are invariant under the natural $Sp(1)$ action, E is hyper-holomorphic.

If the holomorphic bundle is hyper-holomorphic then

$$QA = d_A \Lambda, \quad \Lambda = \eta^{\bar{I}} T_{\bar{I}}^{J} \omega_J, \tag{23}$$

and, consequently, a Q transformation is equivalent to a gauge transformation.

Remark: Q and $\overline{Q}$ are the components of the $sp(1)$ doublet BRST operator Q_A in the preferred complex structure for X (For this see equations (2.17) to (2.22) in [RW].) Invariance under both of these operators means invariance under the action of Q_A. However, Q and $\overline{Q}$ are essentially to be identified with the twisted Dolbeault operators, ∂_ω and $\overline{\partial}$ respectively, in the preferred complex structure. Invariance under Q_A means that if we choose a different complex structure (say I') then the knot observables will be invariant under the corresponding BRST operators Q' and $\overline{Q}'$, which in turn are to be identified with ∂'_ω and $\overline{\partial}'$.

Remark: If one is only interested in obtaining 3-manifold invariants, then all one really requires is that E be holomorphic with respect to the given complex structure I on X.

3.3 The Meaning of Equation (21) and Hyper-holomorphic Vector Bundles

There is a nice geometric interpretation of the equation (21), already mentioned above, which is that it is the condition for which the holomorphic vector bundle E is holomorphic for the entire sphere's worth of complex structures. Let us see that this is the case in a more mundane manner.

Let I be a given complex structure on X. The complexified tangent and cotangent bundles, $TX_{\mathbb{C}}$ and $T^*X_{\mathbb{C}}$ split into a sum of holomorphic and anti-holomorphic bundles as $T^{(1,0)}X \oplus T^{(0,1)}X$ and $T^{*(1,0)}X \oplus T^{*(0,1)}X$ respectively. The decomposition is such that $(1-iI)/2 : T^*X_{\mathbb{C}} \to T^{*(1,0)}X$.

Since X is Hyper-Kähler , with complex structures I, J, and K satisfying the usual quaternionic rules, $I' = I + \delta I = I + \delta b J + \delta c K$ is an infinitesimally deformed complex structure. Denote the the splitting of $T^*X_{\mathbb{C}}$ with respect to I' by $T^{*(1,0)}X' \oplus T^{*(0,1)}X'$. If dz is a basis for $T^{*(1,0)}X$, $d\overline{z}$ is a basis for $T^{*(0,1)}X$, dw is a basis for $T^{*(1,0)}X'$ and $d\overline{w}$ is a basis for $T^{(0,1)}X'$ we find

$$dw = \left(1 - \alpha \overline{T}\right)\, dz + \overline{\alpha}\, T\, d\overline{z} \tag{24}$$

where $\alpha = (\delta c + i\delta b)/4)$ and $T = (J - iK) : T^{*(0,1)}X \to T^{*(1,0)}X$ or put another way $T d\overline{z} \in \mathrm{H}^1(X, T^{(1,0)}X)$. All of this fits within the Kodaira-

Spencer theory of complex deformations. The spheres worth of complex structures means that we only look at the one (complex) dimensional subspace of $\mathrm{H}^1(X, T^{(1,0)}X)$ spanned by T.

Let us now pass on to the case of holomorphic vector bundles. Our holomorphic bundles over X come equipped with a connection ω that satisfies

$$\omega^{(0,1)} = 0 \tag{25}$$

$$F_\omega^{(2,0)} = 0, \tag{26}$$

where the holomorphic splitting is with respect to the given complex structure, I, on X.

In this section by a hyper-holomorphic vector bundle I will mean a holomorphic vector bundle equipped with a given connection which has curvature of type $(1,1)_{\mathcal{J}}$ for **all** of the $\mathcal{J} \in S^2$ of complex structures on X. Now suppose that we want that E be hyper-holomorphic this means, in particular, that for the deformed complex structure I', that F_ω be of type $(1,1)'$. Given that F_ω with respect to I' should be of type $(1,1)'$ but of type $(1,1)$ with respect to I means that one gets conditions on the $(1,1)$ component of the curvature. These conditions are obtained on perusal of the following

$$\begin{aligned} F_\omega^{(1,1)'} &= F_{I\overline{J}}(\omega)\, dw^I\, d\overline{w}^{\overline{J}} \\ &= F_{I\overline{J}}(\omega) \left(dz - \alpha \overline{T}\, dz + \overline{\alpha}\, T\, d\overline{z}\right)^I \left(d\overline{z} - \overline{\alpha}\, T\, d\overline{z} + \alpha \overline{T}\, dz\right)^{\overline{J}} \\ &= F_{I\overline{J}}(\omega) \left(dz^I\, d\overline{z}^{\overline{J}} - dz^I \overline{\alpha}\, (T\, d\overline{z})^{\overline{J}} + dz^I \alpha (\overline{T}\, dz)^{\overline{J}} \right. \\ &\qquad \left. - \alpha (\overline{T}\, dz)^I d\overline{z}^{\overline{J}} + \overline{\alpha} (T d\overline{z})^I d\overline{z}^{\overline{J}} \right), \end{aligned} \tag{27}$$

the $(0,2)$ and $(2,0)$ components on the right hand side will vanish iff

$$T^I_{\overline{K}} F_{I\overline{J}}(\omega) = T^I_{\overline{J}} F_{I\overline{K}}(\omega). \tag{28}$$

These are precisely the equations (21) that we found in the previous section and so the current definition of a hyper-holomorphic vector bundle agrees with that of Verbitsky given in the previous section.

What is particularly satisfying is that the physics, demanding that (19) also be invariant under Q, leads naturally to this definition of a hyper-holomorphic bundle. This is the way I came to it before I was informed that this definition had already appeared in the mathematics literature [V]. Indeed the definition is older than this reference having already appeared in [MS] and the demonstration that the bundles in question are hyper-holomorphic is attributed, in that reference, to N. Hitchin.

3.4 On The Existence of Hyper-Holomorphic Bundles

Clearly the holomorphic tangent bundle of a Hyper-Kähler manifold is hyper-holomorphic, but apart from on a Hyper-Kähler surface I did not know of any general results on the existence of hyper-holomorphic bundles. N. Hitchin [H] has kindly answered the following question in the affirmative: Are there examples of hyper-holomorphic bundles over a hyper-Kähler X other than its holomorphic tangent bundle? Indeed he shows that there is a procedure for constructing such bundles which follows directly from the hyper-Kähler quotient construction [HKLR]. The details of the construction have also appeared in [GN]. I give a very brief description of the salient features.

Let G be a compact Lie group acting on a hyper-Kähler manifold Y, with either $H^1(Y, \mathbb{Z}) = 0$, or $H^2(G) = 0$, which preserves both the metric and the hyper-Kähler structure. Consequently the group preserves the three Kähler forms, ω_A, corresponding to the three complex structures $A = I$, J, K. For each Kähler form there is an associated moment map, $\mu_A :\to \mathfrak{g}^*$, to the dual vector space $\mathfrak{g}^*$ of the Lie algebra.

Each element ζ of the Lie algebra $\mathfrak{g}$ of G defines a vector field, denoted $\overline{\zeta}$, which generates the action of ζ on Y. Then, up to a constant for connected Y,

$$d\mu_A^{\overline{\zeta}} = i_{\overline{\zeta}}\omega_A, \tag{29}$$

defines $\mu_A^{\overline{\zeta}}$. The moment maps μ_A are defined by

$$\langle \mu_A(m), \zeta \rangle = \mu_A^{\overline{\zeta}}(m), \tag{30}$$

and they can be grouped together into one moment map

$$\mu : Y \to \mathbf{R}^3 \otimes \mathfrak{g}^*. \tag{31}$$

Fact 1 [HKLR]: For any $\zeta^* \in \mathbf{R}^3 \otimes \mathfrak{g}^*$ fixed by the action of G, the quotient space $X = \mu^{-1}(\zeta^*)/G$ has a natural Riemannian metric and hyper-Kähler structure.

Fact 2 [H, HKLR, GN]: Let $\pi : \mu^{-1}(\zeta^*) \to \mu^{-1}(\zeta^*)/G = X$ be the projection. Then, $\pi : \mu^{-1}(\zeta^*) \to X$ is a principal G-bundle which comes equipped with a natural connection Θ, where the horizontal space is the orthogonal complement of the tangent space of the orbit $T_y\mu^{-1}(\zeta^*)$ with $y \in \mu^{-1}(\zeta^*)$.

Fact 3 [H, GN]: The natural connection is hyper-holomorphic.

The upshot is that if the hyper-Kähler manifold of interest comes from a hyper-Kähler quotient construction then it comes equipped with a natural hyper-holomorphic principal bundle. Given a representation of G we can construct an associated hyper-holomorphic vector bundle, which is what we are after. In the case of infinite dimensional quotients (on the space of connections for example) the associated hyper-holomorphic vector bundles are index bundles, universal bundles, etc.. Explicit examples in the case of the monopole moduli space can be found in [MS] while for instantons one may refer to [GN].

4 The STU Relation

In the physics approach to topological field theory it is formally enough that one can exhibit metric independence via standard physics arguments. (A metric variation is BRST exact, for example, which is the case in Chern-Simons theory when one includes gauge fixing terms.) The essence of the argument in the case of Chern-Simons theory for 3-manifold invariants has been distilled, made mathematically precise and then abstracted. The net result is the so called IHX relation.

A crucial feature of the Rozansky-Witten theory is that the IHX relation is satisfied by the weights $b_\Gamma(X)$. A proof of this statement for $\mathbb{Q}\mathrm{H}S$ goes along the following lines (this is taken from [RW]). Vertices in a closed $2n$-vertex graph in this theory carry the curvature tensors $R^I{}_{JK\overline{L}}$. Their holomorphic labels are contracted with ϵ^{IJ} (thanks to the χ propagator). The anti-holomorphic labels are totally anti-symmetrized (since this involves products of $\eta_0^{\overline{I}}$) and from the Bianchi identity one has $\overline{\partial}_{\overline{M}} R^I{}_{JK\overline{L}} = \overline{\partial}_{\overline{L}} R^I{}_{JK\overline{M}}$ and so one obtains a $\overline{\partial}$-closed $(0, 2n)$ form on X, that is a map

$$\Gamma_{n,3} \to \mathrm{H}^{2n}(X). \tag{32}$$

The weight functions $b_\Gamma(X)$ satisfy the IHX relations by virtue of the fact that

$$\begin{aligned}&\overline{\partial}_{\overline{N}} \nabla_M R^I{}_{JK\overline{L}} \eta_0^{\overline{N}} \eta_0^{\overline{L}} \\ &\quad = \left(R^I{}_{PM\overline{N}} R^P{}_{JK\overline{L}} + R^I{}_{PK\overline{N}} R^P{}_{JM\overline{L}} + R^I{}_{JP\overline{N}} R^P{}_{KM\overline{L}}\right) \eta_0^{\overline{N}} \eta_0^{\overline{L}}\end{aligned} \tag{33}$$

This tells us that the right hand side is cohomologous to zero.

Chern-Simons theory has the IHX relation, essentially the Jacobi identity for the Lie algebra used in the definition of the theory, encoded in it in two different ways. Firstly, it is subsumed in the whole construction of gauge theories. Secondly, it is explicitly required in order for the BRST operator, Q, to be nilpotent $Q^2 = 0$. How is the IHX relation "built in" in the Rozansky-Witten theory? More concretely how is (33) manifest from the beginning? It is clear from (8) that it is not required for nilpotency of the operator $\overline{Q}$. However, the IHX relation (33) follows from the Bianchi identity for the curvature form and it is the Bianchi identity that ensures that the action (3) is BRST invariant. So in this sense the IHX relation is subsumed from the start. It is in this way that the formal physics proofs of metric independence and the mathematical proofs are connected.

Now in order to have a good knot or link invariant one would like the analogue of the STU relation (see for example [B]). In Chern-Simons theory this amounts to the Lie algebra commutation rules. Let T_a be a basis of generators for the Lie algebra in the $\mathbf{T}$ representation. Basically the STU relations says, $[T_a, T_b] = f^c_{ab}T_c$. The representation matrices are attached to the 3-point vertices in the loop observables. In the present context the STU relation is the following,

$$\overline{\partial}_{\overline{K}}\nabla_L F_{I\overline{J}}\,\eta_0^{\overline{J}}\eta_0^{\overline{K}} = -\left(R^N_{IL\overline{K}}F_{N\overline{J}} + [F_{L\overline{K}}, F_{I\overline{J}}]\right)\eta_0^{\overline{J}}\eta_0^{\overline{K}}, \tag{34}$$

again this equation tells us that the right hand side is cohomologous to zero. In analogy to the Chern-Simons theory the curvature tensor plays the role of the structure constants and the curvature 2-form the role of the representation matrices.

One derives (34) from the Bianchi identity for the curvature two form of the holomorphic vector bundle, as follows: Let E be a holomorphic vector bundle, choose the connection so that $\overline{\partial}_\omega = \overline{\partial}$, then[2]

$$F^{(0,2)}_\omega = 0. \tag{35}$$

The Bianchi identity $d_\omega F_\omega = 0$ tells us that

$$\partial_\omega F^{(2,0)}_\omega = 0, \quad \partial_\omega F^{(1,1)}_\omega = \overline{\partial} F^{(2,0)}_\omega, \quad \overline{\partial} F^{(1,1)}_\omega = 0. \tag{36}$$

We want to get a formula for

$$\begin{aligned}&\overline{\partial}_{\overline{K}}\nabla_L(\omega)F_{I\overline{J}} - (\overline{K}\leftrightarrow\overline{J})\\ &\quad = \left[\overline{\partial}_{\overline{K}}, \nabla_L(\omega)\right]F_{I\overline{J}} + \nabla_L(\omega)\overline{\partial}_{\overline{K}}F_{I\overline{J}} - (\overline{K}\leftrightarrow\overline{J}).\end{aligned} \tag{37}$$

The last term in this equation vanishes by virtue of the last equality in (36), so that

$$\begin{aligned}&\overline{\partial}_{\overline{K}}\nabla_L(\omega)F_{I\overline{J}} - (\overline{K}\leftrightarrow\overline{J})\\ &\quad = \left[\overline{\partial}_{\overline{K}}, \nabla_L(\omega)\right]F_{I\overline{J}} - (\overline{K}\leftrightarrow\overline{J})\\ &\quad = -R^N_{IL\overline{K}}F_{N\overline{J}} - [F_{L\overline{K}}, F_{I\overline{J}}] - (\overline{K}\leftrightarrow\overline{J}),\end{aligned} \tag{38}$$

[2]If we choose a Hermitian structure we could then fix on the unique Hermitian connection for which $\overline{\partial}_\omega = \overline{\partial}$ and $F^{(2,0)}_\omega = F^{(0,2)}_\omega = 0$.

as required.

5 Claims

We are interested in evaluating, for a knot K and holomorphic vector bundle E,

$$Z_X[M,\, \mathcal{O}_E(K)] = \int D\Phi\, \mathrm{e}^{-S(\Phi)}\, \mathcal{O}_E(K), \tag{39}$$

and, more generally, for a link made up of a union of non-intersecting knots K_i with a holomorphic vector bundle E_i associated to each knot

$$Z_X[M,\, \prod_i \mathcal{O}_{E_i}(K_i)] = \int D\Phi\, \mathrm{e}^{-S(\Phi)} \prod_i \mathcal{O}_{E_i}(K_i). \tag{40}$$

The linking number, $\mathrm{L}ink(K_i, K_j)$, of the knots K_i and K_j in a $\mathbb{Q}\mathrm{H}S$ makes an appearance and I use a definition tailored to our present needs. Let K_i denote the $i-$th knot in a $\mathbb{Q}$HS M. Since $H_1(M,\mathbb{Q}) = 0$, we have that $\mathrm{H}_1(M,\mathbb{Z})$ is a finite group and the integral homology classes represented by the K_i are of finite order, say of order m_i, so that $m_i K_i$ (no sum over i) is null-homologous. Let $m_i J_i$ be the de Rham 2-currents Poincare dual to the $m_i K_i$. Then we have that $m_i J_i$ is trivial and so there exist μ_i such that

$$d\mu_i = m_i J_i. \tag{41}$$

Observe that the singular support of $m_i J_i$ (resp. μ_i) does not intersect the singular support of $d\mu_k$ (resp. $dJ_k = 0$) for $i \neq k$ (see [dR] §20 (e) for details). Set $\lambda_i = \mu_i/m_i$. The linking number is now defined to be

$$\mathrm{L}ink(K_i, K_j) = \int_M \lambda_i\, J_j = \int_M \lambda_j\, J_i = \mathrm{L}ink(K_j, K_i). \tag{42}$$

In section 7 I will prove some of the following claims. M is a 3-manifold and, for the first three claims, X is a Hyper-Kähler manifold and the E_i are holomorphic vector bundles over X associated to a link.

Claim 5.1. *If* $b_1(M) \geq 2$ *then*

$$Z_X^{RW}[M, \prod_i \mathcal{O}_{E_i}(K_i)] = \alpha\, .\, Z_X^{RW}[M], \tag{43}$$

with

$$\alpha = \prod_i \mathrm{r}ank(E_i). \tag{44}$$

We need some more notation. Let F_{ω_i} denote the curvature 2-form of E_i,

$$ch(E_i, t) = \mathrm{Tr}_{V_i}\, \mathrm{e}^{\frac{tF_{\omega_i}}{2\pi\sqrt{-1}}}. \tag{45}$$

For a holomorphic line bundle L and $t \in \mathbb{Z}$ one has $ch(L, t) = ch(L^{\otimes t})$.

When X is Hyper-Kähler denote the Chern roots of the holomorphic tangent bundle by $\pm x_1, \ldots, \pm x_n$. Denote by $\Delta_M(t)$ the Alexander polynomial of M normalized so as to be symmetric in t and t^{-1} and so that $\Delta_M(1) = |\mathrm{T}orH_1(M, \mathbb{Z})|$ and set

$$\Delta_M(X) = \prod_{i=1}^{n} \Delta_M\left(\mathrm{e}^{x_i}\right). \tag{46}$$

Claim 5.2. *If $b_1(M) = 1$ then*

$$Z_X^{RW}[M, \prod_i \mathcal{O}_{E_i}(K_i)] = -\int_X \hat{A}(X)\, \Delta_M(X) \prod_i ch\left(E_i, \omega(K_i)\right), \tag{47}$$

where ω is the generator of $\mathrm{H}_1(M, \mathbb{Z})$ and

$$\omega(K_i) = \int_{K_i} \omega, \tag{48}$$

is the intersection of the Poincare dual of the knot with ω.

Claim 5.3. *If M is a $\mathbb{Q}\mathrm{H}S$ and $\dim_{\mathbb{C}} X = 2$ we have that*

$$\begin{aligned} &Z_X^{RW}[M, \prod_i \mathcal{O}_{E_i}(K_i)] \\ &\quad = \alpha \,.\, Z_X^{RW}[M] + |\mathrm{H}_1(M, \mathbb{Z})| \left(\frac{1}{2} \sum_i \alpha_i\, \mathrm{Link}(K_i, K_i) \int_X \mathrm{ch}(E_i) \right. \\ &\qquad \left. + \sum_{i<j} \alpha_{ij}\, \mathrm{Link}(K_i, K_j) \int_X c_1(E_i) c_1(E_j) \right), \end{aligned} \tag{49}$$

where

$$\alpha_j = \prod_{k \neq j} \mathrm{rank}(E_k), \quad \alpha_{ij} = \prod_{k \neq i,j} \mathrm{rank}(E_k). \tag{50}$$

Remark: For $S^2 \times S^1$, a knot can wrap say k times around the S^1, so that $\omega(K_i) = k_i$ and we have

$$Z_X^{RW}[S^2 \times S^1, \prod_i \mathcal{O}_{E_i}(K_i)] = -\int_X \hat{A}(X) \prod_{i=1} ch\left(E_i, k_i\right). \tag{51}$$

This partition function with knot observables can be understood, for compact X and $k_i = 1$, as the index of the twisted Dolbeault operator, coupled to $\prod_i \otimes E_i$ (see the appendix). In fact the Rozansky-Witten path integral yields a proof of the Riemann-Roch formula for the index of the twisted Dolbeault operator.

There are two more claims that I will not prove, but that can be established by slight variations of the proofs for the claims above. In these claims X is a holomorphic symplectic manifold, M a 3-manifold and the E_i are holomorphic vector bundles over X associated to a link. The Hyper-Kähler condition on X is dropped.

Claim 5.4. *For $b_1(M) > 3$*

$$Z_X^{RW}[M] = 0. \tag{52}$$

Claim 5.5. *If $b_1(M) \geq 2$ then*

$$Z_X^{RW}[M, \prod_i \mathcal{O}_{E_i}(K_i)] = \left(\prod_i rank(E_i)\right) . Z_X^{RW}[M]. \tag{53}$$

Remark: Once more we see that these invariants, for $b_1(M) > 0$, are essentially classical invariants of the 3-manifold. To get something new one must take M to be a $\mathbb{Q}\mathrm{H}S$.

6 Some Observations on the Invariants for X a 4n-Torus

At first sight it is quite odd to realize that while the Rozansky-Witten invariants vanish for any 4n-torus (since the curvature tensor vanishes) this is not true for the link invariants. A glance at claim **5.2** shows us that instead, providing $b_1(M) \leq 1$, that $Z_{T^{4n}}^{RW}[M, \prod_i \mathcal{O}_{E_i}(K_i)]$ need not vanish. Indeed for $b_1(M) = 1$ we have

$$Z_{T^{4n}}^{RW}[M, \prod_i \mathcal{O}_{E_i}(K_i)] = -\int_X \prod_i ch\left(E_i, \omega(K_i)\right), \tag{54}$$

though the right hand side of this expression has very little dependence on the 3-manifold M. We do not fare much better with M a $\mathbb{Q}\mathrm{H}S$ either as we see next.

6.1 The Rozansky-Witten Path Integral For X a 4n-Torus and M a $\mathbb{Q}$HS

For T^{4n} the path integral can be exactly performed since the theory is a "Gaussian". Fix on the standard flat metric on T^{4n}. With this choice the metric connection on the holomorphic tangent bundle and the corresponding Riemann curvature tensor vanish. Consequently, the path integral becomes

$$Z_{T^{4n}}[M, \prod_i \mathcal{O}_{E_i}(K_i)] = \int D\Phi \, \mathrm{e}^{-S_0(\Phi)} \prod_i \mathcal{O}_{E_i}(K_i), \tag{55}$$

where

$$S_0 = \int_M \left(\frac{1}{2}\delta_{ij}\, d\phi^i * d\phi^j + \epsilon_{IJ}\chi^I * d\eta^J + \frac{1}{2}\epsilon_{IJ}\chi^I d\chi^J \right). \tag{56}$$

The fields that appear in the link observable are the the field χ^I the constant map ϕ_0^i and the constant $\eta_0^{\overline{J}}$.

The STU relation (34) for tori reads (in Dolbeault cohomology)

$$[F_{I\overline{J}}, F_{J\overline{K}}]\, \eta_0^{\overline{J}} \eta_0^{\overline{K}} \sim 0, \tag{57}$$

which means that the matrices (irrespective of the holomorphic label), when evaluated in the path integral, are essentially commuting. Consequently one can drop the path ordering and simply use the exponential

$$\mathrm{Tr}_{V_i}\, \mathrm{P}\exp\left(i \oint_{K_i} A\right) \sim \mathrm{Tr}_{V_i}\, \exp\left(i \oint_{K_i} A\right). \tag{58}$$

In order to proceed I use a standard 'trick'. Write

$$\mathrm{Tr}_V\, \exp\left(i \oint_K A\right) = \sum_D \langle C_D|\exp\left(i\oint_K A^A_B\, \overline{C}^B\, C_A\right)|\overline{C}^D\rangle, \tag{59}$$

where C_A and $\overline{C}^A$ are Grassmann odd operators with values in V and V^* (the dual vector space) respectively. The operators C_A and $\overline{C}^A$ satisfy the usual algebra

$$\left\{C_A, \overline{C}^B\right\} = \delta_A^B, \tag{60}$$

and the states are defined by

$$C_A|0\rangle = 0, \quad \mathit{and}\ \ \overline{C}^B|0\rangle = |\overline{C}^B\rangle. \tag{61}$$

Introduce such variables for each knot K_i and index them also with the label i. Then the effective action in the path integral is

$$S = S_0 + i \sum_i \int_M \overline{C}_i\, A_i\, C_i\, J_i. \tag{62}$$

We can perform the path integral to obtain, up to an integration over the zero modes,

$$\sum_D \langle C_D | \exp\left(-\frac{1}{2} \sum_{i,j} \mathrm{L}ink(K_i, K_j)\, \epsilon^{IJ}\, (A_{Ji}\overline{C}_i C_i)\, (A_{Ij}\overline{C}_j C_j) \right) |\overline{C}^D\rangle, \tag{63}$$

where,

$$A_{I\,j} = \eta^{\overline{K}}\, \overline{\partial}_{\overline{K}}\, \omega_{I\,j}, \tag{64}$$

with

$$\omega_j, \tag{65}$$

the connection on the bundle E_j. A quick way to arrive at this formula is to use the equation of motion

$$d\chi^I = i\epsilon^{IJ} \sum_i A_{J\,i}\, J_i\, \overline{C}_i\, C_i, \tag{66}$$

the fact that the equation of motion saturates a Gaussian integral and to recall (42).

Since the self linking number appears in the formula (63) one must fix on some framing of the knots that form the link. Preliminary calculations [HT2] indicate that the theory comes prepared with the framing for which the self linking numbers are zero. If one takes, for simplicity, the framing for which $\mathrm{L}ink(K_i, K_i) = 0$, (63) becomes

$$\mathrm{T}r_{\otimes_i V_i}\, \exp\left(- \sum_{i<j} \mathrm{L}ink(K_i, K_j)\, \epsilon^{IJ}\, A_{Ji} \otimes A_{Ij} \right), \tag{67}$$

in general one has

$$\mathrm{T}r_{\otimes_i V_i}\, \exp\left(-\frac{1}{2} \sum_{ij} \mathrm{L}ink(K_i, K_j)\, \epsilon^{IJ}\, A_{Ji} \otimes A_{Ij} \right), \tag{68}$$

with some framing understood. Only the n-th term in the expansion of the exponential will survive the $\eta_0^{\bar{I}}$ integration.

For example, consider a single knot with self-linking number L, then (68), after integration over all the modes, becomes

$$|\mathrm{H}_1(M,\mathbb{Z})|^n L^n \int_X ch(E), \tag{69}$$

so that the invariant enters in a trivial way, meaning that it is not so interesting as an invariant for Hyper-Kähler manifolds. Recall that, the Rozansky-Witten invariants for a rank zero 3-manifold (i.e. $b_1(M) = 0$) do not depend on X simply through its Chern numbers. If they did there would be precious few invariants. In the example that we have just considered we have seen that for a knot and X a torus the holomorphic bundle enters only through its Chern numbers.

6.2 Comparing with Chern-Simons Theory

Now consider the $U(1)$ Chern-Simons theory. There is no perturbation expansion beyond the lowest order (the theory is quadratic). The lowest order term is essentially the square root of the inverse of the Ray-Singer Torsion of M. On the Rozansky-Witten side the compact manifolds for which the Rozansky-Witten invariant vanishes are clearly 4-tori, since the curvature tensor vanishes, and products of compact Hyper-Kähler manifolds with a 4-torus. From this point of view the lowest order in perturbation theory is the 0-torus (point), but this is hardly insightful.

It is possible to compare not just the "pure" theories but also those with knot or link observables as well. For Chern-Simons theory one can introduce Wilson loops

$$\prod_j \exp\left(iq_j \oint_{K_j} A\right), \tag{70}$$

where the q_j are charges. The path integral can again be evaluated directly and yields, up to a factor of the Ray-Singer torsion,

$$\exp\left(-i\frac{2\pi}{k}\sum_j q_j^2\,\mathrm{L}(K_j,K_j) - \frac{4\pi}{k}\sum_{i<j} q_i q_j\,\mathrm{L}(K_i,K_j)\right). \tag{71}$$

As before some framing must be chosen for the self linking numbers $\mathrm{L}(K_j, K_j)$. Expand the exponential (71) out to n-th order. Let products of the linking numbers be a "basis" for which to group the terms that arise in such an expansion. The coefficients will be certain polynomials in the charges. The comparison with (68) can now be completed, the only difference is in the coefficients, here they are polynomials in the charges while in the Rozansky-Witten theory they are integrals of products of curvature 2-forms.

As an example let $n = 1$. Then on the Chern-Simons front we get

$$-i\frac{2\pi}{k}\sum_j q_j^2 \cdot \mathrm{L}(K_j, K_j) - i\frac{4\pi}{k}\sum_{i<j} q_i q_j \cdot \mathrm{L}(K_i, K_j), \tag{72}$$

while on the Rozansky-Witten side we have, up to a factor of the first homology group of M,

$$\frac{1}{2}\sum_j \alpha_j \int_X ch(E_j) \cdot \mathrm{L}(K_j, K_j) - \sum_{i<j} \alpha_{ij} \int_X c_1(E_i)\, c_1(E_j) \cdot \mathrm{L}(K_i, K_j), \tag{73}$$

where

$$\alpha_j = \prod_{k \neq j} rank(E_k), \quad \alpha_{ij} = \prod_{k \neq i,j} rank(E_k). \tag{74}$$

This example exhibits the general nature of the expansion of the invariants and the fact that the basis (3-manifold information) is the same for the Chern-Simons theory and for the Rozansky-Witten model while the weights (products of charges for the $U(1)$ Chern-Simons theory, Casimirs for the non-Abelian Chern-Simons theory, integrals of Chern classes and perhaps other objects in the Rozansky-Witten theory) encode the differences.

7 Calculations

In this section I will calculate the invariants for 3-manifolds with $b_1(M) \geq 1$. One can do certain calculations for rational homology spheres, especially for low dimensional X, and I will present some of those here as well. Many of the details of the calculations are variations on themes taken from [RW], [T], [HT] and so I will be somewhat brief here and refer the reader to the references for more detail.

7.1 Zero Mode Counting

Various arguments, (see [RW], [T] and [HT]) allow one to conclude that the only relevant parts of the connections that appear in perturbative calculations of Feynman diagrams from L_1 (4), L_2 (5) and (40) are

$$V_1 = g_{I\overline{J}}(\phi_0)\, R^{\overline{J}}_{\ \overline{K}\overline{L}M}(\phi_0)\chi^I \eta_0^{\overline{L}} d\phi_\perp^{\overline{K}}\, \phi_\perp^M \tag{75}$$

$$V_2 = -\frac{1}{6}\epsilon_{IJ}(\phi_0)\, R^J_{\ KL\overline{M}}(\phi_0)\, \chi^I \chi^K \chi^L \eta_0^{\overline{M}} \tag{76}$$

$$V_E = \overline{\partial}_{\overline{J}}\omega_I(\phi_0)\, \chi^I \eta_0^{\overline{J}}, \tag{77}$$

respectively. In (77) the 0 subscript means the harmonic part of the field while the $\perp$ subscript means modes orthogonal to the harmonic part. To ease the burden of notation, from now on all tensors are understood to be evaluated on the constant map ϕ_0.

The Feynman diagrams that need to be evaluated now arise from contractions of all of the possible vertices

$$\langle V_1^r V_2^s V_E^t \rangle. \tag{78}$$

A constraint comes from the fact that for $\dim_{\mathbb{C}} X = 2n$ the maximal possible product of $\eta_0^{\overline{I}}$ is $2n$. Consequently,

$$r + s + t = 2n, \tag{79}$$

in order to soak up the $\eta_0^{\overline{I}}$ zero modes. In the following we will look at constraints that arise by counting χ^I zero modes. The χ^I zero modes can only appear in the vertices (75-77) with at most one such mode in V_1, three in V_2 and one in V_E. The number of χ^I zero modes is $2n \times b_1(M)$ (2n because of the holomorphic tangent space label and $b_1(M)$ as it must be a harmonic 1-form on M). The largest number of zero modes that can be soaked up arises when all the χ^I appearing in the vertices are zero modes that is

$$r + 3s + t = 2n \times b_1(M). \tag{80}$$

Together with (79) this implies

$$s = n \times (b_1(M) - 1)\,, \tag{81}$$

but is incompatible with (79) if $b_1(M) > 3$. Consequently

$$Z_X^{RW}[M, \prod_i \mathcal{O}_{E_i}(K_i)] = 0, \quad if \;\; b_1(M) > 3. \tag{82}$$

It is easy to see that for $b_1(M) \geq 1$ that the χ^I that appears in (75) and (77) must be a zero-mode. In the following I will take this for granted.

7.2 Proof of Claim: 5.1

Since, in any case $Z_X^{RW}[M] = 0$ if $b_1(M) > 3$, (82) partially establishes the claim. When $b_1(M) = 3$, one finds from the discussion above that $s = 2n$ and $r = t = 0$ which means that no vertices from the link observables can participate in the calculation of the expectation value of the link observable. So we have established the claim for $b_1(M) = 3$.

If $b_1(M) = 2$, the rule (80) does not hold (since one cannot have all three χ^I being harmonic), rather, one can have at most two χ^I harmonic in V_2 hence,

$$r + ps + t = 4n, \tag{83}$$

where $p = 0$, 1 or 2. $p = 0$ and 1 are ruled out by (79) leaving only $p = 2$, $s = 2n$ and $r = t = 0$. Once more the vertices V_E do not make an appearance, and so we have established claim 5.1.

7.3 Proof of Claim: 5.2

For $b_1(M) = 1$ the counting of χ^I zero modes tells us that indeed one of the χ^I appearing in V_2 must be a zero mode χ_0^I. Such a mode is actually decomposable as

$$\chi_0^I = c^I\,\omega \tag{84}$$

where c^I is an anti-commuting scalar (on M) and ω is the generator of $\mathrm{H}^1(M,\mathbb{Z})$. Which means that for a given knot K_i and associated holomorphic vector bundle E_i that

$$\begin{aligned} \mathcal{O}_{E_i}(K_i) &= \mathrm{Tr}_{V_i}\,\mathrm{P}\,\exp\left(c^I A_{Ii} \oint_{Ki} \omega\right) \\ &= \mathrm{Tr}_{V_i}\,\exp\left(c^I A_{Ii}\,\omega(K_i)\right). \end{aligned} \tag{85}$$

The second equality in (85) comes about as follows. Since the matrix $c^I A_{Ii}$ is position independent one can drop the path ordering. Without the path ordering the integral in the exponent is really $\oint_{K_i} \omega$ which is the linking number between the knot K_i and the fundamental cycle Poincare dual to ω.

The path integral is still to be performed. However, a glance at (85) tells us that the insertion of these observables only effects the zero mode integration of the path integral. The integration over the other modes has been performed in some generality in [HT] the result being equation (8.34) in that reference. One now needs to multiply that result with products of (85) and integrate over the zero modes of the theory. The integration over the zero modes turns the objects that appear into differential forms (the emergence of factors of 2π is explained in [HT]). After these gymnastics one obtains claim 5.2.

7.4 Proof of Claim: 5.3

Here we are interested in $b_1(M) = 0$ and $n = 1$. From the selection rule (79) we see that

$$r + s + t = 2. \tag{86}$$

Let us write the final result as a sum of three terms. The first is, $t = 0$, the second $t = 1$ and the third comes from $t = 2$.

When $t = 0$, we have $r + s = 2$ and the only vertices that appear are those in the calculation of $Z_X^{RW}[M]$, so from these diagrams we get

$$\left(\prod_i \mathrm{r}ankE_i\right) Z_X^{RW}[M]. \tag{87}$$

When $t = 1$, $r = 1$ and $s = 0$ or $r = 0$ and $s = 1$. In the first case the vertex V_1 is contracted with itself along the ϕ^i legs. But this vanishes because the ϕ^i propagator contains a $g^{I\overline{J}}$ which when contracted with the vertex yields $R^{\overline{K}}_{\overline{J}LI}g^{I\overline{J}} = 0$ since X is Ricci flat. In the second case the vertex V_2 is contracted with itself along the two of the three χ^I legs. This vanishes as well and for the same reason as for the V_1 vertex. So there is no contribution from the $t = 1$ diagrams.

For $t = 2$ we necessarily have $r = s = 0$. This means that we may as well set the curvature term in the Rozansky-Witten theory to zero for the purposes of the present calculation. But then the calculation is the same as that for the $4n$-torus of the previous section. In fact the answer, for $n = 1$ is given in (73), thus completing the proof of the claim.

A Coupling to Supersymmetric Quantum Mechanics

In this appendix I would like to mention one small generalization that can be made with regards knot invariants. Witten [W] suggested that the correct way to treat knot observables, Wilson loops, in Chern-Simons theory is by making use of the Borel-Weil-Bott theorem to replace Wilson lines by functional integrals over maps from S^1 into G/T. In the present setting a functional integral formulation of the knot observables is also available. This path integral representation has a number of uses.

Let E be a holomorphic vector bundle over X. One can add to the Rozansky-Witten action the following supersymmetric action

$$\oint \overline{C}\left(\frac{d}{dt} + d\phi^I\,\omega_I - F_{I\overline{J}}\chi^I\,\eta^{\overline{J}}\right)C \tag{88}$$

where C and $\overline{C}$ are Grassmann odd maps from the knot K to sections of E and $\overline{E}$ respectively. This action is also $\overline{Q}$ invariant if we set

$$\overline{Q}\,C = 0 = \overline{Q}\,\overline{C}. \tag{89}$$

If we would like this also to exhibit Q invariance then we must take E to be hyper-holomorphic. If E is hyper-holomorphic then, since Q acts by a gauge transformation (23), invariance of (88) is guaranteed if we perform a gauge transformation on C and $\overline{C}$, that is,

$$\begin{aligned} Q\,C &= -\eta^{\overline{I}}\,T_{\overline{I}}^{J}\,\omega_J\,C, \\ Q\,\overline{C} &= \overline{C}\,\eta^{\overline{I}}\,T_{\overline{I}}^{J}\,\omega_J. \end{aligned} \tag{90}$$

One picks out the path ordered exponential by projecting onto the one particle sector of the theory. This is the equivalent of (59) and can be achieved by placing a projection operator in the path integral over C and $\overline{C}$. But one is not restricted to this, rather, one is free to look at any sector of the Hilbert space that one likes. Consequently, there are many more objects that one can associate to a knot (and hence to a link).

Note also that on the 3-manifold $S^2 \times S^1$ one can essentially squeeze away the non-harmonic modes to be left with a theory on S^1 [T]. If one picks the knot K to be $\{x\} \times S^1$ for some, immaterial, point $\{x\} \in S^2$ then the combined theory, (3) together with (88), is a standard supersymmetric quantum mechanics which represents the index of the Dolbeault operator coupled to a holomorphic bundle [AG].

It would be interesting to have a topological field theory whose bosonic field is a section of $TX \otimes E$ and not just to couple E to a knot.

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