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A rudimentary theory of topological 4D gravity

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Abstract

A theory of topological gravity is a homotopy-theoretic representation of the Segal-Tillmann topologification of a two-category with cobordisms as morphisms. This note describes some relatively accessible examples of such a thing, suggested by the wall-crossing formulas of Donaldson theory.

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1 Gravity categories

A **cobordism category** has manifolds as objects, and cobordisms as morphisms. Such categories were introduced by Milnor [22], but following Segal's definition of conformal field theory [29] and Atiyah's subsequent abstraction of the notion of topological quantum field theory [1] they have been studied very widely. Recently, Tillmann [31] has shown the utility in this context of certain closely related **two**-categories (which generalize the classical notion of category, by admitting morphism-objects which are themselves categories). The following definition is based on her ideas.

Definition A **gravity** two-category has

• (closed) **manifolds** as objects,

• **cobordisms** as morphisms, and

• **isomorphisms** of these cobordisms, equal to the identity on the boundary, as **two**-morphisms.

There are many possible variations on this theme, and I will not try for maximal generality. If the objects of the category have dimension d (so the cobordisms are $(d+1)$-dimensional) then I will say that the gravity category is $(d+1)$-dimensional. I will assume that manifolds are smooth, compact and oriented, but not necessarily connected, and (following Segal) I understand the empty set to be a manifold of any dimension.

1.1 If V and V' are d-manifolds, a morphism

$$W : V \to V'$$

is (the germ of) an orientation-preserving diffeomorphism

$$(V_{op} \cup V') \times [0,1) \cong \nu(\partial W)$$

of the manifold on the left with a collar neighborhood of the boundary of the $(d+1)$-manifold W; the subscript *op* signifies reversed orientation. The morphism category $Mor(V,V')$ has such cobordisms as its objects; it is a topological category, in which the space of morphisms between two cobordisms W and $\tilde{W}$ consists of orientation- and boundary-identification-preserving diffeomorphisms $W \cong \tilde{W}$. Gluing along the boundary defines a continuous composition functor

$$W, W' \mapsto W \circ W' : Mor(V,V') \times Mor(V',V'') \to Mor(V,V'') \ ,$$

while disjoint union of objects gives this two-category a monoidal structure, with the empty set as identity object.

By replacing $Mor(V, V')$ with its set $\pi_0 Mor(V, V')$ of equivalence classes of objects, we obtain the category employed by Atiyah to define a topological quantum field theory; in other words, we can pass from a gravity two-category, in which the morphism objects are enriched by a categorical structure, to a classical category, in which the morphism objects are simply sets. Tillmann's more perspicacious alternative is to interpret the topological category $Mor(V, V')$ as a simplicial topological space and to replace it with its geometric realization $\mathrm{Mor}(V, V')$. This construction preserves Cartesian products (as does π_0: indeed the set of equivalence classes of objects in Mor is the set of components of the space Mor), defining a **topological** gravity category (i.e., a category in which the morphism objects are topological spaces, and the composition maps are continuous). A topological quantum field theory in the sense of Atiyah [12 §1.7] is thus a (continous) monoidal functor from a topological gravity category to the (topological) category of modules over a **discrete** topological ring.

However, we can consider monoidal functors to more general categories: for example, the singular chains on the morphism spaces of a gravity category define a monoidal category enriched over chain complexes, whose representations are the (co)homological field theories of physics. In the language of homotopy theory, these are representations in a category of modules over some Eilenberg-MacLane ring-spectrum. In general, I will call any monoidal functor from a topological gravity category to the category of dualizeable objects over a ring-spectrum, a **theory of topological gravity**. One of the points of this paper is that there is a rich supply of such things.

1.2 This terminology needs some explanation. If W is a manifold with boundary, let $\mathrm{Diff}_+(W)$ be the topological group of orientation-preserving diffeomorphisms of W which restrict to the identity in some neighborhood of ∂W. The components of $\mathrm{Mor}(V, V')$ are indexed by equivalence classes of cobordisms $W : V \to V'$, and the components themselves are the classifying spaces $B\mathrm{Diff}_+(W)$. Gluing [20] defines a continuous homomorphism

$$\mathrm{Diff}_+(W) \times \mathrm{Diff}_+(W') \to \mathrm{Diff}_+(W \circ W') \;;$$

thus the (components of the) composition map in the topological gravity category are the maps these compositions induce on classifying spaces.

On the other hand, a fundamental tautology of Riemannian geometry asserts that an isometry of a complete connected Riemannian manifold which fixes a frame at some point is the identity: such a map preserves the geodesics out of the framed point, and any other point in the manifold can be reached by such a geodesic. It follows that group of diffeomorphisms framing some basepoint will act **freely** on the (contractible) space of Riemannian metrics on a compact connected manifold. The space $B\mathrm{Diff}_+(W)$ is the homotopy quotient of the space of metrics [10, 11] on W by the diffeomorphism group and we can think of morphisms in the $(d+1)$-dimensional gravity category as cobordisms between d-manifolds, together with a choice of equivalence class of Riemannian metric on the cobordism. Riemannian geometry thus provides the gravity category with a smooth structure.

A (projective) Hilbert-space representation of a topological gravity category, along the lines considered by Segal in his definition of a conformal field theory, is thus very close to a quantum theory of gravity. When $d = 1$ we can see this more explicitly: the Riemann moduli space is the quotient of the space of conformal structures on a closed connected surface by the group of its orientation-preserving diffeomorphisms, which acts with finite isotropy when the genus exceeds one. This defines a monoidal functor from the two-dimensional gravity category to Segal's, which (away from closed surfaces of low genus) is a rational homology isomorphism on morphism spaces. Consequently, any conformal field theory in Segal's sense defines a quantum theory of two-dimensional gravity.

1.3 Examples:

i) From this point of view, there is no *a priori* reason to limit ourselves to smooth manifolds. We could begin with a two-category of topological manifolds, and replace its morphism categories by their classifying spaces, as before: there are plenty of non-smoothable four-manifolds!

ii) In higher dimensions, the category of manifolds and equivalence classes of h-cobordisms is a groupoid, with the Whitehead group of an object as its automorphisms. In low dimensions these categories are still quite mysterious.

iii) We can consider classes of manifolds with extra structure: for example, by requiring that the Stiefel-Whitney class w_2 vanish, we can define a gravity category of four-dimensional Spin-manifolds. [The set of Spin-structures on such a manifold is a principal homogeneous space over its first mod two cohomology group, but is not naturally isomorphic to that group.]

iv) Similarly, the four-dimensional gravity category of $\text{Spin}^{\mathbb{C}}$-manifolds is defined by cobordisms endowed with a complex line bundle with Chern class lifting w_2.

Ex. iii) can be regarded as the subcategory of Ex. iv) defined by objects with trivial Chern class. It is natural to think of the morphism categories in Ex. iii) as graded by elements of the middle homology lattice; for example, algebraic surfaces lie on the quadric $c_1^2 = 2\chi + 3\sigma$. [Note that reversing orientation changes the signature, but not the Euler characteristic.]

When d is *odd*, the morphisms of a $d+1$-dimensional gravity category are naturally graded by Euler characteristic: the correction term in the formula

$$\chi(W \circ W') = \chi(W) + \chi(W') - \chi(W \cap W')$$

is zero. When d is one, the Euler characteristic counts the number of handles or loops in the usual quantum or genus expansion; it defines a zeroth Mumford class κ_0. If we exclude closed manifolds from our morphism spaces, and thus do not admit the empty set as a plausible object, this grading is bounded below. The signature defines a similar grading, when $d = 3$.

Many interesting decorations of gravity categories are possible: Lorentz cobordism [28, 33], defined by a nowhere-vanishing vector field oriented suitably at the boundary, is one example. Restricting the objects (e.g. to be unions of (standard, or homology) spheres, or contact manifolds [19]) is another alternative. Witten's original two-dimensional theory [34] admits singular (stable) algebraic curves as morphisms; this compactifies its morphism spaces, and Kontsevich has shown (as Witten conjectured) that the resulting theory has a well-behaved vacuum state.

2 Pretty good theories of topological gravity

A Riemannian metric g on an oriented closed connected two-manifold Σ defines a Hodge operator $*_g$ on its harmonic forms. This operator squares to -1 on one-forms, and so defines a complex structure on the de Rham cohomology $H^1_{dR}(\Sigma)$. The space of isomorphism classes of complex structures on a real Euclidean space of dimension $2g$ is the quotient $\text{SO}(2g)/\text{U}(g)$, so we get a map

$$\tau : B\text{Diff}_+(\Sigma) \to (\text{Metrics})/(\text{Diff}_+(\Sigma)) \to \text{SO}/\text{U}$$

in the large genus limit. This can be constructed more generally by working with differential forms which vanish on the boundary. Orthogonal sum of vector spaces makes an H-space of the target of τ, and it is not hard to see that if Σ and Σ' are surfaces with geodesic boundaries, then gluing them c times along some sets of compatible boundary components defines a homotopy-commutative diagram

$$\begin{array}{ccc} B\mathrm{Diff}_+(\Sigma) \times B\mathrm{Diff}_+(\Sigma') & \longrightarrow & B\mathrm{Diff}_+(\Sigma \circ \Sigma') \\ \downarrow{\scriptstyle \tau\times\tau} & & \downarrow{\scriptstyle \tau} \\ \mathrm{SO/U} \times \mathrm{SO/U} & \xrightarrow{\oplus} & \mathrm{SO}/U\ . \end{array}$$

[The intersection form on the middle homology of $\Sigma \circ \Sigma'$ is the direct sum of the intersection forms of Σ and Σ', together with a **split hyperbolic** intersection form of rank $c-1$, which has a canonical complex structure [32 IV §4].]

This is perhaps the simplest example of a theory of two-dimensional topological gravity: it is a monoidal homotopy-functor to a topological category with one object and the H-space SO/U of morphisms [25]. The functor is a version of the Jacobian, which refines the infinite symmetric product construction (which takes disjoint union to Cartesian product). The Siegel moduli space for abelian varieties has the rational cohomology of an integral symplectic group, and a version of Hirzebruch's proportionality principle implies that the stable rational cohomology of this moduli space agrees with the cohomology of SO/U.

2.1.1 In general, a topological quantum field theory **HF** (with values in some category of modules over a ringspectrum **k**) assigns to a suitable d-manifold V, a module-spectrum $\mathbf{HF}(V)$, such that

i) the construction is exponential, in the sense that

$$\mathbf{HF}(V \sqcup V') \cong \mathbf{HF}(V) \wedge \mathbf{HF}(V')\ ;$$

ii) there is a pairing

$$\text{Trace}\ : \mathbf{HF}(V_{op}) \wedge \mathbf{HF}(V) \to \mathbf{k}$$

which is nondegenerate, in the sense that the induced map from $\mathbf{HF}(V_{op})$ to the functional **k**-dual of $\mathbf{HF}(V)$ is an isomorphism;

iii) there is a natural transformation

$$\tau_W : B\mathrm{Diff}_+(W) \to \mathbf{HF}(\partial W)$$

subject to a **monoidal axiom**: if $\partial W = V_{op} \sqcup V'$, etc., then the diagram

$$\begin{array}{ccc} B\mathrm{Diff}_+(W) \times B\mathrm{Diff}_+(W') & \longrightarrow & B\mathrm{Diff}_+(W \circ W') \\ \downarrow{\scriptstyle \tau\times\tau} & & \downarrow{\scriptstyle \tau} \\ \mathbf{HF}(V_{op}) \wedge (\mathbf{HF}(V') \wedge \mathbf{HF}(V'_{op})) \wedge \mathbf{HF}(V'') & \longrightarrow & \mathbf{HF}(V_{op}) \wedge \mathbf{HF}(V'') \, . \end{array}$$

commutes up to homotopy.

The smash product of two such functors yields another.

2.1.2 Objects in the two-dimensional gravity category are just collections of circles, which can be indexed by nonnegative integers. In this case, a theory is defined by a dualizable **k**-module spectrum **M**, together with a system

$$\tau_q^p \in (M^{\wedge(p+q)})^*(B\mathrm{Diff}_+(\Sigma)) = [B\mathrm{Diff}_+(\Sigma), \mathbf{M} \wedge_{\mathbf{k}} \dots \wedge_{\mathbf{k}} \mathbf{M}]^*$$

of characteristic classes for bundles of connected surfaces Σ with p incoming and q outgoing boundary components, which behave compatibly under gluing. The example above is deceptive, for in that case **M** agrees with the group ring $\mathbf{k} = \mathbb{S}[\mathrm{SO/U}]$, so the multiple smash product simplifies. The topological category with one object, and Tillmann's group-completion

$$\coprod_{g \geq 0} B\mathrm{Diff}_+(\Sigma_g) \to \mathbb{Z} \times B\Gamma_\infty^+$$

as its space of morphisms, defines the universal example of a theory of this type; the cohomology homomorphism defined by the induced map

$$\mathbb{Z} \times B\Gamma_\infty^+ \to \mathrm{SO/U}$$

factors through the classical map which kills the Mumford classes in degree divisible by four. In more general cases related to quantum cohomology [20, 24], **M** will be a Frobenius object in the category of spectra, and the theory can be reformulated in terms of a family of natural transformations

$$\otimes^{p+q} H^*(\mathbf{M}) \to H^*(B\mathrm{Diff}_+(\Sigma)) \, .$$

2.2 The Hodge-theoretic construction described above has a close analogue for four-manifolds, which is also classical in a way: the wall-crossing formulas [17] of Donaldson theory are its descendants. As in dimension two, its construction is based on properties of the intersection form on middle cohomology:

If W is an compact connected oriented four-manifold with ∂W a union of homology spheres then the intersection form

$$x, y \mapsto \langle x, y \rangle = (x \cup y)[W, \partial W]$$

on the integral lattice $B = H^2(W, \partial W, \mathbb{Z})$ is unimodular. In dimension four, Wu's formula implies that

$$q(x) = \langle x, x \rangle \equiv \langle x, w_2 \rangle$$

modulo two, so the form q is even if the manifold admits a Spin-structure [16 §5.7.6]. On a $\mathrm{Spin}^{\mathbb{C}}$-manifold the intersection form is even or odd depending on the parity of the Chern class of its associated complex line bundle.

By a fundamental theorem of Freedman [13] any unimodular quadratic form can arise as the intersection form of a closed topological four-manifold; but by similarly fundamental results of Donaldson [6, 9] the intersection form of a closed **smooth** four-manifold is either indefinite, or diagonalizable over the integers. As in two dimensions, the action of a diffeomorphism on homology defines a monodromy representation

$$\mathrm{Diff}_+(W) \to \mathrm{Aut}_+(B, q) = \mathrm{SO}(B)$$

which factors through $\pi_0(\mathrm{Diff}_+(W))$; it is convenient to think of its kernel [18] as an analogue, for four-manifolds, of the Torelli group of surface theory.

2.3 Let $b = b_+ + b_-$ be the rank, and $\sigma = b_+ - b_-$ the signature, of the inner product space defined by q on $B \otimes \mathbb{R}$. For our purposes the **indefinite** lattices are the most interesting: these are classified by their rank, signature, and type (even if $q(x) \equiv 0$ mod two, otherwise odd). In the indefinite case, the manifold $\mathrm{Grass}^-(B)$ of maximal negative-definite subspaces of $B \otimes \mathbb{R}$ is a noncompact (contractible) symmetric space defined by a cell of dimension b_+b_- in the usual Grassmannian of b_--planes in b-space. The orthogonal group of the lattice acts on this cell with finite isotropy, so the canonical homotopy-to-geometric quotient map

$$B\mathrm{SO}(B) \to \mathrm{Grass}^-(B)/\mathrm{SO}(B)$$

is a rational homology isomorphism. If B and B' are indefinite lattices, then the construction which sends a pair of negative definite subspaces in the real span of each, to their orthogonal sum in the real span of the direct sum lattice, defines a map

$$\mathrm{Grass}^-(B) \times \mathrm{Grass}^-(B') \to \mathrm{Grass}^-(B \oplus B')$$

which is equivariant with respect to the Whitney sum homomorphism

$$\mathrm{SO}(B) \times \mathrm{SO}(B') \to \mathrm{SO}(B \oplus B')$$

The Grothendieck group of the category of even indefinite unimodular lattices is free abelian on two generators, corresponding to the hyperbolic plane and the E_8 lattice [30 V §2]. The 'Hasse-Minkowski' spectrum $\mathbf{K}_{\mathrm{EIU}}$ defined by the algebraic K-theory of the category of such lattices is the group completion of the monoid constructed from the disjoint union of the classifying spaces of their orthogonal groups; the tensor product of two such lattices defines another, making this a commutative ring-spectrum.

2.4 A Riemannian metric g on W defines a Hodge operator $*_g$ on harmonic forms, but now this operator squares to $+1$ on the middle cohomology. The function which assigns to g, the $*_g = -1$-eigenspace of harmonic two-forms vanishing on ∂W, maps the space of Riemannian metrics to the negative-definite Grassmannian $\mathrm{Grass}^-(B)$ equivariantly with respect to the action of $\mathrm{Diff}_+(W)$.

If W and W' are four-manifolds bounded (as above) by homology spheres, and if $W \circ W'$ results from gluing these manifolds along a collection of compatible boundary components, then the quadratic module of $W \circ W'$ is canonically isomorphic to $B \oplus B'$; hence the cohomology representation of the diffeomorphism group defines a monoidal functor from the gravity category of Spin four-manifolds bounded by standard spheres, to the topological category with one object, and the Hasse-Minkowski spectrum as morphisms. There is a similar functor defined on the category with homology spheres as objects, but the resulting lattice is no longer necessarily indefinite [9 §1.2.3].

The higher algebraic K-theory of such lattices has apparently not received much attention. It is remarkable that the relatively naive constructions sketched above already define pretty good theories of topological gravity. The η-invariant of Atiyah-Patodi-Singer [3] is much more sophisticated; to find an interpretation in these terms for it, analogous to the way Floer homology globalizes the Casson invariant, would be extremely interesting.

3 Toward a parametrized Donaldson theory

A good theory of gravity shouldn't exist in a vacuum: it deserves to be coupled to some nontrivial matter. Donaldson [8] and Moore and Witten [23] have suggested the study of equivariant supersymmetric Yang-Mills theory parameterized by classifying spaces of diffeomorphism groups. A fragment of such a theory is sketched below.

3.1 Suppose for simplicity that W is closed. The graded space $\mathrm{Bun}_*(W)$ of gauge equivalence classes of connections on SU(2)-bundles over W has components indexed by the second Chern class of the bundle. Let $\mathbf{D}_*$ be the subspace of Metrics $\times$ $\mathrm{Bun}_*(W)$ consisting of pairs (g, A), where A is a connection on an SU(2)-bundle over W with curvature two-form

$$*_g(F_A) = -F_A$$

antiselfdual with respect to the metric g. The standard transversality arguments of Donaldson theory [9 §4.3] imply that this space is a manifold, with fiber of dimension $8c_2 - \frac{3}{2}(\sigma + \chi)$ above the metric g; at least, provided this metric admits no **reducible** antiselfdual connections. These reducible connections define an interesting kind of distinguished boundary for the space of antiselfdual connections.

3.2 Reducible connections on W are parametrized by the wall arrangement

$$\mathrm{Wall}(B) = \{H \in \mathrm{Grass}^-(B) \mid H \cap B \neq \{0\}\ \}$$

of the lattice B: it is the set of maximal negative-definite subspaces of $B \otimes \mathbb{R}$ containing a lattice point. This is a union of smooth submanifolds of codimension b_-, filtered by the increasing family $\mathrm{Wall}_d(B)$ of subspaces consisting of maximal negative-definite H containing a lattice point x with $0 > q(x) \geq -d$ (which is a locally finite union of manifolds [14]). The orthogonal group of B acts naturally on these arrangements, as well as on the quotient spaces

$$\mathrm{Wall}_d^S(B) = \mathrm{Grass}^-(B)/\mathrm{Wall}_d(B)$$

(which are roughly the S-duals of the wall arrangements). If B and B' are two indefinite lattices, then the orthogonal direct sum map defines a commutative diagram

$$\begin{array}{ccc}
\mathrm{Grass}^-(B) \times \mathrm{Grass}^-(B') & \longrightarrow & \mathrm{Grass}^-(B \oplus B') \\
\downarrow & & \downarrow \\
\mathrm{Wall}_d^S(B) \wedge \mathrm{Wall}_{d'}^S(B') & \longrightarrow & \mathrm{Wall}_{d+d'}^S(B \oplus B')
\end{array}$$

which is equivariant, with respect to the Whitney sum on orthogonal groups. The equivariant cohomology $H^*_{\rm SO}(\mathrm{Wall}^S_*)$ defines yet another variant of a topological gravity theory, but there seems to be little known about such essentially arithmetic invariants.

3.3 If g is in the complement of the preimage $\mathrm{Metrics}^0_d$ of Wall_d in the space Metrics of metrics on W, then no SU(2)-bundle with Chern class less than $-d$ admits a connection with $*_g$-antiselfdual curvature. Thus if $\mathbf{D}^0_d$ denotes the space of pairs (g, A) such that A is gauge equivalent to a connection induced from a line bundle with curvature antiselfdual with respect to g, then

$$(\mathbf{D}_d, \mathbf{D}^0_d) \to (\mathrm{Metrics}, \mathrm{Metrics}^0_d) \times \mathrm{Bun}_d(W)$$

is a kind of $\mathrm{Diff}_+(W)$-equivariant cycle, of relative finite dimension above the space of metrics. It cannot be expected to be proper, but Donaldson theory has developed sophisticated methods to deal with such issues [7]: let $\mathrm{SP}^\infty_d(W_+)$ be the space of finitely supported functions f from W to the integers, such that

$$\sum_{x \in W} f(x) = d \,,$$

and let

$$\overline{\mathbf{D}}_d = \coprod_{0 \leq i \leq d} \mathbf{D}_i \times \mathrm{SP}^\infty_{d-i}(W_+)$$

be the analogue of the Uhlenbeck-Donaldson compactification of $\mathbf{D}_d$ in the stratified space

$$\mathrm{Metrics} \times (\coprod_{0 \leq i \leq d} \mathrm{Bun}_i(W) \times \mathrm{SP}^\infty_{d-i}(W_+)) = \mathrm{Metrics} \times \overline{\mathrm{Bun}}_d(W) \,.$$

Completing the subspace $\mathbf{D}^0_d$ of reducible connections analogously defines a candidate

$$(\overline{\mathbf{D}}_d, \overline{\mathbf{D}}^0_d) \to (\mathrm{Metrics}, \mathrm{Metrics}^0_d) \times \overline{\mathrm{Bun}}_d(W)$$

for a $\mathrm{Diff}_+(W)$-equivariant Donaldson cycle.

To extract homological information from this construction, note that a class z of dimension $*$ in the rational homology of $B\mathrm{Diff}_+(W)$ maps to a sum, with rational coefficients, of homology classes defined by maps

$$Z \to \mathrm{Metrics} \times_{\mathrm{Diff}_+} \mathrm{pt}$$

of **smooth** manifolds Z. The fiber product of such a map with the projection

$$\overline{\mathbf{D}}_d \to \mathrm{Metrics} \times_{\mathrm{Diff}_+} \overline{\mathrm{Bun}}_d(W) \to \mathrm{Metrics} \times_{\mathrm{Diff}_+} \mathrm{pt}$$

defines a class of dimension $* + 8d - \frac{3}{2}(\sigma + \chi)$ in the rational homology of

$$(\mathrm{Metrics}, \mathrm{Metrics}_d^0) \times_{\mathrm{Diff}_+} \overline{\mathrm{Bun}}_d(W) ;$$

note that this admits a canonical map to the space

$$\mathrm{Wall}_d^S \wedge_{\mathrm{SO}(B)} \mathrm{SP}_d^\infty(W_+) ,$$

which depends only on the lattice B.

3.4 The homotopy-to-geometric quotient map for the space of connections is a rational homology equivalence of $\mathrm{Bun}_*(W)$ with the space of based smooth maps from W_+ to $B\mathrm{SU}(2)$ [9 §5.1.15], and the Pontrjagin class defines a rational homology isomorphism of the space of maps with the Eilenberg-MacLane space $H(\mathbb{Z}, 4)$. By the Dold-Thom theorem,

$$\pi_i \mathrm{Maps}(W_+, H(\mathbb{Z}, 4)) \cong H^{4-i}(W, \mathbb{Z}) \cong H_i(W, \mathbb{Z}) \cong \pi_i(\mathrm{SP}^\infty(W_+))$$

so as far as rational (co)homology is concerned, we can replace the space $\overline{\mathrm{Bun}}_*(W)$ with the free topological abelian group on W. [This identification uses Poincaré duality, and hence requires a choice of orientation: the space of bundles is a contravariant functor, but the infinite symmetric product is covariant.] Combined with the constructions outlined above, this defines a generalized Donaldson invariant as a homomorphism

$$\mathcal{D}_d : H_*(B\mathrm{Diff}_+, \mathbb{Q})) \to H_{*+8d-\frac{3}{2}(\sigma+\chi)}(\mathrm{Wall}_d^S \wedge_{\mathrm{SO}} \mathrm{SP}_d^\infty, \mathbb{Q})$$

with values in a group which depends only on the cohomology lattice B; indeed the rational homology of $\mathrm{SP}^\infty(W_+)$ is the symmetric algebra on the homology of W, and the automorphic cohomology

$$H^*_{\mathrm{SO}(B)}(\mathrm{SP}^\infty(W_+), \mathbb{Q}) = H^*(\mathrm{SO}(B), \mathrm{Sym}(H^*(W)))$$

contains the classical ring of automorphic forms for the orthogonal group [5] as the invariant elements of the symmetric algebra on B.

This invariant generalizes the usual one, in the sense that $\mathcal{D}_d$ on a generator of the zero-dimensional homology of $B\mathrm{Diff}_+$ is the classical invariant. [The usual convention is to interpret the antiselfdual cycle as a function on the **co**homology of W, by taking its Kronecker product with $\exp(x), x \in H^*(W)$.] A four-manifold is said to be of **simple** type, if the behavior of its classical invariant as a function of charge is not too complicated: in the present formalism, the condition is that

$$\mathcal{D}_{d+1}(1) \mapsto w_0 w_4^2 \mathcal{D}_d(1)$$

under the homomorphism induced by the restriction map from Wall^S_{d+1} to Wall^S_d (where w_0 and w_4 generate the homology in degrees zero and four of W). This suggests

$$\tilde{\mathfrak{D}}_d = (w_0 w_4^2)^{-d} \mathfrak{D}_d \in \mathrm{Hom}^{-\frac{3}{2}(\sigma+\chi)}(H_*(B\mathrm{Diff}_+), H_*(\mathrm{Wall}^S_d \wedge_{\mathrm{SO}} \mathrm{SP}^\infty_0))$$

as the natural normalization for the generalized invariant.

4 On the inadequacy of the foregoing

The preceding sketch defines at best a **piece** of a topological gravity functor. It is defined only for manifolds without boundary, but it behaves correctly under disjoint union: if W_0 and W_1 are two closed four-manifolds, then

$$\sum_{d=d_0+d_1} \mathfrak{D}_{d_0}(W_0) \otimes \mathfrak{D}_{d_1}(W_1) \mapsto \mathfrak{D}_d(W_0 \cup W_1)$$

under the maps of §3.2; this is nothing but a definition of the generalized invariant for non-connected manifolds.

In fact there is reason to think that these constructions may have wider validity. Some years ago, Atiyah [2] proposed a unification of the invariants of Donaldson and Floer, based on a theory of semi-infinite cycles in the polarized manifold of connections on a three-manifold. A theory of such cycles which behaves naturally under variation of the metric on a bounding four-manifold would yield a topological gravity theory for four-manifolds, taking values in generalized automorphic forms with coefficients in Floer homology.

Many results which follow from Atiyah's program are known now to be true; but (mostly because of difficulty with compactifications), work on these questions has advanced without using his cycle calculus. I am told, however, that recently there has been progress along the lines he suggested [26], though in Seiberg-Witten rather than Floer-Donaldson theory. Meanwhile, Bauer [4] and Furuta [15] have studied generalized Seiberg-Witten invariants from a homotopy-theoretic point of view, and Bauer has shown that his invariant behaves nicely under connected sum. The hope that these new developments can be extended to the context proposed in this paper has encouraged me to write this incomplete and probably naive account.

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