Painlevé III Equation and Bianchi VII$_0$ Model

Nenad Manojlović and Aleksandar Miković[1]

Área Departamental de Matemática, UCEH
Universidade do Algarve, Campus de Gambelas
8000 Faro, Portugal

nmanoj@ualg.pt

amikovic@ualg.pt

Abstract

We examine the reduced phase space of the Bianchi VII$_0$ cosmological model, including the moduli sector. We show that the dynamics of the relevant sector of local degrees of freedom is given by a Painlevé III equation. We then obtain a zero-curvature representation of this Painlevé III equation by applying the Belinskii-Zakharov method to the Bianchi VII$_0$ model.

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1 Introduction

Bianchi VII_0 cosmological model is a finite-dimensional dynamical system with non-trivial dynamics. It can be considered as a symmetry reduction of the Gowdy model [1] whose spatial hypersurface is the three torus T^3. The Gowdy models represent examples of two commuting Killing reductions of general relativity [2], whose space-times have compact spatial hypersurfaces. Although the two commuting Killing vector system is integrable [3], [4], [5], [6], [7], it is not known how to extend the corresponding methods of constructing solutions [8], [9], [10], [11] to the compact case topology. Therefore Bianchi VII_0 model can serve as a toy model for exploring the issue of integrability of the Gowdy model on the three torus.

Belinskii and Francaviglia showed, within a more general framework of Belinskii and Zakharov inverse scattering method [5], that the Einstein equations for some Bianchi models admit a zero-curvature representation, indicating that these are solvable dynamical systems [12]. However, their analysis was not complete in two aspects. The first aspect is related to the fact that their considerations were only local, since they ignored topological obstructions coming from the non-trivial global topology of the spatial hypersurface, the three torus T^3. The problem is that spacetimes with compact spatial sections do not allow in general global Bianchi metrics, and one can put only locally homogeneous metrics [13], [14]. Related to that is that compact spatial manifolds have non-trivial topology, and locally diffeomorphic metrics are not necessarily globally diffeomorphic, which means that there are global degrees of freedom in the metric, beside the usual local ones. It has been shown recently that the moduli parameters enter non-trivially in the diffeomorphism invariant symplectic form, and hence they could change the dynamics of the local degrees of freedom [15], [16]. The second unexplored aspect is to find out what kind of integrable nonlinear dynamical equation can be obtained.

In order to explore these issues, it is convenient to study the dynamics of Bianchi VII_0 model in the canonical formalism. We perform constraint and gauge-fixing analysis and show that the dynamics of a generic sector of local degrees of freedom can be reduced to that of a Painlevé III equation. There is also a special sector with enhanced symmetry, which has a linear dynamical equation. By using the results

of Kodama [16] we show that the moduli parameters do not change the dynamics of generic local degrees of freedom. We then show how Bianchi VII_0 model appears in the Belinskii-Zakharov approach, and how to obtain Painlevé III equation. We then use these results to obtain a linear system whose zero-curvature condition is a Painlevé III equation.

2   The Class A Bianchi Models

Bianchi models are spatially homogeneous spacetimes which admit a three dimensional isometry Lie group G that acts simply transitively on each leaf Σ of the homogeneous foliation, for a review and references see [17]. As a consequence, there exists for each of these models a set of three left-invariant vector fields L_I on Σ which form the Lie algebra of the group G:

$$[L_I, L_J] = C^K{}_{IJ} L_K, \tag{2.1}$$

where $C^I{}_{JK}$ are the structure constants of the Lie group.

Dual to the the vector fields L_I, one can introduce a set of three left-invariant one-forms χ^I which satisfy the Maurer-Cartan equations

$$\mathrm{d}\chi^I + \frac{1}{2} C^I{}_{JK} \; \chi^J \wedge \chi^K = 0. \tag{2.2}$$

If the trace $C^I{}_{IJ}$ of the structure constants is equal to zero, the Bianchi model is said to belong to Bianchi class **A**. For this class of models, the spacetime admits foliations by compact slices.

The structure constants for the class **A** Bianchi models can always be written in the form

$$C^I{}_{JK} = \epsilon_{JKL} S^{LI}, \tag{2.3}$$

where ϵ_{JKL} is the totally antisymmetric symbol, and S^{IL} is a symmetric tensor density of weight one over the Lie algebra of G. Further classification of the class **A** Bianchi models is defined with respect to the signature of the symmetric tensor density S^{IJ}. The type VII Bianchi model of class **A** is denoted as Bianchi VII_0, and it is characterized by

the signature $(+, +, 0)$. Hence the structure constants for this model are given by

$$C^I{}_{JK} = \epsilon_{JK1}\,\delta_1^I + \epsilon_{JK2}\,\delta_2^I. \qquad (2.4)$$

Bianchi models can be considered as the homogeneous sector of general relativity. The dynamics can be obtained by performing the corresponding reduction of the canonical formulation of general relativity. The canonical variables of general relativity are the three metric $g_{ij}(t, x^i)$ on the spatial section Σ, and its canonically conjugate momenta $\pi^{ij}(t, x^i)$, where x^i are coordinates on Σ. The action for a Bianchi model can be obtained by inserting the expressions

$$g_{ij}(t, x) = g_{IJ}(t)\chi_i^I(x)\chi_j^J(x), \quad \pi^{ij}(t, x) = \pi^{IJ}(t)L_I^i(x)L_J^j(x) \qquad (2.5)$$

into the canonical form of the Einstein-Hilbert action. This gives

$$S_{BA} = \int_{t_0}^t dt\left(\pi^{IJ}\dot{g}_{IJ} - N^I H_I - N\,H_0\right), \qquad (2.6)$$

where the canonical variables (g_{IJ}, π^{IJ}) have the Poisson brackets

$$\{\,g_{IJ},\,\pi^{KL}\} = \frac{1}{2}\left(\delta_I^K\,\delta_J^L + \delta_I^L\,\delta_J^K\right). \qquad (2.7)$$

The vector constraint H_I, $I = 1, 2, 3$, is the reduction of the diffeomorphism constraint, and H_I is given by

$$H_I = 2\,C^J{}_{KI}\,g_{JL}\,\pi^{LK} = 2\,\epsilon_{KIM}S^{MJ}\,g_{JL}\,\pi^{LK} \approx 0. \qquad (2.8)$$

The reduction of the Hamiltonian constraint is H_0, and it is given by

$$H_0 = \frac{1}{\sqrt{\det g}}\left(\mathrm{Tr}\,(g\pi g\pi) - \frac{1}{2}\mathrm{Tr}^2(g\pi) + \mathrm{Tr}\,(gSgS) - \frac{1}{2}\mathrm{Tr}^2(gS)\right) \approx 0. \qquad (2.9)$$

N^I and N are the corresponding Lagrange multipliers. The constraints (2.8) and (2.9) form a closed Poisson algebra

$$\{H_I,\,H_J\} = C^K{}_{IJ}\,H_K, \qquad (2.10)$$

$$\{H_0,\,H_I\} = 0, \qquad (2.11)$$

and therefore they constitute a set of first-class constraints. Furthermore, the equation (2.11) implies that the hamiltonian constraint is invariant under the transformations generated by the vector constraint. This fact can be used to go to the parametrized particle form of the action, defined by the diffeomorphism invariant variables and the Hamiltonian constraint.

3   Vector Constraint

The standard approach to finding the dynamics of Bianchi models is to solve first the vector constraint. This requires the corresponding gauge-fixing, and because of (2.11), one can even find the diffeomorphism invariant variables explicitly.

In the case of Bianchi VII_0 model the topology of Σ is fixed to be the three torus T^3 and the coordinates $x^i = (x, y, z)$ can be chosen such that χ^1, χ^2, χ^3 have the canonical form

$$\chi^1 = \cos z \, dx + \sin z \, dy \,, \quad \chi^2 = -\sin z \, dx + \cos z \, dy \,, \qquad \chi^3 = dz \,.$$
$$(3.1)$$

The three left-invariant one-forms χ^I satisfy the Maurer-Cartan equations for the type VII Bianchi model

$$d\chi^1 + \chi^2 \wedge \chi^3 = 0 \,, \quad d\chi^2 + \chi^3 \wedge \chi^1 = 0 \,, \quad d\chi^3 = 0 \,. \qquad (3.2)$$

Note that every Bianchi model has a symmetry group M, which is the automorphism group of the Lie algebra of G. M can can be realized as a subgroup of $\mathbf{GL}(3, \mathbb{R})$ in the following way. Let us consider a set of the left-invariant vector fields L_I. They form the Lie algebra of G through the commutation relations (2.1). An invertible matrix $M^I{}_J$ yields a new set of left-invariant vector fields $\tilde{L}_J = L_I M^I_J$, whose commutation relations will have the same structure constants as L_I if the following identity is satisfied

$$C^I{}_{JK} = (M^{-1})^I{}_L \, C^L{}_{MN} \, M^M{}_J \, M^N{}_K \,, \qquad (3.3)$$

where $(M^{-1})^I{}_L$ is the inverse of the matrix $M^L{}_I$. The matrices M^I_J with the condition (3.3) define the symmetry group M.

Equivalently, with the help of the identity (2.3), we can define M as

$$S^{IJ} = (\det M)^{-1} \, M^I{}_K \, S^{KL} \, (M^T)_L{}^J \,, \qquad (3.4)$$

where $(M^T)_L{}^J$ is the transpose of the matrix $M^J{}_L$. In particular, in the case of the Bianchi-VII model the tensor density S^{IJ} has the signature $(+, +, 0)$ (see equation (2.4)), and hence the condition (3.4) implies the

following form of the matrix M

$$M = M_D \, M_E = \begin{pmatrix} e^{\frac{3}{2}c_0} & 0 & 0 \\ 0 & \pm e^{\frac{3}{2}c_0} & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & u \\ -\sin\theta & \cos\theta & v \\ 0 & 0 & 1 \end{pmatrix} . \qquad (3.5)$$

Let us consider the following change of variables

$$g_{IJ}(t) = \left(M_E^{\,T}\right)_I^{\ K}(t) \, Q_{KL}(t) \, \left(M_E\right)^L_{\ J}(t), \qquad (3.6)$$

where $(M_E)^I_{\ J}(t)$ is given by

$$M_E(t) = \begin{pmatrix} \cos\theta(t) & \sin\theta(t) & u(t) \\ -\sin\theta(t) & \cos\theta(t) & v(t) \\ 0 & 0 & 1 \end{pmatrix} , \qquad (3.7)$$

and $Q_{IJ} = \mathrm{diag}(Q_1, Q_2, Q_3)$ is a diagonal matrix. It is useful to introduce new variables $(\beta_0, \beta_+, \beta_-)$ as

$$Q_1 = e^{2(\beta^0 + \beta^+ + \sqrt{3}\beta^-)}, \ \ Q_2 = e^{2(\beta^0 + \beta^+ - \sqrt{3}\beta^-)}, \ \ Q_3 = e^{2(\beta^0 - 2\beta^+)}. \qquad (3.8)$$

The definition (3.6) involves only M_E, because by rescaling of Q by constants we can always put $M_D = Id$.

We have two alternative ways of imposing the constraints H_I (2.8). One way is to complete the canonical transformation (3.6) by calculating the conjugate momenta

$$p_\theta = \frac{\partial g_{IJ}}{\partial \theta} \pi^{IJ}, \quad p_u = \frac{\partial g_{IJ}}{\partial u} \pi^{IJ}, \quad p_v = \frac{\partial g_{IJ}}{\partial v} \pi^{IJ}, \qquad (3.9)$$

$$p_0 = \frac{\partial g_{IJ}}{\partial \beta^0} \pi^{IJ}, \quad p_+ = \frac{\partial g_{IJ}}{\partial \beta^+} \pi^{IJ}, \quad p_- = \frac{\partial g_{IJ}}{\partial \beta^-} \pi^{IJ}, \qquad (3.10)$$

and then expressing the constraints (2.8) in terms of the new canonical pairs $(\theta, p_\theta, u, p_u, v, p_v, \beta^0, p_0, \beta^+, p_+, \beta^-, p_-)$. This is the approach taken by Kodama [16]. Alternatively, we can calculate π^{IJ} as a function of $(\theta, u, v, \beta^0, \beta^\pm)$ and their time derivatives $(\dot\theta, \dot u, \dot v, \dot\beta^0, \dot\beta^+, \dot\beta^-)$ and then impose the constraints H_I. Once the constraints are imposed we can calculate the pre-symplectic form and determine the canonically conjugate momenta for the diffeomorphism invariant variables. We will choose the second alternative, because it is simpler and it gives an independent check of Kodama's results.

Our first step is to express the conjugate momenta π^{IJ} in terms of the new variables

$$\pi^{IJ} = -\sqrt{\det g}\left(g^{IM} K_{MN} g^{NJ} - g^{IJ} g^{MN} K_{MN}\right), \tag{3.11}$$

where the extrinsic curvature K_{IJ} is defined by

$$K_{IJ} = \frac{1}{2N}\left(-\dot{g}_{IJ} + (L_{\vec{N}}g)_{IJ}\right). \tag{3.12}$$

Our choice of foliation is such that $N^I = 0$, so that

$$K_{IJ} = -\frac{1}{2N}\,\dot{g}_{IJ}. \tag{3.13}$$

The vector constraint (2.8) is expressed in terms of the product $g_{IK}\pi^{KJ}$ and from the equations (3.11) and (3.13) we calculate

$$g_{IK}\,\pi^{KJ} = \frac{\sqrt{\det g}}{2N}\left(\dot{g}_{IK}g^{KJ} - \delta_I^J\left(\dot{g}_{LK}g^{KL}\right)\right). \tag{3.14}$$

Thus,

$$H_1 = 2\,g_{2K}\,\pi^{K3} \sim \dot{g}_{2K}g^{K3}, \tag{3.15}$$

$$H_2 = -2\,g_{1K}\,\pi^{K3} \sim \dot{g}_{1K}g^{K3}, \tag{3.16}$$

$$H_3 = 2\left(g_{1K}\,\pi^{K2} - g_{2K}\,\pi^{K1}\right) \sim \left(\dot{g}_{1K}g^{K2} - \dot{g}_{2K}g^{K1}\right). \tag{3.17}$$

In order to impose the vector constraint, we only have to calculate the components of the product $\dot{g}_{IK}g^{KJ}$ in terms of our new variables (3.6)

$$\dot{g}_{IK}g^{KJ} = \left(M_E^T Q\,\dot{M}_E M_E^{-1} Q^{-1}\left(M_E^T\right)^{-1}\right)_I^{\ J}$$
$$+ \left(M_E^T\,\dot{Q}\,Q^{-1}\left(M_E^T\right)^{-1}\right)_I^{\ J} + \left(\dot{M}_E^{\ T}\left(M_E^T\right)^{-1}\right)_I^{\ J}. \tag{3.18}$$

We substitute (3.7) and (3.8) into (3.18) and after a straightforward calculation we find that $H_I = 0$ is equivalent to

$$\dot{u} - v\dot{\theta} = 0 \quad,\quad \dot{v} + u\dot{\theta} = 0 \quad,\quad \sinh^2(2\sqrt{3}\beta^-)\dot{\theta} = 0. \tag{3.19}$$

Now we insert (3.6) into (3.11), and by taking into account (3.19), we obtain

$$\pi^{IJ}(t) = (M_E^{-1})_I^{\ K} P_{KL}(t)\,((M_E^{-1})^T)^L_{\ J}, \tag{3.20}$$

where the matrix $P = \text{diag}(P_1, P_2, P_3)$ and it is given by

$$P_1 = \frac{\sqrt{\det g}}{N} e^{-2(\beta^0 + \beta^+ + \sqrt{3}\beta^-)} \left(-2\dot{\beta}^0 + \dot{\beta}^+ + \sqrt{3}\dot{\beta}^-\right),$$

$$P_2 = \frac{\sqrt{\det g}}{N} e^{-2(\beta^0 + \beta^+ - \sqrt{3}\beta^-)} \left(-2\dot{\beta}^0 + \dot{\beta}^+ - \sqrt{3}\dot{\beta}^-\right),$$

$$P_3 = -2\frac{\sqrt{\det g}}{N} e^{-2(\beta^0 - 2\beta^+)} \left(\dot{\beta}^0 + \dot{\beta}^+\right). \tag{3.21}$$

Note that in our calculation of P we have used the constraints (3.19), and therefore the expression (3.21) is valid for all solutions of the constraints. The equations (3.19) have two classes of solutions. One class is given by

$$\dot{u} = \dot{v} = \dot{\theta} = 0,$$

which corresponds to the sector $\beta^- \neq 0, \dot{\beta}^- \neq 0$. This is a generic sector invariant under the Bianchi-VII group [16]. In this sector the vector constraint gives that u, v and θ are constants and hence P is automatically diagonal. The second class of solutions is

$$\dot{u} - v\dot{\theta} = \dot{v} + u\dot{\theta} = \beta^- = 0,$$

and it is less obvious that P is diagonal in this case, but it is a consequence of the constraints (3.19). Namely, $\dot{\theta}$ appears in the off-diagonal part of P, but it is multiplied by a factor $Q_1^{-1} - Q_2^{-1}$, which vanishes for $\beta^- = 0$. This is the sector which is invariant under a group larger then the Bianchi-VII group [16]. We will concentrate on the generic sector, although we will give some brief comments about the enhanced symmetry sector.

The pre-symplectic structure α by definition (2.6) is

$$\alpha = \pi^{IJ} \, \mathrm{d}g_{IJ}. \tag{3.22}$$

In order to calculate α in terms of the new variables we substitute the expressions (3.20) and (3.21) for the three metric g_{IJ} and its conjugate momenta π^{IJ} into the equation (3.22), and we obtain

$$\alpha = \text{Tr}\left(P\mathrm{d}Q\right) + 2\text{Tr}\left(PQ \, \mathrm{d}M_E \, M_E^{-1}\right). \tag{3.23}$$

The second term in (3.23) can be calculated from the definition of the matrix M_E (3.7), and we get

$$\mathrm{d}M_E \, M_E^{-1} = \begin{pmatrix} 0 & \mathrm{d}\theta & \mathrm{d}u - v\mathrm{d}\theta \\ -\mathrm{d}\theta & 0 & \mathrm{d}v + u\mathrm{d}\theta \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.24}$$

From (3.20) and (3.21) we obtain that the components of the diagonal matrix $\left(PQ\right)^I{}_J$ are

$$\left(PQ\right)^1{}_1 = \frac{\sqrt{\det g}}{N}\left(-2\dot{\beta}^0 + \dot{\beta}^+ + \sqrt{3}\,\dot{\beta}^-\right),\qquad (3.25)$$

$$\left(PQ\right)^2{}_2 = \frac{\sqrt{\det g}}{N}\left(-2\dot{\beta}^0 + \dot{\beta}^+ - \sqrt{3}\,\dot{\beta}^-\right),\qquad (3.26)$$

$$\left(PQ\right)^3{}_3 = -2\,\frac{\sqrt{\det g}}{N}\left(\dot{\beta}^0 + \dot{\beta}^+\right).\qquad (3.27)$$

Since PQ is a diagonal matrix, it is obvious that

$$\mathrm{Tr}\left(PQ\,\mathrm{d}M_E\,M_E^{-1}\right) = 0.\qquad (3.28)$$

Thus, the second term in (3.23) is identically equal to zero and α is given by

$$\alpha = \mathrm{Tr}\left(P\mathrm{d}Q\right).\qquad (3.29)$$

From

$$\mathrm{d}Q_1 = e^{2\,(\beta^0+\beta^++\sqrt{3}\,\beta^-)}2(\mathrm{d}\beta^0 + \mathrm{d}\beta^+ + \sqrt{3}\,\mathrm{d}\beta^-),\qquad (3.30)$$

$$\mathrm{d}Q_2 = e^{2\,(\beta^0+\beta^+-\sqrt{3}\,\beta^-)}2(\mathrm{d}\beta^0 + \mathrm{d}\beta^+ - \sqrt{3}\,\mathrm{d}\beta^-),\qquad (3.31)$$

$$\mathrm{d}Q_3 = e^{2\,(\beta^0-2\beta^+)}2(\mathrm{d}\beta^0 - 2\,\mathrm{d}\beta^+),\qquad (3.32)$$

and from (3.21) we obtain

$$\alpha = 12\,\frac{\sqrt{\det g}}{N}\left(-\dot{\beta}^0\mathrm{d}\beta^0 + \dot{\beta}^+\mathrm{d}\beta^+ + \dot{\beta}^-\mathrm{d}\beta^-\right).\qquad (3.33)$$

Since

$$p_0 = -\left(12\,\frac{\sqrt{\det g}}{N}\right)\dot{\beta}^0,\qquad p_+ = \left(12\,\frac{\sqrt{\det g}}{N}\right)\dot{\beta}^+,$$

$$p_- = \left(12\,\frac{\sqrt{\det g}}{N}\right)\dot{\beta}^-,\qquad (3.34)$$

the pre-symplectic form becomes

$$\alpha = p_0\mathrm{d}\beta^0 + p_+\mathrm{d}\beta^+ + p_-\mathrm{d}\beta^-.\qquad (3.35)$$

Note that (3.35) is the pre-symplectic form for the generic sector. The enhanced symmetry sector is a special case where $p_- = \beta^- = 0$, so that the pre-symplectic form becomes

$$\alpha = p_0\mathrm{d}\beta^0 + p_+\mathrm{d}\beta^+.\qquad (3.36)$$

Dynamics of the canonical pairs $(\beta^0, p_0, \beta^+, p_+, \beta^-, p_-)$ is defined by the Hamiltonian constraint (2.9). To calculate the expression for the Hamiltonian constraint in terms of the diffeomorphism invariant variables we substitute (3.20) and (3.21) into (2.9) and obtain

$$H_0 = \frac{1}{\sqrt{\det g}}\Big(\mathrm{Tr}\,(PQPQ) - \frac{1}{2}\mathrm{Tr}^2(PQ) + \mathrm{Tr}\,(SQSQ) - \frac{1}{2}\mathrm{Tr}^2(SQ)\Big). \tag{3.37}$$

We notice that the Hamiltonian constraint is independent of the unphysical variables (θ, u, v). In addition, we already have the components of the product $\big(PQ\big)^I{}_J$, which are given by the equations (3.25), (3.26) and (3.27), and SQ is a diagonal matrix

$$\big(SQ\big)^1{}_1 = e^{2(\beta^0+\beta^++\sqrt{3}\beta^-)}, \quad \big(SQ\big)^2{}_2 = e^{2(\beta^0+\beta^+-\sqrt{3}\beta^-)}, \quad \big(SQ\big)^3{}_3 = 0. \tag{3.38}$$

From this we obtain that the Hamiltonian constraint for the generic sector is given by

$$H_0 = \frac{6\sqrt{\det g}}{N^2}\big(-(\dot{\beta^0})^2 + (\dot{\beta^+})^2 + (\dot{\beta^-})^2\big) + \frac{2e^{4(\beta^0+\beta^+)}}{\sqrt{\det g}}\sinh^2(2\sqrt{3}\beta^-), \tag{3.39}$$

or in terms of the canonical pairs (3.34)

$$H_0 = \frac{1}{24\sqrt{\det g}}\big(-(p_0)^2 + (p_+)^2 + (p_-)^2 + 48e^{4(\beta^0+\beta^+)}\sinh^2(2\sqrt{3}\beta^-)\big). \tag{3.40}$$

The previous calculation applies also to the enhanced symmetry sector, but one must take into account that $p_- = \beta^- = 0$. The Hamiltonian constraint then becomes

$$H_0 = \frac{1}{24\sqrt{\det g}}\big(-(p_0)^2 + (p_+)^2\big), \tag{3.41}$$

which corresponds to the dynamics of a two-dimensional relativistic free particle.

4 Diffeomorphism invariant phase space

Note that the reduced pre-symplectic form (3.35) implies that the diffeomorphism invariant phase space is given by (p_μ, β^μ) canonical pairs.

However, this is not correct, since it has been shown recently that the diffeomorphism invariant subspace is larger, because it contains the moduli parameters, which are associated with the global degrees of freedom of the metric [15], [16]. This is a consequence of the fact that Bianchi metrics on compact spatial slices represent compact Reimannian manifolds with non-trivial topology, and such manifolds can be locally diffeomorphic, but not globally diffeomorphic.

As a result one cannot define globally a Bianchi metric, so that one needs a notion of locally homogeneous spacetime [13], [14]. A locally homogeneous Bianchi spacetime M with a symmetry group G can be represented as $\tilde{M}/K$, where $\tilde{M}$ is a homogeneous Bianchi space-time with a simply connected spatial section, and K is a discrete subgroup of G [15]. In the following we will review Kodama's results [16], since his approach is suitable for the Hamiltonian formalism.

The symmetry group $\tilde{G}$ of the covering space $\tilde{M}$ is defined as

$$\tilde{G} = \{f \in Diff(\tilde{M}) \mid f_*\tilde{\Phi} = \tilde{\Phi}\},$$

such that it is isomorphic to G, or it is the smallest possible group which contains G, and it acts transitively on $\tilde{M}$. $\tilde{\Phi} = j^*\Phi$ is the pullback of the canonical data Φ on M and $j : \tilde{M} \to M$ is the covering map.

The moduli space can be parametrized with finitely many parameters $\{\lambda^a\}$, which are determined from the covering maps $j_\lambda : \tilde{M} \to M$ such that $j_\lambda^*(\pi_1(M)) = K_\lambda$ is isomorphic to K and it is a subgroup of $\tilde{G}$, where $\pi_1(M)$ is the fundamental group of M. One then has to find conjugacy classes of K_λ under the homogeneity preserving diffeomorphisms HPDG

$$HPDG = \{\tilde{f} \in Diff(\tilde{M}) \mid \tilde{f}\tilde{G}\tilde{f}^{-1} = \tilde{G}\} \quad .$$

This procedure gives that $K_\lambda = f_\lambda K_0 f_\lambda^{-1}$, where K_0 is a reference point, such that $\tilde{M}/K_0$ is identified with M, and f_λ is a linear transformation on $\tilde{M}$ such that

$$
\begin{aligned}
f_\lambda^* \chi^I &= F_J^I \chi^J \\
f_\lambda^* g_{IJ} &= (F^T)_I^K g_{KL} F_L^K \\
f_\lambda^* \pi^{IJ} &= (F^{-1})_K^I \pi^{KL}((F^{-1})^T)_L^J \\
f_\lambda^* \sqrt{\det g} &= (\det F)(\det \chi)\sqrt{\det g_{IJ}}\,.
\end{aligned}
\tag{4.1}
$$

From (4.1) one obtains that the pre-symplectic form (3.23) becomes

$$\alpha = \frac{\Omega(\lambda)}{V}(\pi^{IJ}dg_{IJ} + 2C_a(\lambda)\pi^{IJ}g_{IJ}d\lambda^a),$$

where $V = \int_{D_0} d^3x \det \chi$,

$$\Omega(\lambda) = \int_{D_0} d^3x (\det \chi)(\det F),$$

$$C_a(\lambda) = \frac{1}{\Omega}\int_{D_0} d^3x (\det \chi)(\det F)\partial_a F F^{-1},$$

and D_0 is the fundamental region of the action of K_0 on $\tilde{M}$. The Hamiltonian constraint is rescaled by $V/\Omega(\lambda)$.

In the case of Bianchi VII model, $G = VII_0^+$, where $+$ fixes the orientation of the spatial section. M is a three-torus T^3, which can be represented as E^3/K where E^3 is the Euclidian space, and $K = \mathbf{Z}^3$. $\tilde{G} = G\tilde{\times}D_2$, where D_2 is the dihedral group and $\tilde{\times}$ denotes a semi-direct product. K_λ are represented by $GL(3,\mathbf{R})$ matrices, whose last row is given by $(l\pi, m\pi, n\pi)$, $l,m,n \in \mathbf{Z}$. By using HPDG and modular transformations, K_λ can be put into form

$$K_\lambda = \begin{pmatrix} X & Y & Z \\ 0 & X^{-1} & W \\ 0 & 0 & n\pi \end{pmatrix}, \quad X > 0, \quad n \geq 0. \tag{4.2}$$

Therefore X, Y, Z, W and n represent the modular parameters. The K_0 is associated with $X = 1$, $Y = Z = W = 0$, and the deformation map f_λ is given by the matrix

$$A = \begin{pmatrix} X & Y & Z \\ 0 & X^{-1} & W \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.3}$$

The fundamental region is $D_0 = \{0 \leq x, y \leq 1, 0 \leq z \leq n\pi\}$.

From (4.3) one obtains

$$F = R_3(z)AR_3(-z), \tag{4.4}$$

where R_3 is the rotational matrix around z-axes. The correction in the pre-symplectic form due to modular parameters is given by

$$Tr(\dot{F}F^{-1}pg) = \left[\frac{\dot{X}}{X}\cos(2z) - \frac{1}{2}(X\dot{Y} - Y\dot{X})\sin(2z)\right](Q_1P_1 - Q_2P_2). \tag{4.5}$$

However, when (4.5) is integrated over D_0, one obtains zero, and hence $\delta\alpha$ vanishes [16]. Therefore the moduli parameters do not enter into the reduced pre-symplectic form, and hence they will not influence the dynamics of the local degrees of freedom.

5   Hamiltonian Constraint

The action for the generic sector of the Bianchi VII$_0$ model can now be expressed in terms of the canonical variables (p_μ, β^μ) as

$$S = \int_{t_0}^{t} dt \left(p_\mu \dot\beta^\mu - \tilde N \, \tilde H_0 \right), \qquad (5.1)$$

were $\tilde N$ is the Lagrange multiplier and $\tilde H_0 = \sqrt{\det g}\, H_0 / 4$ is the rescaled Hamiltonian constraint (3.40). This system is reparametrization invariant i.e., it has a symmetry generated by the constraint $\tilde H_0$

$$\delta p = \epsilon \left\{ \tilde H_0, p \right\}, \quad \delta q = \epsilon \left\{ \tilde H_0, q \right\}, \quad \delta \tilde N = \frac{d\tilde N}{dt}, \qquad (5.2)$$

where ϵ is the parameter of the transformation.

In order to find the dynamics of the physical degrees of freedom, we need to fix the reparametrization gauge symmetry (5.2). As discussed in [18], this type of gauge-fixing requires the specification of the time variable, in addition to the usual requirement that the Faddeev-Popov determinant is non-zero. In the case of our system, the analysis simplifies if we introduce new canonical coordinates

$$T = 4\left(\beta^0 + \beta^+\right) - \ln 16, \quad p_T = \frac{1}{8}\left(p_0 + p_+\right),$$

$$q^1 = 6(\beta^0 - \beta^+), \quad p_1 = \frac{1}{12}\left(p_0 - p_+\right),$$

$$q = 4\sqrt{3}\,\beta^-, \quad p = \frac{1}{4\sqrt{3}}\, p_-. \qquad (5.3)$$

The Hamiltonian constraint now becomes

$$\tilde H_0 = -p_T\, p_1 + \frac{1}{2} p^2 + 8 e^T \sinh^2\left(\frac{q}{2}\right), \qquad (5.4)$$

Since (5.4) is independent of q^1, it follows that p_1 is a constant c. On the other hand, the Lagrange multiplier $\tilde{N}$ plays the role of the one-dimensional metric on the world-line parametrized by t, and hence it can be set to a positive constant via (5.2). Since the equation for T is given by

$$\dot{T} + c\tilde{N} = 0, \tag{5.5}$$

then one can choose the following gauge

$$T = t, \quad \tilde{N} = -1/c. \tag{5.6}$$

Note that the gauge choice (5.6) requires $p_1 = c < 0$, in order for $\tilde{N}$ to be positive and finite. Configurations with $p_1 = -c$ are physically equivalent to $p_1 = c$ configurations, because our system is invariant under time-reversal (configurations with $p_1 = -c$ have $T = -t$). The configurations with $p_1 = 0$ are excluded, since they belong to the enhanced-symmetry sector $p_- = \beta^- = 0$.

The equation (5.6) defines the required gauge choice, and therefore the Hamiltonian of the physical degrees of freedom is given by solving the Hamiltonian constraint for p_T

$$p_T = \frac{H^*}{c} = \frac{1}{c}\left(\frac{1}{2}p^2 + 8e^t \sinh^2\left(\frac{q}{2}\right)\right). \tag{5.7}$$

Hence the physical phase space is given by the (p, q) canonical pairs, and the corresponding Hamiltonian is given by H^*. The dynamical consistency of the gauge choice (5.6) can be checked explicitly, by comparing the equations of motion for p_T, p, q from (5.4) with the corresponding equations coming from the reduced Hamiltonian H^*. For example

$$\dot{p_T} = -8\tilde{N}e^T \sinh^2\left(\frac{q}{2}\right) = \frac{\partial p_T}{\partial t} + \{H^*, p_T\}^* = \frac{\partial p_T}{\partial t}, \tag{5.8}$$

where $\{,\}^*$ is the Poisson bracket with respect to the reduced phase space variables (p, q).

From H^* we get $p = \dot{q}$ and $\dot{p} = -4e^t \sinh q$, so that

$$\frac{d^2q}{dt^2} + 4e^t \sinh q = 0. \tag{5.9}$$

This is a Painlevé III equation [22]. One can put (5.9) into the standard form via time redefinition $\tau = e^t$

$$\frac{d}{d\tau}\left(\tau\frac{dq}{d\tau}\right) + 4\sinh q = 0. \tag{5.10}$$

Note that the standard form (5.10) could have been also obtained directly by choosing the gauge $T = \log t$, $p_1 = -c$ and $\tilde{N} = -1/(ct)$.

6 Zero-curvature representation of Painlevé III

By using a more general framework of Belinskii-Zakharov inverse scattering method for the spacetimes admitting two commuting spacelike Killing vectors [5], Belinskii and Francaviglia showed that the Einstein equations for certain Bianchi models admit a zero-curvature representation [12]. In this section we will review their approach for Bianchi I, II, VI$_0$ and VII$_0$ models. A particular attention will be given to the Bianchi VII$_0$ model. We will show that within the framework of the inverse scattering method, the dynamics of the Bianchi VII$_0$ model is given by the Painlevé III equation (5.10). This is in complete accordance with the results from the previous section. In addition, we will derive a zero-curvature representation for this Painlevé III equation.

We begin with a brief discussion of Belinskii and Zakharov method for the midi-superspace models that are characterized by the existence of a two-parameter Abelian group of motions with two spacelike Killing vectors [5]. Let us choose coordinates adapted to the action of the symmetry group so that the metric assumes the following form

$$ds^2 = -f\, dt^2 + f\, dz^2 + g_{ab}\, dx^a dx^b \,, \qquad (6.1)$$

where $a, b = 1, 2$, $\{x^0, x^1, x^2, x^3\} = \{t, x, y, z\}$, f is a positive function and g_{ab} is a symmetric two-by-two matrix. The function f and the matrix g_{ab} depend only on the co-ordinates $\{t, z\}$, or equivalently on the null co-ordinates $\{\xi, \eta\} = \{\frac{1}{2}(z+t), \frac{1}{2}(z-t)\}$. There is a freedom to perform the co-ordinate transformations

$$\{\xi, \eta\} \to \{\tilde{\xi}(\xi), \tilde{\eta}(\eta)\} \,. \qquad (6.2)$$

It is easy to see that the transformations (6.2) preserve both the conformally flat two-metric $f(-dt^2 + dz^2)$ and the positivity of the function f if $\partial_\xi \tilde{\xi}\, \partial_\eta \tilde{\eta} > 0$.

The complete set of vacuum Einstein equations for the metric (6.1) decomposes into two groups of equations [5]. The first group determines

the matrix g_{ab} and can be written as a single matrix equation

$$\partial_\eta \left(\alpha \, \partial_\xi g \, g^{-1} \right) + \partial_\xi \left(\alpha \, \partial_\eta g \, g^{-1} \right) = 0 \,, \tag{6.3}$$

where $\alpha^2 = \det g$ and $\{\xi, \eta\}$ are the null co-ordinates. The second group of equations determines the function $f(\xi, \eta)$ in terms of a given solution of (6.3):

$$\partial_\xi (\ln f) = \frac{\partial_\xi^2 (\ln \alpha)}{\partial_\xi (\ln \alpha)} + \frac{1}{4\alpha \, \alpha_\xi} \operatorname{tr} A^2 \,, \tag{6.4}$$

$$\partial_\eta (\ln f) = \frac{\partial_\eta^2 (\ln \alpha)}{\partial_\eta (\ln \alpha)} + \frac{1}{4\alpha \, \alpha_\eta} \operatorname{tr} B^2 \,, \tag{6.5}$$

where $\alpha_\xi = \partial_\xi \alpha$, $\alpha_\eta = \partial_\eta \alpha$ and the matrices A and B are defined by

$$A = -\alpha \, \partial_\xi g \, g^{-1}, \qquad B = \alpha \, \partial_\eta g \, g^{-1} \,. \tag{6.6}$$

The dynamics of the system is thus essentially determined by the equation (6.3). By taking the trace of the equation (6.3) and by using the definition for α, we obtain

$$\alpha_{\xi\eta} = 0 \,. \tag{6.7}$$

The two independent solutions of this equation are

$$\alpha = c(\xi) + d(\eta) \,, \quad \beta = c(\xi) - d(\eta) \,. \tag{6.8}$$

By using the transformations (6.2), one can bring the functions $c(\xi)$ and $d(\eta)$ to a prescribed form. However, we will consider the general form without specifying the functions $c(\xi)$ and $d(\eta)$ in advance.

The crucial step in the inverse scattering method is to define the linearized system whose integrability conditions are the equations of interest, in our case the equation (6.3). Following ref. [5], we define the two differential operators

$$D_1 = \partial_\xi \quad - \frac{2\alpha_\xi \lambda}{\lambda - \alpha} \partial_\lambda \,, \tag{6.9}$$

$$D_2 = \partial_\eta \quad + \frac{2\alpha_\eta \lambda}{\lambda + \alpha} \partial_\lambda \,, \tag{6.10}$$

where λ is a complex parameter independent of the co-ordinates $\{\xi, \eta\}$. It is straightforward to see that the differential operators D_1 and D_2 commute since α satisfies the wave equation (6.7)

$$[D_1, D_2] = \alpha_{\xi\eta} \frac{(2\lambda)^2}{\lambda^2 - \alpha^2} \partial_\lambda = 0 \,. \tag{6.11}$$

The next step is to consider the following linear system

$$D_1\psi = \frac{A}{\lambda - \alpha}\psi\,, \tag{6.12}$$

$$D_2\psi = \frac{B}{\lambda + \alpha}\psi\,, \tag{6.13}$$

where $\psi(\lambda, \xi, \eta)$ is a complex matrix function, and the real matrices A, B and the real function α do not depend on the complex parameter λ. The integrability conditions for the system (6.12) and (6.13) are given by the equation (6.3). Furthermore, a solution $\psi(\lambda, \xi, \eta)$ yields a matrix $g(\xi, \eta)$ that satisfies the original equation (6.3). Namely, the matrix $g(\xi, \eta)$ is given by

$$g(\xi, \eta) = \psi(\lambda, \xi, \eta)\big|_{\lambda=0}\,. \tag{6.14}$$

In order to take into account that $g(\xi, \eta)$ is real and symmetric we have to impose two additional conditions, see [5]. Also, it is easy to see that the equations (6.12) and (6.13) for $\lambda = 0$, imply equations (6.6).

Although Belinskii and Francaviglia formulation is more general [12], we will discuss only type **A** Bianchi models which are compatible with the inverse scattering method. It is not difficult to show that Bianchi types I, II, VI$_0$ and VII$_0$ admit the representation (6.1). Namely, the metric for these Bianchi types has the form

$$ds^2 = -dT^2 + g_{ij}\,dx^i dx^j\,, \tag{6.15}$$

where

$$g_{ij} = g_{IJ}\,\chi^I{}_i\,\chi^J{}_j\,. \tag{6.16}$$

For these models it is always possible to have the one forms χ^I in the following form

$$\chi^1 = l^1{}_1\,\mathrm{d}x + l^1{}_2\,\mathrm{d}y\,, \quad \chi^2 = l^2{}_1\,\mathrm{d}x + l^2{}_2\,\mathrm{d}y\,, \quad \chi^3 = \mathrm{d}z\,, \tag{6.17}$$

where $l^a{}_b$ are functions of z only. Let us consider the two-by-two matrix

$$l = \begin{pmatrix} l^1{}_1 & l^1{}_2 \\ l^2{}_1 & l^2{}_2 \end{pmatrix}\,. \tag{6.18}$$

An important consequence of the Maurer-Cartan equations for the one forms χ^I is that the matrix l satisfies the following linear differential equation

$$\frac{dl}{dz} = C^T \epsilon l\,, \tag{6.19}$$

where the matrix C is the same matrix as the upper two-by-two block on the principal diagonal of the matrix S^{IJ} defined in the equation (2.3). ϵ is the antisymmetric matrix with $\epsilon_{12} = 1$.

After a time redefinition $t = t(T)$, the metric (6.15) can be written in the form

$$ds^2 = f(t) \left(-dt^2 + dz^2\right) + g_{ab}(t, z)\, dx^a dx^b\,. \qquad (6.20)$$

Here f is a function of t only, and

$$g(t, z) = l^T(z)\gamma(t)l(z)\,, \qquad (6.21)$$

where l is given by (6.18) and γ is a two-by-two symmetric matrix. Notice that now

$$\alpha^2 = (\det l)^2 \det \gamma\,. \qquad (6.22)$$

Moreover, for these models, the determinant of the matrix l is always equal to one, i.e., $\det l = 1$, so that

$$\alpha^2(t) = \det \gamma(t)\,. \qquad (6.23)$$

In addition, α has to satisfy the equation (6.7), which now reads

$$\ddot{\alpha}(t) = 0\,. \qquad (6.24)$$

Hence, α can only be a linear function of time.

As Belinskii and Francaviglia have showed [12], the linearized system (6.12) and (6.13) can be simplified for the models described by the metric (6.20). The first step is to define a two-by-two matrix function φ by

$$\psi = l^T \varphi\, l\,, \qquad (6.25)$$

and a constant two-by-two matrix

$$R = \epsilon\, C\,. \qquad (6.26)$$

The second step is to substitute (6.21) into (6.6) and use the definition of the coordinates ξ and η. Then the results of these calculations, together with the definition (6.25), can be used to simplify the equations (6.12) and (6.13). The crucial step in which a simplification occurs is the coordinate transformation $\{t, z, \lambda\} \rightarrow \{t, w, \lambda\}$, where w is given by

$$w = \frac{1}{2}\left(\frac{\alpha^2}{\lambda} + 2\beta + \lambda\right)\,. \qquad (6.27)$$

To perform this co-ordinate transformation we can use α and β as given by (6.8). The linear system after this co-ordinate transformation involves only derivatives in t and λ since all the terms involving derivatives in w are canceled. Finally, it is useful to make some simple linear combinations of the two equations and to use the fact that α is a linear function of time. In this way one obtains a new linear system

$$\partial_t \varphi = \frac{\alpha}{\lambda}\left(\gamma R^T \gamma^{-1} \varphi - \varphi R^T\right),$$

$$\partial_\lambda \varphi = \frac{1}{2\dot{\alpha}}\left(-R\varphi - \varphi R^T + \frac{\alpha}{\lambda}\dot{\gamma}\gamma^{-1}\varphi + \frac{\alpha^2}{\lambda^2}\varphi R^T - \frac{\alpha^2}{\lambda^2}\gamma R^T \gamma^{-1}\varphi\right).$$

$$(6.28)$$

Although the matrix function $\varphi(t, \lambda, w)$ depends on all three variables, the right-hand side of the system (6.28) does not have any w dependence.

The integrability condition for the system (6.28) is

$$\frac{1}{\alpha}\frac{d}{dt}\left(\alpha\dot{\gamma}\gamma^{-1}\right) = R\gamma R^T \gamma^{-1} - \gamma R^T \gamma^{-1} R. \qquad (6.29)$$

To derive the equation (6.29) from the system (6.28) it is necessary to use the fact that for these models α is a linear function of time.

Equivalently, one can derive the equation (6.29) by a direct substitution of the formula (6.21) into equation (6.3). A straightforward calculation, using the definition of ξ, η, the equation (6.19) and the fact that α is a function of time only, yields the equation (6.29). Thus the dynamics of the these Bianchi models is essentially determined by the equation (6.29). Furthermore we have confirmed that the linear system (6.28) corresponds to the Bianchi models under consideration.

Let us now consider Bianchi VII_0 model. In that case the spatial hyper-surface is a three torus T^3. As we have shown, the modular parameters do not affect the dynamics of the local degrees of freedom, and hence the equations (6.29) and (6.28) will be correct dynamical equations.

The matrix l for Bianchi VII_0 model is given by

$$l = \begin{pmatrix} \cos z & \sin z \\ -\sin z & \cos z \end{pmatrix}, \qquad (6.30)$$

and therefore R is

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6.31}$$

We can take γ to be diagonal

$$\gamma = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}. \tag{6.32}$$

We also choose $\alpha = t$, thus the following relation between the functions a and b

$$\alpha = a\,b = t. \tag{6.33}$$

In order to derive the differential equation which defines the dynamics for this model we substitute formulas (6.31) and (6.32) into equation (6.29) and use the relation (6.33) in order to eliminate the function b. A straightforward calculation yields the following scalar equation

$$\frac{1}{t}\frac{d}{dt}\big(2t\dot{a}/a\big) = t^2 a^{-4} - a^4 t^{-2}. \tag{6.34}$$

The redefinitions $\tau = t^2/4$ and $e^q = a^4/4\tau$ then give the Painlevé III equation (5.10). If we define $f = e^q$ then the equation (5.10) becomes

$$\frac{d^2 f}{d\tau^2} = \frac{1}{f}\Big(\frac{df}{d\tau}\Big)^2 - \frac{1}{\tau}\Big(\frac{df}{d\tau}\Big) + \frac{2}{\tau}\big(-f^2 + 1\big). \tag{6.35}$$

The equation (6.35) is the canonical form of the Painlevé III equation with the coefficients $\alpha = -2$, $\beta = 2$, $\gamma = \delta = 0$, see [22].

7   Conclusions

The dynamics of the generic sector of the Bianchi VII$_0$ model is given by a Painlevé III equation, and therefore it is an integrable model. The moduli parameters do not affect the dynamics of this sector, and hence do not spoil the integrability. Therefore the original claim by Belinskii and Francaviglia that the Bianchi VII$_0$ model is integrable is shown to be correct. Furthermore, by using their results, we have found a zero-curvature representation of the corresponding Painlevé III equation.

In the enhanced symmetry sector we have obtained a linear dynamical equation. We expect that the moduli parameters would not spoil this, although a thorough investigation of this point would be necessary.

The result that a Painlevé III equation appears as the dynamical equation of the local degrees of freedom in the Bianchi VII_0 model can be used to obtain information both about the Painlevé III equation and about the physical properties of the model.

As far as the theory of Painlevé III equation is considered, the linear system (6.28) represents a new tool for the study of Painlevé III equation. This linear system is different from the linear system which is used for the study of Painlevé III equation within the isomonodromic deformation method [19], [20], [21]. However, it remains to be explored what are the advantages of the new linear system.

On the cosmology side, one can now examine the physical properties of the solutions, like small and large time asymptotic, as well as the singularities, since these properties of the Painlevé III solutions have been thoroughly studied [21]. In addition, the quantization of this model should be straightforward, since the reduced phase space Hamiltonian can be promoted into a Hermitian operator. However, it is not clear whether one can find exact solutions for the quantum dynamics, since the Hamiltonians at different times do not commute.

Acknowledgements

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