Lie Groups, Calabi-Yau Threefolds, and F-Theory

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Abstract

The F-theory vacuum constructed from an elliptic Calabi–Yau threefold with section yields an effective six-dimensional theory. The Lie algebra of the gauge sector of this theory and its representation on the space of massless hypermultiplets are

shown to be determined by the intersection theory of the homology of the Calabi–Yau threefold. (Similar statements hold for M-theory and the type IIA string compactified on the threefold, where there is also a dependence on the expectation values of the Ramond–Ramond fields.) We describe general rules for computing the hypermultiplet spectrum of any F-theory vacuum, including vacua with non-simply-laced gauge groups. The case of monodromy acting on a curve of A_{even} singularities is shown to be particularly interesting and leads to some unexpected rules for how 2-branes are allowed to wrap certain 2-cycles. We also review the peculiar numerical predictions for the geometry of elliptic Calabi–Yau threefolds with section which arise from anomaly cancellation in six dimensions.

1   Introduction

The F-theory vacuum [1] constructed from an elliptically fibered Calabi–Yau threefold X with section determines an effective theory with $N = (1, 0)$ supersymmetry in six dimensions. Such supersymmetric theories will have fields in hypermultiplets, vector supermultiplets and tensor supermultiplets. (See, for example, [2] for a discussion of such theories.)

For any particular F-theory vacuum, the taxonomy of the supermultiplets may be derived from the geometry of X as an elliptic fibration via seemingly straightforward methods in the case of the vector and tensor multiplets [3, 4]. The classification of the hypermultiplet content has always been a little harder to carry out. Many methods have been proposed which allow the hypermultiplets to be determined from the geometry in certain cases [5, 6, 7, 8, 9, 10, 11, 12, 13].

The purpose of this paper is to outline a systematic approach to the problem of determining the gauge symmetry and hypermultiplet content of a given six-dimensional theory obtained from F-theory. (Note that as far as the moduli space of hypermultiplets in concerned, our methods utilize the associated type IIA compactification and thus also apply directly to the compactification of M-theory on X giving an $N = 1$ theory in five dimensions and to the compactification of the type IIA string on X to yield an $N = 2$ theory in four dimensions, pro-

vided that the expectation values of certain Ramond–Ramond fields have been tuned appropriately.)

The methods we employ will not be particularly new but we will see that the process of analyzing the gauge group and matter content can be quite a bit more subtle than had previously been appreciated. In particular, the case of monodromy of the fibration leading to non-simply-laced Lie algebras requires some care. A particularly awkward case which has caused some confusion is when a $\mathbb{Z}_2$ monodromy acts on a curve of A_{even} singularities, i.e., a curve of I_{odd} fibers in F-theory language. In this paper we resolve this problem in agreement with an observation by Intriligator and Rajesh in [14] concerning anomaly cancellation.

In section 2 we will show how many features of a Lie algebra structure arise naturally from an elliptically fibered Calabi-Yau threefold. This will allow us to elucidate the method for determining the gauge algebra. In section 3 we discuss exactly how to analyze the hypermultiplet content in the cases where the associated curves and surfaces within the Calabi-Yau threefold are smooth. We discuss the cases where these curves and surfaces are singular in section 4. This section includes some unexpected rules we are forced to adopt for 2-brane wrapping. Although the results of this section are less rigorous than the preceding section, we are able to give precise results in many instances which can be extended to the general case under the fairly conservative assumption that the relevant physics is determined locally from the geometry of the singularities.

Finally in section 5 we emphasize the peculiar numerical predictions which arise from anomaly cancellation in the F-theory compactification on X.

2   Lie algebras and Calabi-Yau threefolds

We begin with a Calabi-Yau threefold X which admits an elliptic fibration $\pi : X \to \Sigma$, where Σ is a complex surface, and also assume that

this elliptic fibration has a section.[1] The type IIA string compactified on X yields an effective four-dimensional theory with $N = 2$ supersymmetry; its strong-coupling limit, known as "M-theory compactified on X," yields an effective five-dimensional theory. One more effective spatial dimension is obtained in a limit in which the areas of all components of the elliptic fibers shrink to zero—this is the "F-theory limit." See, for example, [3, 18] for an explanation of this.

We point out that most of the following analysis does not really depend upon this elliptic fibration structure and applies to M-theory and type IIA compactifications of X. We use the F-theory language as an organizational tool to give examples later on. One also has the advantage in F-theory of being able to use anomaly cancellation as a powerful tool in checking the consistency of results concerning spectra of massless particles. In the F-theory context we can freely exchange the notion of, say, an I_n fiber and an A_{n-1} singularity. The former is the elliptic fibration description for the latter. Recall [4] that this is because although I_n is really the *extended* Dynkin diagram of A_{n-1} and that one always ignores the components of the fiber which hit the chosen section of the elliptic fibration.[2] Thus, in the zero-area fiber limit of F-theory, a shrunken I_n fiber gives the same physics as an A_{n-1} singularity one dimension lower.

Whenever rational curves in X are shrunk down to zero size we expect 2-branes of the type IIA string wrapped around these curves to contribute massless particles to the spectrum. It is precisely these massless states which are the focus of our interest in this paper.

Actually we need to be careful with the statement that massless states appear automatically when a brane wraps a vanishing cycle. There is always the subtlety of B-fields and R-R fields which should be tuned to the right value (usually denoted "zero" by convention) to really obtain a massless state. As emphasized in [19] the relevant parameters to worry about in this context are the R-R fields. We may see this as follows. If one considers the type IIA string compactified on X

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[1]The F-theory limit cannot be taken unless either the fibration has a section, or a B-field has been turned on in the base [15, 16, 17].

[2]In fact, the F-theory limit should really be taken in two steps: First, shrink to zero area all fiber components not meeting the chosen section, producing M-theory or the type IIA string compactified on a space with ADE singularities; then shrink the remaining component of each fiber down to zero area.

then deformations of the Kähler form (and B-field) on X are given by vector moduli. Suppose we use these Kähler moduli to shrink down a holomorphic 2-cycle to obtain an enhanced gauge symmetry. Once we reach this point of enhanced symmetry we may have a phase transition releasing new hypermultiplet degrees of freedom. Thus at the point of phase transition, these new parameters, which include R-R fields, are fixed at some value. Reversing this point of view, we may tune parameters in the hypermultiplet moduli space to achieve an enhanced gauge symmetry but these parameters include R-R fields. Thus we need to assume always that the R-R parameters have been tuned to the appropriate values required to obtain the enhanced gauge symmetries we discuss below.

Witten [6] analyzed how to determine the massless particle content for a given configuration of rational curves. Let us assume that a given rational curve lives in a family parametrized by a moduli space M. In the simplest case one has an embedding $M \times \mathbb{P}^1 \subset X$. An isolated rational curve is a trivial example of this where M is simply a point.

According to Witten's calculation, one half-hypermultiplet may be associated to the fact that a 2-brane breaks half of the supersymmetry. This half-hypermultiplet is then tensored with the total cohomology of M in an appropriate sense. The result is that if M is a point, then we simply obtain a single half-hypermultiplet. If M is an algebraic curve of genus g then we obtain a single vector multiplet and g hypermultiplets. This was also argued by a different method in [5]. Note that for any wrapping we may also wrap with the opposite orientation to double this spectrum.

Of central interest to us is the fact that compactifying a type IIA string theory (and thus M-theory and F-theory) on a Calabi-Yau threefold X produces a theory with a Yang–Mills sector. The gauge fields may be viewed as arising from integrating the R-R 3-form of the type IIA string over 2-cycles in X to produce 1-forms.[3] These 1-forms play the role of the Yang-Mills connection. In addition, the 4-cycles in X which are dual (via intersection theory) to these 2-cycles will play an important role. Let F denote the 4-form field strength of the R-R 3-form in the type IIA string. Note that the 2-branes of the type IIA

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[3]In M-theory, one likewise integrates the M-theory 3-form field over 2-cycles.

theory are *electrically* charged under this field—that is

$$\int_{M_6} *F = 1, \tag{1}$$

for a 6-dimensional shape M_6 (such as a six-sphere) enclosing the seven directions transverse to a fundamental 2-brane.

Upon compactification we will be wrapping 2-branes around a 2-cycle in X to produce a point particle in four-dimensional space-time. To find the charge of this resulting particle we may take $M_6 = S^2 \times S_i$, where S^2 is a sphere in four dimensional space-time enclosing the particle and S_i is a 4-cycle within X.

It follows that in the type IIA compactification

1. We have $b_2(X) = b_4(X)$ gauge symmetries of the type U(1), each labelled by an element of $H_4(X)$, in addition to the $U(1)$ gauge symmetry coming from the R-R 1-form (whose charge is measured using $M_6 = X$, the generator of $H_6(X)$).

2. If a 2-brane wraps a 2-cycle C_a to produce a particle then the "electric" charge of this particle under the U(1) symmetry associated to a 4-cycle S_i will be the intersection number $(S_i \cap C_a)$.

We thus obtain a perturbative U(1)$^{b_4(X)+1}$ gauge symmetry in type IIA. In the M-theory compactification, there is no R-R 1-form, and the eight transverse directions to the M-theory 2-brane are enclosed by $M_7 = S^3 \times S_i$, so the total "perturbative" gauge symmetry is given by U(1)$^{b_4(X)}$. In the F-theory limit, the only 4-cycles which contribute gauge fields are those with intersection number zero with the elliptic fiber; moreover, 4-cycles which are the inverse images of 2-cycles in the base Σ are associated to tensor multiplets rather than gauge fields. Thus, in the F-theory limit, we get a "perturbative" gauge symmetry group of U(1)$^{b_4(X)-b_2(\Sigma)-1}$.

As is now well-known, and as we will discuss, the wrapped 2-branes will elevate this U(1)$^{b_4(X)+\varepsilon}$ gauge symmetry to a non-abelian Lie group (since certain wrapped branes include vector multiplets in their spectra), where $\varepsilon = 1$, 0, or $-b_2(\Sigma) - 1$ for IIA, M-theory or F-theory, respectively. From now on we will concern ourselves only with the Lie

algebra of the gauge symmetry. It was noted in [20] that, at least in F-theory, the global structure of the gauge group may be recovered from the Mordell-Weil group of X as an elliptic fibration.[4]

If this $\mathfrak{u}(1)^{\oplus(b_4(X)+\varepsilon)}$ appears as the Cartan subalgebra of our gauge algebra then the discussion above implies that we may make the following identifications. Let $\mathfrak{h}$ be the (real) Cartan subalgebra, let $\mathfrak{h}^*$ be the dual space, and let $\Lambda \subset \mathfrak{h}$ be the coroot lattice and $\Lambda^* \subset \mathfrak{h}^*$ be the weight lattice so that the Cartan subgroup $U(1)^{b_4(X)+\varepsilon}$ is naturally identified with $\mathfrak{h}/\Lambda$. For the IIA compactification, we take $\Lambda = H_4(X, \mathbb{Z}) \oplus H_6(X, \mathbb{Z})$ and $\Lambda^* = H_0(X, \mathbb{Z}) \oplus H_2(X, \mathbb{Z})$, and in M-theory, we take $\Lambda = H_4(X, \mathbb{Z})$ and $\Lambda^* = H_2(X, \mathbb{Z})$. In F-theory, we begin with the orthogonal complement within $H_4(X)$ of the elliptic fiber E, and then we mod out by $\pi^{-1}H_2(\Sigma)$ (that is, $\Lambda = [E]^\perp/\pi^{-1}H_2(\Sigma) \subset H_4(X)/\pi^{-1}H_2(\Sigma)$); we then take $\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z})$ to be the dual lattice of Λ. In each case, a 2-brane wrapped around a particular 2-cycle is then naturally associated with an element of the weight lattice and its charges under the Cartan subalgebra are given in the standard way.

We work this out in detail in several particular cases. Consider first the case that X contains a "ruled" complex surface S admitting a fibration $\pi : S \to M$, for some M, where all fibers are isomorphic to $\mathbb{P}^1$. The fibers will shrink down to zero size in the F-theory limit. The simplest example of this is $M \times C_1$ where M is a Riemann surface of genus g and that $C_1 \cong \mathbb{P}^1$ is in the fiber direction. That is to say, in our elliptic fibration $\pi : X \to \Sigma$ we have a curve $M \subset \Sigma$ over which the fiber is I_2. Clearly we have massless states appearing for the 2-branes wrapped around C_1. We also have a $\mathfrak{u}(1)$ symmetry associated to $S_1 \cong M \times C_1$. Let us consider the normal bundle of a single C_1 curve. This normal bundle may be written as $\mathcal{O}(a) \oplus \mathcal{O}(b)$ where $a + b = -2$ by the adjunction formula and the fact that X is a Calabi-Yau space. Since this curve may be translated along the M direction one of these line bundles must be trivial. Thus the normal bundle is $\mathcal{O} \oplus \mathcal{O}(-2)$ where the $\mathcal{O}(-2)$ describes the normal bundle direction which is also normal to S_1. Therefore $(S_1 \cap C_1) = -2$. This tells us that we have a vector supermultiplet and g hypermultiplets from wrapping 2-branes around $M \times \mathbb{P}^1$ all with charge -2 with respect to the $\mathfrak{u}(1)$ gauge symmetry associated to this divisor. Similarly by wrapping with the

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[4]Indeed π_1 of the gauge group is equal to the Mordell-Weil group (including both the free and torsion parts).

opposite orientation we obtain a copy of this except with charge $+2$.

These vector supermultiplets enhance the $\mathfrak{u}(1)$ symmetry to $\mathfrak{su}(2)$ in the usual way and we have an additional g hypermultiplets in the adjoint representation. The key point is to notice in this construction that the condition

$$(S_1 \cap C_1) = -2, \tag{2}$$

has played the role of the *Cartan matrix* of $\mathfrak{su}(2)$.[5]

The next simplest case is where we have a set of curves $C_1, \ldots, C_n$ which may intersect each other and are each isomorphic to $\mathbb{P}^1$ and lying in the fiber direction. We assume that $M \times (\bigcup_i C_i)$ embeds algebraically into X.[6] We now have a Cartan matrix given purely by the configuration of $C_1, \ldots, C_n$. Applying the above method we obtain the standard F-theory result of a simply-laced enhanced gauge symmetry as listed, for example, in table 4 of [18].

As noted first in [11] the real power of this Cartan matrix approach is that it gives a clear way of describing *non*-simply-laced gauge algebras. Consider a less trivial example of ruled surfaces as shown in Figure 1. In this example the moduli space M_1 of the curve C_1 is different from the moduli space M_2 the curve C_2. Think of M_1 as the vertical direction in the figure. We obtain ruled surfaces $S_1 = M_1 \times C_1$ and $S_2 = M_2 \times C_2$. We have a two-fold cover $M_2 \to M_1$ branched at one point in Figure 1. Any other branch points are not shown in the figure.

The intersection matrix of this configuration may be written

$$(S_i \cap C_j) = \begin{pmatrix} -2 & 1 \\ 2 & -2 \end{pmatrix}. \tag{3}$$

This is the Cartan matrix for $\mathfrak{sp}(2)$ (or $\mathfrak{so}(5)$) and so our enhanced gauge algebra should be $\mathfrak{sp}(2)$.

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[5]Of course there is a sign difference here compared to usual Lie algebra theory. This sign difference is purely due to the convention that Lie algebra theorists insist on the Cartan matrix being positive definite, rather than negative definite. If string theory had been studied before Lie algebras then the sign would be the other way!

[6]We can consider more generally a situation where we glue together n distinct $\mathbb{P}^1$ fibrations over M along appropriate disjoint sections, forming a chain. In the remainder of this paper, we will continue to explain by example and will not explicitly state the most general form of the algebraic surfaces which contract to M in the F-theory limit.
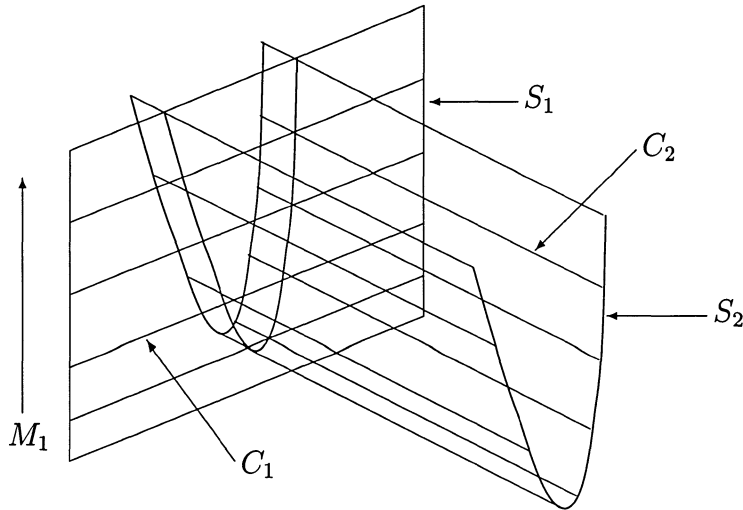
Figure 1: Ruled surfaces producing $\mathfrak{sp}(2)$.

This phenomenon of obtaining a non-simply-laced symmetry algebra was first noted in [7] inspired by the construction of [21]. There it was explained by monodromy acting on the fibers as follows. Let M_1 be embedded in Σ and let M_2 be a two-fold cover of M_1 (branched at various points). Over a generic point in M_1 we see that, ignoring the component meeting the chosen section, the Kodaira fiber consists of one line from S_1 and two lines from S_2 forming the (dual) Dynkin diagram of $\mathfrak{su}(4)$. Moving along a closed path in the complement of the set of branch points of $M_2 \to M_1$ we will exchange the two lines in S_2. This action on the Dynkin diagram is induced by an *outer automorphism* of $\mathfrak{su}(4)$ and the invariant subgroup under this outer automorphism can be taken to be $\mathfrak{sp}(2)$.

One might therefore suspect that the effective gauge algebra is the monodromy-invariant subalgebra of the simply-laced gauge symmetry generated locally by the vanishing cycles. This was the assertion in [7]. Unfortunately it is an ambiguous statement.[7]

Let us analyze carefully all possible outer automorphisms of $SU(2k)$. An element g of $SU(2k)$ satisfies $(^T\bar{g})g = 1$. Complex conjugation t :

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[7]It should be possible to resolve this ambiguity by exhibiting the gauge algebra structure itself (and not just the Cartan matrix) along the lines of [22]. We leave this for future work, and in this paper we shall resort to less direct arguments.

$g \mapsto \bar{g}$ is an example of an outer automorphism. Indeed this acts on the Dynkin diagram of SU(2k) by reflection about the middle node. Clearly the invariant subgroup under this outer automorphism is given by g real. But this yields the group SO(2k)—not what we were expecting!

A general outer automorphism of SU(2k) can be obtained by combining complex conjugation with an arbitrary inner automorphism, yielding $g \mapsto h^{-1}\bar{g}h$, where $h \in$ SU(2k) (there are no other possibilities since that would imply further symmetries of the Dynkin diagram). Since this outer automorphism acts on the Dynkin diagram as the reflection, it is also a viable candidate for the monodromy action on the gauge group. In this general situation, the invariant subalgebra satisfies

$$({}^T g)hg = h. \tag{4}$$

The case $h = 1$ yields SO(2k) as stated before. Now if we put

$$h = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{5}$$

where I is the $k \times k$ identity matrix, we obtain the group Sp(k)—as desired. In this case, the outer automorphism is an involution, but this is not a requirement in general.

We see then that the method of directly working out the Cartan matrix from intersection theory is a better way to determine the effective gauge algebra in F-theory than trying to find subalgebras invariant under outer automorphisms. The latter method is ambiguous. One might try to assert that F-theory picks out the "maximal" invariant subalgebra under all possible outer automorphisms. Indeed, $\mathfrak{sp}(k)$ is "bigger" than $\mathfrak{so}(2k)$ in as much as it has a larger dimension (although in general $\mathfrak{so}(2k) \not\subset \mathfrak{sp}(k)$). However, even this approach is inadequate as will be shown by examples in section 4.

Note that in the M-theory or type IIA compactifications, an ambiguity of the sort we have discovered is actually to be expected. As we have already pointed out, if the Ramond–Ramond fields have non-zero expectation values then some of the non-abelian gauge fields will become massive; when these are integrated out, the gauge group becomes smaller. This is precisely what happens when the outer automorphism of the covering Lie group is varied in the construction above. The gauge algebra which we wish to determine is the one in which these effects

have been turned off so that the F-theory limit can be taken. (A similar phenomenon of variable gauge group depending on the precise value of an outer automorphism has been observed in a closely related context by Witten [23], and applied in [24, 25]).

3   Counting hypermultiplets

In the last section we described how to determine the gauge algebra in F-theory (or M-theory or IIA string theory) by determining the Cartan matrix from intersection theory. Similar methods will in principle determine the hypermultiplet spectrum completely as we now discuss.

First there can be the case of a family of rational curves acquiring extra rational curves at certain points in the family. In the context of elliptic fibrations this can be seen as collisions of curves in Σ over which there are singular fibers. The simplest example is a transverse collision of I_n and I_m.

The resolution of singularities associated to this collision was explained in [26], and applied to the case of an I_n-I_1 collision in the context of string theory in section 8.2 of [11]. The key point is that there exist rational curves within the collision with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. One of these curves C is the intersection of two ruled 4-cycles, one lying over the curve of I_n fibers, and the other lying over the curve of I_m fibers. The normal bundle of C in X is naturally the direct sum of the normal bundles of C in each of these 4-cycles, and each of these is $\mathcal{O}(-1)$. Thus this curve appears as (minus) a fundamental weight. The above rules imply that we have found a curve representing the (lowest) weight of the $(\mathbf{n}, \mathbf{m})$ representation of $\mathfrak{su}(n) \oplus \mathfrak{su}(m)$. By adding other (possibly reducible) curves in the collision fiber we may indeed build up the full $(\mathbf{n}, \mathbf{m})$ representation. Thus *the transverse collision of a curve of I_n and I_m fibers yields a hypermultiplet in the $(\mathbf{n}, \mathbf{m})$ representation of $\mathfrak{su}(n) \oplus \mathfrak{su}(m)$.* (This same result had earlier been determined using quite different methods in [27]. Another approach which is closer to ours appeared in [10].)

Many other "collisions" can be explained in similar ways. However, if the extra rational curves at the collision point have normal bundles

other than $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, then Witten's calculation does not directly apply. General methods for evaluating the corresponding contribution to the hypermultiplet spectrum are not known.

The case of non-simply-laced symmetry algebras raises even more complicated possibilities. Some of the hypermultiplet matter can appear in a somewhat "non-local" manner as we now explain. Suppose we are in a situation analogous to Figure 1. Let us consider the example of a type I_{2k} fiber (where we again ignore the component passing through the chosen section). Let the middle component in the chain have a moduli space given by M_1 and the other components have moduli space M_2 where $M_2 \to M_1$ is a double cover. That is, we have a $\mathbb{Z}_2$-monodromy acting on the I_{2k} fiber (in the only possible way). Figure 1 is the case $k = 2$.

According to [6] we should obtain $g(M_1)$ hypermultiplets for 2-branes wrapping the middle component and $g(M_2)$ hypermultiplets for 2-branes wrapping each of the other components. Note that $g(M_2) \geq g(M_1)$ from the double cover. There are additional hypermultiplets arising from wrapping connected unions of these components. In fact, each of the positive roots of the covering algebra $\mathfrak{su}(2k)$ is represented by such a connected union, some of which are fixed by the monodromy, and others of which are exchanged in pairs under the monodromy. The ones which are fixed under monodromy have M_1 as moduli space, while those which are exchanged in pairs have M_2 as their moduli space.

When we organize these weights in terms of representations of $\mathfrak{sp}(k)$, we find that the invariant subspace describes the adjoint of $\mathfrak{sp}(k)$ while the anti-invariant subspace describes the remaining weights in the adjoint of $\mathfrak{su}(2k)$. On the other hand, each invariant positive root contributes to the invariant subspace, while the roots exchanged in pairs contribute to both the invariant and anti-invariant subspaces. We conclude that the adjoint of $\mathfrak{sp}(k)$ occurs $g(M_1)$ times while the weights in the anti-invariant subspace each occur $g(M_2) - g(M_1)$ times.

We demonstrate which weights appear in the example of $k = 2$ in Figure 2. We show the weights of the adjoint representation. The dots represent weights associated to M_1, i.e., from C_1. The circles represent weights associated to M_2, i.e., from C_2. It is important to note that *reducible* curves may also be wrapped by 2-branes. That is, two rational curves intersecting transversely at a point may be viewed together as a
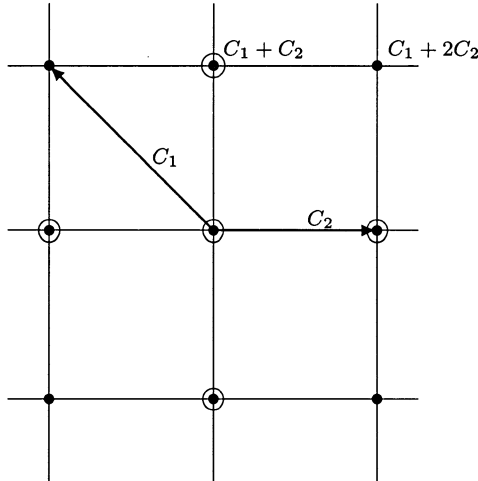
Figure 2: The adjoint of $\mathfrak{sp}(2)$.

nodal rational curve. These wrappings of reducible curves are required to obtain all the adjoint weights of the vector multiplets of the previous section. The reducible curve $C_1 + C_2$ has moduli space given by M_2—since C_2 has a moduli space given by M_2. Looking at Figure 1 we see that there is also a chain of rational curves in the class $C_1 + 2C_2$ but note this this combination is invariant under the $\mathbb{Z}_2$ monodromy and so has moduli space given by M_1. These circles form the weights of **5** of $\mathfrak{sp}(2)$. Ignoring the zero weights for now we see that the adjoint appears $g(M_1)$ times and the **5** of $\mathfrak{sp}(2)$ appears $g(M_2) - g(M_1)$ times.

Indeed the zero weights also work out correctly. A zero weight must represent an uncharged hypermultiplet and therefore a modulus. We may use the work of Wilson [28] to demonstrate this. Wilson showed that a Calabi-Yau threefold containing a ruled surface $M \times \mathbb{P}^1$ has a moduli space which preserves this ruled surface only in codimension $g(M)$. That is, there are $g(M)$ deformations of the Calabi-Yau threefold which destroy this ruled surface. Applying this to both ruled surfaces, we get $g(M_1) + g(M_2)$ deformations. On the other hand, each $\mathfrak{sp}(2)$ adjoint contains a two-dimensional weight zero eigenspace while each **5** contains a one-dimensional weight zero eigenspace. Thus the dimension of the weight zero eigenspace is $2(g(M_1)) + (g(M_2) - g(M_1))$, which simplifies to $g(M_1) + g(M_2)$, as claimed.

The above construction may be easily generalized to $\mathfrak{sp}(k)$:[8]

Theorem 1. *Let $\mathbb{Z}_2$ monodromy act on an I_{2k} fiber in F-theory so that the central component of the fiber has moduli space M_1 and the outer components have moduli space M_2. Thus $M_2 \to M_1$ is a double cover. Then the resulting gauge algebra is $\mathfrak{sp}(k)$ and we have $g(M_1)$ hypermultiplets in the adjoint representation and $g(M_2) - g(M_1)$ hypermultiplets in the Λ^2 representation (which has dimension $k(2k-1) - 1$).*

Similarly $\mathfrak{e}_6$ with $\mathbb{Z}_2$ monodromy will yield an $\mathfrak{f}_4$ gauge algebra with $g(M_2) - g(M_1)$ hypermultiplets in the **26** representation (in addition to the usual $g(M_1)$ adjoints). Also $\mathfrak{so}(2k)$ with $\mathbb{Z}_2$ monodromy will yield an $\mathfrak{so}(2k - 1)$ gauge algebra with $g(M_2) - g(M_1)$ hypermultiplets in the vector **2k − 1** representation. In the case of $\mathfrak{so}(8)$ with $\mathbb{Z}_3$ or $\mathfrak{S}_3$ monodromy, a similar analysis yields $g(M_2) - g(M_1)$ hypermultiplets in the **7** representation of $\mathfrak{g}_2$.

This agrees with the various computations in [8] where $M_1 \cong \mathbb{P}^1$. Let M_2 be the double cover of M_1 branched at b points. Thus $g(M_2) = \frac{1}{2}b - 1$. Then, for example, in the $\mathfrak{f}_4$ case we should have $\frac{1}{2}b - 1$ **26**'s. This agrees with section 4.3 of [8] by identifying the branch points with the $b = 2n + 12$ zeroes of g_{2n+12}.

One might note that the above cases with $\mathbb{Z}_2$ monodromy may be combined into a simple rule as follows. Let $\mathfrak{s}$ be the simply-laced Lie algebra which contains the actual gauge symmetry algebra $\mathfrak{g}$ as a subalgebra invariant under an outer automorphism given by the monodromy action. (In fact, in each of the above cases, the outer automorphism which we use actually has order 2 as an automorphism of $\mathfrak{s}$, not merely order 2 as an automorphism of the Dynkin diagram.) We may then decompose the adjoint representation of $\mathfrak{s}$ as follows

$$\mathrm{Ad}(\mathfrak{s}) = \mathrm{Ad}(\mathfrak{g}) \oplus V_- \qquad (6)$$

where V_- is a (possibly reducible) representation of $\mathfrak{g}$ on which the generator of the $\mathbb{Z}_2$ outer automorphism acts as -1. The above rules may be combined to say that we obtain $g(M_2) - g(M_1)$ hypermultiplets in the V_- representation. As we have already noted above in the case $\mathfrak{g} = \mathfrak{g}_2$, the rule will be different if the monodromy group is not $\mathbb{Z}_2$.

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[8]The explanation given here was applied in [11] to obtain a detailed picture of the surfaces which collapse as the gauge symmetry is enhanced.
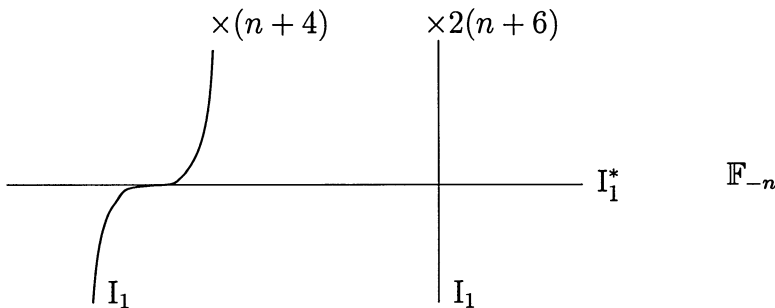
Figure 3: The generic case of Spin(9) gauge symmetry.

In fact, we will see more generally in section 4.1 that if the outer automorphism representing the monodromy has higher order, the simple rule expressed in equation (6) must be modified.

In addition to these "non-local" hypermultiplets coming from rational curves moving in families one may also obtain further hypermultiplets from collisions of curves of reducible fibers as in the I_n-I_m collision discussed above.[9] Note that some simple collisions may just induce monodromy without further contributions (that is, their contributions are completely accounted for by the representation V_- obtained in eq. (6)). As an example we show in Figure 3 the generic case of a Spin(9) gauge symmetry in F-theory.

This figure shows an I_1^* fiber along a section of the Hirzebruch surface $\mathbb{F}_n$. This section has self-intersection $+n$ and is denoted C_0 in the notation of [18]. In the most generic situation, the rest of the discriminant locus of the elliptic fibration will consist of I_1 fibers along curves which intersect C_0 as shown in Figure 3. Generically there are two types of collisions occurring with the frequencies shown. A lengthy computation shows that the $n+4$ cubic collisions[10] produce extra rational curves in the fiber but no monodromy while the $2(n+6)$ transverse collisions produce monodromy but no extra rational curves. Thus the $2(n+6)$ collisions produce $n+5$ of the vector 9's of $\mathfrak{so}(9)$. An analysis of the rational curves in the cubic collision shows that we have $n+4$ spinor

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[9]An argument for why certain hypermultiplets appear to be "local"—i.e., tied to isolated rational curves—or "non-local" was given in [29].

[10]Locally these cubic collisions may be written in Weierstrass form as $y^2 = x^3 - 3s^2t^2x + 2s^3(s + t^3)$ where s and t are affine coordinates in $\mathbb{F}_{-n}$.

16's. Assuming $n \neq -4$, the existence of these spinors shows that the gauge group must be Spin(9). This agrees perfectly with section 4.6 of [8]. Similarly all the other results of [8] may be confirmed.

Finally in this section let us return to the case of $\mathbb{Z}_2$ monodromy acting on a curve of $\mathfrak{su}(2k)$ singularities to give an effective $\mathfrak{sp}(k)$ gauge symmetry. We will consider the Higgs branch in which we give expectation values to the hypermultiplets so as to break completely this $\mathfrak{sp}(k)$ gauge symmetry. Recall that the geometry of moduli spaces of supersymmetric field theories in question imply that the dimension of this Higgs branch should equal the total dimension of the representations of charged hypermultiplets minus the dimension of the gauge group which is broken. We will observe that the geometry is in accord with Theorem 1. We do this by describing the deformations after shrinking all of the curves in the fibers to zero volume. In section 4.2 we will use the ideas introduced here towards the justification of our Main Assertion stated in the next section, which states that the gauge algebra in the case of $\mathfrak{su}(2k + 1)$ with $\mathbb{Z}_2$ monodromy is $\mathfrak{sp}(k)$.

We let $\pi : M_2 \to M_1$ be an unramified (for simplicity) double cover of M_1. In addition, we denote by $\iota : M_2 \to M_2$ the involution which exchanges sheets of the double cover. We now describe a local Calabi-Yau threefold X containing the geometry of $\mathfrak{su}(2k)$ with $\mathbb{Z}_2$ monodromy over M_1. First we construct a Calabi-Yau threefold Y with an $\mathfrak{su}(2k)$ fibration over M_2 without monodromy. Then X will be constructed as a $\mathbb{Z}_2$ quotient of Y.

We construct a singular threefold inside the bundle $V = K_{M_2}^k \oplus K_{M_2}^k \oplus K_{M_2}$ as the variety defined by the equation

$$xy = z^{2k} \tag{7}$$

where x and y are in $K_{M_2}^k$ and z is in K_{M_2}. This threefold has an A_{2k-1} singularity along M_2, which is identified with the zero section of V. It has trivial canonical bundle by the adjunction formula. In a moment we will construct a nowhere vanishing holomorphic 3-form on it, giving independent verification of this fact. The desired threefold Y is obtained by blowing up the singular locus k times in the usual way to obtain a chain of $2k - 1$ ruled surfaces over M_2.

To obtain the desired geometry, we take the quotient of Y by the fixed point free involution obtained by using ι on M_2 while sending

(x, y, z) to $(y, x, -z)$. Note that the fibers of K_{M_2} over $p \in M_2$ and $\iota(p) \in M_2$ are canonically identified with the fiber of K_{M_1} over $\pi(p)$, so this map makes sense. Using the explicit description of the blowup and the fact that x and y are interchanged, it follows that there is $\mathbb{Z}_2$ monodromy.

To show that the quotient X by this involution has trivial canonical bundle, it suffices to show that the involution preserves the holomorphic 3-form on Y. It suffices for our purposes to compute on the singular model. Let ω be any holomorphic 1-form on M_2. Then

$$\omega \wedge dx \wedge dy \wedge dz \tag{8}$$

is a holomorphic 4-form on V with values in K_M^{2k+1}. Now, thinking of ω as a section of K_{M_2}, we divide by ω to obtain the nowhere vanishing 4-form

$$(\omega \wedge dx \wedge dy \wedge dz)\, /\omega \tag{9}$$

on V with values in $K_{M_2}^{2k}$ which is independent of ω. Finally, the residue

$$\mathrm{Res} \left(\frac{(\omega \wedge dx \wedge dy \wedge dz)\, /\omega}{xy - z^{2k}} \right) \tag{10}$$

is the holomorphic 3-form on the singular model of Y. It is clearly invariant under the involution.

The deformations of X may be described as the deformations of Y in V which preserve the involution. The general deformation of Y (up to change of coordinates) is given by

$$xy = z^{2k} + \sum_{i=2}^{2k} f_i z^{2k-i}, \tag{11}$$

where the f_i are sections of $K_{M_2}^i$. Note that we are implicitly assuming $g(M_2) > 0$ to construct these deformations. The invariance condition is that f_i lies in the $(-1)^i$-eigenspace of ι.

We now count parameters. The $+1$-eigenspace of $H^0(K_{M_2}^i)$ has dimension $(2i - 1)(g(M_1) - 1)$, while the -1-eigenspace has dimension $(2i - 1)(g(M_2) - g(M_1))$. Thus the dimension of the Higgs branch is

$$(3 + 7 + \cdots + (4k - 1))(g(M_1) - 1)$$
$$+ (5 + 9 + \cdots + (4k - 3))(g(M_2) - g(M_1)) \tag{12}$$
$$= k(2k + 1)(g(M_1) - 1) + (k(2k - 1) - 1)(g(M_2) - g(M_1)).$$

This is exactly the number of parameters freed up from the $g(M_1)$ adjoints and $g(M_2) - g(M_1)$ copies of the Λ^2 representation by Higgsing an $\mathfrak{sp}(k)$, as expected. We have implicitly assumed that there are no global obstructions to the local deformations that we have constructed above, and our parameter count is consistent with this assumption.

4   The case of monodromy on $\mathfrak{su}(\text{odd})$.

While we appear to have given fairly general rules in the previous section for computing the massless particle spectrum of an F-theory compactification, there are actually many cases where the rules we have given so far become difficult to apply.

In particular, Witten's analysis of the moduli space of rational curves in [6] assumes that everything is smooth (and reduced). This need not be the case. We will discuss some awkward cases which appear quite commonly in F-theory.

We begin with a discussion of a case which has caused some confusion in the literature—that of $\mathbb{Z}_2$ monodromy acting on a gauge algebra of $\mathfrak{su}(2k + 1)$. The approach of asking for the largest subalgebra invariant under an outer automorphism is not that helpful in this case. Putting $h = 1$ in eq. (4) shows that $\mathfrak{so}(2k + 1)$ is one possibility. Decomposing $2k + 1$ as $k + 1 + k$ and putting

$$h = \begin{pmatrix} 0 & 0 & I \\ 0 & 1 & 0 \\ -I & 0 & 0 \end{pmatrix}, \tag{13}$$

(where I is the $k \times k$ identity matrix) shows that $\mathfrak{sp}(k)$ is another possibility. (This form of h is nicely adapted to the action on the Dynkin diagram.) Now it so happens that $\dim(\mathfrak{so}(2k + 1)) = \dim(\mathfrak{sp}(k))$ (and that this is the largest dimension which can occur). So which is the gauge algebra that F-theory actually wants?

Using the approach of section 2 we immediately run into a problem. One of the ruled surfaces, which we will denote S_1, swept out by the reducible components of the fibers will look inevitably locally like the surface
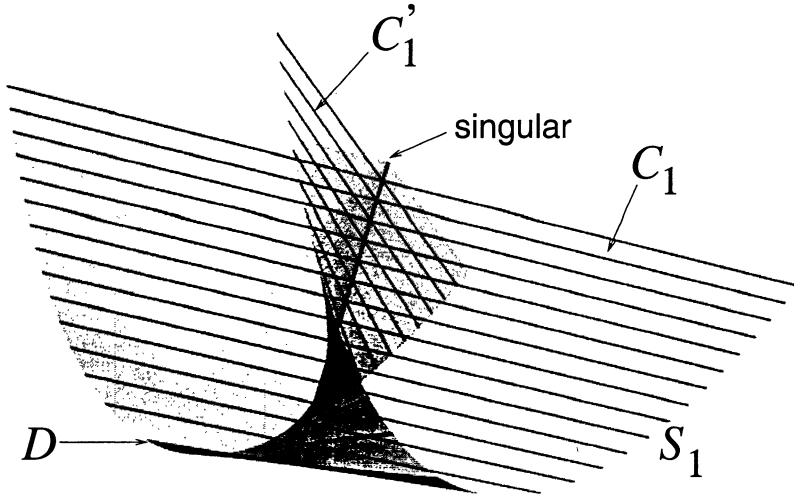
$$y^2 - x^2 z = 0 \tag{14}$$

Figure 4: The singular surface for monodromy in $\mathfrak{su}(3)$.

in $\mathbb{C}^3$. We show a sketch of (the real version of) this surface in Figure 4. Each line C_1 in this surface crosses another line C_1' in the same class. In the case of $\mathfrak{su}(2k+1)$ for $k > 1$ there will be other smooth surfaces. This case is a little hard to visualize. In Figure 5 we show the case of monodromy acting on $\mathfrak{su}(5)$. (In this case S_2 is the surface $z = x^2$.) The thick lines at the bottom of this sketch show the fiber over a branch point of $M_2 \to M_1$.

The problem is that it is not clear what value we should give to $S_1 \cap C_1$ since C_1 meets the singular line in S_1. The most naïve interpretation of Figure 5 is to completely ignore the fact that S_1 is singular and from the figure read off the intersection matrix

$$(S_i \cap C_j) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}. \qquad (15)$$

This would imply that the gauge algebra is $\mathfrak{su}(3)$. In general, according to this argument, $\mathbb{Z}_2$ monodromy acting on $\mathfrak{su}(2k+1)$ would produce $\mathfrak{su}(k+1)$—neither of the possibilities suggested above! It would be the most obvious algebra suggested by "folding the Dynkin diagram up" by the outer automorphism.

We could get an $\mathfrak{sp}(k)$ Cartan matrix from the case in question if we could somehow tie C_1 and C_1' in Figure 4 together. That is, somehow the rules of 2-brane wrapping would have to assert that C_1
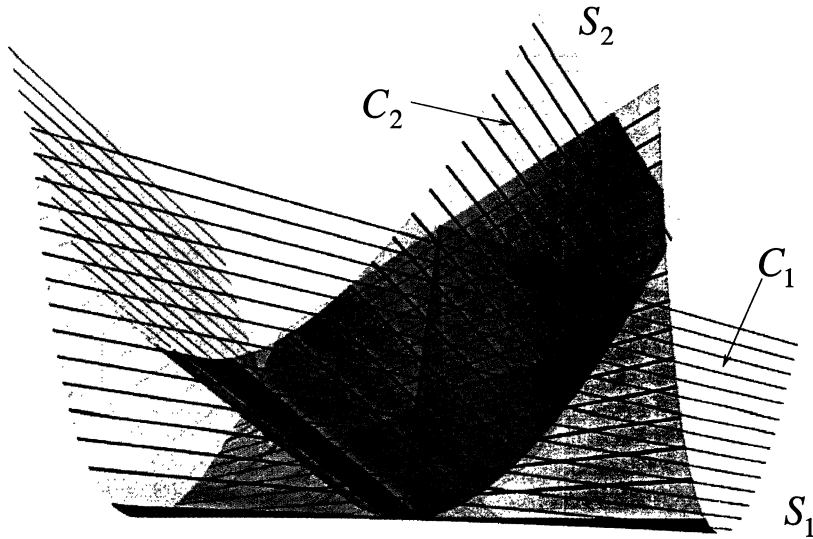
Figure 5: The singular surface for monodromy in $\mathfrak{su}(5)$.

may not be wrapped alone—one must also wrap the intersecting curve C_1' simultaneously. Since S_1 is singular, there is no known reason for ruling such a possibility out. By considering the reducible curve $C_1 + C_1'$ as a single curve, we effective replace S_1 by a simple ruled surface. Thus we would reduce Figure 5 to Figure 1. That is, the case $\mathfrak{su}(2k+1)$ is reduced to $\mathfrak{su}(2k)$ and so we get $\mathfrak{sp}(k)$ under monodromy.

At this point therefore we do not really seem to know what the gauge algebra is. The geometry seems to suggest $\mathfrak{su}(k+1)$ or $\mathfrak{sp}(k)$ while the outer automorphism argument suggests $\mathfrak{sp}(k)$ or $\mathfrak{so}(2k+1)$. We will now give various arguments in support of the following assertion.

Main Assertion . *For an F-theory compactification on an elliptic threefold with a curve of* I_{2k+1} *fibers (which locally suggests a symmetry of* $\mathfrak{su}(2k+1)$*) with* $\mathbb{Z}_2$ *monodromy, the resulting gauge symmetry is* $\mathfrak{sp}(k)$ *(provided that the R-R fields are set to "zero").*

This assertion corrects some statements which appeared in earlier literature where it had been assumed the resulting gauge symmetry was $\mathfrak{so}(2k+1)$ for reasons we discussed above. As mentioned in the introduction one can show that the spectrum of various F-theory models, such as point-like instantons on a D_n singularity, is anomaly free for $\mathfrak{sp}(k)$ but inevitably would have anomalies in some cases had the gauge

algebra contained $\mathfrak{so}(2k+1)$.[11]

Note that the outer automorphism of $SU(2k+1)$ which yields $\mathfrak{sp}(k)$ actually has order 4, since its square is conjugation by the matrix

$$\bar{h}h = h^2 = \begin{pmatrix} -I & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -I \end{pmatrix}, \qquad (16)$$

where I is the $k \times k$ identity matrix; h^2 is not a central element of $SU(2k+1)$. This outer automorphism of course still induces the required reflection of the Dynkin diagram, as we explained near the end of section 2.

This modifies the analysis which led to eq. (6) as follows. We let the outer automorphism of order 4 act on $\mathfrak{s}$ and decompose into eigenspaces:

$$\mathrm{Ad}(\mathfrak{s}) = \mathrm{Ad}(\mathfrak{g}) \oplus V_- \oplus V_i \oplus V_{-i}, \qquad (17)$$

each of which will be a representation of $\mathfrak{g}$ (possibly reducible). As before, the eigenspaces for eigenvalues ± 1 can be accounted for by certain positive roots which are left invariant under the involution and by certain pairs of positive roots which are exchanged under that involution. The moduli space for the former is M_1 and for the latter is M_2; when we consider the quantization of the D2-branes wrapped on the corresponding curves, we find $2g(M_1)$ half-hypermultiplets for each of the invariant roots, and $2g(M_2)$ half-hypermultiplets for each of the pairs. Since each pair contributes to both the $+1$ and -1 eigenspace, this adds up to a total of $2g(M_1)$ half-hypermultiplets in the adjoint representation of $\mathfrak{g}$, and $2g(M_2) - 2g(M_1)$ half-hypermultiplets in the representation V_-.

We have yet to account for the representations V_i and V_{-i}. In fact, these are the roots which contain either C or C', which—as we have argued above—cannot occur as wrapped D2-branes at the generic point of the parameter curve M_1 if we are to reproduce the Cartan matrix compatible with our Main Assertion (or at least, such wrapped branes cannot produce vector multiplets).[12] However, as we will see in section 4.1, the representations V_i and V_{-i} *do* occur in the hypermultiplet

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[11]See the footnote in section 4 of [14] for a full description; further calculations of anomaly cancellation conditions in [30] also support our Main Assertion.

[12]We remain mystified as to the exact mechanism which obstructs D2-branes from

spectrum—perhaps because at the branch points of the map $M_2 \to M_1$, C and C' are identified and there is no apparent obstruction to wrapping the D2-brane there.

To be more concrete concerning the case at hand, with $\mathfrak{s} = \mathfrak{su}(2k + 1)$, $\mathfrak{g} = \mathfrak{sp}(k)$, and the outer automorphism determined by the h in eq. (13), we have

$$V_- = \Lambda^2 \mathbb{C}^{2k} = (\Lambda^2 \mathbb{C}^{2k})_0 \oplus \mathbb{C}, \tag{18}$$

the second exterior power of the fundamental of $\mathfrak{sp}(k)$ (which has a trivial one-dimensional summand), and

$$V_i \cong V_{-i} \cong \mathbb{C}^{2k}, \tag{19}$$

the fundamental representation of $\mathfrak{sp}(k)$.

Thus, the predicted spectrum is:

- $g(M_1)$ hypermultiplets in the adjoint representation

- $g(M_2) - g(M_1)$ hypermultiplets in the second exterior power representation (including its trivial summand), and

- additional hypermultiplets in the fundamental representation.

In fact, an anomaly calculation [30] predicts that there will be precisely $2g(M_1) - 2 + \frac{3}{2}b = 2(g(M_2) - g(M_1)) + \frac{1}{2}b$ such hypermultiplets. One possible interpretation of this formula is that there are two fundamentals (V_i and V_{-i}) associated to the parameter curve M_2 and an additional half-fundamental at each branch point.[13]

4.1   The case of $\mathfrak{su}(3) \to \mathfrak{sp}(1)$.

A Kodaira type IV fiber would intrinsically produce an $\mathfrak{su}(3)$ gauge symmetry but monodromy may act on this fiber producing the case of

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wrapping these unions of curves, or which removes the vector multiplets from the spectrum of the wrapped branes. Note that Freed and Witten [31] have observed obstructions in D-branes related to anomalies.

[13]There are other possible interpretations; for example, one can form the degree four cover M_3 of M_1 which corresponds to the order four element of $SU(2k + 1)$, and express things in terms of the genus of M_3.

interest. At first sight this might not look like such a good candidate for examination since $\mathfrak{su}(2) \cong \mathfrak{sp}(1) \cong \mathfrak{so}(3)$! However, the hypermultiplet spectrum will allow us to distinguish the cases.

Consider the case of amassing point-like E_8-instantons on an orbifold point of a K3 surface along the lines analyzed in [32]. We will be interested in the case of four instantons and six instantons on a $\mathbb{C}^3/\mathbb{Z}_3$ quotient singularity. From result 3 and figure 7 of [32] we may deduce the spectrum without encountering any difficulties. In the case of four instantons, the $\mathbb{Z}_3$ singularity may actually be partially resolved to a $\mathbb{Z}_2$ singularity without affecting the particle spectrum. This $\mathbb{Z}_2$ singularity may then be effected by a "vertical" line of I_2 fibers (in the notation of [32]). The six instanton case is effected by a vertical line of I_3 fibers. The results are

- For four point-like E_8-instantons on a $\mathbb{Z}_3$ singularity we have a nonperturbative enhanced gauge algebra of $\mathfrak{su}(2)$ with hypermultiplets in four **2** representations.

- For six point-like E_8-instantons on a $\mathbb{Z}_3$ singularity we have a nonperturbative enhanced gauge algebra of $\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2)$ with hypermultiplets as $(\mathbf{2}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2})$.

We may also produce exactly the same physics by using a vertical line of type IV fibers. The configurations of curves of Kodaira fibers in the base of the elliptic fibration is shown in Figure 6 for the cases of four and six instantons respectively. These diagrams are again similar to those presented in [32] and represent the situation after the base has been blown up the requisite number of times. The short curved lines represent fragments of the the curve of I_1 fibers.

Let us begin with the case of six instantons on the right of Figure 6. The lines of type II fibers produce no gauge symmetry enhancement. The upper and lower diagonal lines of type IV fibers each collide once transversely with a line of type II fibers and once non-transversely with the curve of I_1 fibers. Actually these collisions are very similar.[14] Each of these collisions produces $\mathbb{Z}_2$ monodromy in the type IV fiber

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[14]Indeed for a special choice of moduli, the line of I_1 fibers can be turned into a line of type II fibers intersecting the line of type IV fibers transversely.
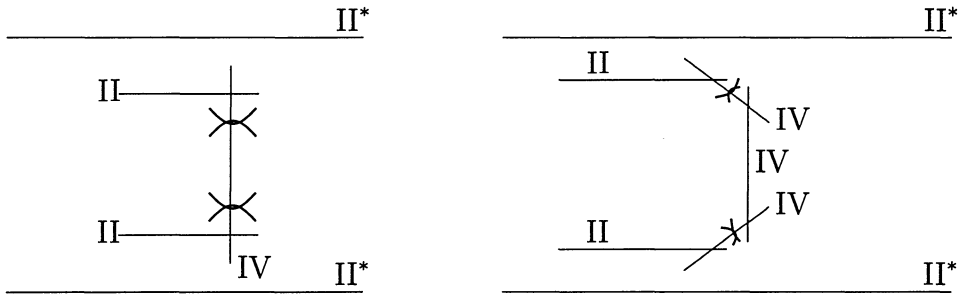
Figure 6: Four and Six Point-like instantons on $\mathbb{Z}_3$ orbifold points.

producing the geometry of Figure 4. Thus each of these diagonal lines of type IV fibers produce an $\mathfrak{su}(2)$ (or $\mathfrak{sp}(1)$) gauge symmetry.

The remaining vertical line of type IV fibers collides with the other two lines of type IV fibers. Resolving this collision shows that no monodromy is induced. Thus this vertical line represents an $\mathfrak{su}(3)$ gauge symmetry. An analysis of the collisions shows that there would be an induced hypermultiplet in the $(\mathbf{3}, \mathbf{3})$ of $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$ for each collision *if there were no monodromy.* Clearly from the desired spectrum above, this $(\mathbf{3}, \mathbf{3})$ must break up as $(\mathbf{2}, \mathbf{3}) \oplus (\mathbf{1}, \mathbf{3})$ of $\mathfrak{su}(2) \oplus \mathfrak{su}(3)$. This tells us immediately that the inclusion $\mathfrak{su}(3) \supset \mathfrak{su}(2)$ produced by the action of the monodromy produces a decomposition of the fundamental of $\mathfrak{su}(3)$ via $\mathbf{3} \to \mathbf{2} \oplus \mathbf{1}$. *This rules out the natural embedding* $\mathfrak{su}(3) \supset \mathfrak{so}(3) \cong \mathfrak{su}(2)$ for which $\mathbf{3} \to \mathbf{3}$.

We are left with having to account for a hypermultiplet $\mathbf{2}$ in each of the $\mathfrak{su}(2)$'s. This must come from the monodromy-inducing collisions of the diagonal lines of type IV fibers with the lines of type II and type I_1 fibers. As these collisions are all the isomorphic locally, each collision must produce a half-hypermultiplet $\mathbf{2}$. This is in agreement with our comments concerning the V_i and V_{-i} representations at the end of the previous subsection. The collision point is the point around which the monodromy acts and so it associated with the location of the curve denoted D in Figure 4.

The choice of associating this $\mathfrak{su}(2)$ as the $k = 1$ case of $\mathfrak{sp}(k)$ or $\mathfrak{su}(k+1)$ differs as explained above by whether we view the positive root of $\mathfrak{su}(2)$ as being associated to C_1 or to $C_1 + C_1'$. Clearly in the latter case we have $2D = C + C_1$ as divisor classes and so D naturally generates

the **2** as required. If only C_1 were identified as the positive root then D would produce nothing new. Therefore we can only correctly identify the spectrum F-theory in the case of the geometry on the right-hand side of Figure 6 if we take one of roots of the gauge algebra to be C_1+C_1'. That is, there really does appear to be a rule in string theory which allows 2-branes to wrap $C_1 + C_1'$ together but not C_1 or C_1' individually.

We can further verify our picture by considering the spectrum for four instantons on the left of Figure 6. There are four collisions with the vertical line of type IV fibers, each producing monodromy. Thus, $g(M_2) = 1$ and there are $b = 4$ branch points. Following the arguments at the end of the previous subsection, we thus predict a spectrum consisting of one hypermultiplet in the $\Lambda^2\mathbb{C}^2 \cong \mathbb{C}$ representation (from V_-), and $2(g(M_2) - g(M_1)) + \frac{1}{2}b = 4$ hypermultiplets in the fundamental representation (i.e. $V_{\pm i}$). This precisely agrees with the spectrum found above: there are four **2**'s of $\mathfrak{su}(2)$. Even the V_- representation "**1**" occurs correctly: it is the deformation *à la* Wilson [28] of M_1, or in physical terms of the heterotic string, it is the deformation of the $\mathbb{Z}_3$ singularity to a $\mathbb{Z}_2$ singularity which does not affect the spectrum as noted earlier.

The rules of 2-brane wrapping are therefore rather unusual for the curves C_1 and C_1'. As observed previously, away from the branch points, a 2-brane can *never* wrap C_1 or C_1' individually. However, as we have just seen, at the branch points where C_1 and C_1' coincide, the 2-brane is allowed to wrap the curve. In fact, when this wrapping is taken with both orientations, a hypermultiplet in the **2** of $\mathfrak{su}(2)$ is produced for each branch point.

Since C_1 lies in the singular surface S_1 it is perhaps not surprising that the usual rules of 2-brane wrapping appear to break down. Anyway, since this same S_1 appears as the "end" component for the higher rank gauge groups of this type, assuming string theory wraps branes around curves in S_1 in a similar way in that context, we arrive at our Main Assertion.

4.2   Deformation to the $\mathfrak{su}$(even) case.

We can also give a different argument in favor of the $\mathfrak{sp}(k)$ gauge group in case M_1 has positive genus. Let us start with the case of I_{2k+2} fibers with $\mathbb{Z}_2$ monodromy. By Theorem 1, this leads to a $\mathfrak{sp}(k+1)$ gauge group and at least one adjoint hypermultiplet. We will show in a moment that the corresponding A_{2k+1} singularity can be smoothed to an A_{2k} singularity with $\mathbb{Z}_2$ monodromy. This corresponds to giving a nonzero vev to a semisimple element of the adjoint hypermultiplet, and the $\mathfrak{sp}(k+1)$ gauge group gets Higgsed to some rank k subgroup $\mathfrak{g} \subset \mathfrak{sp}(k+1)$. We still have to determine what $\mathfrak{g}$ without knowledge of the $\mathfrak{u}(1)\,\mathfrak{sp}(k+1)$ that acquires a vev. Clearly $\mathfrak{sp}(k) \subset \mathfrak{sp}(k+1)$ is possible, so we could have $\mathfrak{g} = \mathfrak{sp}(k)$. We now argue that $\mathfrak{g} = \mathfrak{so}(2k+1)$ is impossible.

Lemma 1. *There is no embedding of $\mathfrak{so}(2k+1)$ in $\mathfrak{sp}(k+1)$ for $k > 1$.*

Proof This argument is due to R. Zierau. Suppose that there were an embedding of $\mathfrak{so}(2k+1)$ in $\mathfrak{sp}(k+1)$. Then the fundamental $2k+2$ dimensional representation V_{2k+2} of $\mathfrak{sp}(k+1)$ would restrict to a representation of $\mathfrak{so}(2k+1)$, which necessarily decomposes as a fundamental representation V_{2k+1} of $\mathfrak{so}(2k+1)$ plus a trivial representation. The alternating form on V_{2k+2} restricts to a alternating form on V_{2k+1}. Since V_{2k+1} is odd dimensional, this form is degenerate. It's nullspace $W \subset V_{2k+1}$ is invariant under $\mathfrak{so}(2k+1)$, and is a proper subspace since the alternating form on V_{2k+1} is not identically zero. This is a contradiction.

The singular surface in question given by eq. (14) may be written as

$$y^2 = x^2 z, \qquad (20)$$

and thought of as a double cover of the xz-plane branched along $z = 0$ and doubly along $x = 0$. It is the double branching that makes the surface singular. We may smooth the surface by deforming to

$$y^2 = x(x - \epsilon)z. \qquad (21)$$

Now the double branching has been split to $x = 0$ and $x = \epsilon$. For a fixed value of z this process replaces a nodal rational curve by a

smooth rational curve, where the nodal rational curve can be viewed as two rational curves intersecting transversely at a point. That is, each pair of intersecting lines in Figure 4 is replaced by a single line and the surface is smoothed.

This smoothing process is remarkably benign at the level of global geometry. It is often possible to perform it even when the geometry of the ambient threefold, X, is completely smooth at all times.

We can then derive the $\mathfrak{sp}(k)$ gauge symmetry indirectly as follows. The existence of the deformation shows that 2-branes are not allowed to wrap the individual lines of Figure 4. The deformation converts each pair of lines into a single new line. Thus if physics is not discontinuously affected by the deformation, the 2-branes contributing to vector particles must only be allowed to wrap the pairs of line in Figure 4 together. As we have observed in the discussion immediately preceding the Main Assertion, we can now conclude that we do indeed obtain $\mathfrak{sp}(k)$.[15]

There is of course a problem with the proof of our Main Assertion by this argument—there may be global obstructions to such a deformation. Wilson's criterion suggests that such deformations only occur when $g(M_1) > 0$, where M_1 is the base of the fibration of Figure 4 as a ruled surface. We address this in part by giving an example of a deformation when $g(M_1) > 0$.

We return to the setup introduced at the end of section 3. Using the notation leading to eq. (11), we write the equation

$$xy = z^{2k+2} + f_1 z^{2k+1}. \tag{22}$$

This gives $\mathfrak{su}(2k + 1)$ with $\mathbb{Z}_2$ monodromy at the generic point of M_1. The deformation is simply

$$xy = z^{2k+2} + f_1 z^{2k+1} + \epsilon f_2 z^{2k}. \tag{23}$$

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[15]Note that the fact that hypermultiplets may arise from wrapping 2-branes around the individual lines is not compromised by this argument. When we deform the curve of A_{2k} singularities to a curve of A_{2k-1} singularities we may affect the geometry of some points on this curve. Thus hypermultiplets which were "spread" over the whole curve of singularities may be localized to isolated rational curves by this deformation process. Massless vectors cannot come from such isolated curves.

To make sense of this, we have to say a little more about the blowup. The A_{2k} blowups are determined by a procedure given in [33] after choosing an ordering of the $2k + 1$ factors of $z^{2k+2} + f_1 z^{2k+1} + \epsilon f_2 z^{2k} = z^{2k}(z^2 + f_1 z + \epsilon f_2)$. Choosing the $(z^2 + f_1 z + \epsilon f_2)$ factor to be in the middle, we obtain the desired geometry. The last blowup creates a single ruled surface, which smooths out the singular component. It is immediate to see from the description in [33] that for generic f_i this is a smooth deformation of the desired type.

In this model, we have placed a restriction on the genus and have introduced localized matter at the zeros of f_1. However, if we are willing to accept that the process of gauge symmetry enhancement is dictated by local geometry then this example is enough to justify our Main Assertion.

5   Numerical Oddities

Finally we close with a note on the peculiar numerical predictions dictated by anomaly cancellation in the six-dimensional physics produced by F-theory compactified on X. This has been discussed in many places before (for example [34]) and is often used as a method of enumerating the spectrum of hypermultiplets. Here we have outlined a systematic way of constructing the hypermultiplet spectrum and so the anomaly constraint becomes a peculiar numerical property of the geometry of an elliptic Calabi-Yau threefold.

For completeness we will repeat the anomaly condition here. We consider an elliptic fibration $\pi : X \to \Sigma$ with a section. Let G be the gauge group (or algebra) in six dimensions and $\rho(\Sigma)$ the Picard number of Σ. Then anomaly cancellation along the lines of [35] yields the following

$$\dim \mathrm{G} - \sum_i \varepsilon_i \dim R_i = 29\rho(\Sigma) - 302, \qquad (24)$$

where the hypermultiplets fall into representations R_i of G and ε_i is equal to 1 if the representation is real or $\frac{1}{2}$ if the representation is complex or quaternionic (pseudoreal). Note that the trivial representations also contribute to the sum. These can be determined from the fact that the number of neutral hypermultiplet moduli are equal to $h^{2,1}(X) + 1$.

As an example consider the extreme case of $G \cong E_8^{17} \times F_4^{16} \times G_2^{32} \times$ SU(2)^{32} corresponding to 24 point-like E_8-instantons on a binary icosahedral quotient singularity in the heterotic string [32]. The Calabi-Yau threefold for the F-theory description of this has $\rho(\Sigma) = 194$. Applying the methods of sections 3 and 4.1 to this threefold we also arrive at a spectrum of hypermultiplets of a $(\mathbf{1}, \mathbf{2}) \oplus (\mathbf{7}, \mathbf{2})$ for each of the 32 copies of $G_2 \times$ SU(2).[16] These representations are quaternionic. Equation (24) then reads

$$5592 - (\tfrac{1}{2} \times 32 \times 16 + 12) = 29 \times 194 - 302. \qquad (25)$$

The anomaly condition in eq. (24) has been verified in situations illustrating Theorem 1 and our Main Assertion (see the footnote in Section 4 of [14]).

Note that one may obtain further conditions from the anomaly cancellation condition. For example one may require the vanishing of coefficient of each "tr(F^4)" term in the anomaly. See for example [34, 9].

It would be very satisfying to give a purely geometric proof of eq. (24) and the other anomaly conditions. (A geometric proof which covers a wide variety of cases has recently been constructed [30].) Sadly at present the origin of this formula without using string theory is something of a mystery. Note that the existence of a section in the fibration $\pi : X \to \Sigma$ is necessary for this to work. If this requirement is not satisfied then there is no six-dimensional physics and the condition need not be satisfied [18] (for an example, see [36]).

Acknowledgements

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[16]The fact that this hypermultiplet spectrum canceled the anomalies was noted in [12].

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