D–brane Physics and Noncommutative Yang–Mills Theory

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Abstract

We discuss the physics of a single Dp–brane in the presence of a background electromagnetic field B_{ij}. It has recently been shown [1] that, in a specific $\alpha' \to 0$ limit, the physics of the brane is correctly described by noncommutative Yang–Mills theory, where the noncommutative gauge potential is given explicitly in terms of the ordinary $U(1)$ field. In a previous paper [2] the physics of a $D2$–brane was analyzed in the Sen–Seiberg limit of M(atrix) theory by considering a specific coordinate change on the brane world–volume.

We show in this note that the limit considered in [2] is the same as the one described in [1], in the specific case $p = 2$, rank $B_{ij} = 2$. Moreover we show that the coordinate change in [2] can be reinterpreted, in the spirit of [1], as a field redefinition of the ordinary Yang–Mills field, and we prove that the transformations agree for large backgrounds. The results are finally used to considerably streamline the proof of the equivalence of the standard Born–Infeld action with noncommutative Yang–Mills theory, in the large wave–length regime.

1   Introduction

In this short note we are going to examine the physics of a single flat D–brane in the presence of a large background electromagnetic field. The problem has received lately considerable attention and has been analyzed in complete detail in the recent work [1]. In particular it has been shown in [1] that, in order to describe the physics of the fluctuating electromagnetic field, one has two equivalent options. On one side, one can follow the standard treatment of the subject in terms of an ordinary $U(1)$ gauge field in the presence of the background field strength. On the other side, one can fully include the effects of the background by rewriting the action in terms of a $U(1)$ gauge field of noncommutative gauge theory. In this formulation, the background is used to define a Poisson structure and a star product on the brane world–volume, which are then used to define the noncommutative gauge theory. Ordinary gauge transformations, which form an abelian group, are replaced by noncommutative gauge transformations, which give rise to a non abelian group. Nonetheless gauge equivalent configurations are so in both descriptions, and therefore the gauge orbits are the same in both cases. Finally, in [1], the authors identify a specific $\alpha' \to 0$ limit in which the noncommutative description simplifies considerably and reduces to standard noncommutative Yang–Mills theory.

In a previous work [2], a similar setting and limit was considered in the analysis of the physics of $D2$–branes in Type IIA string theory. The motivations for the analysis in [2] are quite different from the ones in [1], and the discussion is driven by the attempt to connect the physics of a $D2$–brane to that of an $M2$–brane in the 11–dimensional strong

coupling limit of Type IIA. In particular, in order to connect with the 11–dimensional interpretation of the theory, it was crucial in [2] to treat on the same footing the directions transverse and parallel to the brane. This can be achieved by changing coordinates on the world–volume of the brane, so as to eliminate any fluctuation of the field strength in favor of fluctuations of the induced metric. Since the action governing the brane is invariant under diffeomorphisms, the coordinate change can be considered as a field redefinition, and does not alter the physics. On the other hand, in terms of the new fields, the action for a *D*2–brane possesses a smooth polynomial limit in the 11–dimensional limit of Type IIA.

In this note we will show that the limit considered in [2] is that of [1] in the specific case of a *D*2–brane. Moreover, we will show that the coordinate change considered in [2] is the same, to leading order in derivatives, to the one considered in [1]. The result can be understood as follows. As we have briefly described above, and as we shall explain in detail later, it is convenient to change coordinates in order to eliminate any fluctuations of the $U(1)$ field strength. This change of coordinates is not unique, but is defined up to diffeomorphisms of the world–volume of the brane which leave invariant the background field strength 2–form. The group of such diffeomorphisms is non abelian, and has as infinitesimal generators the Hamiltonian vector fields defined in terms of the background, now considered as a symplectic structure on the world–volume. These diffeomorphisms replace the ordinary abelian gauge invariance of the original theory, as in the case of [1]. Moreover, to leading order in derivatives, commutators in terms of the star product considered in [1] are nothing but Poisson brackets with respect to the symplectic structure defined by the background. This explains why the change of coordinates is the same to leading order.

Given the above facts, we show that the results of [2], generalized to the case of a generic *Dp*–brane, can be used to streamline the proof of the equivalence of the standard Born–Infeld action with noncommutative Yang–Mills theory, in the $\alpha' \to 0$ limit considered in [1]. Moreover, in the large wave–length regime, the results in this note give a clear geometric interpretation to the field redefinition given in [1].

We are going to use primarily the notation of [1], in order to allow a quick comparison of the equations. Moreover, given the nature of this

short note, we are not including an extensive reference list. For a more complete bibliography we refer the reader to the papers [1] and [2].

2   The Coordinate Redefinition

In this section we are going to analyze the physics of a flat Dp–brane in Type II string theory, whose world–volume is extended in the spacetime directions $0, \cdots, p$. We are not going to consider the transverse motions of the brane, and therefore we will limit our attention exclusively to the degrees of freedom of the $U(1)$ gauge potential living on the brane world–volume. In particular we will analyze the dynamics of the Dp–brane in the presence of a large background electromagnetic field.

We will denote with X^i, $i = 0, \cdots, p$, the coordinates on the target spacetime which are parallel to the Dp–brane. The world–volume $\Sigma = \mathbf{R}^{p+1}$ of the brane can be parameterized with the natural coordinates x^i $(i = 0, \ldots, p)$ inherited from the target, which are defined by

$$X^i(x) = x^i.$$

The embedding functions are therefore non–dynamical, and all of the physics of the Dp–brane (recall that we are neglecting transverse motions) is described by the fluctuating $U(1)$ gauge potential. As we already mentioned, we are going to work in the presence of a large constant background electromagnetic field B_{ij}. We will assume in this paper that the constant matrix B is of maximal rank. For the most part of what follows, we shall actually assume that p is odd, and therefore that B is invertible (rk $B = p + 1$). Following the notation of [1], we shall in this case denote with $\theta = B^{-1}$ the inverse matrix. At the end of the section we will briefly return to the case of even p (rk $B = p$).

Let us denote with $a = a_i(x)dx^i$ the fluctuating gauge potential, and with $f = da$ the corresponding field strength. The total field strength is then given by

$$F_{ij}(x) = B_{ij} + f_{ij}(x) = B_{ij} + \partial_i a_j(x) - \partial_j a_i(x). \qquad (1)$$

We now change parameterization on the world–volume Σ by choosing new coordinates σ^i. Clearly the target–space embedding coordi-

nates are now given by

$$X^i(\sigma) = x^i(\sigma).$$

Moreover the new field strength is now given by

$$\widetilde{F}_{ij}(\sigma) = \frac{\partial x^k}{\partial \sigma^i} \frac{\partial x^l}{\partial \sigma^j} F_{kl}(x(\sigma)). \tag{2}$$

We claim that we can choose the coordinates σ so that the the field strength $\widetilde{F}_{ij}$ is given by

$$\widetilde{F}_{ij}(\sigma) = B_{ij}. \tag{3}$$

In other words, we can eliminate any fluctuation of the electromagnetic field by a simple coordinate redefinition. In the new coordinate system σ^i the dynamics of the brane is not described by the $U(1)$ gauge potential $a_i(x)$, but is now equivalently described by the embedding functions $x^i(\sigma)$, which are now the dynamical fields.

Let us start our analysis by considering small fluctuations, and therefore by working to first order in a_i. First let us define the displacement functions

$$x^i(\sigma) = \sigma^i + d^i(\sigma).$$

For small fluctuations, d^i is of order $o(a)$. To first order in fluctuations equation (2) becomes

$$\widetilde{F}_{ij} = F_{ij} + \mathcal{L}_d F_{ij} = F_{ij} + d^k \partial_k F_{ij} + F_{kj} \partial_i d^k + F_{ik} \partial_j d^k,$$

where $\mathcal{L}_d$ is the Lie derivative in the direction of the vector–field d^i. Using equations (3) and (1), and recalling the antisymmetry of B_{ij}, we can rewrite the above (to order $o(a)$) as

$$f_{ij} = \partial_i a_j - \partial_j a_i = B_{jk} \partial_i d^k - B_{ik} \partial_j d^k.$$

The above is an identity if and only if

$$a_i = B_{ij} d^j + \partial_i \lambda \tag{4}$$

for some scalar λ. Therefore

$$d^i = \theta^{ij} a_j$$

up to transformations of the form

$$d^i \to d^i + \theta^{ij} \partial_j \lambda.$$

Let us now move to the general case. Suppose that we are given a generic gauge field $a_i(x)$, and a corresponding coordinate change $x^i(\sigma)$ which satisfies (3). We can then analyze, given an infinitesimal change $a_i \to a_i + \delta a_i$, the corresponding variation $x^i \to x^i + \delta x^i$. In fact we can use the results just derived above in the linear approximation. First let us change coordinates from x^i to σ^i. The gauge field fluctuation $\delta a_i(x)$ transforms as $\delta a_i(x) \to \widetilde{\delta a}_i(\sigma) = \delta a_j(x(\sigma))\partial_i x^j$. We are now in the condition analyzed previously, since, by construction, the field strength in the coordinates σ^i is equal to B_{ij}, and we are considering an additional infinitesimal change $\widetilde{\delta a}_i$ to the gauge potential. We can now use the above results on infinitesimal fluctuations to conclude that $x^i(\sigma) \to x^i(\sigma + \eta)$, where $\eta^i(\sigma) = \theta^{ij}\widetilde{\delta a}_j(\sigma)$. Therefore $\delta x^i = \eta^j \partial_j x^i$, and we have the relation

$$\delta x^i(\sigma) = \theta^{jk}\partial_j x^i \partial_k x^l \, \delta a_l(x(\sigma)). \tag{5}$$

We have seen before that, to first order in a_i, $x^i = \sigma^i + \theta^{ij}a_j$. Using this result and integrating the above equation we conclude that, to second order in a_i, we have

$$x^i = \sigma^i + \theta^{ij}a_j + \frac{1}{2}\theta^{ij}\theta^{kl}\left(2a_l\partial_k a_j + a_k\partial_j a_l\right) + \cdots.$$

To make contact with the work of [1], let us define the following variable
$$\hat{A}_i(\sigma) = B_{ij}d^j(\sigma)$$
which is given, in terms of a, by the formula

$$\hat{A}_i = a_i + \frac{1}{2}\theta^{kl}\left(2a_l\partial_k a_i + a_k\partial_i a_l\right) + \cdots. \tag{6}$$

We clearly see that, to first non–trivial order in θ, the field $\hat{A}_i$ corresponds to the one described in [1]. We will now better analyze this correspondence by describing how the original gauge invariance of the theory manifests itself in terms of the new dynamical fields d^i or, alternatively, $\hat{A}_i$.

The coordinate change from the variables x^i to the variables σ^i is defined so that the electromagnetic field, in the coordinates σ^i, is given by the constant matrix B_{ij}. Clearly the choice of coordinates σ^i is not unique. In fact, given an infinitesimal vector field $V^i(\sigma)$, we may

define new coordinates $\sigma^i + V^i(\sigma)$ which are equally valid if $\mathcal{L}_V B_{ij} = B_{kj}\partial_i V^k + B_{ik}\partial_j V^k = 0$. This is equivalent to the statement that the 1–form $B_{ij}V^j$ is closed, or that $V^i = \theta^{ij}\partial_j\rho$. The reparameterization of the world–volume Σ can be equivalently represented by a change in the functions $x^i(\sigma)$ given by

$$x^i \to x^i - \theta^{jk}\partial_j\rho\partial_k x^i = x^i + i\{\rho, x^i\}, \tag{7}$$

where we have introduced the Poisson bracket

$$\{A, B\} = i\,\theta^{ij}\,\partial_i A\partial_j B$$

on the brane world–volume. Note that

$$\{\sigma^i, \sigma^j\} = i\,\theta^{ij},$$

and more generally that

$$\{\sigma^i, A\} = i\,\theta^{ij}\partial_j A.$$

Written in terms of $\hat{A}_i$, equation (7) reads

$$\hat{A}_i \to \hat{A}_i + \partial_i\rho + i\{\rho, \hat{A}_i\}. \tag{8}$$

Let us mention, for completeness, the following consistency check. Consider, in equation (5), a variation $\delta a_i(x)$ which is pure gauge $\delta a_i(x) = \partial_i\lambda(x)$. It is then clear that, if we define $\rho(\sigma) = \lambda(x(\sigma))$, we have that $\partial_k\rho(\sigma) = \partial_k x^l \delta a_l(x(\sigma))$. Therefore the variation of $x^i(\sigma)$ is given by

$$\delta x^i = \theta^{jk}\partial_j x^i \partial_k\rho = i\{\rho, x^i\}$$

and is therefore pure gauge.

In the last part of this section we wish to connect the above discussion to that of [1]. In [1] the relation between a and $\hat{A}$ is derived starting from the knowledge of the correct non abelian gauge invariance $\hat{A}_i \to \hat{A}_i + \partial_i\rho + i[\rho, \hat{A}_i]$, where $[A, B] = A \star B - B \star A = \{A, B\} + o(\theta^2)$. Equivalently, we can start with the gauge transformation (8) and follow the argument of [1] to derive (6). In fact, the computation is exactly the same, since all the formulae are identical to first non–trivial order in θ. In the treatment in this note the noncommutative gauge group is the set of diffeomorphisms of the brane world–volume which leave the 2–form B_{ij} invariant. The group is infinitesimally generated by

the vector fields of the form $V^i = \theta^{ij}\partial_j\rho$, which are nothing but the Hamiltonian vector fields defined in terms of B_{ij}, now considered as a symplectic structure on Σ.

We have worked in the case of p odd. Let us now very briefly discuss the case of even p, when B_{ij} cannot be invertible. Divide the coordinates X^i in X^0 and X^a, $a = 1, \cdots, p$ and assume that $B_{0a} = 0$ and that B_{ab} is invertible. Let us consider equation (4). For a correct choice of λ we can work under the assumption that $a_0 = 0$. It is then clear that we can solve equation (4) by imposing

$$d^0 = 0 \qquad x^0(\sigma) = \sigma^0 \tag{9}$$

and by choosing $d^a = \theta^{ab}a_b$, where θ^{ab} is the inverse of the invertible part B_{ab} of the background field strength. It is also clear that we can impose the constraint (9) not only for small fluctuations of the commutative gauge potential a_i, but, in complete generality, to all orders in a_i (this is the choice discussed in [2], where the attention was focused on the case $p = 2$ for physical reasons). The constraint (9) restricts the noncommutative gauge group to the diffeomorphism which leave the background field strength and (9) invariant. This group is generated by Hamiltonian vector fields V^i with $V^0 = 0$ and which satisfy $\partial_i\rho = B_{ij}V^j$. This means that ρ is time independent and that we can essentially reduce the problem by one dimension, therefore going back to the case previously discussed.

3   The $\alpha' \to 0$ Limit

In this last section we analyze the $\alpha' \to 0$ limit considered in [1]. Following the reasoning in [2] and using the results of the last section we streamline the proof of the equivalence of the standard Born–Infeld action with noncommutative Yang–Mills theory in the large wave–length regime. We will work again for convenience in the case p odd. This is done both for notational simplicity and since the case $p = 2$ was treated in detail in [2]. In what follows we shall closely follow the notation of [1] in order to make quick contact with the results of that paper.

As in the last section we fix the $U(1)$ field strength to be B_{ij} and consider as dynamical fields the embedding functions $x^i(\sigma)$. Following

[1] we assume that the metric in the target spacetime is given by

$$g_{ij}.$$

The induced metric on the brane is then given by

$$h_{ij} = \partial_i X^k \partial_j X^l \, g_{kl}.$$

Again as in [1], we consider the following limit[1] (recall that we are considering the case of B_{ij} of maximal rank $p+1$)

$$g_{ij} \sim \epsilon \qquad \alpha' \sim \epsilon^{1/2} \qquad g_s \sim \epsilon,$$

where we take $\epsilon \to 0$. The tension of the Dp–brane is given by $T \propto 1/(g_s l_s^{p+1})$ and therefore scales as

$$T \sim \epsilon^{-\frac{p+5}{4}}.$$

We can then expand the Born–Infeld action as

$$\begin{aligned}
S &= T \int_\Sigma d^{p+1}\sigma \sqrt{\det\left(h_{ij} + 2\pi\alpha' B_{ij}\right)} \\
&= T \int_\Sigma d^{p+1}\sigma \sqrt{\det\left(2\pi\alpha' B_{ij}\right)} \\
&\quad \cdot \left(1 - \frac{1}{4}\left(\frac{1}{2\pi\alpha'}\right)^2 \theta^{ij} \partial_j X^a \partial_k X^b g_{ab} \theta^{kl} \partial_l X^c \partial_i X^d g_{cd}\right) + \cdots \\
&= \text{const.} - \frac{T}{4}\left(\frac{1}{2\pi\alpha'}\right)^2 \int_\Sigma d^{p+1}\sigma \sqrt{\det(2\pi\alpha' B_{ij})} \\
&\quad \cdot \{X^a, X^c\}\{X^b, X^d\} g_{ab} g_{cd} + \cdots.
\end{aligned}$$

The action scales as $T(\alpha')^{\frac{p-3}{2}}(g_{ab})^2$, and is therefore finite in the $\epsilon \to 0$ limit, thus showing that the Born–Infeld action does have a smooth polynomial limit, if it is written in terms of the correct variables.

To make contact with [1], let us define, as in the last section, the noncommutative gauge potential $\hat{A}_i$ by the equation

$$X^i = \sigma^i + \theta^{ij} \hat{A}_j.$$

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[1]The constant η in [2] is related to ϵ in [1] by $\epsilon = \eta^4$. With this redefinition one can easily check that the limit considered in [2] is exactly the same as that analyzed in [1], in the particular case of $p = 2$ and rk $B_{ij} = 2$.

If we define the noncommutative field strength by

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i - i\{\hat{A}_i, \hat{A}_j\}$$

we can readily check that

$$-i\{X^i, X^j\} = \theta^{ij} + \theta^{ik}\theta^{jl}\hat{F}_{kl}.$$

Define as in [1] the open–string metric

$$
\begin{aligned}
G_{ij} &= -(2\pi\alpha')^2 B_{ik} g^{kl} B_{lj} \\
G^{ij} &= -\frac{1}{(2\pi\alpha')^2} \theta^{ik} g_{kl} \theta^{lj}
\end{aligned}
$$

and the open–string coupling constant

$$G_s = g_s \det{}^{1/2}\left(2\pi\alpha' B_{ik} g^{kj}\right) \sim \epsilon^{\frac{3-p}{4}}.$$

It is then quick to check that

$$
-\left(\frac{1}{2\pi\alpha'}\right)^2 \{X^a, X^c\}\{X^b, X^d\} g_{ab} g_{cd}
$$
$$
= (2\pi\alpha')^2 G^{ab} G^{cd} \hat{F}_{ac} \hat{F}_{bd} + \text{const.} + \text{total derivative.}
$$

Therefore the action S becomes

$$S = \text{const.} + \frac{1}{4G_{YM}^2} \int_\Sigma d^{p+1}\sigma \sqrt{\det G_{ij}} G^{ab} G^{cd} \hat{F}_{ac} \hat{F}_{bd},$$

where

$$G^2{}_{YM} = \frac{1}{(2\pi)^{p-2} G_s l_s^{p-3}}$$

is the noncommutative Yang–Mills coupling, which is independent of ϵ.

   This then concludes the proof of the equivalence of the commutative and noncommutative descriptions in the large wave–length approximation. It is uniquely based on the invariance under diffeomorphisms of the Born–Infeld action. This fact both simplifies the proof considerably and also clarifies the geometric nature of the field redefinition to leading order in the derivative expansion. Let we also note, as a conclusion, that the analysis and the results in [1], restricted to the description of flat $D2$–branes, do give a formal and complete proof of the conjecture stated in [2].

Acknowledgments

References

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