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Massey and Fukaya products on elliptic curves

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Abstract

We compare some triple Massey products on elliptic curve with the corresponding Fukaya products on the symplectic torus and recover the classical identity due to Kronecker. We also express triple Fukaya products corresponding to generic configurations of 4 circles on the symplectic torus in terms of indefinite theta series. The A_∞-constraint for these products leads to a 5-term identity between these series.

This note is a complement to [10]. One of its goals is to show that some higher Massey products on an elliptic curve can be computed as higher compositions in Fukaya category of the dual symplectic torus in accordance with the homological mirror conjecture of M. Kontsevich [8]. Namely, we consider triple Massey products of very simple type which are uniquely defined, compute them in terms of theta-functions and compare the result with the series one obtains in Fukaya picture. The identity we get in this way was first discovered by Kronecker.

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An interesting phenomenon is that although Massey products on an elliptic curve are partially defined and multivaled, one always has the corresponding univalued Fukaya product. Thus, the equivalence with Fukaya category equips the derived category of coherent sheaves on an elliptic curve with some additional structure. In order to understand this structure we study the relation between the triple products m_3 and the triangulated structure. It turns out that using m_3 one can define homotopy operators on cohomological long exact sequences associated with a generic distinguished triangle. Furthermore, higher products m_k with $k \geq 4$ define higher homotopy operators on these exact sequences. It seems that most of the A_∞-structure can be recovered from these homotopy operators.

The rest of the paper is devoted to explicit computations of higher compositions m_3 in Fukaya category of a torus corresponding to four lines with rational slopes. It turns out that the answer is given in terms of theta series associated with not necessarily definite quadratic forms on rank-2 lattices. Such series were introduced by L. Göttsche and D. Zagier in [6]. The idea is that when the quadratic form on the lattice is indefinite one has to restrict the summation over the lattice to the cone where the form is positive (introducing signs for different connected components of the cone). In the case when one takes the maximal cone on which the quadratic form is positive such a series is especially nice: as shown in [6] it is a Jacoby form. In the case when the quadratic form factors over $\mathbb{Q}$ into product of two linear forms, indefinite theta series for arbitrary cones can be expressed via the function

$$\kappa(y,x;\tau) = \sum_{n\in\mathbb{Z}} \frac{\exp(2\pi i(\tau n^2/2 + nx))}{\exp(2\pi i n\tau) - \exp(2\pi i y)}$$

where τ is in the upper half-plane. The latter function was introduced (with slightly different notation) by M. P. Appell in his study of doubly-periodic functions of the third kind in [1]. We write explicitly the A_∞-constraints between m_2 and m_3 in Fukaya category of a torus as an identity between indefinite theta series. In particular, we recover some non-trivial identities involving κ and theta-functions.

Our computation of univalued triple Massey products on elliptic curve can be generalized to the case of higher genus curves. The answer is always given as certain ratio of theta-functions. We expect that these products can be compared with Fukaya compositions on the symplectic torus, which is mirror dual to the Jacobian of a curve. This generalization will be considered in a future paper.

Throughout this paper we use the notation $\mathrm{e}(z) = \exp(2\pi i z)$.

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1 Triple Massey products

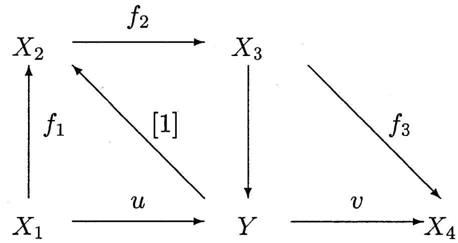
1.1 General construction

(cf. [3]) Let $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4$ be a sequence of morphisms in a triangulated category such that $f_2 \circ f_1 = 0$ and $f_3 \circ f_2 = 0$. Then one can construct the subset of elements $MP(f_1, f_2, f_3)$ in $\mathrm{Hom}^{-1}(X_1, X_4)$ which is a coset by the sum of the images of the maps

$$\mathrm{Hom}^{-1}(X_2, X_4) \to \mathrm{Hom}^{-1}(X_1, X_4) : g \mapsto g \circ f_1,$$

$$\mathrm{Hom}^{-1}(X_1, X_3) \to \mathrm{Hom}^{-1}(X_1, X_4) : h \mapsto f_3 \circ h.$$

Namely, $k \in MP(f_1, f_2, f_3)$ iff $k = v \circ u$ and the following diagram is commutative



(1.1.1)

where $X_2 \to X_3 \to Y \to X_2[1]$ is an exact triangle. In particular, if $\mathrm{Hom}^i(X_1, X_3) = \mathrm{Hom}^i(X_2, X_4) = 0$ for $i = -1, 0$ then the compositions $f_2 \circ f_1$ and $f_3 \circ f_2$ are always zero, hence, the Massey product $MP(f_1, f_2, f_3)$ contains one element, and we obtain the linear map

$$m_3 : \mathrm{Hom}(X_1, X_2) \otimes \mathrm{Hom}(X_2, X_3) \otimes \mathrm{Hom}(X_3, X_4) \to \mathrm{Hom}^{-1}(X_1, X_4).$$

Here is a more concrete description of this map. Consider the exact triangle

$$K \to \mathrm{Hom}(X_2, X_3) \otimes X_2 \to X_3 \to K[1]$$

Then our assumptions imply that the following natural maps are isomorphisms:

$$\alpha : \mathrm{Hom}(X_1, K) \to \mathrm{Hom}(X_1, X_2) \otimes \mathrm{Hom}(X_2, X_3),$$

$$\beta : \mathrm{Hom}^{-1}(K, X_4) \to \mathrm{Hom}(X_3, X_4).$$

Now m_3 is equal to the following composition

$$\mathrm{Hom}(X_1, X_2) \otimes \mathrm{Hom}(X_2, X_3) \otimes \mathrm{Hom}(X_3, X_4) \xrightarrow{\alpha^{-1} \otimes \beta^{-1}}$$
$$\mathrm{Hom}(X_1, K) \otimes \mathrm{Hom}^{-1}(K, X_4) \to \mathrm{Hom}^{-1}(X_1, X_4)$$

where the last arrow is the natural composition map.

1.2 Some Massey products on elliptic curve

Let us consider the simplest example of triple Massey product on elliptic curve E over a field k. Namely, we want to descibe the Massey product

$$\mathrm{Hom}(\mathcal{O},\mathcal{O}_{x_0})\otimes\mathrm{Hom}(\mathcal{O}_{x_0},\mathcal{L}[1])\otimes\mathrm{Hom}(\mathcal{L}[1],\mathcal{O}_x[1])\to\mathrm{Hom}(\mathcal{O},\mathcal{O}_x)$$

where $x\neq x_0$, $\mathcal{L}\neq\mathcal{O}$, $\deg\mathcal{L}=0$. If $\mathcal{L}|_{x_0}$ is trivialized then using the Serre duality the source of this arrow can be identified with $(\mathcal{L}|_x\otimes\omega)^*$ where ω is the stalk of the canonical bundle of E at zero, while the target is k. The dual map gives a canonical element $s(\mathcal{L},x)\in\mathcal{L}|_x\otimes\omega$. Note that $\mathcal{L}|_x$ is a stalk of the Poincaré line bundle $\mathcal{P}$ on $E\times\hat{E}$ where $\hat{E}$ is the dual elliptic curve. It is easy to see the above Massey product is the value at the point $(\mathcal{L},x)$ of the canonical rational section of $\mathcal{P}\otimes\omega$ (with poles at $x=x_0$ and $\mathcal{L}=\mathcal{O}$).

Now assume that $k=\mathbb{C}$ and the elliptic curve E is $\mathbb{C}/\Gamma$ where $\Gamma=\Gamma_\tau=\mathbb{Z}+\mathbb{Z}\tau$, τ is in the upper-half plane. We want to express the above Massey product in terms of theta-function. Let us denote by $\pi:\mathbb{C}\to E$ the canonical projection. Consider the line bundle L on E (equipped with a trivialization of its pull-back to $\mathbb{C}/\mathbb{Z}$) such that the classical theta-function

$$\theta(z,\tau)=\sum_n \mathrm{e}(\tau n^2/2+nz)$$

is a section of L (as a function of z), so $L\simeq\mathcal{O}(\xi)$ where $\xi=(\tau+1)/2$. Now set $\mathcal{L}=t_y^*L\otimes L^{-1}$ where $y\in\mathbb{C}$ is not a lattice point. We set $x_0=\pi(0)$ and fix a lifting of the second point x to $\mathbb{C}$ (abusing the notation we denote this lifting by $x\in\mathbb{C}$). Then the trivialization of π^*L induces trivializations of $\mathcal{L}_{x_0}$ and $\mathcal{L}_x$, so we have canonical generators $f_1\in\mathrm{Hom}(\mathcal{O},\mathcal{O}_{x_0})$, $f_2\in\mathrm{Hom}(\mathcal{L},\mathcal{O}_{x_0})^*$ and $f_3\in\mathrm{Hom}(\mathcal{L},\mathcal{O}_x)$. For a non-zero global holomorphic 1-form $\alpha\in H^0(E,\omega)$ we can consider the corresponding isomorphism of functors

$$S_\alpha:\mathrm{Hom}(A,B)^*\to\mathrm{Hom}^1(B,A) \tag{1.2.1}$$

derived from the Serre duality. Then we have an element $S_\alpha(f_2)\in\mathrm{Hom}(\mathcal{O}_{x_0},\mathcal{L}[1])$, and we can consider the triple Massey product $MP(f_1,S_\alpha(f_2),f_3)$.

Lemma 1.1. *Let $f\in\mathrm{Hom}(\mathcal{O},\mathcal{O}_\xi)$ be the canonical generator, let $\alpha=\theta'(\frac{\tau+1}{2})dz$. Then the element $S_\alpha(f)\in\mathrm{Ext}^1(\mathcal{O}_\xi,\mathcal{O})$ is represented by the extension*

$$0\to\mathcal{O}\xrightarrow{\theta}L\to\mathcal{O}_\xi\to 0. \tag{1.2.2}$$

Proof. We have the canonical extension

$$0\to\omega\to\omega(\xi)\xrightarrow{\mathrm{Res}}\mathcal{O}_\xi\to 0. \tag{1.2.3}$$

Via the isomorphism $\mathcal{O}\simeq\omega$ induced by α this extension represents $S_\alpha(f)$. Now we claim that the map

$$L\to\omega(\xi):s\mapsto\frac{s}{\theta}\cdot\alpha$$

extends to the isomorphism between (1.2.2) and (1.2.3). Indeed, this follows from the fact that $\mathrm{Res}_\xi(\frac{\alpha}{\theta})=1$. □

Proposition 1.2. *Let* $\alpha = \theta'(\frac{\tau+1}{2})dz$. *Then*

$$MP(f_1, S_\alpha(f_2), f_3) = \frac{\theta(x+y+\xi)}{\theta(x+\xi)\theta(y+\xi)} \cdot f_x \tag{1.2.4}$$

where $f_x \in \mathrm{Hom}(\mathcal{O}, \mathcal{O}_x)$ *is the canonical generator,* $\xi = \frac{\tau+1}{2}$.

Proof. It follows from the above Lemma that the element $S_\alpha(f_2)$ correspond to the extension

$$0 \to \mathcal{L} \xrightarrow{t_\xi^*\theta} t_{y+\xi}^* L \to \mathcal{O}_{x_0} \to 0.$$

The recipe for computation of Massey products is: first lift f_1 to the section s of $t_{y+\xi}^* L$; then find the morphism $g : t_{y+\xi}^* L \to \mathcal{O}_x$ such that the diagram

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{t_\xi^*\theta} & t_{y+\xi}^* L \\ & \searrow^{f_3} & \downarrow g \\ & & \mathcal{O}_x \end{array} \tag{1.2.5}$$

commutes; and finally apply g to s. We have $s = \frac{t_{y+\xi}^*\theta}{\theta(y+\xi)}$, g is $\frac{1}{\theta(x+\xi)}$ times the canonical generator of $\mathrm{Hom}(t_{y+\xi}^* L, \mathcal{O}_x)$ (induced by the trivialization of $\pi^* L$), so we get (1.2.4). □

2 Comparison with Fukaya composition

2.1 Triple Fukaya composition and triangulated structure

In [10] we have constructed an equivalence between the derived category of coherent sheaves on an elliptic curve E and the Fukaya category of the corresponding 2-dimensional torus with complexified symplectic form. An object of the latter category consists of the following data: a geodesic circle $\Lambda \in T = \mathbb{R}^2/\mathbb{Z}^2$, an angle $\phi \in \mathbb{R}$ such that $\mathbb{R}\mathrm{e}(\phi)$ is parallel to Λ, and a local system $\mathcal{L}$ on Λ. More precisely, one can consider formal direct sums of such objects. Note that the change of ϕ by $\phi + 1$ corresponds to the translation functor on the derived category. Morphisms from $(\Lambda, \phi, \mathcal{L})$ to $(\Lambda', \phi', \mathcal{L}')$ (where $\Lambda \neq \Lambda'$) can be non-zero only if $\phi < \phi' < \phi + 1$. In this case the Hom-space is

$$\sum_{x \in \Lambda \cap \Lambda'} \mathcal{L}_x^* \otimes \mathcal{L}_x.$$

The composition is defined using holomorphic triangles bounding three given geodesics circles (see [10]). There are also higher compositions m_k, $k \geq 3$ which are defined using holomorphic $(k+1)$-gons (see [5]). They satisfy A_∞-axioms (with $m_1 = 0$) of which the following is an example:

$$m_3(m_2(a_1,a_2),a_3,a_4) \pm m_3(a_1,m_2(a_2,a_3),a_4) \pm m_3(a_1,a_2,m_2(a_3,a_4)) \\ \pm m_2(m_3(a_1,a_2,a_3),a_4) \pm m_2(a_1,m_3(a_2,a_3,a_4)) = 0 \quad (2.1.1)$$

where $a_1,\ldots,a_4$ is the sequence of composable morphisms, the signs depend on degrees of a_i's. Note that the proof of A_∞-identities for the Fukaya compositions in our case is easy. Basically they follow from additivity of the area of plane figures. For example, the usual associativity of m_2 (which is equivalent to the addition formula for theta function) can be proved by considering two ways of cutting a non-convex quadrangle into two triangles and converting this into the identity for the corresponding generating series.

The functor from the Fukaya category to the derived category of coherent sheaves on elliptic curve is set up in such a way that one gets objects of abelian category (i.e. complexes concentrated in degree zero) for $\phi \in (-1/2, 1/2]$. The functor is constructed on objects from this subcategory using theta-functions. To obtain the entire derived category $\mathcal{D}^b(E)$ one has to deal with Ext^1. We fix a generator $\alpha \in H^0(E,\omega)$ and use the corresponding duality isomorphism (1.2.1) (on Fukaya side the corresponding isomorphism is obvious). Then we get a functor Φ_α from the Fukaya category to $\mathcal{D}^b(E)$.

In [5] Fukaya constructs the functor on Ext^1 in a different way. Namely, he starts with the same functor Φ on abelian category and then proceeds as follows. Let $a : X \to Y$ be a morphism of degree 1 in Fukaya category. To construct the corresponding morphism $\Phi(a) : \Phi(X) \to \Phi(Y)[1]$ one has to choose a resolution

$$0 \to \Phi(Y) \stackrel{\Phi(d_0)}{\to} \Phi(Z_0) \stackrel{\Phi(d_1)}{\to} \Phi(Z_1) \stackrel{\Phi(d_2)}{\to} \Phi(Z_2)$$

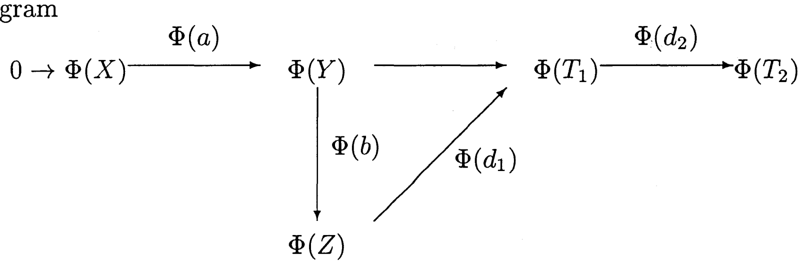
such that $\mathrm{Hom}^1(X, Z_0) = 0$. Now consider the triple composition $m_3(a, d_0, d_1) \in \mathrm{Hom}^0(X, Z_1)$. Then A_∞-axioms imply that $m_2(m_3(a, d_0, d_1), d_2) = 0$, hence, $-\Phi(m_3(a, d_0, d_1))$ factors through $\ker \Phi(d_2) = \mathrm{im}\, \Phi(d_1)$, so it defines an element in $\mathrm{Hom}^1(\Phi(X), \Phi(Y))$ which we take to be $\Phi(a)$ (we have changed the sign compare to Fukaya's definition in order for Proposition 2.1 below to be true). In fact, in the above construction the condition $\mathrm{Hom}^1(X, Z_1) = 0$ can be relaxed to the requirement that $m_2(a, d_0) = 0$. The independence on a choice of a resolution is proven by providing an alternative definition via a resolution for $\Phi(X)$. Let us denote by Φ the obtained functor from the Fukaya category to $\mathcal{D}^b(E)$. Fukaya showed in [5] that Φ is an equivalence so it should coincide with Φ_α for certain 1-form α. To find α we will use the following characterizing property of the functor F.

Proposition 2.1. *Let $a \in \mathrm{Hom}(X,Y)$, $b \in \mathrm{Hom}(Y,Z)$, $c \in \mathrm{Hom}(Z, X[1])$ be morphisms in Fukaya category such that $\Phi(a)$, $\Phi(b)$ and $\Phi(c)$ form an exact triangle. Assume in addition that $\mathrm{Hom}(Y,X) = 0$. Then $m_3(a,b,c) = \mathrm{id}_X$.*

Proof. Let us use the above definition to compute $\Phi(c)$. Namely, let us choose a resolution for $\Phi(Z)$ of the form

$$0 \to \Phi(Z) \stackrel{\Phi(d_1)}{\to} \Phi(T_1) \stackrel{\Phi(d_2)}{\to} \Phi(T_2).$$

Then we have the induced resolution for $\Phi(X)$ which fits into the commutative diagram



(2.1.2)

Then by definition $\Phi(c)$ is induced by the morphism $-\Phi(m_3(c, a, m_2(b, d_1)))$. Now the A_∞-constraint implies that

$$m_3(c, a, m_2(b, d_1)) = m_2(m_3(c, a, b), d_1).$$

It follows that $\Phi(c)$ is the composition of $-\Phi(m_3(c, a, b))$ with $\Phi(c)$, i.e. $m_2(m_3(c, a, b), c) = -c$. Now again by A_∞-axiom we deduce that

$$m_2(c, m_3(a, b, c)) = -m_2(m_3(c, a, b), c) = c.$$

Since $\mathrm{Hom}(Y, X) = 0$ this implies that $m_3(a, b, c) = \mathrm{id}_X$. □

The following corollary can be found in Fukaya's paper [5].

Corollary 2.2. *Let $a \in \mathrm{Hom}(X, Y)$, $b \in \mathrm{Hom}(Y, Z)$, $c \in \mathrm{Hom}(Z, X)$ be a triple of morphisms in Fukaya category such that $b \circ a = 0$ and $c \circ b = 0$. Then $\pm\Phi(m_3(a, b, c)) \in MP(\Phi(a), \Phi(b), \Phi(c))$.*

Corollary 2.3. *Let*

$$X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} X[1]$$

be a distinguished triangle in $\mathcal{D}^b(E)$ identified with the Fukaya category using the functor Φ. Assume that

$$\mathrm{Hom}^i(X, Y) = \mathrm{Hom}^i(Y, Z) = \mathrm{Hom}^{i+1}(Z, X) = 0$$

for $i \neq 0$. Then for every object $T \in \mathcal{D}^b(E)$ there is a canonical homotopy operator H_1 on the corresponding long exact sequence of morphisms from T to the above triangle, i.e. $H_1\partial + \partial H_1 = \mathrm{id}$ where ∂ is the differential in the latter exact sequence. Namely, for $f \in \mathrm{Hom}(T, X)$ one should take $H_1(f) = \pm m_3(f, a, b) \in \mathrm{Hom}(T, Z[-1])$, etc. Furthermore, one can define higher homotopy operators by setting $H_2(f) = \pm m_4(f, a, b, c)$, $H_3(f) = \pm m_5(f, a, b, c, a)$, etc., such that $H_1^2 = H_2\partial + \partial H_2$, $H_1H_2 - H_2H_1 = H_3\partial + \partial H_3$, etc. Similarly, there are canonical homotopy operators $H_i^!$ for the long exact sequence of morphisms from the triangle to T.

The proofs of both corollaries are easy exercises in applying A_∞-axioms which we leave to the reader. In particular, for the second corollary one has to use that fact that higher products containing an identity morphism vanish.

Proposition 2.4. *One has* $\Phi = \Phi_{2\pi i dz}$.

Proof. Let us consider the following three of objects in Fukaya category: $\Lambda_1 = (t, 0)$ with trivial local system, $\Lambda_2 = (t, t)$ with trivial local system, $\Lambda_3 = (1/2, t)$ with the connection $\pi i dt$. Then we have canonical morphisms e_i from Λ_i to Λ_{i+1} such that $\deg(e_1) = \deg(e_2) = 0$ and $\deg(e_3) = 1$. It follows easily from Lemma 1.1 that the morphisms $\Phi(e_1)$, $\Phi(e_2)$ and $\Phi_{\theta'(\xi)dz}(e_3)$ form an exact triangle

$$\mathcal{O} \to L \to \mathcal{O}_\xi \to \mathcal{O}[1]$$

where $\xi = \frac{\tau+1}{2}$. On the other hand, it is easy to compute that

$$m_3(e_1, e_2, e_3) = \frac{\theta'(\xi)}{2\pi i}.$$

It follows that $m_3(e_1, e_2, \frac{2\pi i}{\theta'(\xi)} e_3) = 1$ while $\Phi(e_1)$, $\Phi(e_2)$ and $\Phi_{2\pi i dz}(\frac{2\pi i}{\theta'(\xi)} e_3)$ form an exact triangle, hence $\Phi_{2\pi i dz} = \Phi$. □

2.2 Fukaya series

Let us consider the triple Fukaya composition that corresponds to the Massey product considered in 1.2. The corresponding four objects of the Fukaya category of a torus are: $\Lambda_1 = (t, 0)$ with trivial connection, $\Lambda_2 = (0, t)$ with trivial connection, $\Lambda_3 = (t, -\alpha_1)$ with the connection $(-2\pi i \beta_1)dx$, $\Lambda_4 = (-\alpha_2, t)$ with the connection $2\pi i \beta_2 dy$. Here $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real numbers. There is an essentially unique choice of logarithms of slopes for Λ_i such that $\operatorname{Hom}^0(\Lambda_i, \Lambda_{i+1}) \neq 0$ and with such a choice one has $\operatorname{Hom}^{-1}(\Lambda_1, \Lambda_4) \neq 0$. Moreover, all these spaces are one-dimensional so the corresponding Fukaya composition m_3 is just a number. This composition is defined only if α_1 and α_2 are not integers. Then it is given by the following series

$$\sum_{(\alpha_1+m)(\alpha_2+n)>0} \operatorname{sign}(\alpha_1 + m)\, \mathrm{e}(\tau(\alpha_1 + m)(\alpha_2 + n) + (\alpha_1 + m)\beta_2 + (\alpha_2 + n)\beta_1)$$

where we denote $\operatorname{sign}(t) = 1$ for $t > 0$, $\operatorname{sign}(t) = -1$ for $t < 0$, the sum is over integers m and n subject to the condition that $(\alpha_1 + m)$ has the same sign as $(\alpha_2 + n)$, The restriction on m and n is imposed by the condition for maps from the disk to be holomorphic in the definition of Fukaya composition while the sign comes from the canonical orientaion of the corresponding moduli space. We can write the above expression in the form

$$\mathrm{e}(\tau\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1) f(\alpha_1\tau + \beta_1, \alpha_2\tau + \beta_2; \tau)$$

where

$$f(z_1, z_2; \tau) = \sum_{(\alpha(z_1)+m)(\alpha(z_2)+n)>0} \operatorname{sign}(\alpha(z_1) + m)\, \mathrm{e}(\tau mn + nz_1 + mz_2)$$

is a holomorphic function of z_1 and z_2 defined for $\operatorname{Im} z_i \notin \mathbb{Z}(\operatorname{Im}\tau)$, where $\alpha(z) = \operatorname{Im}(z)/\operatorname{Im}(\tau)$.

2.3 The function $f(z_1, z_2, \tau)$

The function $f(z_1, z_2; \tau)$ is well-known (cf. [9], [11],[12]). It extends to a meromorphic function in z_1, z_2 with poles at the lattice points $z_1 \in \Gamma_\tau$ or $z_2 \in \Gamma_\tau$ where $\Gamma_\tau = \mathbb{Z} + \mathbb{Z}\tau$, and satisfies the following identities:

$$f(z_2, z_1; \tau) = f(z_1, z_2; \tau), \tag{2.3.1}$$

$$f(z_1 + m + n\tau, z_2; \tau) = \mathrm{e}(-nz_2) f(z_1, z_2; \tau), \tag{2.3.2}$$

$$f(z_1, z_2 + m + n\tau; \tau) = \mathrm{e}(-nz_1) f(z_1, z_2; \tau). \tag{2.3.3}$$

Using this quasi-periodicity properties of f it is easy to derive the following identity (2.3.4) which was first discovered by Kronecker [9] (see also [11], ch. VIII):

$$f(z_1, z_2; \tau) = \frac{\theta'((\tau+1)/2, \tau)}{2\pi i} \cdot \frac{\theta(z_1 + z_2 - (\tau+1)/2, \tau)}{\theta(z_1 - (\tau+1)/2, \tau)\theta(z_2 - (\tau+1)/2, \tau)} \tag{2.3.4}$$

Note that $\frac{\theta'((\tau+1)/2,\tau)}{2\pi i} = \prod_{n \geq 1}(1 - q^n)^3$ where $q = \mathrm{e}(\tau)$.

It is clear from the definition that

$$f(z_1, z_2; \tau + 1) = f(z_1, z_2; \tau).$$

Now using the identity (2.3.4) one can easily deduce from the functional equation for theta function that $f(z_1, z_2; \tau)$ satisfies the functional equation of the form

$$f(z_1/\tau, z_2/\tau; -\tau^{-1}) = \zeta \cdot \tau \cdot \mathrm{e}(z_1 z_2/\tau) f(z_1, z_2; \tau)$$

where ζ is a root of unity. Using the property

$$f(-z_1, -z_2; \tau) = -f(z_1, z_2; \tau)$$

we immediately conclude that $\zeta^2 = 1$, so $\zeta = \pm 1$. Finally substituting $z_1 = (\tau+1)/2$, $z_2 = \tau/2$ and looking at the sign of both sides when $\tau = it$, $t \in \mathbb{R}$ and $t \to +\infty$ we find that $\zeta = 1$. So the functional equation for f becomes

$$f(z_1/\tau, z_2/\tau; -\tau^{-1}) = \tau \cdot \mathrm{e}(z_1 z_2/\tau) f(z_1, z_2; \tau).$$

In fact, $f(z_1, z_2, \tau)$ is a meromorphic Jacobi form of weight 1 for the lattice $\mathbb{Z}^2$ with the quadratic form $Q(m, n) = mn$ (cf. [6]).

This equation can also be derived from the representation of f in the following form (cf. [11], ch.VIII):

$$f(z_1, z_2; \tau) = -\frac{\mathrm{e}(-\alpha(z_2)z_1)}{2\pi i} \sum\nolimits_e \frac{\chi(w)}{z_1 + w}$$

provided that $0 < \alpha(z_i) < 1$ for $i = 1, 2$. Here $\sum_e$ denotes the Eisenstein summation over the lattice Γ_τ, χ is the character of Γ_τ such that $\chi(1) = \mathrm{e}(-\alpha(z_2))$, $\chi(\tau) = \mathrm{e}(z_2 - \alpha(z_2)\tau)$.

2.4 Comparison

Now we want to interpret the identity (2.3.4) as an equality of the triple Fukaya product with the corresponding Massey product asserted in Corollary 2.2. Using the explicit construction of the functor Φ (cf. [10]) we compute that $\Phi(\Lambda_1) = \mathcal{O}$, $\Phi(\Lambda_2) = \mathcal{O}_{x_0}$, $\Phi(\Lambda_3) = t_y^*L \otimes L^{-1}$ where $y = \alpha_1\tau + \beta_1$, $\Phi(\Lambda_4) = \mathcal{O}_x$ where $x = \alpha_2\tau + \beta_2$. Let $e_i \in \mathrm{Hom}^*(\Lambda_i, \Lambda_{i+1})$, $i = 1, 2, 3$, be canonical generators, where $\deg(e_1) = \deg(e_3) = 0$, $\deg(e_2) = 1$. Then using the notation of 1.2 we have $\Phi(e_1) = f_1$, $\Phi_\alpha(e_2) = S_\alpha(f_2)$, and $\Phi(e_3) = \mathrm{e}(\tau\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1)f_3$. Also for a canonical generator $e \in \mathrm{Hom}^0(\Lambda_1, \Lambda_4)$ we have $\Phi(e) = f_x$. Now for $\alpha = \theta'(\xi)$, where $\xi = (\tau + 1)/2$ we derive from (1.2.4) that

$$MP(\Phi(e_1), \Phi_{\theta'(\xi)dz}(e_2), \Phi(e_3)) = \mathrm{e}(\tau\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1) \cdot \frac{\theta(x+y+\xi)}{\theta(x+\xi)\theta(y+\xi)} \cdot f_x$$

Thus using the above computation of $m_3(e_1, e_2, e_3)$ and Proposition 2.4 we get

$$\mathrm{e}(\tau\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1)f(x,y) \cdot f_x = \Phi(m_3(e_1, e_2, e_3)) =$$
$$MP(\Phi(e_1), \Phi_{2\pi i dz}(e_2), \Phi(e_3)) = \frac{\theta'(\xi)}{2\pi i} MP(\Phi(e_1), \Phi_{\theta'(\xi)dz}(e_2), \Phi(e_3)).$$

Using the above expression for the Massey product in the RHS we obtain the identity (2.3.4).

Remark. It is not hard to see that Fukaya triple products involving four lines forming any parallelogram with sides of rational slopes are expressed via the function f and the equality with the corresponding (univalued) Massey products follows from the identity (2.3.4).

3 Higher compositions in Fukaya category

3.1 Trapezoid compositions

Now we are going to consider some compositions m_3 in Fukaya category of a torus such that the corresponding triple Massey products on elliptic curve are not well-defined. Namely, consider four lagrangians: $\Lambda_1 = (t, -t)$ and $\Lambda_4 = (t, 0)$ with trivial connections, $\Lambda_2 = (t, -\alpha_2)$ with the connection $-2\pi i\beta_2 dx$, and $\Lambda_3 = (-\alpha_1, t)$ with the connection $2\pi i\beta_1 dy$, where α_i, β_i are real numbers, $\alpha_i \notin \mathbb{Z}$. There is a natural choice of logarithms of slopes so that $\mathrm{Hom}^0(\Lambda_i, \Lambda_{i+1}) \simeq \mathbb{C}$ and $\mathrm{Hom}^{-1}(\Lambda_1, \Lambda_4) \simeq \mathbb{C}$. Note that $\mathrm{Hom}^0(\Lambda_1, \Lambda_3) \neq 0$, so the corresponding triple Massey product on elliptic curve is not defined. The Fukaya composition

$$m_3 : \mathrm{Hom}(\Lambda_1, \Lambda_2) \otimes \mathrm{Hom}(\Lambda_2, \Lambda_3) \otimes \mathrm{Hom}(\Lambda_3, \Lambda_4) \to \mathrm{Hom}^{-1}(\Lambda_1, \Lambda_4)$$

is just the number given by the series

$$\sum_{(n+\alpha_1)(m+\alpha_2)>0} \mathrm{sign}(m+\alpha_2)\, \mathrm{e}((n+\alpha_1+\frac{m+\alpha_2}{2})(m+\alpha_2)\tau + (m+\alpha_2)\beta_1 + (m+\alpha_2+n+\alpha_1)\beta_2).$$

Let us define

$$g(z_1, z_2; \tau) = \sum_{(n+\alpha(z_1))(m+\alpha(z_2))>0} \operatorname{sign}(m+\alpha(z_2)) \operatorname{e}((n+m/2)m\tau + mz_1 + (m+n)z_2)$$

where as before $\alpha(z) = \operatorname{Im}(z)/\operatorname{Im}(\tau)$, $\alpha(z_1) \notin \mathbb{Z}$, $\alpha(z_2) \notin \mathbb{Z}$. Then the above composition is equal to

$$\operatorname{e}((\alpha_1 + \alpha_2/2)\alpha_2\tau + \alpha_2\beta_1 + (\alpha_1 + \alpha_2)\beta_2) g(\alpha_1\tau + \beta_1, \alpha_2\tau + \beta_2; \tau).$$

3.2 Properties of $g(z_1, z_2; \tau)$

The function g is holomorphic for $\alpha(z_i) \notin \mathbb{Z}$, $i = 1, 2$, and satisfies the following quasi-periodicity identities

$$g(z_1 + m + n\tau, z_2; \tau) = \operatorname{e}(-nz_2) g(z_1, z_2; \tau), \quad (3.2.1)$$

$$g(z_1, z_2 + m + n\tau; \tau) = \operatorname{e}(-n^2\tau/2 - n(z_1 + z_2)) g(z_1, z_2; \tau). \quad (3.2.2)$$

In other words, g can be considered as a holomorphic section of a line bundle on E^2 over the open subset $(E \setminus S)^2$ where $S = \mathbb{R}/\mathbb{Z} \subset \mathbb{C}/\Gamma_\tau$. However, it doesn't extend to a meromorphic section on E^2. On the other hand, let us denote by g_0 the restriction of g to the region $0 < \alpha(z_1) < 1$, $\alpha(z_2) \notin \mathbb{Z}$. Then we claim that g_0 extends to a meromorphic function on $\mathbb{C}^2$ with poles of order 1 at $z_2 \in \Gamma_\tau$. Indeed, if we sum first over n in the series defining g we get

$$g_0(z_1, z_2; \tau) = \sum_{m \in \mathbb{Z}} \frac{\operatorname{e}(m^2\tau/2 + m(z_1 + z_2))}{1 - \operatorname{e}(m\tau + z_2)}.$$

The latter series clearly satisfies the properties we claimed. However, g_0 lacks the quasi-periodicity of g. More precisely, it is easy to see that g and g_0 are related as follows:

$$g_0(z_1, z_2; \tau) - g(z_1, z_2; \tau) = p(z_1, \tau)\theta(z_1 + z_2; \tau)$$

where $p(z, \tau)$ is the following piecewise polynomial function of $\operatorname{e}(z_1)$:

$$p(z, \tau) = \begin{cases} -\sum_{0 < n \le \alpha(z)} \operatorname{e}(-n^2\tau/2 + nz_1), & \alpha(z) \ge 0, \\ \sum_{\alpha(z) < n \le 0} \operatorname{e}(-n^2\tau/2 + nz_1), & \alpha(z) < 0 \end{cases}$$

Let us consider the following series

$$\kappa(y, x; \tau) = \sum_{n \in \mathbb{Z}} \frac{\operatorname{e}(n^2/2\tau + nx)}{\operatorname{e}(n\tau) - \operatorname{e}(y)}. \quad (3.2.3)$$

This function is holomorphic for $y \notin \Gamma_\tau$ and satisfies the difference equation

$$\kappa(y, x + m + \tau; \tau) = \operatorname{e}(y)\kappa(y, x; \tau) + \theta(x, \tau)$$

where $m \in \mathbb{Z}$. Then we have

$$g_0(z_1, z_2; \tau) = -\operatorname{e}(-z_2)\kappa(-z_2, z_1 + z_2; \tau) = \kappa(z_2, \tau - z_1 - z_2; \tau). \quad (3.2.4)$$

Remark. The function κ and its derivatives were used by M. P. Appell to represent an arbitrary doubly-periodic function of the third kind as a sum of simple elements (cf. [1],[7]). On the other hand, κ can be expressed via the bilateral basic hypergeometric series. Namely, using notation of [2] we have

$$\kappa(y,x;\tau) = (1-\mathrm{e}(-y))^{-1}\cdot {}_1\psi_2(\mathrm{e}(-y);0,\mathrm{e}(\tau-y);\mathrm{e}(\tau),\mathrm{e}(x+(\tau+1)/2)).$$

3.3 Trapezoid Massey products

The geometric meaning of the function κ above is the following: the pair $(\kappa(y,x;\tau), \theta(x,\tau))$ determines a global section of a rank 2 bundle F_y on E which is a non-trivial extension of L by the line bundle of degree 0 corresponding to y. Namely, the bundle F_y is defined as follows

$$F_y = \mathbb{C}^* \times \mathbb{C}^2/(z,v) \mapsto (z\cdot \mathrm{e}(\tau), A_y(z)v)$$

where

$$A_y(z) = \begin{pmatrix} \mathrm{e}(y) & 1 \\ 0 & \mathrm{e}(-\tau/2)z^{-1} \end{pmatrix}.$$

We claim that some of triple Massey products corresponding to the trapezoid Fukaya products considered above are univalued and are also expressed via κ. Namely, consider the triple Massey product for $X_1 = \mathcal{O}$, $X_2 = L$, $X_3 = \mathcal{L}[1]$, $X_4 = \mathcal{O}_\xi[1]$ where $\mathcal{L}$ is a non-trivial line bundle of degree 0, $\xi = (\tau+1)/2$. Then $\mathrm{Hom}^*(X_1,X_3) = 0$ and $\mathrm{Hom}^0(X_2,X_4) = 0$. Furthermore, the composition map

$$\mathrm{Hom}(X_1,X_2)\otimes \mathrm{Hom}^{-1}(X_2,X_4) \to \mathrm{Hom}^{-1}(X_1,X_4)$$

is zero since the unique section of L vanishes at ξ. It follows that the Massey product is well-defined and univalued in this case. To compute it one should include the non-zero morphism $X_2 \to X_3$ into an exact triangle

$$X_2 \to X_3 \to C \to X_2[1]$$

then lift the morphisms $X_1 \to X_2$ and $X_3 \to X_4$ to morphisms $X_1 \to C[-1]$ and $C \to X_4$ and compose the obtained two morphisms. In our case one starts with a global section $s : \mathcal{O} \to L$, then s can be lifted canonically to a section $\widetilde{s}$ of F_y where $\mathcal{L} \simeq t_y L^{-1}\otimes L$. Now one chooses a splitting $r : (F_y)_\xi \to \mathbb{C}$ of the embedding $\mathbb{C} \simeq \mathcal{L}_\xi \to (F_y)$ and applies r to $\widetilde{s}(\xi)$. The result doesn't depend on a choice of splitting at ξ since $s(\xi) = 0$. More concretely, for $s = \theta(x,\tau)$ we have $\widetilde{s} = \kappa(y,x;\tau)$, hence, the above Massey product is given by $\kappa(y,(\tau+1)/2;\tau)$. One can easily check that this answer agrees with the corresponding Fukaya product. Note also that in fact this Massey product can still be expressed via theta functions due to the identity

$$\kappa(y,\frac{\tau+1}{2}) = \frac{\theta'(\frac{\tau+1}{2})}{2\pi i\theta(y-\frac{\tau+1}{2})}$$

If we replace ξ by another point on E the Massey product will no longer be univalued. However, if one chooses a splitting of the embedding $\mathcal{L} \to F_y$ over some open subset $U \subset E$ and a trivialization of $\mathcal{L}|_U$ then replacing r by this splitting we get a univalued operation. The function $\kappa(y, x; \tau)$ appears as such operation corresponding to the choice of a splitting over $E \setminus S$ coming from the trivialization of the pull-back of F_y to $\mathbb{C}^*$.

Another example of well-defined Massey products that are expressed in terms of the function κ is the following. Consider an extension

$$0 \to \mathcal{L} \xrightarrow{a} F \xrightarrow{b} M \to 0$$

where M and $\mathcal{L}$ are line bundles, $\deg M > 0$, $\deg \mathcal{L} = 0$, $\mathcal{L} \not\simeq \mathcal{O}$. Then as we have seen before for any section $s : \mathcal{O} \to M$ the triple product $m_3(s, c, a)$, where $c : M \to \mathcal{L}[1]$ is represented by the above extension, is a lifting of s to a section of F. The corresponding Massey product is well-defined so we have $MP(s, c, a) = m_3(s, c, a)$. The latter Fukaya product is of trapezoid type since $\deg \mathcal{L} = \deg \mathcal{O} = 0$, so it can be expressed via κ.

3.4 Associativity constraint

Let us consider an example of associativity constraint for Fukaya's A_∞-category of a torus involving triple products computed above.

Let us consider the following five lagrangians in $\mathbb{R}^2/\mathbb{Z}^2$: $\Lambda_1 = (t, -t)$ with trivial connection, $\Lambda_2 = (t, \alpha_2)$ with $2\pi i \beta_2 dx$, $\Lambda_3 = (-\alpha_1 - \alpha_2, t)$ with $2\pi i(\beta_1 + \beta_2)dy$, $\Lambda_4 = (t, \alpha_1 + \alpha_2 + \alpha_3)$ with $2\pi i(\beta_1 + \beta_2 + \beta_3)dx$, $\Lambda_5 = (0, t)$ with trivial connection. Here α_i, β_i are real numbers, $\alpha_1 + \alpha_2 \notin \mathbb{Z}$, $\alpha_1 + \alpha_3 \notin \mathbb{Z}$. We choose liftings of Λ_i to objects in Fukaya category in such a way that there is a non-zero morphism a_i of degree 0 from Λ_i to Λ_{i+1}. We want to write the above A_∞-identity for these morphisms. Note that all the Hom-spaces between our objects are either zero or one-dimensional, so the relevant compositions m_2 and m_3 are just numbers. Taking into account the fact that $\Lambda_2 \cap \Lambda_4 = \Lambda_3 \cap \Lambda_5 = \emptyset$ the identity boils down to

$$\begin{aligned} &m_2(\Lambda_1, \Lambda_2, \Lambda_3) m_3(\Lambda_1, \Lambda_3, \Lambda_4, \Lambda_5) + m_2(\Lambda_1, \Lambda_4, \Lambda_5) m_3(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) - \\ &m_2(\Lambda_1, \Lambda_2, \Lambda_5) m_3(\Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5) = 0 \end{aligned} \tag{3.4.1}$$

where for example $m_2(\Lambda_1, \Lambda_2, \Lambda_3)$ is the unique non-zero m_2-composition of morphisms between $\Lambda_1, \Lambda_2, \Lambda_3$, etc.

Now we can express all ingredients of (3.4.1) in terms of theta-functions, and functions f and g introduced above. Namely, denoting $z_i = \alpha_i \tau + \beta_i$ for $i = 1, 2, 3$

we obtain

$$m_2(\Lambda_1,\Lambda_2,\Lambda_3)=\mathrm{e}(\alpha_1^2\frac{\tau}{2}+\alpha_1\beta_1)\theta(z_1,\tau),$$
$$m_2(\Lambda_1,\Lambda_4,\Lambda_5)=$$
$$\mathrm{e}((\alpha_1+\alpha_2+\alpha_3)^2\frac{\tau}{2}+(\alpha_1+\alpha_2+\alpha_3)(\beta_1+\beta_2+\beta_3))\theta(z_1+z_2+z_3,\tau),$$
$$m_2(\Lambda_1,\Lambda_2,\Lambda_5)=\mathrm{e}(\alpha_2^2\frac{\tau}{2}+\alpha_2\beta_2)\theta(z_2,\tau),$$
$$m_3(\Lambda_1,\Lambda_3,\Lambda_4,\Lambda_5)=$$
$$\mathrm{e}((\frac{\alpha_1+\alpha_2}{2}+\alpha_3)(\alpha_1+\alpha_2)\tau+(\alpha_1+\alpha_2)(\beta_1+\beta_2+\beta_3)+\alpha_3(\beta_1+\beta_2))g(z_3,z_1+z_2;\tau),$$
$$m_3(\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4)=\mathrm{e}((\frac{\alpha_1^2}{2}-\frac{\alpha_3^2}{2})\tau+\alpha_1\beta_1-\alpha_3\beta_3)g(-z_3,z_1+z_3;\tau),$$
$$m_3(\Lambda_2,\Lambda_3,\Lambda_4,\Lambda_5)=$$
$$\mathrm{e}((\alpha_1+\alpha_2)(\alpha_1+\alpha_3)\tau+(\alpha_1+\alpha_2)(\beta_1+\beta_3)+(\alpha_1+\alpha_3)(\beta_1+\beta_2))f(z_1+z_2,z_1+z_3;\tau).$$

Substituting this into (3.4.1) and deleting similar terms we obtain the following identity

$$\begin{aligned}&\theta(z_1,\tau)g(z_3,z_1+z_2;\tau)+\theta(z_1+z_2+z_3,\tau)g(-z_3,z_1+z_3;\tau)=\\&\theta(z_2,\tau)f(z_1+z_2,z_1+z_3;\tau).\end{aligned}\tag{3.4.2}$$

By (3.2.4) this implies the following identity between meromorphic functions of $\mathbb{C}^3$:

$$\mathrm{e}(y)\theta(y+z,\tau)\kappa(y,z-x;\tau)-\mathrm{e}(-x)\theta(x-z,\tau)\kappa(-x,y+z;\tau)=\theta(z,\tau)f(x,y;\tau)\tag{3.4.3}$$

where we have put $x=z_1+z_2$, $y=z_1+z_3$, $z=z_2$.

We used the fact that Fukaya composition satisfies axioms of A_∞-category to derive the above identity. However, it can be also proved in a straightforward way comparing residues of both sides at poles and using the difference equation for κ. It appears in a slightly different form in Halphen's book [7] (p.481, formula (45) and the next one).

3.5 More Fukaya products

Let us consider another example of triple Fukaya products where none of the four lines are parallel. Namely, let $\Lambda_1=(\alpha_1+t,-\alpha_1+t)$, $\Lambda_2=(-\alpha_2,t)$, $\Lambda_3=(t,0)$, $\Lambda_4=(t,-t)$. Then we can choose (essentially uniquely) lifts of the corresponding lagrangain circles in $\mathbb{R}^2/\mathbb{Z}^2$ to objects of Fukaya category in such a way that there is a non-zero morphism of degree zero from Λ_i to Λ_{i+1}. All the Hom-spaces $\mathrm{Hom}(\Lambda_i,\Lambda_{i+1})$ are 1-dimensional, however, $\mathrm{Hom}^{-1}(\Lambda_1,\Lambda_4)$ is 2-dimensional since the corresponding circles have 2 points of intersection. Let us consider the component of the triple product corresponding to the point $(\alpha_1,-\alpha_1)\in\Lambda_1\cap\Lambda_4$. Then as before its value is expressed via certain holomorphic function of $z_1=\alpha_1\tau$ and

$z_2 = \alpha_2\tau$ (one could add some monodromies in the above picture to get all values of complex variables z_1 and z_2). This function has the following form

$$h(z_1, z_2; \tau) = \sum_{(m+\alpha(z_1))(n+\alpha(z_2))>0} \operatorname{sign}(m + \alpha(z_1))\, \mathrm{e}\left(\frac{\tau}{2}\left(2m^2 + 4mn + n^2\right) + 2(m+n)z_1 + (2m+n)z_2\right).$$

It is holomorphic in the region $\alpha(z_i) \notin \mathbb{Z}$ and can be considered as a section of a line bundle on E^2 over the corresponding open subset of E^2 because of the following quasi-periodicity:

$$h(z_1 + m + \tau, z_2; \tau) = \mathrm{e}(-\tau - 2z_1 - 2z_2)h(z_1, z_2; \tau), \tag{3.5.1}$$

$$h(z_1, z_2 + m + \tau; \tau) = \mathrm{e}(-\tau/2 - 2z_1 - z_2)h(z_1, z_2; \tau). \tag{3.5.2}$$

Let us denote by h_0 the restriction of h to the region $0 < \alpha(z_i) < 1$ for $i = 1, 2$. Then it is easy to see that h_0 extends to a holomorphic function on $\mathbb{C}^2$ (the series converges absolutely).

As before we can translate certain A_∞-associativity axiom into an identity involving $h(z_1, z_2)$. More precisely, we can consider five lines: two with slope 0, one with slope 1, one with slope -1, and one with slope ∞. Their relative position is described by three parameters. Adding monodromies we get the following identity with three complex variables z_1, z_2, z_3:

$$\begin{aligned}&\theta(2z_1 + z_3, \tau)h(z_1 + z_3, -z_1 + z_2 - z_3; \tau) + \theta(z_1 + z_2 + z_3, \tau)h(-z_1 - z_3, z_3; \tau) = \\ &= \theta(2z_2, 2\tau)g(-z_1 + z_2 - z_3, -z_1 - z_2; \tau) + \theta(2z_1, 2\tau)g(z_3, z_1 + z_2)\end{aligned} \tag{3.5.3}$$

Using difference equations for h and g we can further transform this into the following identity involving h_0 and κ:

$$\begin{aligned}&\theta(2x + y, \tau)h_0(x, z; \tau) - \theta(2x + z, \tau)h_0(x, y; \tau) = \\ &= \theta(2(x + z), 2\tau)\kappa(-2x - y - z, 2x + y + \tau; \tau) - \\ &- \theta(2(x + y), 2\tau)\kappa(-2x - y - z, 2x + z + \tau; \tau)\end{aligned} \tag{3.5.4}$$

Substituting $y = -x$ in this identity we can express $h_0(x, z; \tau)$ via $h_0(x, -x; \tau)$ and functions κ and θ. We claim that $h_0(x) := h_0(x, -x; \tau)$ can also be expressed as a rational function of κ and θ. Indeed, we have

$$h_0(x) = \sum_{(m+1/2)(n+1/2)>0} \operatorname{sign}(m + 1/2)\, \mathrm{e}\left(\frac{\tau}{2}\left(2m^2 + 4mn + n^2\right) + nx\right).$$

Hence,

$$\begin{aligned}h_0(x + \tau) = \mathrm{e}(\frac{\tau}{2} + x) \times &\sum_{(m+1/2)(n+1/2)>0} \operatorname{sign}(m + 1/2)\, \mathrm{e} \\ &\left(\frac{\tau}{2}\left(2(m+1)^2 + 4(m+1)(n-1) + (n-1)^2\right) + (n-1)x\right) = \\ &= \mathrm{e}(\frac{\tau}{2} + x)\left(h_0(x) - \theta(x, \tau)\right) + \theta(0, 2\tau).\end{aligned}$$

Let us denote $\psi(x)=\theta(x-(\tau+1)/2,\tau)h_0(x)$. Then ψ satisfies the equation

$$\psi(x+\tau)=\mathrm{e}(\frac{\tau+1}{2})\psi(x)+\mathrm{e}(\frac{\tau}{2})\theta(x,\tau)\theta(x-\frac{\tau+1}{2},\tau)+\theta(0,2\tau)\theta(x+\frac{\tau+1}{2},\tau).$$

It is easy to see that this equation has the unique holomorphic solution, namely,

$$\psi(x)=\theta(0,2\tau)\kappa(\frac{\tau+1}{2},x+\frac{\tau+1}{2};\tau)+$$
$$\theta(\frac{\tau+1}{2},2\tau)\cdot\left(\mathrm{e}(\frac{\tau}{2})\kappa(\frac{\tau+1}{2},2x-\frac{\tau+1}{2};2\tau)-\mathrm{e}(x-\frac{\tau}{2})\kappa(-\frac{\tau+1}{2},2x+\frac{\tau+1}{2};2\tau)\right)$$

3.6 Generic case

Let us consider four lines $L_i(y_i)=\{(\frac{y_i+t}{\lambda_i},t)\}$, $i=1,2,3,4$, where the slopes λ_i are distinct rational numbers, y_i are fixed real numbers. Let us denote by $\overline{L}_i(y_i)$ the image of L_i in $\mathbb{R}^2/\mathbb{Z}^2$. The intersection of $\overline{L}_i(y_i)$ and $\overline{L}_j(y_j)$ consists of the following set of points:

$$e_{a,b}(y_i,y_j)=(y_{ij},y'_{ij})+\frac{a\lambda_j+b}{\lambda_j-\lambda_i}(1,\lambda_i)\mod \mathbb{Z}^2$$

where a,b are integers, $y_{ij}=\frac{y_j-y_i}{\lambda_j-\lambda_i}$, $y'_{ij}=\frac{\lambda_i y_j-\lambda_j y_i}{\lambda_j-\lambda_i}$. Note that if a and b are integers then $\overline{L}_i(y_i+a\lambda_i+b)=\overline{L}(y_i)$. Furthermore, we have

$$e_{a,b}(y_i,y_j)=e_{0,0}(y_i-a\lambda_i-b,y_j)=e_{0,0}(y_i,y_j+a\lambda_j+b).$$

For every $i<j$ and an intersection point $p\in\overline{L}_i(y_i)\cap\overline{L}_j(y_j)$ we denote by $[p]$ the corresponding morphism from $\overline{L}_i(y_i)$ to $\overline{L}_j(y_j)$ in the Fukaya category. Also let us denote $e_{ij}=e_{0,0}(y_i,y_j)$. Note that $\deg[e_{a,b}(y_i,y_j)]=\deg(i,j)$ where $\deg(i,j)=0$ if $\lambda_i<\lambda_j$ and $\deg(i,j)=1$ otherwise. We want to compute the Fukaya product $m_3([e_{12}],[e_{23}],[e_{34}])$, so we assume that

$$\sum_{i=1}^{3}\deg(i,i+1)=\deg(1,4)+1 \tag{3.6.1}$$

(otherwise this product is zero).

For every rational number λ let us denote by $I_\lambda\subset\mathbb{Z}$ the subset of $n\in\mathbb{Z}$ such that $n\lambda\in\mathbb{Z}$. To each quadruple of distinct rational numbers $\lambda_1,\dots,\lambda_4$ we associate a lattice $\Lambda=\Lambda(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ and a sublattice $\Lambda^+=\Lambda^+(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ as follows:

$$\Lambda(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=$$
$$=\{\overline{n}=(n_1,n_2,n_3,n_4)\in\mathbb{Q}^4\mid\sum_{i=1}^{4}n_i=0,\sum_{i=1}^{4}\lambda_i n_i=0;n_2\in I_{\lambda_2},n_3\in I_{\lambda_3}\},$$
$$\Lambda^+(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=\{\overline{n}\in\Lambda(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\mid n_1\in I_{\lambda_1}\}.$$

Equivalently, the sublattice Λ^+ is distinguished in Λ by the condition $n_4 \in I_{\lambda_4}$. Consider the following quadratic form on $\Lambda \otimes \mathbb{R}$:

$$Q(x) = Q_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(x) = (\lambda_3 - \lambda_4)x_3x_4 + (\lambda_1 - \lambda_2)x_1x_2.$$

Clearly, Q takes integer values on Λ^+. However, it is in general indefinite. Let $C \subset \Lambda \otimes \mathbb{R} \simeq \mathbb{R}^2$ be the subset consisting of x such that $(\lambda_{i-1} - \lambda_i)x_ix_{i-1} > 0$ for all $i = 1, 2, 3, 4$ (where we set $x_0 = x_4$). Then clearly $Q(x) > 0$ for any $x \in C$. Note that our assumption (3.6.1) implies that C is non-empty. It is easy to see that two of the four inequalities defining C are redundant, so C is always a region in the plane bounded by 2 lines. Let $C = C^+ \cup C^-$ be a decomposition of C into connected components, $\varepsilon : C \to \pm 1$ be the function assigning 1 (resp. -1) to points of C^+ (resp. C^-). Note that for any quadruple of real numbers $y = (y_1, y_2, y_3, y_4)$ the vector

$$v(y) = (y_{14} - y_{12}, y_{12} - y_{23}, y_{23} - y_{34}, y_{34} - y_{14})$$

(where $y_{ij} = \frac{y_j - y_i}{\lambda_j - \lambda_i}$) belongs to the subspace $\Lambda \otimes \mathbb{R} \subset \mathbb{R}^4$. Now for $\overline{n}_0 \in \Lambda \otimes \mathbb{Q}$ and $z \in \mathbb{C}^4$ we set

$$F_{\lambda_1,\lambda_2,\lambda_3,\lambda_4;\overline{n}_0}(z) = \\ = \sum_{\overline{n} \in (\Lambda^+ + \overline{n}_0) \cap (C - v(\alpha(z)))} \varepsilon(\overline{n} + v(\alpha(z))) \cdot \mathrm{e}(\frac{\tau}{2} Q(\overline{n}) + n_1z_1 + n_2z_2 + n_3z_3 + n_4z_4))$$

where $\tau \cdot \alpha$ is the first projection of the direct sum decomposition $\mathbb{C}^4 = \tau\mathbb{R}^4 \oplus \mathbb{R}^4$. For z varying in some open subset of $\mathbb{C}^4$ (in fact, in the complement to codimension one analytic subset), this is a holomorphic function of z. Note that linear relations between n_i imply that

$$n_1z_1 + n_2z_2 + n_3z_3 + n_4z_4 = (\lambda_1 - \lambda_2)(z_{14} - z_{12})n_2 + (\lambda_1 - \lambda_3)(z_{14} - z_{13})n_3,$$

where $z_{ij} = \frac{z_j - z_i}{\lambda_j - \lambda_i}$. On the other hand, the vector $v(\alpha(z))$ is determined by its first two components $\alpha(z_{14} - z_{12})$ and $\alpha(z_{12} - z_{23}) = \frac{\lambda_3 - \lambda_1}{\lambda_3 - \lambda_2}\alpha(z_{12} - z_{13})$. Thus, F actually depends on two holomorphic variables $z_{13} - z_{12}$ and $z_{14} - z_{12}$.

Proposition 3.1. *Assume that* $\sum_{i=1}^{3} \deg[e_{i,i+1}] = \deg[e_{1,4}] + 1$. *Then one has*

$$m_3([e_{12}], [e_{23}], [e_{34}]) = \pm\, \mathrm{e}(\frac{\tau}{2}\Delta(y_1, y_2, y_3, y_4)) \cdot \\ \sum_{\overline{k} \in \Lambda/\Lambda^+} F_{\lambda_1,\lambda_2,\lambda_3,\lambda_4;\overline{k}}(\tau y)[e_{-k_2-k_3,\lambda_2k_2+\lambda_3k_3}(y_1, y_4)]. \tag{3.6.2}$$

where

$$\Delta(y_1, y_2, y_3, y_4) = \det \begin{pmatrix} y_{34} - y_{12} & y_{23} - y_{14} \\ y'_{34} - y'_{12} & y'_{23} - y'_{14} \end{pmatrix}$$

Proof. The idea is to parametrize the quadrangles contributing to m_3 by elements of Λ. Namely, let $p_{i,i+1}$, $i = 1, \dots, 4$ be the vertices of such a quadrangle (where the edge between $p_{i-1,i}$ and $p_{i,i+1}$ belongs to $\overline{L}_i(y_i)$ modulo $\mathbb{Z}^2$). Then denoting the difference between the first coordinates of $p_{i-1,i}$ and $p_{i,i+1}$ by n_i we find that

(n_1, n_2, n_3, n_4) belongs to $\Lambda(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. The condition $n+v(y) \in C$ is equivalent to the requirement that the lagrangians come in the correct order when one goes clockwise along the quadrangle. □

In the above proposition we consider $\overline{L}_i(y_i)$ equipped with trivial local systems. If we add non-trivial connections along $\overline{L}_i(y_i)$ the formula (3.6.2) will change by adding some linear combinations of these connections to the arguments of the function F.

For five lines $L(y_i)$, $i = 1, \dots, 5$ we can consider the identity obtained by comparing the coefficients with $[e_{15}]$ in the A_∞-constraint (2.1.1) for $a_1 = [e_{12}]$, $a_2 = [e_{23}]$, $a_3 = [e_{34}]$ and $a_4 = [e_{45}]$.

Before writing the corresponding identity let us introduce some more notation. Let $\lambda_1, \lambda_2, \lambda_3$ be a triple of distinct rational numbers. We denote

$$I_{\lambda_1,\lambda_2,\lambda_3} = I_{\lambda_2} \cap \frac{\lambda_3 - \lambda_1}{\lambda_3 - \lambda_2} I_{\lambda_1}.$$

One easily checks that $I_{\lambda_1,\lambda_2,\lambda_3} = I_{\lambda_3,\lambda_2,\lambda_1}$. Assume that

$$\deg(1,2) + \deg(2,3) = \deg(1,3). \tag{3.6.3}$$

Then similarly to the previous proposition one checks that

$$m_2([e_{12}],[e_{23}]) = \\ \mathrm{e}(-\frac{\tau}{2}\Delta(y_1,y_2,y_3)) \cdot \sum_{n_0 \in I_{\lambda_2}/I_{\lambda_1,\lambda_2,\lambda_3}} \theta_{\lambda_1,\lambda_2,\lambda_3;n_0}(\tau(y_1,y_2,y_3))[e_{n_0,-\lambda_2 n_0}(y_1,y_3)]$$

where

$$\Delta(y_1,y_2,y_3) = \det \begin{pmatrix} y_{23} - y_{12} & y_{13} - y_{12} \\ y'_{23} - y'_{12} & y'_{13} - y'_{12} \end{pmatrix},$$

$$\theta_{\lambda_1,\lambda_2,\lambda_3;n_0}(z_1,z_2,z_3) = \\ = \sum_{n \in I_{\lambda_1,\lambda_2,\lambda_3}} \mathrm{e}\left(\frac{(\lambda_3-\lambda_2)(\lambda_2-\lambda_1)\tau}{2(\lambda_3-\lambda_1)}(n+n_0)^2 + (n+n_0)(z_{23}-z_{12})\right).$$

Note that the condition (3.6.3) implies that $(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)^{-1} > 0$ so the above series converges.

Now the A_∞-constraint leads to the following identity.

$$\varepsilon_1 \cdot \sum_{\overline{n}\in\Lambda_{2345}/\Lambda_{2345}^{+}, n_2\in I_2+\frac{\lambda_5-\lambda_1}{\lambda_5-\lambda_2}I_1} F_{2345,\overline{n}}(z)\theta_{125,n_2''}(z) +$$
$$\varepsilon_2 \cdot \sum_{\overline{n}\in\Lambda_{1234}/\Lambda_{1234}^{+}, n_4\in I_4+\frac{\lambda_5-\lambda_1}{\lambda_4-\lambda_1}I_5} F_{1234,\overline{n}}(z)\theta_{145,n_4''}(z) +$$
$$\varepsilon_3 \cdot \sum_{k\in I_2/I_{123}, k\in\frac{\lambda_1-\lambda_3}{\lambda_1-\lambda_2}I_3+\frac{\lambda_1-\lambda_4}{\lambda_1-\lambda_2}I_4+\frac{\lambda_1-\lambda_5}{\lambda_1-\lambda_2}I_5} F_{1345,k''u''+k'''u'''}(z)\theta_{123,k}(z) +$$
$$\varepsilon_4 \cdot \sum_{k\in I_4/I_{345}, k\in\frac{\lambda_5-\lambda_1}{\lambda_5-\lambda_4}I_1+\frac{\lambda_5-\lambda_2}{\lambda_5-\lambda_4}I_2+\frac{\lambda_5-\lambda_3}{\lambda_5-\lambda_4}I_3} F_{1235,k'v'+k''v''}(z)\theta_{345,k}(z) +$$
$$\varepsilon_5 \cdot \sum_{k\in I_3/I_{234}, k\in\frac{\lambda_5-\lambda_1}{\lambda_5-\lambda_3}I_1+\frac{\lambda_5-\lambda_2}{\lambda_5-\lambda_3}I_2+\frac{\lambda_5-\lambda_4}{\lambda_5-\lambda_3}I_4} F_{1245,k'w'+k''w''+k'''w'''}(z)\theta_{234,k}(z) = 0.$$

Here z is in $\mathbb{C}^5$ minus some analytic subset of codimension 1, however, all the functions in this identity actually depend on three holomorphic variables: $z_{13}-z_{12}$, $z_{14}-z_{12}$ and $z_{15}-z_{12}$ (where as before $z_{ij}=(z_j-z_i)/(\lambda_j-\lambda_i)$). For every $1\le i<j<k<l\le 5$ we denote $F_{ijkl,\overline{n}}(z)=F_{\lambda_i,\lambda_j,\lambda_k,\lambda_l;\overline{n}}(z_i,z_j,z_k,z_l)$. Similarly, $\theta_{ijk,n}(z)=\theta_{\lambda_i,\lambda_j,\lambda_k;n}(z_i,z_j,z_k)$. Also we write for simplicity $I_i=I_{\lambda_i}$, $\Lambda_{ijkl}=\Lambda(\lambda_i,\lambda_j,\lambda_k,\lambda_l)$, etc. The elements of $\Lambda_{ijkl}\otimes\mathbb{Q}$ are denoted by $\overline{n}=(n_i,n_j,n_k,n_l)$. The multiple ε_i before each term is ±1 if the conditions on the degrees are satisfied (we will specify the sign later), and 0 otherwise. For example, $\varepsilon_1=0$ unless $\deg(1,2)+\deg(2,5)=\deg(1,5)$ and $\deg(2,3)+\deg(3,4)+\deg(4,5)=\deg(2,5)+1$. In the first two terms of the identity we denote $n_2=n_2'+n_2''$ (resp. $n_4=n_4'+n_4''$) with respect to the inclusion $n_2\in I_2+\frac{\lambda_5-\lambda_1}{\lambda_5-\lambda_2}I_1$ (resp. $n_4\in I_4+\frac{\lambda_5-\lambda_1}{\lambda_4-\lambda_1}I_5$). Similarly, in the last three terms the decomposition $k=k'+k''+k'''$ corresponds to the inclusion of k into sum of three ideals. Note that although k is a representative of a coset, the condition that k belongs to the sum of three ideals is well-defined. In 3rd and 4th term this is clear, while in the 5th term the inclusion

$$I_{234}\subset\frac{\lambda_5-\lambda_2}{\lambda_5-\lambda_3}I_2+\frac{\lambda_5-\lambda_4}{\lambda_5-\lambda_3}I_4$$

follows from the identity

$$k=\frac{\lambda_5-\lambda_2}{\lambda_5-\lambda_3}\cdot(\frac{\lambda_4-\lambda_3}{\lambda_4-\lambda_2}k)+\frac{\lambda_5-\lambda_4}{\lambda_5-\lambda_3}\cdot(\frac{\lambda_3-\lambda_2}{\lambda_4-\lambda_2}k).$$

Finally, we denoted

$$u' = (\frac{(\lambda_4-\lambda_3)(\lambda_2-\lambda_1)}{(\lambda_4-\lambda_1)(\lambda_3-\lambda_1)}, \frac{\lambda_1-\lambda_2}{\lambda_3-\lambda_1}, \frac{\lambda_1-\lambda_2}{\lambda_1-\lambda_4}, 0),$$
$$u'' = (\frac{(\lambda_5-\lambda_3)(\lambda_2-\lambda_1)}{(\lambda_5-\lambda_1)(\lambda_3-\lambda_1)}, \frac{\lambda_1-\lambda_2}{\lambda_3-\lambda_1}, 0, \frac{\lambda_2-\lambda_1}{\lambda_5-\lambda_1}),$$
$$v' = (\frac{\lambda_5-\lambda_4}{\lambda_5-\lambda_1}, 0, \frac{\lambda_5-\lambda_4}{\lambda_3-\lambda_5}, \frac{(\lambda_5-\lambda_4)(\lambda_1-\lambda_3)}{(\lambda_5-\lambda_1)(\lambda_3-\lambda_5)}),$$
$$v'' = (0, \frac{\lambda_5-\lambda_4}{\lambda_5-\lambda_2}, \frac{\lambda_5-\lambda_4}{\lambda_3-\lambda_5}, \frac{(\lambda_5-\lambda_4)(\lambda_2-\lambda_3)}{(\lambda_5-\lambda_2)(\lambda_3-\lambda_5)}),$$
$$w' = (\frac{\lambda_5-\lambda_3}{\lambda_5-\lambda_1}, \frac{\lambda_4-\lambda_3}{\lambda_2-\lambda_4}, \frac{\lambda_3-\lambda_2}{\lambda_2-\lambda_4}, \frac{\lambda_3-\lambda_1}{\lambda_5-\lambda_1}),$$
$$w'' = (0, \frac{(\lambda_5-\lambda_4)(\lambda_2-\lambda_3)}{(\lambda_5-\lambda_2)(\lambda_2-\lambda_4)}, \frac{\lambda_3-\lambda_2}{\lambda_2-\lambda_4}, \frac{\lambda_3-\lambda_2}{\lambda_5-\lambda_2}),$$
$$w''' = (0, \frac{\lambda_4-\lambda_3}{\lambda_2-\lambda_4}, \frac{(\lambda_5-\lambda_2)(\lambda_3-\lambda_4)}{(\lambda_2-\lambda_4)(\lambda_5-\lambda_4)}, \frac{\lambda_3-\lambda_4}{\lambda_5-\lambda_4})$$

Note that there is an ambiguity of sign in the definition of F_{ijkl}. The claim is that for every given configuration of $\deg(e_i, e_j)$ there exists a choice of signs in F_{ijkl} and ε_i which makes the above identity true. For example, let us assume that $\lambda_3 < \lambda_1 < \lambda_4 < \lambda_2 < \lambda_5$. Then all ε_i are non-zero. Let us choose the positive components C^+ of the cones C_{ijkl} entering in the definition of F_{ijkl} as follows:

$$C^+_{2345} : n_2 > 0, n_3 > 0, n_4 < 0, n_5 > 0,$$
$$C^+_{1234} : n_1 > 0, n_2 < 0, n_3 < 0, n_4 > 0,$$
$$C^+_{1345} : n_1 > 0, n_3 > 0, n_4 < 0, n_5 > 0,$$
$$C^+_{1235} : n_1 > 0, n_2 < 0, n_3 < 0, n_5 > 0,$$
$$C^+_{1245} : n_1 > 0, n_2 < 0, n_4 < 0, n_5 > 0.$$

Then the signs should be chosen as follows: $\varepsilon_1 = \varepsilon_2 = \varepsilon_5 = 1$, $\varepsilon_3 = \varepsilon_4 = -1$. To see this we note that for purely imaginary τ we can represent each of five terms of our identity in the form

$$\sum_{\overline{n},m} \varepsilon(\overline{n}) c_{\overline{n},m}\, \mathrm{e}(\sum_{i=1}^{5} l_i(\overline{n}, m) z_i)$$

where the sum is taken over a coset for a lattice in $(\Lambda_{ijkl} \otimes \mathbb{Q}) \times \mathbb{Q}$, l_i are some rational linear functions of $(\overline{n}, m)$, and the coefficients $c_{\overline{n},m}$ are positive. It is easy to see that l_i are linearly independent, so there are no cancellations in the above Fourier series. Now we consider a term corresponding to $\overline{n} \in C^+_{2345}$ and $m >> 0$ (resp. $m << 0$) and find out that the only term it can cancel out with has $\overline{n'} \in C^+_{1235}$ (resp. $\overline{n'} \in C^-_{1245}$), hence, $\varepsilon_4 = -\varepsilon_1$ (resp. $\varepsilon_5 = \varepsilon_1$). On the other hand, a term corresponding to $\overline{n} \in C^+_{1234}$ and $m >> 0$ (resp. $m << 0$) can cancel out only with the term which has $\overline{n'} \in C^+_{1345}$ (resp. $\overline{n'} \in C^-_{1245}$), hence, $\varepsilon_3 = -\varepsilon_2$ (resp. $\varepsilon_5 = \varepsilon_2$).

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