

Partitioning of electric and magnetic energy in $SU(2)$ gauge theory

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Introduction

It has been known for over twenty years that the Coulomb solution in Maxwellian electro/magneto-statics for a point charge at the origin in 3-space is unstable when embedded in a non-abelian gauge theory with large coupling constant. A version of this phenomenon was first seen by Mandula [Ma] and other manifestations of it were described in some detail by Sikivie and Weiss [SW1]-[SW3], Jackiw, Jacobs and Rebbi [JKR], Jackiw and Rossi [JR] and in Jackiw's lecture notes, [J].

¹Supported in part by the National Science Foundation
e-print archive: <http://xxx.lanl.gov/math-ph/0201050>

The instability arises more or less for the following reason: The Maxwellian Coulomb solution has no magnetic field. However, in a non-abelian gauge theory, static electric and magnetic fields interact; and so it is energetically favorable at large coupling to have a magnetic field that suppresses the coulomb like behavior of the electric field. This instability has nothing to do with the singular nature of a point charge distribution; it occurs, for example, when the charge is uniformly distributed over a ball. This said, the Quixotic purpose of this article is to shed light on the dependence of the minimal energy on the coupling constant for the afore mentioned uniform charge distribution. Of particular interest are the relative contributions to this minimal energy from the electric and magnetic fields. Theorem 1, below, reports the results on these questions.

Of further interest are the functional forms of the electric and magnetic fields for an energy minimizer. However, as their detailed behavior resists analysis, the study here concentrates instead on the behavior of these fields for an energy critical point that minimizes energy under a restrictive hypothesis. Even so, the energy of this critical point is very close to the minimum energy and may well equal the minimum energy. In any event, the long range behavior of the electric and magnetic potentials for this other critical point are described below in some detail with Theorem 2 summarizing most of the salient features.

Here is the background for the story: Let $su(2)$ denote the Lie algebra of $SU(2)$, thus, the vector space of 2×2 complex valued, anti-hermitian and traceless matrices. Now, let A denote an $su(2)$ valued 1-form, and let $B_A \equiv *(dA + A \wedge A)$ denote its 'magnetic field'. Here, $*$ is the Hodge star isomorphism from 2-forms to 1-forms; for example $*(dx^1 \wedge dx^2) = dx^3$. Introduce a norm on $su(2)$ by the requirement that its square to send $\tau \in su(2)$ to $|\tau|^2 \equiv -2 \text{ trace } (\tau^2)$; then use the latter with the Euclidean norm on $T^*\mathbb{R}^3$ to defined the norms of $su(2)$ valued 1-forms.

This done, the interest here is with the vector space, \mathcal{A} , of $su(2)$ valued 1-forms A which vanish where $r \leq 1$, are smooth where $r > 1$, have locally square integrable first derivatives and are such that $|B_A|^2$ is integrable over \mathbb{R}^3 . Of special interest are the $su(2)$ -valued 1-forms that minimize the energy functional, $\mathcal{E} : \mathcal{A} \rightarrow (0, \infty)$, whose definition follows.

To define \mathcal{E} , it is necessary to first digress with an introduction to the covariant derivative, ∇_A , defined by $A \in \mathcal{A}$. This derivative sends an $su(2)$ valued function Ψ to the $su(2)$ valued 1-form $\nabla_A \Psi = d\Psi + A\Psi - \Psi A$. The covariant derivative of Ψ in the direction of a vector v is obtained by contracting v with $\nabla_A \Psi$. The covariant derivative defines the covariant Laplacian, ∇_A^2 , which is the sum of the second powers of the covariant derivatives in each of the 3 coordinate directions.

Now, let $\tau^1 \in su(2)$ denote a fixed element with norm 1 and let ρ denote the $su(2)$ valued function that equals $3(4\pi)^{-1}\tau^1$ on the unit ball in \mathbb{R}^3 and vanishes on the complement of this ball. Then each $A \in \mathcal{A}$ determines a unique $su(2)$ valued function, Ψ_A , that obeys

$$-\nabla_A \Psi_A = \rho \quad (1)$$

and is such that both $|\nabla_A \Psi_A|$ and $|\Psi_A|^3$ square integrable over \mathbb{R}^3 . Standard calculus of variation techniques can be used to argue both for the existence and uniqueness of Ψ_A and to study its dependence on A .

Given the preceding, specify a positive number g , the coupling constant; and then define \mathcal{E} by the rule

$$\mathcal{E}(A) \equiv 2^{-1}g^{-2} \int |B_A|^2 + 2^{-1}g^2 \int |\nabla_A \Psi_A|^2. \quad (2)$$

Here, and below the notation is such that the integral sign with no indication of domain or measure denotes integration over \mathbb{R}^3 with the Euclidean volume element.

A critical point of \mathcal{E} is, by definition, an $su(2)$ valued 1-form A with the property that

$$\frac{d}{dt} \mathcal{E}(A + ta)|_{t=0} = 0 \quad (3)$$

for all $a \in \mathcal{A}$. A critical point A is called a local minimizer of \mathcal{E} if

$$\mathcal{E}(A + ta) \geq \mathcal{E}(A) \quad (4)$$

for every a and, given a , for all t in some neighborhood of $0 \in \mathbb{R}$. A global minimizer of \mathcal{E} obeys (4) for all $a \in \mathcal{A}$ and $t \in \mathbb{R}$.

With regards to the choice of ball radius 1 and charge normalization $\int |\rho| = 1$ in the definitions of \mathcal{A} and \mathcal{E} , note that changes of either can be absorbed by a combination of rescaling \mathcal{E} and g . Indeed, if the

corresponding variational problem is posed in the ball of radius R , with charge density ρ such that $\int |\rho| = q$, and with coupling constant g , then the corresponding energy infimum is equal to the product of $q^{1/2}R^{-1}$ times the energy infimum for the $R = 1$, $q = 1$ and coupling constant $gq^{1/2}$ version of the functional in (2). In any event, this rescaling property justifies the focus here on the use of the unit ball and $\int |\rho| = 1$ charge in the definitions of \mathcal{A} and \mathcal{E} .

By the way, the 1-forms in \mathcal{A} are required to vanish inside the unit ball for three reasons of which pay small homage to the underlying physics. First, high energy particle physics experiments observe that the non-abelian magnetic fields associated to the subnuclear forces are small at short distances. Second, physics would, in any event, give the charge density ρ as a functional of some other field (say a spinor with values in \mathbb{C}^2), and the latter would, perforce, obey an equation that also involved the 1-form A . Because such an auxiliary equation is not considered here, a formulation that allows A to vary in the ball lacks a certain logical consistency. Finally, allow A to vary where $\rho \neq 0$ and there is no invariant notion of charge density to allow energy comparisons. The latter point underlies statements in some of the afore-mentioned references to the effect that the Coulomb solution is unstable for all values of the coupling constant.

Note that this variational setting enjoys a restrictive sort of gauge invariance. Indeed, if A is in \mathcal{A} and $h : \mathbb{R}^3 \rightarrow SU(2)$ is the identity matrix on the unit ball, then the 'gauge equivalent' 1-form $h^{-1}Ah + h^{-1}dh$ is also in \mathcal{A} and $\mathcal{E}(A) = \mathcal{E}(h^{-1}Ah + h^{-1}dh)$.

As a parenthetical remark, note that the variational setting here can be formulated in a completely gauge invariant manner. To do so, first change the definition of \mathcal{A} so that the requirement that A vanish in the ball is replaced by the requirement that B_A vanish in the ball. This done, make the following requirements on ρ : First, it should vanish outside the ball and its norm should equal $3/(4\pi)$ inside. Second, require that $\nabla_A \rho = 0$ where $\rho \neq 0$. This done, then the energy \mathcal{E} can be considered to be a function of pairs (A, ρ) , subject to the preceding constraints. As such, this energy is gauge invariant since $(h^{-1}Ah + h^{-1}dh, h^{-1}\rho h)$ and (A, ρ) have the same energy for all smooth $h : \mathbb{R}^3 \rightarrow SU(2)$.

The author knows the explicit form of only one critical point of

\mathcal{E} (up to gauge equivalence), this being the gauge transforms of the ‘coulomb’ solution which has $A = 0$ with the corresponding Ψ_A given in terms of the radial coordinate r on \mathbb{R}^3 as

- $\Psi_{A=0} = (8\pi)^{-1}(3 - r^2)\tau^1$ where $r \leq 1$. (5)
- $\Psi_{A=0} = (4\pi)^{-1}r^{-1}\tau^1$ where $r \geq 1$.

The energy, $\mathcal{E}(0)$, of the Coulomb solution is equal to $3g^2(40\pi)^{-1}$.

With the preceding understood, consider:

Theorem 1. *For each $g > 0$ the corresponding version of the functional \mathcal{E} achieves its global minimum on \mathcal{A} . When $g \leq (\sqrt{2}\pi)^{1/2}$ this minimum is attained only on the Coulomb solution and its gauge transforms. However, when $g > (6\pi)^{1/2}$ then the Coulomb solution is no longer the minimizer of \mathcal{E} . Moreover, there exists a g -independent constant c such that when $g > (6\pi)^{1/2}$, then*

$$g^2(40\pi)^{-1} + c^{-1}g \leq \inf_{\mathcal{A}} \mathcal{E} \leq g^2(40\pi)^{-1} + cg. \quad (6)$$

In addition, if, for such g , the corresponding version of \mathcal{E} achieves its infimum at $A \in \mathcal{A}$, then

- $g^2(20\pi)^{-1} \leq \int_{r \leq 1} |\nabla_A \Psi_A|^2$. (7)
- $c^{-1}g \leq g^2 \int_{r \geq 1} |\nabla_A \Psi_A|^2 \leq cg$.
- $c^{-1}g \leq g^{-2} \int_{1 \leq r \leq 1+c/g} |B_A|^2$ and $g^{-2} \int_{1 \leq r} |B_A|^2 \leq cg$.
- $|\Psi_A| < |\Psi_{A=0}|$ everywhere, and $|\Psi_A| \leq cg^{-1/2}r^{-1}$ where $r \geq 2$.

Remark that the factor $g^2(40\pi)^{-1}$ appears in (6) and in the first point of (7) because $A = 0$ in the unit ball and $(40\pi)^{-1}$ is the minimum over the set of solutions in the unit ball to the equation $-\Delta_{A=0}\Psi = \rho$ of the functional that sends Ψ to $\int_{r \leq 1} |\nabla_{A=0}\Psi|^2$. By the way, note in the second point of (7) that a substantial fraction of the magnetic energy is contributed from the shell of thickness $\mathcal{O}(g^{-1})$ that surrounds the unit ball. Note also that the results in Theorem 1 do not change in any

substantial way if the charge distribution ρ in (1) is allowed to vary in the ball as $\rho_0\tau^1$ with ρ_0 positive with integral 1.

As remarked above, the author has little to say about the detailed point to point behavior of the fields (A, Ψ_A) when A is an absolute minimizer of a large g version of \mathcal{E} . On the otherhand, quite a bit can be said about these fields for the absolute minimizer of \mathcal{E} 's restriction to a certain subset, $\mathcal{A}_0 \subset \mathcal{A}$. Here, $A \in \mathcal{A}_0$ when A is gauge equivalent to $A_0 = \alpha \sin \theta d\varphi\tau^2$ where α is a function on \mathbb{R}^3 , θ and φ are the standard spherical coordinates and $\tau^2 \in su(2)$ has unit length and is orthogonal to τ^1 . In this regard, the critical points of \mathcal{E} 's restriction to \mathcal{A}_0 are also unrestricted critical points of \mathcal{E} . This understood, the next theorem describes the minimizers of \mathcal{E} 's restriction to \mathcal{A}_0 .

Theorem 2. *For each $g > 0$, there is a function α on \mathbb{R}^3 , unique up to multiplication by ± 1 , and characterized by the fact that the global minimizers of \mathcal{E} 's restriction to \mathcal{A}_0 are all gauge equivalent to $A \equiv \alpha \sin \theta d\varphi\tau^2$. In this regard, $\alpha = 0$ when $g \leq (6\pi)^{1/2}$, but not so when $g > (6\pi)^{1/2}$. In addition, there is a constant, $c \geq 1$, with the following significance: For $g > (6\pi)^{1/2}$, the energy $\mathcal{E}(A)$ obeys (6) and A with its corresponding Ψ_A obeys (7). Moreover,*

- $\Psi_A = \Psi\tau^1$ with Ψ a positive function on \mathbb{R}^3 .
- $\alpha = f \sin \theta$ with $|f| > 0$ where $r > 1$.
- At points where $r \geq c$,
 - a) $\Psi \leq \sqrt{2}g^{-2}e^{-g^{1/4}/c}r^{-1}$.
 - b) $\alpha = fr^{-1} \sin \theta$ with $|f| \geq g^{1/2}/c$.
- At points where $r \gg c$,
 - a) $\Psi = \sqrt{2}g^{-2}(e_0 + m_\Psi)r^{-1}$ where e_0 is a positive constant less than $e^{-g^{1/4}/c}$ and $|m_\Psi| \leq cr^{-1}$.
 - b) $\alpha = (c_0 + m_\alpha)r^{-[(9-8e_0^2)^{1/2}-1]} \sin \theta$ where c_0 is a positive constant and $|m_\alpha| \leq cr^{-1}$.

Of particular interest to the author with regards to Theorem 2 is the fact that the electric potential, Ψ , although Coulomb like at large distance, has an effective charge $\sqrt{-2}g^{-2}e_0$ that is absolutely microscopic

at large g . Meanwhile, the magnetic potential, α , falls to zero rather like the field of a dipole charge with dipole moment of size $\mathcal{O}(g^{1/2})$.

As with the statement of Theorem 1, the conclusions of Theorem 2 are not changed substantially if the charge distribution ρ in (1) is allowed to vary in the ball as $\rho_0\tau^1$ with ρ_0 positive with integral 1. This said, only the $\rho_0 = 3(4\pi)^{-1}$ case is discussed below.

The remainder of this article is occupied with the proofs of Theorems 1 and 2. In this regard, the assertions of Theorem 2 are proved first as they are used in part to prove Theorem 1. In particular, the assertions of Theorem 2 are proved as various propositions and lemmas in Sections a-j below. These are all summarized in Section k to tie up the argument for Theorem 2. Section k then ends with the proof of Theorem 1.

Before starting, note that the author benefitted at an early stage of this project from conversations with R. Scott.

a) A reformulation of the variational problem on \mathcal{A}_0

The variational problem given by \mathcal{E} 's restriction to the set \mathcal{A}_0 of $su(2)$ -valued 1-forms gauge equivalent to $\alpha \sin\theta d\varphi\tau^2$ with α a function on \mathbb{R}^3 can be reformulated as follows:

Let $g \geq 0$ be a constant, and, for now, let ρ be a function with compact support where the distance, r , to the origin is less than 1 and with integral equal to 1. Now consider an equation for a C^0 function α of the first two of the spherical coordinates (r, θ) on \mathbb{R}^3 which is given as follows: First, introduce the unique, C^1 solution $\psi \equiv \psi[\alpha]$ on \mathbb{R}^3 to the equations

- $-\Delta\psi + r^{-2}\alpha^2\psi = \rho$ where $r > 1$. (8)

- $\lim_{r \rightarrow \infty} \psi = 0$.

Here, ψ is a function of (r, θ) too, and

$$\Delta = r^{-2}(r^2(\cdot)_r)_r + (\sin \theta)^{-1}(\sin \theta(\cdot)_\theta)_\theta$$

is the standard Laplacian. Then, with ψ understood, require that

- $\alpha = 0$ where $r \leq 1$. (9)
- $-\alpha_{rr} - r^{-2}((\sin \theta)^{-1}(\sin \theta \alpha)_\theta)_\theta - g^4 \psi^2 \alpha = 0$ where $r > 1$.
- $r^{-2} \alpha$ is square integrable on \mathbb{R}^3 .

Here, and also below when ambiguities are unlikely, the partial derivative of a function, f , with respect to r is written as f_r and with respect to θ as f_θ . When ambiguity can arise, these derivatives are written below as $\partial_r f$ and $\partial_\theta f$, respectively.

Of special interest are those functions α for that are absolute minimizers of the functional

$$\begin{aligned} \mathcal{E}_0(\alpha) \equiv & 2^{-1} g^{-2} \int r^{-2} (\alpha_r^2 + r^{-2} (\sin \theta)^{-2} (\sin \theta \alpha)_\theta^2) \\ & + 2^{-1} g^2 \int (|\nabla \psi|^2 + r^{-2} \alpha^2 \psi^2). \end{aligned} \quad (10)$$

Note that $\mathcal{E}_0(\alpha) = \mathcal{E}(\alpha \sin \theta d\varphi \tau^2)$ with \mathcal{E} as in (2), so \mathcal{E}_0 is the restriction of \mathcal{E} to those $A \in \mathcal{A}$ of the form $\alpha \sin \theta d\varphi \tau^2$.

Note that the introduction of the 1-form $a \equiv \alpha \sin \theta d\varphi$ allows the $r \geq 1$ parts of (8) and (9) to be rewritten as

- $-\Delta \psi + |a|^2 \psi = 0$, (11)
- $-\Delta a - g^4 |\psi|^2 a = 0$.

In terms of the 1-form a , the energy \mathcal{E}_0 becomes

$$\mathcal{E}_0 = 2^{-1} g^{-2} \int |\nabla a|^2 + 2^{-1} g^2 \int (|\nabla \psi|^2 + |a|^2 |\psi|^2). \quad (12)$$

Note that \mathcal{E}_0 in (10) defines a bonafide functional on the space, \mathcal{C}_0 , of continuous functions on \mathbb{R}^3 that vanish on the unit ball, are invariant under rotations about the axis defined by the spherical angle φ and are such that $r^{-2}(\alpha_r^2 + r^{-2}(\sin \theta)^{-2}(\sin \theta \alpha_\theta)^2)$ has finite integral. Moreover, techniques from the calculus of variations establish that \mathcal{E}_0 achieves its infimum on \mathcal{C}_0 . In addition, standard elliptic regularity techniques (as found, for example, in Chapter 6 of [Mo]) justify the assertion that any critical point of \mathcal{E}_0 on $c\mathcal{C}_0$ is smooth where $r > 1$.

b) Some simple inequalities

The subsequent analysis exploits some basic observations that concern \mathcal{E}_0 and the solutions to (8) and (9). In this regard, the first observation is obtained by multiplying both sides of the top equation in (11) by ψ and then integrating the result over \mathbb{R}^3 . A subsequent integration by parts then finds that

$$\int (|\nabla \psi|^2 + |a|^2 |\psi|^2) = \int \psi \rho. \quad (13)$$

The second observation follows by contracting both sides of the bottom equation in (11) with the 1-form a and then integrating the result over \mathbb{R}^3 . This done, an integration by parts finds

$$g^{-2} \int |\nabla a|^2 = g^2 \int |a|^2 |\psi|^2. \quad (14)$$

The third observation concerns a pair ψ and ψ' where these functions solve the versions of the second that are defined by a corresponding pair, a and a' . This understood, note the following:

$$\begin{aligned} \text{If } |a| \geq |a'| \text{ everywhere with the inequality strict somewhere,} \\ \text{then } \psi < \psi' \text{ everywhere.} \end{aligned} \quad (15)$$

Indeed, this follows from the maximum principle because the top equation in (8) implies that

$$-\Delta(\psi - \psi') + (|a^2| - |a'^2|)\psi + |a'|^2(\psi - \psi') = 0. \quad (16)$$

In particular, this equation precludes a non-negative maximum for $\psi - \psi'$.

The observation in (15) implies that for any a , the corresponding ψ is sandwiched as

$$\psi_D < \psi \leq \psi_{\text{Coul}}, \quad (17)$$

where ψ_{Coul} is the solution to $-\Delta\sigma = \rho$ on \mathbb{R}^3 which decays to zero as $r \rightarrow \infty$, while ψ_D is the solution to $-\Delta\sigma = \rho$ in the ball where $r \leq 1$ which vanishes at $r = 1$. For example, in the case where $\rho = \rho_0 = 3(4\pi)^{-1}$ where $r \leq 1$,

- $(8\pi)^{-1}(1 - r^2) < \psi \leq (8\pi)^{-1}(3 - r^2)$ where $r \leq 1$. (18)
- $0 < \psi \leq (4\pi)^{-1}r^{-1}$ where $r \geq 1$.

By the way, note that (13) and (18) imply that

$$(40\pi)^{-1} < 2^{-1} \int (|\nabla\psi|^2 + |a|^2|\psi|^2) \leq 3(20\pi)^{-1} \quad (19)$$

in the case where ρ is the constant $3(4\pi)^{-1}$.

c) The coulomb solution

The Coulomb solution to (8) and (9) has $\alpha = 0$ and $\psi \equiv \psi_0$ given by

$$\psi_{\text{Coul}}(x) = (4\pi)^{-1} \int |x - (\cdot)|^{-1} \rho. \quad (20)$$

For example, when ρ has the constant value $3(4\pi)^{-1}$ in the unit ball and is zero outside, then ψ_0 is equal to $(8\pi)^{-1}(3 - r^2)$ in the ball and $(4\pi)^{-1}r^{-1}$ outside. As the following lemma attests, the Coulomb solution, $\alpha \equiv 0$, is the minimizer of \mathcal{E}_0 when g is small and not the minimizer when g is large.

Lemma 3. *When $g \leq g_0$, then $\alpha = 0$ is the absolute minimizer of \mathcal{E}_0 on \mathcal{C}_0 , and when $g > g_0$, it is not. In the case where $\rho = 3(4\pi)^{-1}$ inside the ball and zero outside, $g_0 = (6\pi)^{1/2}$.*

The proof of this lemma exploits two fundamental inequalities that are also used elsewhere in the paper:

$$(21)$$

- Suppose that f is a function of r that vanishes where $r \leq 1$ and is such that f_r is square integrable on $[1, \infty)$ with respect to dr . Then $\int_{r \geq 1} r^{-2} f^2 dr \leq 4 \int_{r \geq 1} f_r^2 dr$.
- Suppose that f is a function of r that vanishes in the limit as $r \rightarrow \infty$ and is such that $r f_r$ is square integrable with respect to dr . Then, $\int_{r \geq 1} f^2 dr \leq 4 \int_{r \geq 1} f_r^2 r^2 dr$.
- Suppose that f is a function of the spherical angle θ and is such that $(\sin \theta)^{-1}(\sin \theta f)_\theta^2$ is integrable on $[0, \pi]$. Then $\int_{0 \leq \theta \leq \pi} \sin \theta f^2 d\theta \leq 2^{-1} \int_{0 \leq \theta \leq \pi} (\sin \theta)^{-1} (\sin \theta f)_\theta^2 d\theta$.

To argue for the first inequality, write $r^{-2} dr$ as $-d(r^{-1})$ on the left hand side of the integral, integrate by parts and then use the triangle inequality. The second inequality follows by directly integrating by parts on the left hand side of the integral and then employing the triangle inequality. The third inequality follows from the fact that the corresponding Sturm-Liouville operator has smallest eigenvalue 2. Note for future reference that $f = \sin \theta$ is the corresponding eigenfunction.

Proof of Lemma 3. Consider first the argument that $\alpha = 0$ is the minimizer of \mathcal{E}_0 when g is small. For this purpose, appeal to the first point in (21) and also (13) to derive the inequality

$$\mathcal{E}_0(\alpha) \geq (9/8)g^{-2} \int r^{-4} \alpha^2 + 2^{-1}g^2 \int_{r \leq 1} \rho \psi. \quad (22)$$

Now, multiply the top line of (8) by the Coulomb solution ψ_{Coul} and integrate the result over \mathbb{R}^3 to find that

$$\int_{r \leq 1} \rho \psi + \int r^{-2} \alpha^2 \psi \psi_{\text{Coul}} = \int_{r \leq 1} \rho \psi_{\text{Coul}}. \quad (23)$$

Then, since $\mathcal{E}(0) = 2^{-1} \int_{r \leq 1} \rho \psi_{\text{Coul}}$, this last equality, (17) and (22) imply that

$$\mathcal{E}_0(\alpha) - \mathcal{E}_0(0) \geq (9/8)g^{-2} \int r^{-4} \alpha^2 - 2^{-1}g^2 \int r^{-2} \alpha^2 \psi_{\text{Coul}}^2. \quad (23)$$

Finally, $\rho \geq 0$, so the maximum principle finds $\psi_{\text{Coul}} > 0$ everywhere and $\psi_{\text{Coul}} \geq mr^{-1}$ where $r \geq 1$; here $m = \min_{r=1} \psi_{\text{Coul}}$. This understood, then (23) asserts that the Coulomb solution is the minimizer

when $g \leq 3^{1/2}(2m)^{-1/2}$. For example, when $\rho = 3(4\pi)^{-1}$, the Coulomb solution is the minimizer when $g \leq (6\pi)^{1/2}$.

To see that $\alpha = 0$ is not the minimizer of \mathcal{E}_0 when g is large, it is necessary only to prove that the Hessian of the large g versions of \mathcal{E}_0 at $\alpha = 0$ is not a positive semi-definite quadratic form on \mathcal{C}_0 . In this regard, note that this Hessian assigns to each $\beta \in \mathcal{C}_0$ the number

$$\mathcal{H}_0(\beta) \equiv g^{-2} \int_{r \geq 1} r^{-2} (\beta_r^2 + r^{-2} (\sin \theta)^{-2} (\sin \theta \beta)_\theta^2) - g^2 \int_{r \geq 1} r^{-2} \beta^2 \psi_{\text{Coul}}^2. \tag{24}$$

Now, according to the maximum principle, $\psi_{\text{Coul}} \leq Mr^{-1}$ where M denotes the maximum value of ψ_{Coul} on the $r = 1$ sphere. This understood, then

$$\mathcal{H}_0(\beta) \leq g^{-2} \int_{r \geq 1} r^{-2} (\beta_r^2 + r^{-2} (\sin \theta)^{-2} (\sin \theta \beta)_\theta^2) - g^2 M^2 \int_{r \geq 1} r^{-4} \beta^2. \tag{25}$$

With (25) understood, fix some small $\varepsilon > 0$ and take β to be zero where $r \leq 1$ and to equal the function $(r^{(1-\varepsilon)/2} - 1) \sin \theta$ where $r \leq 1$. This done, a calculation finds that

$$\mathcal{H}_0(\beta) < (8\pi/3)\varepsilon^{-1}(g^{-2}9/8 - g^2M^2 + \mathcal{O}(\varepsilon)), \tag{26}$$

which is negative for small ε provided that $g \geq 3^{1/2}(2M)^{-1/2}$.

d) Some energy inequalities

The next proposition gives a first indication of the large g behavior of various parts of the infimum of \mathcal{E}_0 . In the statement of this proposition and in the discussions of the subsequent sections of this paper, the function ρ in (8) is implicitly that which vanishes outside the unit ball and equals the constant $3(4\pi)^{-1}$ inside.

Proposition 4. *There is a constant c_1 with the following significance: Given $g > (6\pi)^{1/2}$, suppose that α is a minimizer of \mathcal{E}_0 . Then α and its corresponding ψ obey*

- $(20\pi)^{-1} < \int_{r \leq 1} |\nabla \psi|^2 < (20\pi)^{-1} + c_1 g^{-1}$.

- $\int_{r>1} |\nabla\psi|^2 < c_1g^{-1}$.
- $\int |a|^2\psi^2 < c_1g^{-1}$.
- $\psi < c_1g^{-1/2}r^{-1}$ where $r > 2$.

The remainder of this section is occupied with the

Proof of Proposition 4. The proof starts with the following observation: Let σ be a function on the unit ball that obeys $-\Delta\sigma = 3(4\pi)^{-1}$. Then,

$$\int_{r\leq 1} |\nabla\sigma|^2 \geq (20\pi)^{-1}, \quad (27)$$

and is an equality if and only if $\sigma = (8\pi)^{-1}(m - r^2)$ with $m \in \mathbb{R}$. Indeed, this follows by writing σ as a sum of products of functions of r times spherical harmonics. The preceding inequality implies the left hand inequality in the first point of the proposition.

There are two parts to the proofs of the remaining assertions. The first part below proves the right hand inequality in the first point, and both the second and third points of Proposition 4. The second part proves the final point.

Part 1. Let $E(g)$ denote the infimum of the g -version of \mathcal{E}_0 . What with (14), a bound by $g^2(40\pi)^{-1} + c_1g$ on $E(g)$ gives the right hand inequality in the first point of Proposition 4 plus the next two points in the proposition. This step establish such an upper bound for $E(g)$.

For this purpose, fix, $\varepsilon > 0$ and a non-decreasing function β on $[0, \infty)$ that has value 0 on $[0, 1]$, equals 1 on $[1 + \varepsilon, \infty)$ and obeys $\beta' < 2/\varepsilon$. This done, fix $\lambda > 1$ and fix a smooth function $\chi : [0, \pi] \rightarrow [0, 1]$ that obeys $\chi(\theta) = \chi(\pi - \theta)$, equals 1 on $[\lambda^{-1}, \pi/2]$, equals $\lambda\theta$ for $\theta < (2\lambda)^{-1}$, and obeys $|\chi'| \leq 4\lambda$.

Now set $a = \lambda\beta(r)\chi(\theta)\sin\theta d\varphi$. Let $\psi \equiv \psi[\lambda, \varepsilon]$ denote the corresponding solution to the a -version of the first equation in (8). Then, a straight forward calculation finds that

$$E(g) \leq cg^{-2}\lambda^2(\varepsilon^{-1} + \ln(\lambda)) + 3/(8\pi)g^2 \int_{r\leq 1} \psi; \quad (28)$$

here c is a constant which is independent of g , ε and λ .

Hold onto (28) for the moment, and introduce the piece-wise continuous 1-form a' that is equal $\lambda\chi \sin\theta d\varphi$ where $r \geq 1 + \varepsilon$ and equal to 0 where $r \leq 1 + \varepsilon$. Let $\psi' \equiv \psi'[\lambda, \varepsilon]$ denote the corresponding solution to the a' version of (8). By virtue of (15), $\psi < \psi'$ since $|a| \geq |a'|$. Thus, (28) implies that

$$E(g) \leq cg^{-2}\lambda^2(\varepsilon^{-1} + \ln(\lambda)) + 3/(8\pi)g^2 \int_{r \leq 1} \psi'. \quad (29)$$

To make further progress, introduce the function $\underline{\rho}$ which is defined to equal $3/(4\pi)$ where $r \leq 1 + \varepsilon$ and to equal 0 where $r > 1 + \varepsilon$. Let $\underline{\psi}$ denote the corresponding solution to the equation

$$-\Delta\underline{\psi} + |a'|^2\underline{\psi} = \underline{\rho}. \quad (30)$$

on \mathbb{R}^3 which decays to zero as $r \rightarrow \infty$. As ψ' solves the analogous equation with ρ' replacing $\underline{\rho}$ and since $\underline{\rho} \geq \rho'$, one has

$$-\Delta(\psi' - \underline{\psi}) + |a'|^2(\psi' - \underline{\psi}) \leq 0 \quad (31)$$

on the whole of \mathbb{R}^3 . In particular, (31) with the maximum principle implies that $\psi' < \underline{\psi}$. Thus,

$$E(g) \leq cg^{-2}\lambda^2(\varepsilon^{-1} + \ln(\lambda)) + 3/(8\pi)g^2 \int_{r \leq 1} \underline{\psi}. \quad (32)$$

Moreover, since the integral of $\underline{\psi}$ over the radius 1 ball is less than its integral over the ball of radius $1 + \varepsilon$, the preceding inequality immediately give

$$E(g) \leq cg^{-2}\lambda^2(\varepsilon^{-1} + \ln(\lambda)) + 3/(8\pi)g^2 \int_{r \leq 1 + \varepsilon} \underline{\psi}. \quad (33)$$

Now, the next step is to rescale the expression on the far right in (33) so that the integral is over the radius 1 ball again. For this purpose, introduce the function η which assigns to the point $x \in \mathbb{R}^3$ the value $\eta(x) \equiv (1 + \varepsilon)^{-2}\underline{\psi}((1 + \varepsilon)x)$. In terms of η , the inequality in (33) reads

$$E(g) \leq cg^{-2}\lambda^2(\varepsilon^{-1} + \ln(\lambda)) + 3/(8\pi)(1 + \varepsilon)^5g^2 \int_{r \leq 1} \eta. \quad (34)$$

Moreover, the function η obeys the equation

$$-\Delta\eta + |b|^2\eta = \rho, \quad (35)$$

for the case with b given by $\lambda\chi \sin \theta d\varphi$ where $r \geq 1$ and $b = 0$ where $r \leq 1$.

To complete the proof of the second point of Proposition 4, it is necessary to bound the size of $3/(8\pi) \int_{r \leq 1} \eta$. For this purpose, note that this integral is equal to the supremum over all C^1 functions u which decay to zero as $r \rightarrow \infty$ of the functional

$$e(u) \equiv 3/(4\pi) \int_{r \leq 1} u - 2^{-1} \int (|\nabla u|^2 + |b|^2 u^2). \tag{36}$$

In this regard, there is a constant $c_0 > 0$ which is independent of λ and is such that when $\lambda \geq 1/100$, the following is true: Let u be any C^1 function of $\theta \in [0, \pi]$. Then

$$\int_{[0, \pi]} (u_\theta^2 + \lambda^2 \chi^2 u^2) \sin \theta d\theta \geq c_0 \lambda^2 \int_{[0, \pi]} u^2 \sin \theta d\theta. \tag{37}$$

Thus, for any C^1 function u as in (36), one has

$$e(u) \leq 3/(4\pi) \int_{r \leq 1} u - 2^{-1} \int |\partial_r u|^2 - 2^{-1} c_0 \lambda^2 \int_{r \geq 1} r^{-2} u^2. \tag{38}$$

And, (38) implies that

$$\begin{aligned} & 3/(8\pi) \int_{r \leq 1} \eta \\ & \leq \sup_u \left\{ 3/(4\pi) \int_{r \leq 1} u - 2^{-1} \int |\partial_r u|^2 - 2^{-1} c_0 \lambda^2 \int_{r \geq 1} r^{-2} u^2 \right\}. \end{aligned} \tag{39}$$

The point now is that the supremum on the right hand side of (39) can be calculated explicitly because a maximizing function, u_0 , can be exhibited: For this purpose, introduce the number $p = 2^{-1}(1 + (1 + 4c_0\lambda^2)^{1/2})$, and then

- $u_0 = (8\pi)^{-1}(2/p + 1 - r^2)$ where $r \leq 1$. (40)
- $u_0 = (4\pi)^{-1}p^{-1}r^{-p}$ where $r \geq 1$.

Given u_0 , the value of the supremum on the right hand side of (40) can be readily computed to be $(40\pi)^{-1}(1 + 5/p)$. Since $1/p \leq c_0^{1/2}\lambda^{-1}$, this means that

$$3/(8\pi) \int_{r \leq 1} \eta \leq (40\pi)^{-1}(1 + 5c_0^{-1/2}\lambda^{-1}), \tag{41}$$

and

$$E(g) \leq cg^{-2}\lambda^2(\varepsilon^{-1} + \ln(\lambda)) + (40\pi)^{-1}g^2(1 + \varepsilon)^5(1 + 5c_0^{-1/2}\lambda^{-1}). \quad (42)$$

This last inequality holds for any choice of $\varepsilon > 0$ and $\lambda > 1$, and so their values will be chosen to make the left hand side of (42) small. For this purpose, the first observation is that when $\varepsilon < 1/2$ and $\lambda > 5/c_0^{1/2}$, then the expression on the right hand side of (42) is no smaller than

$$E(g) \leq cg^{-2}\lambda^2(\varepsilon^{-1} + \ln(\lambda)) + (40\pi)^{-1}g^2(1 + 5c_0^{-1/2}\lambda^{-1}) + g^2\varepsilon. \quad (43)$$

Moreover, the left hand side of (43) is no greater than its value at $\varepsilon = \sqrt{cg^{-2}\lambda}$; thus

$$E(g) \leq 2\sqrt{2c}\lambda + cg^{-2}\lambda^2 \ln(\lambda) + (8\pi)^{-1}g^2c_0^{-1/2}\lambda^{-1} + (40\pi)^{-1}g^2. \quad (44)$$

Finally, $E(g)$ is no greater than the value of the left hand side of (44) in the case $\lambda = g$ which gives the bound

$$E(g) \leq (40\pi)^{-1}g^2 + c'g. \quad (45)$$

Here, c' is a constant which is independent of g .

Part 2. The assertion in the fourth point of Proposition 4 follows using the maximum principle with the equation in the second point of (8) given that there is a g -independent constant c'_1 such that

$$|\psi|(x) \leq c'_1g^{-1/2} \quad \text{where } r \geq 2, \quad (46)$$

To prove this last inequality, consider that (46) implies the equality

$$\psi(x)^2 = (2\pi)^{-1} \int |x - (\cdot)|^{-1}(\rho\psi - |\nabla\psi|^2 - |a|^2\psi^2). \quad (47)$$

Store this last equation momentarily to fix a smooth function $\beta : [0, \infty) \rightarrow [0, 1]$ which equals 1 on $[0, 1]$ and vanishes on $[3/2, \infty)$. Promote β to a function, $\underline{\beta}$, on \mathbb{R}^3 by setting $\underline{\beta}(y) = \beta(|y|)$.

With $\underline{\beta}$ understood, remark that (47) implies that

$$\psi(x)^2 \leq (2\pi)^{-1} \int |x - (\cdot)|^{-1}\underline{\beta}(\rho\psi - |\nabla\psi|^2 - |a|^2\psi^2) \quad (48)$$

since $\rho = 0$ where $\underline{\beta} \neq 1$. Then, an integration by parts finds

$$\psi^2(x) \leq (4\pi)^{-1} \int |\Delta(\underline{\beta}|x - (\cdot)|^{-1})| \psi^2. \quad (49)$$

Now, if $|x| \geq 2$, then the right hand side of (49) is no greater than

$$c' \int_{1 \leq r \leq 2} \psi^2, \quad (50)$$

where c' is a constant which depends only on the particular choice for β . Meanwhile, the integral in (50) is bounded by $16 \int_{r \geq 1} |\nabla \psi|^2$ by virtue of the second point in (21). Thus, (49) and (50) imply that

$$\psi(x)^2 \leq c'' \int_{r \geq 1} |\nabla \psi|^2 \quad (51)$$

at all points x with $|x| \geq 2$. This last inequality and the inequality in the second point to Proposition 4 completes the argument for the fourth point of Proposition 4.

e) Some preliminary conclusions about α at large r

The purpose of this subsection is to begin the study the pointwise behavior of a minimizer, α , of \mathcal{E}_0 . The particular observation in this section is summarized by

Proposition 5. *If α is a solution to a $g > 0$ version of (9), then $|\alpha|$ is uniformly bounded on \mathbb{R}^3 . Moreover, given $\varepsilon > 0$, there exists r_ε such that $\alpha < \varepsilon$ when $r > r_\varepsilon$. Said differently, $a = \alpha \sin \theta d\varphi$ obeys $|a| \leq \varepsilon r^{-1}$ when $r > r_\varepsilon$.*

Proof of Proposition 5. The derivation of the asserted bound requires a four step argument.

Step 1. Fix $r \geq 1$ and integrate both sides of the equation in the first point of either (8) or (11) with respect to the standard spherical

measure $d\Omega \equiv \sin\theta d\theta d\varphi$ on the unit sphere. After an integration by parts and multiplication by r^2 , the resulting equation reads:

$$-\partial_r r^2 \partial_r \int_{S^2} \psi(r, \cdot) d\Omega + \int_{S^2} r^2 (|a|^2 \psi)(r, \cdot) d\Omega = 0. \quad (52)$$

Apply the maximum principle to this last equation to conclude that

$$\partial_r \int \psi(r, \cdot) d\Omega < 0 \quad (53)$$

where $r \geq 1$.

Step 2. Now, integrate both sides of (52) with respect to the measure dr over the interval $[1, R]$ for any chosen $R \geq 1$. The result is

$$-\left(r^2 \partial_r \int \psi(r, \cdot) d\Omega \right)_{r=R} + \int_{1 \leq r \leq R} |a|^2 \psi = - \int \psi_r(1, \cdot) d\Omega = 1. \quad (54)$$

Here, the right hand equality follows by integrating both sides of the equation in the first point of (8) over the radius 1 ball.

This last equation implies that

$$\int |a|^2 \psi \leq 1. \quad (55)$$

Step 3. Equation (55) implies that $|x - (\cdot)|^{-1} |a| \psi^2$ is integrable for any choice of $x \in \mathbb{R}^3$ with $|x| \geq 1$. Indeed, as $\psi \leq (4\pi)^{-1} r^{-1}$ where $r \geq 1$ by virtue of (18), a version of Holder's inequality gives the bound

$$\begin{aligned} \int_{r \geq 1} |x - (\cdot)|^{-1} |a| \psi^2 &\leq \left(\int_{r \geq 1} |a|^2 \psi \right)^{1/2} \left(\int_{r \geq 1} |x - (\cdot)|^{-2} \psi^3 \right)^{1/2} \\ &\leq (4\pi)^{-3/2} |x|^{-1} (\ln |x|)^{1/2}. \end{aligned} \quad (56)$$

In particular, as $|x - (\cdot)|^{-1} |a| \psi^2$ is integrable for any $x \in \mathbb{R}^3$, and $|a|/r$ is square integrable, the solution a to the second equation in (9) is given by

$$a|_x = (4\pi)^{-1} g^4 \int_{r \geq 1} |x - (\cdot)|^{-1} a \psi^2. \quad (57)$$

Note that, (56) and (57) imply that $|\alpha| = r|a| \leq \zeta g^4 (\ln r)^{1/2}$ with ζ a constant that is independent of α .

Step 4. To remove the factor of $(\ln r)^{1/2}$ from this estimate, first fix $\delta \in (0, 1/4)$; its value is determined by the chosen ε . Next, write the integral in (57) as a sum of two parts, $s_+ + s_-$ where s_+ is given by (57) with the integration domain now the region where $r \geq \delta|x|$; correspondingly, s_- is given by (57) with the integration domain the ball of radius $\delta|x|$ centered at the origin. Note that $|s_+|$ is bounded by

$$g^4 \zeta \left(\int_{r > \delta|x|} |a|^2 \psi \right)^{1/2} \left(\int_{r > \delta|x|} |x - (\cdot)|^{-2} r^{-3} \right)^{1/2} \tag{58}$$

$$\leq \zeta_+ \sigma(|x|) |x|^{-1} (\ln \delta)^{1/2},$$

where ζ_+ is independent of r, δ, α and g , while $\sigma(s)^2 \equiv \int_{r > \delta s} |a|^2 \psi$. In particular, note that

$$\lim_{s \rightarrow \infty} \sigma(s) = 0. \tag{59}$$

Meanwhile, the necessary bound for s_- requires two preliminary observations. The first is that

$$\int_{r < R} a \psi^2 = 0 \tag{60}$$

for any $R \geq 1$. Indeed, this follows from symmetry considerations after writing $a = \alpha \sin \theta d\varphi$ with respect to the Cartesian differentials $\{dx, dy, dz\}$ as $a = \alpha(r, \theta)(\cos \varphi dy - \sin \varphi dx)$. The second key observation is that when $|y| \leq 2^{-1}|x|$, then

$$|x - y|^{-1} = |x|^{-1} + |x|^{-2} \wp_1(x, y), \tag{61}$$

where $|\wp(x, y)| \leq \zeta|y|$.

Together, (60) and (61) imply that

$$a_- = (4\pi)^{-1} g^4 |x|^{-2} \int_{r \leq \delta|x|} \wp_1 a \psi^2. \tag{62}$$

In particular, (62) implies that

$$|a_-| \leq \zeta |x|^{-2} \left(\int |a|^2 \psi \right)^{1/2} \left(\int_{r \leq \delta|x|} r^{-1} d^3 y \right)^{1/2} \leq \zeta_- \delta |x|^{-1}, \tag{63}$$

where ζ_- is independent of both $|x|$ and δ .

The assertion of Proposition 5 follows directly from (58), (59) and (63) by first choosing $\delta = 2^{-1}(1 + \zeta_-)^{-1}\varepsilon$ to make $|a_-| < 2^{-1}\varepsilon|x|^{-1}$; and then choose $|x|$ large so that $\sigma(|x|)$ in (59) is smaller than $2^{-1}(1 + \zeta_+)^{-1}\varepsilon/\ln(\delta)^{1/2}$.

f) Some refined conclusions about ψ at large r

The purpose of this step is to refine the large r bound of ψ from Proposition 4. Indeed, consider:

Proposition 6. *Suppose that α is a minimizer of a $g > (6\pi)^{1/2}$ version of \mathcal{E}_0 . Then the corresponding ψ obeys $\psi = \psi_0 r^{-1} + o(r^{-1})$ at large r where ψ_0 is a constant that is no larger than $\sqrt{2g^{-2}}$.*

The remainder of this section is occupied with the

Proof of Proposition 6. The proof is divided into 3 steps.

Step 1. This step establishes that $\psi = \psi_0 r^{-1} + o(r^{-1})$ with ψ_0 a constant. For this purpose, note that it is an immediate consequence of (8) that such will be the case provided that

$$\int |x - y|^{-1} |a|^2 \psi d^3 y = m_0 r^{-1} + o(r^{-1}) \quad (64)$$

for r large, where m_0 is a constant. And, this last conclusion, follows directly from (18) and Proposition 5.

Step 2. This step constitutes a digression to establish

Lemma 7. *Suppose that α minimizes a $g > 0$ version of \mathcal{E}_0 . Then either α is non-negative or non-positive.*

Proof of Lemma 7. Since the α and $|\alpha|$ versions of (8) are identical, the corresponding solutions agree, and $\mathcal{E}_0(|\alpha|) = \mathcal{E}_0(\alpha)$. Thus, $|\alpha|$ minimizes \mathcal{E}_0 if α does. As $|\alpha| = \alpha$ where $\alpha \geq 0$, it follows from the unique continuation theorem of Aronszajn [A] that $\alpha = |\alpha|$ everywhere if α is anywhere larger than zero.

With this lemma understood, the on going (and usually implicit) assumption in the remainder of this article is that $\alpha \geq 0$.

Step 3. Now, introduce

$$f(r) \equiv \int_{0 \leq \theta \leq \pi} \alpha(r, \theta) \sin^2 \theta d\theta. \quad (65)$$

As $\alpha \geq 0$, so $f \geq 0$. This function f obeys the following differential equation:

$$-f_{rr} + 2r^{-2}f = g^4 \int_{0 \leq \theta \leq \pi} \psi^2 \alpha \sin^2 \theta d\theta, \tag{66}$$

as can be seen by integrating both sides of the equation in the second point of (9) using the measure $\sin^2 \theta d\theta$ on $[0, \pi]$.

Using the fact that $r\psi$ has limit ψ_0 as $r \rightarrow \infty$, this last equation implies that

$$-f_{rr} + 2r^2f > 2r^{-2}f \tag{67}$$

at large r if $\psi_0 > \text{sur}d2g^{-2}$. This last equation implies that $f_{rr} < 0$ for large r . Now, were f_r ever negative, then f_r would become more negative as r increased and thus the condition $f > 0$ would be violated. Hence, $f_r > 0$ for all sufficiently large r . Of course, this implies that f is increasing as r tends to ∞ , a conclusion which is forbidden by Proposition 5. Thus, $\psi_0 \leq \sqrt{2}g^{-2}$ as claimed.

g) The behavior of α at large r

Take $g > (6\pi)^{1/2}$ in this section. This done, reintroduce the constant ψ_0 from Proposition 6 and agree to write the latter as $\sqrt{2}g^{-2}e_0$ where $e_0 \in [0, 1]$. It then follows from (66) using the maximum principle that the function f in (65) obeys

$$f > r^{-p} \tag{68}$$

for any $p > p_0 \equiv 2^{-1}[(1 + 8(1 - e_0^2))^{1/2} - 1]$. In fact, one can show without much difficulty that there is a constant $c_0 > 0$ such that

$$\alpha = c_0 \sin \theta r^{-p_0} + o(r^{-p_0}) \tag{69}$$

as r gets large. With regard to (68) and (69), note that when $e_0 > 0$, these equations imply that the magnetic field, $B = *da$, falls off at an anomalously slow rate; as the classical dipole field requires $\alpha = \mathcal{O}(r^{-1})$ at large r . In any event, as is argued momentarily $e_0 > 0$ as asserted in the third point of Theorem 2.

The claim that $e_0 > 0$ follows from the existence of a bounded function $\sigma = \sigma(r, \theta)$ that obeys the four conditions

- $-\Delta\sigma + r^{-2}\alpha^2\sigma = 0$ (70)
- $\lim_{r \rightarrow \infty} \sigma$ exists and equals 1.
- $\sigma > 0$ everywhere.
- $\sigma(0) = \sqrt{2}g^{-2}e_0$.

To belabor the obvious, the last two lines of (70) are compatible only if $e_0 > 0$.

To consider the existence issue for σ , fix a smooth function $\beta : [0, \infty) \rightarrow [0, 1]$ which vanishes on $[2, \infty)$, equals one on $[0, 1]$ and is non-increasing. Given $R > 1$, promote β to a smooth function, β_R , on \mathbb{R}^3 by setting $\beta_R(x) \equiv \beta(|x|/R)$. This done, consider solving for a function σ_R that obeys

- $-\Delta\sigma_R + \beta_R r^{-2}\alpha^2\sigma_R = 0$. (71)
- $\lim_{r \rightarrow \infty} \sigma_R$ exists and equals 1.

The existence of a unique such function is not hard to establish and the latter task is left to the reader. Here are some of the properties of σ_R : First, the maximum principle guarantees that $\sigma_R > 0$ and, as α is not zero identically, that $\sigma_R < 1$. Second, $\sigma_R \leq 1 - \mathcal{O}(r^{-1})$. Third, $|\nabla\sigma_R| = \mathcal{O}(r^{-2})$. Indeed, these last two properties are consequence of the integral equation equivalent of (71):

$$\sigma_R(x) = 1 - (4\pi)^{-1} \int |x - (\cdot)|^{-1} \beta_R r^{-2} \alpha^2 \sigma_R. \quad (72)$$

The maximum principle also guarantees that $\sigma_R < \sigma_{R'}$ when $R > R'$. This understood, it follows that the sequence $\{\sigma_R\}_{R \rightarrow \infty}$ is decreasing pointwise and so there is a unique limit, σ . Moreover, by virtue of the first line in (71) and the bound $\sigma_R < 1$ the functions in the set $\{\sigma_R\}$ are uniformly continuous with bounded first derivatives on any given compact set. Moreover, they have uniformly continuous derivatives to any order on any compact set that avoids the unit sphere. Thus, the limit function σ is smooth where $r \geq 1$ and obeys the equation in the first line of (70). Also, $\sigma < 1$ everywhere and $\sigma \geq 0$ everywhere with equality only if $\sigma \equiv 0$.

Can σ vanish identically? To prove that $\sigma > 0$ in the case $e_0 < 1$, note first that under this assumption, $r^{-2}\alpha^2 \leq c\beta_1 r^{-2-2p_0}$ with $p_0 > 0$ as in (69) and with $c > 0$ a constant. This understood, then (72) implies the existence of some constant c' , independent of R , and such that $\sigma_R \geq 1c'$. As σ_R converges pointwise to σ , this last equation implies that $\sigma > 0$ somewhere and hence everywhere on \mathbb{R}^3 when $e_0 < 1$.

In the case where $e_0 = 1$, a non-zero lower bound for σ can still be deduced from (72), albeit with more effort. The start of this task uses the fact that σ_R converges pointwise to σ to deduce from (72) that

$$\sigma(x) \geq 1 - (4\pi)^{-1} \int |x - (\cdot)|^{-1} r^{-2} \alpha^2. \quad (73)$$

Now, to estimate the integral, consider breaking the integration domain into three regions: Region 1 has $r > 2|x|$, Region 2 has $|x|/2 \leq r \leq 2|x|$ and Region 3 has $r \leq |x|/2$.

The contribution to the integral in (73) from Region 1 is no more than

$$(4\pi)^{-1} \int_{r>2|x|} r^{-3} \alpha^2 \leq \zeta \int_{r>2|x|} |a|^2 \psi, \quad (74)$$

where ζ is an x -independent constant. As the integral of $|a|^2 \psi$ is finite (as asserted by (55)), so the function of $|x|$ defined by the right hand integral in (74) tends uniformly to zero as $|x|$ tends to infinity. Meanwhile, the contribution to the integral on the right hand side of (73) from Region 2 is no greater than

$$\begin{aligned} & (4\pi)^{-1} \left(\sup_{|x|/2 \leq r \leq 2|x|} \alpha^2(y) \right) \int_{|x|/2 \leq r \leq 2|x|} |x - (\cdot)|^{-1} r^{-2} \\ & \leq \zeta \sup_{|x|/2 \leq r \leq 2|x|} \alpha^2(y), \end{aligned} \quad (75)$$

where ζ is, again, and x -independent constant. Note Proposition 5 asserts that the right hand side of (75) tends uniformly to zero as $|x|$ tends to infinity.

Finally, consider the contribution to the integral on the right hand side of (73) from Region 3. This contribution is no greater than

$$c|x|^{-1} \int_{r<|x|} r^{-2} \alpha^2. \quad (76)$$

To bound the latter expression, divide the region of integration into the domains $\{A_n : 1 \leq n \leq |x|\}$ where the index n is an integer and $A_n = \{y : n \leq |y| \leq n + 1\}$. Meanwhile, let s_n denote the integral of $r^{-3}\alpha^2$ over A_n . In this regard, note that $r^{-3}\alpha^2$ is integrable over all of \mathbb{R}^3 by virtue of (55), and so $\sum_{1 \leq n < \infty} s_n < \infty$. Let S denote the value of this infinite sum. It then follows that the expression in (76) is no greater than

$$c'|x|^{-1} \sum_{1 \leq n \leq |x|} ns_n. \tag{77}$$

Now, as the $\sum_{1 \leq n < \infty} s_n$ is finite, given ε , there exists N such that $\sum_{n \geq N} s_n < \varepsilon$. This understood, then (77) is no greater than

$$\zeta \left(\sum_{n \geq N} s_n + N|x|^{-1} \sum_{1 \leq n \leq N} s_n \right) \leq \zeta'(\varepsilon + N|x|^{-1}). \tag{78}$$

Here, both ζ and ζ' are x -independent constants. This last bound implies that Region 3's contribution to the integral on the right hand side of (73) tends uniformly to zero as $|x|$ tends to infinity.

Thus, the analysis for the three regions has established the following: Give $\varepsilon > 0$, there exists $r_\varepsilon > 1$ such that

$$\sigma(x) \geq 1 - \varepsilon \quad \text{when } |x| > r_\varepsilon. \tag{79}$$

This last inequality establishes that $\sigma > 0$ even when $e_0 = 1$. By the way, this last inequality also establishes that σ tends uniformly to 1 as r tends to infinity.

Now turn to the final point in (70). To establish this point, multiply both sides of the equation $-\Delta\psi + r^{-2}\alpha^2\psi = \rho$ by σ_R and integrate the resulting equation over \mathbb{R}^3 . Integrate by parts twice to place the Laplacian on σ_R . Then, invoke (71) to obtain the equality

$$4\pi\sqrt{2}g^{-2}e_0 + \int (1 - \beta_R)r^{-2}\alpha^2\psi\sigma_R = \sigma_R(0). \tag{80}$$

Here, the fact that σ_R is harmonic in the ball has been used to identify $\int \rho\sigma_R$ with $\sigma_R(0)$. In any event, this last equation asserts that $\sigma_R(0) > 4\pi\sqrt{2}g^{-2}e_0$ for all R , and so $\sigma(0) > 0$ for all R . This proves that σ is not identically zero when $e_0 > 0$ and thus when $e_0 = 1$.

Equation (80) implies the final line of (70) by virtue of (55), for the latter implies that the integral term on the left side of (80) converges uniformly to zero as $R \rightarrow \infty$.

h) A lower bound for the magnetic energy

The purpose of this subsection is to derive a lower bound for $\int |\nabla a|^2$. Here is the precise statement:

Proposition 8. *There exists a constant $c \geq 1$ with the following significance: Fix $g > (6\pi)^{1/2}$ and let α be a minimizer of the g version of \mathcal{E}_0 in (10). Set $a = \alpha \sin \theta d\varphi$. Let U denote the annulus where $1 \leq r \leq 1 + cg^{-1}$. Then*

$$g^{-2} \int_U |\nabla a|^2 \geq c^{-1}g. \quad (81)$$

The remainder of this section is occupied with the

Proof of Proposition 8. The proof is divided into four steps.

Step 1. First, let c_1 denote the constant which appears in Proposition 4, and introduce \mathcal{C} to denote the set of Sobolev class L_1^2 functions u on \mathbb{R}^3 which vanish where $r \geq 1 + 64\pi c_1 g^{-1}$. The claim here is that if $g \geq 64\pi c_1$, then there exists a unique $\kappa \equiv \kappa(\alpha) \in (0, \infty)$ with the property that

$$\begin{aligned} & 3(8\pi)^{-1} \int_{r \leq 1} \psi \\ &= \sup_{u \in \mathcal{C}} \left\{ 3(4\pi)^{-1} \int_{r \leq 1} u - 2^{-1} \left(\int |\nabla u|^2 + \kappa^2 g^2 \int_{r \geq 1} r^{-2} u^2 \right) \right\}. \end{aligned} \quad (82)$$

To see that κ exists, remark first that for any choice of κ , the supremum on the right hand side of (82) is achieved by a unique function, u_κ , which can be written down in closed form. To write u_κ , first introduce

- $d \equiv 64\pi c_1 g^{-1}$. (83)
- $p \equiv 2^{-1}((1 + 4\kappa^2 g^2)^{1/2} + 1)$.
- $p' \equiv 2^{-1}((1 + 4\kappa^2 g^2)^{1/2} - 1)$.

With these definitions understood, here is u_κ :

(84)

- $u_\kappa = (8\pi)^{-1}(1 - r^2 + [(1 + d)^{p+p'} - 1]/[p(1 + d)^{p+p'} + p'])$ where $r \leq 1$.
- $u_\kappa = (4\pi)^{-1}[(1 + d)^{p+p'}r^{-p} - r^{p'}]/[p(1 + d)^{p+p'} + p']$ where $1 \leq r \leq 1 + d$.

Since the supremum on the right hand side of (82) is $3(8\pi)^{-1} \int_{r \leq 1} u_\kappa$, integration of the first line in (82) finds the supremum on the right hand side of (82) equal to

$$f(\kappa) \equiv (40\pi)^{-1} + (16\pi)^{-1}[(1 + d)^{p+p'} - 1][p(1 + d)^{p+p'} + p']^{-1}. \quad (85)$$

Now, there are four key observations about $f(\cdot)$ in (85): First, f is a continuous function of $\kappa \in [0, \infty)$. Second, f is a monotonically decreasing function on $[0, \infty)$. Third, $\lim_{\kappa \rightarrow \infty} f(\kappa) = (40\pi)^{-1}$. Finally,

$$f(0) = (40\pi)^{-1} + (16\pi)^{-1}(1 + (64\pi c_1)^{-1}g)^{-1}. \quad (86)$$

If the right hand side of (86) is greater than $3(8\pi)^{-1} \int_{r \leq 1} \psi$, then it follows from (13) and the first two points of Proposition 4 that $3(8\pi)^{-1} \int_{r \leq 1} \psi$ lies in the range of f and so there exists some $\kappa \in (0, \infty)$ which makes (82) true. Moreover, there will be a unique such κ since f is monotonically decreasing.

Now, due to (13) and Proposition 4, the right hand side of (86) can be guaranteed greater than $3(8\pi)^{-1} \int_{r \leq 1} \psi$ when

$$(16\pi)^{-1}(1 + (64\pi c_1)^{-1}g)^{-1} \geq 2c_1/g, \quad (87)$$

which occurs when $g \geq 64\pi c_1$.

Step 2. Here is the next point:

$$\text{There exists } K \geq 1 \text{ such that if } g \geq 64\pi c_1, \text{ then } \kappa(\alpha) \geq 1/K. \quad (88)$$

Indeed, suppose that $\kappa = 10^{-5}(64\pi c_1)^{-1}$. Also, assume that $g > 10^{10}64\pi c_1$. Thus $\kappa g > 10^5$. With this understood, $p, p' = \kappa g + o(10^{-5})$ and $(1 + d)^{p+p'} = 1 + (128\pi c_1)\kappa + o(10^{-5})$. This implies that

$$f(\kappa) \geq (40\pi)^{-1} + (32\pi)^{-1}(128\pi c_1 \kappa)/(\kappa g + \kappa g)^{-1} = (40\pi)^{-1} + 2c_1 g^{-1}. \quad (89)$$

Because of (13) and the assertions of Proposition 4, the right hand side of (89) can not be smaller than $3(8\pi)^{-1} \int_{r \leq 1} \psi$. Thus, since f is monotonically decreasing, it follows that $\kappa(\alpha)$ in (82) is larger than $10^{-5}(64\pi c_1)^{-1}$ when $g > 10^{10}(64\pi c_1)$. This fact implies the existence of K which makes (88) hold for the larger range of g .

Step 3: Now, remember that

$$3(8\pi)^{-1} \int_{r \leq 1} \psi = \sup_u \left\{ 3(4\pi)^{-1} \int_{r \leq 1} u - 2^{-1} \int (|\nabla u|^2 + |a|^2 u^2) \right\}, \tag{90}$$

where the supremum on the right hand side is taken over the set of functions u on \mathbb{R}^3 with both ∇u and $r^{-1}u$ square integrable. Note that u_κ is such a function for any κ . In particular, take

$$\kappa = \min(1/K, 10^{-5}(64\pi c_1)^{-1}).$$

In this case, the fact that $f(\kappa)$ is decreasing, (82) and (90) imply that

$$\int |a|^2 u_\kappa^2 \geq \kappa^2 g^2 \int_{r \geq 1} r^{-2} u_\kappa^2. \tag{91}$$

This last inequality with the second point in (84) imply that there is a constant $C \geq 1$ such that when $g \geq C$, then

$$\int_U |a|^2 \geq C^{-2} g. \tag{93}$$

Since $a|_{r=1} = 0$, this last inequality implies the existence of the constant c which makes (81) hold.

i) The value of e_0 and the form of ψ and α where $r \geq 10$

The previous sections found a constant $e_0 \in (0, 1]$ that controls the large r asymptotics of both ψ and α in as much as $\psi \sim \sqrt{2}g^{-2}e_0 r^{-1}$ and $\alpha \sim c_0 \sin r^{-[(9-8e_0^2)^{1/2}-1]/2}$ as $r \rightarrow \infty$. In particular, these observations imply that $\psi \leq \sqrt{2}g^{-2}e_1 r^{-1}$ and $\alpha \geq c_1 \sin \theta r^{-1}$ at large r with e_1 and

c_1 positive constants. This understood, the purpose of this section is to bound e_1 from above and c_1 from below and to establish a lower bound for the radius r where this upper bound for ψ and lower bound for α are reasonable. The fruits of this labor are summarized by

Proposition 9. *There exists $\gamma \geq 1$ with the following significance: Suppose that $g > (6\pi)^{1/2}$ and that α minimizes the g -version of \mathcal{E}_0 . At points where $r \geq \gamma$, this α and its associated ψ obey*

- $\psi \leq \sqrt{2}g^{-2}e^{-g^{1/4}/\gamma r^{-1}}$.
- $\alpha \geq \gamma^{-1}g^{1/2} \sin \theta r^{-1}$.

The remainder of this section is dedicated to the

Proof of Proposition 9. As the proof is long, it is broken into twelve steps. Take $g > (6\pi)^{1/2}$ in all steps below.

Step 1. According to (93), there is a positive, g -independent constant c such that $|a|^2$ has integral greater than $c^{-1}g$ over the region where $r < 1 + c/g$. This step provides a refinement with a proof that the mass of this integral can not concentrate where $\sin \theta$ is zero. Here is the precise statement:

Lemma 10. *There exists $c > 1$ such that when $g > c$, then*

$$\int_{r \leq 1+c/g, \sin \theta > 1/c} |a|^2 \geq c^{-1}g. \quad (94)$$

Proof of Lemma 10. Suppose that the lemma were false. Then, given $R \geq 1$ and $\varepsilon > 0$, there would exist arbitrarily large values for g for which

$$\int_{r \leq 1+R/g, \sin \theta > 1/R^2} |a|^2 \leq \varepsilon g/R. \quad (95)$$

Given that such is the case, fix $\delta \in (0, 1/1000)$ and let u_δ denote the following function:

$$u_\delta \equiv \max(0, 3(8\pi)^{-1}[(1 - r^2) + \delta(2 + (1 - 3 \cos^2 \theta)r^2)]). \quad (96)$$

Note that $u_\delta < \delta$ where $r \geq 1$ and $u_\delta = 0$ where $r > 1 + 2\delta$. Moreover,

$$u_\delta < 9(8\pi)^{-1}\delta \sin^2 \theta \quad (97)$$

where $r \geq 1$ and $\sin^2 \theta < 2/3$.

The preceding properties of u_δ imply that

$$\begin{aligned} & \int \rho u_\delta - 2^{-1} \int (|\nabla u_\delta|^2 + |a|^2 u_\delta^2) \\ & \geq (40\pi)^{-1} + 2\delta - c_1 \delta^2 \\ & \quad - \delta^2 \left(\int_{r \leq 1+R/g, \sin \theta > 1/R^2} |a|^2 + R^{-8} \int_{r < 1+R/g} |a|^2 \right). \end{aligned} \quad (98)$$

With (98) understood, take $\delta = R/(2g)$. This done, the right hand side of (98) is no smaller than

$$(40\pi)^{-1} + R/g(1 - \zeta(\varepsilon + R^{-4} + R/g)), \quad (99)$$

where ζ is a g -independent constant. Indeed, (99) follows immediately from (98) and (95) given that $\int_{r < 1+R/g} |a|^2$ is bounded by a g -independent multiple of $R^2 g$. In this regard, the latter bound is obtained as follows: Note first that $\int_{r < r+R/g} |a|^2$ is bounded by $(R/g)^2 \int_{r < 1+R/g} |\nabla a|^2$ since $a = 0$ where $r = 1$. Meanwhile, according to (14), the integral of $|\nabla a|^2$ over the whole of \mathbb{R}^3 is equal to g^4 times that of $|a|^2 |\psi|^2$, and the latter, by virtue of Proposition 4, is no greater than $c_1 g^{-1}$.

Now, the point is that when R is large, the lower bound in (99) is not compatible with Proposition 4 because ψ and not $u_{R/(2g)}$ maximizes the expression $\int \rho u d^3 x - 2^{-1} \int (|\nabla u|^2 + |a|^2 u^2)$ as u ranges over the smooth functions on \mathbb{R}^3 for which $|u|/r$ is square integrable.

Step 2. This step proves that there is a g -independent constant $\underline{c} > 1$ and, for each $g > \underline{c}$, there is a number $r_0 \in (1, 1 + \underline{c}/g)$ such that the function f in (65) obeys

$$f(r_0) \geq \underline{c}^{-1} g^{1/2}. \quad (100)$$

In this regard, note that (100) and (66) imply that

$$f(r) \geq \underline{c}^{-1} g^{1/2} r^{-1} \quad (101)$$

for all $r \geq r_0$.

The proof for existence of such constant \underline{c} and r_0 uses three previously established facts: The first fact is the assertion in Lemma 10. The second, a converse of sorts, was derived to end the proof of Lemma 10: The constant c in Lemma 10 can be chosen so that $\int_{1 \leq r \leq 1+c/g} |a|^2 \leq cg$. The third, a consequence of (9) and Proposition 4, is that c can be chosen so that $\int_{1 < r < 1+c/g} |\nabla a|^2 < cg^3$. By virtue of these three facts, there exists a g -independent constant $c' \geq 1$ and a possibly g -dependent number $r_0 \in (1, 1 + c'/g)$ such that $m \equiv |a| \Big|_{r=r_0}$ obeys

$$\bullet \quad c'^{-1}g \leq \int_{\sin \theta > 1/c'} m^2 d\theta \leq c'g^2. \quad (102)$$

$$\bullet \quad \int_{\sin \theta > 1/c'} m_\theta^2 d\theta \leq c'g^4.$$

As argued momentarily, these inequalities imply that

$$\int_{\sin \theta > 1/c'} m d\theta \geq 2^{-3/4} c'^{-3/2} g^{1/2}. \quad (103)$$

The desired inequality in (100) follows from (103) with $\underline{c} = 2^{-3/4} c'^{-7/2}$.

To establish (103), note first that $\int_{\sin \theta > 1/c'} m^2 d\theta \leq \hat{m} \int_{\sin \theta > 1/c'} m d\theta$ where \hat{m} denotes the maximum value of m where $\sin \theta > 1/c'$. Thus, the left most inequality in the first point of (102) requires

$$c'^{-1} \hat{m}^{-1} g^2 \leq \int_{\sin \theta > 1/c'} m d\theta. \quad (104)$$

To obtain the requisite upper bound on \hat{m} , first note that when $m(\theta) = \hat{m}$, then $m(\theta') \geq 2^{-1}$ provided that

$$|\theta - \theta'| < 4^{-1} c'^{-1} g^{-4} \hat{m}^2. \quad (105)$$

Indeed, this follows from the bottom point in (102). Second, observe that (105) is consistent with the right hand inequality in the top point of (102) only if

$$2^{-1} \hat{m}^2 (4^{-1} c'^{-1} g^{-4}) \hat{m}^2 \leq c' g^2. \quad (106)$$

Thus, $\hat{m} \leq (8c'^2)^{1/4} g^{3/2}$. Insert this last bound into (104) to obtain (103).

Step 3. The step constitutes a digression of sorts to state and then prove

Lemma 11. *Let β be a function of the spherical coordinates (r, θ) that vanishes where $r \leq 1$, has limit zero as $r \rightarrow \infty$ and such that $r^{-2}\beta, r^{-1}\beta_r$ and $r^{-2}(\sin \theta)^{-1}(\sin \theta \beta)_\theta$ are all square integrable. Then*

$$\int r^{-2}(\beta_r^2 + r^{-2}(\sin \theta)^{-2}(\sin \theta \beta)_\theta^2) \geq g^4 \int r^{-2}\beta^2\psi^2 \tag{107}$$

with equality if and only if β is a multiple of α .

Proof of Lemma 11. Suppose that (107) is violated by some non-trivial β . Then, as is explained momentarily, there exists a non-negative violator, β , with

$$-\beta_{rr} - r^{-2}((\sin \theta)^{-1}(\sin \theta \beta)_\theta)_\theta - g^4\psi^2\beta \leq 0 \tag{108}$$

where $r \geq 1$ and with a strict inequality on some open set. Take this last equation and multiply by α , then integrate the resulting inequality over the $r \geq 1$ portion of \mathbb{R}^3 . As $\alpha > 0$, the result is a negative number. However, as α also obeys (9), two applications of integration by parts contradict this last assertion.

To prove (108), note first that if there exists β that violates (107), there exists such a function β which is positive and has compact support in some very large radius ball. This said, fix $R \geq 1$ and let $B_R \subset \mathbb{R}^3$ denote the ball of radius R . When R is large, standard potential theory finds a unique function β_R on $B_R - B_1$, vanishing on the boundary of this domain, positive on its interior with maximum 1 and satisfying

$$-(\beta_R)_{rr} - r^{-2}((\sin \theta)^{-1}(\sin \theta \beta_R)_\theta)_\theta - g^4\psi^2\beta_R = -\lambda_R\beta_R \quad \text{where } r \geq 1 \tag{109}$$

with $\lambda_R > 0$. Potential theory can also be used to prove that λ_R increases with R . Meanwhile, view (109) at a local maximum of β_R to see that $\lambda_R \leq g^4 \max_{r \geq 1} \psi^2$.

Now, consider that $\psi = \sqrt{2}g^{-2}e_0r^{-1} + o(r^{-1})$. This understood, it follows from (109) that there exists $r_0 \geq 1$ such that when R is large, then β_R has no local maxima where $r \geq r_0$. In fact, this last equation implies that $\beta_R \leq e^{-\lambda_R r/2}$ where $r \geq r_0$. In any event, as $\beta_R = 1$ at some $r < r_0$ for all large R , and as λ_R is increasing and bounded, (109) coupled with standard elliptic regularity theory finds that $\lim R \rightarrow \infty \beta_R \equiv \beta$ exists and has the following properties: First, β is a smooth function where $r > 1$ and vanishes at $r = 1$. Second,

β has maximum 1. Third, all of the following are square integrable: $r^{-1}\beta$, $r^{-2}(\sin \theta)^{-1}(\sin \theta \beta)_\theta$ and $r^{-2}\beta$. Finally, β is a violator of (107) and obeys (108) with the inequality holding on a non-empty set. (In fact, β obeys (109) with λ_R replaced by $\lambda \equiv \lim_{R \rightarrow \infty} \lambda_R$.)

Step 4. This step uses (107) to obtain an upper bound on the size of ψ where $r \geq 4$. The precise statement is

Lemma 12. *There exists a g -independent constant ξ with the following significance: At points where $r \geq 4$, the function ψ obeys $\psi \leq \xi g^{-2} r^{-1}$.*

Proof of Lemma 12. To begin, take β in (107) as follows:

- $\beta(r, \theta) = (r - 1) \sin^{1/4} \theta$ where $r \leq 2$. (110)
- $\beta(r, \theta) = 4r^{-2} \sin^{1/4} \theta$ where $r \geq 2$.

The left hand side of (107) for this choice of β is finite, some constant. This understood, then (107) implies the existence of a g -independent constant ξ_1 such that

$$\int_{2 \leq r \leq 8} \psi^2 \sin^{1/2} \theta \leq g^{-4} \xi_1. \quad (111)$$

Next, use the fact that $-\Delta \psi \leq 0$ where $r \geq 1$ and standard Green's function techniques to find a g -independent constant such that

$$\psi \leq \kappa r^{-1} \int_{2 \leq r \leq 3} \psi. \quad (112)$$

at points where $r \geq 4$. (Remember when deriving (112) that $\psi = |\psi|$.) With (111) and (112), the lemma follows using Holder's inequality and the fact that $\sin^{-1/2} \theta$ is locally integrable on \mathbb{R}^3 .

Step 5. This step uses the bound in Lemma 12 to obtain an upper bound on the integral over the region where $r \geq 8$ of $r^{-2} \alpha^2 \psi^2$. To obtain this bound, fix a function, χ , of r that equals 1 where $r \geq 8$ and zero where $r \leq 4$. This done, multiply both sides of the equation in the first point of (8) by $\chi \psi$ and then integrate the result of \mathbb{R}^3 . Two

applications of integration by parts and an appeal to Lemma 12 yields an inequality of the form

$$\int_{8 \leq r} (|\nabla \psi|^2 + r^{-2} \alpha^2 \psi^2) \leq \xi_1 \int_{4 \leq r \leq 8} \psi^2 \leq \xi_2 g^{-4}, \quad (113)$$

where ξ_1 and ξ_2 are g -independent constants.

Step 6. To start this step, decompose α as $\alpha = \hat{\alpha} + \beta$, where $\hat{\alpha} = \alpha$ at $r = 8$ but otherwise solves the equation

$$-(\hat{\alpha}_{rr} + r^{-2}((\sin \theta)^{-1}(\sin \theta \hat{\alpha})_{\theta})) = 0, \quad (114)$$

while $\beta = 0$ at $r = 8$ and obeys

$$-(\beta_{rr} + r^{-2}((\sin \theta)^{-1}(\sin \theta \beta)_{\theta})) = g^4 \psi^2 \alpha. \quad (115)$$

Both $\hat{\alpha}$ and β are solutions to their respective equations on \mathbb{R}^3 that limit to zero as $r \rightarrow \infty$. In this regard, note that such a decomposition can be found using standard properties of the Laplacian on \mathbb{R}^3 as $\hat{\alpha}$ and β are obtained by first solving the equations

- $-\Delta \hat{a} = 0$ where $r \geq 8$ with $\hat{a}|_{r=16} = a|_{r=16}$, (116)
- $-\Delta b = g^4 \psi^2 a$ where $r \geq 8$ with $b|_{r=16} = 0$,

and then writing $\hat{a} = \hat{\alpha} \sin \theta d\varphi$ and $b = \beta \sin \theta d\varphi$.

In any event, the purpose of this step is to obtain a pointwise bound on β . In particular, as $|b| = r^{-1}|\beta|$, such a bound can be obtained using the Dirichlet Green's function for Δ in conjunction with (116). Indeed, this strategy finds

$$|x|^{-1}|\beta(x)| \leq (4\pi)^{-1} g^4 \int_{8 \leq r} |x - (\cdot)|^{-1} r^{-1} \psi^2 \alpha. \quad (117)$$

This last inequality understood, use Lemma 12 to eliminate one power of ψ and so bound the right side of (117)

$$\begin{aligned} & g^4 \int_{8 \leq r} |x - (\cdot)|^{-1} r^{-1} \psi^2 \alpha \\ & \leq \xi g^2 \int_{8 \leq r} |x - (\cdot)|^{-1} r^{-2} \psi \alpha \\ & \leq \xi g^2 \left(\int_{8 \leq r} |x - (\cdot)|^{-2} r^{-2} \right)^{1/2} \left(\int_{8 \leq r} r^{-2} \psi^2 \alpha^2 \right)^{1/2}. \end{aligned} \quad (118)$$

Here, the left most inequality results from an application of Holder's inequality. Next, plug (113) into this last inequality to discover a g -independent constant ξ such that

$$|\beta| \leq \xi r^{1/2} \quad (119)$$

at points x with $|x| = r$.

Step 7. This step studies the behavior of $\hat{\alpha}$. For this purpose, it proves useful to reintroduce the function $f(r)$ from (65) and (101). This done, the resulting analysis of $\hat{\alpha}$ is then summarized by the following assertion: There exists a g -independent constant ξ such that

$$|\hat{\alpha}|_{(r,\theta)} - r^{-1}f|_{r=16} \sin \theta| \leq \xi r^{-2}f|_{r=16} \sin \theta \quad (120)$$

where $r \geq 16$. With regards to this last equation, note that the maximum principle guarantees that is non-negative.

To prove the assertion, and for use subsequently, it proves useful to introduce the functions $\nu \equiv (r^2 \sin \theta)^{-1}\alpha$ and $\hat{\nu} \equiv (r^2 \sin \theta)^{-1}\hat{\alpha}$. Next, interpret ν and $\hat{\nu}$ as functions on \mathbb{R}^5 by writing r and θ in terms of standard Cartesian coordinates as $r = (x_1^2 + \dots + x_5^2)^{1/2}$ and $\theta = \text{Arccos}(x_5/r)$. This done, then the second line in (9) reads $-\Delta_5\nu - g^4\psi^2\nu$ and (114) becomes

- $-\Delta_5\hat{\nu} = 0$ where $r > 8$. (121)
- $\hat{\nu}|_{r=8} = \nu|_{r=8}$.
- $\hat{\nu} \rightarrow 0$ as $r \rightarrow \infty$.

Here, Δ_5 denotes the standard Laplacian on \mathbb{R}^5 . All this understood, it follows from (121) using standard Green's function techniques that

$$\hat{\nu} \leq \xi_1 r^{-3} \int_0^\pi \nu_{r=8} \sin^3 \theta d\theta = \xi_2 r^{-3} f_{r=8} \quad (122)$$

where $\xi_{1,2}$ are g -independent constants and where f is the function of r from (65).

Put this last bound in the bank temporarily and consider the expansion of $\hat{\nu}$ as a sum of spherical harmonics. That is, write $\hat{\nu} = \sum_\lambda \hat{\nu}^\lambda \kappa^\lambda$ where κ^λ is a function only of θ and solves the eigenvalue equation

$$-(\sin \theta)^{-3}((\sin \theta)^3 \kappa_\theta^\lambda)_\theta = \lambda \kappa^\lambda. \quad (123)$$

In this regard, the lowest eigenvalue is $\lambda = 0$ with the constants as eigenfunctions. The next lowest eigenvalue is $\lambda = 4$ with $\cos \theta$ as the eigenvalue. Meanwhile, \hat{v}^λ is a constant multiple of $r^{-(3+\sqrt{3+4\lambda})/2}$. This last point understood, then (120) follows directly.

Step 8. This step uses (119) and (120) to prove the first assertion of Proposition 9. For this purpose, note that these two inequalities directly imply the existence of g -independent constants $\hat{m} \geq 1$ and $r_* \geq 16$ such that

$$\alpha \geq \hat{m}^{-1}g^{1/2}r^{-1} \sin \theta - \hat{m}r^{1/2}. \tag{124}$$

where $r \geq r_*$. This understood, return now to the equation in the first point of (8) for ψ . Multiply both sides of this equation by ψ and then, at each fixed $r \geq r_*$, integrate the result over the constant r sphere to obtain the following equation for the function $q(r) \equiv \int_0^\pi \psi^2|_r \sin \theta d\theta$:

$$-r^{-2}(r^2q_r)_r + 2 \int_0^\pi (\psi_r^2 + r^{-2}(\psi_\theta^2 + \alpha^2\psi^2)) \sin \theta d\theta = 0. \tag{125}$$

This last equation implies the inequality

$$-r^{-2}(r_2q_r)_r + r^{-2}\lambda(r)q \leq 0, \tag{126}$$

where $\lambda(r)$ is the smallest eigenvalue for the operator

$$L|_r \equiv -(\sin \theta)^{-1}(\sin \theta(\cdot)_\theta)_\theta + \alpha^2|_r.$$

In this regard, use (124) with standard eigenvalue estimation techniques to find a g -independent constant $\xi \geq 1$ such that

$$\lambda(r) \geq \xi^{-1}g^{1/2}r^{-2} - \xi r. \tag{127}$$

Next, use (127) to find g_1 and a g -independent constant $m_* \geq 1$ such that $\lambda(r) \geq m_*^{-1}g^{1/2}$ when $g \geq g_1$ and $m_* \leq r \leq m_* + 1$.

Given this lower bound on λ where $m_* \leq r \leq m_* + 1$ and Lemma 12's bound for q at $r = m_*$, the differential inequality in (126) implies the existence of a g -independent constant $\mu \geq 1$ such that when $g \geq g_1$, then

$$q(r) \leq \mu g^{-4}e^{-g^{1/4}/\mu}. \tag{128}$$

where $r = m_* + 1$. This last bound and the maximum principle then bound $q(r)$ by μg^{-4} at all $r \in [m_* + 1, m_* + 3]$. Finally, with the latter bound in hand, the arguments from Step 4 provide a g -independent constant $\mu' \geq 1$ and the pointwise bound of ψ by $g^{-2}e^{-g^{1/4}/\mu'}r$ where $r \geq m_* + 2$. This pointwise bound directly gives the first assertion of Proposition 9.

Step 9. This last step uses the first assertion in Proposition 9 to obtain the second. To begin, reintroduce $\hat{\alpha}$ and β from Step 6. Then, the following assertion restates some facts from previous steps: There exists a g -independent constants $\mu \geq 1$ and μ' such that when $g \geq \mu$, then

- $\psi \leq \mu g^{-2}r^{-1}e^{-g^{1/4}/\mu}$ at all points with $r \geq \mu$. (129)
- $\hat{\alpha} \geq \mu^{-1}g^{1/2}r^{-1}\sin\theta$ at all points with $r \geq \mu$.
- $|\beta| \leq \mu'$ at all points with $\mu \leq r \leq \mu + 1$.

This last point understood, introduce $\nu \equiv (r^2 \sin \theta)^{-1}\alpha$ and decompose the latter where $r \geq \mu$ as the sum $\nu_0 + \nu_1$ where these new functions obey

- $-\Delta_S \nu_0 = 0$ with $\nu_0|_{r=\mu} = \nu$. (130)
- $-\Delta_S \nu_1 = g^4 \psi^2 \nu$ with $\nu_1|_{r=\mu} = 0$.

To proceed, observe from (69) that $r^p \alpha$ is bounded on \mathbb{R}^3 as long as $p \leq p_0$ with p_0 as described just prior to (69). Thus, said, then it follows from (128) that with $\varepsilon > 0$ specified, there exists g_ε such that when $g > g_\varepsilon$, then $r^{1-\varepsilon}\alpha$ is bounded. This said, then $r^{3-\varepsilon}\nu$ is bounded on \mathbb{R}^5 . Let $z(\nu) \equiv \sup_{r \geq \mu}(r^{3-\varepsilon}\nu)$ and define $z(\nu_1)$ analogously. Then, use the Green's function of $-\Delta_S$ to conclude from the second line of (130) and the top line in (129)

$$\begin{aligned} |\nu_1|(x) &\leq (2\pi^2)^{-1}g^4 \int_{r \geq \mu} |x - (\cdot)|^{-3} \psi^2 \nu d\text{vol}_5 & (131) \\ &\leq (2\pi^2)^{-1}\mu^2 e^{-2g^{1/4}/\mu} z(\nu) \int_{r \geq \mu} |x - (\cdot)|^{-3} r^{-5+\varepsilon} d\text{vol}_5. \end{aligned}$$

Here, $dvol_5$ denotes the Euclidean volume element on \mathbb{R}^5 . Now, the right most integral in (131) is no smaller than $(2\pi^2)\varepsilon^{-1}|x|^{-3+\varepsilon}$, and so

$$|x|^{3-\varepsilon}|\nu_1|(x) \leq \varepsilon^{-1}\mu^2e^{-2g^{1/4}/\mu}z(\nu). \tag{132}$$

Then, as $z(\nu) \leq z(\nu_0) + z(\nu_1)$, this last inequality implies that

$$\sup_{r \geq \mu} r^{3-\varepsilon}|\nu_1| = z(\nu_1) \leq 2\varepsilon^{-1}\mu^2e^{-2g^{1/4}/\mu}z(\nu_0) \tag{133}$$

as long as $\varepsilon \geq 2\mu^2e^{-2g^{1/4}/\mu}$.

This last point with (129) implies that there is a g -independent constant $c \geq 1$ such that when $g \geq c$ then

$$\alpha|_{r=2\mu} \geq c^{-1}g^{1/2} \sin \theta. \tag{134}$$

This understood, (9) plus the maximum principle implies that

$$\alpha \geq c^{-1}g^{1/2}r^{-1} \sin \theta \tag{135}$$

at all $r \geq 2\mu$.

j) The uniqueness of solutions

The purpose of this section is to consider the following question: Are there two distinct, non-negative functions which are both absolute minima of \mathcal{E}_0 in (10)? Here is the answer:

Proposition 13. *For any $g > 0$, the functional \mathcal{E}_0 has a unique, non-negative minimizer.*

The remainder of this section is occupied with the proof of this proposition.

Proof of Proposition 13. The first step in the proof is to elaborate on the conclusions of Lemma 7. This is provided by

Lemma 14. *If an absolute minimizer, α , to \mathcal{E}_0 is somewhere positive, then α is bounded from below where $r \geq 1$ by a constant, non-zero multiple of $r^{-2}(r - 1) \sin \theta$.*

Proof of Lemma 14. First, let $\nu \equiv (r^2 \sin \theta)^{-1} \alpha$ and consider ν and ψ as in Step 7 of the previous section to be functions on \mathbb{R}^5 via identifications $r = (x_1^2 + \dots + x_5^2)^{1/2}$ and $\theta = \text{Arccos}(x_5/r)$ on \mathbb{R}^5 . As previously noted, the function ν obeys

- $\nu = 0$ where $r \leq 1$. (136)
- $\Delta_5 \nu - g^4 \psi^2 \nu = 0$ where $r > 1$.
- ν is square integrable over \mathbb{R}^5 .

As before Δ_5 denotes the standard Laplacian on \mathbb{R}^5 . Now, when $y \in \mathbb{R}^5$ has $|y| > 1$, use $G(\cdot; y)$ denote the Green's function on the complement of the unit ball in \mathbb{R}^5 with Dirichlet boundary conditions on the surface of the ball and with pole at y . Thus,

$$G(x; y) = (2\pi^2)^{-1} (|x - y|^{-3} - ||y|x - y||^{-3}). \quad (137)$$

By inspection, $G(x; y) > 0$ where $|x| > 1$, and $\partial_r G(\cdot; y)|_x > 0$ where $|x| = 1$. This said, then ν in (136) can be written in terms of G as

$$\nu(x) = g^4 \int G(x; y) (\psi^2 \nu)|_y d^5 y. \quad (138)$$

The claims in the lemma follows directly from this representation of ν .

With Lemma 14 in hand, suppose now that α is an absolute minimizer of e_0 and is not identically zero. Let α' be any other function in the domain of \mathcal{E}_0 . By virtue of Lemma 14, the function α' can be written as $\alpha' = h\alpha$, where h is smooth where $r > 1$ and bounded on sets where r is bounded. As is demonstrated below, $\mathcal{E}_0(\alpha')$ can be written in terms of h as

$$\begin{aligned} \mathcal{E}_0(\alpha') &= \mathcal{E}_0(\alpha) + 2^{-1} g^{-2} \int (h_r^2 + r^{-2} h_\theta^2) r^{-2} \alpha^2 d^3 y \\ &\quad + 2^{-1} g^2 \int (h^2 - 1) r^{-2} \alpha^2 \psi (\psi - \psi') d^3 y, \end{aligned} \quad (139)$$

where ψ' is the solution to the α' version of (8). To see the significance of (139), first note that the primed and unprimed versions of (8) together imply that

$$-\Delta(\psi - \psi) + r^{-2} \alpha'^2 (\psi - \psi') + r^{-2} (1 - h^2) \alpha^2 \psi = 0. \quad (140)$$

Now, multiply both sides of this last equation by $(\psi - \psi')$ and then integrate over \mathbb{R}^3 . An integration by parts results in the equality

$$\int (|\nabla(\psi - \psi')|^2 + r^{-2}\alpha^2(\psi - \psi')^2)d^3y + \int r^2(1 - h^2)\alpha^2\psi(\psi - \psi')d^3y = 0. \quad (141)$$

This last equation precludes a negative value for the second term on the right hand side of (141); in fact, said term is positive unless $\psi = \psi'$ and thus $h = 1$. Hence, $\mathcal{E}_0(\alpha') > \mathcal{E}_0(\alpha)$ unless $\alpha' = \alpha$.

By way of tying loose ends, what follows next is the derivation of (139). To begin, multiply both sides of the equation in the middle point of (9) by $h^2\alpha$ and then move one factor of h through the various derivatives on α to write the leading order derivatives in terms of $\alpha' = h\alpha$. This done, integrate the result over \mathbb{R}^3 and then integrate once by parts to see that the difference between the α' and α versions of the first integral in (10) is equal to

$$2^{-1}g^{-2} \int (h_r^2 + r^{-2}h_\theta^2)r^{-2}\alpha^2 + 2^{-1}g^2 \int (h^2 - 1)r^2\alpha^2\psi^2. \quad (142)$$

Next, note that the difference between α' and α versions of the second integral in (10) is equal to

$$\int (\psi' - \psi)\rho. \quad (143)$$

The latter is equal to the integral that is obtained by multiplying both sides of the equation in the top point of (8) by $(\psi' - \psi)$ and then integrating over \mathbb{R}^3 . Two applications of integration by parts then equates (143) with

$$\int (\psi(-\Delta(\psi' - \psi) + r^{-2}\alpha^2(\psi - \psi'))))d^3y. \quad (144)$$

To utilize this last identity, note that (140) can be rewritten as

$$-\Delta(\psi - \psi') + r^{-2}\alpha^2(\psi - \psi') + r^{-2}(1 - h^2)\alpha^2\psi' = 0. \quad (145)$$

This last equation implies that the integral in (143) is equal to

$$\int r^{-2}(1 - h^2)\alpha^2\psi\psi'd^3y. \quad (146)$$

Thus, the contribution to $\mathcal{E}_0(\alpha') - \mathcal{E}_0(\alpha)$ from the second integral on the right side of (10) is equal to the expression in (146) times $2^{-1}g^2$. Adding the latter to (142) produces the desired right hand side of (139).

k) The proofs of Theorems 1 and 2

The purpose of this final section is to present the proofs of the two theorems in the introduction. As the proof of Theorem 2 requires little more than the collation of results from the preceding sections, it is given first.

Proof of Theorem 2. The fact that minimizer, α , of \mathcal{E}_0 is unique up to multiplication by ± 1 is proved as Proposition 13. Lemma 3 proves that the minimizer is $\alpha \equiv 0$ when $g \leq (6\pi)^{1/2}$ and not so otherwise. The bounds in (6) for $\mathcal{E}_0(\alpha)$ follow from Propositions 4 and 8 using (14). The bounds in (7) also follow from these propositions with the help of (14) and (17). The positivity of ψ asserted by the first point of Theorem 2 follows by applying the maximum principle to the first equation in (8). Lemma 14 asserts the conclusion of the second point of Theorem 2, and the third point of Theorem 2 is a restatement of the conclusions of Proposition 9. The fact that $r\psi$ has a limit as $r \rightarrow \infty$ equal to $\sqrt{2}g^{-2}e_0$ with $e_0 \leq 1$ is proved above as Proposition 6. The fact that $e_0 > 0$ is proved in Section g. The bound $e_0 e^{-g^{1/4}/c}$ follows from the assertion in Part a of the third point of Theorem 2 using (8) and the maximum principle. Meanwhile, the assertion from Part b that $\alpha \sim c_0 r^{-[(9-8e_0^2)^{1/2}-1]/2} \sin \theta$ with c_0 bounded is discussed in Section g.

To complete the proof of the final point of Theorem 2, here is the argument for the assertion that $r|m_\psi|$ is bounded: First, write ψ using (8) as

$$\psi(x) = (4\pi)^{-1}|x|^{-1} - (4\pi)^{-1} \int |x - (\cdot)|^{-1} \alpha^2 r^{-2} \psi. \quad (147)$$

As $\psi \leq r^{-1}$ and $|\alpha| \leq r^{-3/4}$ at large r , it follows that both $\alpha^2 r^{-2} \psi$ and $\alpha^2 r^{-1} \psi$ are integrable. This said, the bound on $|m_\psi|$ follows from (147) since $|x - y|^{-1} - |x|^{-1} = \mathcal{O}(|y|/|x|^2)$ when $|y| < |x|/8$.

Finally, here is the argument for the bounded behavior of $r|m_\alpha|$: Write α in terms of the function ν that appears in (138). It follows from (138) and the bound just proved for m_ψ that the function m_α obeys a fixed point equation of the form $m = T(m)$ with

$$T(m) = e^{-g^{1/4}/c} |x|^2 \int_{|y| > |x|/4} G(x, y) |y|^{-4 - [(9-8e_0^2)^{1/2}-1]/2} m(y) d^5 y + h(x). \quad (148)$$

Here, h is a function that satisfies $|h(x)| \leq c|x|^{-1}$. Now, $T(m)$ is uniformly contracting on the Banach space of bounded functions which vanish where $|x| \leq 1$, and so it has a unique fixed point on this space. This fixed point is m_α . This said, introduce the Banach space \mathcal{B} consisting of those functions m that vanish where $|x| \leq 1$ and are such that $|x||m|$ is bounded. Here, the norm of $m \in \mathcal{B}$ is the supremum on \mathbb{R}^3 of $|x||m|$. As T also maps \mathcal{B} to itself as a uniformly contracting operator, so it has a unique fixed point in \mathcal{B} . Since the functions in \mathcal{B} are bounded on \mathbb{R}^3 , the latter fixed point is the same as the former; thus $m_\alpha \in \mathcal{B}$.

Proof of Theorem 1. The assertion that the coulomb solution is the minimizer of the small g versions of \mathcal{E} is proved last. The coulomb solution is not the absolute minimum of any $g > (6\pi)^{1/2}$ version of \mathcal{E} since it is not the minimizer of the same g version of \mathcal{E}_0 . As for the other assertions, the upper bound in (6) follows from the analogous, \mathcal{E}_0 version. Meanwhile, the lower bound in (6) is a consequence of the first point and the lower bound in either the second or third points of (7). In this regard, the argument for the first point of (7) appears just after Theorem 2. Meanwhile, the upper bounds in the second and third points of (7) follow given the first point and the upper bound in (6). As the lower bounds in the second and third points of (7) are proved momentarily, consider in the mean time the arguments for the fourth point. In particular, the pointwise bound on $|\Psi_A|$ by $|\Psi_{A=0}|$ follows via the maximum principle since $u \equiv |\Psi_A|$ satisfies the differential inequality $-\Delta u \leq \rho_0$ where ρ_0 is zero where $r \geq 1$ and equals $3/4\pi$ where $r \leq 1$. The bound on $|\Psi_A|$ by $cg^{-1/2}r^{-1}$ where $r \geq 2$ is proved by essentially the same argument that proves the final point of Proposition 4.

Given the upper bounds in the second and third points of Theorem 2, the corresponding lower bounds in these points follow as corollaries to

Proposition 15. *There is a constant $c \geq 1$ with the following significance: Suppose that $A \in \mathcal{A}$, that $\gamma \geq 1$ and that the corresponding Ψ_A obeys $\int_{r \geq 1} |\nabla_A \Psi_A|^2 \leq \gamma^{-1}$. Then, $\int_{1 \leq r \leq c/\gamma} |B_A|^2 \geq c^{-1}\gamma^3$.*

Proof of Proposition 15. Before starting, note that the argument is very much the same as the one above that proved Proposition 8. In

any event, to start, let ψ now denote -2 trace $(\tau^1 \Psi_A)$. The argument given in Steps 1-2 of the proof of Proposition 8 find γ -independent constants c_1, c_2 and K , all greater than 1, and a unique $\kappa \geq 1/K$ such that the following is true: Let \mathcal{C} denote the space of Sobolev class L^2 functions on \mathbb{R}^3 that vanish where $r \geq 1 + c_2 \gamma^{-1}$. Then,

$$\begin{aligned} & 3(8\pi)^{-1} \int_{r \leq 1} \psi \\ &= \sup_{u \in \mathcal{C}} \left\{ 3(4\pi)^{-1} \int_{r \leq 1} u - 2^{-1} \left(\int |\nabla u|^2 + \kappa^2 \gamma^2 \int_{r \geq 1} r^{-2} u^2 \right) \right\}. \end{aligned} \quad (149)$$

holds when $\gamma \geq 64\pi c_1$. Moreover, the supremum on the right hand side of (149) is achieved by the function u_κ from (84) where p, p' and d are by the formulas in (83) after replacing g with γ . This understood, note that

$$\begin{aligned} & 3(8\pi)^{-1} \int_{r \leq 1} \psi \\ &= \sup_u \left\{ 3(4\pi)^{-1} \int_{r \leq 1} (-2 \text{ trace } (\hat{u} \tau^{-1})) - 2^{-1} \int |\nabla_A \hat{u}|^2 \right\}, \end{aligned} \quad (150)$$

where the supremum is taken over all $su(2)$ -valued functions \hat{u} with both $\nabla_A \hat{u}$ and $r^{-1} \hat{u}$ square integrable. This understood, remark next that $\hat{u} \equiv u_\kappa \tau^1$ is such a function and as

$$\nabla_A \hat{u} = \nabla u_\kappa \tau^1 + [A, \tau^1] u_\kappa, \quad (151)$$

so (150) implies that

$$\int |[A, \tau^1]|^2 u_\kappa^2 \geq \kappa^2 g^2 \int_{r \geq 1} r^{-2} u_\kappa^2. \quad (152)$$

As in the derivation of (93), this last inequality implies the existence of a γ -independent constant $C \geq 1$ such that

$$\int_U |A|^2 \geq C^{-2} \gamma, \quad (153)$$

where U is the portion of \mathbb{R}^3 where $1 \leq r \leq 1 + c_2 \gamma^{-1}$. Since $A|_{r=1} = 0$, this last equation implies that $\int_U |\partial_r A|^2 \geq C' \gamma^3$ where C' is a positive, γ -independent constant. To finish the argument, note that every

element in \mathcal{A} is gauge equivalent to some 1-form that annihilates the vector field ∂_r ; and for such A , the inequality $|B_A| \geq |\partial_r A|$ holds.

The proof of Theorem 1 now lacks only the justification of the assertion that the Coulomb solution is the minimizer when g is less than the given bound. To argue this point, agree, first of all, to consider only those $su(2)$ valued 1-forms $A \in \mathcal{A}$ that annihilate the vector field ∂_r . As just noted at the end of the proof of the preceding proposition, every element in \mathcal{A} is gauge equivalent to such a 1-form. (In fact, the gauge transformation is unique if required to be the identity on the radius 1 ball.)

Now, to start the argument, use (1) to conclude that

$$\int |\nabla_A \Psi_A|^2 = 3(4\pi)^{-1} \int_{r \leq 1} -2 \text{trace}(\tau^1 \Psi_A). \tag{154}$$

Meanwhile, (1) also implies that

$$\int \langle \nabla_A \Psi_{A=0}, \nabla_A \Psi_A \rangle = 3(4\pi)^{-1} \int_{r \leq 1} -2 \text{trace}(\tau^1 \Psi_{A=0}) = 2\mathcal{E}(0). \tag{155}$$

Here, \langle, \rangle is shorthand for the inner product on $su(2)$ -valued 1-forms. Now, as A has no dr component and $\nabla \Psi_{A=0}$ has only a dr component, the left hand side of (155) is equal to

$$\int \langle -\Delta_{A=0} \Psi_{A=0}, \Psi_A \rangle + \int \langle [A, \Psi_{A=0}], \nabla_A \Psi_A \rangle. \tag{156}$$

Furthermore, as $-\Delta_{A=0} \Psi_{A=0} = \rho$, this last equation and (154) imply that

$$2^{-1} \int |\nabla_A \Psi_A|^2 - \mathcal{E}(0) = -2^{-1} \int \langle [A, \Psi_{A=0}], \nabla_A \Psi_A \rangle. \tag{157}$$

To proceed, use the lack of dr component in A and the lack of other components in $\nabla \Psi_{A=0}$ to equate the right hand side of (157) with

$$-2^{-1} \int \langle [A, \Psi_{A=0}], \nabla_A (\Psi_A - \Psi_{A=0}) \rangle - 2^{-1} \int |[A, \Psi_{A=0}]|^2. \tag{158}$$

This done, the triangle inequality finds

$$\begin{aligned} & 2^{-1} \int |\nabla_A \Psi_A|^2 - \mathcal{E}(0) \\ & \geq -4^{-1} \int |\nabla_A (\Psi_A - \Psi_{A=0})|^2 - (3/4) \int |[A, \Psi_{A=0}]|^2. \end{aligned} \tag{159}$$

The first term on the right hand sider of (159) is awkward for the purposes at hand. To replace it, note that (1) and it's $A = 0$ version imply that

$$-\Delta_A(\Psi_A - \Psi_{A=0}) = \nabla_A \bullet [A, \Psi_{A=0}] \quad (160)$$

where \bullet signifies the contraction of 1-form indices. Here again, use has been made of the lack of a dr component of A and the lack of the other components of $\nabla\Psi_{A=0}$. Contract both sides of this last equation with $\Psi_A - \Psi_{A=0}$, integrate the result over \mathbb{R}^3 , integrate by parts and, finally, employ the triangle inequality to discover that

$$\int |\nabla_A(\Psi_A - \Psi_{A=0})|^2 \leq \int |[A, \Psi_{A=0}]|^2. \quad (161)$$

Thus, (159) implies that

$$2^{-1} \int |\nabla_A \Psi_A|^2 - \mathcal{E}(0) \geq - \int |[A, \Psi_0]|^2 \geq -(4\pi)^{-2} \int r^{-2} |A|^2. \quad (162)$$

Meanwhile, as the lack of dr component in A also implies that $|B_A| \geq |\partial_r A|$, so it follows, after appeal to the first line in (21), that

$$\mathcal{E}(A) - \mathcal{E}(0) \geq 8^{-1} g^{-2} \int r^{-2} |A|^2 - (4\pi)^{-2} g^2 \int r^{-2} |A|^2. \quad (163)$$

Thus, $\mathcal{E}(A) \geq \mathcal{E}(0)$ when $g^2 \leq \sqrt{2\pi}$.

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