M5-branes, Special Lagrangian Submanifolds and σ-models

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Abstract

We study M-theory fivebranes wrapped on Special Lagrangian submanifolds (Σ_n) in Calabi-Yau three- and fourfolds. When the M5 wraps a four-cycle, the resulting theory is a two-dimensional domain wall embedded in three-dimensional bulk with four supercharges. The theory on the wall is specified in terms of the geometry of the CY manifold and the cycle Σ_4. It is chiral and anomalous, however the presence of a three-dimensional gravitational Chern-Simons term with a coefficient that jumps when crossing the wall allows to cancel the anomaly by inflow. Kähler manifolds of special type, where the potential depends only on the real part of the complex coordinate, are shown to emerge as the target spaces of two-dimensional σ-models when the M5 is wrapped on $\Sigma_3 \times S^1$, thus providing

a physical realization of some recent symplectic construction by Hitchin.

1   Introduction and summary

Calibrated geometries [1] (see also [2], [3]) have been studied in the context of mirror symmetry [4], [5] and intersecting branes (see e.g., [6] for a review and references). Our focus is on Special Lagrangian (SL) submanifolds in Calabi-Yau three- and fourfolds, and we mostly concentrate on the deformation properties of these submanifolds.

We study a class of two-dimensional σ-models originating from M-theory fivebranes wrapped on four-manifolds. Fivebranes wrapped on holomorphic divisors in a Calabi-Yau threefold were a subject for recent investigation [7], [8]. Here we turn to the case of M5 wrapped on SL four-cycles, when M theory is compactified on a Calabi-Yau fourfold. While the amount of supersymmetry can be figured out by general arguments [9], the actual counting of the multiplets is somewhat involved. Such a counting was done for a (very ample) divisor $\mathcal{P}$ in a CY three-fold, in [7]. The geometry of the resulting $(0,4)$ σ-model was discussed in detail in [8]. The understanding of cases with lower supersymmetries, arising when M5 and D5 are wrapped on a SL four-cycle in a Calabi-Yau four-fold, requires some results from McLean's deformation theory [10]. Recent symplectic constructions by Hitchin also enter in a natural way [11], [12]. Indeed, there is a natural extension to the symplectic case of many results on deformations of complex submanifolds, and thus some of the results of [7], [8] (where the submanifold was taken to be a divisor in a CY threefold) can be extended as well. Our study shows yet another aspect of the intimate relation between calibrated geometries and supersymmetry. Even though the main focus of our attention is on SL submanifolds, other calibrations can be studied along similar lines.

The wrapped fivebranes and the resulting σ-models are our main concern, but it should be noted that the study of cycles is relevant for D-brane moduli spaces, in particular instanton effects in type IIb (F-theory) compactifications when the D3-branes are completely wrapped. The wrapping of D3-branes on supersymmetric cycles in four-folds was

considered in [13], and for the case of SL submanifolds such configurations were shown to preserve half of supersymmetry.

In principle, there are few σ-models obtained by M5 wrapping SL submanifolds in four-folds. The summary is presented in the table (X_n denotes a (complex) n-fold of $SU(n)$ holonomy, while Σ_k denotes a SL cycle of real dimension k):

Four-fold	SLS	Preserved Susy	σ-model
T^8	T^4	1	$(8,8)$
$T^4 \times K3$	$T^2 \times \Sigma_2$	$1/2$	$(4,4)$
$K3 \times K3$	$\Sigma_2 \times \Sigma_2'$	$1/4$	$(2,2)$
$T^2 \times X_3$	$S^1 \times \Sigma_3$	$1/4$	$(2,2)$
X_4	Σ_4	$1/8$	$(0,2)$

Table 1.

In this note, we discuss only the multiplet structure and the classical target spaces. In cases with lower supersymmetries that are discussed here these will not be protected against quantum corrections and there can be world-sheet renormalizations.

First, we will analyze the cases where the submanifolds are complex concentrating on the fourfolds with $SU(4)$ holonomy[1]. Before going into details of the multiplet structure, we can have a look at the field content after the reduction. As in the case of [8] the target space classically factorizes into two sectors, the "universal" and the "entropic" (to borrow the terminology from [8]). The universal sector consists of two real scalars, one coming from the coordinate parametrizing the position of the M5 in three dimensional spacetime and one coming from the component of the β-field along the Kähler form (this will be better explained in section 2). In the entropic sector the two sources for the two-dimensional scalar fields are the self-dual tensor field on the fivebrane worldvolume, and the deformations of the cycle $\mathcal{C}$ inside the Calabi-Yau. The position of the cycle inside the fourfold is parametrized by the four (out of five) scalars in the $(0,2)$ tensor multiplet on the fivebrane worldvolume. After the reduction these will yield $d_{\mathcal{C}}$ real scalars

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[1]$K3 \times K3$ compactifications can be discussed as a special case.

on the world-sheet (the notation will be justified shortly). The self-dual β-field gives rise to b_2^+ and b_2^- right- and left-movers respectively (we have already mentioned that one of the scalars coming from the β field belongs to the "universal" sector). The excess of left-movers tells us that the resulting σ-model is of heterotic type, and there is a coupling to a gauge field. The numbers of right- and left-moving fermions are given by the twisted Dirac index. A very quick analysis of the multiplets reveals the connection between supersymmetry and McLean's deformation theory [10]: It is required by supersymmetry (simply by boson-fermion matching) that the first Betti number of the submanifold $b_1(\mathcal{C}) = \dim H^1(\mathcal{C}, \mathbb{R})$ be equal to the dimension of the moduli space of deformations of $\mathcal{C}$ inside X, $d_\mathcal{C}$. Indeed it is known that the tangent space of the local moduli space at $\mathcal{C}$ of Special Lagrangian submanifolds can be canonically identified with the space of harmonic one-forms on $\mathcal{C}$ [10]. This identification is crucial for the supersymmetry of the two-dimensional σ-model. The idea of using supersymmetry transformations to give a physical derivation of McLean's deformation theorems was explained to us by J. Harvey and G. Moore (see also [14]).

Since the resulting two-dimensional theory is chiral, it can suffer from anomalies. In section 3, we discuss the cancellation mechanism. It turns out that the three dimensional bulk theory has a gravitational Chern-Simons term with a discontinuity in its coefficient. The variation of this term cancels the two-dimensional anomaly by inflow.

In section 4, we will turn to M5 wrapped on a real manifold $S^1 \times \Sigma_3$. By duality, this case will be related to the study of moduli spaces of D-branes, and we will be able to generalize some of the results here to D-branes wrapped on SL cycles. Using the self-duality of the β field on the worldvolume of M5 and the fact that $b_1(\Sigma_3) = b_2(\Sigma_3)$, it is easy to see that the resulting number of zero-modes from the tensor field is b_1. Just like for D-branes, these modes are paired with b_1 zero-modes arising from deformations. However supersymmetry predicts the stronger result that the target space is Kähler. In constructing the metric, we find many analogies with the c-map construction of Cecotti, Ferrara and Girardello [15] and establish a *weak c-map*. The result is in agreement with the one obtained for D-brane moduli spaces [11]. A special care is needed for the treatment of the normal directions.

2 Wrapped M5: Complex Special Lagrangian cycles

Here we will perform the reduction to two dimensions of the 6D supersymmetry. As explained in [8], it suffices to consider the reduction of the supersymmetry transformations of the tensor multiplet from flat space. For this purpose we will need the zero-mode expansion of the 6D fields in the fivebrane action.

The bosonic zero-modes

The Kaluza-Klein ansatz for the chiral two-form β is:

$$\beta = \rho^a \omega_a + (\pi^I \omega_I + c.c.) + uJ, \qquad (2.1)$$

where J is the Kähler form on $\mathcal{C}$, $\{\omega_I, I = 1, \dots b_{20}(\mathcal{C})\}$ is a basis of $H^{2,0}(\mathcal{C})$ and $\{\omega_a, a = 1, \dots b_2^-(\mathcal{C})\}$ is a basis of anti-self-dual $(1,1)$ forms on $\mathcal{C}$. The scalars ρ^a, u (π^I) are real (complex). We therefore get b_2^- left-moving and b_2^+ right-moving real scalars. We have implicitly used the fact that due to the Kählerity of $\mathcal{C}$ we have the decomposition

$$H^{2+}(\mathcal{C}) = H^{2,0}(\mathcal{C}) \oplus H^{0,2}(\mathcal{C}) \oplus J; \quad H^2(\mathcal{C}) = H^{2+}(\mathcal{C}) \oplus H^{2-}(\mathcal{C}) \quad (2.2)$$

The five scalars of the 6D $(2,0)$ tensor multiplet parametrize the position of the fivebrane in transverse five-dimensional space. When the fivebrane is wrapped, four of them (say X^{6-9}) parametrize the position of the four-cycle $\mathcal{C}$ inside the Calabi-Yau four-fold X while the fifth (X^{10}) describes the motion of the string in the three non-compact dimensions. The massless modes arise from deformations of $W_2 \times \mathcal{C}$ preserving supersymmetry. Let $\mathcal{M}$ be the space of deformations of $\mathcal{C}$. The tangent space to $\mathcal{M}$ at $\mathcal{C}$ is

$$T_{\mathcal{C}}\mathcal{M} = H^0(\mathcal{C}, \mathcal{N}) \qquad (2.3)$$

where $\mathcal{N}$ is the normal bundle of $\mathcal{C}$ inside X. For $\mathcal{C}$ a special lagrangian submanifold the following equivalence holds [10]:

$$\mathcal{N} \cong T^*\mathcal{C} \qquad (2.4)$$

We will rely heavily on this relation in the following. Taking into account that $H^0(\mathcal{C}, T^*\mathcal{C}) \cong H^{1,0}(\mathcal{C})$ we see that (2.3) implies

$$dim_{\mathbb{C}}\mathcal{M} = b_{10}(\mathcal{C}) = \frac{1}{2}b_1(\mathcal{C}) \qquad (2.5)$$

Let us choose a complex basis $\{v_{\widetilde{I}}^m, m = 1, 2, \widetilde{I} = 1, \ldots b_{10}(\mathcal{C})\}$ of sections of $\mathcal{N}$. Due to the equivalence (2.4) we may identify

$$v_{\widetilde{I}}^{1,2} = \omega_{\widetilde{I}}^{1,2},\tag{2.6}$$

where $\{\omega_{\widetilde{I}\overline{m}}dz^{\overline{m}},\}$ is a basis of $H^{0,1}(\mathcal{C})$ and raising/lowering of the m index is possible due to the existence of a metric on $T\mathcal{C}$. We will expand to first-order in $\varphi^{\widetilde{I}}$

$$X^6 + iX^7 = 2v_{\widetilde{I}}^1\varphi^{\widetilde{I}}; \quad X^8 + iX^9 = -2v_{\widetilde{I}}^2(\varphi^{\widetilde{I}})^*\tag{2.7}$$

where $\varphi^{\widetilde{I}}$, $\widetilde{I} = 1, \ldots \frac{1}{2}b_1$ are two-dimensional complex massless unconstrained bosons. The (world-sheet) scalar X^{10} accounts for one real boson.

Altogether the number of right-, left-moving real bosonic degrees of freedom is

$$N_L^B = b_1 + b_2^- + 1; \quad N_R^B = b_1 + b_2^+ + 1.\tag{2.8}$$

The fermionic zero-modes

The fivebrane breaks the Lorentz invariance of W_6 down to $Spin(1,1)_{W_2} \times Spin(4)_{\mathcal{C}}$. The fermions of the tensor multiplet transform in the $(+, \mathbf{2}_+) \oplus (-, \mathbf{2}_-)$. Moreover the $Spin(5) \cong USp(2)$ $\mathcal{R}$-symmetry of the 6D tensor multiplet is broken by the Calabi-Yau to $Spin(4)_{\mathcal{N}}$, where $SO(4)_{\mathcal{N}}$ is the structure group of $\mathcal{N}$. From the 2D point of view there is a $Spin(4)_{\mathcal{C}} \times Spin(4)_{\mathcal{N}}$ $\mathcal{R}$-symmetry and fermions in the $(\mathbf{2}_-, \mathbf{2}_{\pm})$, $(\mathbf{2}_+, \mathbf{2}_{\pm})$ give rise to left-, right-movers on W_2 respectively. Due to (2.4) the fermions can be thought of as bispinors on $\mathcal{C}$ and are in one-to-one correspondence with (p, q) forms on $\mathcal{C}$: Since $\mathcal{C}$ is Kähler, one can construct two sets of gamma matrices $\{\gamma^m, \gamma^{\overline{n}}\} = \{\widetilde{\gamma}^m, \widetilde{\gamma}^{\overline{n}}\} = g^{m\overline{n}}$, $[\gamma, \widetilde{\gamma}] = 0$. We can take $\widetilde{\gamma}^m = i\rho^{(\mathcal{C})}\gamma^m$, where $\rho^{(\mathcal{C})}$ is the chirality matrix on $\mathcal{C}$. We will regard $\gamma^m, \widetilde{\gamma}^m$ as creation operators and let us denote the "Fock vacuum" by $|0\rangle$. Let ψ^R (ψ^L) be a bispinor in the $(\mathbf{2}_+, \mathbf{2}_+) \oplus (\mathbf{2}_+, \mathbf{2}_-)$ $((\mathbf{2}_-, \mathbf{2}_+) \oplus (\mathbf{2}_-, \mathbf{2}_-))$ of $Spin(4)_{\mathcal{C}} \times Spin(4)_{\mathcal{N}}$. One has

$$\psi^R \sim (\Omega^{(0,0)} + \Omega_{\overline{m}}^{(0,1)}\widetilde{\gamma}^{\overline{m}} + \Omega_{mn}^{(2,0)}\gamma^{mn} + \Omega_{\overline{mn}}^{(0,2)}\widetilde{\gamma}^{\overline{mn}}$$
$$+ \Omega_{mn\overline{k}}^{(2,1)}\gamma^{mn}\widetilde{\gamma}^{\overline{k}} + \Omega_{mn\overline{kl}}^{(2,2)}\gamma^{mn}\widetilde{\gamma}^{\overline{kl}})|0\rangle$$
$$\psi^L \sim (\Omega_m^{(1,0)}\gamma^m + \Omega_{m\overline{n}}^{(1,1)}\gamma^m\widetilde{\gamma}^{\overline{n}} + \Omega_{m\overline{nk}}^{(1,2)}\gamma^m\widetilde{\gamma}^{\overline{nk}})|0\rangle\tag{2.9}$$

where $\Omega^{(p,q)}$ is a (p,q) form on $\mathcal{C}$. To see this note that because $\mathcal{C}$ is Kähler the positive and negative spin bundles on $\mathcal{C}$ (due to (2.4) a similar result holds for $\mathcal{N}$) decompose as:

$$S^+(T\mathcal{C}) \otimes K^{1/2} \cong \Omega^{0,0} \oplus \Omega^{2,0}; \quad S^-(T\mathcal{C}) \otimes K^{-1/2} \cong \Omega^{1,0} \qquad (2.10)$$

and we can expand the fermionic zeromodes in terms of forms on $\mathcal{C}$. Also note that $\Omega^{(0,1)}_{\overline{m}}\widetilde{\gamma}^{\overline{m}}|0\rangle$ $(\Omega^{(1,0)}_{m}\gamma^{m}|0\rangle)$ transforms in the $\mathbf{2_+}$ $(\mathbf{2_-})$ of $Spin(4)_{\mathcal{N}}$ $(Spin(4)_{\mathcal{C}})$. Taking into account (2.2) and the isomorphism $H^{p,q}(\mathcal{C}) \cong H^{2-p,2-q}(\mathcal{C})$ we see that from the 2D point of view the number of (real) left- and right-moving fermionic degrees of freedom is

$$N_L^F = b_1 + b_2^- + 1; \quad N_R^F = b_1 + b_2^+ + 1. \qquad (2.11)$$

In order to reduce the supersymmetry, we need to be slightly more explicit than in (2.9). Let us choose our ten-dimensional matrices to locally decompose as

$$\Gamma^{0,1} = \gamma^{0,1} \otimes \rho^{(\mathcal{C})} \otimes \rho^{(\mathcal{N})}$$
$$\Gamma^{2,3,4,5} = \mathbb{I}_2 \otimes \gamma^{2,3,4,5} \otimes \mathbb{I}^{(\mathcal{N})}$$
$$\Gamma^{6,7,8,9} = \mathbb{I}_2 \otimes \rho^{(\mathcal{C})} \otimes \gamma^{6,7,8,9} \qquad (2.12)$$

where

$$\gamma^0 = i\sigma^2; \quad \gamma^1 = \sigma^1$$
$$\gamma^{2,3,4} = \begin{pmatrix} 0 & \sigma^{1,2,3} \\ \sigma^{1,2,3} & 0 \end{pmatrix}; \quad \gamma^5 = \begin{pmatrix} 0 & i\mathbb{I}_2 \\ -i\mathbb{I}_2 & 0 \end{pmatrix} \qquad (2.13)$$

and $\gamma^i = \gamma^{i+4}$, $i = 2, \ldots 5$.

We locally decompose the covariantly constant spinor of the Calabi-Yau as $\xi^{(8)} = \xi^{(\mathcal{N})} \otimes \xi$ and we take $\xi^{(8)}$, ξ to be of positive chirality. If we write $\xi^{(8)}$ as $\xi_{(i)}$ with $i = 1, \ldots 4$ the symplectic index, the statement about the chirality corresponds according to our conventions to setting $\xi_{i=3} = \xi_{i=4} = 0$. The fermions are expanded in terms of 2D right- and left-movers as

$$\psi_i^{(6)} = \psi_{i-}^I \otimes \Delta_{(i)}^I + \psi_{j-}^{\widetilde{I}} \otimes \Delta_{(ij)}^{\widetilde{I}} + \psi_{i-}^0 \otimes \xi_{(i)} + \text{left movers}, \qquad (2.14)$$

where

$$\Delta_{I(i)} = \begin{cases} \omega^{I\overline{mn}}\gamma^{\overline{mn}}\xi & \text{for } i = 1 \\ \omega_{Imn}\gamma^{mn}\xi^* & \text{for } i = 2 \\ 0 & \text{for } i = 3, 4 \end{cases} ; \quad \Delta_{\widetilde{I}(ij)} = (\omega_{\widetilde{I}\overline{m}}\gamma^{\overline{m}} + \omega_{\widetilde{I}m}\gamma^m)_{ij}\xi_{(j)}$$

$$(2.15)$$

and

$$
\xi_{(i)} = \begin{cases} \xi & \text{for } i = 1 \\ \xi^* & \text{for } i = 2 \\ 0 & \text{for } i = 3, 4. \end{cases}
\tag{2.16}
$$

The right-moving zero modes $\psi^I_{1,2}$, $\psi^{\widetilde{I}}_{1,2}$, $\psi^0_{1,2}$ are complex antichiral 2D fermions. Only half of those are independent since the 6D symplectic reality condition implies $\psi^x_1 = i(\psi^x_2)^*$, where $x = 0, \widetilde{I}$, or I. In the following we set $\psi^x := \psi^x_1$.

2.1   The 2d supersymmetry

Substituting the expansions (2.1), (2.7), (2.14) into the supersymmetry transformations of the tensor multiplet we get

$$
\begin{aligned}
\delta\varphi^{\widetilde{I}} &= -\varepsilon^*_+ \psi^{\widetilde{I}}_-; & \delta\psi^{\widetilde{I}}_- &= \partial_- \varphi^{\widetilde{I}} \varepsilon_+ \\
\delta\pi^I &= -\varepsilon^*_+ \psi^I_-; & \delta\psi^I_- &= \partial_- \pi^I \varepsilon_+ \\
\delta\rho^a &= 0; & \delta\psi^a_+ &= 0
\end{aligned}
\tag{2.17}
$$

where ψ^a_+ are the left-moving fermions and the supersymmetry parameter ε_+ is a chiral complex 2D spinor. These are the supersymmetry transformations, written in complex notation, of the 2D fields of a $(0, 2)$ σ-model. In order to compare with the standard $(0, 1)$-superspace description [16] and read off the complex structure of the target space of the σ-model, we pass to a real basis: $\pi^I = \phi^{1I} + i\phi^{2I}$; $\varphi^{\widetilde{I}} = \phi^{1\widetilde{I}} + i\phi^{2\widetilde{I}}$. Similarly for the fermions: $\psi^I_- = \chi^{1I}_- + i\chi^{2I}_-$; $\psi^{\widetilde{I}}_- = \chi^{1\widetilde{I}}_- + i\chi^{2\widetilde{I}}_-$; $\varepsilon_+ = \zeta_+ + i\eta_+$. The susy transformations become

$$
\begin{aligned}
\delta\phi^{ix} &= (\zeta_+ \delta^{ix}_{jy} + \eta_+ J^{ix}_{jy})\chi^{jy}_- \\
\delta\chi^{ix}_- &= (\zeta_+ \delta^{ix}_{jy} - \eta_+ J^{ix}_{jy})\partial_- \phi^{jy}; \quad i = 1, 2; \quad x, y = I \text{ or } \widetilde{I}
\end{aligned}
\tag{2.18}
$$

where $J^{ix}_{jy} := i[\sigma^2]^i_{\ j}\delta^x_{\ y}$ satisfies $J^2 = -\mathbb{1}$. Moreover the transformations (2.18) and the fact that $(0, 2)$ supersymmetry is unbroken implies that J has to be covariantly constant, giving a Kähler target space.

The transformation of ψ^0 can be joined with the transformations for u and X^{10} to give

$$
\delta\psi^0_- = \partial_- U \varepsilon_+; \quad \delta U = -\varepsilon^*_+ \psi^0
\tag{2.19}
$$

where $U := X^{10} + iu$. This "universal" factor, consisting of the fields $\{u, X^{10}, \psi^0_-\}$, possesses separately $(0, 2)$ supersymmetry.

2.2 The structure of the σ-model

As we have seen, the spectrum on the two-dimensional worldvolume is determined by three numbers b_2^+, b_2^- and b_1. Similarly to [7], one can express $\sigma(\mathcal{C}) = b_2^+ - b_2^-$ and $\chi(\mathcal{C}) = 2 + b_2^+ + b_2^- - 2b_1$ in terms of Calabi-Yau quantities

$$\chi(\mathcal{C}) = \eta^2; \quad \sigma(\mathcal{C}) = -\frac{1}{3}c_2\eta \qquad (2.20)$$

where

$$c_2\eta := \int_X c_2(X) \wedge \eta; \quad \eta^2 := \int_X \eta \wedge \eta \qquad (2.21)$$

and η is the Poincaré dual to the cycle $\mathcal{C}$. It is an element of $H^4(X, \mathbb{Z})$ [2] and restricts to $\eta = c_2(\mathcal{N}) = c_2(\mathcal{C})$ on $\mathcal{C}$ [18], [19], where for the last equality we used (2.4). Equation (2.4) and the exact sequence

$$0 \to T\mathcal{C} \to TX \to \mathcal{N} \to 0 \qquad (2.22)$$

were also used in order to derive (2.20).

Since $\mathcal{C}$ is holomorphically embedded in X, η is of type $(2, 2)$. The latter requirement (the existence of a holomorphic four cycle) puts a restriction to the complex structure of X. Note also that η is primitive[3]. In the next section we will explain the relation between η and the cohomology class of the four-form field strength G_4 of eleven dimensional supergravity. The conditions that G_4 is integral, $(2, 2)$ and primitive are precisely the requirements for compactification of M-theory on a manifold of eight (real) dimensions preserving four supercharges in three dimensional Minkowski spacetime [20].

Finally putting everything together, we see that the total number of $(0, 2)$ multiplets is $b_1 + b_2^+ + 1$, while the rank of the vector bundle over the target space is

$$\text{rank}V = \frac{1}{2}(b_1 + b_2^- + 1) + |\sigma(\mathcal{C})|. \qquad (2.23)$$

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[2]Strictly speaking, η is an element of the cohomology with compact support $H^4_{cpct}(X, \mathbb{Z})$ which in general for noncompact X forms a sublattice of $H^4(X, \mathbb{Z})$. For a detailed discussion in a similar context see [17].

[3]This can be seen roughly as follows: We locally parametrize X by (l_a, n^a), $a = 1, \ldots 4$ so that $\mathcal{C}$ is given by $n^a = 0$. Then η is schematically given by $\delta(n)dn^1 \wedge \ldots dn^4$. Since $\mathcal{C}$ is SL we can locally write the Kähler class as $\omega = dl_a \wedge dn^a$. We thus see that $\omega \wedge \eta = 0$

Note that the model is chiral even when the numbers of left- and right-moving modes coming from the β-field are the same ($\sigma(\mathcal{C}) = 0$). The standard formulation with $(0,1)$ supersymmetry [16] couples the gauge field to the fermionic current. The latter can be bosonized as in [8]. Here the left-moving (bosonized) current comes from both the bosonization of the left moving fermions and from the $|\sigma(\mathcal{C})|$ left-moving bosons.

As in [8], in reducing the fivebrane action to the worldsheet of the σ-model, both the β-field and the gauge connection of the vector bundle will appear flat at least to this approximation. Repeating the analysis of [8] to extract information for the metric, will involve variations of Hodge structures of weights one and two. However the classical geometry of the model is not protected by supersymmetry from quantum corrections [21] and it is not clear to what extent such an analysis should be trusted.

When the fourfold is of the form $K3 \times K3$, the corresponding SL submanifold is still complex, and the resulting two-dimensional theory can be discussed as a special case of the more general model presented above. Now one has to bear in mind that the fermions on the CY are no longer chiral. As before, for each cycle Σ_2^i, $i = 1, 2$ we can take $\eta_i = c_1(N^i) = -c_1(\Sigma^i)$ and have $b_1^i = 2D_i + 2$ where $D_i = \frac{1}{2} \int_{X_i} \eta_i^2$. Then for $\Sigma_2^1 \times \Sigma_2^2$,

$$b_1 = 4 + 2D_1 + 2D_2$$
$$b_2 = 2 + (2D_1 + 2)(2D_2 + 2) \qquad (2.24)$$

and $b_2^+ = b_2^-$. We can see that for the left-movers the gauge bundle can be identified with the tangent bundle, and the resulting model has $(2,2)$ supersymmetry in accordance with the expectations from the general supersymmetry analysis.

3 Inflow in 3 dimensions

Following [22], we would like to present an alternative way of counting of the zero-modes on the string. This counting is based on using the fivebrane anomalies. Here as well we can reduce the fivebrane anomaly given by [23]

$$I_8(TW, \mathcal{N}) = \frac{1}{48} \left(\frac{1}{4}(p_1(TW) - p_1(\mathcal{N}))^2 + p_2(\mathcal{N}) - p_2(TW) \right) \quad (3.1)$$

by using $TW_6|_C = TC \times TW_2$ and $\mathcal{N} = \mathcal{N}(C \hookrightarrow X)$. On the other hand we know that the total two-dimensional anomaly (note that there is no anomaly of the normal bundle anymore) is given simply by the difference of central charges of left- and right-movers, $I_{strng} = (c_L - c_R)p_1(TW_2)/24$. Comparing with the reduction of the fivebrane anomaly (3.1), we recover

$$c_L - c_R = \tfrac{1}{2}c_2\eta = -\frac{3}{2}\sigma(C) \qquad (3.2)$$

in agreement with (2.8), (2.11) and (2.20).

We now turn to the anomaly cancellation mechanism and find some new twists here. In particular, the discussion of anomaly cancellation is closely related to the jump in the value of a certain cohomology class (to be identified momentarily) recently discussed in [17]. Since in eleven dimensions a complete cancellation of the anomalies on the fivebrane worldvolume occurs [24], we expect the same to happen after the compactification/wrapping. Note that in three dimensions, the string is a domain-wall type of object and thus it is magnetically charged with respect to the zero-form field strength whose value jumps when crossing the wall. The source equation for such a string can be written as

$$d\Lambda = \delta(W_2 \hookrightarrow M_3) \qquad (3.3)$$

where $\delta(W_2 \hookrightarrow M_3)$ is the Poincaré dual to the string worldvolume. With some abuse of notation we will call Λ a "cosmological constant"; it takes different values on the different sides of wall[4].

Equation (3.3) can be seen as simply the reduction of the fivebrane source equation. Somewhat schematically, in this background the fivebrane source equation becomes

$$dG_4 = 2\pi\delta(x)dx \wedge \eta, \qquad (3.4)$$

where x is a coordinate along the direction transverse to W_2 in M_3 so that $\delta(x)dx$ represents the Poincaré dual of $W_2 \hookrightarrow M_3$. Similarly η is

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[4]One should not confuse this with the case where there is an actual cosmological constant in the uncompactified dimensions, corresponding to the geometry of M-theory on $AdS_3 \times X_4$ [17]. There the cosmological constant turns out to be proportional to the projection of the four-form field strength of 11D supergravity to the $(4, 0)$ part of the cohomology of X_4.

the Poincaré dual of $\mathcal{C}$ inside X and can be represented as a four-form with compact support $\delta(\mathcal{C}_4 \hookrightarrow X)$. We see that the cohomology class of $G_4/2\pi$ jumps by η when crossing the wall at $x = 0$ in M_3.

As seen in [22], when G_4 has non-zero fluxes through a four-dimensional surface, the term in the effective action [25],

$$\Delta_{11} = G_4 \wedge X_7^{(0)}(TM), \qquad (3.5)$$

where

$$dX_7^{(0)}(TM) = -\frac{1}{48}\left(\frac{1}{4}p_1^2(TM) - p_2(TM)\right), \qquad (3.6)$$

can give rise to lower-dimensional Chern-Simons terms. Indeed there is a flux of the four-form field strength through a four-cycle and the reduction yields a three-dimensional gravitational Chern-Simons term:

$$\sim \frac{1}{48}\int_X \eta \wedge p_1(T\mathcal{C}) \cdot p_1^{(0)}(TM) \sim \frac{3}{2}\sigma(\mathcal{C})\frac{p_1^{(0)}(TM)}{24}, \qquad (3.7)$$

where

$$dp_1^{(0)}(TM) = p_1(TM) \qquad (3.8)$$

As we have already seen, the string anomaly is $\sim \frac{3}{2}\sigma(\mathcal{C})$ as well. To complete the discussion we need to examine the flux quantization.

The compactification of Δ_{11} and the Chern-Simons coupling of the eleven-dimensional supergravity on fourfolds leads to a tadpole in three dimensions [26], [27] proportional to $(\chi/24 - \int_X G_4^2/(8\pi^2))$ where χ is the Euler number of the fourfold X. The coefficient of the three-dimensional gravitational Chern-Simons term is expected to be the same by supersymmetry, and we argue that the coupling is of the form:

$$\Delta_3 = \frac{3}{2}\Lambda\,\sigma(\mathcal{C})p_1^{(0)}(TM) \qquad (3.9)$$

which taking (3.3) into account cancels by inflow the string anomaly. It would be interesting to check the three-dimensional supersymmetry of these terms directly. Note that a similar structure of a gravitational Chern-Simons term with discontinuity in the coefficient is also expected from the general discussion of inflow on domain walls [28].

4 $(2,2)$ σ-models, D-brane moduli spaces and the weak c-map

By wrapping the $M5$ on Σ_3 in a CY threefold X_3, we arrive at a three dimensional theory with four supercharges ($N = 2$). Further wrapping on S^1 gives a $(2,2)$ two-dimensional model that can be directly obtained from a D4 wrapped on Σ_3. As we will see the results easily generalize to all D-branes.

Because of the self-duality of the β-field on the worldvolume of M5 and due to the fact that $b_1(\Sigma_3) = b_2(\Sigma_3)$, we can get either a three-dimensional theory with b_1 vectors or a theory with b_1 scalars, which are dual to each other in three dimensions. In addition there are $d_\mathcal{C} = b_1$ zero modes, coming from deformations of the SL submanifold inside the CY threefold, and a "universal" sector consisting of two real bosons (and their susy partners completing a $(2,2)$ multiplet) coming from the coordinates parametrizing the position of $W_2 \times S^1$ inside $\mathbb{R}^{1,2} \times T^2$. When $b_1(\Sigma_3) = 0$, the cycle is rigid, and the only surviving degrees of freedom in two dimensions are the two real scalars of the universal sector. The resulting CFT with $\hat{c} = 3$ can be written down explicitly [5]. In direct analogy with the construction of [15], we see that the dualization of vectors leads to a doubling of the scalar "entropic" coordinates. It is clear that this doubling should work in such a fashion to ensure the 3d $N = 2$ supersymmetry. We see that indeed the (real) moduli space of the deformations $\mathcal{M}$ is such that $T\mathcal{M}$ is a Kähler manifold. A Kähler metric on this manifold was constructed by Hitchin in [11]. We can now see a physical construction.

Fix a point $\{\phi^i\}$ in the moduli space $\mathcal{M}$. At this point the tangent space is

$$T_\phi \mathcal{M} = H^1(\mathcal{C}, \mathbb{R}) \oplus H^1(\mathcal{C}, \mathbb{R}) \cong H^1(\mathcal{C}, \mathbb{C}) \qquad (4.1)$$

Let $(l_a, n^a(\phi))$ be a parametrization of X such that $\mathcal{C}_\phi \hookrightarrow X$ is given by $n^a(\phi) = 0$ and $\{l_a\}$ are coordinates along $\mathcal{C}_\phi$. The normal directions

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[5]It is probably interesting to note that this theory corresponds to a SL three-manifold with vanishing first and second Betti numbers. Such manifolds can be constructed by quotiening S^3 by discrete groups, and they have non-trivial first homotopy groups. It is an open conjecture that any closed simply-connected (and therefore with vanishing H_1) three-manifold is homeomorphic to S^3 (Poincaré conjecture).

X^a are KK expanded in terms of the basis $\{v_i^a := \partial n^a/\partial \phi^i\}$ of sections of the normal bundle of $\mathcal{C}_\phi \hookrightarrow X$ as

$$\partial_\mu X^a = v_i^a \partial_\mu \phi^i, \qquad (4.2)$$

where $\partial_\mu := \partial/\partial\sigma^\mu$ $\partial_i := \partial/\partial\phi^i$, and σ^μ; $\mu = 0,1,2$ are coordinates on W_3. The ϕ^i's are real.

Using the special lagrangian property of $\mathcal{C}$ one can show [11] that $\{\omega_i; \ i = 1, \ldots b_1(\mathcal{C})\}$, where $\omega_i := dl_a v_i^a$, is a basis of $H^1(\mathcal{C}_\phi, \mathbb{R})$. The hermitian metric on $H^1(\mathcal{C}_\phi, \mathbb{R})$ defines a hermitian metric on $T\mathcal{M}$:

$$D_{ij} := \int_\mathcal{C} \omega_i \wedge *\omega_j = \int_\mathcal{C} dV_3 \sqrt{g} g_{ab} v_i^a v_j^b, \qquad (4.3)$$

where g_{ab} is the metric on $\mathcal{C}$. Let us define "inertial" bases $\{\alpha_I\}$ of $H^1(\mathcal{C}, \mathbb{R})$ and $\{\beta_I\}$ of $H^2(\mathcal{C}, \mathbb{R})$ by

$$\omega_i = \Lambda_i{}^I \alpha_I; \quad *\omega_i = M_i{}^I \beta_I \qquad (4.4)$$

And let $\{A_I\}$, $\{B_I\}$ be the dual bases so that

$$\Lambda_i{}^I = \int_{A_I} \omega_i; \quad M_i{}^I = \int_{B_I} *\omega_i \qquad (4.5)$$

Hence the metric can be rewritten as

$$D_{ij} = \delta_{IJ} \Lambda_i{}^I M_j{}^J \qquad (4.6)$$

Since $\omega_i d\phi^i$, $*\omega_i d\phi^i$ are closed as forms on $\mathcal{M}$ [11], we have

$$\partial_{[i} \Lambda_{j]}{}^I = 0; \quad \partial_{[i} M_{j]}{}^I = 0 \qquad (4.7)$$

and therefore we can locally write

$$D_{ij} = \partial_i \partial_j K(\phi) \qquad (4.8)$$

for some scalar function $K(\phi)$.

Let us now use the above results to reduce the 6D theory of the M5 to W_3. Like before (see [8] for detailed discussion of the reduction and for references) it turns out that it suffices to work with unconstrained β-field in the action and impose the self-duality condition as an extra constraint. The KK ansatz for the tensor field is

$$\beta_2 = \omega_{ia} A_\mu^i d\sigma^\mu \wedge dl^a + \frac{1}{2}(*\omega_i)_{ab}\lambda^i dl^a \wedge dl^b + \beta_{\mu\nu} d\sigma^\mu \wedge d\sigma^\nu \qquad (4.9)$$

Suppressing internal derivatives first and then imposing the constraint $d\beta_2 = *d\beta_2$ we get

$$d\beta_2 = d\sigma^\mu \wedge \partial_\mu \beta_2$$
$$= \frac{1}{2}\epsilon_{\mu\nu}{}^\rho (\nabla_\rho \lambda)^i \omega_{ia} d\sigma^\mu \wedge d\sigma^\nu \wedge dl^a + \frac{1}{2}(\nabla_\mu \lambda)^i (*\omega_i)_{ab} d\sigma^\mu \wedge dl^a \wedge dl^b \cdot$$

(4.10)

where

$$(\nabla_\mu)^i{}_j := \delta^i{}_j \partial_\mu + \partial_\mu \phi^k \Gamma^i_{jk}$$ (4.11)

and

$$\Gamma^i_{jk} := \Lambda^i{}_I \partial_j M_k{}^I.$$ (4.12)

Conceptually this is the connection, with respect to the basis $\{*\omega_i\}$, on a fibre bundle with fibre $H^2(\mathcal{C}_\phi, \mathbb{R})$ over the point $\phi^i \in \mathcal{M}$. The action is [6]

$$S = \int_{W_6} d\beta_2 \wedge *d\beta_2 + \int_{W_3} d^3\sigma \int_C dV_3 \sqrt{g} g_{ab} \partial_\mu X^a \partial^\mu X^b.$$ (4.13)

Plugging (4.2), (4.10) above [7] and taking (4.3) into account we get

$$S = \int_{W_3} d^3\sigma D_{ij}(\phi)(\partial_\mu \phi^i \partial^\mu \phi^j + (\nabla_\mu \lambda)^i (\nabla^\mu \lambda)^j).$$ (4.14)

From (4.7) we see that $\Lambda_i{}^I d\phi^i$ is a closed 1-form on $\mathcal{M}$ and therefore we can introduce the coordinate u^I such that

$$du^I := \Lambda_i{}^I d\phi^i.$$ (4.15)

We moreover define

$$y^I := M_i{}^I \lambda^i.$$ (4.16)

Using (4.6), (4.15), (4.16) it is easy to see that

$$z^I := u^I + iy^I$$ (4.17)

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[6]As we already noted above (4.9) the β-field should be thought of as being unconstrained, otherwise the first term on the rhs of (4.13) vanishes identically. The self-duality is imposed as a constraint on the equations of motion deriving from (4.13).

[7]In [8] it was shown that the reduction of the fivebrane action in arbitrary curved background is recovered (at least to leading order) by first reducing the action in flat background and then covariantizing.

defines an almost complex structure on $T\mathcal{M}$, which can be shown to be integrable, reproducing the construction of [11] in a physical context.

Eq. (4.15) defines a set of coordinates which can be shown to be equivalent to the special coordinates of [29]. Indeed we have

$$du^I = \int_{A_I} \Omega \qquad (4.18)$$

where $\Omega := \omega_j \wedge d\phi^j$. We can "normalize" our coordinates ϕ^i so that at $\phi^i = 0$, $\omega_i = \alpha_I$ and therefore $d\phi^i = du^I$ (see (4.4), (4.15)). The supersymmetry transformations of section 2 are worked out at $\phi^i = 0$. Note that the resulting space $T\mathcal{M}$ is Kähler, as follows from (4.8), and has the special property that the potential depends only on the real part of the complex coordinates. Due to this very restricted nature of the target space one may even expect some non-renormalization theorems in spite of having little supersymmetry. It will be interesting to investigate this question in more detail.

To summarize, the target space of the σ-model obtained by wrapping M5 on $\Sigma_3 \times S^1$ is given by the moduli space of a D3-brane wrapped completely on such a cycle. Due to the presence of the S^1 and the equivalence of 3D vectors and scalars (T-duality), it is clear that wrapping a D5 would produce a similar target space.

We conclude with two remarks on four-cycles. The case of a D5 wrapped on a cycle Σ_4 in a fourfold is somewhat puzzling: it would produce a space-filling two-dimensional theory that by general arguments [9] is expected to have very low supersymmetry. From the other side as seen from [12], in this case the cycle is complex and Kähler and the cotangent bundle construction yields a hyper-Kähler target space indicating at least four supercharges, and thus a supersymmetry enhancement.

Considering wrapped D-branes on SL leads to an identification [4] of the complexified space $H^1(\mathcal{C})$ and the moduli spaces of stable bundles V on the mirror manifold Y, $H^1(EndV, Y)$. It was conjectured in [29] that this relation is expected to generalize to $H^k(\mathcal{C}) = H^k(EndV, Y)$, for all k. A higher-rank space, $H^2(\mathcal{C})$, appears for the first time at $dim_{\mathbb{R}}\mathcal{C} = 4$ and is relevant only for wrapped M5-branes (while D-branes don't "see" this space). It would be interesting to understand if the M5-brane can play any role in testing the extended mirror conjecture.

Acknowledgments

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