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Instantons in N=4 Sp(N) and SO(N) theories and the AdS/CFT correspondence

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Abstract

Following work on theories with SU(N) gauge groups, we perform a large-N saddle-point approximation of the measure

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for ADHM multi-instantons in $\mathcal{N} = 4$ supersymmetric gauge theories with symplectic or orthogonal gauge groups. For Sp(N) we find that a saddle-point only exists in the even instanton charge sector. For either Sp(N) or SO(N) the saddle-point solution parametrizes $AdS_5 \times \mathbb{R}P^5$, the dual supergravity geometry in the AdS/CFT correspondence for these theories. The instanton measure at large-N has the form of the partition function of ten-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory with a unitary gauge group dimensionally reduced to zero dimensions.

1 Introduction

This paper is concerned with the remarkable series of conjectures that have been made postulating a duality in the 't Hooft coupling of conformally invariant gauge theories and type IIB string theory on a background of the form $AdS_5 \times M_5$, where M_5 is a certain five dimensional compact space. In the simplest case—the original conjecture of Maldacena [1]—the gauge theory has $\mathcal{N} = 4$ supersymmetry with SU(N) gauge group in which case M_5 is S^5 (see also [2, 3, 4]). When the gauge group is one of the other classical groups, the five sphere is replaced by the projective space $S^5/\mathbb{Z}_2 \simeq \mathbb{R}P^5$ [5, 6]. There are many other generalizations; however, in this paper we concentrate on the $\mathcal{N} = 4$ theories with gauge groups Sp(N) and SO(N). In this sense this is a companion to [7] which considered the SU(N) theories.

Refs. [8, 7] uncovered striking relations between the multi-instantons of $\mathcal{N} = 4$ supersymmetric gauge theory and D-instantons in type IIB supergravity building on the ideas of [9, 10]. The first relation involves the theory at finite N and the second at large-N:

(i) At finite N, we showed that the measure for ADHM instantons was precisely equal to the partition function of D-instantons moving in a background of N coincident D3-branes in the decoupling limit $\alpha' \to 0$. This partition function is precisely the dimensional reduction to zero dimensions of the pure $\mathcal{N} = (1,1)$ supersymmetric U(k) gauge theory in six dimensions with N additional hypermultiplets (and so the resulting theory actually only has $\mathcal{N} = (1,0)$ supersymmetry). What is particularly striking is that the auxiliary scalars χ introduced in [7] to bi-linearize a certain four-fermion interaction in the instanton action arises in a very natural way as the scalars corresponding to the six-dimensional gauge field and describe the freedom for the D-instantons to be ejected from the D3-branes.

(ii) At large N, the ADHM measure can be approximated by an expansion around a saddle-point and is proportional to the partition function of the six-dimensional pure $\mathcal{N} = (1,1)$ supersymmetric $\mathrm{U}(k)$ gauge theory—with no additional matter—dimensionally reduced to zero dimensions. Alternatively, this can be described as the ten dimensional $\mathcal{N} = 1$ supersymmetric $\mathrm{U}(k)$ gauge theory dimensionally reduced to zero dimensions. This is precisely what one expects for D-instantons in a flat background with no D3 branes present where the $\mathrm{U}(1)^k \subset \mathrm{U}(k)$ components of the gauge field are interpreted as the position of the charge-k D-instanton in $\mathbb{R}^{10}$. The only difference with the large-N measure is that the abelian component of the gauge field is now interpreted as the position in $AdS_5 \times S^5$: the near horizon geometry of the $N \to \infty$ D3-branes.

The large-N approximation of the measure described in (ii) was performed by a steepest-decent method. This gives a way to probe the ten-dimensional geometry directly since the solution of the saddle-point equations could be interpreted as the position of a point-like object in $AdS_5 \times S^5$. From the gauge theory side the solution represents the simple configuration where all the instantons are at the same point in $\mathbb{R}^4$ with the same scale size and in mutually commuting $\mathrm{SU}(2)$ subgroups of the gauge group. The other five coordinates arise from auxiliary scalar variables that are used to bi-linearize a certain four-fermion interaction mediated by the Yukawa couplings in the gauge theory. These additional variables are $\mathrm{SO}(6)$, the R-symmetry of the $\mathcal{N} = 4$ theory, vector-valued and at the saddle-point they are constrained to lie on S^5.

This qualitative relation between large-N ADHM instantons and D-instantons in $AdS_5 \times S^5$ was made quantitative since D-instantons contribute terms to the type IIB effective [11, 9, 10] action that imply very particular ADHM instanton contributions to certain correlation functions. In Ref. [7] we found precise agreement between the D-instanton induced effects and ADHM contributions to the relevant correlation functions. This paper investigates the extent to which this relation between D-instantons and ADHM instantons persists when the gauge group of the theory is either SO(N) or Sp(N). Other generalizations involving $\mathcal{N} < 4$ supersymmetric theories either based on orbifolds of the SU(N) theory or the finite $\mathcal{N} = 2$ Sp(N) theory have been considered in Refs. [12, 13]. We will not consider the finite-N relation spelled out in (i) above; rather we shall concentrate on the large-N situation. Our results are as follows:

(i) For gauge group Sp(N) the ADHM construction at charge k involves an auxiliary group O(k). In this case the large-N saddle-point equations only have a solution for even instanton number. The saddle-point solution in the instanton charge $2k$ sector describes the positions of k point-like objects, the D-instantons of the string theory, in $AdS_5 \times \mathbb{R}P^5$. The expansion around the general saddle-point solution can be captured by an expansion around the maximally degenerate solution where all the D-instantons are at the same point in $AdS_5 \times \mathbb{R}P^5$. This solution is invariant under U(k) $\subset$ O($2k$).

(ii) For gauge group SO(N) the ADHM construction at charge k involves an auxiliary group Sp(k). In this case the large-N saddle-point equations have a solution for all instanton numbers k which describes the positions of k point-like objects, the D-instantons of the string theory, in $AdS_5 \times \mathbb{R}P^5$. In this case the maximally degenerate saddle-point solution is invariant under U(k) $\subset$ Sp(k).

In both cases (i) and (ii) the large-N ADHM instanton measure has the same form as that of the instanton charge k sector of the $\mathcal{N} = 4$ SU(N) theory. In other words at leading order the measure is equal to the partition function of ten-dimensional $\mathcal{N} = 1$ supersymmetric U(k) dimensionally reduced to zero dimensions (the unbroken residual symmetry group in both cases). The abelian components correspond to the overall position of the configuration in $AdS_5 \times \mathbb{R}P^5$ along with the fermionic collective coordinates corresponding to the 16 supersymmetric and superconformal fermion zero modes. The remaining non-abelian SU(k) partition function can then be explicitly computed to give a factor proportional to $\sum_{d|k} d^{-2}$, a sum over over the positive integer divisors d of k. In addition to this there is an overall numerical factor $\sqrt{N} g^8 k^{-7/2}$ identical to the SU(N) case.

These relations imply that instanton contribution at charge $2k$ and k, respectively, to various correlation function in the Sp(N) and SO(N) theories match precisely the charge k instanton contributions to the same correlators in the SU(N) theory.

2 The ADHM Formalism for the Classical Groups

The ADHM formalism [14] for constructing instanton solutions was adapted for dealing with arbitrary classical gauge groups in the early instanton literature. The method adopted was to consider the construction for one of the series of classical groups, e.g. symplectic groups in Ref. [15] and orthogonal groups in Ref. [16], and then embed the other two series in this series. Our approach will be no different, although we will start from the unitary series. This has the advantage of avoiding the language of quaterions but naturally our construction will be equivalent to those in the previously mentioned references.

After briefly reviewing the ADHM formalism an $\mathcal{N}=4$ gauge theory with gauge group SU(N), we will then show how the orthogonal and symplectic cases may be embedded therein. Full details of the SU(N) can be found in [17, 7].

The instanton solution at charge k, is described by an $(N+2k)\times 2k$ dimensional matrix a, and its conjugate $\bar{a}$, with the form

$$a = \begin{pmatrix} w_{\dot{\alpha}} \\ a'_{\alpha\dot{\alpha}} \end{pmatrix} , \qquad \bar{a} = \begin{pmatrix} \bar{w}^{\dot{\alpha}} & \bar{a}'^{\dot{\alpha}\alpha} \end{pmatrix} . \tag{2.1}$$

Here $w_{\dot{\alpha}}$ is a (spacetime) Weyl-spinor-valued $N\times k$ matrix and $a'_{\alpha\dot{\alpha}} = a'_n \sigma^n_{\alpha\dot{\alpha}}$ where a'_n is a (spacetime) vector-valued $k\times k$ matrix. The conjugates are defined as

$$\bar{w}^{\dot{\alpha}} \equiv (w_{\dot{\alpha}})^\dagger , \qquad \bar{a}'^{\dot{\alpha}\alpha} \equiv (a'_{\alpha\dot{\alpha}})^\dagger . \tag{2.2}$$

The matrices a'_n are restricted to be hermitian: $(a'_n)^\dagger = a'_n$. The remaining $4k(N+k)$ variables are still an over-parametrization of the instanton moduli space which is obtained by a hyper-Kähler quotient construction. One first imposes the ADHM constraints:

$$D^{\dot{\alpha}}{}_{\dot{\beta}} \equiv \bar{w}^{\dot{\alpha}} w_{\dot{\beta}} + \bar{a}'^{\dot{\alpha}\alpha} a'_{\alpha\dot{\beta}} = \lambda \delta^{\dot{\alpha}}{}_{\dot{\beta}} , \tag{2.3}$$

where λ is an arbitrary constant matrix. The ADHM moduli space is then identified with the space of a's subject to (2.3) modulo the action of an U(k) symmetry which acts on the instanton indices of the variables as follows

$$w_{\dot\alpha} \to w_{\dot\alpha} U \ , \qquad a'_{\alpha\dot\alpha} \to U^\dagger a'_{\alpha\dot\alpha} U \ , \quad U \in \mathrm{U}(k). \tag{2.4}$$

The final piece of the story is the explicit construction of the self-dual gauge field itself. To this end we define the matrix

$$\Delta(x) = \begin{pmatrix} w_{\dot\alpha} \\ x_{\alpha\dot\alpha} 1_{[k]\times[k]} + a'_{\alpha\dot\alpha} \end{pmatrix} \ , \tag{2.5}$$

where $x_{\alpha\dot\alpha}$, or equivalently x_n, related via $x_{\alpha\dot\alpha} = x_n \sigma^n_{\alpha\dot\alpha}$, is a point in spacetime. For generic x, the $(N+2k)\times N$ dimensional complex-valued matrix $U(x)$ is a basis for $\ker(\bar\Delta)$:

$$\bar\Delta U = 0 = \bar U \Delta \ , \tag{2.6}$$

where U is orthonormalized according to

$$\bar U U = \ 1_{[N]\times[N]} \, . \tag{2.7}$$

The self-dual gauge field is then simply

$$v_n = \bar U \partial_n U \ . \tag{2.8}$$

The fermions in the background of the instanton lead to new Grassmann collective coordinates. Associated to the fermions in the adjoint and anti-symmetric tensor representations, λ^A, these collective coordinates are described by the $(N+2k)\times k$ matrices $\mathcal{M}^A$, and their conjugates[1]

$$\mathcal{M}^A = \begin{pmatrix} \mu^A \\ \mathcal{M}'^A_\alpha \end{pmatrix} \ , \qquad \bar{\mathcal{M}}^A = \begin{pmatrix} \bar\mu^A & \bar{\mathcal{M}}'^{\alpha A} \end{pmatrix} \ , \tag{2.9}$$

where μ^A are $N \times k$ matrices and $\mathcal{M}'^A_\alpha$ are Weyl-spinor-valued $k \times k$ matrices. The conjugates are defined by

$$\bar\mu^A \equiv (\mu^A)^\dagger \ , \qquad \bar{\mathcal{M}}'^{\alpha A} \equiv (\mathcal{M}'^A_\alpha)^\dagger \ , \tag{2.10}$$

and the constraint $\bar{\mathcal{M}}'^A_\alpha = \mathcal{M}'^A_\alpha$ is imposed. These fermionic collective coordinates are subject to analogues of the ADHM constraints (2.3):

$$\lambda^A_{\dot\alpha} \equiv \bar w_{\dot\alpha} \mu^A + \bar\mu^A w_{\dot\alpha} + [\mathcal{M}'^{\alpha A}, a'_{\alpha\dot\alpha}] = 0 \ . \tag{2.11}$$

[1]Here $A = 1,2,3,4$ is the spinor index of the $\mathrm{SU}(4)_R$ symmetry.

After this brief description of the SU(N) ADHM formalism we now turn to the other classical groups Sp(N) and SO(N).[2] In order to construct instanton solutions for gauge theories with these groups we can use the embeddings

$$\mathrm{Sp}(N) \subset \mathrm{SU}(2N) \ , \qquad \mathrm{SO}(N) \subset \mathrm{SU}(N) \ , \tag{2.12}$$

to extract the ADHM formalism for these groups in terms of the SU(N) ADHM construction. The surprising feature of the resulting formalisms is that the auxiliary group, U(k), in the SU(N) case and denoted generally as H(k), at instanton number k, in the general case is *not* in the same classical series as the gauge group G. Table 1 shows the auxiliary groups and defines the quantities N' and k' which allow us to present a unified treatment of Sp(N) and SO(N).

G	SU(N)	Sp(N)	SO(N)
H(k)	U(k)	O(k)	Sp(k)
N'	N	$2N$	N
k'	k	k	$2k$

Table 1. Gauge and associated auxiliary groups.

To describe the other classical groups we start with the theory with gauge group SU(N') at instanton instanton number k'. Instanton solutions in the Sp(N) and SO(N) theories follow by simply imposing certain reality conditions on the ADHM construction of the SU(N') theory which ensures that the gauge field lies in the correct $sp(N)$ and $so(N)$ subalgebra of $su(N')$. In order to deal with both the Sp(N) and SO(N) case at the same time it is useful to define the notion of a generalized transpose operation denoted t which acts either on gauge or instanton indices. Specifically, on Sp(n) group indices t acts as as a symplectic transpose, i.e. on a column vector v, $v^t = v^T J^T$, where J is $2n \times 2n$ the symplectic matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ ; \tag{2.13}$$

[2]For the orthogonal groups we restrict $N \geq 4$.

while on O(n) group indices t is a conventional transpose $t \equiv T$. The adjoint representations of both groups are hermitian t-anti-symmetric matrices with dimensions $n(2n+1)$ and $n(n-1)/2$, respectively. Hermitian t-symmetric matrices correspond to the *anti-symmetric* representation of Sp(n), with dimension $n(2n-1)$ and the symmetric representation of SO(N), with dimensions $n(n+1)/2$. For the symplectic groups t-(anti-)symmetric matrices are conventionally called (anti-)self-dual [18].

The reality conditions on the bosonic collective coordinates can then be written compactly as

$$\bar{w}^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}(w_{\dot\beta})^t \ , \qquad (a'_{\alpha\dot\alpha})^t = a'_{\alpha\dot\alpha} \ , \tag{2.14}$$

These reality conditions are only preserved by the subgroup H(k) $\subset$ U(k') of the auxiliary symmetry group. Given (2.2) and (2.14), one can see that the matrices a'_n are hermitian and t-symmetric, i.e., real symmetric in the case of auxiliary group O(k), and symplectic anti-symmetric in the case of auxiliary group Sp(k). It is easy to verify that the ADHM constraints (2.3) themselves are anti-hermitian t-anti-symmetric, in other words H(k) adjoint-valued. It is straightforward to show that these reality conditions are precisely what is required to render the gauge field t-anti-symmetric, in other words to restrict it to an $sp(N)$ and $so(N)$ subalgebra of $su(N')$.

The fermionic collective coordinates are subject to a similar set of reality conditions:

$$\bar\mu^A = (\mu^A)^t \ , \qquad (\mathcal{M}'^A_\alpha)^t = \mathcal{M}'^A_\alpha \ . \tag{2.15}$$

So $\mathcal{M}'^A_\alpha$ is t-symmetric. The fermionic ADHM constraints (2.11) are, like their bosonic counterparts t-anti-symmetric.[3]

We can now count the number of real physical bosonic and fermionic collective coordinates. For both Sp(N) and SO(N) at instanton number k there are $4kN$ independent w variables, taking into account the reality conditions. The number of a'_n variables is $4\times k(k+1)/2$ and $4\times k(2k-1)$, for Sp(N) and SO(N), respectively. The physical moduli space is then the space of these variables modulo the three H(k)-valued ADHM constraints (2.3) and auxiliary H(k) symmetry. Hence the dimension

[3] In proving this it is useful to notice that $(w^t_{\dot\alpha})^t = -w_{\dot\alpha}$ and $((\mu^A)^t)^t = -\mu^A$.

of the physical moduli space is $4k(N+1)$ and $4k(N-2)$, for $\mathrm{Sp}(N)$ and $\mathrm{SO}(N)$, respectively. This agrees with the counting via the index theorem. The counting of the fermionic sector of the physical moduli space goes as follows. For each A, there are $2kN$ real degrees-of-freedom in μ^A and $2\times k(k+1)/2$ and $2\times k(2k-1)$, in $\mathcal{M}'^A_\alpha$, for $\mathrm{Sp}(N)$ and $\mathrm{SO}(N)$, respectively. The ADHM constraints then impose $2\times k(k-1)/2$ and $2\times k(2k+1)$ conditions, for $\mathrm{Sp}(N)$ and $\mathrm{SO}(N)$, respectively. Hence there are $2k(N+1)$ and $2k(N-2)$ real physical fermionic collective coordinates for $\mathrm{Sp}(N)$ and $\mathrm{SO}(N)$, respectively, for each A. Again this agrees with the counting via the index theorem.

3 The Multi-Instanton Collective Coordinate Measure

In this section we write down the measure on the space of ADHM collective coordinates and then show how, for N large enough, both the bosonic and fermionic ADHM constraints can be explicitly resolved. This leads to a gauge invariant form for the measure which is then amenable to a large-N limit.

3.1 The ADHM multi-instanton 'flat' measure

In order to calculate physical quantities we need to know how to integrate on the space of ADHM variables. This is the measure induced from the full functional integral of the field theory. Thankfully, it turns out the measure is remarkably simple when written in terms of the ADHM variables: it is just the flat measure for all the variables with the ADHM constraints imposed via explicit delta functions [19, 20, 7]. In order to define the physical measure, we must divide by the volume of the auxiliary group $\mathrm{H}(k)$:

$$\int d\mu^k_{\rm phys} = \frac{a_k\,(C_1'')^k}{\mathrm{Vol}\,\mathrm{H}(k)}\int da'\,dw\prod_{A=1,2,3,4} d\mathcal{M}'^A\,d\mu^A\prod_{B=2,3,4} d\mathcal{A}^{1B}$$
$$\times\prod_{c=1,2,3}\delta\big((\tau^c)^{\dot\beta}{}_{\dot\alpha}D^{\dot\alpha}{}_{\dot\beta}\big)\prod_{A=1,2,3,4}\delta(\lambda^A_{\dot\alpha})\prod_{B=2,3,4}\delta(\mathbf{L}\cdot\mathcal{A}^{1B}-\Lambda^{1B}). \tag{3.1}$$

The form of the measure is an obvious generalization of the SU(N) measure [7, 20] to the other groups. The following points are worthy of mention:

(i) The integrals over the t-symmetric hermitian matrices a'_n and $\mathcal{M}'^A_\alpha$ are defined as the integrals over the components with respect to a basis R^r of hermitian t-symmetric matrices, normalized so that $\mathrm{tr}_{k'}\, R^r R^s = \delta^{rs}$.

(ii) The delta functions for the ADHM constraints and the integrals over the pseudo collective coordinates $\mathcal{A}^{AB}$ are defined with respect to a basis of t-anti-symmetric hermitian matrices, i.e. the Lie algebra of H(k), normalized so that $\mathrm{tr}_{k'}\, T^r T^s = \delta^{rs}$.

(iii) The constant factor a_k is fixed by clustering decomposition as discussed in the Appendix. Up to a factor of c^k, which can absorbed into $(C''_1)^k$, we prove

$$a_k = \begin{cases} 2^{-k^2/4} & \mathrm{Sp}(N)\ , \\ 2^{-2k^2} & \mathrm{SO}(N)\ . \end{cases} \tag{3.2}$$

(iv) The factor involving the k^{th} power of the constant C''_1 is not fixed by clustering. It can be fixed by a comparison with the explicit one instanton measure that can be deduced in a similar way to that of the one instanton SU(N) measure in [21] to yield

$$C''_1 = (a_1)^{-1} \times \begin{cases} \left(\frac{g^2}{2\pi^3}\right)^{2(N+1)} & \mathrm{Sp}(N) \\ \left(\frac{g^2}{2\pi^3}\right)^{2(N-2)} & \mathrm{SO}(N)\ . \end{cases} \tag{3.3}$$

(v) The pseudo collective coordinates $\mathcal{A}^{AB}$ are t-anti-symmetric, or H(k) Lie algebra-valued, matrices associated to the scalar fields of the theory. The operator $\mathbf{L}$ is a linear operator on the Lie algebra of H(k) with a positive spectrum, defined by

$$\mathbf{L}\cdot\Omega = \frac{1}{2}\{\Omega, W^0\} + [a'_n, [a'_n, \Omega]]\ . \tag{3.4}$$

The integrals over the $\mathcal{A}^{AB}$ can be done explicitly to yield

$$\int \prod_{B=2,3,4} d\mathcal{A}^{1B} \prod_{B=2,3,4} \delta\big(\mathbf{L}\cdot\mathcal{A}^{1B} - \Lambda^{1B}\big) = \big(\det \mathbf{L}\big)^{-3}\ . \tag{3.5}$$

However, the un-integrated form (3.1) makes the supersymmetry more manifest [20].

3.2 The multi-instanton action

The action for the theory evaluated on the instanton solution at leading order has the form

$$S^k_{\rm inst} \;=\; \frac{8\pi^2 k}{g^2}\; - ik\theta \;+\; S^k_{\rm quad}\;. \tag{3.6}$$

Here $S^k_{\rm quad}$ is a particular fermion quadrilinear term, with one fermion collective coordinate chosen from each of the four gaugino sectors $A = 1,2,3,4$ [7]:

$$S^k_{\rm quad} \;=\; \frac{\pi^2}{g^2}\,\epsilon_{ABCD}\,{\rm tr}_k\,\Lambda^{AB}\mathcal{A}^{CD} \;=\; \frac{\pi^2}{g^2}\,\epsilon_{ABCD}\,{\rm tr}_k\,\Lambda^{AB}\mathbf{L}^{-1}\Lambda^{CD}\;. \tag{3.7}$$

Here

$$\Lambda^{AB} \;=\; \frac{1}{2\sqrt{2}}\left(\bar{\mathcal{M}}^A\mathcal{M}^B - \bar{\mathcal{M}}^B\mathcal{M}^A\right)\,. \tag{3.8}$$

It is straightforward to show that Λ_{AB} is H(k) adjoint-valued, i.e., t-anti-symmetric.[4]

As in [7], the four-fermion interaction can be bi-linearized by introducing an SO(6) R-symmetry vector of hermitian t-anti-symmetric matrices χ_a (i.e. H(k) adjoint-valued), $a = 1,\ldots,6$. We can re-write χ_a as an anti-symmetric tensor by introducing the Clebsch-Gordon coefficients Σ^a_{AB} (see [7] for definitions):

$$\chi_{AB} = \frac{1}{\sqrt{8}}\Sigma^a_{AB}\chi_a\;. \tag{3.9}$$

The identity we need is[5]

$$\begin{aligned}&(\det\mathbf{L})^{-3}\exp\left(-S^k_{\rm quad}\right)\\ &= \pi^{-3k'(k'\mp1)}\int d\chi\,\exp\left[-{\rm tr}_{k'}\,\chi_a\mathbf{L}\chi_a + 4\pi i g^{-1}{\rm tr}_{k'}\,\chi_{AB}\Lambda^{AB}\right].\end{aligned} \tag{3.10}$$

[4] In order to show that Λ_{AB} is t-anti-symmetric, it is important to bear in mind that t just acts on gauge and instanton indices. Hence on a product of Grassmann quantities, say A and B, there is an extra minus sign under t-conjugation: $(AB)^t = -B^tA^t$.

[5] In this and subsequent equations the upper signs will correspond to gauge group Sp(N) and the lower signs to gauge group SO(N).

Here, the integral is defined with respect to the t-anti-symmetric basis: $\chi_a = \chi_a^r T^r$. Notice that this bi-linearization absorbs the $(\det \mathbf{L})^{-3}$ factor from the $\mathcal{A}^{AB}$ integrals (3.5).

3.3 The gauge invariant measure

It is convenient to change variables to a gauge invariant set. Since we will be integrating gauge invariant quantities, the integral over gauge transformation yields a volume factor. The gauge invariant variables are encoded in the $2k' \times 2k'$ matrix W:

$$W^{\dot\alpha}{}_{\dot\beta} = \bar w^{\dot\alpha} w_{\dot\beta} \ . \tag{3.11}$$

Associated to this are four $k' \times k'$ hermitian matrices defined via

$$W^0 = \mathrm{tr}_2\, W, \quad W^c = \mathrm{tr}_2\, \tau^c W, \quad c = 1,2,3 \ . \tag{3.12}$$

Using the reality conditions (2.14) one finds that in addition to being hermitian

$$(W^0)^t = W^0 \ , \qquad (W^c)^t = -W^c \ . \tag{3.13}$$

The bosonic ADHM (2.3) constraints are then linear in W^c:

$$0 = W^c + i[a'_n, a'_m]\mathrm{tr}_2\, \tau^c \bar\sigma^{nm} = W^c - i[a'_n, a'_m]\bar\eta^c_{nm} \ , \tag{3.14}$$

where $\bar\eta^c_{nm}$ is a 't Hooft eta-symbol [7].

When $N' \geq 2k'$, we can change variables from the w's to the W's yielding a Jacobian as well as a numerical factor that reflects the volume of the gauge group action:

$$\int_{\substack{\text{gauge}\\\text{coset}}} dw \ = \ c_{k,N} \int (\det_{2k'} W)^{N'/2 - k'^2/2 \pm 1/4} \, dW^0 \prod_{c=1,2,3} dW^c \ . \tag{3.15}$$

We will only need the numerical pre-factor in the large N limit which can be evaluated by integrating both sides of (3.15) again a suitable exponential test function to give

$$c_{k,N} \underset{N\to\infty}{=} 2^{-2k'^2 \pm k'} (N'/\pi)^{-2kN + k'^2 \mp k'/2} e^{2kN} \ . \tag{3.16}$$

This change of variables brings with it a significant advantage: we can integrate out the W^c variables using the delta functions in (3.1) since the latter are linear in the former.

On the fermionic side we need to integrate out the superpartners of the gauge degrees-of-freedom. To isolate these variables we expand

$$\mu^A = w_{\dot\alpha}\zeta^{\dot\alpha A} + w_{\dot\alpha}\sigma^{\dot\alpha A} + \nu^A \ , \tag{3.17}$$

where the ν^A modes—the ones we need to integrate out—are in the subspace orthogonal to the w's, i.e., $\bar w^{\dot\alpha}\nu^A = 0$, and the variables $\zeta^{\dot\alpha A}$ and $\sigma^{\dot\alpha A}$ are t-symmetric and t-anti-symmetric $k' \times k'$ matrices, respectively. Notice that the ν^A modes do not appear in the fermionic ADHM constraints (2.11). The change of variables from μ^A to $\{\zeta^A, \sigma^A, \nu^A\}$ involves a Jacobian:

$$\int \prod_{A=1,2,3,4} d\mu^A = \int \left(\det{}_{2k'} W\right)^{-2\hat k} \prod_{A=1,2,3,4} d\zeta^A \, d\sigma^A \, d\nu^A \ . \tag{3.18}$$

We can now integrate out the ν^A variables:

$$\begin{aligned}&\int \prod_{A=1,2,3,4} d\nu^A \, \exp\left[\sqrt{8}\pi i g^{-1} \mathrm{tr}_{k'}\, \chi_{AB} (\nu^A)^t \nu^B\right] \\ &= 2^{6kN-6k'^2} (\pi/g)^{4kN-4k'^2} \left(\det{}_{4k'}\chi\right)^{N'/2-k'} \ .\end{aligned} \tag{3.19}$$

Furthermore, the fermionic ADHM constraints can then be used to integrate out the t-anti-symmetric variables $\sigma^{\dot\alpha A}$, however the result is rather cumbersome to write down and is in any case is not needed at this stage. Fortunately in the large-N limit to be described shortly it simplifies considerably.

4 The Measure in the Large-N Limit

The gauge invariant form of the measure is now amenable to a large N limit. As in [7], we write terms which have the form of "something to the power N" as $\exp(-N' S_{\text{eff}}/2)$, for some "effective large-N action" S_{eff}, and then perform a saddle-point approximation by minimizing the action with respect to the variables. In order to achieve this, we have to re-scale the χ variables:

$$\chi_a \to \sqrt{N'/2}\,\chi_a. \tag{4.1}$$

The effective action has three contributions

$$S_{\text{eff}} = -\log\det{}_{2k'} W - \log\det{}_{4k'}\chi + \mathrm{tr}_{k'}\,\chi_a \mathbf{L} \chi_a \, , \tag{4.2}$$

up to constant which we keep track of separately.

The resulting saddle-point equations are identical to those in [7] for the SU(N) case:

$$\epsilon^{ABCD}\left(\mathbf{L}\chi_{AB}\right)\chi_{CE} = \tfrac{1}{2}\delta^D_E 1_{[k']\times[k']} \ , \tag{4.3a}$$
$$\chi_a\chi_a = \tfrac{1}{2}(W^{-1})^0 \ , \tag{4.3b}$$
$$[\chi_a,[\chi_a,a'_n]] = i\bar\eta^c_{nm}[a'_m,(W^{-1})^c] \ . \tag{4.3c}$$

Using the SU(N) case [7] as a guide, we can write down the solutions to these equations. As in the $\mathcal{N}=4$ case we look for a solution with $W^c=0$, $c=1,2,3$, which means that the instantons are embedded in mutually commuting SU(2) subgroups of the gauge group. In this case equations (4.3c) are equivalent to

$$[a'_n,a'_m] = [a'_n,\chi_a] = [\chi_a,\chi_b] = 0 \ , \quad W^0 = \tfrac{1}{2}(\chi_a\chi_a)^{-1} \ . \tag{4.4}$$

The final equation can be viewed as giving the value of W^0, whose eigenvalues are the instanton sizes at the saddle-point. Clearly $\chi_a\chi_a$ and W^0 must be non-degenerate.

For Sp(N) it is easy to see that a well-defined solution only exists for k even. The point is that the set of commuting anti-symmetric matrices χ_a of odd dimension have a common null eigenvector and so $\chi_a\chi_a$ has no inverse contrary to hypothesis.[6] From now on, for Sp(N) we shift $k\to 2k$ and so $k'=k$ for both Sp(N) and SO(N) and we will replace k' with k. The general solution with instanton number $2k$, up to action of the auxiliary O($2k$) symmetry, is in block form

$$\begin{aligned} W^0 &= \mathrm{diag}\big(2\rho_1,\ldots,2\rho_k\big)\otimes 1_{[2]\times[2]} \ , \\ a'_n &= \mathrm{diag}\big(-X^1_n,\ldots,-X^k_n\big)\otimes 1_{[2]\times[2]} \ , \\ \chi_a &= \mathrm{diag}\big(\rho_1^{-1}\hat\Omega^1_a,\ldots,\rho_k^{-1}\hat\Omega^k_a\big)\otimes\tau^2 \ . \end{aligned} \tag{4.5}$$

In the above the $\hat\Omega^i_a$ are unit $SO(6)$ vectors. The form of this solution is preserved by the following discrete element of the auxiliary group:

$$1_{[k]\times[k]}\otimes\tau^1 \in \mathrm{O}(2k) \ , \tag{4.6}$$

[6] For k odd one of the eigenvalues of W^0 has to be infinite and although the action is formally finite the N-independent pre-factor in the measure is zero. We do investigate this rather pathological solution in any more detail.

which has the effect of changing the sign of $\hat{\Omega}^i_a$. In other words $\{\rho_i, X^i_n, \hat{\Omega}^i_a\}$ parametrizes the positions of k point-like objects, to be identified with the D-instantons of the dual string theory, in $AdS_5 \times S^5/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ acts by inversion. This latter space is nothing but the five-dimensional projective space $\mathbb{R}P^5$.

For SO(N) a solution exists for all instanton numbers and, up to the auxiliary Sp(k) symmetry, is of the block form

$$\begin{aligned} W^0 &= \mathrm{diag}\big(2\rho_1, \dots, 2\rho_k\big) \otimes 1_{[2]\times[2]} \ , \\ a'_n &= \mathrm{diag}\big(-X^1_n, \dots, -X^k_n\big) \otimes 1_{[2]\times[2]} \ , \\ \chi_a &= \mathrm{diag}\big(\rho_1^{-1}\hat{\Omega}^1_a, \dots, \rho_k^{-1}\hat{\Omega}^k_a\big) \otimes \tau^3 \ . \end{aligned} \tag{4.7}$$

As in the Sp(N) case there is a discrete transformation

$$1_{[k]\times[k]} \otimes i\tau^2 \in \mathrm{Sp}(k) \ , \tag{4.8}$$

that fixes the form of the solution but reverses the signs of the unit six-vectors $\hat{\Omega}^i_a$. So the solution describes the position of k D-instantons in $AdS_5 \times \mathbb{R}P^5$.

In principle, we have to expand the effective action around the general solutions written down in the last section to sufficient order to ensure that the fluctuation integrals converge. In general because the Gaussian form has zeroes whenever two D-instantons coincide one has to go to quartic order in the fluctuations. Fortunately, as explained in [7], we do not need to expand about the most general solution to the saddle-point equations to quartic order since this is equivalent to expanding to the same order around the most degenerate solution where all the D-instantons are at the same point in $AdS_5 \times \mathbb{R}P^5$. The resulting quartic action has flat directions corresponding to the relative positions of the D-instantons. However, when the fermionic integrals are taken into account the integrals over these relative positions turn out to be convergent and hence these degrees-of-freedom should be viewed as fluctuations around rather than facets of the maximally degenerate solution. The variables left un-integrated, since they are not convergent, can be viewed as centre-of-mass coordinates.

The maximally degenerate solution for both cases can be written as

$$W^0 = 2\rho^2\, 1_{[2k]\times[2k]} \ , \qquad \chi_a = \rho^{-1}\hat{\Omega}_a\, S \ , \qquad a'_n = -X_n\, 1_{[2k]\times[2k]} \ , \tag{4.9}$$

where

$$S = 1_{[k]\times[k]} \otimes \begin{cases} \tau^2 & \mathrm{Sp}(N) \ , \\ \tau^3 & \mathrm{SO}(N) \ . \end{cases} \tag{4.10}$$

The parameters $\{\rho, X_n, \hat{\Omega}_a\}$ will be identified with centre-of-mass coordinates. This solution is left invariant by a subgroup of $\mathrm{U}(k) \subset \mathrm{H}(k)$ consisting of transformations that fix S. For $\mathrm{Sp}(N)$ it is generated by the elements

$$\mathrm{Re}(a_{[k]\times[k]}) \otimes \tau^2 \quad \text{and} \quad i\,\mathrm{Im}(a_{[k]\times[k]}) \otimes 1_{[2]\times[2]} \tag{4.11}$$

and for $\mathrm{SO}(N)$ by elements

$$a_{[k]\times[k]} \otimes \tau^3 \ . \tag{4.12}$$

In both (4.11) and (4.12) a is a hermitian $k \times k$ matrix.

To construct the final form of the measure at large N, we have to expand the effective action around the maximally degenerate solution to sufficient order to ensure that the fluctuation integrals are convergent. As in the $\mathrm{SU}(N)$ case [7], some variables need only to be expanded to Gaussian order whilst for the remainder, we must go to quartic order. Although many details are identical to the $\mathrm{SU}(N)$ case in Ref. [7] there are differences which we explain below:

(i) As in [7], we separate out the non-convergent integrals over the centre-of-mass coordinates $(\rho, X_n, \hat{\Omega}_a)$ and expand the ADHM variables around the maximally degenerate saddle-point solution as

$$a'_n = -X_n 1_{[2k]\times[2k]} + \delta a'_n \ , \tag{4.13a}$$

$$W^0 = 2\rho^2 1_{[2k]\times[2k]} + \delta W^0 \ , \tag{4.13b}$$

$$\chi_a = \rho^{-1}\hat{\Omega}_a S + \delta\chi_a \ , \tag{4.13c}$$

It is useful to employ the further decomposition

$$\delta a'_n = \tilde{a}'_n + \hat{a}'_n \ , \qquad \delta\chi_a = \tilde{\chi}_a + \hat{\chi}_a \ , \tag{4.14}$$

where $\tilde{a}'_n$ and $\tilde{\chi}_a$ are the fluctuations which do not commute with S (in fact they anti-commute) and $\hat{a}'_n$ and $\hat{\chi}_a$ are the remainder that do commute with S.

Similarly, we separate out the integrals over the 16 supersymmetric and super-conformal zero-modes ξ^A_α and $\bar{\eta}^{\dot{\alpha}A}$, respectively, that are not lifted by the quadrilinear term (3.7), and expand the fermionic variables in a similar way to the bosonic variables:

$$\mathcal{M}'^{A}_{\alpha} = 4\xi^{A}_{\alpha}1_{[2k]\times[2k]} + \tilde{\mathcal{M}}'^{A}_{\alpha} + \hat{\mathcal{M}}'^{A}_{\alpha} \,, \qquad (4.15a)$$

$$\zeta^{\dot\alpha A} = 4\bar\eta^{\dot\alpha A}1_{[2k]\times[2k]} + \tilde\zeta^{\dot\alpha A} + \hat\zeta^{\dot\alpha A} \,, \qquad (4.15b)$$

where we have introduced the same decomposition for the fermions as for the bosons in Eq. (4.14).

(ii) The expansion of the $\log\det_{8k}\chi$ term in the saddle-point action can be read-off from the SU(N) expression [7] with the following observation. In the present situation χ is being expanded around S, rather than the identity in the U(N) case, but since $S^2 = 1$ we can write

$$\chi_a = \rho^{-1}\hat\Omega_a S + \delta\chi_a = S(\rho^{-1}\hat\Omega_a 1_{[2k]\times[2k]} + S\delta\chi_a) \,. \qquad (4.16)$$

Moreover $\det_{8k}\chi = \det_{8k}(S\chi)$ and so we can use our previous SU(N) formulae for expanding $\det_{8k}\chi$ with the replacement $\delta\chi_a \to S\delta\chi_a$.

(iii) The expansion of the effective action to Gaussian order around the saddle-point solution is

$$S^{(2)} = -\frac{1}{\rho^2}\mathrm{tr}_{2k}\,[S,\delta a'_n]^2 - \rho^2\mathrm{tr}_{2k}\,[S,\delta\chi^{\perp}_a]^2 + \mathrm{tr}_{2k}\left((2\rho^2)^{-1}\delta W^0 + \rho\{S,\hat\Omega_a\delta\chi_a\}\right)^2. \qquad (4.17)$$

In the above $\delta\chi^{\perp}_a$ is the component of $\delta\chi_a$ orthogonal to the unit vector $\hat\Omega_a$. Writing the fluctuations in terms of the decompositions (4.14), this becomes

$$S^{(2)} = \frac{4}{\rho^2}\mathrm{tr}_{2k}\,(\tilde a'_n)^2 + 4\rho^2\mathrm{tr}_{2k}\,(\tilde\chi^{\perp}_a)^2 + \mathrm{tr}_{2k}\,\varphi^2, \qquad (4.18)$$

where

$$\varphi = 2\{S,\hat\Omega_a\delta\chi_a\} + \frac{1}{2\rho^2}\delta W^0. \qquad (4.19)$$

Notice that the Gaussian terms lift all the fluctuations δW^0, $\tilde a'_n$ and $\tilde\chi_a$ except the component of $\tilde\chi_a$ along $\hat\Omega_a$, which we denote $\tilde\chi^{\parallel}_a$. In fact this variation corresponds precisely to the action of infinitesimal H(k) transformations on the maximally degenerate solution (4.9) although notice that as stated above the subgroup U(k) leaves the solution invariant. The observation is that any fluctuation $\tilde\chi^{\parallel}_a$ can be written as $\hat\Omega_a[S,\epsilon]$

for some $\epsilon \in \mathrm{H}(k)$ (the Lie algebra of $\mathrm{H}(k)$) and conversely, infinitesimal $\mathrm{H}(k)$ transformations generate all such $\tilde{\chi}_a^{\|}$. We can 'gauge-fix' this symmetry by the condition $\tilde{\chi}_a^{\|} = 0$ which results in a Jacobian factor that involves the volume of the coset $\mathrm{H}(k)/\mathrm{U}(k)$.

Since $\mathrm{tr}_{2k}(\tilde{x}\hat{y}\hat{z}) = 0$, for three matrix quantities x, y and z with the decomposition (4.14), at leading order in $1/N$ the variables $\tilde{a}_n'$ and $\tilde{\chi}_a$ are completely decoupled from the other fluctuations and we can proceed to integrate them. Taking into account the gauge-fixing Jacobian we have

$$\begin{aligned}&\int d\tilde{a}_n'\, d\tilde{\chi}_a \, \exp\big(-2N'\mathrm{tr}_{2k}\,[\rho^{-2}(\tilde{a}_n')^2 + \rho^2(\tilde{\chi}_a^{\perp})^2]\big)\\ &= 2^{9k^2/2\mp k/2}(a_k')^{-1}\frac{\mathrm{Vol}\,\mathrm{H}(k)}{\mathrm{Vol}\,\mathrm{U}(k)}\left(\frac{N'}{\pi}\right)^{-5k^2} .\end{aligned} \tag{4.20}$$

Here the constant a_k' is related to one defined in (3.2) with $a_k' = a_{2k}$, for $Sp(N)$, and $a_k' = a_k$, for $SO(N)$.

(iv) The situations for the fermions is very similar. First of all we fulfill our promise to deal with the fermionic ADHM constraints. To leading order in $1/N$, these constraints read

$$2\rho^2\sigma_{\dot\alpha}^A = -\tfrac{1}{2}[\delta W^0, \zeta_{\dot\alpha}^A] - [\mathcal{M}'^{\alpha A}, a_{\alpha\dot\alpha}'] . \tag{4.21}$$

So the integrals over the $\sigma^{\dot\alpha A}$ variables soak up the delta-functions imposing the fermionic ADHM constraints, as promised. In (4.21) δW^0 are the fluctuations in W^0 all of which are lifted at Gaussian order. Due to a cross term we can effectively replace δW^0 with $-4\rho^3\hat\Theta\cdot\chi$ at leading order (see [7]). Collecting all the leading order terms, the fermion couplings are

$$\begin{aligned}S_{\mathrm{f}} = i\Big(\frac{8\pi^2N'}{g^2}\Big)^{1/2}\mathrm{tr}_{2k}\Big[&-2\rho^2(\hat\Theta\cdot\chi)\hat\Theta_{AB}\zeta^{\dot\alpha A}\zeta_{\dot\alpha}^B\\ &+\rho^{-1}\hat\Theta_{AB}\big[a_{\alpha\dot\alpha}', \mathcal{M}'^{\alpha A}\big]\zeta^{\dot\alpha B} + \chi_{AB}\big(\rho^2\zeta^{\dot\alpha A}\zeta_{\dot\alpha}^B + \mathcal{M}'^{\alpha A}\mathcal{M}'^B_{\alpha}\big)\Big] .\end{aligned} \tag{4.22}$$

It is straightforward to see that the variables $\tilde{\mathcal{M}}'^A$ and $\tilde\zeta^A$ couple directly to the saddle-point solution and can be integrated-out directly:

$$\int \prod_{A=1,2,3,4} d\tilde{\mathcal{M}}'^{A}\, d\tilde{\zeta}^{A} \cdot \exp\left(2N'^{1/2}\pi g^{-1}\hat{\Omega}_{AB}\mathrm{tr}_{2k}[\rho^{-1}S\tilde{\mathcal{M}}'^{A}_{\alpha}\tilde{\mathcal{M}}'^{\alpha B} + \rho S\tilde{\zeta}^{A}_{\dot\alpha}\tilde{\zeta}^{\dot\alpha B}]\right) = 2^{k^2\pm 2k}\left(\frac{N'}{g^2\pi^2}\right)^{k^2\pm 2k} . \tag{4.23}$$

(v) The remaining variables on the bosonic side are φ, $\hat{a}'_n$ and $\hat{\chi}_a$. Fortunately the action for these variables, to leading order is exactly as in the SU(N) case [7] with the replacement, pointed out in (ii), $\hat{\chi}_a \to S\hat{\chi}_a$. However, since all the hatted variables commute with S in the final expressions this replacement is equivalent to to leaving $\hat{\chi}_a$ unchanged and instead taking $\hat{a}'_n \to S\hat{a}'_n$, $\hat{\mathcal{M}}'^{\alpha A} \to S\hat{\mathcal{M}}'^{\alpha A}$ and $\hat{\zeta}^{\dot\alpha A} \to S\hat{\zeta}^{\dot\alpha A}$. This is more convenient because then all the quantities $\{S\hat{a}'_n, \hat{\chi}_a, S\hat{\mathcal{M}}'^{\alpha A}, S\hat{\zeta}^{\dot\alpha A}\}$ are adjoint-valued in SU(k) $\subset$ H(k). The variables φ can be integrated out at Gaussian order, as is evident from (4.18), however as in [7] we have to take account of a coupling between φ and a bi-linear in the remaining variables $\hat{a}'_n$ and $\hat{\chi}_a$. Thus the φ integral yields a quartic term in these variables which along with the quartic terms in the expansions of the two determinant factors in (4.2) gives a remarkably simple final result. As in [7] the remaining fluctuations can be assembled into a ten-dimensional SU(k) gauge field with components

$$A_\mu = (N'/4)^{1/4}\big(\rho^{-1}S\hat{a}'_n, \rho\hat{\chi}_a\big), \tag{4.24}$$

and the quartic action for these variables is precisely action of ten-dimensional SU(k) gauge theory dimensionally reduced to zero dimensions:

$$NS_{\mathrm{b}}(A_\mu) \;=\; -\mathrm{tr}_{2k}\,[A_\mu, A_\nu]^2 \; . \tag{4.25}$$

(vi) Not surprisingly the remaining fermionic variables fermions can be assembled into a Majorana-Weyl fermion of the dimensionally reduced ten-dimensional theory with sixteen components

$$\Psi = \sqrt{\frac{\pi}{2g}}(N'/4)^{1/8}\big(\rho^{-1/2}S\hat{\mathcal{M}}'^{A}_{\alpha}, \rho^{1/2}S\hat{\zeta}^{\dot\alpha A}\big) \; . \tag{4.26}$$

The coupling of these fermions to the gauge field A_μ completes the action (4.25) to that of a ten-dimensional $\mathcal{N}=1$ supersymmetric SU(k)

gauge theory dimensionally reduced to zero dimensions:

$$S(A_\mu,\Psi) = -\mathrm{tr}_{2k}\,[A_\mu,A_\nu]^2 + 2\mathrm{tr}_{2k}\,\bar\Psi\Gamma_\mu\,[A_\mu,\Psi]\ , \tag{4.27}$$

where, in the way described in [7], the ten-dimensional Gamma matrices Γ_μ depend explicitly on the unit vector $\hat\Omega_a$.

Our final result for the measure in the large-N limit is[7]

$$\int d\mu^k_{\rm phys}\, e^{-S^k_{\rm inst}} \underset{N\to\infty}{=} \frac{\sqrt{N}g^8e^{4\pi ik\tau}}{k^3\pi^{9k^2/2+8}\mathrm{Vol}\,\mathrm{U}(k)}\int\frac{d\rho\,d^4X}{\rho^5}\,d^5\hat\Omega\prod_{A=1,2,3,4}d^2\xi^A d^2\bar\eta^A\cdot\hat{\mathscr{Z}}_k\ . \tag{4.28}$$

Here $\hat{\mathscr{Z}}_k$ is the partition function of ten-dimensional $\mathcal{N}=1$ supersymmetric SU(k) gauge theory dimensionally reduced to zero dimensions:[8]

$$\hat{\mathscr{Z}}_k \ = \int_{\mathrm{SU}(k)} d^{10}A\,d^{16}\Psi\,e^{-S(A_\mu,\Psi)}\ . \tag{4.29}$$

To emphasize the above result pertains to (i) Sp(N) in the sector with even instanton charge $2k$ and (ii) SO(N) in all the charge sectors with charge k. Notice that the form of the large-N measure is identical to that of of the charge k sector of the SU(N) theory.

At leading order in $1/N$ where any operator insertions take their saddle-point values the SU(k) partition function $\hat{\mathscr{Z}}_k$ is simply an overall constant factor that has been evaluated in Ref. [22]:[9]

$$\hat{\mathscr{Z}}_k = \big(\sqrt{2\pi}\big)^{10(k^2-1)}\big(\sqrt{2}\big)^{(16-10)(k^2-1)}\cdot\frac{2^{k(k+1)/2}\pi^{(k-1)/2}}{2\sqrt{k}\prod_{i=1}^{k-1}i!}\cdot\sum_{d|k}\frac{1}{d^2}\ . \tag{4.30}$$

[7]In this result we have not kept track of factors of 2 of the form $2^{c_1+c_2k}$ (c_1 and c_2 being constants) since 2^{c_1} only affects the overall normalization of the answer and 2^{c_2k} can obviously be absorbed into a multiplicative redefinition of the modular parameter $q=e^{2\pi i\tau}$; henceforth such a redefinition is understood.

[8]Our normalization for the partition is different from that in [7].

[9]We have written the result in a way which allows an easy comparison with [22]. The factors of $\sqrt{2\pi}$ and $\sqrt{2}$ arise, respectively, from the difference in the definition of the bosonic integrals and the normalization of the generators: we have $\mathrm{tr}_k\,T^rT^s=\delta^{rs}$ rather than $\frac12\delta^{rs}$. The remaining factors are the result of [22].

Effectively for calculating correlation functions, we can take the large-N measure to be

$$\int d\mu^k_{\rm phys}\, e^{-S^k_{\rm inst}} \underset{N\to\infty}{=} \frac{\sqrt{N}g^8}{\pi^{27/2}} k^{-7/2} e^{4\pi i k\tau} \cdot \sum_{d|k} \frac{1}{d^2} \int \frac{d^4X\, d\rho}{\rho^5}\, d^5\hat{\Omega} \prod_{A=1,2,3,4} d^2\xi^A d^2\bar{\eta}^A \ . \tag{4.31}$$

To summarize, the large-N instanton measures for the charge $2k$ sector of the Sp(N) theory and the charge k sector of the SO(N) theory, have an identical form to that for the charge-k sector in the SU(N) theory. So to leading order in $1/N$, correlation functions will receive identical instanton contributions from these sectors.

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Appendix A: Cluster Decomposition

The clustering property of the k-instanton measure, namely the degeneration of the measure into the $k-1$ and 1 instanton measures when one instanton is far separated from the others, is crucial for determining the correct k dependent normalization in front of the measure (up to the constant $(C_1'')^k$). The clustering property has been extensively discussed for the case of gauge groups Sp(1) $\simeq$ SU(2) [19] and SU(N) in [17], so our discussion will be brief.

We will consider the complete clustering limit which describes a region of the moduli space where all the instantons are well separated from each other. In this limit, we require

$$\int d\mu^k_{\rm phys} \longrightarrow \frac{1}{k!}\int d\mu^1_{\rm phys} \times \cdots \times d\mu^1_{\rm phys} \ . \tag{A.1}$$

Intuitively, the complete clustering limit is the region of moduli space where the matrices $[a'_n, a'_m] \approx 0$ and where their eigenvalues, which specify the positions of the individual instantons are sufficiently part apart. We shall discover exactly what "sufficiently part apart" actually means as we proceed.

It is useful to use the auxiliary $\mathrm{H}(k)$ symmetry to simultaneously diagonalize the t-symmetric matrices a'_n matrices in the clustering limit. This amounts to "gauge-fixing" the coset $\mathrm{H}(k)/\mathrm{H}(1)^k$ of the auxiliary $\mathrm{H}(k)$ symmetry.[10] The remaining $\mathrm{H}(1)^k$ symmetry corresponds to the auxiliary symmetry groups of the k individual instantons. The diagonal elements of $X^i_n = (a'_n)_{ii}$ then specify the positions of the instantons (with $X^i = X^{i+k}$ for $\mathrm{Sp}(k)$). The gauge fixing involves a Jacobian factor:

$$\frac{1}{\mathrm{Vol}\,\mathrm{H}(k)}\int da' \longrightarrow \frac{c_k}{k!(\mathrm{Vol}\,\mathrm{H}(1))^k}\int\prod_{i=1}^k d^4X^i\,da'^{\perp}\prod_{1\le i<j\le k}|X^i-X^j|^{\beta}\,. \tag{A.2}$$

where $\beta = 1$, 2 and 4, for $\mathrm{O}(k)$, $\mathrm{U}(k)$ and $\mathrm{Sp}(k)$, respectively, and the constant c_k is

$$c_k = \begin{cases} 2^{k^2/4-3k/2} & \mathrm{O}(k)\ , \\ 2^{k(k+1)/2} & \mathrm{U}(k)\ , \\ 2^{2k^2-3k/2} & \mathrm{Sp}(k)\ , \end{cases} \tag{A.3}$$

and where [23]

$$\mathrm{Vol}\,\mathrm{H}(k) = \prod_{j=1}^k \frac{2\pi^{j\beta/2}}{\Gamma(j\beta/2)}\ . \tag{A.4}$$

In (A.2) $a'^{\perp}_n$ are the off-diagonal components of a'_n that are not generated by the adjoint action of $\mathrm{H}(k)$ on the diagonal matrices $\mathrm{diag}(X^1_n,\ldots,X^k_n)$, for $\mathrm{O}(k)$ and $\mathrm{U}(k)$, and $\mathrm{diag}(X^1_n,\ldots,X^k_n)\otimes 1_{[2]\times[2]}$ for $\mathrm{Sp}(k)$. Explicitly, these are the projections

$$(a'^{\perp}_n)_{ij} = (a'_n)_{ij} - \frac{(X^i-X^j)_m(a'_m)_{ij}}{|X^i-X^j|^2}(X^i-X^j)_n\ . \tag{A.5}$$

The relation (A.2) can easily be derived from the well-known Jacobian that arises from changing variables for an integral over the elements of

[10]This gauge fixing is a valid procedure because the measure is only used to integrate $\mathrm{H}(k)$ invariant functions.

a (i) real symmetric (ii) Hermitian or a (iii) Hermitian self-dual matrix A, for $O(k)$, $U(k)$ and $Sp(k)$, respectively (normalized with respect to the basis R^r), to its eigenvalues a^i (see for example [18]):

$$\int dA = \frac{c_k}{k!} \frac{\mathrm{Vol}\,\mathrm{H}(k)}{(\mathrm{Vol}\,\mathrm{H}(1))^k} \int \prod_{i=1}^{k} da^i \prod_{1 \le i < j \le k} |a^i - a^j|^\beta \ . \tag{A.6}$$

The clustering limit is defined as the limit where the off-diagonal bosonic ADHM constraints can be approximated by

$$(X^i - X^j)^{\dot{\alpha}\alpha} (a'^{\perp}_{\alpha\dot{\beta}})_{ij} + (\bar{w}^{\dot{\alpha}} w_{\dot{\beta}})_{ij} = \lambda_{ij} \delta^{\dot{\alpha}}_{\ \dot{\beta}} \ , \tag{A.7}$$

for arbitrary λ_{ij}, i.e., to order $\mathcal{O}(|X^i - X^j|^{-2})$ they are approximately linear in the elements $(a'^{\perp}_n)_{ij}$. In this limit, we can then easily integrate out these elements. Similarly, in the clustering limit, the off-diagonal fermionic ADHM constraints are approximately linear in the elements $(\mathcal{M}'_\alpha)_{ij}$; to order $\mathcal{O}(|X^i - X^j|^{-1})$

$$(X^i - X^j)_{\alpha\dot{\alpha}} (\mathcal{M}'^{\alpha})_{ij} + (\bar{\mu} w_{\dot{\alpha}})_{ij} + (\bar{w}_{\dot{\alpha}} \mu)_{ij} = 0 \ . \tag{A.8}$$

In the clustering limit, we can then easily integrate out the off-diagonal elements $(\mathcal{M}'_\alpha)_{ij}$.

After integrating-out the off-diagonal components of the ADHM constraints using (A.7) and (A.8), the k independent diagonal components of the constraints are the ADHM constraints for the k individual instantons. Taking careful account of numerical factors one finds that the measure in (3.1) must include the factor a_k, defined in (3.2) to cancel the factor of 2^{k^2} in (A.2). Notice that the factor of 2^k can be absorbed into the constant $(C''_1)^k$.

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