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Quantum Anti–de Sitter space and sphere at roots of unity

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Abstract

An algebra of functions on q–deformed Anti–de Sitter space AdS_q^D is defined which is covariant under $U_q(so(2, D-1))$, for q a root of unity. The star–structure is studied in detail. The scalar fields have an intrinsic high–energy cutoff, and arise most naturally as fields on orbifolds $AdS_q^D \times S^D/\Gamma$ if D is odd, and $AdS_q^D \times S_\chi^{2D-1}/\Gamma$ if D is even. Here Γ is a finite abelian group, and S_χ is a certain "chiral sector" of the classical sphere. Hilbert spaces of square integrable functions are discussed. Analogous results are found for the q–deformed sphere S_q^D.

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1 Introduction

The D–dimensional Anti–de Sitter space AdS^D is a homogeneous space with constant negative curvature and cosmological constant. Its symmetry group $SO(2, D-1)$ plays the role of the D–dimensional Poincaré group, which is recovered in the flat limit by a contraction. It is of considerable interest in theoretical physics for several reasons. For example, it can be used as a simple model for field theory on curved spaces [11], and it arises naturally in the context of supergravity [40]. Recently, an interesting conjecture relating string or M theory on $AdS^D \times W$ with (super)conformal field theories on the boundary has been proposed [26], where W is a certain sphere or a product space containing a sphere. Moreover, there is some evidence that a full quantum treatment would lead to some non–classical version of the manifolds. This includes the appearance of a "stringy exclusion principle" [27] in the spectrum of fields on AdS space.

In this paper, we study a non–commutative version of the AdS space, which is covariant under the standard Drinfeld–Jimbo quantum group $SO_q(2, D-1)$. It can be understood as a quantization of a certain Poisson structure on the classical AdS space, where $q-1$ is a deformation parameter which plays the role of the Planck constant. In principle, this is can be done for real q and q a phase. For real q, the qualitative features of quantum groups and spaces are typically similar to the classical case; in particular, no cutoff is expected.

Here we consider the case where q is a root of unity. It is well–known that then quantum groups show completely new, "non–perturbative" features; roughly speaking, phenomena which are typical for infinite–dimensional representations of classical non–compact groups occur already with finite–dimensional representations. In particular, it has been shown that there exist finite–dimensional unitary representations of the quantum AdS groups at roots of unity [37, 6], where all the features of the classical case are consistently combined with a cutoff.

The correct definition of quantum–AdS space for q a phase is not obvious; different versions have been proposed in the literature [4, 13], which are not very satisfactory or incomplete. The first goal of this paper is to clarify this situation, and to give a precise definition in terms of operators on Hilbert spaces. To find the proper definition, we make 2 basic assumptions: 1) covariance under the q–deformed universal enveloping algebra $U_q(so(2, D-1))$, and 2) allowing only finite–dimensional representations, hence insisting on a full regularization and avoiding "q–analysis". It is very remarkable that this is indeed possible, while maintaining the correct low–energy limit.

As we will show, these assumptions lead to an algebra of functions on the complexified quantum sphere, which decomposes into different sectors corresponding to different real forms. They describe the compact sphere S_q^D and certain noncompact forms, in particular the quantum Anti–de Sitter space AdS_q^D. This will provide us with scalar fields which are unitary representations of $U_q(so(2, D-1))$, and correspond to the classical square–integrable scalar fields on AdS space, describing spin 0 elementary particles. The remarkable difference to the classical case is that all this happens within the framework of polynomial functions, whose properties are completely different from the classical case. Nevertheless, the classical fields are recovered in the limit of q approaching 1. In particular, this allows to study questions of functional analysis in the classical case with purely algebraic methods.

Moreover, it will turn out that the definition of AdS_q^D implies a number of additional, unexpected features. They include the appearance of an additional undeformed symmetry group $SO(D+1)$ if D is odd and $Sp(D)$ if D is even, which are in some sense spontaneously broken [37]. Moreover, it turns out that the quantum spaces are obtained most naturally as product of the quantum AdS space (or sphere) with a classical sphere. More precisely, one obtains the products $AdS_q^{2r+1} \times S^{2r+1}/\Gamma$ and $AdS_q^{2r} \times S_\chi^{4r-1}/\Gamma$, where $\Gamma = (Z_2)^r$, and S_χ^{4r-1} is a certain "chiral" sector of S^{4r-1}. The quotients of the classical spaces are actually twisted sectors of orbifolds. It should be emphasized that no specific assumptions have been made here, it is simply a consequence of the remarkable structures that appear at roots of unity. Of course, this is quite intriguing in the context of the AdS–CFT correspondence mentioned above, since we obtain $AdS_q^3 \times S^3, AdS_q^5 \times S^5$ and $AdS_q^4 \times S_\chi^7$, which are precisely cases of interest there (apart from the "chiral sector" of S^7, whose meaning is not entirely clear). These and other physical aspects will be discussed further in Section 7.

This paper is organized as follows. In Section 2, some basic facts about quantum groups and spaces are reviewed, including aspects of the representation theory at roots of unity which will be needed. In Section 3, we discuss in detail the meaning of reality structures, and determine the real form of the quantum AdS group $\mathcal{U}_q(so(2, D-2))$. Section 4 is devoted to a closer analysis of the structure of polynomial functions on the complex quantum spaces at roots of unity. In Section 5, we identify different noncompact sectors, which leads to the definition of Hilbert spaces of scalar fields. Their product structure with classical spheres is analyzed in Section 5.2. Sections 5.3 and 5.4 are mathematical interludes, and will allow us to write down explicitly the star structure of the real quantum spaces in Theorems 5.3 and 5.4, which are some of the main results of this work. In Section 6, we comment on further developments towards formulating physical models, and propose an on–shell condition which is somewhat reminiscent of string theory. Some physical aspects are discussed in Section 7. The Appendices include several proofs that were omitted in the text, as well an exposition of the vector representations of $so(D)$ for convenience.

Some advice to the reader: In Sections 5.3 and 5.4, the star structure is defined in several steps, and considerable effort is made to give the precise mathematical definitions and to explain why it is the correct one. However the final result, Theorems 5.3 and 5.4 can be stated very briefly. Thus the reader who is not interested in the mathematical details may skip much of these sections and simply accept the results.

2 The basic algebras

We first recall the classical Anti–de Sitter space AdS^{D-1}, which is a $(D-1)$ –dimensional manifold with constant negative curvature and signature $(+,-,\dots,-)$. It can be embedded in a D–dimensional flat space with signature $(+,+,-,\dots,-)$ by

$$z_1^2 + z_D^2 - z_2^2 - \cdots - z_{D-1}^2 = R^2, \tag{2.1}$$

where R will be called the "radius" of the AdS space. The group of isometries of this space is $SO(2, D-2)$, which plays the role of the $(D-1)$–dimensional Poincaré group.

This space has some rather peculiar features. Its time–like geodesics are finite and closed, and the time "translations" is the $U(1)$ subgroup of rotations in the (z_1, z_D)–plane. The space–like geodesics are unbounded. There exist nice unitary positive–energy representations of $SO(2, D-2)$ which correspond to elementary particles with arbitrary spin. It is also worth recalling that $SO(2, D-2)$ is the conformal group in $D-2$ dimensions acting on $(D-2)$ –dimensional Minkowski space, which can be interpreted as the boundary of AdS^{D-1}.

To define the noncommutative version, we first review some basic facts about the q–deformed orthogonal group and Euclidean space [8]; for a more detailed discussion see e.g. [9, 35]. The algebra of functions $\text{Fun}_q(SO(D, \mathbb{C}))$ on the orthogonal quantum group is generated by matrix elements A^i_j with relations

$$\hat{R}^{ik}_{mn} A^m_j A^n_l = A^i_n A^k_m \hat{R}^{nm}_{jl}, \tag{2.2}$$

where the matrix $\hat{R}^{ik}_{mn}$ is explained below. $\text{Fun}_q(SO(D, \mathbb{C}))$ is the Hopf algebra dual to the *quantized universal enveloping algebra* $U_q(so(D, \mathbb{C}))$, which is easier to work with in practice. Given a root system of a simple Lie group $\mathfrak{g}$ with Killing metric (,) and Cartan matrix A_{ij}, $\mathcal{U}_q := U_q(\mathfrak{g})$ is the Hopf algebra with generators $\{X^{\pm}_i, H_i;\ i = 1, \ldots, r\}$ and relations [16, 7, 8]

$$\begin{aligned}
[H_i, H_j] &= 0, && (2.3)\\
[H_i, X^{\pm}_j] &= \pm A_{ji} X^{\pm}_j, && (2.4)\\
[X^+_i, X^-_j] &= \delta_{i,j} \frac{q^{d_i H_i} - q^{-d_i H_i}}{q^{d_i} - q^{-d_i}} = \delta_{i,j} [H_i]_{q_i}, && (2.5)
\end{aligned}$$

plus a quantum version of the Serre relations. Here q is a complex number such that $q \neq \pm q^{-1}$, $q_i = q^{d_i}$ where $d_i = (\alpha_i, \alpha_i)/2$ are relatively prime, and $[n]_{q_i} = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$ approaches n as $q \to 1$. The comultiplication is

$$\begin{aligned}
\Delta(H_i) &= H_i \otimes 1 + 1 \otimes H_i \\
\Delta(X^{\pm}_i) &= X^{\pm}_i \otimes q^{-d_i H_i/2} + q^{d_i H_i/2} \otimes X^{\pm}_i, && (2.6)
\end{aligned}$$

antipode and counit are

$$\begin{aligned}
S(H_i) &= -H_i, \\
S(X^+_i) &= -q^{-d_i} X^+_i, \quad S(X^-_i) = -q^{d_i} X^-_i, \\
\varepsilon(H_i) &= \varepsilon(X^{\pm}_i) = 0. && (2.7)
\end{aligned}$$

The classical case is obtained by taking $q = 1$. The consistency of this definition can be checked explicitly.

The Cartan–Weyl involution is defined as

$$\theta(X_i^\pm) = X_i^\mp, \quad \theta(H_i) = H_i, \tag{2.8}$$

extended as a linear anti–algebra map; in particular, $\theta(q) = q$ for any $q \in \mathbb{C}$. It is obviously consistent with the algebra, and one can check that

$$(\theta\otimes\theta)\Delta(x) = \Delta(\theta(x)), \tag{2.9}$$
$$S(\theta(x)) = \theta(S^{-1}(x)). \tag{2.10}$$

Borel subalgebras $\mathcal{U}_q^{\pm,0}$ can be defined in the obvious way. This defines a quasitriangular Hopf algebra, which means that there exists a special element $\mathcal{R} \in \mathcal{U}_q \otimes \mathcal{U}_q$ which satisfies

$$\Delta'(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1} \tag{2.11}$$

for any $x \in \mathcal{U}_q$, and other properties which will not be used explicitly. Here $\Delta'(x) = \tau \circ \Delta(x)$ is the flipped coproduct. There are explicit formulas for $\mathcal{R}$, of the form [20, 19]

$$\mathcal{R} = q^{\sum \alpha_{ij}^{-1} d_i H_i \otimes d_j H_j}\left(1\otimes 1 + \sum \mathcal{U}_q^{-res} \otimes \mathcal{U}_q^{+res}\right) \tag{2.12}$$

where $\alpha_{ij} = (\alpha_i, \alpha_j)$. In this paper, we consider $\mathfrak{g} = so(2r+1) = B_r$ and $\mathfrak{g} = so(2r) = D_r$.

As was shown in [7], the following remarkable element

$$v = (S\mathcal{R}_2)\mathcal{R}_1 q^{-2\tilde{\rho}} \tag{2.13}$$

is in the center of $\mathcal{U}_q$, and will be called Drinfeld–Casimir. Here $\tilde{\rho}$ is dual to the Weyl vector $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$. It satisfies

$$\Delta(v) = \mathcal{R}^{-1}\mathcal{R}_{21}^{-1} v \otimes v, \tag{2.14}$$
$$v^{-1} = q^{2\tilde{\rho}}\mathcal{R}_2 S^2(\mathcal{R}_1), \tag{2.15}$$

where $\mathcal{R}_{12} = \mathcal{R}$ and $\mathcal{R}_{21} = \tau \circ \mathcal{R}$.

Consider the matrix $R^{ij}_{kl} = \pi^i_k \otimes \pi^j_l(\mathcal{R})$ where π^i_k is the D-dimensional ("vector") representation of $\mathcal{U}_q$, and let $q_S = q$ for $so(2r)$, and[1] $q_S = q^2$

[1] This is necessary to be compatible with $\mathcal{U}_q$, as will be checked below. In [8], the conventions are such that q_S is replaced by q.

for $so(2r+1)$. Then $\hat{R}^{ij}_{kl} = R^{ji}_{kl}$ decomposes as $\hat{R}^{ij}_{kl} = (q_S P^+ - q_S^{-1} P^- + q_S^{1-D} P^0)^{ij}_{kl}$, where P^+, P^- and P^0 are the invariant projectors on the traceless symmetric, the antisymmetric, and the singlet component in the tensor product of 2 vector representations, respectively. The invariant tensor g^{ij} is given by $(P^0)^{ij}_{kl} = \frac{q_S^2-1}{(q_S^D-1)(q_S^{2-D}+1)} g^{ij} g_{kl}$ where $g_{ik}g^{kj} = \delta^j_i$, and can be normalized such that [8]

$$g^{ij} = g_{ij} = \delta_{i,j'} q^{-\rho_i} \tag{2.16}$$

where $j' = D+1-j$, and ρ_i is given in Appendix A.

There is a canonical way to generalize the classical algebra of coordinate functions on $\mathbb{C}^D$. The algebra $\mathbb{C}^D_q$ defining the *complex quantum Euclidean space* [8] is generated by x_i, with commutation relations

$$(P^-)^{ij}_{kl} x_i x_j = 0 \tag{2.17}$$

which are invariant under $\mathcal{U}_q$. Its center is generated by 1 and

$$x^2 := g^{ij} x_i x_j. \tag{2.18}$$

The *complex quantum sphere* $S^{D-1}_{q,\mathbb{C}}$ is generated by t_i which satisfy the same relations (2.17) as x_i, and in addition [8]

$$t^2 = g^{ij} t_i t_j = 1. \tag{2.19}$$

Explicitly, the commutation relations for x_i (and t_i) are [8]

$$\begin{aligned} x_i x_j &= q_S x_j x_i, \qquad i \neq j',\ i<j \\ [x_{i'}, x_i] &= (q_S^2 - 1) \sum_{j=1}^{i'-1} q^{\rho_{i'}-\rho_j} x_j x_{j'} - \frac{q_S^2-1}{1+q_S^{D-2}}\, q^{\rho_{i'}} x^2, \qquad i<i' \end{aligned} \tag{2.20}$$

Both algebras are covariant under the right coaction $x_i \to x_j \otimes A^j_i$ of $\mathrm{Fun}_q(SO(D,\mathbb{C}))$, which is equivalent to a left action

$$x_i \to u \cdot x_i = x_j \pi^j_i(u) \tag{2.21}$$

for $u \in U_q(so(D,\mathbb{C}))$. We will usually work with the latter, which is more familiar from the classical Lie algebras; then $\pi^i_j(uu') = \pi^i_k(u)\pi^k_j(u')$. We can use this for a quick check of the first relation in (2.20):

$$X_1^+ \cdot (x_1 x_2 - q_S x_2 x_1) = q_S^{1/2} x_1 x_1 - q_S q_S^{-1/2} x_1 x_1 = 0 \tag{2.22}$$

using (2.6) and Appendix A, as it must be. The other relations can be checked similarly.

The algebra of functions AdS_q^{D-1} defining the quantum Anti–de Sitter space will be defined as a real form of this complex quantum sphere, with a (co)action of the quantum Anti-de Sitter group. Therefore as an algebra, it is again defined by $(P^-)^{ij}_{kl} t_i t_j = 0$, $t^2 = 1$. One could introduce a physical scale by setting

$$y_i := t_i R \tag{2.23}$$

for a constant $R > 0$, so that $y^2 = R^2$. We will simply use the units defined by $R = 1$; physically speaking, the scale will be set by the "radius" of AdS space.

So far, all these spaces are complex. The crucial issue is to find the correct definition of the corresponding *real* quantum spaces. This is not obvious especially if q is a phase, and in fact different possibilities have been proposed in the literature [4, 13]. This will be discussed in detail in later sections.

2.1 Roots of Unity and Representations

Since we are primarily interested in the case where q is a root of unity, we will consider a more powerful version of the above, the so–called "restricted specialization" $\mathcal{U}_q^{res} := U_q^{res}(so(D,\mathbb{C}))$ [22] with generators $X_i^{\pm(k)} = \frac{(X_i^{\pm})^k}{[k]_{q_i}!}$ for $k \in \mathbb{N}$ as well as H_i. For generic q, i.e., q not a root of unity, this is the same as before. However if q is a root of unity, the situation is very different. With hindsight, we restrict ourselves to roots of unity of the form

$$q = e^{i\pi/M}, \tag{2.24}$$

where M is even if D is odd; the reason for this will become clear later. Then $k = M$ is the smallest positive integer such that $[k]_q = 0$, since $q^M = -1$. We also define $M_i = M/d_i$, and $M_\alpha = M/d_\alpha$ for any root α. Then M_i is the smallest integer such that

$$[M_i]_{q_i} = 0, \tag{2.25}$$

since $q_i = e^{i\pi d_i/M}$.

The important point is that the additional generators $X_i^{\pm(M_i)}$ in $\mathcal{U}_q^{res}$ are nevertheless well–defined with a well–defined coproduct, and therefore act on tensor products of representations. In particular, $(X_i^{\pm})^{M_i} = 0$ in $\mathcal{U}_q^{res}$. Therefore $\mathcal{U}_q^{res}$ contains a remarkable sub–Hopf algebra $\mathcal{U}_q^{fin}$ (the "small quantum group") generated by $X_i^{\pm}$ and H_i. Here we have slightly changed the standard convention and included the H_i as well, slightly abusing the name "finite". This is more appropriate for our purpose.

We will only consider finite–dimensional representations in this paper. The Cartan generators can then be diagonalized, with eigenvalues

$$\langle H_i, \lambda\rangle = \frac{(\alpha_i, \lambda)}{d_i} = (\alpha_i^\vee, \lambda), \tag{2.26}$$

on a weight λ, where as usual $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ is the coroot of α. The fundamental weights Λ_i satisfy $(\Lambda_i, \alpha_j^\vee) = \delta_{i,j}$, therefore

$$\langle H_i, \Lambda_j\rangle = \delta_{ij}, \tag{2.27}$$

and span the lattice of integral weights. The irreducible highest–weight representations with highest weight λ will be denoted by $L^{res}(\lambda)$.

The vector representation V_D of $\mathcal{U}_q(so(D, \mathbb{C}))$ is the representation $L^{res}(\lambda_1)$ with basis x_i (or t_i) for $i = 1, \ldots, D$. Their weights λ_i are given explicitly in Appendix A. If $D > 4$, then the highest weight λ_1 is equal to Λ_1. We also define

$$d_S = (\lambda_1, \lambda_1), \quad M_S = M/d_S, \tag{2.28}$$

so that $q_S = q^{d_S}$ which was used above.

It is well–known [31] that for generic q, the representation theory is essentially the same as in the classical case. In particular, all finite–dimensional representations (=modules) of $\mathcal{U}_q^{res}$ are direct sums of some $L^{res}(\lambda)$. Their character

$$\chi(L^{res}(\lambda)) = e^\lambda \sum_{\eta>0} \dim L^{res}(\lambda)_\eta e^{-\eta} =: \chi(\lambda) \tag{2.29}$$

is given by Weyls formula. Here $L^{res}(\lambda)_\eta$ is the weight space of $L^{res}(\lambda)$ with weight $\lambda - \eta$. The irreducible highest weight representations of $\mathcal{U}_q^{fin}$ will be denoted by $L^{fin}(\lambda)$.

The value of the Drinfeld–Casimir v (2.13) on $L^{res}(\lambda)$ (and on any highest–weight module with highest weight λ) was first determined in [30]:

$$v \cdot w = q^{-c_\lambda} w \quad \text{for } \; w \in L^{res}(\lambda), \tag{2.30}$$

where $c_\lambda = (\lambda, \lambda + 2\rho)$ is the value of the *classical* quadratic Casimir on $L(\lambda)$. In particular for highest weights of the form $\lambda = k\lambda_1$, the classical Casimir for $so(2r+1)$ is

$$c_{k\lambda_1} = 2k^2 + 2k(D-2), \tag{2.31}$$

and for $so(2r)$ it is

$$c_{k\lambda_1} = k^2 + k(D-2). \tag{2.32}$$

Finally, we quote a few important facts about irreducible representations at roots of unity. The first one [2, 5] states that the structure of L^{res} with "small" highest weight λ is the same as classically:

Theorem 2.1. *Assume that λ is a dominant integral weight with $(\lambda + \rho, \alpha) \leq M$ for all positive roots α. Then the highest weight representation $L^{res}(\lambda)$ has the same character χ as in the classical case, given by Weyl's character formula.*

This follows from the strong linkage principle, which was first shown in [2]; for a more elementary approach, see [38]. Moreover, $L^{fin}(\lambda) = L^{res}(\lambda)$ for these weights λ, since the $X_i^{\pm(M_i)}$ act trivially.

For general λ, the structure of $L^{res}(\lambda)$ is difficult to analyze. However for the "special weights"

$$\lambda_z = \sum M_i z_i \Lambda_i \quad \text{for } \; z_i \in \mathbb{Z} \tag{2.33}$$

it can be understood easily, and this will be the key for much of the following. The relation (2.5) together with (2.25) implies that for any highest weight module $\mathcal{U}_q^{res} \cdot w_{\lambda_z}$ with highest weight λ_z,

$$\varphi_i := X_i^- \cdot w_{\lambda_z} \tag{2.34}$$

is a highest weight vector (possibly zero) for any i, i.e., $X_i^+ \cdot \varphi_j = 0$ for all i, j. Because $L^{res}(\lambda_z)$ is irreducible by definition, it follows that

$$X_i^- \cdot w_{\lambda_z} = 0 \;\; \text{for all } i \tag{2.35}$$

in $L^{res}(\lambda_z)$. In particular, the irreducible representations $L^{fin}(\lambda_z)$ of $\mathcal{U}_q^{fin}(so(D))$ are one–dimensional. However the "large" generators $X_i^{\pm(M_i)} \in \mathcal{U}_q^{res}$ do act nontrivially on $L^{res}(\lambda_z)$, as we will see next.

Using the commutation relations, any element of $\mathcal{U}_q^{-res}(so(D))$ can be written as a sum of terms of the form $X_{i_1}^{-(M_{i_1})} \dots X_{i_k}^{-(M_{i_k})}\, \mathcal{U}_q^{-fin}$. It follows that all weights of $L^{res}(\lambda_z)$ have the form $\lambda_{z'} = \lambda_z - \sum_i n_i M_i \alpha_i$ with $n_i \in \mathbb{N}$. In other words, $L^{res}(\lambda_z)$ is a direct sum of one–dimensional representations $L^{fin}(\lambda_{z'})$ of $\mathcal{U}_q^{fin}$, since all $\lambda_{z'}$ are special points. In fact, the weights λ_z have the structure of a weight lattice with "fundamental weights" $M_i\Lambda_i$. This turns out to be the rescaled lattice of a dual Lie algebra $\tilde{\mathfrak{g}}$, with Cartan matrix $\tilde{A}_{ij} = A_{ji}$ provided (2.24) holds [37]. In the present case, $\tilde{\mathfrak{g}} = so(D)$ if D is even, and $\tilde{\mathfrak{g}} = sp(D-1)$ if D is odd. In fact, $\mathcal{U}_q^{res}$ "contains" a corresponding classical Lie algebra as a quotient. This is the essence of a remarkable result of Lusztig [23], and can be made explicit as follows ([37], Theorem 4.2):

Let $a_i \in \{0,1\}$ such that $a_i + a_j = 1$ if $A_{ij} \neq 0$ and $i \neq j$; this is always possible. Define $\tilde{K}_i = q^{d_i M_i H_i}$, and

$$\begin{aligned} \tilde{X}_i^+ &= X_i^{+(M_i)} \tilde{K}_i^{a_i}, \\ \tilde{X}_i^- &= X_i^{-(M_i)} \tilde{K}_i^{1-a_i} q_i^{M_i^2}, \\ \tilde{H}_i &= [\tilde{X}_i^+, \tilde{X}_i^-]. \end{aligned} \tag{2.36}$$

Then one can show the following:

Theorem 2.2. *For all special weights λ_z, $L^{res}(\lambda_z)$ is an irreducible highest–weight representation of the classical universal enveloping algebra $U(\tilde{\mathfrak{g}})$, with generators $\tilde{X}_i^\pm$ and $\tilde{H}_i$. If $v_{z'} \in L^{res}(\lambda_z)$ has weight $\sum_j z'_j M_j \Lambda_j$, then $\tilde{H}_i \cdot v_{z'} = z'_i v_{z'}$.*

In particular if λ_z is a dominant weight, then the character of $L^{res}(\lambda_z)$ is invariant under the Weyl group, and can be obtained from Weyl's formula by rescaling the weights accordingly.

Using this, the structure of $L^{res}(\lambda)$ with "large" dominant integral λ can be described as follows [5]:

Theorem 2.3. *Let λ_z as in (2.33) and λ_0 be an integral weights with $0 \leq (\lambda_0, \alpha_i^\vee) < M_i$ for all i. Then*

$$L^{res}(\lambda_0 + \lambda_z) = L^{res}(\lambda_0) \otimes L^{res}(\lambda_z). \tag{2.37}$$

This is not hard to prove, see [5] or [37]. Moreover, the generators (2.36) essentially act on the second factor in (2.37) and $\mathcal{U}_q^{fin}$ on the first, but with a certain "twisting" [37].

3 Reality structures and symmetry algebra

Since the proper choice of the reality structure is crucial in the following, we first discuss the relation of the real structures on the spaces with their symmetry algebras.

The algebra of functions on both classical and quantum D-dimensional complex Euclidean space is generated by coordinate functions x_i, which transform in the vector representation V_D of $U_q(so(D,\mathbb{C}))$. The tensor product of 2 such representations contains a unique trivial representation; in other words, there is an invariant bilinear form $\langle\ ,\ \rangle : V_D \times V_D \to \mathbb{C}$. In Hopf–algebra language, invariance means $\langle f, g\rangle =< u_{(1)} \cdot f, u_{(2)} \cdot g >$ for $f, g \in V_D$, where $\Delta(u) = u \otimes 1 + 1 \otimes u = u_{(1)} \otimes u_{(2)}$ denotes the coproduct of $u \in U(so(D,\mathbb{C}))$ or $\mathcal{U}_q$. This extends immediately to polynomial functions.

In the classical case, the algebra of complex functions on *real* Euclidean space (or any real manifold) is equipped with a $*$–structure, i.e., an antilinear map whose square is the identity, defined by the complex conjugation. The above invariant bilinear form then induces a *hermitian* inner product by

$$(f, g) = \langle f^*, g\rangle, \tag{3.1}$$

which satisfies $(f, g)^* = (g, f)$. The signature $(p, D-p)$ of this inner product is the signature of the (pseudo)Euclidean real space, and it is positive definite only in the Euclidean case. But in any case, it defines a $*$–structure on the algebra $U(so(D,\mathbb{C}))$, i.e., an antilinear anti–algebra map, by

$$(f, u \cdot g) = (u^* \cdot f, g) \tag{3.2}$$

for all $u \in U(so(D,\mathbb{C}))$. This defines the real form $U(so(p, D-p))$, and one can then study its unitary representations, which are infinite–dimensional except in the euclidean case.

In the q–deformed case, we use this connection between the real form of a function space and its symmetry algebra in the other direction: there will be a clear choice of the real form of $\mathcal{U}_q$, which then determines the real form of the function algebras. A *real form* or $*$ *–structure* of $\mathcal{U}_q$ is an antilinear anti–algebra map $*$ on $\mathcal{U}_q$ whose square is the identity.

An *invariant inner product* on a representation of $\mathcal{U}_q$ is a hermitian inner product which satisfies (3.2) for all $u \in \mathcal{U}_q$; in other words, the star on $\mathcal{U}_q$ is implemented by the adjoint, which is well–defined for non-degenerate inner products. This is particularly intuitive for elements of $\mathcal{U}_q$ of the form $g = exp(itu)$ with $u^* = u$, since then $g^* = g^{-1}$.

A representation of $\mathcal{U}_q$ is *unitary* with respect to a real form of $\mathcal{U}_q$ if it has a positive definite invariant inner product. Then

$$(\pi(u)^j_i)^* = \pi^i_j(u^*) \tag{3.3}$$

for any $u \in \mathcal{U}_q$, in an orthonormal basis.

Our guiding principle to find the appropriate q–deformed real spaces is that the star structure on $\mathcal{U}_q$ should admit a sufficiently large class of unitary representations of the quantum AdS group in order to describe elementary particles, in the spirit of Wigner. The AdS group is particularly well suited for such an approach, because it has unitary representations for any half–integer spin corresponding to massive as well as massless particles with positive energy, in any dimension. In fact, one can choose the Cartan subalgebra such that the energy is one of its generators, and the unitary representations are then lowest–weight representations with positive energy and discrete spectrum. The unitary representations of the Poincaré group are recovered in the flat limit. We want to maintain these features in the q–deformed case. This will uniquely select the real structure.

3.1 Star structures on $\mathcal{U}_q$

There are essentially 2 types of star structures[2] on $\mathcal{U}_q$ [37], the first of the form

$$X_i^{\pm *} = s_i X_i^{\mp}, \quad H_i^* = H_i, \tag{3.4}$$

[2]There are additional ones if the Dynkin diagram has automophisms, such as for $so(2r)$; they however do not correspond to Anti–de Sitter groups, and do not admit finite–dimensional unitary representations.

and the second of the form

$$X_i^{\pm *} = s_i X_i^{\pm}, \quad H_i^* = -H_i, \tag{3.5}$$

with $s_i = \pm 1$. They define consistent star algebras for both real q and q a root of unity. The compatibility conditions with the coproduct are different for real q and q a phase, which will show up in different reality structures of the associated quantum spaces.

It is known that there exist finite–dimensional unitary representations of the first type (3.4) if q is a root of unity [38, 37], which have the desired properties including a high–energy cutoff in the AdS case. For $q \in \mathbb{R}$, unitary representations of noncompact forms also exist, but they have no cutoff. There are also certain unitary representations of the second type for q a phase, e.g. for $U_q(so(2,1))$ [32], but they are not highest–weight representations; in particular, the Cartan subalgebra cannot be diagonalized, and the energy in the AdS case is not positive definite (notice that the Cartan subalgebra is distinguished for $q \neq 1$, unlike in the classical case). Finite–dimensional unitary representations of the second type cannot exist, since then H_i could be diagonalized with purely imaginary spectrum, which is in contradiction with the commutation relation (2.4); Therefore we only consider star structures on $\mathcal{U}_q$ of the first type (3.4) from now on, with q a root of unity.

Consider the vector representation V_D of $\mathcal{U}_q$, with basis x_i and weights λ_i for $i = 1, \ldots, D$. V_D is unitary with respect to the compact form

$$X_i^{\pm *} = X_i^{\mp}, \quad H_i^* = H_i, \tag{3.6}$$

which defines $U_q(so(D))$ for both real q and q a phase. In general, there is a unique invariant inner product on highest weight modules satisfying (3.2) for star structures of the type (3.4). The weight vectors x_i are then orthogonal for different weights, and they can defined to be orthonormal, i.e.,

$$(x_i, x_j) = \delta_{ij}. \tag{3.7}$$

This is the standard convention in the literature.

Now it is easy to find the definition of $U_q(so(2, D-2))$. Let E be the element of the Cartan subalgebra which is dual to λ_1/d_S, so that $\langle E, \lambda\rangle = (\lambda_1, \lambda)/d_S$. Then the eigenvalues of E on V_D are $E_i := (\lambda_1, \lambda_i)/d_S = (1, 0, \ldots, 0, -1)$; E will turn out to be the energy. We claim that the star structure defining $U_q(so(2, D-2))$ is

$$X_i^{\pm *} = (-1)^E \theta(X_i^{\pm})(-1)^E = s_i X_i^{\mp},$$
$$H_i^* = H_i \qquad \text{with } s_i = (-1)^{\langle E, \alpha_i \rangle}, \quad (3.8)$$

where $s_i = (-1, 1 \ldots, 1)$ for $D \neq 4$, and $s_i = (-1, -1)$ for $D = 4$. This is a star algebra of the first type (3.4) which can be considered for both $q \in \mathbb{R}$ and q a root of unity, and there exist unitary representations in both cases. The maximal compact subalgebra is $U_q(so(D-2)) \times U_q(so(2))$ where $U_q(so(D-2))$ generated by $X_2^{\pm}, \ldots, X_r^{\pm}, H_2, \ldots, H_r$ (for $D \neq 4$), and $U_q(so(2))$ by E; these subalgebras commute. The unique (up to normalization) corresponding invariant inner product on V_D which satisfies (3.2) is

$$(x_i, x_j) = (-1)^{E_i} \delta_{ij} \quad (3.9)$$

in the above basis, and has the correct signature for the *AdS* case. We will also find the desired unitary representations of $U_q^{fin}(so(2, D-2))$, provided (2.24) holds. Therefore we define the "physical" quantum Anti–de Sitter group at roots of unity to be the real form (3.8) of $U_q^{fin}(so(2, D-2))$.

3.2 Quantum Euclidean and Anti–de Sitter space for real q

For real q, there is a standard star structure on the algebra $\mathbb{C}_q^D$. The algebra $\mathbb{R}_q^D$ on the *real quantum Euclidean space* is defined by [8]

$$x_i^* = x_j g^{ji}. \quad (3.10)$$

This is a consistent star algebra, which indeed leads to the invariant inner product $(x_i, x_j) = \delta_{ij}$, and by (3.1) corresponds to the real form (3.6) on $\mathcal{U}_q$. It is also consistent with the constraint $x_i x_j g^{ij} = 1$, thereby defining the *quantum Euclidean sphere* S_q^{D-1} for real q.

The algebra AdS_q^{D-1} on *quantum Anti–de Sitter space* for real q is similarly defined by

$$t_i^* = -(-1)^{E_i} t_j g^{ji}, \quad (3.11)$$

together with

$$t^2 = t_i t_j g^{ij} = 1. \quad (3.12)$$

This is consistent with (2.17) because E is in the Cartan subalgebra, and has the correct classical limit. By (3.1), this leads to the invariant inner product $(t_i, t_j) = (-1)^{E_i}\delta_{ij}$.

If q is a phase, (3.10) does not extend as a star on the algebra $\mathbb{C}_q^D$. In this case, star structures different from (3.10) have been proposed, of the type $x_i^* = \pm x_i$ [8, 4]. While $\mathbb{C}_q^D$ then becomes a star algebra, they lead to star structures on $\mathcal{U}_q$ of the second type (3.5), which we have discarded above. However, we will see below that there is a star on the semidirect product algebra $\mathcal{U}_q \ltimes \mathbb{C}_q^D$ which extends (3.8), and agrees with (3.10) or (3.11) in the classical limit.

4 Scalar fields at roots of unity

In order to construct scalar fields on classical AdS space which are unitary representations, one has to consider square–integrable functions; they are of course not polynomials. The situation for $q \in \mathbb{R}$ is completely analogous. At roots of unity however, the structure of polynomials changes completely, and we will see that the analogues of the classical scalar fields can in fact be written as polynomials in the t_i.

To see this, we have to study $\mathbb{C}_q^D$ in more detail. Consider the set of homogeneous polynomials $M_q^k \subset \mathbb{C}_q^D$ with degree k in the x_i, which forms a submodule of the k–fold tensor product representation $V_D^{\otimes k}$ of $\mathcal{U}_q^{res}$. M_q^k is not irreducible, because the metric projector may be nontrivial. Clearly $x_1 ... x_1 = (x_1)^k \in M_q^k$ is a highest weight vector with weight $k\lambda_1$. It generates the highest weight module

$$\mathcal{F}(k) := \mathcal{U}_q^{res} \cdot (x_1)^k \subset M_q^k. \tag{4.1}$$

If q is generic, then $\mathcal{F}(k)$ is a irreducible representation with highest weight $k\lambda_1$ corresponding to a totally (q–)symmetric traceless tensor, and $M_q^k = \mathcal{F}(k) \oplus x^2\mathcal{F}(k-2) \oplus \cdots$ as classically. If q is a root of unity, then M_q^k is not completely reducible any more, which is a typical phenomenon at roots of unity. The full structure of M_q^k is quite complicated, and will be discussed elsewhere. Here we only consider those modes which will be needed for the Hilbert space of scalar fields on AdS_q.

First, we identify the polynomials $\mathcal{F}(k)$ in $\mathbb{C}_q^D$ which have essentially the same structure as classically, which means the character of $L^{res}(k\lambda_1)$ is the same as classically. Using (2.24), Theorem 2.1 implies that $\chi(L^{res}(k\lambda_1)) = \chi(k\lambda_1)$ for $k \leq M_S - (D-3)$ (assuming $D \geq 4$; indeed, any positive root α can then be written as $\alpha = \sum n_i\alpha_i$ with $n_i \leq a_i$, where a_i are the Coxeter labels. Hence $(k\lambda_1 + \rho, \alpha) \leq (k + D - 3)d_1$, which is $\leq M = d_1 M_1$ provided $k \leq M_1 - (D-3)$. For $D = 3$, the bound is $k \leq M_S - 1/2$).

However, this bound can be improved using the strong linkage principle, see e.g., [38]. We can assume that $k < M_S$; then it implies that the character of $L^{res}(k\lambda_1) = L^{fin}(k\lambda_1)$ is the same as classically, unless it contains a dominant integral weight μ which is in the orbit of $k\lambda_1$ under the Weyl group acting with center $M_S\lambda_1 - \rho$ (since $M_s\lambda_1$ is a special weight). Since $(\frac{\lambda_1}{d_S}, \rho) = \langle E, \rho\rangle = (D-2)/2$ by Appendix A, this is only possible for $k > k_S$ where

$$k_S = M_S - (D-2)/2. \tag{4.2}$$

Hence

$$\chi(L^{res}(k\lambda_1)) = \chi(k\lambda_1) \quad \text{for } \ k \leq k_S. \tag{4.3}$$

In fact, it turns out that the smallest k where $\chi(L^{res}(k\lambda_1))$ is non–classical for even $D > 4$ is $k = k_S + 1$, where

$$\chi(L^{res}((k_S+1)\lambda_1)) = \chi((k_S+1)\lambda_1) - \chi((k_S-1)\lambda_1). \tag{4.4}$$

(For odd D, there is a similar phenomenon). This is related to the scalar singleton field on the AdS space, as we will see in the next section. One can check that e.g., the Casimir v (2.13) indeed becomes degenerate for these weights.

To summarize, the structure of M_q^k is the same as classically for small k:

Theorem 4.1. *For $n_0 \leq k_S$ and roots of unity of the form (2.24),*

$$\mathcal{F}(n_0) \cong L^{res}(n_0\lambda_1) = L^{fin}(n_0\lambda_1) \tag{4.5}$$

has the same character $\chi(n_0\lambda_1)$ as classically. Moreover, $M_q^{n_0}$ is the direct sum

$$M_q^{n_0} = \bigoplus_{0 \leq k \leq n_0/2} (x^2)^k \mathcal{F}(n_0 - 2k). \tag{4.6}$$

The proof is completed in Appendix C.

Next consider $\mathcal{F}(kM_S)$, which is of central importance to us. Its highest weight $kM_S\lambda_1$ is a special weight (2.33). Using (2.6) and the commutation relations (2.20), one has

$$\begin{aligned} X_1^- \cdot (x_1)^n &= q_S^{-(n-1)/2} x_2 (x_1)^{n-1} + q_S^{-(n-3)/2} x_1 x_2 (x_1)^{n-2} \\ &\quad + \cdots + q_S^{(n-1)/2} (x_1)^{n-1} x_2 \\ &= q_S^{-(n-1)/2} (1 + q_S^2 + \cdots + q_S^{2(n-1)}) x_2 (x_1)^{n-1} \\ &= q_S^{(n-1)/2} [n]_{q_S} x_2 (x_1)^{n-1}. \end{aligned} \tag{4.7}$$

Since $[kM_S]_{q_S} = 0$, it follows that $X_i^- \cdot (x_1)^{kM_S} = 0$ for all i, hence $(x_1)^{kM_S}$ is a one–dimensional representation of $\mathcal{U}_q^{fin}$. As discussed in Section 2.1 below (2.35), this implies that all weights of $\mathcal{F}(kM_S)$ have the form $\lambda_z = \sum z_i M_i \Lambda_i$, and $\mathcal{F}(kM_S)$ is a representation of the classical universal enveloping algebra $U(\tilde{\mathfrak{g}})$ with generators $\tilde{X}_i^\pm$ and $\tilde{H}_i$. Moreover, it is a highest weight module with highest weight vector $(x_1)^{kM_S}$. By the classical representation theory, it follows that it is irreducible, hence

$$\mathcal{F}(kM_S) \cong L^{res}(kM_S\lambda_1), \tag{4.8}$$

which is essentially $L(k\lambda_1)$ of $\tilde{\mathfrak{g}}$ (except for $D = 3$, where it is $L(k\Lambda_1)$).

Now consider more generally $n = n_0 + kM_S$ with $0 \le n_0 < k_S$ and $k \in \mathbb{N}$. Then both $\mathcal{F}(n)$ and $\mathcal{F}(n_0) \otimes \mathcal{F}(kM_S)$ are highest weight modules with highest weight $n\lambda_1$. Clearly $\mathcal{F}(n) \subset \mathcal{F}(n_0)\mathcal{F}(kM_S)$, by (4.1) and (2.6). On the other hand, $\mathcal{F}(n_0)\mathcal{F}(kM_S) \subset \mathcal{F}(n_0) \otimes \mathcal{F}(kM_S)$, which is isomorphic to $L^{res}(n\lambda_1)$ by Theorem 2.3. Hence we have shown that

Theorem 4.2. *For $n = n_0 + kM_S$ with $0 \le n_0 < k_S$ and $k \in \mathbb{N}$,*

$$\mathcal{F}(n) = \mathcal{F}(n_0)\mathcal{F}(kM_S) \cong \mathcal{F}(n_0) \otimes \mathcal{F}(kM_S) \cong L^{res}(n\lambda_1). \tag{4.9}$$

In particular, it is essentially the tensor product of $\mathcal{F}(n_0)$ with the irreducible representation $L(k\lambda_1)$ (or $L(k\Lambda_1)$ for $D = 3$) of the classical algebra $\tilde{\mathfrak{g}}$.

5 Real forms and Hilbert space representations of the quantum spaces

5.1 Hilbert spaces for S_q^{D-1} and AdS_q^{D-1}

Now we are ready to discuss the reality structure at roots of unity. As was pointed out before, (3.10) and (3.11) are not consistent with the algebra for q a phase. To find the correct definition, we first construct the Hilbert spaces, and then simply calculate the adjoint of the operators of interest. We want to emphasize that the inner products on irreducible representations are determined uniquely by (3.2). Indeed on any highest weight module, there exists a unique (up to normalization) invariant inner product for a given star structure of the form (3.4). This is because once the inner product is defined on the highest weight vector, it can be calculated for all descendant vectors using (3.2); in general, it is not unitary. The resulting invariant inner product is non–degenerate if the representation is irreducible.

We first discuss the quantum sphere. As representation of $\mathcal{U}_q^{res}$, we can consider

$$\mathcal{F}(n) = \mathcal{U}_q^{res} \cdot (t_1)^n \subset S_{q,\mathbb{C}}^{D-1} \tag{5.1}$$

instead of (4.1). However, not all these $\mathcal{F}(n)$ should be considered as fields on the "real" quantum sphere, only those which are unitary representations of the compact form

$$X_i^{\pm *} = X_i^{\mp}, \quad H_i^* = H_i, \tag{5.2}$$

of $U_q^{fin}(so(D))$, with the natural inner product discussed above. It was proved in [37] that this certainly holds for those $L^{res}(k\lambda_1) = L^{fin}(k\lambda_1)$ with $k \leq k_S$ (4.2). Here the assumption (2.24) is important.

Hence we could define a Hilbert space of functions on the real quantum sphere to be the direct sum of all $\mathcal{F}(k)$ with $k \leq k_S$. In order to obtain a simple definition of position operators in Section 5.5, we impose the slightly stronger bound $k < k_S$, and define the Hilbert space of functions on the *real quantum sphere* to be

$$S_q^{D-1} := \bigoplus_{0 \leq n_0 < k_S} \mathcal{F}(n_0) = \bigoplus_{0 \leq n_0 < k_S} \mathcal{U}_q^{fin} \cdot (t_1)^{n_0}. \tag{5.3}$$

The position operators will be discussed in Section 5.5. Its generators are essentially the t_i as in Section 2; they will have to be slightly modified in order to account for the cutoff. Their star structure is determined uniquely by the adjoint on the Hilbert space, and will be given explicitly in Section 5.5. S_q^{D-1} is a truncated version of the classical sphere $S^{D-1} = \oplus_{n\geq 0} L(n\lambda_1)$, which is recovered in the limit $M \to \infty$, or $q \to 1$.

Now consider more generally $n = n_0 + kM_S$ with $0 \leq n_0 < k_S$, so that $\mathcal{F}(n) = \mathcal{F}(n_0) \otimes \mathcal{F}(kM_S)$ according to (4.9). Since $\mathcal{F}(kM_S) \cong L^{res}(kM_S\lambda_1)$ is a finite–dimensional irreducible representation of the classical $\tilde{\mathfrak{g}}$, it has a standard positive–definite invariant inner product. We just discussed the inner product on $\mathcal{F}(n_0)$. This suggests a natural positive definite inner product on $\mathcal{F}(n)$ as the tensor product, so that

$$(f\rho_{(k)}, f'\rho'_{(k)}) := (f, f')(\rho_{(k)}, \rho'_{(k)}) \tag{5.4}$$

where $f, f' \in \mathcal{F}(n_0)$ and $\rho_{(k)}, \rho'_{(k)} \in \mathcal{F}(kM_S)$. This leads to an interesting physical interpretation, as we will see. We will always use an orthonormal basis corresponding to this inner product from now on.

We have seen in the previous section that all weights of $L^{res}(kM_S\lambda_1)$ have the form $\lambda_z = \sum_i z_i M_i \Lambda_i$. Therefore $\mathcal{F}(n)$ is the direct sum of irreducible representations $L^{fin}(n_0\lambda_1 + \lambda_z)$ of $\mathcal{U}_q^{fin}$ for various λ_z; if the multiplicity of a certain $L^{fin}(n_0\lambda_1 + \lambda_z)$ in $\mathcal{F}(n)$ is larger than one, then the corresponding subspace can be decomposed as an orthogonal sum of irreducible components. Hence $\mathcal{F}(n)$ is a direct orthogonal sum of Hilbert spaces of type $L^{fin}(n_0\lambda_1 + \lambda_z)$, with the induced inner product (5.4), and the different components are related by the action of the classical generators $\tilde{X}_i^{\pm}$ (2.36). One can now calculate the adjoint of the generators of $\mathcal{U}_q^{fin}$ on $L^{fin}(n_0\lambda_1 + \lambda_z)$ with respect to this inner product. The result is (see [37], Theorem 5.1)

$$H_i^* = H_i, \quad X_i^{\pm *} = s_i X_i^{\mp}, \quad \text{where} \quad s_i = (-1)^{z_i}. \tag{5.5}$$

If $\pi_i^j(u)$ is the representation of $u \in \mathcal{U}_q^{fin}$ on $L^{fin}(n_0\lambda_1 + \lambda_z)$ with respect to an orthonormal basis, this means that

$$(\pi(u)^\dagger)_i^j = \pi_j^i(u)^* = \pi_i^j(u^*) \tag{5.6}$$

as in (3.3); here $*$ really depends on z as in (5.5), which will however be suppressed in the following. Hence $L^{fin}(n_0\lambda_1 + \lambda_z)$ is a unitary representation of a certain real form of the type (3.4) of $U_q^{fin}(so(D, \mathbb{C}))$, and the "sectors" with different λ_z but the same s_i are unitary with respect to the same real form form. Comparing with Section 3.1, we conclude that if all z_i are even, then $L^{fin}(n_0\lambda_1 + \lambda_z)$ is a unitary representation of the compact form (3.6) and hence a scalar field on S_q^{D-1}. If the z_i are such that $s_i = (-1)^{z_i}$ is as in (3.8), then it is a unitary representation of the Anti–de Sitter group $\mathcal{U}_q^{fin}(so(2, D-2))$, and should in fact be viewed as a scalar field on AdS_q^{D-1}, corresponding to a square–integrable scalar field in the classical case. To understand this, consider the $L^{fin}((2M_S - k)\lambda_1)$ as lowest–weight representations $L_{fin}(k\lambda_1)$ of $\mathcal{U}_q^{fin}$ with lowest weight $k\lambda_1$. For low energies, they have the same structure as the scalar fields on the classical AdS space, which are irreducible unitary lowest–weight representations of $SO(2, D-2)$ realized in terms of square–integrable functions. It is very remarkable that they are realized here in terms of polynomials in the coordinate functions. Therefore we define the Hilbert space of functions on the *real quantum Anti–de Sitter space* to be

$$\begin{aligned} AdS_q^{D-1} := & \bigoplus_{M_S \leq n < M_S + k_S} \mathcal{U}_q^{fin} \cdot (t_1)^n \\ = & \bigoplus_{M_S \leq n < M_S + k_S} L^{fin}(n\lambda_1) = S_q^{D-1}(t_1)^{M_S}. \end{aligned} \tag{5.7}$$

In terms of lowest–weight representations $L_{fin}(n\lambda_1) = \mathcal{U}_q^{fin} \cdot (t_D^{M_S - n} t_1^{M_S})$ of $\mathcal{U}_q^{fin}$, this can be written as

$$AdS_q^{D-1} = \bigoplus_{(D-2)/2 < n \leq M_S} L_{fin}(n\lambda_1). \tag{5.8}$$

For energies less than M_S, the states in the Hilbert spaces are the same as classically, and the action of $\mathcal{U}_q^{fin}$ approaches the classical one for any given weight as $M \to \infty$. The energy of all states is less than $2M_S$. The precise definition of the position operators will be given in Section 5.5. In the classical case, the lower bound $(D-2)/2$ can be seen by a simple dimensional argument; however it can be violated slightly.

For even $D > 4$, the irreducible quotient of $\mathcal{U}_q^{fin} \cdot (t_D^{M_S-(D-4)/2} t_1^{M_S})$ is the scalar singleton $L_{fin}((D-4)/2\lambda_1)$, with lowest weight $(D-4)/2\ \lambda_1$; we will not consider it any more here.

5.2 Product spaces

From the above discussion, it would seem much more natural to consider all polynomials in the t_i instead of just certain $\mathcal{U}_q^{fin} \cdot (t_1)^n$. For simplicity, we will restrict ourselves to the polynomials of the form $\mathcal{F}(n) = \mathcal{U}_q^{res} \cdot (t_1)^n$ for all $kM_S < n < kM_S + k_S$ and all $k \in \mathbb{N}$, and study their field content. Let $\widetilde{\lambda_1} = \lambda_1$ for $D > 3$, and $\widetilde{\lambda_1} = \lambda_1/2 = \Lambda_1$ for $D = 3$. Then $\mathcal{F}(n) = \mathcal{F}(n_0)\mathcal{F}(kM_S)$ using (4.9), where the second factor is essentially the representation $L(k\widetilde{\lambda_1})$ of the classical symmetry algebra $\tilde{\mathfrak{g}}$, which connects the various components with different $\lambda_{z'}$. Hence

$$\oplus\mathcal{F}(n) = \Big(\bigoplus_{0 \leq n_0 < k_S} \mathcal{F}(n_0)\Big) \otimes \Big(\bigoplus_{k \in \mathbb{N}} L(k\widetilde{\lambda_1})\Big), \qquad (5.9)$$

where certain modes were omitted as in Section 5.1 for simplicity. For the moment we ignore the reality structure.

Observe that the $L(k\widetilde{\lambda_1})$ are very particular representations of the classical $\tilde{\mathfrak{g}}$, which have a nice interpretation. Consider first $D = 2r$. Then the dual algebra is $\widetilde{so}(2r) = so(2r)$ as shown in [37], and $L(k\widetilde{\lambda_1})$ can be viewed as a function on the classical sphere S^{D-1}. Hence $\oplus_{k \in \mathbb{N}} L(k\widetilde{\lambda_1}) \cong Fun(S^{D-1})$, which is the space of polynomial functions on the classical sphere S^{D-1}.

Next, consider $D = 2r+1$, which is less obvious. Then $\widetilde{\lambda_1} = \Lambda_1$, and $\mathcal{F}(kM_S)$ is the highest–weight representation $L(k\Lambda_1)$ of $\widetilde{so}(2r+1) = sp(2r)$ with highest weight $k\Lambda_1$. Observe that the $2r$–dimensional representation $L(\Lambda_1)$ is not real, in the sense that the $2r$ variables z_i are necessarily complex, and can be considered as $4r$ real variables x_i. The compact form $USp(2r)$ acts by multiplying the z_i with a unitary matrix. Therefore the radius $x^2 = \sum_i z_i \overline{z_i}$ is invariant under $USp(2r)$, and the orbit of $USp(2r)$ in $L(\Lambda_1) = \mathbb{C}^{2r}$ turns out to be sphere S^{4r-1} (to see this, one can show that the stabilizer of $(1, 0, \ldots, 0)$ is $USp(2r-2)$, hence $USp(2r)/USp(2r-2) \cong S^{4r-1}$ on dimensional grounds). The set of polynomials of degree k in the z_i is precisely the representation $L(k\Lambda_1)$, hence $\oplus_{k \in \mathbb{N}} L(k\Lambda_1)$ is the space of those functions on S^{4r-1} which are induced by holomorphic functions on $\mathbb{C}^{2r}$; for lack of a better name, we call it the "chiral sector" S_χ^{4r-1} of S^{4r-1}. The full sphere could be obtained by tensoring S_χ with another (conjugate) chiral sphere.

In summary, we found (ignoring the reality structure) $\oplus_{n\in\mathbb{N}}\mathcal{F}(n) \cong S_q^{D-1} \otimes Fun(M)$, where

$$M = \begin{cases} S^{D-1}, & D-1 \text{ odd} \\ S_\chi^{2D-3}, & D-1 \text{ even}, \end{cases} \tag{5.10}$$

which should be interpreted as the functions on the product of the quantum sphere with the classical sphere, or the "chiral sector" of the classical sphere.

Now we take into account the star structure (5.5) induced on the various $L^{fin}(n_0\lambda_1 + \lambda_{z'}) \subset \mathcal{F}(n)$. As we have seen before, certain sectors $\lambda_{z'}$ of the modules $\mathcal{F}(n)$ correspond to S_q^{D-1}, others correspond to AdS_q^{D-1}, and others yet have not been identified here. This means that we obtain the product of AdS_q^{D-1} or S_q^{D-1} with certain quotients (=orbifolds) of the classical spaces. The functions on the orbifolds are described by functions on the covering space which are invariant under the action of a certain classical group, possibly modulo signs (for twisted orbifolds).

To make this more specific, recall that $v_{z'} \in L^{fin}(n_0\lambda_1+\lambda_{z'}) \subset \mathcal{F}(n)$ is an element of AdS_q^{D-1} if $s_j = (-1)^{z'_j}$ is as stated in (3.8). By Theorem 2.2, this condition can be restated as

$$e^{i\pi\tilde{H}_j} \cdot v_{z'} = s_j v_{z'}. \tag{5.11}$$

Here the $e^{i\pi\tilde{H}_j}$ are elements of the classical group corresponding to $\tilde{\mathfrak{g}}$, whose square is the identity; they can be viewed as rotations by π. They generate the abelian group

$$\Gamma = (Z_2)^r. \tag{5.12}$$

This means that the classical functions $v_{z'}$ on $M = S^{D-1}$ or S_χ^{2D-3} are really functions on the (twisted) orbifold M/Γ. Hence

$$\oplus\mathcal{F}(n) = \left(AdS_q^{D-1} \times M/\Gamma\right) \;\oplus\; \left(S_q^{D-1} \times M/\Gamma\right) \;\oplus \ldots \tag{5.13}$$

in somewhat sloppy notation, where the different summands correspond to different twisted sectors of the orbifold M/Γ. In fact, some of these other sectors (which were not considered so far) are again equivalent to AdS_q^{D-1}. For example for $D = 5$, all sectors turn out to be either S_q^4

or AdS^4_q. Hence they might recombine, and $(Z_2)^r$ might effectively be smaller.

The residual symmetry of $\tilde{\mathfrak{g}}$ on these orbifolds is [37] $su(2)^r$ for $D = 2r+1$, and to $u(1)^r$ for $D = 2r$; in a sense, $\tilde{\mathfrak{g}}$ is spontaneously broken.

5.3 The Universal Weyl Element ω

The appropriate mathematical tool to describe the involution is an element of an extension of $\mathcal{U}_q$ by generators ω_i of the braid group, introduced in [19] and [21]. The ω_i act on representations of $\mathcal{U}_q^{res}$, and define a braid group action on $\mathcal{U}_q$ via $T_i(u) = \omega_i u \omega_i^{-1}$ for $u \in \mathcal{U}_q$. All we need is the generator corresponding to the longest element of the Weyl group, ω. Acting on an irreducible representation, ω maps the highest weight vector into the lowest weight vector of the contragredient (dual) representation, as classically. It has the following important properties [19, 21]:

$$\begin{aligned} \Delta(\omega) &= \mathcal{R}^{-1}\omega\otimes\omega = \omega\otimes\omega\mathcal{R}_{21}^{-1}, && (5.14)\\ \omega u\omega^{-1} &= \theta S\gamma(u) = S^{-1}\theta\gamma(u), && (5.15)\\ \omega^2 &= v\epsilon, && (5.16)\end{aligned}$$

for $u \in \mathcal{U}_q$, where $\gamma = \mathrm{id}$ in the case $so(2r+1)$, and γ is the automorphism of the Dynkin diagram[3] in the case $so(2r)$. ϵ is a Casimir with

$$\Delta(\epsilon) = \epsilon\otimes\epsilon \qquad (5.17)$$

which takes the values ± 1. It turns out that $\epsilon = -1$ only for spinorial representations which are not considered in this paper, thus we will put $\epsilon = 1$ from now on. (5.14) justifies the name "universal Weyl element" for ω. One can also define antipode and counit for ω, and explicit formulas for ω in terms of formal sums in $\mathcal{U}_q$ can be given [14]. Since explicit proofs of (5.15) and (5.16) have only been given for the rank one case in the literature, we will sketch short proofs for the general case in Appendix B.

[3]for $so(8)$, it is the automorphism which interchanges α_3 with α_4 and leaves Λ_1 invariant, as can be seen from V_D.

For our purpose, we define a slightly modified element

$$\tilde{\omega} := q^{2\tilde{\rho}}\omega, \tag{5.18}$$

with the same properties as ω except

$$\tilde{\omega} u \tilde{\omega}^{-1} = \theta S^{-1}\gamma(u) = S\theta\gamma(u). \tag{5.19}$$

ω or $\tilde{\omega}$ has a simple interpretation for real representations. A representation V of $\mathcal{U}_q^{res}$ will be called *real* here if there exists an intertwiner

$$g : V \otimes V \to \mathbb{C}, \tag{5.20}$$

or in other words an invariant tensor on $V \otimes V$. If g_{ij} is the matrix of this intertwiner, then

$$g_{ij}\pi^i_k(u_1)\pi^j_l(u_2) = \varepsilon(u)g_{kl}, \tag{5.21}$$

for all $u \in \mathcal{U}_q^{res}$, which by standard Hopf algebra identities is equivalent to

$$g_{ij}\pi^i_k(u) = g_{kl}\pi(S(u))^l_j, \tag{5.22}$$

or $\pi(u)^T g = g\pi(S(u))$ in matrix language. Now by (5.19), one can write $S(u) = \tilde{\omega}\theta(\gamma(u))\tilde{\omega}^{-1}$, and one obtains

$$\pi(u)^T = A^{-1}\pi(\theta(\gamma(u)))A, \tag{5.23}$$

where $A^{-1} = g\pi(\tilde{\omega})$.

In the classical case, it is known that the "real" representations of a Lie group can be chosen to be orthogonal or symplectic matrices, in a suitable basis. Moreover, a highest weight representation is real if the highest weight is invariant under the automorphism γ. We can show the following related statement in the q–deformed case:

Lemma 5.1. *If γ acts trivially on the dominant integral weight λ, i.e., $\langle H_i, \lambda\rangle = \langle \gamma(H_i), \lambda\rangle$ for all i, then the automorphism γ can be realized as a conjugation on $L^{res}(\lambda)$,*

$$\pi(\gamma(u)) = C\pi(u)C \tag{5.24}$$

with $C^2 = \mathbf{1}$, and $L^{res}(\lambda)$ is real. If $L^{res}(\lambda)$ is in addition a unitary representation of the compact form (3.6) of $\mathcal{U}_q^{res}$, then there exists an orthonormal basis of $L^{res}(\lambda)$ such that the matrices $\pi(u)^i_j$ satisfy

$$\pi(u)^T = J^{-1}\pi(\theta(u))J \tag{5.25}$$

and (5.6) for all $u \in \mathcal{U}_q^{res}$, where $J = \mathbf{1}$ if $L(\lambda)$ is orthogonal at $q = 1$, and J is block–diagonal with blocks of the form $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ if $L(\lambda)$ is symplectic at $q = 1$. In this basis, $C^\dagger = C$ and $C^T = JCJ^{-1}$.

This is proved in Appendix C. An alternative approach has been given in [35]. We will always use such a basis from now on. Even though these properties will be used only for the vector representation V_D in this paper where they can be checked directly, they hold much more generally. In particular, (5.25) together with (5.23) implies by Schur's Lemma that the invariant tensor is given by

$$\hat{g} = J^T C \pi(\tilde{\omega}^{-1}), \tag{5.26}$$

where the hat on $\hat{g}$ will indicate the particular normalization implied by this equation. Further consistency relations of the $\hat{R}$–matrix with the invariant tensor follow from (5.14), see [30] or [35].

Now we concentrate on the case of interest here, which is the vector representation $V_D \cong L^{res}(\lambda_1)$. Since γ acts trivially on the highest weight, V_D is real, and the invariant tensor g is nothing but the metric (2.16). Moreover V_D is orthogonal classically, hence $J = 1$, and

$$\pi(u)^T = \pi(\theta(u)) \tag{5.27}$$

for all $u \in \mathcal{U}_q^{res}$. Therefore $\hat{g}_{ij} = C^i_k \pi^k_j(\tilde{\omega}^{-1})$, and $C^i_j = \delta^i_j$ for $so(2n+1)$.

Finally, the star structure (3.6) on $\mathcal{U}_q$ can be extended by $\omega^* = \omega^{-1}$. This is consistent with (5.14), (5.15) and (5.16) for all real forms (5.5) of $\mathcal{U}_q$, and also with the explicit formulas for ω. Acting with (5.15) on a unitary irreducible representation, it follows by Schur's Lemma that $\pi(\omega)^\dagger \pi(\omega) = 1$, hence $\pi(\omega)^\dagger = \pi(\omega^*)$, as it should be. By (5.26), this implies that

$$\hat{g}^\dagger = \hat{g}^{-1}. \tag{5.28}$$

For V_D, this means $\hat{g}^*_{ij} = \hat{g}^{ji}$, which is well-known.

5.4 Pre–involution on the extended coordinate algebras

In this section, we construct an auxiliary "pre–involution" on certain algebras of coordinate functions extended by $\mathcal{U}_q^{res}$, which will be used in the next section to write down the star structure on the (extended) coordinate algebra.

Let $\mathcal{F}$ be a $\mathcal{U}_q^{res}$–module algebra, which in the present paper will be either $\mathbb{C}_q^D$ or $S_{q,\mathbb{C}}^{D-1}$. The following constructions are however much more general.

It turns out one has to extend $\mathcal{F}$ in order to write down the involution. Consider the *semi–direct product (cross product) algebra* $\mathcal{U}_q^{res} \ltimes \mathcal{F}$. As a vector space, this is nothing but $\mathcal{U}_q^{res} \otimes \mathcal{F}$, with the relations

$$ux_i = (u_1 \cdot x_i)u_2 \tag{5.29}$$

for $x_i \in \mathcal{F}$ and $\Delta(u) = u_1 \otimes u_2$ for $u \in \mathcal{U}_q$. This defines a consistent algebra, because $\mathcal{U}_q^{res}$ is a Hopf algebra and $\mathcal{F}$ is a (left) $\mathcal{U}_q^{res}$ –module algebra.

Let $\hat{g}^{ij}$ be the inverse of the metric, i.e., $\hat{g}^{ij}\hat{g}_{jk} = \delta^i_k$, and define as usual [8]

$$L^{-i}_{\ \ j} = \pi^i_j(\mathcal{R}_1^{-1})\mathcal{R}_2^{-1}. \tag{5.30}$$

Then we can define an auxiliary "pre–involution" on $\mathcal{U}_q^{res} \ltimes \mathcal{F}$ as follows.

Proposition 5.2. *Let* $\mathcal{F}$ *be* $\mathbb{C}_q^D$ *or* $S_{q,\mathbb{C}}^{D-1}$, *with generators* x_i. *The map* $\overline{(.)}$ *on* $\mathcal{U}_q^{res} \ltimes \mathcal{F}$ *defined by*

$$\begin{aligned} \overline{x_i} &= \tilde{\omega} x_j C^j_i \tilde{\omega}^{-1} = x_l L^{-l}_{\ \ k} \hat{g}^{ki} \\ \overline{H_i} &= H_i, \ \ \overline{X_i^{\pm}} = X_i^{\mp} \end{aligned} \tag{5.31}$$

can be consistently extended as an antilinear anti–algebra map on $\mathcal{U}_q^{res} \ltimes \mathcal{F}$, *which satisfies*

$$\overline{\overline{x}} = vxv^{-1} \tag{5.32}$$

for any $x \in \mathcal{U}_q^{res} \ltimes \mathcal{F}$. *Moreover,*

$$\overline{x^2} = \overline{x_i x_j g^{ij}} = x^2, \tag{5.33}$$

if the normalization of $g^{ij} = q_S^{(D-1)/2}\hat{g}^{ij}$ *is such that* $g^{ij} = g_{ij}$.

On $\mathcal{U}_q^{res}$, this is just the compact involution considered before; notice that v is central in $\mathcal{U}_q^{res}$. For $q = 1$, $L^{-i}_{\ \ j} = \delta^i_j$, and (5.31) agrees with

(3.10) in the classical limit. In particular, the use of $\tilde{\omega}$ is not essential, but a considerable simplification.

This generalizes easily for more general algebras generated by generators x_i in bigger, real representations. The $\mathcal{R}$–matrix involved in $\overline{x_i}$ essentially corrects the flip in the conjugation of the coproduct of $\mathcal{U}_q$ which occurs if q is a phase. In the form with L^-, a somewhat similar formula was proposed in [24] in a different context. However, we use it as an intermediate step towards the correct "physical" involution. The two formulas for $\overline{x_i}$ are identical, because

$$\begin{aligned} \tilde{\omega} x_j C^j_i \tilde{\omega}^{-1} &= x_l \pi^l_k(\mathcal{R}_1^{-1}) \pi^k_j(\tilde{\omega}) C^j_i \mathcal{R}_2^{-1} \\ &= x_l L^{-l}{}_k \hat{g}^{ki} \end{aligned} \tag{5.34}$$

where we used (5.14) and $\pi^k_j(\tilde{\omega})C^j_i = (\hat{g}^{-1})^k_i = \hat{g}^{ki}$, since $J = 1$ here.

Proof One has to verify that the map is consistent with (5.29); this is done in Appendix C. The verification that $\overline{\overline{x_i}} = v x_i v^{-1}$ is immediate if one uses $\overline{\tilde{\omega}} = \tilde{\omega}^{-1}$ and $C^* = C^{\dagger T} = C$. Recall that we do not consider spinorial representations here, so $\epsilon = 1$ in (5.16). In the more "down–to–earth" version with L^-, the calculation is more complicated, and sketched in the Appendix C.

The bar is compatible with $(P^-)^{ij}_{kl} x_i x_j = 0$, using the fact that $R^* = R^{-1}$; note that for even D, $\hat{R}$ commutes with $C \otimes C$, because $\mathcal{R}$ is invariant under the automorphism γ.

To see the last relation, observe that $C^i_k C^j_l g_{ij}$ is also an invariant tensor, hence it is proportonal to g_{kl}, and in fact it is equal to g_{kl} (consider e.g. the highest weight state, where $C = 1$). By (5.26), this implies that $C\pi(\tilde{\omega}) = \pi(\tilde{\omega})C$ and therefore $g^2 = q_S^{-(D-1)} \pi(v^{-1}) = \mathbf{1}$, so that g^{-1} is equal to g. This is known explicitly for V_D. Hence (5.33) follows from $\tilde{\omega} x^2 \tilde{\omega}^{-1} = x^2$, since x^2 is a singlet. Therefore the bar is well–defined on $S^{D-1}_{q,\mathbb{C}}$.

5.5 Position operators and star structure

We concentrate on $S^{D-1}_{q,\mathbb{C}}$ from now on. The generators t_i define linear operators on $S^{D-1}_{q,\mathbb{C}}$, and we now restrict them to the real sphere $S^{D-1}_q = \oplus_{0 \leq n_0 < k_S} \mathcal{F}(n_0)$. Since $t_i \mathcal{F}(k) \in \mathcal{F}(k+1) \oplus \mathcal{F}(k-1)$ and using the fact that all $\mathcal{F}(k)$ for $0 \leq k < k_S + 1$ are linearly independent by Theorem 4.6, the projector

$$P_c : \bigoplus_{0 \leq n_0 < k_S+1} \mathcal{F}(n_0) \to \bigoplus_{0 \leq n_0 < k_S} \mathcal{F}(n_0) = S_q^{D-1} \tag{5.35}$$

is well–defined[4], commutes with $\mathcal{U}_q^{res}$, and maps $t_i S_q^{D-1} \subset S_{q,\mathbb{C}}^{D-1}$ into S_q^{D-1}. Hence the operator

$$\hat{t}_i := P_c t_i P_c \tag{5.36}$$

acts on S_q^{D-1}, and generates the algebra of functions on the real quantum sphere. Of course for $k < k_S - 1$, it coincides with t_i. Hence from now on, we can consider S_q^{D-1} as an algebra, generated by the $\hat{t}_i$. As before, it can be extended to $\mathcal{U}_q^{res} \ltimes S_q^{D-1}$, which we consider as an *operator algebra* acting on S_q^{D-1}. This will be understood from now on. Since S_q^{D-1} is a Hilbert space, we can calculate the star on that algebra, which is given by the adjoint. We will do this by constructing the inner product on the Hilbert space explicitly.

Observe that the "pre–involution" defined in Proposition 5.2 induces a "pre–involution" on $\mathcal{U}_q^{res} \ltimes S_q^{D-1}$ by

$$\overline{\hat{t}_i} = \tilde{\omega} \hat{t}_j C^j_i \tilde{\omega}^{-1}, \tag{5.37}$$

since

$$\begin{aligned} P_c \overline{t_i} P_c &= P_c \tilde{\omega} t_j C^j_i \tilde{\omega}^{-1} P_c = \tilde{\omega} P_c t_j P_c C^j_i \tilde{\omega}^{-1} \\ &= \tilde{\omega} \hat{t}_j C^j_i \tilde{\omega}^{-1} = \overline{\hat{t}_i} \end{aligned} \tag{5.38}$$

if acting on S_q^{D-1}. Moreover, there exists a unique invariant state $< >$ on S_q^{D-1}, given by the projection on the invariant component (singlet) in the decomposition (5.3), i.e.,

$$\langle f(\hat{t}) \rangle = f_0 \in \mathbb{C}, \tag{5.39}$$

where f_0 is the unique trivial component of $f(\hat{t}) \in S_q^{D-1}$. Because it is invariant, i.e., $\langle u \cdot f \rangle = \varepsilon(u) \langle f \rangle$, this state extends naturally to a state on $\mathcal{U}_q^{res} \ltimes S_q^{D-1}$, so that $\langle uf \rangle = \varepsilon(u) \langle f \rangle = \langle fu \rangle$. It satisfies

$$\langle f \rangle^* = \langle \overline{f} \rangle \tag{5.40}$$

[4]this is no longer true if we include more modes in S_q^{D-1} than in (5.3), since then non–decomposable modules appear.

for $f \in S_q^{D-1}$, which again extends to $f \in \mathcal{U}_q^{res} \ltimes S_q^{D-1}$. This can be used to define an inner product $(\ ,\ )_0$ on S_q^{D-1},

$$(f, g)_0 := \langle \overline{f} g \rangle, \tag{5.41}$$

which however is not hermitian, rather

$$(f, g)_0^* = \langle \overline{\overline{f} g} \rangle = \langle \overline{g} v f v^{-1} \rangle = (g, v \cdot f)_0. \tag{5.42}$$

It satisfies furthermore

$$(f, u \cdot g)_0 = \langle \overline{f} u g \rangle = \langle \overline{u^* f} g \rangle = (u^* \cdot f, g)_0 \tag{5.43}$$

where u^* is the compact involution (3.6), by (5.31).

Now it is easy to write down an invariant hermitian inner product on S_q^{D-1}. We need the square–root of the Casimir v (2.13) acting on the representation S_q^{D-1}. Since v is diagonal on S_q^{D-1} with eigenvalues q^{-c_λ} by Theorem 4.1, we can simply define the operator $\sqrt{v}$ acting on S_q^{D-1} by

$$\sqrt{v} := q^{-\frac{1}{2} c_{n\lambda_1}} \quad \text{on } \mathcal{F}(n), \tag{5.44}$$

and similarly $\sqrt{v^{-1}}$. Here $\mathcal{U}_q^{res}$ is considered as an algebra of operators rather than an abstract (Hopf) algebra, hence there is no problem adding this new operator $\sqrt{v}$. One may also give a more formal definition in terms of a rational function of v, see [35]. This allows to define

$$\Omega := \tilde{\omega} \sqrt{v^{-1}}, \tag{5.45}$$

again as an operator on S_q^{D-1} (or similar spaces). It satisfies $\Omega u \Omega = \theta S^{-1} \gamma(u) = S \theta \gamma(u)$, and

$$\Omega^2 = 1 \tag{5.46}$$

if acting on S_q^{D-1}, by (5.16). Then the following holds:

Theorem 5.3 ("Quantum Euclidean sphere"). *For any $f, g \in S_q^{D-1}$,*

$$(f, g) := (f, \sqrt{v} \cdot g)_0 \tag{5.47}$$

defines a positive definite hermitian inner product on S_q^{D-1}. It satisfies

$$(u \cdot f, g) = (f, u^* \cdot g) \tag{5.48}$$

$$(\hat{t}_i f, g) = (f, \hat{t}_i^* g) \tag{5.49}$$

where u^* is the compact involution (3.6) on $\mathcal{U}_q^{res}$, and

$$\hat{t}_i^* = \Omega \hat{t}_j C_j^i \Omega = \sqrt{v^{-1}}\, \hat{t}_j (L^-)^j_k\, \hat{g}^{ki} \sqrt{v} \tag{5.50}$$

as operators on S_q^{D-1}.

Of course, $(f, \hat{t}_i g) = (\hat{t}_i^* f, g)$ also holds by hermiticity. In particular, this defines a star structure on the *operator* algebra generated by $\hat{t}_i$ and $\mathcal{U}_q^{res}$ acting on S_q^{D-1}.

Proof (5.49) follows from

$$\begin{aligned} (\hat{t}_i f, g) &= (\hat{t}_i f, \sqrt{v} \cdot g)_0 \\ &= \langle \overline{f \hat{t}_i} \sqrt{v} \cdot f \rangle \\ &= \langle \overline{f} \tilde{\omega} \hat{t}_j C_i^j \tilde{\omega}^{-1} \sqrt{v} \cdot g \rangle \\ &= \langle \overline{f} \sqrt{v} \Omega \hat{t}_j C_i^j \Omega^{-1} g \rangle \\ &= (f, \Omega \hat{t}_j C_i^j \Omega^{-1} g). \end{aligned} \tag{5.51}$$

(5.48) follows immediately from (5.43), and hermiticity is seen easily using (5.42):

$$\begin{aligned} (f,g)^* &= (f, \sqrt{v} \cdot g)_0^* = (\sqrt{v^{-1}} \cdot f, g)_0^* \\ &= (g, v\sqrt{v^{-1}} \cdot f)_0 \\ &= (g, f). \end{aligned} \tag{5.52}$$

It only remains to prove that it is positive definite. For this, it is enough to check that $((t_1)^n, (t_1)^n) > 0$ for all $n < k_S$, using the invariance of $(\ ,\ )$ and unitarity of $L^{res}(n\lambda_1)$ for $n < k_S$. This follows either by a continuity argument for q' on the unit circle between 1 and q using (2.24), or by a direct calculation, which is done in Appendix C. There, the bound $n < k_S$ will seen explicitly (the reason is that for larger n, $t_i \mathcal{F}(n)$ becomes indecomposable).

This completes the definition of the real quantum sphere S_q^{D-1}. Together with Ω or $\mathcal{U}_q^{res}$, it forms a star algebra. In the classical limit $q \to 1$, the star becomes the correct involution $t_i^* = t_j g^{ji}$ (3.10) for any given mode $\mathcal{F}(n)$, as was pointed out below Proposition 5.2. Moreover, $S_q^{D-1} \subset S_{q'}^{D-1}$ naturally as a vector space if $q' = e^{i\pi/M'}$ with $M' > M$, since the irreducible representations $L^{res}(n\lambda_1)$ of $\mathcal{U}_q^{res}$ and $\mathcal{U}_{q'}^{res}$ with degrees $n \leq k_S$ are isomorphic. Real, self–adjoint position operators can be defined for example as $\hat{t}_i + \hat{t}_i^*$. This is similar to the "fuzzy sphere" [25] in the sense that there is a cutoff in the set of representations of the symmetry group, but we do not insist here that the algebra of functions should be a simple matrix algebra; instead, we consider a larger algebra including Ω or $\mathcal{U}_q$.

Finally we define the algebra of coordinate functions or position operators on the real Anti–de Sitter space AdS_q^{D-1}, which was our main goal. Since the Hilbert space of scalar fields (5.7) (corresponding to square–integrable functions) has the structure

$$AdS_q^{D-1} = S_q^{D-1} \otimes L^{fin}(M_S\lambda_1) = S_q^{D-1}\rho, \tag{5.53}$$

where $\rho = (t_1)^{M_S}$, we can simply define the operators

$$\hat{t}_i \cdot (f\ \rho) := (\hat{t}_i \cdot f)\ \rho \tag{5.54}$$

for $f \in S_q^{D-1}$, using the definition (5.36). The positive definite inner product on AdS_q^{D-1} is defined as

$$(f\ \rho, g\ \rho) = (f,g) \tag{5.55}$$

where (f,g) is given by (5.47). Using (5.50), the adjoint of $\hat{t}_i$ is

$$(\hat{t}_i)^* \cdot (f\ \rho) = (\Omega \hat{t}_j C^j_i \Omega \cdot f)\ \rho; \tag{5.56}$$

However, we want to express this directly in terms of the Hilbert space AdS_q^{D-1}, without explicitly using the above decomposition. The result is

Theorem 5.4 ("Quantum Anti-de Sitter space"). *The positive definite inner product (5.55) on AdS_q^{D-1} satisfies*

$$\begin{aligned} (u \cdot f, g) &= (f, u^* \cdot g) && (5.57)\\ (\hat{t}_i f, g) &= (f, \hat{t}_i^* g) && (5.58) \end{aligned}$$

where u^* is the star (3.8) on $\mathcal{U}_q^{res}(so(2, D-2))$, and

$$(\hat{t}_i)^* = -\Omega(-1)^E\ \hat{t}_j C^j_i \Omega(-1)^E = -(-1)^{E_j}\sqrt{v^{-1}}\ \hat{t}_j (L^-)^j_k \sqrt{v}\hat{g}^{ki}, \tag{5.59}$$

where E is the energy operator defined in Section (3.1).

Proof Using (2.12), one can see that for $\rho \in L^{fin}(\sum z_i M_i \Lambda_i)$,

$$(\mathcal{R}_1 \cdot f) \otimes (\mathcal{R}_2 \cdot \rho) = q^{(\lambda_f, \lambda_z)} f \otimes \rho \tag{5.60}$$

where λ_f is the weight of f and $\lambda_z = \sum z_i M_i \Lambda_i$; this is because the term $\sum \mathcal{U}_q^{+res} \otimes \mathcal{U}_q^{-res}$ in (2.12) does not contribute since $X_i^{\pm} \cdot \rho = 0$, and the "large" generators vanish as well, by the more explicit formulas in [20, 19]. On AdS_q^{D-1}, one has $q^{d_i M_i z_i} = s_i$ (no sum) with s_i as in (3.8). Hence $q^{(\lambda_f, \lambda_z)} f = (-1)^E \cdot f$, where E is the energy operator as defined in Section 3.1. Therefore

$$(\mathcal{R}_1 \cdot f) \otimes (\mathcal{R}_2 \cdot \rho) = ((-1)^E \cdot f) \otimes \rho = (\mathcal{R}_2 \cdot f) \otimes (\mathcal{R}_1 \cdot \rho). \tag{5.61}$$

Now

$$\begin{aligned}
&\Omega(-1)^E \hat{t}_j C_i^j \Omega(-1)^E \cdot (f\rho) \\
&= \Omega(-1)^E \hat{t}_j C_i^j \sqrt{v^{-1}} (\mathcal{R}_1^{-1} \tilde{\omega}(-1)^E \cdot f)(\mathcal{R}_2^{-1} \tilde{\omega}(-1)^E \cdot \rho) \\
&= \Omega(-1)^E \hat{t}_j C_i^j \sqrt{v^{-1}} (\tilde{\omega} \cdot f)(\tilde{\omega}(-1)^E \cdot \rho) \\
&= -(\sqrt{v} \mathcal{R}_2 \tilde{\omega}^{-1} (-1)^E \hat{t}_j C_i^j \sqrt{v^{-1}} \tilde{\omega} \cdot f)(\mathcal{R}_1 \tilde{\omega}^{-1} (-1)^E \tilde{\omega}(-1)^E \cdot \rho) \\
&= -(\sqrt{v} \tilde{\omega}^{-1} \hat{t}_j C_i^j \sqrt{v^{-1}} \tilde{\omega} \cdot f)\rho \\
&= -(\Omega \hat{t}_j C_i^j \Omega \cdot f)\rho.
\end{aligned} \tag{5.62}$$

Here we used $\Omega = \sqrt{v^{-1}}\tilde{\omega} = \sqrt{v}\tilde{\omega}^{-1}$, $\Delta(\tilde{\omega}^{-1}) = \mathcal{R}_{21}\tilde{\omega}^{-1} \otimes \tilde{\omega}^{-1}$, and the fact that

$$\sqrt{v}\, \hat{t}_i \sqrt{v^{-1}} (f \otimes \rho) = -(\sqrt{v}\, \hat{t}_i \sqrt{v^{-1}} f) \otimes \rho \tag{5.63}$$

for $f\rho \in AdS_q^{D-1}$, which is not hard to see.

This completes the definition of the star–algebra of coordinate functions on the real quantum Anti–de Sitter space AdS_q^{D-1}, extended by Ω or $\mathcal{U}_q^{res}$. In the second form of (5.59), the classical limit $q \to 1$ is easy to understand for low–energy states (i.e., states with weight close to the origin), since then $(L^-)^j_k$ becomes δ^j_k, and the star approaches the classical form $\hat{t}_i^* = -\hat{t}_j (-1)^{E_j} g^{ji}$ in the limit, see (3.11). For energies $E < M_S$, the scalar fields have the same structure as classically. Self–adjoint position operators can be defined as $\hat{t}_i + \hat{t}_i^*$, and together with $\mathcal{U}_q$ we have a complete framework for quantum mechanics on the q–deformed AdS space.

It should be noticed that the operators $\hat{t}_i$ satisfy the algebra (2.20) on all $\mathcal{F}(n)$ except the one with maximal n. This is related to the classical issue of the domain of definition of the position operators on AdS space, which are then unbounded operators, and can be obtained as the limit of $\hat{t}_i$ for $q \to 1$. Then the "maximal" $\mathcal{F}(n)$ becomes the lowest–weight module with minimal mass, and could be called the "massless" field (even though the definition of "massless" is somewhat ambiguous on the AdS space). In the classical case, the product of this field with the coordinate function t_i is not in the Hilbert space any more, and one might simply say that this field is not in the domain of definition of the position operator. However its domain can be extended, and the definition (5.36) of the operator $\hat{t}_i$ amounts to an extension such that (5.59) still holds, at the expense of the commutation relations. Hence we enter the nontrivial subject of extensions of the domain of definition of symmetric operators. This shows that even issues of operator analysis can be addressed in our "regularized", finite–dimensional version, which might be of interest even from the purely classical point of view.

6 Further developments

6.1 Calculus

It is well–known that there exists a q–analogue of the usual differential calculus, differential forms etc. on the quantum Euclidean space; for an extensive review see e.g. [9]. Even though this is usually done for generic q, there is no problem extending it to roots of unity. Moreover, the reality structures defined in Section 5 immediately generalize to these algebras. The basic relations are as follows [39]:

$$\partial_i x_j = g_{ij} + q_S (\hat{R}^{-1})^{kl}_{ij} x_k \partial_l, \tag{6.1}$$

with

$$(P^-)^{ij}_{kl} \partial_i \partial_j = 0. \tag{6.2}$$

The one–forms dx_i satisfy

$$dx_i dx_j = -q_S \hat{R}^{kl}_{ij} dx_k dx_l, \quad x_i dx_j = q_S \hat{R}^{kl}_{ij} dx_k x_l. \tag{6.3}$$

All this is covariant under the symmetry algebra $\mathcal{U}_q^{res}$.

The differential forms that should be used on the sphere or Anti–de Sitter space are $\frac{1}{r} dx_i$, where $r^2 = x^2$. However, the number of independent one–forms remains D instead of $D-1$, since $d(x^2)$ cannot be omitted (i.e., factored out) consistently. This may in fact be quite interesting from a physical point of view, see Section 6.3. From a formal point of view, this is related to the fact that there exists a certain analog of the Dirac–operator in the sense of Connes,

$$w = \frac{q_S^2}{(q_S+1)r^2} d(r^2) \tag{6.4}$$

which generates the calculus on these quantum spaces by

$$[w, f]_\pm = (1-q_S) df \tag{6.5}$$

for any form f with the appropriate grading. This was discovered by Bruno Zumino [41]. It satisfies

$$dw = w^2 = 0. \tag{6.6}$$

It is also worth observing that $d(x_1)^{M_S} = 0$. One could go on and define an analog of the Laplace operator by $\Delta = g^{ij}\partial_i\partial_j$, and write down wave equations. However at roots of unity, this is perhaps a bit too simple–minded. A different proposal will be made in Section 6.3.

6.2 Integral

The invariant integral of polynomial functions on the quantum sphere is uniquely given by the projection on the invariant component (5.39), as in [34, 12, 9] for real q. To make the notation more intuitive, we can write

$$\int_{S_q^{D-1}} f(\hat{t}) = \langle f(\hat{t}) \rangle \tag{6.7}$$

where S_q^{D-1} is considered as the algebra generated by $\hat{t}_i$. One can give an explicit formula, using the calculus discussed in the previous section: Consider the invariant linear functional [34, 9] defined on M_q^{2k} by $\langle f(x) \rangle_\Delta := c_k^{-1} \Delta^k f(x)\big|$ for $k < k_S$, where c_k is given by (11.21) in the Appendix, and the vertical bar means "evaluation", i.e., it anihilates the derivatives. We define c_k such that $\langle x^2 g(x) \rangle_\Delta = \langle g(x) \rangle_\Delta$, hence $\langle\ \rangle_\Delta$ is a functional on S_q^{D-1}. Since it is also invariant, it agrees with the unique integral,

$$\int_{S_q^{D-1}} f(\hat{t}) = c_k^{-1} \Delta^k f(x)\big| \tag{6.8}$$

if $f(\hat{t})$ is given by a reduced polynomial of degree $2k$; of course, $\int_{S_q^{D-1}} f(\hat{t}) = 0$ if $f(\hat{t})$ is odd. (5.40) implies that

$$\left(\int_{S_q^{D-1}} f(\hat{t})\right)^* = \int_{S_q^{D-1}} f(\hat{t})^*, \tag{6.9}$$

with the involution (5.50); note that even though $f(\hat{t})^*$ is an element of $\mathcal{U}_q^{res} \ltimes S_q^{D-1}$, the projection on the invariant component of S_q^{D-1} unambiguously reduces $\int_{S_q^{D-1}} f(\hat{t})^*$ to a complex number. In particular,

$$\int_{S_q^{D-1}} f(\hat{t})^* g(\hat{t}) \tag{6.10}$$

agrees with the positive–definite inner product on S_q^{D-1} defined in Theorem 5.3.

On AdS_q^{D-1}, the situation is similar. The algebra of $S_{q,\mathbb{C}}^{D-1}$ implies for example

$$\begin{aligned} (t_1)^{M_S} t_i &= -t_i (t_1)^{M_S}, \quad i \neq 1, D \\ (t_1)^{M_S} t_D &= t_D (t_1)^{M_S} \end{aligned} \tag{6.11}$$

(the minus could be eliminated by defining $\zeta_i = q_S^E t_i$). Similarly, one obtains commutation relations between the other elements of $\mathcal{F}(M_S)$ and the generators t_i, and hence between the elements of $\mathcal{F}(M_S)$ themselves. This algebra is consistent with the action of $\mathcal{U}_q^{res}$, which is that of $\tilde{\mathfrak{g}}$ as we have seen earlier. Moreover, it is naturally a Hilbert space as we have used many times, hence the algebra generated by $\mathcal{F}(M_S)$ is naturally a star algebra, using the adjoint. The inner product can again be obtained from a unique invariant state. Combining this with S_q^{D-1}, $\mathcal{A} := (\oplus_k \mathcal{F}(kM_S)) \otimes S_q^{D-1}$ becomes a star–algebra. The unique invariant state defines an integral on $\mathcal{A}$, so that the inner product (5.4) of this Hilbert space can be written in the form

$$(f, g) = \int_{\mathcal{A}} f^* g. \tag{6.12}$$

In particular, this defines a positive–definite integral on AdS_q^{D-1}.

6.3 Towards formulating physical models

We have already discussed the one–particle states for scalar particles in detail: they are just the irreducible representations $\mathcal{F}(n) \cong L^{fin}(n\lambda_1) \in AdS_q^{D-1}$ for $M_S \leq n < M_S + k_S$, or equivalently the lowest weight modules $L_{fin}(k\lambda_1)$ for $(D-2)/2 < k \leq M_S$. The integer k characterizes the "mass" of the particle, in the Anti–de Sitter sense. How it should be determined is a dynamical question; one might e.g. write down Lagrangians and try to formulate a variational principle. In particular, this would allow to consider interaction terms, since functions on quantum Anti–de Sitter space can be multiplied. We will come back to this in a moment.

Spin one particles or vector fields are irreducible representations $L^{fin}(k\lambda_1 - \alpha_1)$, where k satisfies certain unitarity bounds. They can be described as one–forms on AdS_q^{D-1}, as discussed in Section 6.1. As we pointed out, there are now D components instead of $D-1$, the additional one corresponding to "radial" forms $f(t)dr$. In the classical limit, they simply become scalar fields on AdS space, which are intimately connected to their vector partners. In fact in the massless case, they turn out to be ghosts of the gauge fields. Moreover, there is a natural BRST operator which arises from $\mathcal{U}_q^{res}$ [36, 35]. All these surprising features show that the remarkable structures that arise at roots of unity may have very interesting physical interpretations. We should also mention the classical symmetry $\tilde{\mathfrak{g}}$ here, which appears to be spontaneously broken in a natural way to $su(2)^r$ for $D = 2r+1$, and to $u(1)^r$ for $D = 2r$ [37]. Of course, fields with different spin can also be described.

However, the correct formulation of a dynamical principle on AdS_q is not clear, and deserves some discussion. The traditional approach would be to write down a Lagrangian, which determines a Hamiltonian, which in turn dictates the dynamics of a model. On AdS_q^{D-1} however, the energy is given a priori; it is simply the element E of the Cartan subalgebra of $\mathcal{U}_q(so(2, D-2))$ defined in Section 3.1. Hence it seems that the dynamics should be formulated here by a principle which determines the physical, "on–shell" degrees of freedom, as in string theory. This is quite trivial in a non–interacting model, since it just amounts to picking an irreducible representation. In an interacting model, determining a consistent physical Hilbert space in terms of the representations we have at our disposal seems very nontrivial. In particular, it involves tensor products of the one–particle states, which is a rather complicated and rich topic for quantum groups at roots of unity [5]. This is also related to knot theory via the $\hat{R}$ –matrix and the Drinfeld Casimir v (2.13), which has proved to be very important in earlier sections. In view of this and other considerations, we propose the following "on–shell" condition:

$$v = 1 \tag{6.13}$$

or perhaps $v = c$ for a constant c, as an operator identity on the physical many–particle Hilbert space, supplemented by other conditions. This is somewhat reminiscent of the on–shell condition in string theory, $L_0 = a$ (among others), see also [1]. It will be discussed further in the next section.

Nevertheless, there are obstacles if one tries to perform a "second quantization". In particular, it is neither clear how to impose a symmetrization postulate on AdS_q^{D-1}, nor how to define an invariant path–integral. Consider many–particle states, which should be described by tensor products of the one–particle irreducible representations $L^{fin}(k\lambda_1)$. A first complication is that at roots of unity, such tensor products are generally not completely reducible. However, there exists a reduced tensor product, which yields precisely the correct "physical" many–particle states, as a direct sum of unitary irreducible representations. This was defined in the case of $\mathcal{U}_q^{fin}(so(2,3))$ in [38], and will be discussed more generally elsewhere[5]. The more difficult problem is that there seems to be no natural definition of a totally symmetric or antisymmetric tensor product for more than 2 factors, because the $\hat{R}$–matrix represents (a quotient of) the braid group rather than the symmetric group. A related difficulty arises in the context of path integrals: if one expands the fields as $\psi(t) = \sum a_{(\underline{k})} f^{(\underline{k})}(t)$ in the usual way, then the coefficients $a_{(\underline{k})}$ should be covariant under $\mathcal{U}_q^{fin}$ as well, hence they cannot be ordinary numbers, and there is no obvious definition of an integral. Some new ideas are needed here, presumably related to the natural concepts of braiding, links and vertices associated to the fields $L^{fin}(k\lambda_1)$. A proposal in this direction has been given recently in a somewhat different context [28], which is however not consistent with a Hilbert space formulation.

[5]it is different from the usual "truncated" tensor product [18, 5, 24] defined in the context of conformal field theory.

Finally, we want to mention that it is easily possible to describe fields on covering spaces of AdS_q^{D-1}. This is done by giving up the quantization of the eigenvalues of the Energy E to (half)integers, which arises from the compact time–like curves on Anti–de Sitter space. The corresponding unitary representations of $\mathcal{U}_q^{fin}(so(2, D-2))$ with non–integral weights indeed exist [37]. However, $\mathcal{U}_q^{res}$ is then not admissible any more, which means that the classical symmetry $\tilde{\mathfrak{g}}$ disappears. The fields cannot be described simply by polynomials in the t_i any more, but they are still finite–dimensional with similar properties.

7 Discussion of physical aspects

Some remarks on the physical aspects of the suggested framework are in order.

Hilbert space First and foremost, the Hilbert spaces of particles respectively fields on AdS_q^{D-1} are finite–dimensional, even though they are covariant under the noncompact symmetry algebra $\mathcal{U}_q(so(2, D-2))$. Since all local aspects of such a theory should have a smooth limit to the flat, undeformed case, this can nevertheless be viewed as a regularization of the classical, flat case if one takes the limit $R \to \infty$. All states have manifestly positive energy, with a high–energy cutoff of the order of M, in units where $R = 1$. Moreover, there exists only a finite number of inequivalent one–particle representations.

The total number of states in AdS_q^{D-1} as defined in (5.7) can be calculated easily. For simplicity we calculate

$$\begin{aligned}\dim(AdS_q^{D-1}) &= \dim(S_q^{D-1}) \approx \dim\left(\sum_{0\leq k<M_S} \mathcal{F}(k)\right) \\ &= \dim(M_q^{M_S-1}) + \dim(M_q^{M_S-2}) \\ &= \begin{pmatrix} M_S + D - 2 \\ D-1 \end{pmatrix} + \begin{pmatrix} M_S + D - 3 \\ D-1 \end{pmatrix} \\ &\approx 2\frac{M_S^{D-1}}{(D-1)!},\end{aligned} \tag{7.1}$$

for large M_S. This means that if we assign the classical area to the sphere S_q^{D-1}, then the number of states per area is approximately

$$\frac{\#\text{ states}}{\text{area}} \approx c\left(\frac{M_S}{R}\right)^{D-1}, \tag{7.2}$$

where $c \approx \sqrt{2}\left(\frac{e\pi}{2D}\right)^{D/2}$. This corresponds to a "quantum length" of $L_{min} = R/M_S$, and the same should hold on AdS_q^{D-1}.

However, the physical interpretation of the cutoff M_S is not entirely clear. At first sight, on may be tempted to relate L_{min} to the Planck scale, even though we have not attempted to formulate gravity here. On the other hand, the commutation relations (2.20) together with (2.19) are inhomogeneous and suggest the existence of another, larger length scale given by $L_{NC}^2 \approx R^2 i(q-q^{-1})$ or $L_{NC}/R \approx \sqrt{M^{-1}}$, where one would expect the noncommutativity to become important. This implies a certain quantization of the radius R of AdS space in terms of L_{NC} and M. Of course this argument may be questioned because the coordinate functions t_i are not real. However this scale is relevant whenever functions on AdS_q^{D-1} are multiplied, for example if one would write down interaction terms in Lagrangians, as discussed in the previous section. We will see further evidence for the importance of this scale in the next paragraph. Hence L_{NC} is expected to be the relevant "non–commutative" scale where the "fuzzyness" of the space becomes important, while M_S/R is the maximal energy of the entire many–particle Hilbert space. Of course, these issues can only be settled once a physical model has been formulated and studied. It is nevertheless quite interesting that in the context of loop quantum gravity and spin networks [3], a similar quantization of the cosmological constant in terms of the Planck length and the root of unity of a quantum group was found.

At this point, we can get some insight into the proposed on–shell condition $v = 1$ (6.13), for the one–particle sector. Consider the unitary representations $L^{fin}(\lambda) = L_{fin}(\mu)$ for $\lambda = 2M_S\lambda_1 - \mu$, which has lowest weight μ. It satisfies $v = 1$ if the classical Casimir c_λ is divisible by $2M$, by (2.30). This holds in particular if $c_\lambda = 0$, which is a good definition of massless fields on AdS space; for example, it means $\mu = (D-2)\lambda_1$ for scalar fields. It also characterizes certain gauge fields with higher spin [10], for example the massless vector fields on AdS_q^4, which contain the correct degrees of freedom of gauge fields [38]. This will be discussed in more detail elsewhere; in any case, the constraint $v = 1$ yields certain massless fields, and a series of massive fields with masses of the order of $\sqrt{M}$, since c_λ is quadratic in λ. The latter is essentially L_{NC}^{-1}, the scale found above where noncommutative effects are expected to become important. Therefore one gets qualitatively a similar picture as in string theory. However, the tower of massive fields terminates here.

Anti–particles Another interesting issue is the physical meaning of certain observables, such as the operator E defined in Section 3.1. For E much smaller than M_S, this is clearly the generalization of the classical energy. However, note that the spectrum (=the weights) of the irreducible representations $L^{fin}((M_S + k)\lambda_1)$ for $k < M_S$ is symmetric around $M_S\lambda_1$, and there are as many states with $E = 2M_S - n$ as there are with $E = n$. Hence one could interpret $[E]_{q_S} \approx \frac{M_S}{\pi}\sin(\frac{E\pi}{M_S})$ as the physical energy which has a maximum at $E = M_S$, and interpret the states with $E > M_S$ as anti–particle states. This is actually more natural from the mathematical point of view. Then particle and anti–particle states could be unified in one irreducible representation, which is a fascinating picture. Alternatively, one may interpret $L^{fin}((-M_S + k)\lambda_1) \subset L^{res}((M_S + k)\lambda_1)$ as the anti-particle states corresponding to $L^{fin}((M_S + k)\lambda_1)$. Then the operator Ω defined in (5.45) plays the role of a CPT operator. It could also be redefined using $U(\tilde{\mathfrak{g}})$ so that it leaves $L^{fin}((M_S + k)\lambda_1)$ invariant, if latter is interpreted as containing both particle and antiparticle states as just discussed.

Comments on the AdS–CFT correspondence Recently, an interesting conjecture between string or M theory on $AdS^n \times W$ and (super)conformal field theories in $n-1$ dimensions has been proposed [26]. This includes the cases $AdS^5 \times S^5$, $AdS^3 \times S^3 \times M$, and $AdS^4 \times S^7$. Moreover, this has been extended to certain quotients (orbifolds) S^n/Γ [17], which can break supersymmetry. Even though no precise definition of the string theory or M theory side is known, it has been argued that it implies a "stringy exclusion principle" [27], which may indicate that the "effective" space–time is noncommutative. One may try to identify the integer M with the number of D–branes, as was already suggested in the literature [13, 15]. Hence it is quite striking that the mere definition of quantized AdS spaces at roots of unity presented here leads naturally to such product spaces with spheres, as explained in Section 5.2. In particular, the appearance of $AdS^4 \times S^7_\chi$ was completely unexpected, and occurs without making any choices. Another feature of the AdS–CFT correspondence is the quantization of the radius of *AdS* space in terms of the Planck scale and the number of D–branes [26]. We found a similar quantization above, where the relevant integer is M.

One may object that there is no reason why precisely the Drinfeld–Jimbo quantum groups at roots of unity should play a role in the AdS–CFT correspondence. On the other hand, it has been known for a long time [1] that they do play a role in certain conformal field theories, such as the WZW models. Since the AdS space can be viewed as a coset of the AdS group, it is plausible that the quantum AdS group at roots of unity should be important in the above context. Hence it seems worthwhile to study the consequences of that assumption; as we pointed out, they are quite striking and rich. However at present, it is difficult to study these issues in more detail, since little is known about both the string theory side and the quantum group side.

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9 Appendix A:Vector representations

We review the vector representations for convenience. First consider $so(D)$ with $D = 2r + 1$, and Dynkin diagram as in figure 9.

Figure 1: $so(2r + 1)$.

Then $V_D = L^{res}(\Lambda_1)$ has a basis x_i for $i = 1, \ldots, D$ with weights $\lambda_i = \sum_j (\lambda_i, \alpha_j^\vee)\Lambda_j$. The coordinates $(\lambda_i, \alpha_j^\vee) \in \mathbb{Z}$ will be listed below as row vectors. For $D > 3$, the weights are in descending order

$$\begin{array}{lll}
\lambda_1 & = \Lambda_1 & = (1, 0, 0, \ldots, 0) \\
\lambda_2 & = \Lambda_1 - \alpha_1 & = (-1, 1, 0, \ldots, 0) \\
\lambda_3 & = \Lambda_1 - \alpha_1 - \alpha_2 & = (0, -1, 1, 0, \ldots, 0) \\
\ldots & & \\
\lambda_r & = \Lambda_1 - \sum_{i=1}^{r-1} \alpha_i & = (0, \ldots, -1, 2) \\
\lambda_{r+1} & = \Lambda_1 - \sum_{i=1}^{r} \alpha_i & = (0, \ldots, 0) \\
\lambda_{r+2} & = \Lambda_1 - \sum_{i=1}^{r} \alpha_i - \alpha_r & = (0, \ldots, 1, -2) \\
\lambda_{r+3} & = \Lambda_1 - \sum_{i=1}^{r} \alpha_i - \alpha_r - \alpha_{r-1} & = (0, \ldots, 1, -1, 0) \\
\ldots & & \\
\lambda_{2r+1} & = \Lambda_1 - 2\sum_{i=1}^{r} \alpha_i & = (-1, 0, \ldots, 0) = -\Lambda_1.
\end{array} \tag{9.1}$$

If $D = 3$, then $\lambda_1 = 2\Lambda_1 = \alpha$. In any case, $\lambda_1 = \sum_{i=1}^{r} \alpha_i$, with $(\lambda_1, \lambda_1) = 2$. α_r is the short root, and[6] $\rho_i = (\rho, \lambda_i) = (2r-1, 2r-3, \ldots, 1, 0, -1, .., -2r+1)$.

Next consider $so(D)$ with $D = 2r$ and Dynkin diagram as in figure 9.

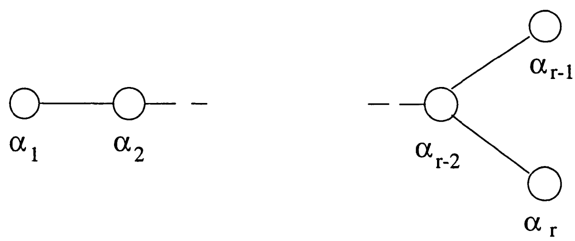

Figure 2: $so(2r)$.

The vector representation V_D has a basis x_i for $i = 1, \ldots, D$ with

[6] in [8], the conventions are such that there is an additional factor 1/2.

weights λ_i. For $D > 4$, the weights in descending order are now

$$\begin{array}{lll}
\lambda_1 & = \Lambda_1 & = (1,0,0,\dots,0) \\
\lambda_2 & = \Lambda_1 - \alpha_1 & = (-1,1,0,\dots,0) \\
\lambda_3 & = \Lambda_1 - \alpha_1 - \alpha_2 & = (0,-1,1,0,\dots,0) \\
\dots & & \\
\lambda_{r-1} & = \Lambda_1 - \sum_{i=1}^{r-2} \alpha_i & = (0,\dots,-1,1,1) \\
\lambda_r & = \Lambda_1 - \sum_{i=1}^{r-2} \alpha_i - \alpha_{r-1} & = (0,\dots,-1,1) \\
\lambda_{r+1} & = \Lambda_1 - \sum_{i=1}^{r-2} \alpha_i - \alpha_r & = (0,\dots,1,-1) \\
\lambda_{r+2} & = \Lambda_1 - \sum_{i=1}^{r} \alpha_i & = (0,\dots,1,-1,-1) \\
\lambda_{r+3} & = \Lambda_1 - \sum_{i=1}^{r} \alpha_i - \alpha_{r-2} & = (0,\dots,1,-1,0,0) \\
\dots & & \\
\lambda_{2r} & = \Lambda_1 - \sum_{i=1}^{r} \alpha_i - \sum_{i=1}^{r-2} \alpha_i & = (-1,0,\dots,0) = -\Lambda_1.
\end{array} \tag{9.2}$$

If $D = 4$, then $\lambda_1 = \Lambda_1 + \Lambda_2 = \frac{1}{2}(\alpha_1 + \alpha_2)$. In any case, $\Lambda_1 = \sum_{i=1}^{r} \alpha_i - \frac{1}{2}(\alpha_{r-1} + \alpha_r)$, and $(\lambda_1, \lambda_1) = 1$. The Weyl vector is invariant under the automorphism γ, and $\rho_i = (\rho, \lambda_i) = (r-1, r-2, \dots, 1, 0, 0, -1, \dots, -r+1)$.

10 Appendix B: On the Weyl Element ω

We want to give some explanations to the remarkable formulas (5.14), (5.16) and (5.15).

To get some confidence in these formulas, notice that if

$$\Delta(\omega) = \mathcal{R}^{-1} \omega \otimes \omega \tag{10.1}$$

holds on the tensor product of 2 fundamental representations, then it holds for any representations. This can be seen inductively: if (10.1) holds on $V \otimes V'$, then it also holds on the representations in $V \otimes V' \otimes V''$. Indeed by coassociativity,

$$(\mathrm{id} \otimes \Delta)\Delta(\omega) \overset{!}{=} \mathcal{R}^{-1}_{1,(23)}(\omega \otimes \omega) \tag{10.2}$$

acting on $V \otimes (V' \otimes V'')$ should be the same as

$$(\Delta \otimes \mathrm{id})\Delta(\omega) \overset{!}{=} \mathcal{R}^{-1}_{(12),3}(\omega \otimes \omega) \tag{10.3}$$

acting on $(V \otimes V') \otimes V''$. The rhs agree indeed since $\mathcal{U}_q$ is quasitriangular, see e.g. Lemma 1.1.1 in [35]; thus the lhs agree as they should. Furthermore, since the action of ω on any given finite–dimensional representation can be expressed in terms of generators of $\mathcal{U}_q$, it satisfies $\Delta'(\omega) = \mathcal{R}\Delta(\omega)\mathcal{R}^{-1}$, and together with (10.1) this implies $\Delta(\omega) = \omega \otimes \omega \mathcal{R}_{21}$. (10.1) also shows that ω is well-defined at roots of unity; this can alternatively be seen from the explicit formulas given e.g. in [21].

The action of ω on the fundamental representations can be found explicitly, see [19]. If they are real, it is essentially the invariant metric g_{ij}, by (5.26).

(5.16) together with the statement that $\epsilon = \pm 1$ on any irreducible representation now follows easily, since $\epsilon := v^{-1}\omega^2$ is grouplike, i.e., $\Delta(\epsilon) = \epsilon \otimes \epsilon$, and it only remains to check that ϵ is ± 1 on the fundamental representations, where it agrees with the classical limit.

Finally, we show $\omega x \omega^{-1} = \theta S \gamma(x)$. Again, this can be verified for the fundamental representations, and it follows for tensor product representations by (10.1) and the fact that both sides of (5.15) are algebra maps.

11 Appendix C: some proofs

Proof of Theorem 4.1 The fact that $\mathcal{F}(k) \cong L^{res}(k\lambda_1) = L^{fin}(k\lambda_1)$ have the same character as classically for k as stated was already explained in the text; this can also be seen to a large extent by considering various Casimirs. It remains to show that they are linearly independent at roots of unity, and that (4.6) holds. Linear independence is clear: if there were a nontrivial linear combination of vectors in the $\mathcal{F}(k)$ which is zero, then there would exists a nontrivial submodule of the form $\mathcal{F}(k_0) \cap (\oplus \mathcal{F}(k_i)) \neq 0$. This would mean that at least one $F(k)$ contains a highest weight vector, which is in contradiction to $\mathcal{F}(k)$ being irreducible (this also follows from well–known facts about representations at roots of unity [5]). Now (4.6) follows immediately, because the characters and hence the dimensions of the $\mathcal{F}(k)$ are the same as classically.

Proof of Lemma 5.1 Assume that γ acts trivially on λ. Since $L^{res}(\lambda)$ is a highest weight module, every $v \in L^{res}(\lambda)$ can be written in the form $v = u^- \cdot w_\lambda$ for $u^- \in \mathcal{U}_q^{-res}$. Define $T_\gamma \cdot v := \gamma(u^-) \cdot w_\lambda$. This is well–defined, because $T_\gamma \cdot (H_i \cdot w_\lambda) = \gamma(H_i) \cdot w_\lambda = H_i \cdot w_\lambda$ by the assumption. Clearly $T_\gamma^2 = 1$, and if C is the matrix representing T_γ, then $C^2 = 1$ and (5.24) follow.

To see that $L^{res}(\lambda)$ is real, observe that $L^{res}(\lambda)$ contains a vector $\omega \cdot w_\lambda$ with weight $-\lambda$, since $\omega H_i \omega^{-1} = -\gamma(H_i)$ and γ acts trivially on λ. This implies (e.g. by the method of characters) that for generic q, the representation $L^{res}(\lambda) \otimes L^{res}(\lambda)$ has precisely one trivial component, which defines the invariant tensor. It can be viewed as a solution of the equations $X_i^+ = H_i = 0$ which are analytic in q. Therefore this invariant tensor is smooth, and induces an invariant tensor also at roots of unity, where the irreducible representation $L^{res}(\lambda)$ can be obtained as quotient of the generic ones. Hence $L^{res}(\lambda)$ is real. This can also be seen from Theorem 2.3.

Next we prove (5.25). (5.23) implies

$$\pi(u)^T = B^{-1}\pi(\theta(u))B \tag{11.1}$$

with $B = CA$, and applying this twice yields

$$\pi(u) = (B^T B^{-1})\pi(u)(B^T B^{-1})^{-1}, \tag{11.2}$$

using $\theta^2 = \text{id}$. Since the representation is irreducible, Schur's Lemma implies that $B^T B^{-1} = \alpha\mathbf{1}$, or equivalently $B^T = \alpha B$ for some $\alpha \in \mathbb{C}$. This is possible only for $\alpha = \pm 1$, thus

$$B^T = \pm B. \tag{11.3}$$

Similarly, taking the complex conjugate of (11.1), we obtain

$$\begin{aligned} \pi(u)^\dagger &= B^{-1*}\pi(\theta(u))^* B^* \\ &= B^{-1*}\pi(\theta(u^*))^T B^* = B^{-1*}B^{-1}\pi(u^*)BB^*, \end{aligned} \tag{11.4}$$

using (5.6) and $\theta(u)^* = \theta(u^*)$. On the other hand, $\pi(u)^\dagger = \pi(u^*)$, and it follows as above that $BB^* = \beta\mathbf{1}$. Together with (11.3), this implies that $BB^\dagger = \pm\beta\mathbf{1}$. The constant $\pm\beta$ must be real and positive because $BB^\dagger$ is a positive definite matrix. We can assume that the determinant of B (or g) has modulus one, hence

$$BB^\dagger = \mathbf{1}. \tag{11.5}$$

Decomposing $B = X + iY$ with real matrices X, Y, it follows that $[X, Y] = 0$. Thus X and Y can be simultaneously (block–)diagonalized with a real, orthogonal matrix O, and B can be written in the form $B = O^T J O$ with J diagonal if $B^T = B$, or J block–diagonal with blocks proportional to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ if $B^T = -B$. Changing the basis using the matrix O yields $\pi(u)^T = J^{-1}\pi(\theta(u))J$ while preserving (5.6), and J has the desired form after a transformation with a unitary diagonal matrix. In particular, (5.25) together with $J^T C = g\pi(\tilde{\omega})$ (see below (5.23)) implies that

$$\hat{g} = J^T C \pi(\tilde{\omega}^{-1}). \tag{11.6}$$

Next, we show that $J = 1$ if the representation is orthogonal at $q = 1$, and $J^T = -J$ if it is symplectic. Equation (5.21) holds also for $u = \tilde{\omega}$, where $\varepsilon(\tilde{\omega}) = 1$ (this can be understood e.g. from the explicit form [14] of ω as a power series in $\mathcal{U}_q^{res}$, or simply by the fact that $\tilde{\omega}$ acts trivially on trivial representations). Using (5.14), this implies

$$g_{ij}(R^{-1})^{ij}_{mn}\pi^m_k(\tilde{\omega})\pi^n_l(\tilde{\omega}) = g_{kl} \tag{11.7}$$

Since $L^{res}(\lambda)$ is real, $L^{res}(\lambda) \otimes L^{res}(\lambda)$ contains precisely one trivial module for generic q, as classically. In that case, it is known [30] that

$$g_{ij}(R^{-1})^{ij}_{mn} = g_{ij}(\hat{R}^{-1})^{ij}_{nm} = \sigma q^{c_\lambda} g_{nm} \tag{11.8}$$

where c_λ is the classical quadratic Casimir. Now for $q = 1$, $(\hat{R})^{ij}_{nm} = \delta^{ij}_{mn}$ is the permutation operator, hence $\sigma = 1$ if $L(\lambda)$ is orthogonal at $q = 1$, and $\sigma = -1$ if it is symplectic. In any case, $\sigma q^{c_\lambda} g_{nm}\pi^m_k(\tilde{\omega})\pi^n_l(\tilde{\omega}) = g_{kl}$, and $\sigma q^{c_\lambda} g_{nm}\pi^m_k(\tilde{\omega}) = g_{kl}\pi^l_n(\tilde{\omega}^{-1})$. Now $\tilde{\omega}^{-1} = \tilde{\omega}v^{-1}$ by (5.16), and together with (2.30) we obtain $g_{nm}\pi^m_k(\tilde{\omega}) = \sigma g_{kl}\pi^l_n(\tilde{\omega})$, or $g\pi(\tilde{\omega}) = \sigma(g\pi(\tilde{\omega}))^T$. By (11.6), this implies $JC = \sigma(JC)^T$. By definition, C leaves the subspace with weight $\pm\lambda$ invariant, where $C = 1$. Hence $J^T = \sigma J$, and the claim follows.

In particular, we have shown $C^T = JCJ^{-1}$, which can also be seen by calculating $\pi(\gamma(u))^T$ in 2 different ways and using Schur's Lemma. Finally, taking the adjoint of (5.24) yields $\pi(\gamma(u))^\dagger = \pi(\gamma(u^*)) = C\pi(u)^\dagger C$. On the other hand, $\pi(\gamma(u))^\dagger = C^\dagger \pi(u)^\dagger C^\dagger$, and Schur's Lemma implies that $CC^\dagger = \alpha \mathbf{1}$ for some $\alpha \in \mathbb{C}$. Since $C = 1$ on the highest weight vector, we obtain $C^\dagger = C$.

(One can show similarly that there exists another basis such that the representation satisfies both (5.6) and $\pi(u)^T = -J^{-1}\pi(u)J$ for $u \in \{X_i^{\pm}, H_i\}$.)

Proof of Proposition 5.2 First we check the compatibility with the cross product algebra. Applying the bar to $ux_i = u_1 \cdot x_i u_2$ where $\Delta(u) = u_1 \otimes u_2$ is the Sweedler notation for the coproduct, we get

$$\begin{aligned} \tilde{\omega} x_k C^k_i \tilde{\omega}^{-1}\overline{u} &= \overline{u_2}\, \overline{x_l} \pi^l_i(u_1)^* \\ &= \overline{u_2}\tilde{\omega} x_k C^k_l \tilde{\omega}^{-1}\pi^i_l(\overline{u_1}), \end{aligned} \tag{11.9}$$

using the fact that V_D is a unitary representation of the compact form. Multiplying from the left with $\tilde{\omega}^{-1}$ and from the right with $\tilde{\omega}$, we have to show

$$x_k C^k_i \tilde{\omega}^{-1}\overline{u}\tilde{\omega} \stackrel{!}{=} \tilde{\omega}^{-1}(\overline{u})_1 \tilde{\omega} x_k C^k_l \pi^i_l((\overline{u})_2), \tag{11.10}$$

or using (5.19)

$$\begin{aligned} x_k C^k_i \theta S^{-1}\gamma(\overline{u}) &\stackrel{!}{=} (\theta S^{-1}\gamma(\overline{u})_1)_1 \cdot x_k C^k_l (\theta S^{-1}\gamma(\overline{u})_1)_2 \pi^i_l((\overline{u})_2) \\ &= (\theta S^{-1}\gamma(\overline{u})_{12}) \cdot x_k C^k_l \theta S^{-1}\gamma(\overline{u})_{11}\pi^i_l((\overline{u})_2) \\ &= x_n C^k_l \pi^k_n(S^{-1}\gamma(\overline{u})_{12})\pi^i_l((\overline{u})_2)\theta S^{-1}\gamma(\overline{u})_{11} \\ &= x_n \pi^i_m((\overline{u})_2 S^{-1}(\overline{u})_{12}) C^m_n \theta S^{-1}\gamma(\overline{u})_{11} \\ &= x_n \pi^i_m(\varepsilon((\overline{u})_2)) C^m_n \theta S^{-1}\gamma(\overline{u})_1 \\ &= x_n \theta S^{-1}\gamma(\overline{u}) C^i_n \\ &= x_n \theta S^{-1}\gamma(\overline{u}) C^n_i \end{aligned} \tag{11.11}$$

as desired, using Lemma 5.1, $C = C^T$ and standard properties of Hopf algebras (recall that $C = 1$ for B_n, which simplifies this calculation).

Next we verify that $\overline{\overline{x_i}}$ is given by the conjugation with v, using the form (5.34). Applying the bar to (5.34), we find using $\overline{L^{-i}_{\ j}} = L^{+j}_{\ i}$ (which follows from $\overline{\mathcal{R}}_1 \otimes \overline{\mathcal{R}}_2 = \mathcal{R}_2^{-1} \otimes \mathcal{R}_1^{-1}$) and $g^\dagger = g^{-1}$

$$\begin{aligned} \overline{\overline{x_i}} &= \hat{g}_{ik} L^{+k}_{\ l} x_t L^{-t}_{\ u} \hat{g}^{ul} \\ &= \hat{g}_{ik} x_r \pi^r_t(L^{+k}_{\ s}) L^{+s}_{\ l} L^{-t}_{\ u} \hat{g}^{ul} \\ &= \hat{g}_{ik} x_r R^{rk}_{ts} L^{+s}_{\ l} L^{-t}_{\ u} \hat{g}^{ul} \\ &= \hat{g}_{ik} x_r L^{-r}_{\ t} L^{+k}_{\ s} R^{ts}_{ul} \hat{g}^{ul} \\ &= \hat{g}_{ik} x_r L^{-r}_{\ t} L^{+k}_{\ s} \hat{g}^{st} q^{-c_{\Lambda_1}}, \end{aligned} \tag{11.12}$$

where we used the standard commutation relations of the $L^{\pm}$–matrices [8], and $R^{ts}_{ul}\hat{g}^{ul} = \hat{R}^{st}_{ul}\hat{g}^{ul} = q^{-c_{\Lambda_1}}\hat{g}^{st}$. On the other hand, using (2.14) one has

$$\begin{aligned} vx_iv^{-1} &= x_j\pi^j_k(\mathcal{R}_1^{-1})\pi^k_i(\mathcal{R}_b^{-1})\mathcal{R}_2^{-1}\mathcal{R}_a^{-1}q^{-c_{\Lambda_1}} \\ &= x_jL^{-j}{}_kSL^{+k}{}_iq^{-c_{\Lambda_1}} \\ &= x_jL^{-j}{}_k\hat{g}_{it}L^{+t}{}_l\hat{g}^{lk}q^{-c_{\Lambda_1}}, \end{aligned} \tag{11.13}$$

since $(1\otimes\pi^k_i)\mathcal{R}^{-1} = SL^{+k}_{\ \ i} = \hat{g}_{it}L^{+t}{}_l\hat{g}^{lk}$, see [8]. Clearly both forms agree.

Positivity of (5.47) Consider

$$\begin{aligned} ((t_1)^n,(t_1)^n) &= <\overline{(t_1)^n}\sqrt{v}\cdot(t_1)^n> \\ &= <(t_1)^n\tilde{\omega}^{-1}\sqrt{v}\cdot(t_1)^n> \end{aligned} \tag{11.14}$$

where we used $C^i_1 = \delta^i_1$ since t_1 is the highest–weight vector of V_D (we omit the hat on the t_i since $n < k_S$). Using $(\mathcal{R}_1^{-1}\cdot t_1)(\mathcal{R}_2^{-1}\cdot t_1) = q^{-(\lambda_1,\lambda_1)}(t_1)^2$, one can show that $\tilde{\omega}\cdot(t_1)^n = q_S^{-n(n-1)/2}(\tilde{\omega}\cdot t_1)^n$, hence

$$\begin{aligned} ((t_1)^n,(t_1)^n) &= q_S^{n(n-1)/2} < (t_1)^n\sqrt{v}(\tilde{\omega}^{-1}\cdot t_1)^n > \\ &= q_S^{n(n-1)/2}q_S^{-(n^2+n(D-2))/2} < (t_1)^n(\tilde{\omega}^{-1}\cdot t_1)^n > \\ &= q_S^{-n(D-1)/2} < (t_1)^n(t_kC^k_l\hat{g}_{l1})^n > \\ &= q_S^{-n(D-1)/2} < (t_1)^n(t_D\hat{g}_{D1})^n > \\ &= (g_{D1})^n < (t_1)^n(t_D)^n > \end{aligned} \tag{11.15}$$

where $\hat{g}_{ij} = g_{ij}q_S^{(D-1)/2}$, see Proposition 5.2. To proceed, we use the formula $\langle(t_1)^n(t_D)^n\rangle = c_n^{-1}\Delta^n(x_1)^n(x_D)^n\big|$ of Section 6.2, as well as $\Delta x_i = q_S^2x_i\Delta + \mu\partial_i$ where $\mu = 1 + q_S^{2-D}$. Using $\hat{R}^{11}_{11} = q_S$ [8], (6.1) implies $\partial_1x_1 = x_1\partial_1$, and therefore

$$\Delta x_1^n = q_S^{2n}x_1^n\Delta + q_S^{n-1}[n]_{q_S}\mu x_1^{n-1}\partial_1. \tag{11.16}$$

Now observe that $\partial_ix_D^n\big| = 0$ unless $i = 1$, hence $\partial_1x_D^n\big| = (g_{1D} + q_S^2x_D\partial_1)x_D^{n-1}\big|$, using $(\hat{R}^{-1})^{D1}_{1D} = q_S$. Therefore

$$\partial_1x_D^n\big| = q_S^{n-1}[n]_{q_S}g_{1D}x_D^{n-1}. \tag{11.17}$$

Moreover $\Delta x_D^n\Big| = 0$, and it follows

$$\Delta x_1^n x_D^n\Big| = q_S^{2(n-1)}[n]_{q_S}^2 \mu g_{1D}(x_1^{n-1}x_D^{n-1}), \tag{11.18}$$

thus

$$\langle (t_1)^n (t_D)^n\rangle = c_n^{-1} q_S^{n(n-1)}([n]_{q_S}!)^2(\mu g_{1D})^n \tag{11.19}$$

where $[n]_{q_S}! = [n]_{q_S}[n-1]_{q_S}\cdots[1]_{q_S}$. Now $g_{D1}g_{1D} = 1$, therefore

$$((t_1)^n,(t_1)^n) = q_S^{n(n-1)}(\mu)^n c_n^{-1}([n]_{q_S}!)^2. \tag{11.20}$$

By another tedious, but straightforward calculation one finds

$$\begin{aligned} c_n &= \Delta^n(x^2)^n\Big| \\ &= \mu^{2n} q_S^{n(n-1)+n(D-2)/2}[n]_{q_S}![n+D/2-1]_{q_S}!([D/2-1]_{q_S}!)^{-1}, \end{aligned} \tag{11.21}$$

hence

$$\begin{aligned} ((t_1)^n,(t_1)^n) &= \mu^{-n} q_S^{-n(D-2)/2}[D/2-1]_{q_S}![n]_{q_S}!([n+D/2-1]_{q_S}!)^{-1} \\ &= (q_S^{(D-2)/2}+q_S^{-(D-2)/2})^{-n}\left[\begin{array}{c} n+D/2-1 \\ D/2-1 \end{array}\right]_{q_S}^{-1}, \end{aligned} \tag{11.22}$$

where $\left[\begin{array}{c} n \\ m \end{array}\right]_q = \frac{[n]_q!}{[m]_q![n-m]_q!}$. This is manifestly positive for $n < M_S - (D-2)/2 = k_S$.

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