Boundaries of zero scalar curvature in the AdS/CFT Correspondence

Mingliang Cai and Gregory J. Galloway[1]

University of Miami, Department of Mathematics
Coral Gables, FL, 33124

mcai@math.miami.edu
galloway@math.miami.edu

1   Introduction

In [16], Witten and Yau consider the AdS/CFT correspondence in the context of a Riemannian Einstein manifold M^{n+1} of negative Ricci curvature which admits a conformal compactification (in the sense of Penrose [15]) with conformal boundary N^n. As discussed in [16],

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a conformal field theory on N relevant to this correspondence is stable (with respect to the brane action on M) if the conformal class of the boundary contains a metric of positive scalar curvature and is unstable if it contains a metric of negative scalar curvature. In the borderline case of zero scalar curvature, the theory may be stable or unstable. Witten and Yau go on to prove that if the conformal class of N contains a metric of positive scalar curvature, then M and N have several desirable properties: (1) N is connected, which avoids the difficulty of coupling seemingly independent conformal field theories, (2) the nth homology of M vanishes, in particular M has no wormholes and (3) at the fundamental group level, the topology of M is "bounded by" the topology of N.

The aim of the present paper is to show that all of these results extend to the case where the conformal class of the boundary contains a metric of nonnegative scalar curvature. By a well known result of Kazdan and Warner [13], if N has a metric of nonnegative scalar curvature, and if the scalar curvature is positive at some point, then N has a conformally related metric of positive scalar curvature. Hence, the essential case handled here is the case in which the conformal class of the boundary contains a metric of zero scalar curvature. The proof method used in this paper is quite different from, and in some sense dual to, that used in [16]. The method of [16] involves minimizing the co-dimension one brane action on M, and uses the machinery of geometric measure theory, while the arguments presented here use only geodesic geometry. We proceed to a precise statement of our results.

Let M^{n+1} be a complete Riemannian manifold, with metric g, and suppose M admits a conformal compactification, with conformal boundary (or conformal infinity) N. Thus it is assumed that M is the interior of a compact manifold-with-boundary $\overline{M}^{n+1}$ and that there exists a smooth function r on $\overline{M}$ such that (1) $r > 0$ on M, (2) $r = 0$ and $dr \neq 0$ along $N = \partial\overline{M}$, and (3) $r^2 g$ extends smoothly to a Riemannian metric $\bar{g}$ on $\overline{M}$. The induced metric $h = \bar{g}|_{TN}$ on N changes by a conformal factor with a change in the *defining function* r, and so N has a well defined conformal structure. If the conformal class of metrics on N contains a metric of positive (resp., nonnegative, zero, etc.) scalar curvature we say that N has positive (resp., nonnegative, zero, etc.) scalar curvature.

In [16], Witten and Yau consider conformally compactified orientable Einstein manifolds M^{n+1} which satisfy Ric $= -ng$. More generally, their results allow Ric $\geq -ng$ (i.e., Ric$(X,X) \geq -ng(X,X) = -n$ for all unit vectors X) provided Ric $\to -ng$ sufficiently fast as one approaches the conformal boundary N. (Regarding this fall-off, it is sufficient to require Ric $= -ng + o(r^2)$, where r is a suitably chosen defining function for the conformal boundary. This is discussed in more detail below.) In this setting they prove that if N has a component of positive scalar curvature then (1) N is connected, (2) $H_n(\overline{M}, \mathbb{Z}) = 0$, and (3) the map $i_* : \Pi_1(N) \to \Pi_1(\overline{M})$ induced by inclusion $i : N \to \overline{M}$ is onto. The last result says that any loop in M can be deformed to a loop in N. Thus, at the fundamental group level, the topology of M can be no more complicated than the topology of N. In particular, if N is simply connected, so is M. The following theorem generalizes these results by weakening the scalar curvature condition on N.

Theorem 1. *Let M^{n+1} ($n \geq 2$) be a complete Riemannian manifold which admits a conformal compactification, with conformal boundary N^n, and suppose the Ricci tensor of M satisfies, Ric $\geq -ng$, such that Ric $\to -ng$ sufficiently fast on approach to conformal infinity. If N has a component of nonnegative scalar curvature then the following properties hold.*

(a) N is connected.

(b) If M is orientable, the nth homology of $\overline{M}$ vanishes, $H_n(\overline{M}, \mathbb{Z}) = 0$.

(c) The map $i_* : \Pi_1(N) \to \Pi_1(\overline{M})$ (i = inclusion) is onto.

The essential step in the proof is to establish part (a). Part (c) then follows from part (a) by a covering space argument, as noted in [16]. In turn, as will be shown here, part (b) follows from part (c) via basic homology theory (essentially Poincaré duality; cf. [7, 8], where similar arguments have been used). In order to give a flavor of the sorts of techniques that will be used to prove part (a), we will first give a proof of the connectedness of N in the setting of Witten and Yau [16]; i.e., under the assumption that N has a component of positive scalar curvature.

The result of Witten and Yau on the connectedness of the boundary is easily derived from the following proposition.

Proposition 2. *Suppose M^{n+1} is a complete Riemannian manifold-with-boundary having Ricci curvature greater than or equal to $-n$. If the boundary ∂M is compact and has mean curvature $H > n$ then M is compact.*

By our conventions, $H = \mathrm{div}_{\partial M}\, X$, where X is the outward pointing unit normal along ∂M. Results similar to Proposition 2 are obtained in [16] by minimizing the brane action and making use of the machinery of geometric measure theory. Here we give a proof of Proposition 2 using basic techniques in geodesic geometry. These arguments are reminiscent of the kinds of arguments used in the proof of the classical Hawking-Penrose singularity theorems.

Proof of Proposition 2. Suppose M is noncompact. Then we can find a point $q \in M$ such that the distance from q to ∂M is greater than $\coth^{-1}(1 + \delta)$. Here $\delta > 0$ is chosen so that the mean curvature H of ∂M satisfies $H \geq n(1 + \delta)$. Let p be a point on ∂M closest to q, and let $\sigma : [0, \ell] \to M$ be a unit speed minimal geodesic from $p = \sigma(0)$ to $q = \sigma(\ell)$. Let $\rho : M \to \mathbb{R}$ be the distance function to the boundary,

$$\rho(x) = d(x, \partial M) = \inf_{y \in \partial M} d(x, y)\,.$$

In general, ρ is continuous on M and smooth outside the focal cut locus of ∂M. In particular, since σ realizes the distance to ∂M, ρ is smooth on an open set U containing $\sigma \setminus \{q\}$.

For $0 \leq s < \ell$, let $H(s) = -\triangle \rho(\sigma(s)) = \mathrm{div}(-\nabla \rho)(\sigma(s))$. Geometrically, $H(s)$ is the mean curvature of the slice $\rho = s$, with respect to the unit normal $-\nabla \rho$, at the point $\sigma(s)$. $H = H(s)$ obeys the well known traced Riccati equation [11],

$$H' = \mathrm{Ric}(\sigma', \sigma') + |B|^2\,, \tag{1}$$

where $' = d/ds$ and $B(s) = -\mathrm{Hess}(\rho)(\sigma(s)) = \nabla(-\nabla \rho)(\sigma(s))$. Since $H(s)$ is the trace of $B(s)$, the Schwarz inequality implies, $|B|^2 \geq H^2/n$. Equation 1, taken together with this inequality and the inequalities

$\mathrm{Ric}(\sigma', \sigma') \geq -n$ and $H(0) = H_{\partial M}(p) \geq n(1 + \delta)$, implies that $\mathcal{H}(s) := H(s)/n$ satisfies,

$$\mathcal{H}' \geq \mathcal{H}^2 - 1, \qquad \mathcal{H}(0) \geq 1 + \delta.$$

By comparison with the unique solution to: $h' = h^2 - 1$, $h(0) = 1 + \delta$, we obtain,

$$\mathcal{H}(s) \geq \coth(a - s),$$

where $a = \coth^{-1}(1 + \delta) < \ell = d(q, \partial M)$. This inequality implies that $\mathcal{H} = \mathcal{H}(s)$ is unbounded on $[0, a)$, which contradicts the fact that it is smooth on $[0, \ell)$. Thus, M must be compact. $\qquad\square$

We remark that the rigid version of Proposition 2, in which one assumes the weak inequality, $H \geq n$, holds, has previously been treated in the literature, cf., [12, 5].

Return to the setting of Theorem 1, but in the case considered by Witten and Yau [16] in which some component N_0, say, of N has positive scalar curvature. We indicate how the connectedness of N in this case follows from Proposition 2. Let r be a defining function for the conformal boundary N, and let U be a neighborhood of N_0 which does not meet any other components of N. Let $M_t = M \setminus \{x \in U : r(x) < t\}$. For t sufficiently small, M_t is a manifold-with-boundary, with boundary ∂M_t diffeomorphic to N_0, which satisfies the hypotheses of Proposition 2. (That the mean curvature of ∂M_t satisfies $H > n$ uses the assumption of positive scalar curvature on N_0. It also uses the fall-off condition on the Ricci curvature; without that, there are simple counter-examples to the Witten-Yau result.) We conclude that M_t is compact, from which it follows that N has no other components; i.e., $N = N_0$, and hence is connected.

We now describe briefly our approach to the proof of part (a) of Theorem 1. The idea, roughly, is as follows: First we show that if there is a sequence of compact hypersurfaces Σ_k in M going to infinity in some end such that the mean curvature H_k of Σ_k approaches n "fast enough" then M has only one end. Then we show that this rate is realized if the end admits a conformal compactification such that the conformal class of the boundary has a metric of zero scalar curvature. The first, and main, step is to establish a suitable refinement of Proposition 2.

Theorem 3. *Let M^{n+1} be a complete Riemannian manifold having Ricci curvature greater than or equal to $-n$. Fix a base point $o \in M$. Suppose there exists a sequence of compact hypersurfaces $\{\Sigma_k\}$ satisfying the following conditions.*

(a) *Each Σ_k separates M. We call the component of $M \setminus \Sigma_k$ containing o the inside of Σ_k and the other component the outside.*

(b) *$d(o, \Sigma_k) \to \infty$ as $k \to \infty$.*

(c) *Denote by H_k the mean curvature of Σ_k with respect to the outward normal, and let h_k be the smaller of $\min\{H_k(x) : x \in \Sigma_k\}$ and n. Assume that*

$$\lim_{k \to \infty} (n - h_k)e^{2d(o, \Sigma_k)} = 0. \tag{2}$$

Then M has only one or two ends. If M has two ends then M is isometric to $\mathbb{R} \times \Sigma$, with warped product metric $dr^2 + e^{2r}g_0$, where Σ is compact and g_0 is a metric of nonnegative Ricci curvature on Σ.

An *end* of M is, roughly speaking, an unbounded component of the complement of a sufficiently large compact subset of M, cf. e.g., [1] for a precise definition. If M admits a conformal compactification then each of its ends is diffeomorphic to $\mathbb{R} \times \Sigma$, where Σ is a component of the conformal boundary. In the special case of two ends in the theorem, M has a *cusp* at one end, and hence does not admit a conformal compactification. (Regardless of curvature conditions, the ends of a conformally compactified manifold must have positive mean curvature near infinity.)

We recall an example considered in [16] which satisfies all the hypotheses of Theorem 3 except for the mean curvature decay condition (2). Let (Σ, g_0) be any compact negatively curved (Ric $= -(n-1)g$) Einstein manifold of dimension n. Then $M = \mathbb{R} \times \Sigma$, with warped product metric $g = dr^2 + \cosh^2(r)g_0$ is an Einstein manifold satisfying, Ric $= -ng$. The slices $\Sigma_r = \{r\} \times \Sigma$, $r > 0$, have mean curvature $H(r) = n \tanh r$. Hence $H(r) \to n$ as $r \to \infty$, but $\lim_{r \to \infty} e^{2r}(n - H(r)) = 2n$. This shows that the mean curvature condition in Theorem 3 is in some sense optimal.

In the next section we present the proof of Theorem 3, and in the final section we present the proof of Theorem 1.

2   Proof of Theorem 3

The proof of Theorem 3 is similar in spirit to the proof of the Cheeger-Gromoll splitting theorem [4, 6], and makes use of (generalized) Busemann functions and the method of support functions [17, 6]. The method of support functions provides an elementary way to work with the Laplacian of certain geometrically defined functions, such as Busemann functions, which are in general only C^0.

Let M be a Riemannian manifold and let $f \in C^0(M)$ be a continuous function on M. A *lower support* function for f at $p \in M$ is a function ϕ defined and continuous on a neighborhood U of p such that $\phi \leq f$ on U and $\phi(p) = f(p)$. We say that $f \in C^0(M)$ satisfies $\triangle f \geq a$ $(a \in \mathbb{R})$ in the *support sense* provided for each $p \in M$ and every $\epsilon > 0$ there exists a C^2 lower support function $\phi_{p,\epsilon}$ for f at p such that $\triangle \phi_{p,\epsilon}(p) \geq a - \epsilon$. The Hopf-Calabi maximum principle asserts that if M is connected and $f \in C^0(M)$ satisfies $\triangle f \geq 0$ in the support sense then f cannot attain a maximum unless it is constant. The proof of the Hopf-Calabi maximum principle is completely elementary; a short elegant proof is given in [6] (cf., also [3]). We will need to make use of a slightly more general version of the Hopf-Calabi maximum principle.

Definition 1. *A function $f \in C^0(M)$ satisfies $\triangle f \geq a$, $a \in \mathbb{R}$, in the* generalized support sense *provided for each $p \in M$, there is a neighborhood U of p such that the following conditions hold.*

  (a) *There exists a sequence $\{f_k\}$, $f_k \in C^0(U)$, such that $f_k \to f$ uniformly on U.*

  (b) $\triangle f_k \geq a_k$ *on U in the support sense, and $a_k \to a$.*

Lemma 4 (Generalized maximum principle). *Suppose M is a connected Riemannian manifold, and $f \in C^0(M)$ satisfies $\triangle f \geq 0$ in the generalized support sense. Then, if f attains a maximum, it is constant.*

Proof. The proof is a simple modification of the proof of the Hopf-Calabi maximum principle given in [6]. We omit the details. □

One defines $\triangle f \leq a$ in the generalized support sense in a similar way, using C^2 *upper* support functions. By definition, $\triangle f = a$ in the generalized support sense provided $\triangle f \geq a$ and $\triangle f \leq a$ in the generalized support sense. If $\triangle f = a$ in the generalized support sense then $f \in C^\infty(M)$ and $\triangle f = a$ in the usual sense. Indeed, for any small geodesic ball B, basic elliptic theory [9] guarantees that the Dirichlet problem: $\triangle h = a$, $h|_{\partial B} = f|_{\partial B}$, has a solution $h \in C^\infty(B) \cap C^0(\overline{B})$. Then $\triangle(f - h) = 0$ on B in the generalized support sense, and the generalized maximum principle applied to $\pm(f - h)$ implies that $f|_B = h$.

Proof of Theorem 3. Suppose that M has more than one end. Then there is a compact set K such that $M \setminus K$ has at least two unbounded components E_1 and E_2, say. Since $M \setminus K$ has at most finitely many components we may assume without loss of generality that $\Sigma_k \subset E_1$ for all k. We now construct a *line* in M. (Recall, a line is a complete unit speed geodesic, each segment of which realizes the distance between its endpoints.) Let $\{q_k\}$ be a sequence in E_2 going to infinity, $d(o, q_k) \to \infty$. Let p_k be a point on Σ_k closest to q_k, and let $\sigma_k : [-a_k, b_k] \to M$ be a unit speed minimal geodesic from p_k to q_k. Since σ_k meets K, we may parameterize σ_k so that $\sigma_k(0) \in K$. By passing to a subsequence if necessary we have $\sigma_k(0) \to \bar{o} \in M$ and $\sigma'_k(0) \to X \in T_{\bar{o}}M$. Let $\sigma : \mathbb{R} \to M$ be the geodesic satisfying $\sigma(0) = \bar{o}$ and $\sigma'(0) = X$. As σ is the limit of minimal segments, it is a line in M.

We consider two Busemann functions on M, a Busemann function associated with the sets Σ_k (in the sense decribed in [17]) and the standard Busemann function associated with the ray (half-line) $\sigma|_{[0,\infty)}$. For each k, let $\beta_k : M \to \mathbb{R}$ be the function defined by, $\beta_k(x) = d(\bar{o}, \Sigma_k) - d(x, \Sigma_k)$. The triangle inequality implies that β_k is Lipschitz continuous, with Lipschitz constant one, and satisfies $|\beta_k(x)| \leq d(x, \bar{o})$. Hence, the family of functions $\{\beta_k\}$ is equicontinuous and uniformly bounded on compact subsets. Thus, by Ascoli's theorem, and passing to a subsequence if necessary, we have that β_k converges on compact subsets to a continuous function $\beta : M \to \mathbb{R}$, called the Busemann function associated with $\{\Sigma_k\}$. We will ultimately show that $\beta \in C^\infty(M)$

and satisfies $\triangle \beta = n$, from which the special form of (M, g) in the statement of Theorem 3 will readily follow. The first step is to establish the following.

Claim. $\triangle \beta \geq n$ *in the generalized support sense.*

Let p be any point in M and let $B = B(p, r)$ be a small geodesic ball centered at p of radius r. To prove the claim we show that $\triangle \beta_k \geq n_k$ on B in the support sense, where $n_k \to n$. Given $q \in B$ and $\epsilon > 0$, we construct a support function $\beta_k^{q,\epsilon}$ for β_k at q as follows. Let z be a point on Σ_k closest to q, and let $\gamma : [0, \ell] : M \to \mathbb{R}$ be a unit speed minimal geodesic from $z = \gamma(0)$ to $q = \gamma(\ell)$. Let V be a small neighborhood of z in Σ_k. By bending V slightly toward the outside of Σ_k we obtain a smooth hypersurface V' with the following properties: (1) $z \in V'$ is the unique closest point in V' to q, (2) the second fundamental form of V' at z (with respect to the outward normal) is strictly less than that of V, and (3) the mean curvature of V' at z satisfies $H_{V'}(z) \geq H_V(z) - \epsilon \geq h_k - \epsilon$. By construction, γ minimizes the distance from q to V', and there are no focal cut points to V' on γ (in particular, q is not a focal cut point). Hence the function $\beta_k^{q,\epsilon}(x) = d(\bar{o}, \Sigma_k) - d(x, V')$ is a lower support function for β_k at q which is smooth on a neighborhood of γ.

For $0 \leq s \leq \ell$, let $H(s) = \triangle \beta_k^{q,\epsilon}(\gamma(s))$. Arguing as in Proposition 2, $\mathcal{H}(s) = H(s)/n$ satisfies, $\mathcal{H}' \geq \mathcal{H}^2 - 1$, $\mathcal{H}(0) \geq (h_k - \epsilon)/n$. Since $(h_k - \epsilon)/n < 1$, by comparing with the unique solution to $h' = h^2 - 1$, $h(0) = (h_k - \epsilon)/n$, we obtain, $\mathcal{H}(s) \geq \tanh(a - s)$, where $a = \tanh^{-1}((h_k - \epsilon)/n) = \frac{1}{2} \ln(\frac{n+h_k-\epsilon}{n-h_k+\epsilon})$. Setting $s = \ell$ in this inequality, we obtain,

$$\triangle \beta_k^{q,\epsilon}(q) = H(\ell) \geq n \tanh(a - \ell) = n \frac{e^{2a} - e^{2\ell}}{e^{2a} + e^{2\ell}} \tag{3}$$

$$= n \frac{(n + h_k - \epsilon) - (n - h_k + \epsilon)e^{2\ell}}{(n + h_k - \epsilon) + (n - h_k + \epsilon)e^{2\ell}}.$$

Now, by the triangle inequality, $\ell = d(q, \Sigma_k) \leq r + d(o, p) + d(o, \Sigma_k)$, and hence $e^{2\ell} \leq Ce^{2d(o,\Sigma_k)}$, where $C = e^{2(r+d(o,p))}$. Making use of this latter inequality in (3) we conclude that $\triangle \beta_k \geq n_k$ on B in the support sense, where

$$n_k = n \frac{(n + h_k) - C(n - h_k)e^{2d(o,\Sigma_k)}}{(n + h_k) + C(n - h_k)e^{2d(o,\Sigma_k)}}. \tag{4}$$

Invoking the mean curvature condition (2), we see that $n_k \to n$. This yields the claim.

We now consider the standard Busemann function associated to the ray $\sigma|_{[0,\infty)}$. For each $s > 0$, define the function $b_s : M \to \mathbb{R}$ by, $b_s(x) = d(\bar{o}, \sigma(s)) - d(x, \sigma(s)) = s - d(x, \sigma(s))$. For each $x \in M$, $b_s(x)$ is increasing in s and bounded by $d(\bar{o}, x)$. The Busemann function $b : M \to \mathbb{R}$ of $\sigma|_{[0,\infty)}$ is defined to be the limit function, $b(x) = \lim_{s \to \infty} b_s(x)$. Because the family $\{b_s\}$ is equicontinuous, b is continuous.

In the present situation in which the Ricci curvature is greater than or equal to $-n$, it is known that $\triangle b \geq -n$ in the generalized support sense (in fact, in the support sense, cf. [5]). As the arguments involved to show this are similar to (but simpler than) the arguments used in the proof of the claim, we make only a few brief comments. The relevant support functions $b_s^{q,\epsilon}$ for b_s are defined as follows. Let $\gamma : [0, \ell] \to M$ be a unit speed minimal geodesic from q to $\sigma(s)$. The function $b_s^{q,\epsilon}$ defined by, $b_s^{q,\epsilon}(x) = s - (\epsilon + d(x, \gamma(\ell - \epsilon)))$ is a lower support function for b_s at q which is smooth near q. By standard comparison techniques [11] like those used above, $b_s^{q,\epsilon}$ satisfies $\triangle b_s^{q,\epsilon}(q) \geq -n \coth(\ell - \epsilon)$. From this it easily follows that $\triangle b \geq -n$ in the generalized support sense.

To summarize, we have shown that the Busemann functions β and b satisfy $\triangle \beta \geq n$ and $\triangle b \geq -n$ in the generalized support sense. Hence the sum $f = \beta + b$ satisfies $\triangle f \geq 0$ in the generalized support sense. Moreover, f satisfies, $f \leq 0$ on M. Indeed, we have,

$$
\begin{aligned}
\beta(x) + b_s(x) &= \lim_{k \to \infty} [d(\sigma(0), \Sigma_k) - d(x, \Sigma_k)] + s - d(x, \sigma(s)) \qquad (5)\\
&= \lim_{k \to \infty} [d(\sigma_k(0), \Sigma_k) - d(x, \Sigma_k)] + \lim_{k \to \infty} [s - d(x, \sigma_k(s))]\\
&= \lim_{k \to \infty} [d(\sigma_k(0), \Sigma_k) + s - (d(\sigma_k(s), x) + d(x, \Sigma_k))] .
\end{aligned}
$$

By the triangle inequality,

$$
d(\sigma_k(s), x) + d(x, \Sigma_k) \geq d(\sigma_k(s), \Sigma_k) = d(\sigma_k(0), \Sigma_k) + s . \qquad (6)
$$

The inequalities (5) and (6) imply $\beta(x) + b_s(x) \leq 0$. Letting $s \to \infty$, we obtain $f = \beta + b \leq 0$. But note that $f(\bar{o}) = \beta(\bar{o}) + b(\bar{o}) = \lim_{k \to \infty} \beta_k(\bar{o}) + \lim_{s \to \infty} b_s(\bar{o}) = 0 + 0 = 0$. Thus, by the generalized maximum principle $f \equiv 0$. Hence, $\beta = -b$ and so satisfies $\triangle \beta \leq n$ in the generalized

support sense. Since $\triangle\beta \geq n$ in the generalized support sense, as well, we conclude from the discussion after Lemma 4 that β is smooth and satisfies $\triangle\beta = n$ in the usual sense.

It is known that Busemann functions, where differentiable, have unit gradient, and hence $|\nabla\beta| = 1$ everywhere. (Briefly, this follows from the fact that β satisfies, $|\beta(q) - \beta(p)| \leq d(p,q)$, with equality holding when p and q are on an *asymptotic ray*, cf. Lemma 6 in [17].) This has as well known consequences the fact that $\nabla\beta$ is a geodesic vector field, i.e., its integral curves are unit speed geodesics, and that β satisfies ([4, 6]),

$$\nabla\beta(\triangle\beta) = \text{Ric}(\nabla\beta, \nabla\beta) + |\text{Hess}\,\beta|^2 \,. \tag{7}$$

(Compare with Equation 1.) Since the left hand side vanishes, we have $|\text{Hess}\,\beta|^2 = -\text{Ric}(\nabla\beta, \nabla\beta) \leq n$. But from the Schwarz inequality, $|\text{Hess}\,\beta|^2 \geq |\triangle\beta|^2/n = n$. Hence, equality holds, which implies,

$$\text{Hess}\,\beta|_{\nabla\beta^\perp} = g|_{\nabla\beta^\perp} \quad \text{and} \quad \text{Ric}(\nabla\beta, \nabla\beta) = -n \,. \tag{8}$$

Exponentiating out from the slice $\Sigma = \beta^{-1}(0)$ along its normal geodesics ($=$ integral curves of β) establishes a global diffeomorphism $M \approx \mathbb{R} \times \Sigma$, with respect to which g takes the form

$$g = dr^2 + g_{ij}(r,x)dx^i dx^j \,, \tag{9}$$

where $\partial/\partial r = \nabla\beta$ and $g_r = g_{ij}(r,x)dx^i dx^j$ is the induced metric on the slice $\Sigma^r = \beta^{-1}(r) \approx \{r\} \times \Sigma$. Along Σ^r, $\text{Hess}\,\beta(\partial_i, \partial_j) = \frac{1}{2}\partial_r g_{ij} = g_{ij}$ (by the first equation in (8)), which gives,

$$g = dr^2 + e^{2r}g_{ij}(0,x)dx^i dx^j \,, \tag{10}$$

as required. The second equation in (8) and a calculation show that $g_0 = g_{ij}(0,x)dx^i dx^j$ is a metric of nonnegative Ricci curvature. Finally, Σ is compact, otherwise $M \approx \mathbb{R} \times \Sigma$ has only one end, contrary to assumption. This concludes the proof of Theorem 3. $\qquad\square$

3   Proof of Theorem 1

Let N_0 be the component in the statement of the theorem which admits in its conformal class a metric h of nonnegative scalar curvature. As

discussed in the introduction, we may assume, in fact, that h is a metric of zero scalar curvature. Then there is a defining function r such that near N_0, M has the form, $M = [0, r_0) \times N_0$, with metric g of the form,

$$g = \frac{1}{r^2}\bar{g} = \frac{1}{r^2}\left(\frac{1}{\bar{g}(\bar{\nabla}r, \bar{\nabla}r)}dr^2 + g_r\right),$$

where g_r is the metric induced on $N_r = \{r\} \times N_0$ from $\bar{g}$, such that $g_0 = h$. Assume that (M, g) is Einstein, with Ric $= -ng$, or more generally that (M, g) satisfies, Ric $\geq -ng$ such that, as part of our fall-off assumption, the scalar curvature S of (M, g) satisfies,

$$S \to -n(n+1) \quad \text{as} \ r \to 0.$$

As a computation shows, this implies that $\bar{g}(\bar{\nabla}r, \bar{\nabla}r) = 1$ along N_0. Then as described in [10] (cf., Lemma 2.1) the defining function r can be chosen uniquely in a neighborhood of N_0 so that $\bar{g}(\bar{\nabla}r, \bar{\nabla}r) = 1$ in this neighborhood. Thus, we may assume that in $[0, r_0) \times N_0$, g has the form,

$$g = \frac{1}{r^2}(dr^2 + g_r). \tag{11}$$

With respect to this distinguished defining function we impose the following fall-off requirement,

$$r^{-2}(\text{Ric} + ng) \to 0 \ \text{uniformly as} \ r \to 0. \tag{12}$$

This condition is compatible with the fall-off condition considered in [2]. Let us also emphasize that these fall-off conditions are automatically satisfied when M is Einstein with Ric $= -ng$.

The Gauss equation in (M, g) implies,

$$H^2 = \hat{S} - S + \text{Ric}(X, X) + |B|^2, \tag{13}$$

where, for each r, $\hat{S}$, B and H are, respectively, the scalar curvature (in the induced metric), second fundamental form and mean curvature of N_r, and $X = -r\partial/\partial r$ is the outward unit normal to N_r. For each r, let $\bar{S}$ denote the scalar curvature of N_r in the metric g_r; $\bar{S}$ and $\hat{S}$ are related by $\hat{S} = r^2\bar{S}$. Using this and the inequality $|B|^2 \geq H^2/n$, (13) implies the inequality,

$$n^2 - H^2 \leq -\frac{n}{n-1}(r^2\bar{S} + \kappa), \tag{14}$$

where $\kappa = 2\operatorname{Ric}(X, X) - S - n(n-1)$. (Note that in arriving at (14) we have used $n \neq 1$. In fact there are simple counterexamples to Theorem 1 if one allows $n = 1$.) It follows from (12) that $r^{-2}\kappa \to 0$ uniformly as $r \to 0$. Since $H > 0$ for r sufficiently small, we have $n^2 - H^2 \geq n(n-H)$ at points where $H \leq n$. This, together with (14) implies,

$$r^{-2}(n - H) \leq -\frac{1}{n-1}(\bar{S} + r^{-2}\kappa) \quad \text{where } H \leq n. \tag{15}$$

Pick a sequence $r_k \to 0$, and set $\hat{N}_k = N_{r_k}$. Given $o \in \hat{N}_{k_0}$, we have,

$$e^{d(o, \hat{N}_k)} = e^{\int_{r_k}^{r_{k_0}} \frac{1}{r} dr} = r_{k_0} r_k^{-1}. \tag{16}$$

Let $(r_k, x_k) \in \hat{N}_k$ be a point where the mean curvature of $\hat{N}_k$ achieves a minimum. We may assume this minimum mean curvature $h_k = H(r_k, x_k)$ is less than or equal to n, otherwise by the Witten-Yau result we are done. By passing to a subsequence we may further assume $(r_k, x_k) \to (0, x_0) \in N_0$. Then, setting $(r, x) = (r_k, x_k)$ in (15) we obtain,

$$e^{2d(o, \hat{N}_k)}(n - h_k) \leq -\frac{r_{k_0}^2}{n-1}(\bar{S}(r_k, x_k) + r_k^{-2}\kappa(r_k, x_k)). \tag{17}$$

Since, as $k \to \infty$, $\bar{S}(r_k, x_k) \to \bar{S}(0, x_0) = 0$ and $r_k^{-2}\kappa(r_k, x_k) \to 0$, we have that condition (2) in Theorem 3 is satisfied. Then by Theorem 3 and remarks in the paragraph following its statement, M has only one end and hence $N = N_0$, i.e., N is connected.

This concludes the proof of part (a). Part (c) follows from part (a), as indicated in the proof of Theorem 3.3 in [16]. One passes to the covering space $\overline{M}'$ of $\overline{M}$ associated with the subgroup $i_*(\Pi_1(N))$ of $\Pi_1(\overline{M})$. The boundary $\partial \overline{M}'$ contains a copy of N and has more than one component if $i_* : \Pi_1(N) \to \Pi_1(\overline{M})$ is not onto, contradicting part (a) applied to $\overline{M}'$. (Although a priori $\partial \overline{M}'$ need not be compact, note that the proof of part (a) only requires that the assumed component of nonnegative scalar curvature be compact.) Part (b) follows from part (c) by some basic homology theory, as we now describe. Similar arguments have been used in [7, 8] where related results in the spacetime setting have been obtained.

To prove part (c) consider the relative homology sequence for the pair $\overline{M} \supset N$ (all homology is over $\mathbb{Z}$),

$$\cdots \ \to H_1(N) \xrightarrow{\alpha} H_1(\overline{M}) \xrightarrow{\beta} H_1(\overline{M}, N) \xrightarrow{\partial} \tilde{H}_0(N) = 0 \quad . \qquad (18)$$

(Here $\tilde{H}_0(N)$ is the reduced zeroth dimensional homology group.) To make use of part (c), we use the fact that the first integral homology of a space is isomorphic to the fundamental group modded out by its commutator subgroup. Hence, modding out by the commutator subgroups of $\Pi_1(N)$ and $\Pi_1(\overline{M})$, we obtain a surjective linear map from $H_1(N)$ to $H_1(\overline{M})$, i.e., α in (18) is onto. Since α is onto, $\ker \beta = \operatorname{im} \alpha = H_1(\overline{M})$ which implies $\beta \equiv 0$. Hence $\ker \partial = \operatorname{im} \beta = 0$, and thus ∂ is injective. This implies that $H_1(\overline{M}, N) = 0$. But by Poincaré duality for manifolds-with-boundary, $H_1(\overline{M}, N) \cong H^n(\overline{M}) \cong H_n(\overline{M})$, where for the second isomorphism we have used the fact that $H_n(\overline{M})$ is free (cf., [14]). We conclude that $H_n(\overline{M}, \mathbb{Z}) = 0$.

If M is nonorientable (which, by part (a) and a covering space argument, can happen if and only if N is nonorientable), essentially the same argument shows $H_n(\overline{M}, \mathbb{Z}/2)$ vanishes.

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