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Blowup formulae in Donaldson-Witten theory and integrable hierarchies

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Abstract

We investigate blowup formulae in Donaldson-Witten theory with gauge group $SU(N)$, using the theory of hyperelliptic Kleinian functions. We find that the blowup function is a hyperelliptic σ-function and we describe an explicit procedure to expand it in terms of the Casimirs of the gauge group up to arbitrary order. As a corollary, we obtain a new expression for the contact terms and we show that the correlation functions involving the exceptional divisor are governed by the KdV hierarchy. We also show that, for manifolds of simple type, the blowup function becomes a τ-function for a multisoliton solution.

1 Introduction

Blowup formulae [1] have played an important role in Donaldson-Witten theory. First of all, they relate the Donaldson invariants of a manifold X with those of its blownup $\widehat{X}$, and they have been a crucial ingredient in the derivation of explicit expressions for these invariants, their wall-crossings [2], [3], and their structural properties in the case of non-simple type manifolds [4]. Another important aspect of these formulae is that they give an explicit connection between the mathematical and the physical approach to Donaldson invariants. For example, in the derivation of the blowup formula for $SU(2)$ Donaldson invariants given in [1], the elliptic curve of the Seiberg-Witten solution [5], [6] appears in a natural way. Conversely, the result of [1] can be derived in a very elegant way within the framework of the u-plane integral of Moore and Witten [7].

Donaldson-Witten theory can be generalized to higher rank gauge groups using the approach of [7]. A detailed analysis of this theory for $SU(N)$ has been made in [8], and also in [9] from a slightly different point of view. In particular, one of the results of [8], [9] is a blowup formula for $SU(N)$ Donaldson theory, which is written in terms of theta functions[1]. It was already noticed in [8] that the blowup function

[1]In [9], the blowup formula was also derived in the $SU(2)$ case. The general formula for $SU(N)$ is implicit in the results presented there, and it was in fact used to obtain expressions for the contact terms.

is essentially a τ-function of the Toda-KP hierarchy, and reflects the underlying integrable structure of the low-energy effective theory [10], [11], [12], [13]. This relation between blowup functions and integrable hierarchies has been explored in [14], [15], [16].

In this paper, we shall analyze in full detail the properties and structure of the blowup formulae in $SU(N)$ Donaldson-Witten theory. As we will review below, an important aspect of blowup functions is that they must admit an expansion whose coefficients are polynomials in the Casimirs of the gauge group (equivalently, in the local observables of the corresponding topological theory). In the case of $SU(2)$, the fact that the expression for the blowup formula in terms of theta functions admits such an expansion is a result of the theory of elliptic functions, which also provides an explicit way of performing the expansion by using elliptic σ-functions.

In the case of $SU(N)$, it was argued in [8] that such an expansion should exist on physical grounds, but no recipe was given to perform the expansion. In this paper we solve this problem by using the hyperelliptic generalization of σ-functions and the theory of hyperelliptic Kleinian functions. This theory was developed at the end of nineteenth century by Klein, Baker, Bolza, and many others, but has completely dropped out of the textbooks. There has been recently some revival of this theory in connection with the algebro-geometric approach to integrable hierarchies [17], [18], [19], and as we will show in this paper, the theory of hyperelliptic Kleinian functions is the right framework to address the properties of the blowup functions in $SU(N)$ Donaldson-Witten theory. For example, the contact terms of two-observables are deeply related to the blowup function, as it was first realized in [9]. We will show that the theory of hyperelliptic Kleinian functions gives a simple expression for these contact terms as periods of certain meromorphic forms.

Another interesting aspect of this approach is that it makes possible to clarify further the connection to integrable hierarchies. We will show in detail that the blowup function, after a linear transformation of the coupling constants appearing in the u-plane integral, satisfies the differential equations of the KdV integrable system. As a corollary, the correlation functions involving the exceptional divisor on the blownup manifold are governed by the KdV hierarchy. This gives a formal connection to two-dimensional topological gravity [20].

As it is well-known, in the $SU(2)$ case the blowup formula has a simple structure when the manifold is of simple type, and it corresponds to the degeneration of elliptic functions to trigonometric functions [1]. In the $SU(N)$ case, the simple type condition corresponds to a maximal degeneration of the hyperelliptic curve. These degenerations are well-known in the algebro-geometric approach to integrable systems, and correspond to multisoliton solutions of the hierarchy (see, for example, [21], [22]). We will then show that the blowup function of $SU(N)$ becomes a τ-function for an $(N-1)$-soliton solution of the underlying KdV hierarchy. As a corollary of this analysis we will give explicit expressions for some physical quantities at the $\mathcal{N}=1$ points of $\mathcal{N}=2$ $SU(N)$ Yang-Mills theory.

The paper is organized as follows: in section 2, we review the basic results on blowup formulae in Donaldson-Witten theory for the gauge group $SU(N)$, following the results of [8], [9], [16]. In section 3, we introduce Kleinian functions and hyperelliptic σ-functions and some of their properties. In particular, we give a detailed account of the differential equations that they satisfy and we present a systematic way to solve them for any genus g. We apply these results to the Seiberg-Witten curve for $SU(N)$ in section 4, and we derive some new results on the contact terms of the twisted theory. We present explicit results for the expansion of the blowup functions for $g=2$ and $g=3$. In section 5, we explain the relation between the blowup function and the KdV hierarchy. We then consider, in section 6, the important case of manifolds of simple type, and we compute in full detail the blowup function at the $\mathcal{N}=1$ points. Finally, in section 7 we state our conclusions and prospects for future research in this subject.

2 The blowup function in twisted $\mathcal{N}=2$ super Yang-Mills

In this section we give a brief review of the blowup formula in twisted $\mathcal{N}=2$ Yang-Mills theory. A detailed account can be found in [8], [16].

Twisted $\mathcal{N}=2$ theories have a finite set of gauge-invariant operators called observables which can be understood as BRST cohomology classes. For $SU(N)$ gauge theories, the simplest observables are the $N-1$ Casimirs of the gauge group, which give a basis for the ring of local, BRST invariant operators of the theory. We will take these observables to be the elementary symmetric polynomials in the eigenvalues of the complex scalar field ϕ in the $\mathcal{N}=2$ vector multiplet:

$$\mathcal{O}_k = S_k(\phi_i) = \frac{1}{k}\mathrm{Tr}\phi^k + \cdots \qquad k=2,\ldots,N\ . \tag{2.1}$$

The advantage of these operators is that their vacuum expectation values are precisely the u_k that parametrize the Coulomb branch of the physical theory.

From the above operators one can generate the rest of the observables using the descent procedure. We will consider only simply connected manifolds, for simplicity. In this case, the other observables of the theory are associated to integrals over two-cycles S in the manifold X of differential forms constructed by acting on the Casimirs with a spin one (descent) operator G_μ,

$$I_k(S) = \int_S G^2\mathcal{O}_k = \frac{1}{k}\int_S \mathrm{Tr}(\phi^{k-1}F) + \cdots \tag{2.2}$$

Here, F is the Yang-Mills field strength. In general, S will be an arbitrary linear combination of basic two-cycles S_i, $i=1,\ldots,b_2(X)$, i.e., $S=\sum_{i=1}^{b_2(X)} t_i S_i$, therefore

$$I_k(S) = \sum_{i=1}^{b_2(X)} t_i I_k(S_i)\ . \tag{2.3}$$

In total, we have $(N-1)\cdot b_2(X)$ independent operators $I_k(S_i)$. The basic problem now is to compute the generating function for correlators involving the observables that we have just described, that is:

$$Z(p_k,f_k,S) = \left\langle \exp\left[\sum_k (f_k I_k(S) + p_k\mathcal{O}_k)\right]\right\rangle_X\ . \tag{2.4}$$

As it has been explained in [7] for $SU(2)$, and generalized in [8] to $SU(N)$, the computation of (2.4) can be done by using the low-energy exact solution of $\mathcal{N}=2$, $SU(N)$ Yang-Mills theory. This solution is encoded in the hyperelliptic curve describing a genus $g = N - 1$ Riemann surface Σ_g [23], [24]:

$$y^2 = P_N^2(x) - 4\Lambda^{2N} , \tag{2.5}$$

where

$$P_N(x) = x^N - \sum_{k=2}^{N} u_k x^{N-k} \tag{2.6}$$

is the characteristic polynomial of $SU(N)$, and $u_k = \langle \mathcal{O}_k \rangle$ are the VEVs of the Casimir operators (2.1). Associated to this curve there is a meromorphic differential of the second kind (also known as Seiberg-Witten differential), with a double pole at infinity, that can be explicitly written as:

$$dS_{SW} = P_N'(x) \frac{x dx}{y} . \tag{2.7}$$

This one-form satisfies the equation:

$$\frac{\partial dS_{SW}}{\partial u_{k+1}} = dv_k , \tag{2.8}$$

where

$$dv_k = \frac{x^{g-k} dx}{y} , \qquad k = 1, \ldots, g , \tag{2.9}$$

is a basis of holomorphic differentials for hyperelliptic curves of genus g. Given a symplectic basis of homology cycles $A^i, B_i \in H_1(\Sigma_g, \mathbb{Z})$ one may compute the period integrals of these differentials:

$$A^i{}_k = \frac{1}{2\pi i} \oint_{A^i} dv_k , \qquad B_{ik} = \frac{1}{2\pi i} \oint_{B_i} dv_k . \tag{2.10}$$

(Notice that, in contrast to [25], [26], [27], we have explicitly included the $2\pi i$ factors). Using these quantities we can define the period matrix of Σ_g as

$$\tau_{ij} = B_{ik} (A^{-1})^k{}_j . \tag{2.11}$$

The low-energy $\mathcal{N} = 2$ theory is described by a prepotential $\mathcal{F}(a^i, \Lambda)$, where the a^i variables, associated to the cycles A^i, are given by the integrals over these cycles of dS_{SW}

$$a^i(u_k, \Lambda) = \frac{1}{2\pi i} \oint_{A^i} \frac{x P_N'(x)}{\sqrt{P_N^2(x) - 4\Lambda^{2N}}} dx . \tag{2.12}$$

The same expression holds for the dual variables $a_{D,i} \equiv \partial\mathcal{F}/\partial a^i$, with B_i instead of A^i. The effective gauge couplings are given by (2.11). It follows from (2.8), (2.10) and (2.12) that

$$\frac{\partial a^i}{\partial u_{k+1}} = A^i{}_k\ , \qquad \frac{\partial a_{D,i}}{\partial u_{k+1}} = B_{ik}\ . \tag{2.13}$$

The blowup formula arises in the following context. Suppose that we have a four-manifold X, and we consider the blownup manifold at a point p, $\hat{X} = \mathrm{Bl}_p(X)$. Under this operation, the homology changes as follows (see, for example, [28]):

$$H_2(X) \to H_2(\hat{X}) = H_2(X) \oplus \mathbf{Z} \cdot B\ , \tag{2.14}$$

where B, the class of the exceptional divisor, satisfies $B^2 = -1$. Since the blownup manifold $\hat{X}$ has an extra two-homology class, there are extra operators $I_k(B)$ that must be included in the generating function. We will then write $\hat{S} = S + tB$. There is also the possibility of having a non-Abelian magnetic flux through the new divisor, and this flux is specified by a vector $\vec{\beta}$ with components of the form [8]:

$$\beta^i = (C^{-1})^i_j n^j, \tag{2.15}$$

where the n^j are integers, and $(C^{-1})^i_j$ is the inverse of the Cartan matrix for $SU(N)$. The generating function for the correlation functions on $\hat{X}$ is

$$\widehat{Z}_{\vec{\beta}}(p_k, f_k, t_k, \widehat{S}) = \left\langle \exp\left[\sum_k (f_k I_k(S) + t_k I_k(B) + p_k \mathcal{O}_k)\right]\right\rangle_{\hat{X},\vec{\beta}}\ , \tag{2.16}$$

where $t_k = t \cdot f_k$. The blowup formula states that this generating function is given by

$$\widehat{Z}_{\vec{\beta}}(p_k, f_k, t_k, \widehat{S}) = \left\langle \exp\left[\sum_k (f_k I_k(S) + p_k \mathcal{O}_k)\right] \tau_{\vec{\beta}}(t_k|\mathcal{O}_k)\right\rangle_X\ , \tag{2.17}$$

where $\tau_{\vec{\beta}}(t_k|\mathcal{O}_k)$ will be called the *blowup function.* This function is a series in the t_k whose coefficients are polynomials in the operators $\mathcal{O}_k$:

$$\tau_{\vec{\beta}}(t_k|\mathcal{O}_k) = \sum_{\vec{n}} t^{\vec{n}} \mathcal{B}_{\vec{n},\vec{\beta}}(\mathcal{O}_2, \ldots, \mathcal{O}_N)\ , \tag{2.18}$$

where $\vec{n} = (n_2, \ldots, n_N)$ is an $(N-1)$-uple of nonnegative integers, and $t^{\vec{n}} \equiv t_2^{n_2} \cdots t_N^{n_N}$. The order of the terms in the expansion (2.18) is given by $|\vec{n}| = \sum_i n_i$. The fact that such a formula exists can be justified intuitively by thinking about the blowup as a punctual defect which can be represented by an infinite series of local operators [7]. Since the ring of local, BRST invariant operators is generated by the $\mathcal{O}_k$, one would expect a factor like (2.18) relating the generating functions.

The precise expression for the blowup function $\tau_{\vec{\beta}}(t_k|\mathcal{O}_k)$ was derived in [7] for the gauge group $SU(2)$, and in [8], [9] in the general case of $SU(N)$, using the u-plane integral. To write the formula for this function, we will need to introduce the Riemann theta function $\Theta[\vec{\alpha}, \vec{\beta}](\vec{z}|\tau)$ with characteristics $\vec{\alpha} = (\alpha_1, \ldots, \alpha_g)$ and $\vec{\beta} = (\beta_1, \ldots, \beta_g)$, which we will take as:

$$\Theta[\vec{\alpha}, \vec{\beta}](\vec{z}|\tau) = \sum_{n_i \in \mathbb{Z}} \exp\left[i\pi\tau_{ij}(n_i + \beta_i)(n_j + \beta_j) + 2\pi i(n_i + \beta_i)(z_i + \alpha_i)\right]. \quad (2.19)$$

Then, the blowup function has the following form:

$$\tau_{\vec{\beta}}(t_i|u_i) = \mathrm{e}^{-\sum_{k,l} t_k t_l \mathcal{T}_{k,l}} \frac{\Theta[\vec{\Delta}, \vec{\beta}](\vec{\xi}|\tau)}{\Theta[\vec{\Delta}, \vec{0}](0|\tau)}, \quad (2.20)$$

where

$$\xi_i = \sum_{k=2}^{N} \frac{t_k}{2\pi} \frac{\partial u_k}{\partial a^i}, \qquad i = 1, \ldots, N-1. \quad (2.21)$$

We will consider $\vec{\beta} = \vec{0}$ most of the time (notice that, in general, the β_i won't be half-integers). The corresponding blowup function will be simply denoted by $\tau(t_i|u_i)$. In (2.20), we have introduced the symbol $\mathcal{T}_{k,l}$ to denote the contact terms associated to the observables $I_k(S)$. They are given by [9]:

$$\mathcal{T}_{k,l} = -\frac{1}{2\pi i} \partial_{\tau_{ij}} \log \Theta[\vec{\Delta}, \vec{0}](0|\tau) \frac{\partial u_k}{\partial a^i} \frac{\partial u_l}{\partial a^j}. \quad (2.22)$$

As first noticed in [9], the explicit expression for the contact terms can be deduced from the blowup function by requiring invariance under $\mathrm{Sp}(2r, \mathbb{Z})$ transformations, and taking also into account that they must vanish semiclassically [7]. In the $SU(2)$ case one recovers precisely the blowup formula of Fintushel and Stern [1]. As remarked in [16], one of the consequences of the semiclassical vanishing of the contact terms (or, equivalently, of the expression (2.22)) is that the quadratic terms in the "times" t_i in the blowup function *vanish* for $\vec{\beta} = \vec{0}$, i.e., the expansion (2.18) has the structure:

$$\tau(t_i|u_i) = 1 + \sum_{|\vec{n}|=4} \mathcal{B}_{(n_2,\dots,n_N)}(u_i)\, t_2^{n_2} \cdots t_N^{n_N} + \cdots . \tag{2.23}$$

This will be important later on.

3 A survey of the theory of hyperelliptic Kleinian functions

In the first half of this section we will review in some detail the basic constructions in the theory of hyperelliptic Kleinian functions. A very good modern survey is [17]. We will also rely heavily on the results by Bolza [29], [30], [31] and Baker [32], [33]. In the last subsection, we will develop a constructive procedure to expand an even half-integer hyperelliptic σ-function up to arbitrary order in the moduli of the curve following the centenarian footsteps of [31].

3.1 Hyperelliptic curves and Abelian differentials

The basic objects we need to develop the theory are Abelian differentials on a hyperelliptic curve. Although we will concentrate most of the time on the curve (2.5), we will attempt to give a summary of the general story and consider hyperelliptic curves of the "even" form

$$y^2 = f(x) = \sum_{i=0}^{2g+2} \lambda_i x^i, \tag{3.1}$$

describing a Riemann surface of genus g. The curve is said to be in *canonical form* when $\lambda_{2g+2} = 0$ and $\lambda_{2g+1} = 4$, and any curve of the form (3.1) can be put in such a form by a fractional linear transformation. A basis of Abelian differentials of the first kind is given by the set of g holomorphic 1-forms (2.9). To construct hyperelliptic σ-functions, we will also need a basis of Abelian differentials of the second kind. To introduce these differentials we construct a generating functional as follows. First, we consider a function $F(x_1, x_2)$ (sometimes called a Weierstrass polynomial) which is at most of degree $g+1$ both in x_1 and x_2, and satisfies the following conditions:

$$F(x_1, x_2) = F(x_2, x_1) \ , \quad F(x, x) = 2f(x) \ ,$$
$$\left(\frac{\partial F(x_1, x_2)}{\partial x_1}\right)_{x_1 = x_2} = f'(x_2) \ . \tag{3.2}$$

One then defines a basis of Abelian differentials of the second kind, $dr^k(x)$ through the identity:

$$\sum_{k=1}^{g} dv_k(x_1)\, dr^k(x_2)$$
$$= -\frac{1}{2y_1}\frac{\partial}{\partial x_2}\left(\frac{y_2}{x_1 - x_2}\right) dx_1\, dx_2 + \frac{F(x_1, x_2)}{4(x_1 - x_2)^2}\frac{dx_1\, dx_2}{y_1 y_2} \ , \tag{3.3}$$

and also a global Abelian differential form of the second kind

$$d\omega(x_1, x_2) = \frac{2y_1 y_2 + F(x_1, x_2)}{4(x_1 - x_2)^2}\frac{dx_1\, dx_2}{y_1\, y_2} \ , \tag{3.4}$$

which has a double pole at $x_1 = x_2$ with coefficient normalized to 1. We will consider three different choices of $F(x_1, x_2)$ in this paper:

1) The function used, for example, in [32], [17] is given by

$$F_{(1)}(x_1, x_2) = 2\lambda_{2g+2} x_1^{g+1} x_2^{g+2} + \sum_{i=0}^{g} x_1^i x_2^i (2\lambda_{2i} + \lambda_{2i+1}(x_1 + x_2)) \ , \tag{3.5}$$

and the corresponding basis is

$$dr^j = \sum_{k=g+1-j}^{g+j} (k + j - g)\ \lambda_{k-j+g+2}\, \frac{x^k dx}{4y} \ , \tag{3.6}$$

where j ranges from 1 to g.

2) A second choice expresses $F(x_1, x_2)$ in a way which is "covariant" with respect to an $\mathrm{Sl}(2,\mathbb{R})$ transformation of the x-coordinate, as we will explain in more detail in section 5. A convenient way to express this polynomial is through the use of a "symbolic" notation as follows. The equation for the hyperelliptic curve (3.1) is written as

$$y^2 = (\alpha_1 + \alpha_2 x)^{2g+2} \ , \tag{3.7}$$

so that

$$\lambda_p = \binom{2g+2}{p} \alpha_1^{2g+2-p} \alpha_2^p \ . \tag{3.8}$$

Of course the notation is symbolic in the sense that α_1 and α_2 are not defined as complex numbers. One now defines the so-called $(g+1)$–polar of the hyperelliptic curve as:

$$\begin{aligned} F_{(2)}(x_1, x_2) &= 2(\alpha_1 + \alpha_2 x_1)^{g+1} (\alpha_1 + \alpha_2 x_2)^{g+1} \\ &= 2 \sum_{p,q=1}^{g+1} \frac{\binom{g+1}{p}\binom{g+1}{q}}{\binom{2g+2}{p+q}} \lambda_{p+q} \, x_1^p x_2^q \ . \end{aligned} \tag{3.9}$$

We are not aware of the existence of a simple and closed expression for the corresponding meromorphic differentials though they can be easily computed case by case.

3) A third choice, due to Baker [34], and studied in detail by Bolza [31], will be particularly useful in this paper. It is well known that, for hyperelliptic curves, even and non-singular half-integer characteristics are in one to one correspondence with the factorizations of (3.1) in two polynomials of degree $g+1$, say $y^2 = Q(x)R(x)$. To this factorization we will associate the Weierstrass polynomial:

$$F_{(3)}(x_1, x_2) = Q(x_1)R(x_2) + Q(x_2)R(x_1) \ . \tag{3.10}$$

In the case of Seiberg-Witten hyperelliptic curves (2.5), the dr^j basis acquires a simple expression that will be discussed below.

It is not difficult to prove that two different choices of Weierstrass polynomial, $F(x_1, x_2)$, $\widehat{F}(x_1, x_2)$, both satisfying (3.2), are related by (see [33], p. 315)

$$F(x_1, x_2) - \widehat{F}(x_1, x_2) = 4(x_1 - x_2)^2 \psi(x_1, x_2) \ , \tag{3.11}$$

where $\psi(x_1, x_2)$ is a polynomial symmetric in x_1, x_2, and of degree at most $g-1$ in each variable. It can therefore be written as

$$\psi(x_1, x_2) = \sum_{i,j=1}^{g} d_{ij} x_1^{g-i} x_2^{g-j} , \tag{3.12}$$

where d_{ij} is symmetric in i, j. Inserting (3.11) in (3.3) we also obtain the relation between the different basis of second kind differentials

$$dr^j = \widehat{dr}^j + \sum_{k=1}^{g} d_{jk} dv_k . \tag{3.13}$$

Given a basis of differentials of the second kind dr^k, constructed from a Weierstrass polynomial $F(x_1, x_2)$, we define the following matrices of periods:

$$\eta^{ki} = -\frac{1}{2\pi i} \oint_{A^i} dr^k , \qquad \eta'^{k}{}_{i} = -\frac{1}{2\pi i} \oint_{B_i} dr^k . \tag{3.14}$$

These matrices generalize the usual $\eta_\alpha = \zeta(\omega_\alpha)$ of an elliptic curve, to a hyperelliptic curve. Notice that the biperiods of the global Abelian differential (3.4) can be written as

$$\oint_{A^i} \oint_{A^j} d\omega = 4\pi^2 \, A^i_{\ k} \, \eta^{kj} . \tag{3.15}$$

One can also prove a generalization of Legendre's relation (see, for example, [17]):

$$\eta = 2\kappa A , \qquad \eta' = 2\kappa B - \frac{1}{2}(A^{-1})^t , \tag{3.16}$$

where κ is a symmetric matrix.

3.2 Hyperelliptic σ-functions and Kleinian functions

We are now ready to introduce the key objects: the hyperelliptic σ-functions. To motivate the definition, recall that the usual elliptic

σ-functions can be written as quotients of theta functions with an extra exponential involving the η-periods (see, for example, [35]). This property suggests to define the hyperelliptic σ-functions in terms of theta functions. We need to choose a characteristic $[\vec{\alpha}, \vec{\beta}]$ for these functions, and a Weierstrass polynomial $F(x_1, x_2)$ to define a set of meromorphic Abelian differentials with their corresponding η-periods. The σ-function is then defined as:

$$\sigma^F[\vec{\alpha}, \vec{\beta}](\vec{v}) = \frac{1}{C} \exp\{v_i \kappa^{il} v_l\} \Theta[\vec{\alpha}, \vec{\beta}]((2\pi i)^{-1} v_l (A^{-1})^l{}_i | \tau) \ . \tag{3.17}$$

In the above equation, the matrix κ (see (3.16)) is given by

$$\kappa^{il} = \frac{1}{2} \eta^{ij} (A^{-1})^l{}_j \ , \tag{3.18}$$

and C is a nonzero modular form of weight $(1/2, 0)$ with respect to the action of $\mathrm{Sp}(2g, \mathbb{Z})$. When the characteristic $[\vec{\alpha}, \vec{\beta}]$ is even and nonsingular, a useful choice is the one made in [29]:

$$C = \Theta[\vec{\alpha}, \vec{\beta}](0|\tau) = (\det(A))^{1/2} \Delta_Q^{1/8} \Delta_R^{1/8} \ , \tag{3.19}$$

where we have used Thomae's formula [36] for the even characteristic associated to the splitting $y^2 = Q(x)R(x)$, and $\Delta_{Q,R}$ are the discriminants of the Q, R factors.

An important property of the σ-functions is that they are invariant under the action of the modular group $\mathrm{Sp}(2g, \mathbb{Z})$. On the other hand, for a fixed characteristic, σ-functions corresponding to different Weierstrass polynomials are related by

$$\sigma^{\widehat{F}}[\vec{\alpha}, \vec{\beta}](\vec{v}) = \exp\left(\frac{1}{2} \sum_{i,j} d_{ij} v_i v_j \right) \sigma^F[\vec{\alpha}, \vec{\beta}](\vec{v}) \ , \tag{3.20}$$

where σ^F has been defined with $F(x_1, x_2)$ and $\sigma^{\widehat{F}}$ has been defined with $\widehat{F}(x_1, x_2)$.

We are now ready to introduce the hyperelliptic *Kleinian functions* as derivatives of the σ-function:

$$\zeta_j^F[\vec{\alpha}, \vec{\beta}](\vec{v}) = \frac{\partial \ln \sigma^F[\vec{\alpha}, \vec{\beta}](\vec{v})}{\partial v_j}, \qquad \wp_{ij}^F[\vec{\alpha}, \vec{\beta}](\vec{v}) = -\frac{\partial^2 \ln \sigma^F[\vec{\alpha}, \vec{\beta}](\vec{v})}{\partial v_i \partial v_j}. \tag{3.21}$$

These functions generalize the Weierstrass $\zeta(z)$ and $\wp(z)$ to the hyperelliptic case, and in some cases they provide an explicit solution for Jacobi's inversion problem. Notice that they depend, again, on the choice of Weierstrass polynomial. In particular, one has that

$$\wp^{\widehat{F}}_{ij}[\vec{\alpha},\vec{\beta}](\vec{v}) = \wp^{F}_{ij}[\vec{\alpha},\vec{\beta}](\vec{v}) - d_{ij} \; . \tag{3.22}$$

One of the key aspects of the hyperelliptic Kleinian functions $\wp^{F}_{ij}[\vec{\alpha},\vec{\beta}](\vec{v})$ and of the σ-functions is that they satisfy differential equations which generalize those of the elliptic case like, for example, Weierstrass' cubic relation $(\wp'(u))^2 = 4\wp(u)^3 - g_2\wp(u) - g_3$. This will be the subject of the next subsection.

3.3 Differential equations for the hyperelliptic Kleinian functions

The relations involving the hyperelliptic Kleinian functions $\wp^{F}_{ij}[\vec{\alpha},\vec{\beta}](\vec{v})$ and their derivatives were originally studied by Baker in [32], [33]. The case of $g = 2$ was investigated in full detail in [32]. A generalization of this construction has been recently worked out in [17]. In this approach, one obtains a set of second order partial differential equations for the $\wp^{F}_{ij}[\vec{\alpha},\vec{\beta}](\vec{v})$ with respect to the "times" v_l, that in principle could be solved in a series expansion. This would give the series expansion for the σ-function in terms of the "times" and the moduli of the curve. The main difficulty to extend this method to higher genus is that the relevant differential equations are given in an implicit way, and even for $g = 3$ a lot of work is needed in order to extract the first few terms of the expansion (see, for example, [17] where the first two terms have been obtained for a special –singular– characteristic). It is important to notice that these differential equations, being of second order, are the same for the different characteristics. The choice of characteristic shows up in the choice of initial conditions for the equations.

For the derivatives of $\wp_{11}$ one can, however, write an explicit equation for arbitrary genus which will be useful later:

$$\begin{aligned}\wp_{111i} &= (6\wp_{11} + \lambda_{2g})\wp_{1i} + \frac{1}{4}\lambda_{2g+1}(6\wp_{i+1,1} - 2\wp_{i2} + \frac{1}{2}\delta_{i1}\lambda_{2g-1}) \\ &\quad + \frac{1}{2}\lambda_{2g+2}(6\wp_{i+2,1} - 6\wp_{i+1,2} + 2\wp_{i3} - \delta_{i1}\lambda_{2g-2} - \frac{1}{2}\delta_{i2}\lambda_{2g-3}) \; .\end{aligned} \tag{3.23}$$

The hyperelliptic Kleinian functions which are used in this equation are defined by means of the Weierstrass polynomial (3.5). Any other choice will amount, by (3.22), to a v-independent shift. In (3.23), the extra subindices denote derivatives with respect to the components v_i. This equation follows from [17], eq.(5.3), and when the curve is written in the canonical way, it reduces to Proposition 4.1 of the same paper.

A different approach to this problem has been taken in a series of papers by Bolza [29], [30], [31], who obtained a partial differential equation for *even* hyperelliptic σ-functions which can be explicitly written for any genus. First, Bolza derived an equation for the logarithmic derivative of the Kleinian functions $\wp^F_{ij}$. Let us consider a σ-function defined by the Weierstrass polynomial F, and by the –even and non-singular– characteristic $[\vec{\alpha},\vec{\beta}]$ associated to the factorization $y^2 = f(x) = Q(x)R(x)$. Then one has [29]:

$$\sum_{i,j} \wp^F_{ij}[\vec{\alpha},\vec{\beta}](0) x_1^{g-i} x_2^{g-j} = \frac{F(x_1,x_2) - Q(x_1)R(x_2) - Q(x_2)R(x_1)}{4(x_1-x_2)^2} \ . \tag{3.24}$$

This equation will be important later in order to identify the σ-function which is relevant to the blowup formula. Notice, in particular, that it tells us that $\wp^F_{ij}[\vec{\alpha},\vec{\beta}](0)$ vanishes when the Weierstrass polynomial is $F_{(3)}$.

We can now state Bolza's differential equation for an even σ-function. Let a be one of the $2g+2$ zeroes of (3.1). We first define the following functions:

$$(x-z)^{g-1} = \sum_j x^{g-j} h_j(z) \ , \tag{3.25}$$

and also the matrices $p^F_{ij}(a)$, $q^F_{ij}(a)$ through the relations:

$$\begin{aligned} \sum_{i,j=1}^{g} p^F_{ij}(a) x^{g-i} h_j(z) &= \frac{1}{2}\frac{(x-z)^{g-1}}{x-a} - \frac{1}{2}\frac{(a-z)^{g-1}}{f'(a)}\frac{F(x,a)}{(x-a)^2} \ , \\ \sum_{i,j=1}^{g} q^F_{ij}(a) x^{g-i} z^{g-j} &= \frac{1}{8}\Big(\frac{1}{x-a} + \frac{1}{z-a}\Big)\frac{F(x,z)}{(x-z)^2} \\ &\quad + \frac{1}{4}\frac{1}{(x-z)^2}\frac{\partial F(x,z)}{\partial a} - \frac{1}{8}\frac{F(x,a)F(z,a)}{f'(a)(x-a)^2(z-a)^2} \ . \end{aligned} \tag{3.26}$$

In this equation, the $'$ denotes derivative w.r.t. x. We can now state the differential equation satisfied by $\sigma^F[\vec{\alpha},\vec{\beta}](\vec{v})$ [29]:

$$\frac{\partial \sigma^F}{\partial a} = -\sum_{i,j=1}^{g} p_{ij}^F(a) v_i \frac{\partial \sigma^F}{\partial v_j} - \frac{1}{2}\sigma^F \sum_{i,j=1}^{g} q_{ij}^F(a) v_i v_j + \sum_{i,j=1}^{g} \frac{a^{2g-i-j}}{f'(a)} \left(\frac{\partial^2 \sigma^F}{\partial v_i \partial v_j} + \sigma^F \wp_{ij}^F(0) \right) , \qquad (3.27)$$

where we have dropped the characteristic to gain in clarity. This equation endowes recursive relations for the Taylor expansion of σ^F. In fact, the appearance of a set of recursive relations is immediate provided we replace our even σ-function by its Taylor expansion

$$\sigma^F[\vec{\alpha},\vec{\beta}](\vec{v}) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \varsigma_n(\vec{v}) , \qquad (3.28)$$

where $\varsigma_n(\vec{v})$ are homogeneous polynomials of degree $2n$ in v_l,

$$\sum_{i=1}^{g} v_i \frac{\partial \varsigma_n(\vec{v})}{\partial v_i} = 2n \, \varsigma_n(\vec{v}) . \qquad (3.29)$$

The recursive relation for the ς_n polynomials reads

$$\sum_{i,j=1}^{g} \frac{a^{2g-i-j}}{f'(a)} \frac{\partial^2 \varsigma_n}{\partial v_i \partial v_j} \qquad (3.30)$$
$$= 2n(2n-1)\left\{ \frac{\partial \varsigma_{n-1}}{\partial a} - \varsigma_{n-1} \sum_{i,j=1}^{g} \frac{a^{2g-i-j}}{f'(a)} \wp_{ij}^F(0) + \sum_{i,j=1}^{g} p_{ij}^F(a) v_i \frac{\partial \varsigma_{n-1}}{\partial v_j} + (n-1)(2n-3)\, \varsigma_{n-2} \sum_{i,j=1}^{g} q_{ij}^F(a) v_i v_j \right\} .$$

The main difficulty of these equations is that they involve the derivatives of σ^F or ς_n with respect to a branch point a, which is of little practical use. However, as we will see in what follows, one can deduce from (3.27) a differential equation involving the coefficients of the curve (3.1), which will allow us to give recursive relations for the expansion of the σ-functions relevant to the blowup formula. A final comment is in order. Due to the fact that $\varsigma_0 = 1$, one can already obtain a set of differential equations for the quadratic term from (3.30) that yields

$$\varsigma_1(\vec{v}) = -\sum_{i,j=1}^{g} \wp^F_{ij}(0) v_i v_j \ . \tag{3.31}$$

Thus, after (3.24), the quadratic contribution to the σ-function vanishes when $F = F_{(3)}$.

Let us fix for future convenience the Weierstrass polynomial to be $F_{(3)}$ (3.10) . This implies no lack of generality as long as (3.20) allows to go from a given polynomial to any other. The corresponding matrices $p^{(3)}_{ij}(a)$ and $q^{(3)}_{ij}(a)$ get further simplified to:

$$\begin{aligned} \sum_{i,j=1}^{g} p^{(3)}_{ij}(a) x^{g-i} h_j(z) &= \frac{(x-z)^{g-1}\Xi(a,a) - (a-z)^{g-1}\Xi(x,a)}{2\ \Xi(a,a)(x-a)} \ , \\ \sum_{i,j=1}^{g} q^{(3)}_{ij}(a) x^{g-i} z^{g-j} &= \frac{\Xi(x,z)\Xi(a,a) - \Xi(x,a)\Xi(z,a)}{8\ \Xi(a,a)(x-a)(z-a)} \ , \end{aligned} \tag{3.32}$$

where we have introduced the quantity $\Xi(x,z)$,

$$\Xi(x,z) = \frac{Q(x)R(z) - Q(z)R(x)}{x-z} \ , \tag{3.33}$$

that can be easily seen to be a symmetric polynomial of degree at most g in its variables. We shall assume in what follows that a is a root of $Q(x)$. Notice then that $\Xi(x,a) = Q(x)R(a)/(x-a) = -R(a)\partial Q(x)/\partial a$ and $\Xi(a,a) = f'(a) = Q'(a)R(a)$. We also have that $\wp^{(3)}_{ij}(0)$ vanishes. The recursive equation (3.30) can be written as

$$\begin{aligned} \sum_{i,j=1}^{g} a^{2g-i-j} \frac{\partial^2 \varsigma_n}{\partial v_i \partial v_j} = 2n(2n-1)\ \Xi(a,a) \left\{ \frac{\partial \varsigma_{n-1}}{\partial a} + \sum_{i,j=1}^{g} p^{(3)}_{ij}(a) v_i \frac{\partial \varsigma_{n-1}}{\partial v_j} \right. \\ \left. + (n-1)(2n-3)\ \varsigma_{n-2} \sum_{i,j=1}^{g} q^{(3)}_{ij}(a) v_i v_j \right\} \ . \end{aligned} \tag{3.34}$$

Now, let $\varphi(x)$ be a polynomial of degree $g+p$. Then, one has

$$\sum_{(a)} \frac{\Xi(x,a)}{\Xi(a,a)}\ \varphi(a) = \varphi(x) - \sum_{i=0}^{p-1} \mu_i x^i Q(x) \ , \tag{3.35}$$

for appropriate μ_i defined in such a way that the polynomial of the r.h.s. has degree g. This result comes immediately from the fact that both sides of (3.35) are equal when evaluated at any of the $g+1$ roots of $Q(x)$. Consider now the function

$$M(x,z) = \Xi(x,z)Q'(z) - \sum_{i=0}^{g-1} \mu_i(x) z^i Q(z) \ , \tag{3.36}$$

where, again, $\mu_i(x)$ are chosen in such a way that $M(x,z)$ is of degree g in variable z,

$$M(x,z) = \sum_{i=1}^{g+1} m_i(x) z^{g+1-i} \ . \tag{3.37}$$

It can be written, after (3.35), as

$$\begin{aligned} M(x,z) = \sum_{(a)} \frac{\Xi(z,a)}{\Xi(a,a)} M(x,a) &= -\sum_{(a)} \Xi(x,a) \frac{\partial Q(z)}{\partial a} \\ &= -\sum_{(a)} \Xi(x,a) \sum_{i=0}^{g+1} \frac{\partial q_i}{\partial a} z^{g+1-i} \ , \end{aligned} \tag{3.38}$$

where q_i are the coefficients of $Q(x)$. For a given function $\mathcal{G}$, we can replace z^{g+1-i} by $\partial \mathcal{G}/\partial q_i$ in (3.37)(3.38) with the result

$$\sum_{i=1}^{g+1} m_i(x) \frac{\partial \mathcal{G}}{\partial q_i} = -\sum_{(a)} \Xi(x,a) \frac{\partial \mathcal{G}}{\partial a} \ . \tag{3.39}$$

We now multiply (3.34) by $\Xi(x,a)/\Xi(a,a)$ and sum over (a). The l.h.s. as well as the last two terms of the r.h.s. are poynomials in a, while the remaining term is the above referred problematic derivative that we can now handle by means of (3.39). Conversely, we can instead consider a root b of the polynomial $R(x)$ and arrive to formulae analogous to (3.35)–(3.39) with b, R and r_i instead of a, Q and q_i, whereas $m_i(x)$ is replaced by, say, $-n_i(x)$ due to the change of sign of $\Xi(x,b)$ with respect to $\Xi(x,a)$. At the end of the day, the recursive relation can be brought to the following form

$$\begin{aligned} \mathcal{Z}[x, \varsigma_n(\vec{v})] = -2n(2n-1)\Big\{ &\Delta\varsigma_{n-1} - \mathcal{P}[x, \varsigma_{n-1}(\vec{v})] \\ &- (n-1)(2n-3)\ \varsigma_{n-2}\ \mathcal{Q}[x,\vec{v}] \Big\} \ , \end{aligned} \tag{3.40}$$

where the polynomials $\mathcal{Z}[x,\varsigma_n(\vec{v})]$, $\mathcal{P}[x,\varsigma_{n-1}(\vec{v})]$ and $\mathcal{Q}[x,\vec{v}]$,

$$\mathcal{Z}[x,\varsigma_n(\vec{v})] \equiv \sum_{(a)} \frac{\Xi(x,a)}{\Xi(a,a)} \sum_{i,j=1}^{g} a^{2g-i-j} \frac{\partial^2 \varsigma_n}{\partial v_i \partial v_j} + (a \to b) \ ,$$

$$\mathcal{P}[x,\varsigma_{n-1}(\vec{v})] \equiv \sum_{(a)} \Xi(x,a) \sum_{i,j=1}^{g} p_{ij}(a) v_i \frac{\partial \varsigma_{n-1}}{\partial v_j} - (a \to b) \ ,$$

$$\mathcal{Q}[x,\vec{v}] \equiv \sum_{(a)} \Xi(x,a) \sum_{i,j=1}^{g} q_{ij}(a) v_i v_j + (a \to b) \ , \tag{3.41}$$

should be computed as explained in (3.35), and the differential operator Δ is given by

$$\Delta\varsigma_{n-1} = \sum_{i=1}^{g+1} m_i(x) \frac{\partial \varsigma_{n-1}}{\partial q_i} + \sum_{i=1}^{g+1} n_i(x) \frac{\partial \varsigma_{n-1}}{\partial r_i} \ . \tag{3.42}$$

Notice that Δ involves derivatives with respect to all the coefficients of the hyperelliptic curve. Thus, when considering the setup provided by the Seiberg-Witten geometry, it will be necessary to retain the dependence of any quantity on the whole set of coefficients of the curve, provided one is interested in higher orders of the Taylor expansion. The procedure described above leads to a recursive computation of the hyperelliptic σ-function up to arbitrary order in time variables.

4 Expansion of the blowup function

We will show in this section that the formalism discussed above is the appropriate framework to address a detailed study of the blowup function.

4.1 The Seiberg-Witten geometry

We will be now more specific and focus on the hyperelliptic curve (2.5) describing the low-energy effective action of $\mathcal{N}=2$, $SU(N)$ super Yang-Mills theory. This curve can be written as follows:

$$y^2(x) = Q(x)R(x) \ , \tag{4.1}$$

where

$$Q(x) = P_N(x) - 2\Lambda^N , \qquad R(x) = P_N(x) + 2\Lambda^N . \tag{4.2}$$

The Weierstrass polynomial which is relevant to our problem is, as will be clear below, $F_{(3)}$ (3.10). It is not difficult to prove that the Abelian differentials of the second kind corresponding to this generating function are given by:

$$dr^j = \frac{1}{2} P_j'(x) P_N(x) \frac{dx}{y} , \qquad j = 1, \dots, N-1 . \tag{4.3}$$

From now on, unless the contrary is stated, the dr^j will denote the above differentials, i.e., we will assume that the basis of Abelian differentials is given by the generating function (3.10) for the specific case of the Seiberg-Witten curve (2.5). Notice that

$$dr^1 = \frac{1}{2N} dS_{SW} - \frac{1}{2N} \sum_{k=1}^{g} (k+1) u_{k+1} dv^k . \tag{4.4}$$

These Abelian differentials of the second kind are associated to the coordinates on the Jacobian v_i, $i = 1, \dots, N-1$, that appear in the expression for the σ-function (3.17). In this sense, they play the role of the differentials that define a Whitham hierarchy and a prepotential theory [37].

The connection to the Whitham hierarchy can be made more concrete by relating the differentials (4.3) to another basis of Abelian differentials of the second kind which will be useful later. This basis was introduced in [25], and is given by

$$d\hat{\Omega}_n = R_{n,N}(x) \frac{P_N'(x) dx}{y} , \tag{4.5}$$

where the polynomials $R_{n,N}(x)$, of degree n, are given by $R_{n,N}(x) = (P_N(x))_+^{\frac{n}{N}}$. In this equation, $(P_N(x))^{\frac{n}{N}}$ denotes the n/N-th power of the polynomial $P_N(x)$ understood as a Laurent series in x

$$P_N(x)^{\frac{n}{N}} = \sum_{m=-\infty}^{n} b_{m,n} x^n , \tag{4.6}$$

and the + suffix means that one only keeps the nonnegative powers of

x. One has, for example [25]:

$$\begin{aligned} R_{1,N}(x) &= x \ , \qquad R_{2,N}(x) = x^2 - \frac{2}{N}u_2 \ , \\ R_{3,N}(x) &= x^3 - \frac{3}{N}u_2 x - \frac{3}{N}u_3 \ . \end{aligned} \tag{4.7}$$

So, in particular, $d\hat{\Omega}_1 = dS_{SW}$. The relation between these polynomials and (4.3) is:

$$d\hat{\Omega}_n = \frac{2N}{n}\sum_{p=1}^{n} b_{n-N,p-N}\ dr^{N-p} - \sum_{m=1}^{N-1} a_{n,m} dv_m \ , \tag{4.8}$$

where

$$\begin{aligned} a_{n,m} = &\sum_{p=0}^{N-m-1} (N-m-p) \\ &\cdot \left(b_{n,p}\ u_{m+p} + \frac{N}{n}\sum_{k=1}^{n} b_{n-N,-k}\ u_{m+p-k}\ u_{N-p} \right) \ . \end{aligned} \tag{4.9}$$

taking $u_0 = -1$, $u_1 = 0$ and $u_{k>N} = u_{k<0} = 0$. For $N = 3$, for example, one finds:

$$\begin{aligned} d\hat{\Omega}_1 &= 6dr_1 + 2u_2 dv_1 + 3u_3 dv_2 \ , \\ d\hat{\Omega}_2 &= 3dr_2 + 3u_3 dv_1 + \frac{2u_2^2}{3} dv_2 \ . \end{aligned} \tag{4.10}$$

It is precisely the basis $d\hat{\Omega}_n$ the one that turns out to be relevant in the study of adiabatic deformations of the Seiberg-Witten solution within the framework of the Whitham hierarchy [25].

4.2 Blowup function and σ-functions. Contact terms revisited

We will only consider in this section the case of zero magnetic flux, so $\vec{\beta} = \vec{0}$ and the characteristic of the theta function is $[\vec{\Delta}, \vec{0}]$. This characteristic is the one associated to the splitting of the Seiberg-Witten curve given in (4.1) (see [25], [26]). In view of (3.17), we see that the blowup function (2.20) has the form of a σ-function. To make this comparison more precise, notice that, in the Seiberg-Witten context,

$$(A^{-1})^l{}_i = \frac{\partial u_{l+1}}{\partial a^i} , \qquad \kappa^{il} = \frac{1}{2}\eta^{ij}\frac{\partial u_{l+1}}{\partial a^j} . \tag{4.11}$$

This means that the "times" of the blowup function are related to the vector $\vec{v}$ in (3.17) just by $v_l = it_{l+1}$. We have to compare now the exponentials in (3.17) and (2.20). As we stressed at the end of section 2, when there is no non-Abelian magnetic flux through the exceptional divisor, i.e., the characteristic is $[\vec{\Delta}, \vec{0}]$, the quadratic terms in the expansion of the blowup function vanish (2.23). But this is precisely the behavior of the σ-function associated to the generating function (3.10), as it follows from (3.24) and (3.31). We then obtain the following results:

- The blowup function of $SU(N)$ Donaldson theory in the absence of magnetic flux is a hyperelliptic σ-function with characteristic $[\vec{\Delta}, \vec{0}]$ and with the Weierstrass polynomial given in (3.10),

$$\tau(t_i|u_i) = \sigma^{F_{(3)}}[\vec{\Delta}, \vec{0}](it_{l+1}) . \tag{4.12}$$

This identity, combined with the results of section 3, gives a rather explicit realization of the expansion (2.18). We will give concrete results for the lower genus hyperelliptic surfaces in the next subsection.

- The contact terms $\mathcal{T}_{k+1,l+1}$ are given by

$$\mathcal{T}_{k+1,l+1} = \kappa^{k,l} = -\frac{1}{8\pi i}\frac{\partial u_{l+1}}{\partial a^i}\oint_{A^i} P'_k(x)P_N(x)\frac{dx}{y} , \tag{4.13}$$

where κ is the matrix introduced in (3.16), and we have used the explicit expression for the dr^k given in (4.3). This result gives yet another remarkably simple form of writing the contact terms of $SU(N)$ twisted Yang-Mills theory, this time in terms of periods of Abelian differentials. Using (4.4), one obtains, for example:

$$\mathcal{T}_{2,\ell} = \frac{1}{4N}\left(\ell u_\ell - a^i\frac{\partial u_\ell}{\partial a^i}\right) , \tag{4.14}$$

for $l = 1, \ldots, N-1$. One can in fact explicitly check some of these expressions by using the results of [25], [26]. The starting point are the Whitham equations

$$\left(\frac{\partial u_k}{\partial \log \Lambda}\right)_{T_{n\geq 2}=0} = k u_k - a^i \frac{\partial u_k}{\partial a^i} , \qquad \left(\frac{\partial u_k}{\partial T_n}\right)_{T_{n\geq 2}=0} = -c^i_{(n)} \frac{\partial u_k}{\partial a^i} , \tag{4.15}$$

where, in the second equation, $n = 2, \ldots, N$, and

$$c^i_{(n)} = \frac{1}{2\pi i} \oint_{A^i} d\widehat{\Omega}_n . \tag{4.16}$$

In (4.15), the slow times T_n are the "hatted times" introduced in [26]. The Whitham equations in the above form can be easily deduced from equation (3.18) of [25] and the redefinition of Whitham times in [26]. Notice that these equations have already the flavor of (4.13), since they express the derivatives of the moduli with respect to the slow times in terms of A-periods of Abelian differentials of the second kind. The derivatives of the moduli entering in (4.15) are in fact closely related to the contact terms. In the formalism of [25], the natural duality-invariant coordinates are not the moduli u_{k+1}, but some combinations thereof:

$$\mathcal{H}_{k+1,l+1} = \frac{N}{kl} \mathrm{res}_{\infty} \left[(P_N(x))^{\frac{k}{N}} d(P_N(x))^{\frac{l}{N}}_{+} \right] . \tag{4.17}$$

The moduli u_{k+1} are substituted in this formalism by:

$$\mathcal{H}_{k+1} \equiv \mathcal{H}_{k+1,2} = u_{k+1} + g_{k+1}(u_2, \ldots, u_{k-1}) . \tag{4.18}$$

One has, for example:

$$\mathcal{H}_2 = u_2 , \qquad \mathcal{H}_3 = u_3 , \qquad \mathcal{H}_{3,3} = u_4 + \frac{N-2}{2N} u_2^2 . \tag{4.19}$$

The RG equations of [25] give explicit results for the derivatives of the $\mathcal{H}_{k+1}$:

$$\begin{aligned} \left(\frac{\partial \mathcal{H}_{k+1}}{\partial \log \Lambda}\right)_{T_{n\geq 2}=0} &= -2N \frac{\partial \mathcal{H}_2}{\partial a^i} \frac{\partial \mathcal{H}_{k+1}}{\partial a^j} \frac{1}{\pi i} \partial_{\tau_{ij}} \log \Theta[\vec{\Delta}, \vec{0}](0|\tau) , \\ \left(\frac{\partial \mathcal{H}_{k+1}}{\partial T_l}\right)_{T_{n\geq 2}=0} &= -(k+l)\mathcal{H}_{k+l} \\ &\quad - \frac{2N}{l} \frac{\partial \mathcal{H}_{k+1}}{\partial a^i} \frac{\partial \mathcal{H}_{l+1}}{\partial a^j} \frac{1}{\pi i} \partial_{\tau_{ij}} \log \Theta[\vec{\Delta}, \vec{0}](0|\tau) . \end{aligned} \tag{4.20}$$

Since the first equation in (4.20) also holds by substituting $\mathcal{H}_{k+1} \to u_{k+1}$, one can combine it with (2.22) and (4.15) to obtain precisely (4.14). In the same way, one can obtain expressions relating the $\mathcal{T}_{k,l}$ to the periods of the family of Abelian differentials (4.5), and then use (4.8) to check (4.13). For example, for $g = 2$ one finds:

$$\mathcal{T}_{3,3} = \frac{u_2^2}{9} - \frac{1}{12}\frac{\partial u_3}{\partial a^i}\, c^i_{(2)} \, . \tag{4.21}$$

Using now (4.10) one can explicitly check (4.13) for $SU(3)$ twisted Yang-Mills theory.

The expression (4.13) for the contact terms turns out to be very useful, since the differentials dr^j are rather explicit in comparison with the Abelian differentials $d\hat{\Omega}_n$ introduced in [25]. In particular, there are some cases in which (4.13) is more effective than the expression (2.22) involving theta functions. We will see an example in section 6. There, we treat in detail the case of manifolds of simple type, where contributions come only from those points of the moduli space where the maximal number of mutually local monopoles (dyons) get simultaneously massless.

4.3 Expansion of the blowup function for lower genus

It is instructive to consider in more detail the way in which the formalism of the previous section leads to an expansion of the blowup function as in (2.18) for hyperelliptic surfaces of lower genus. We already know the answer for the first two terms, since $\tau(t_i|u_i)$ is an even σ-function with generating function $F^{(3)}$: $\varsigma_0 = 1$ and $\varsigma_1 = 0$. The differential equations for the fourth order term are encoded in the relation

$$\mathcal{Z}_{SW}[x, \varsigma_2(\vec{v})] = 12\ \mathcal{Q}_{SW}[x, \vec{v}] \, , \tag{4.22}$$

where we use the subindex SW, to indicate that a given quantity has been evaluated in the Seiberg-Witten curve (4.1). The l.h.s. of (4.22) is given by

$$\mathcal{Z}_{SW}[x, \varsigma_2(\vec{v})] = 2\sum_{i,j=1}^{g} x^{2g-i-j}\varsigma_2^{(ij)} - \sum_{i=0}^{g-3}(\mu_i Q(x) + \nu_i R(x))x^i \, , \tag{4.23}$$

where μ_i and ν_i are constants (with respect to x though functions of the "times" $\vec{v}$) that reduce the degree of (4.23) as explained in (3.35), and we have defined

$$\varsigma_n^{(ij)} \equiv \frac{\partial^2 \varsigma_n}{\partial v_i \partial v_j} \,. \tag{4.24}$$

Notice that the second term of the r.h.s. in (4.23) vanishes for $g = 2$. We obtained, for example,

$$\begin{aligned} \mathcal{Z}_{SW}[x, \varsigma_2(\vec{v})] &= 2\Big[\varsigma_2^{(11)}\, x^2 + 2\varsigma_2^{(12)}\, x + \varsigma_2^{(22)}\Big] \,, \\ \mathcal{Z}_{SW}[x, \varsigma_2(\vec{v})] &= 2\Big[2\varsigma_2^{(12)}\, x^3 + (\varsigma_2^{(22)} + 2\varsigma_2^{(13)} + u_2 \varsigma_2^{(11)})\, x^2 \\ &\quad + (2\varsigma_2^{(23)} + u_3 \varsigma_2^{(11)})\, x + (\varsigma_2^{(33)} + u_4 \varsigma_2^{(11)})\Big] \,, \end{aligned} \tag{4.25}$$

for $g = 2$ and $g = 3$ respectively. Concerning $\mathcal{Q}_{SW}[x, \vec{v}]$, a closed expression does not seem to be feasible. In the cases of lower genus, we found

$$\begin{aligned} \mathcal{Q}_{SW}[x, \vec{v}] &= -4\Lambda^6 (v_2^2\, x^2 + 4 v_1 v_2\, x + v_1^2 + u_2 v_2^2) \,, \\ \mathcal{Q}_{SW}[x, \vec{v}] &= -4\Lambda^8 (4 v_2 v_3\, x^3 + (6 v_1 v_3 + 4 v_2^2 - u_2 v_3^2)\, x^2 \\ &\quad + (4 v_1 v_2 + 3 u_3 v_3^2)\, x + (u_4 + u_2^2) v_3^2 + v_1^2 - 2 v_3 (u_2 v_1 - u_3 v_2)). \end{aligned} \tag{4.26}$$

Inserting these polynomials in (4.22) results in a set of differential equations for ς_2 that can be easily solved. For example, in the case of $g = 3$, i.e., $N = 4$, the resulting expansion for the blowup function is

$$\begin{aligned} \tau_{SU(4)}(t_i | u_i) = 1 - \frac{\Lambda^8}{12} \Big[& u_2^2 t_4^4 - 4 u_2 t_4^3 t_2 + 4 u_3 t_4^3 t_3 \\ & + 6 t_2^2 t_4^2 + 12 t_2 t_3^2 t_4 + 2 t_3^4 \Big] + \cdots \,. \end{aligned} \tag{4.27}$$

In the case $g = 2$, i.e., $N = 3$ it is interesting to work out in detail the next-to-leading order in the expansion. Notice that it is only from the sixth order term in the Taylor expansion of the blowup function that the full complexity of (3.40) enters into the game. Thus, for the sake of checking the recursive procedure that we derived in the previous section we must compute ς_3. The relevant equation is

$$\mathcal{Z}_{SW}[x, \varsigma_3(\vec{v})] = -30 \Big\{ \Delta \varsigma_2 |_{SW} - \mathcal{P}_{SW}[x, \varsigma_2(\vec{v})] \Big\} \,, \tag{4.28}$$

where we must include the full dependence of ς_2 in the coefficients of a generic hyperelliptic curve before applying the differential operator Δ.

The second term of the r.h.s. in (4.28) vanishes for $g = 2$. On the other hand, the term in the l.h.s. is exactly as (4.25) provided we replace ς_2 by ς_3. The final answer for the blowup function up to sixth order in the "times" is

$$\begin{aligned}\tau_{SU(3)}(t_i|u_i) = 1 &- \frac{\Lambda^6}{12}\Big[u_2 t_3^4 + 6t_2^2 t_3^2\Big] \\ &- \frac{\Lambda^6}{360}\Big[3t_2^6 - 15u_2 t_2^4 t_3^2 - 60u_3 t_2^3 t_3^3 - 15u_2^2 t_2^2 t_3^4 \\ &- 12u_2 u_3 t_2 t_3^5 - u_2^3 t_3^6 + 3u_3^2 t_3^6 - 12\Lambda^6 t_3^6\Big] + \cdots . \end{aligned} \tag{4.29}$$

Notice that $\tau(t_i|u_i)$ is homogeneous of degree zero provided we assign a negative weight $1-i$ to variables t_i. We will use the expansions (4.27) and (4.29) in section 6 below to check the expressions for the blowup functions in the case of manifolds of simple type.

5 Relation with the KdV hierarchy

In this section, we will show that the blowup function satisfies the differential equations of the KdV hierarchy. More precisely, we will show that, after redefining the times through a linear transformation, we obtain a g-gap solution of the KdV hierarchy. This is essentially a consequence of Theorem 4.6 of [17] (which we review below), but some extra work is needed in order to adapt it to our context. We first analyze the effect of special linear transformations on the hyperelliptic σ-functions, and then we establish the relation with the KdV hierarchy. A similar relation has been pointed out in [15].

5.1 $\mathrm{Sl}(2,\mathbb{R})$ covariance of the σ-functions

Consider a hyperelliptic curve of degree $2g+2$ written in the symbolic form (3.7), and perform an $\mathrm{Sl}(2,\mathbb{R})$ transformation of the x-variable:

$$x = \frac{a+bt}{c+dt}\,, \qquad bc - ad = 1\ . \tag{5.1}$$

The curve (3.7) becomes

$$Y^2 = (\beta_1 + \beta_2 t)^{2g+2} = \sum_{i=0}^{2g+2} \widehat{\lambda}_i t^i \ , \qquad (5.2)$$

where

$$Y = (c + dt)^{g+1} y \ , \quad \beta_1 = c\alpha_1 + a\alpha_2 \ , \quad \beta_2 = d\alpha_1 + b\alpha_2 \ . \qquad (5.3)$$

It is clear that one can always choose the $\mathrm{Sl}(2, \mathbb{R})$ transformation in such a way that the new curve is in canonical form, i.e., such that

$$\widehat{\lambda}_{2g+2} = \beta_2^{2g+2} = 0, \qquad \widehat{\lambda}_{2g+1} = \beta_1 \beta_2^{2g+1} = 4 \ . \qquad (5.4)$$

We will now analyze the changes induced by this transformation in the rest of the objects defining the σ-functions. First, we consider the Abelian differentials of the first kind (2.9). Since

$$x^{g-i} \frac{dx}{y} = (a + bt)^{g-i} (c + dt)^{i-1} \frac{dt}{Y} \ , \qquad (5.5)$$

for $i = 1, \ldots, g$, it follows that

$$dv_i(x) = \Lambda_i^m d\hat{v}_m(t) \ , \qquad (5.6)$$

where the matrix Λ_i^m can be obtained from the $\mathrm{Sl}(2, \mathbb{R})$ matrix by using (5.5). This matrix is invertible, since one can explicitly construct an inverse by writing $t = (cx - a)/(b - dx)$. It follows from (5.5) that the periods of the Abelian differentials of the first kind transform as:

$$A^i_j = \widehat{A}^i_m \Lambda_j^m \ , \qquad B_{ij} = \widehat{B}_{im} \Lambda_j^m \ , \qquad (5.7)$$

and therefore the period matrix τ remains invariant under this transformation (in the above equations, the hat refers to the periods of the curve (5.2)).

Let us now examine the η-periods. We have to make now a choice of Weierstrass polynomial, and to achieve covariance under $\mathrm{Sl}(2, \mathbb{R})$ we take (3.9). It is easy to check that

$$F_{(2)}(x_1, x_2) = (c + dt_1)^{-g-1} (c + dt_2)^{-g-1} F_{(2)}(t_1, t_2) \ . \qquad (5.8)$$

Therefore, the normalized global Abelian differential of the second kind (3.4) also remains invariant. From (3.15), one can then deduce the transformation properties of the η-periods:

$$\widehat{\eta}^{ij} = \Lambda_k^{\ i} \eta^{kj} \; . \tag{5.9}$$

We can now examine the properties of the σ-function under these transformations. Define

$$\widehat{v}_l = (\Lambda^{-1})_l^{\ m} v_m \; , \tag{5.10}$$

which is nothing but a linear transformation of the "evolution times". Using the above results, we find that

$$\sigma^F[\vec{\alpha}, \vec{\beta}](v_l)_{(x,y)} = \sigma^F[\vec{\alpha}, \vec{\beta}](\widehat{v}_l)_{(t,Y)} \; , \tag{5.11}$$

where F denotes here the polar Weierstrass polynomials associated to the corresponding curves. This is the key result that we will need. An important corollary of (5.11) is that, after substituting $v_l = \Lambda_l^{\ m} \widehat{v}_m$, the σ-function $\sigma^F[\vec{\alpha}, \vec{\beta}](v_l)_{(x,y)}$ satisfies the same differential equations than $\sigma^F[\vec{\alpha}, \vec{\beta}](\widehat{v}_l)_{(t,Y)}$ with respect to the hatted times.

5.2 The KdV hierarchy

One of the key results of [17] is that the hyperelliptic Kleinian functions satisfy the equations of the KdV hierarchy, when the curve is written in a canonical form, and when the Weierstrass polynomial is given by (3.5). This can be easily deduced from (3.23). When $\widehat{\lambda}_{2g+2} = 0$, $\widehat{\lambda}_{2g+1} = 4$, the equation becomes:

$$\wp_{111i} = (6\wp_{11} + \widehat{\lambda}_{2g})\wp_{1i} + 6\wp_{i+1,1} - 2\wp_{2i} + \frac{1}{2}\delta_{i1}\widehat{\lambda}_{2g-1} \; . \tag{5.12}$$

Take now $\mathcal{U} = 2\wp_{11} + \frac{1}{6}\widehat{\lambda}_{2g}$, put $x \equiv v_1$ and let $t_i = v_i$ be the higher evolution times. The equation (5.12) reads:

$$\frac{\partial \mathcal{U}}{\partial t_2} = \frac{1}{4}\mathcal{U}''' - \frac{3}{2}\mathcal{U}\mathcal{U}' \; , \tag{5.13}$$

where $'$ denotes derivatives w.r.t. x. (5.13) is precisely the KdV equation. It is easy to prove that in fact $\mathcal{U}$ solves the KdV hierarchy, or, more precisely, that it is a g-gap solution of the hierarchy. To see this, recall that the higher evolution equations of the KdV hierarchy are (for a review, see Appendix A of [38]):

$$\frac{\partial \mathcal{U}}{\partial t_i} = R_i'(\mathcal{U}, \mathcal{U}', \dots) , \qquad i \geq 3 , \tag{5.14}$$

where the functions in the right hand side are defined recursively as follows:

$$R_{i+1}' = \frac{1}{4} R_i''' - (\mathcal{U} + c) R_i' - \frac{1}{2} \mathcal{U}' R_i , \tag{5.15}$$

and c is a constant. The equations (5.14) and (5.15) with $c = \widehat{\lambda}_{2g}/12$ can be easily checked using again (5.12) and

$$\wp_{111}\wp_{1i} - \wp_{11i}\wp_{11} + \wp_{11,i+1} - \wp_{12i} = 0 , \tag{5.16}$$

which is obtained from (5.12) by imposing $\partial_i \wp_{1111} = \partial_1 \wp_{111i}$.

We can now state our main result about the relation of the blowup function to the KdV hierarchy. Taking into account (4.12) and (5.11), we can write:

$$\tau(v_m = \Lambda_m{}^l \widehat{v}_l | \mathcal{O}_i) = \mathrm{e}^{\sum_{ij} c_{ij} \widehat{v}_i \widehat{v}_j} \sigma^F[\vec{\Delta}, \vec{0}](\widehat{v}_l)_{(t,Y)} , \tag{5.17}$$

where the σ-function in the right hand side has been defined using the Weierstrass function (3.5), and the linear transformation Λ has been chosen in such a way that the hyperelliptic curve (t, Y) is written in a canonical form. The c_{ij} are constants depending on the parameters of the $\mathrm{Sl}(2, \mathbb{R})$ transformation and the moduli of the curve, and they can be computed explicitly. They simply arise as in (3.20), by comparing σ-functions defined for different Weierstrass polynomials. Using the results above, we finally find that

$$\mathcal{U} = -2 \frac{\partial^2 \log \tau}{\partial \widehat{v}_1^2} + 4c_{11} + \frac{1}{6} \widehat{\lambda}_{2g} \tag{5.18}$$

is a g-gap solution of the KdV hierarchy. In other words, the blowup function is, up to a redefinition of the evolution times and the shift in (5.18), a τ-function of the KdV hierarchy. Remember that the blowup function appears in fact in the generating function of the correlation functions involving the exceptional divisor. A corollary of the above is that these correlation functions on the manifold $\widehat{X}$ are governed by the KdV hierarchy, and they have as initial conditions the generating function of the original manifold X.

In [1], the blowup function of $SU(2)$ Donaldson-Witten theory was obtained precisely by solving a differential equation. The above result shows that the generalization to $SU(N)$ involves the KdV hierarchy. In fact, we can now recognize *a posteriori* the differential equation of [1] as the reduction of the KdV equation, whose quasi-periodic solutions are of course elliptic functions. It is interesting to notice that the differential equations governing the blowup behavior of $SU(N)$ topological Yang-Mills theory in four dimensions turn out to be essentially the same than the equations governing the correlation functions of two-dimensional topological gravity [20]. This is yet another manifestation of the intimate relationship between 4d $\mathcal{N} = 2$ theories and 2d physics[2].

6 Manifolds of simple type and multisoliton solutions

6.1 $\mathcal{N} = 1$ points

There are points in the moduli space of the hyperelliptic curve where one has maximal degeneration, *i.e.*, all the B_i cycles collapse. These points are usually called, in the context of $\mathcal{N} = 2$ gauge theories, the $\mathcal{N} = 1$ points, since these are the confining vacua that one obtains after breaking $\mathcal{N} = 2$ down to $\mathcal{N} = 1$. The physics of these points in pure Yang-Mills theory has been studied in detail in [39], and some aspects have been addressed in [40] from the point of view of the Whitham hierarchy. In this subsection we will rederive some of the results of [39], [40] by using the approach of [21], section 4.4. In particular, we will obtain a compact expression for the leading contribution of the off-diagonal magnetic couplings near the $\mathcal{N} = 1$ points.

The $\mathcal{N} = 1$ points of the $\mathcal{N} = 2$ gauge theory are described by Chebyshev polynomials. The polynomial $P_N(x)$ becomes [3], at a point

[2] Although the integrable hierarchy is the same, the generating function of 2d topological gravity and the blowup function are of course very different. For example, the former contains an infinite number of times corresponding to the gravitational cohomology classes, while the latter is a g-gap solution with only a finite number of times turned on.

[3] We set for convenience $\Lambda = 1$ along this section.

of maximal degeneration,

$$P_N(x) = 2\cos\left(N \arccos\frac{x}{2}\right) , \tag{6.1}$$

and the other $\mathcal{N} = 1$ points are obtained using the $\mathbb{Z}_N$ symmetry of the theory. From now on we will focus on the $\mathcal{N} = 1$ point corresponding to (6.1). The branch points of the curve are now the single branch points $e_1 = -e_{2g+2} = 2$, and the double branch points are:

$$e_{2k} = e_{2k+1} = \widehat{\phi}_k = 2\cos\frac{\pi k}{N} , \qquad k = 1, \ldots, g . \tag{6.2}$$

The values of the Casimirs at this degeneration are given by the elementary symmetric polynomials of the eigenvalues $2\cos\frac{\pi(k-1/2)}{N}$, $k = 1, \ldots, N$ [39]. For example,

$$u_2 = N, \quad u_3 = 0, \quad u_4 = \frac{N}{2}(3 - N) . \tag{6.3}$$

When the curve degenerates in the way specified by (6.1), the B_i cycles surround the points $\widehat{\phi}_i$ clockwise, while the A^i cycles become curves going from $\widehat{\phi}_i$ to 2 on the upper sheet and returning to $\widehat{\phi}_i$ on the lower sheet. The hyperelliptic curve (3.1) becomes

$$y = \sqrt{x^2 - 4}\prod_{k=1}^{g}(x - \widehat{\phi}_k) . \tag{6.4}$$

Consider now the normalized "magnetic" holomorphic differentials:

$$\omega^i = (B^{-1})^{ki} dv_k = \frac{\varphi^i(x)dx}{y} . \tag{6.5}$$

Then, it follows from (2.10) that

$$\frac{1}{2\pi i}\oint_{B_j}\omega^i = -\mathrm{res}_{x=\widehat{\phi}_j}\omega^i = \delta^i_j . \tag{6.6}$$

Using the explicit expressions (2.9) and (6.4), we find:

$$\varphi^j(x) = -2i\sin\frac{\pi j}{N}\prod_{l\neq j}(x - \widehat{\phi}_l) , \tag{6.7}$$

and

$$\omega^j = -\frac{2i\sin\frac{\pi j}{N}}{\sqrt{x^2 - 4}\,(x - \widehat{\phi}_j) .} \tag{6.8}$$

Let $S_0 = 1$, $S_j = \sum_{i_1<\cdots<i_j} x_{i_1}\cdots x_{i_j}$ be the elementary symmetric polynomial of degree j. From (6.5), (6.7) and (4.11) one deduces:

$$\frac{\partial u_{\ell+1}}{\partial a_{D,m}} = 2i(-1)^{\ell}\sin\frac{\pi m}{N}\, S_{\ell-1}(\widehat{\phi}_{p\neq m}) \ . \tag{6.9}$$

One can in fact check that this expression agrees with the results of [39]. Indeed, one can rederive from (6.9) equation (5.3) of [26].

Near the $\mathcal{N}=1$ points, the diagonal components of the "magnetic" couplings diverge, but the off-diagonal components are finite. The leading terms of the off-diagonal components have been investigated in [39], where an implicit expression for them was proposed in terms of an integral involving a scaling trajectory. In [40] it was shown that the Whitham hierarchy gives some nontrivial constraints on these terms, and an explicit expression satisfying the constraints was proposed. We will now derive a very simple expression for the leading terms of the off-diagonal couplings. From the above considerations it follows that

$$\tau_D^{k\ell} = \frac{1}{\pi i}\int_{\widehat{\phi}_k}^{2} \omega^{\ell} \ . \tag{6.10}$$

Taking into account (6.8), the computation of (6.10) reduces to an elementary integral [21]. Denoting:

$$\gamma_j = -i\sqrt{\frac{\widehat{\phi}_j - 2}{\widehat{\phi}_j + 2}} = \tan\frac{\pi j}{2N} \ , \tag{6.11}$$

we find

$$\tau_D^{k\ell} = \frac{1}{\pi i}\log\frac{\gamma_\ell - \gamma_k}{\gamma_\ell + \gamma_k} \ , \qquad k < \ell \ . \tag{6.12}$$

We have checked that this expression agrees with the proposal of [40] up to $N = 5$, although (6.12) is considerably simpler. Finally, notice that the diagonal couplings diverge logarithmically $\tau_D^{ii} \to i\infty$ [39].

6.2 The blowup function for manifolds of simple type

We are now ready to compute the blowup function for manifolds of simple type. The first thing we have to do is to rewrite (2.20) in the magnetic frame which is appropriate to the strong coupling regime, as in the related analysis of [40]. Since the blowup function is invariant under duality transformations, the only change will be in the characteristic, which is now $[\vec{0}, \vec{\Delta}]$, and in the substitution of all the variables by their duals (i.e., we will have τ_D^{ij} instead of τ_{ij}, and $\partial u_k/\partial a_{D,i}$ instead of $\partial u_k/\partial a^i$.)

We now have all the ingredients to investigate the blowup function for manifolds of simple type. The dual theta function $\Theta_D[\vec{0}, \vec{\Delta}](\vec{\xi}|\tau)$ vanishes at the $\mathcal{N} = 1$ point, but after quotienting by $\Theta_D[\vec{0}, \vec{\Delta}](0|\tau)$ we get a finite result:

$$\frac{1}{C} \sum_{s_j=\pm 1} \prod_{p<q} \left(\frac{\gamma_q - \gamma_p}{\gamma_q + \gamma_p} \right)^{s_p s_q/2} \exp\left\{ \sum_{l=2}^{N} \frac{i s_j t_l}{2} \frac{\partial u_l}{\partial a_{D,j}} \right\} , \tag{6.13}$$

where

$$C = \sum_{s_p=\pm 1} \prod_{p<q} \left(\frac{\gamma_q - \gamma_p}{\gamma_q + \gamma_p} \right)^{s_p s_q/2} , \tag{6.14}$$

and the values of the B-periods at the $\mathcal{N} = 1$ points are given in (6.9). To derive the above equation, we have used the explicit expression for the offdiagonal couplings (6.12). The values of the contact terms can be obtained from the logarithmic derivatives of (6.13) following (2.22), but it proves to be much more useful to use our new equation for the contact terms (4.13). We just have to compute the B-periods of the Abelian differentials (4.3) at the $\mathcal{N} = 1$ point. This is easy to do by making the change of variables $x = 2\cos\theta$ [39]. One has

$$dr^\ell = i P_\ell'(\theta) \cot N\theta \, \sin\theta \, d\theta , \tag{6.15}$$

with periods:

$$\eta^\ell{}_k = \mathrm{res}_{\theta=\hat{\theta}_k} dr^\ell = \frac{i}{N} P_\ell'(\widehat{\phi}_k) \sin\frac{k\pi}{N} , \tag{6.16}$$

where $\hat{\theta}_k = k\pi/N$. The contact terms are then given by:

$$\mathcal{T}_{k,\ell} = \frac{i}{2N} P_{k-1}'(\widehat{\phi}_m) \sin\frac{m\pi}{N} \frac{\partial u_\ell}{\partial a_{D,m}} . \tag{6.17}$$

One has, for example:

$$
\begin{aligned}
\mathcal{T}_{2,\ell} &= \frac{i}{2N}\sin\frac{m\pi}{N}\frac{\partial u_\ell}{\partial a_{D,m}} = \frac{\ell}{4N}u_\ell\,, \\
\mathcal{T}_{3,\ell} &= \frac{i}{N}\sin\frac{2m\pi}{N}\frac{\partial u_\ell}{\partial a_{D,m}}\,.
\end{aligned} \tag{6.18}
$$

We have checked for low values of N that the expression (6.17) agrees with the one obtained using (2.22). Clearly, (6.17) is much more compact in this case. The last expression in the first line of (6.18) actually follows from (4.14), but can be checked using the results of this section.

Putting all the ingredients together, we find that the blowup function at the $\mathcal{N}=1$ point is given by

$$
\begin{aligned}
\tau(t_i) = \frac{1}{C}\exp\left\{-\sum_{k,\ell} t_k t_\ell \frac{i}{2N} P'_{k-1}(\widehat{\phi}_m)\sin\frac{m\pi}{N}\frac{\partial u_\ell}{\partial a_{D,m}}\right\} \\
\cdot \sum_{s_j=\pm 1}\prod_{p<q}\left(\frac{\gamma_q-\gamma_p}{\gamma_q+\gamma_p}\right)^{s_p s_q/2} \exp\left\{\sum_{l=2}^{N}\frac{i s_j t_l}{2}\frac{\partial u_l}{\partial a_{D,j}}\right\}\,.
\end{aligned} \tag{6.19}
$$

From the point of view of the underlying KdV hierarchy, this blowup function has a very simple interpretation: it is a τ function for an $(N-1)$-soliton solution, after making the linear transformation of times explained in section 5. This is a simple consequence of the fact that quasi-periodic solutions of the KdV hierarchy become multisoliton solutions in the limit of maximal degeneracy of the underlying Riemann surface (see, for example, [21], [22]).

An important consistency check of (6.19) can be made by considering the explicit expression of the Donaldson-Witten generating function for manifolds X of symple type with $b_2^+(X)>1$ obtained in [8], which is trivially extended to include more general descent operators:

$$
\begin{aligned}
&Z(p_k, f_k, S)_X^{\mathcal{N}=1} \\
&= \alpha^\chi \beta^\sigma \sum_{x_j}\left(\prod_{j=1}^{N-1} SW(x_j)\right)\prod_{j<k}\left(\frac{\gamma_k-\gamma_j}{\gamma_j+\gamma_k}\right)^{-(x_j,x_k)/2} \\
&\quad\cdot \exp\left\{\sum_{k=2}^{N}\left(p_k u_k - \frac{i}{2} f_k \frac{\partial u_k}{\partial a_{D,j}}(S,x_j)\right) + S^2\sum_{k,l} f_k f_l \mathcal{T}_{k,l}\right\}\,.
\end{aligned} \tag{6.20}
$$

In this equation, we have only recorded the contribution of one of the $\mathcal{N}=1$ points, since the contributions of the other points follow from $\mathbb{Z}_N$ symmetry. For each $i=1,\dots,N-1$, the sum over x_i is over all the Seiberg-Witten basic classes of the manifold X [41], whose Seiberg-Witten invariants are denoted by $SW(x_j)$. The values of the B-periods and the contact terms are those given in (6.9) and (6.17), respectively. $(\ , \ )$ denotes the product in (co)homology. Finally, α and β are universal constants that only depend on N. If we now perform a blowup, for every basic class x of X we will obtain the basic classes $x \pm B$ in $\widehat{X}$, where x denotes the pullback to $\widehat{X}$ of the basic class of X [42]. The Seiberg-Witten invarians are $SW(x \pm B) = SW(x)$ [42]. If we now consider $Z(p_k, f_k, S)_{\widehat{X}}^{\mathcal{N}=1}$, we will have to substitute $x_i \to x_i + s_i B$ in (6.20), with $s_i = \pm 1$. The sum over basic classes of the blownup manifold $\widehat{X}$ factorizes into a sum over the x_i and a sum over the s_i. Taking into account that $(x, B) = 0$ for any cohomology class x pulled back from X to the blownup manifold, and that $B^2 = -1$, we find that, under blowup, (6.20) gets an extra factor which exactly agrees with (6.19) up to an overall constant[4]. This is an important consistency check of the whole story and in particular of the expression (6.20). The check is not trivial since, when using (6.20), we have to rely on properties of the Seiberg-Witten invariants, while (6.19) was derived by means of the u-plane integral.

Let us finish this section by considering in detail the expression we obtained for the blowup function at the $\mathcal{N}=1$ point (6.19) for the cases of lower genus. For $g=2$, for example, $\gamma_1 = 1/\sqrt{3}$ and $\gamma_2 = \sqrt{3}$. After using the explicit values of the B-periods given in (6.9), we obtained:

$$\tau_{SU(3)}(t_2, t_3) = \frac{1}{3} e^{-\frac{1}{2}t_2^2 - t_3^2} \Big\{ \cosh(\sqrt{3} t_2) + 2\cosh(\sqrt{3} t_3) \Big\} . \qquad (6.21)$$

Notice that the blowup function for simple type manifolds is given by a compact expression as (6.21), in contrast to the case of non-simple type manifolds that we analyzed above. This fact was already observed in the elliptic case [1], and is related to the degeneration of hyperelliptic functions to trigonometric functions. On the other hand, both expressions must coincide as long as the blowup function is duality invariant. This means that the whole expansion (2.18) must reorganize itself into

[4]The overall normalization also agrees if one takes into account the universal constants of the u-plane integral in the definition of the blowup function.

(6.21) when $u_2 = 3$ and $u_3 = 0$. Indeed, in expanding (6.21) up to sixth order in the times

$$\tau_{SU(3)}(t_2,t_3) = 1 - \frac{1}{2}t_2^2t_3^2 - \frac{1}{4}t_3^4 - \frac{1}{120}t_2^6 + \frac{1}{8}t_2^4t_3^2 + \frac{3}{8}t_2^2t_3^4 + \frac{13}{120}t_3^6 + \cdots, \tag{6.22}$$

we find complete agreement with the expansion (4.29) in the nonsimple type case. This is an important consistency check of the results of this paper.

For $g = 3$, one has $\gamma_1 = \sqrt{2}-1$, $\gamma_2 = 1$ and $\gamma_3 = \sqrt{2}+1$, and the blowup function turns out to be:

$$\begin{aligned}
&\tau_{SU(4)}(t_2,t_3,t_4) \\
&= \frac{1}{4\sqrt{2}} e^{-\frac{1}{2}t_2^2 - t_3^2 - 2t_4^2 + t_2t_4} \Big\{ \sqrt{2}\cosh(t_2+2t_3-2t_4) \\
&\quad + \sqrt{2}\cosh(-t_2+2t_3+2t_4) + (\sqrt{2}-1)\cosh((\sqrt{2}+1)t_2+2t_4) \\
&\quad + (\sqrt{2}+1)\cosh((\sqrt{2}-1)t_2-2t_4)\Big\} .
\end{aligned} \tag{6.23}$$

Again, it is immediate to check that the leading terms of its expansion,

$$\tau_{SU(4)}(t_2,t_3,t_4) = 1 - \frac{1}{6}t_3^4 - t_2t_3^2t_4 - \frac{1}{2}t_2^2t_4^2 + \frac{4}{3}t_2t_4^3 - \frac{4}{3}t_4^4 + \cdots, \tag{6.24}$$

are in agreement with the result obtained in the nonsimple type case, after taking into account that $u_2 = 4$ and $u_3 = 0$ at the $\mathcal{N} = 1$ point.

7 Concluding Remarks

In this paper we have carried out a detailed analysis of blowup formulae in $SU(N)$ Donaldson-Witten theory. In particular, we have found an explicit procedure to expand it in terms of the Casimirs of the gauge group up to arbitrary order, by using the theory of hyperelliptic Kleinian functions. This theory clarifies in fact many other aspects of blowup formulae and the u-plane integral, like contact terms and the relation with integrable hierarchies.

Although higher rank generalizations of Donaldson-Witten theory seem to be rather intractable mathematically, it is likely that the behavior of the higher rank invariants under blowup can be determined by using only a limited amount of information, like in the work of Fintushel and Stern [1]. This article gives very precise predictions for this behavior. In particular, it implies that the higher rank generalization of the differential equations studied in [1] will be essentially the KdV hierarchy.

Our work can be generalized in many different directions. First of all, we have analyzed only the case of $\vec{\beta} = \vec{0}$, and certainly this is only one particular case of the general blowup formula. More work is needed along this direction. In particular, it would require a generalization of the procedure developed in section 3 for other kind of σ-functions

It would be also interesting to work out the details for theories including massive hypermultiplets and/or other gauge groups. One of the most interesting aspects of the theories with matter is that the magnetic flux turns out to be fixed by topological constraints, and this gives a nonzero value of $\vec{\beta}$ in the blowup function [7], [43].

Another direction to explore is the relation between the hyperelliptic Kleinian functions and the theory of the prepotential. The blowup function gives a natural set of Abelian differentials of the second kind, and we know from general principles that such a set is one of the basic ingredients in the construction of a Whitham hierarchy [37]. It would be very interesting to develop this relation in general, at least for hierarchies associated to hyperelliptic curves. This would further clarify the relations between blowup functions in generalizations of Donaldson-Witten theory, and the construction of Whitham hierarchies for supersymmetric $\mathcal{N} = 2$ theories in [12], [13], [25], [26], [44].

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