Descendant Gromov-Witten Invariants,

Simple Hurwitz Numbers,

and the Virasoro Conjecture

for $\mathbb{P}^1$

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Abstract

In this "experimental" research, we use known topological recursion relations in genera-zero, -one, and -two to compute the n-point descendant Gromov-Witten invariants of $\mathbb{P}^1$

for arbitrary degrees and low values of n. The results are consistent with the Virasoro conjecture and also lead to explicit computations of all Hodge integrals in these genera. We also derive new recursion relations for simple Hurwitz numbers similar to those of Graber and Pandharipande.

1   Introduction

It is well-known that the intersection theory on the compactified moduli space $\overline{\mathcal{M}}_{g,n}$ of stable n-pointed genus-g curves, or equivalently the two-dimensional pure gravity, is governed by an integrable KdV hierarchy [26, 38]. More precisely, the KdV hierarchy allows one to compute recursively the intersection numbers of tautological divisors on $\overline{\mathcal{M}}_{g,n}$, for arbitrary g and n, in terms of two basic invariants in genus-0 and genus-1. Physically, it means the following: It is a common and useful practice to perturb a given quantum field theory by introducing into the action couplings to physical operators and to study the perturbed partition function which becomes the generating function for the correlators of the original theory. For example, in a topological string theory on a target space V, the physical operators are the cohomology classes γ_a of V and their gravitational descendants $\tau_{m,a}, m \in \mathbb{Z}_{\geq 1}$. In this case, one considers a perturbation by

$$\sum_{m=0}^{\infty} \sum_{\gamma_a \in H^*(V)} t_m^a \, \tau_{m,a} \tag{1.1}$$

where $\tau_{0,a}$ represents the *primary field* associated with the cohomology class γ_a itself. The parameters t_m^a are said to form the coordinates on the so-called *large phase space*[2]. In this setting, the KdV structure implies that the partition function of the perturbed topological string theory on a point target space, in which the puncture or identity operator is the only primary field, is a τ-function of the hierarchy and the τ-function is uniquely fixed by the string equation.

It was soon realized that the statement of the integrable structure can be rephrased in terms of certain differential operators on the large

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[2]Similarly, the *small phase space* refers to a space of deformations by only the primary fields; i.e. the subspace $t_m^a = 0, m > 0$, of the large phase space.

phase space which annihilate the τ-function. It turns out that these operators furnish a representation of a subsector of the Virasoro algebra, and thus the KdV hierarchy of the intersection theory is also known as the Virasoro constraints [6].

One immediate generalization of the above picture is to introduce more primary fields by coupling the two-dimensional topological gravity to topological field theories. For example, coupling the topological minimal models to topological gravity leads to $d < 1$ topological string theories which are governed by W-algebra constraints, generalizing the Virasoro algebra. A more interesting and perhaps more physical way is to consider topological string theories on more general non-trivial target spaces. This approach has led to physical means of studying the Gromov-Witten (GW) invariants, of which the subject of quantum cohomology is a subset, on Fano and Calabi-Yau manifolds. Such physical models describe the intersection theory, sometimes called the gravitational quantum cohomology, on some suitably defined moduli spaces of stable holomorphic maps from Riemann surfaces to target spaces [3, 4, 27, 31]. Based on the previous examples, should one expect some kind of an integrable structure to govern the intersection theory in these cases as well?

Through a series of papers [9, 10, 11, 12, 13, 24], it has been indeed conjectured that there should exist a certain integrable hierarchy which underlies the gravitational quantum cohomology and which manifests itself again in terms of a set of differential operators forming a half branch of the Virasoro algebra:

$$[L_n, L_m] = (n - m)L_{n+m}, \quad n, m \geq -1 . \tag{1.2}$$

This conjecture is now referred to as the Virasoro conjecture and has been proven up to genus-1 by mathematicians for manifolds satisfying certain conditions [7, 32, 33]; in particular, for complex projective spaces $\mathbb{P}^n$. Historically, this conjecture is based on the discovery of a matrix model for the topological string theory on $\mathbb{P}^1$ [9, 10, 24]; the Ward identities for the matrix model form a Virasoro algebra. The authors of [10] have checked for a few cases that the intersection numbers on the moduli space of stable maps indeed satisfy the constraints implied by their conjecture. Despite some curious matchings, there is yet no complete proof[3] of their conjecture even for $\mathbb{P}^1$.

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[3]That is, except for a point and Calabi-Yau varieties of dimension greater than

In this paper, we take a retrograding step towards attempting to unravel the mystery of the Virasoro constraints for $\mathbb{P}^1$. At first sight, this example appears to be the simplest generalization of the pure gravity case, but it turns out that the only interesting GW invariants for $\mathbb{P}^1$ are the gravitational descendants. The reason is that the only primary fields of the theory are the identity and the second cohomology class which can be eliminated via the puncture and divisor equations [24, 38]. We thus cannot obtain any nontrivial information by restricting our attention to the small phase space, as was done in [12, 13], and we would need to consider various descendant GW-invariants to study whether there exist any possible constraints on the theory. Important ingredients in our computations of the GW-invariants are the known topological recursion relations (TRRs) in genera-zero, -one and -two.

Incidentally, the relation between the TRRs and the Virasoro constraints are not clear, even in the present case of $\mathbb{P}^1$. In the pure gravity case, the Virasoro constraints, or the KdV hierarchy, completely determine all the correlators in all genera in terms of $\langle \tau_{0,0}\tau_{0,0}\tau_{0,0}\rangle_0$ and $\langle \tau_{1,0}\rangle_1$, and there is no need for additional TRRs; the TRRs are thus redundant for a point target space. For higher dimensional target spaces, however, the Virasoro constraints by themselves are not powerful enough to determine all the correlators and require the help of additional TRRs. In fact, it is not known whether the Virasoro constraints, even if they were true, together with various TRRs, would be able to determine all the correlators for a non-trivial target space. Interestingly, for $\mathbb{P}^1$, the positive modes of the Virasoro constraints are not needed to compute all the descendant GW-invariants up to genus-2. That is, the TRRs of [11, 17, 38], together with the L_{-1} and L_0 constraints[4], are enough to compute all the correlators in genus-zero, -one, and -two. At least in these low genera, it thus seems that the Virasoro constraints for $\mathbb{P}^1$ are redundant, and indeed, we have checked for many of the GW-invariants which we have obtained via TRRs that they actually do satisfy the constraints.

Before we proceed, it is perhaps necessary to clarify the nature of our work, so as to align the reader's line of thinking with our own.

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or equal to three.

[4]These two constraints are proven to hold for all manifolds. The L_{-1} constraint is the string equation of Witten [38], and L_0 the equation of Hori [24] combining the dilaton, divisor and dimension equations.

The philosophy of this paper is not to prove any parts of the Virasoro conjecture. Instead, we admittedly take an un-innovative approach to computing the descendant GW-invariants of $\mathbb{P}^1$ by using the available topological recursion relations, and the numbers that we thus obtain are independent of the Virasoro conjecture. Since a genus-g recursion relation involves lower genus contributions, a mistake in genera-zero and -one would propagate through any subsequent computations in higher genera. We therefore check many[5] of our results by verifying that they satisfy the Virasoro constraints in genera-zero and -one, which are rigorously proven to hold [7, 32, 33]. Based on those numbers, we are able to compute the genus-2 GW-invariants containing up to three arbitrary descendant fields, and we again check that they satisfy many of the genus-2 Virasoro constraints. As the Virasoro constraints in genus-2 are conjectural, our verification provides a minor support for the claim.

This paper is organized as follows: We first compute the descendant GW-invariants and the Hodge integrals in genera-zero, -one, and -two just by using known topological recursion relations. In §3, we use these results to check the Virasoro conjecture by explicitly checking that the correlators satisfy the constraints. We also comment on the higher-genus cases and on the TRRs of Eguchi and Xiong [13]. In §4, which is independent of other sections, we derive new recursion relations for simple Hurwitz numbers in genera-zero and -one by using TRRs as well as by applying the Virasoro constraints discussed in the previous sections. The paper concludes with speculations and open questions regarding the relation between the TRRs and the Virasoro constraints.

Notations

Many different notations are being used by mathematicians and physicists. Here, we clarify the conventions that we use:

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[5]A computer program has allowed us to check that over 10,000 Virasoro constraints are satisfied.

$\tau_{m,\alpha}$ the m-th descendant of a primary field
$\gamma_\alpha \in H^{2\alpha}(\mathbb{P}^1, \mathbb{C})$. See also (2.3).

$\tau_{0,0}$ the identity element in $H^*(\mathbb{P}^1, \mathbb{C})$.

$\tau_{0,1}$ the basis of $H^2(\mathbb{P}^1, \mathbb{Z})$.

t_m^α the coordinate associated with $\tau_{m,\alpha}$
on the large phase space.

$\langle\!\langle\ \rangle\!\rangle_g$ a genus-g correlator in the large phase space.

$\langle\ \rangle_g$ a genus-g correlator at the origin
of the large phase space, i.e. $t_m^a = 0, \forall a, m$.

$\langle\ \rangle_{g,d}$ a degree-d Gromov-Witten invariant in genus-g
at $t_m^a = 0, \forall a, m$.

c_m $\sum_{k=1}^m 1/k$.

- We call m the *degree*[6] of the descendant $\tau_{m,\alpha}$.

- Following the physics nomenclature, we sometimes call the n-point descendant invariant $\langle \tau_{m_1,\alpha_1} \cdots \tau_{m_n,\alpha_n} \rangle_g$ an n-point correlation function, or simply an n-point functions.

- Technically, an n-point GW-invariant $\langle \tau_{m_1,\alpha_1} \cdots \tau_{m_n,\alpha_n} \rangle_g$ is a sum of Gromov-Witten invariants $\langle \tau_{m_1,\alpha_1} \cdots \tau_{m_n,\alpha_n} \rangle_{g,d}$ in various degrees with coefficients in the Novikov ring of $\mathbb{P}^1$, but since each correlator receives a contribution from only one specific degree, we will often use $\langle \tau_{n_1,\alpha_1} \cdots \tau_{n_k,\alpha_k} \rangle_g$ and $\langle \tau_{n_1,\alpha_1} \cdots \tau_{n_k,\alpha_k} \rangle_{g,d}$ interchangeably. The degree d of the non-vanishing GW-invariant can be worked out easily from the dimension of the virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)]^{\mathrm{vir}}$.

- Many terminologies from algebraic geometry are used in this paper without explanation, and we refer the reader to the available references for their definitions [5, 18, 23].

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[6]This degree should not be confused with the degree of a stable map in $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$.

2   Computations of the Descendant GW-Invariants

In this section, we compute the descendant GW-invariants of $\mathbb{P}^1$ in genera-zero, -one and -two by using the topological recursion relations of Witten [38], Eguchi-Hori-Xiong [11] and Getzler [17]. In general, a TRR in genus-g involves lower genus GW-invariants; as a result, the computational usefulness of a TRR depends on the knowledge of the lower genus invariants. We thus proceed systematically from descendant GW-invariants in genus-0 to those in genus-2.

2.1   Properties of the Descendant GW-Invariants

The descendant GW-invariants, also known as the gravitational correlators, satisfy certain topological axioms which will be used throughout this paper. In this subsection, we briefly review these properties and refer the reader to [5] for details.

Let $\overline{\mathcal{M}}_{g,n}(V,\beta)$ be the compactified moduli space of stable holomorphic maps $f : \Sigma \to V$ of genus-g n-pointed Riemann surfaces Σ to a smooth projective variety V, such that $f_*[\Sigma] = \beta \in H_2(V,\mathbb{Z})$. Let $\pi : \overline{\mathcal{M}}_{g,n+1}(V,\beta) \to \overline{\mathcal{M}}_{g,n}(V,\beta)$ be the universal curve with n natural sections

$$\sigma_i : \overline{\mathcal{M}}_{g,n}(V,\beta) \longrightarrow \overline{\mathcal{M}}_{g,n+1}(V,\beta) \tag{2.1}$$

associated with the n marked points. The tautological line bundle $\mathcal{L}_i \to \overline{\mathcal{M}}_{g,n}(V,\beta)$ is defined to be $\sigma_i^*\omega$, where ω is the relative dualizing sheaf of π, and we denote its first Chern class, called the tautological ψ-class, by ψ_i.

Let $\mathrm{ev} : \overline{\mathcal{M}}_{g,n}(V,\beta) \to V^n$ be the evaluation map defined by

$$\mathrm{ev} : [f : (\Sigma, z_1, \ldots, z_n) \to V] \in \overline{\mathcal{M}}_{g,n}(V,\beta)$$
$$\longmapsto (f(z_1), \ldots, f(z_n)) \in V^n. \tag{2.2}$$

Then, the descendant GW-invariant is defined to be

$$\left\langle \tau_{m_1,\alpha_1} \cdots \tau_{m_n,\alpha_n} \right\rangle_{g,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{vir}}} \psi_1^{m_1} \cdots \psi_n^{m_n} \cup \mathrm{ev}^*(\gamma_{\alpha_1} \otimes \cdots \otimes \gamma_{\alpha_n}). \tag{2.3}$$

These invariants satisfy certain "axioms" which are generalizations of those occurring in the pure gravity case, i.e. in the case of a point target space. Specializing to the case of $\mathbb{P}^1$, they are:

- **Degree Axiom.** The GW-invariant $\langle \tau_{m_1,\alpha_1} \cdots \tau_{m_n,\alpha_n} \rangle_{g,d}$ vanishes if

$$2d + 2g - 2 + n \neq \sum_i^n (m_i + \alpha_i), \tag{2.4}$$

  where $d \in \mathbb{Z}_{\geq 0}$ is the degree of the map, i.e. $f_*[\Sigma] = d\beta$ where β generates the effective cycles of $H_2(\mathbb{P}^1, \mathbb{Z})$.

- **String Axiom.** For either $n + 2g \geq 3$ or $d > 1, n \geq 1$,

$$\langle \tau_{0,0} \tau_{m_1,\alpha_1} \cdots \tau_{m_n,\alpha_n} \rangle_{g,d} \tag{2.5}$$

$$= \sum_{i=1}^n \langle \tau_{m_1,\alpha_1} \cdots \tau_{m_{i-1},\alpha_{i-1}} \tau_{m_i-1,\alpha_i} \tau_{m_{i+1},\alpha_{i+1}} \cdots \tau_{m_n,\alpha_n} \rangle_{g,d}.$$

- **Divisor Axiom.** For either $n + 2g \geq 3$ or $d > 1, n \geq 1$,

$$\langle \tau_{0,1} \tau_{m_1,\alpha_1} \cdots \tau_{m_n,\alpha_n} \rangle_{g,d} \tag{2.6}$$

$$= \sum_i^n \langle \tau_{m_1,\alpha_1} \cdots \tau_{m_{i-1},\alpha_{i-1}} \tau_{m_i-1,1+\alpha_i} \tau_{m_{i+1},\alpha_{i+1}} \cdots \tau_{m_n,\alpha_n} \rangle_{g,d}$$

$$+ d \langle \tau_{m_1,\alpha_1} \cdots \tau_{m_n,\alpha_n} \rangle_{g,d}.$$

- **Dilaton Axiom.** For either $n + 2g \geq 3$ or $d > 1, n \geq 1$,

$$\langle \tau_{1,0} \tau_{m_1,\alpha_1} \cdots \tau_{m_n,\alpha_n} \rangle_{g,d} = (2g - 2 + n) \langle \tau_{m_1,\alpha_1} \cdots \tau_{m_n,\alpha_n} \rangle_{g,d}. \tag{2.7}$$

In degree zero, there are the following exceptional cases:

$$\langle \tau_{0,0} \tau_{0,a} \tau_{0,b} \rangle_{0,0} = \int_{\mathbb{P}^1} \gamma_a \cup \gamma_b$$

$$\langle \tau_{0,1} \rangle_{1,0} = -\frac{1}{24}$$

$$\langle \tau_{1,0} \rangle_{1,0} = \frac{1}{12}$$

The dimension of the virtual fundamental class is

$$\mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)) = 2(g - 1) + 2d + n .$$

Thus, from the degree axiom, we see that the non-vanishing GW-invariants are of the form

$$\left\langle \tau_{n_1,0} \cdots \tau_{n_k,0} \, \tau_{m_1,1} \cdots \tau_{m_\ell,1} \right\rangle_{g,d}$$

where

$$2(g-1) + 2d + k = \sum_{i=1}^{k} n_i + \sum_{i=1}^{\ell} m_i \,.$$

In particular, the only non-vanishing GW-invariants that do not contain the tautological ψ-classes are:

$$\left\langle \tau_{0,1}\tau_{0,1} \cdots \tau_{0,1} \right\rangle_{0,1} = 1 \quad \text{and} \quad \left\langle \tau_{0,0}\tau_{0,0}\tau_{0,1} \right\rangle_{0,0} = 1$$

in genus-0 and

$$\left\langle \tau_{0,1} \right\rangle_{1,0} = -\frac{1}{24}$$

in genus-1. All other non-vanishing GW-invariants thus contain the tautological ψ-classes, and we call them descendant GW-invariants.

2.2   Genus-Zero

Topological recursion relations (TRRs) generally follow from the equality of the tautological classes with boundary classes on the moduli space $\mathcal{M}_{g,n}$, and they are used to reduce the degree of the descendants inside correlators. The TRRs for the generating functions of the GW-invariants of $\mathbb{P}^1$ are given by

$$
\begin{aligned}
\left\langle\!\left\langle \tau_{m_1,\alpha_1}\tau_{m_2,\alpha_2}\tau_{m_3,\alpha_3} \right\rangle\!\right\rangle_0 &= \left\langle\!\left\langle \tau_{m_1-1,\alpha_1}\tau_{0,0} \right\rangle\!\right\rangle_0 \left\langle\!\left\langle \tau_{0,1}\tau_{m_2,\alpha_2}\tau_{m_3,\alpha_3} \right\rangle\!\right\rangle_0 \\
&\quad + \left\langle\!\left\langle \tau_{m_1-1,\alpha_1}\tau_{0,1} \right\rangle\!\right\rangle_0 \left\langle\!\left\langle \tau_{0,0}\tau_{m_2,\alpha_2}\tau_{m_3,\alpha_3} \right\rangle\!\right\rangle_0. \quad (2.8)
\end{aligned}
$$

We can compute the n-point descendant GW-invariants by using this relation and other topological "axioms;" the numbers that we compute are thus independent of the Virasoro conjecture. We will later use these information to test numerically the Virasoro conjecture in genus-2.

We start with the two-point functions $\left\langle \tau_{m_1,\alpha_1}\tau_{m_2,\alpha_2} \right\rangle_0$ by noticing that there are two ways of reducing the invariants $\left\langle \tau_{m_1,\alpha_1}\tau_{m_2,\alpha_2}\tau_{0,0} \right\rangle_0$. That is, we can either use the genus-0 TRR (2.8) to reduce the degree

of τ_{m_1,α_1} or use the string equation to reduce the invariant into a sum of two-point functions:

$$\langle \tau_{m_1-1,\alpha_1} \tau_{m_2,\alpha_2} \rangle_0 + \langle \tau_{m_1,\alpha_1} \tau_{m_2-1,\alpha_2} \rangle_0 \qquad (2.9)$$
$$= \langle \tau_{m_1-1,\alpha_1} \tau_{0,0} \rangle_0 \langle \tau_{0,1} \tau_{m_2,\alpha_2} \rangle_0 + \langle \tau_{m_1-1,\alpha_1} \tau_{0,1} \rangle_0 \langle \tau_{0,0} \tau_{m_2,\alpha_2} \rangle_0$$

The two-point functions of the form $\langle \tau_{m,\alpha} \tau_{0,\beta} \rangle_0$ can be computed by using the TRR of Eguchi-Hori-Xiong (B.1) and are given by (B.4). Now, (2.9) gives us a set of recursive relations among the two-point functions. For example, define

$$X(m) = \langle \tau_{2m-1,0} \tau_{2d-2m+1,0} \rangle_{0,d} \quad \text{and} \quad Y(m) = \langle \tau_{2m,0} \tau_{2d-2m,0} \rangle_{0,d} .$$
$$(2.10)$$

Then, we obtain the relations

$$Y(m) = -X(m) + A(m,d) \qquad \text{and} \qquad (2.11)$$
$$X(m) = -Y(m-1) + A(d-m+1,d) ,$$

where

$$A(m,d) = \frac{(-2c_{m-1} - 1/m)(-2c_{d-m})}{m(m-1)!^2 (d-m)!^2} \qquad (2.12)$$

and

$$c_m = \sum_{k=1}^{m} \frac{1}{k} . \qquad (2.13)$$

We can solve for $X(m)$ and $Y(m)$ recursively and obtain

$$\langle \tau_{2m-1,0} \tau_{2d-2m+1,0} \rangle_{0,d} = X(m) = 2\frac{c_d}{d!^2} + \sum_{k=2}^{m} \widetilde{\Delta_1}(k,d) ,$$

$$\langle \tau_{2m,0} \tau_{2d-2m,0} \rangle_{0,d} = Y(m) = -2\frac{c_d}{d!^2} + \sum_{k=1}^{m} \Delta_1(k,d) , \qquad (2.14)$$

where

$$\Delta_1(k,d) = A(k,d) - A(d-k+1,d) ,$$
$$\widetilde{\Delta_1}(k,d) = A(d-k+1,d) - A(k-1,d) , \qquad (2.15)$$

and the summation is set to zero whenever the lower limit exceeds the upper limit. Other two-point functions are similarly determined, and

we summarize the results in Appendix B. The one-point descendants can be obtained from the two-point functions by using the string equation.

The genus-0 TRR for the three-point GW-invariants are

$$\langle \tau_{m_1,\alpha_1} \tau_{m_2,\alpha_2} \tau_{m_3,\alpha_3} \rangle_{0,d} \qquad (2.16)$$

$$= \begin{cases} \langle \tau_{m_1-1,\alpha_1} \tau_{0,0} \rangle_{0,d'} \langle \tau_{0,1} \tau_{m_2,\alpha_2} \tau_{m_3,\alpha_3} \rangle_{0,d''} & \text{if } m_1 + \alpha_1 \text{ is odd,} \\ \langle \tau_{m_1-1,\alpha_1} \tau_{0,1} \rangle_{0,d'+\frac{1}{2}} \langle \tau_{0,0} \tau_{m_2,\alpha_2} \tau_{m_3,\alpha_3} \rangle_{0,d''-\frac{1}{2}} & \text{if } m_1 + \alpha_1 \text{ is even.} \end{cases}$$

where the degree d of the holomorphic maps must satisfy

$$2d + 1 = \sum_{i=1}^{3} m_i + \sum_{i=1}^{3} \alpha_i ,$$

$$d' = \frac{m_1 + \alpha_1 - 1}{2} \quad \text{and} \quad d'' = d - d' . \qquad (2.17)$$

By the divisor axiom and the string equation, we can further manipulate the above quantities to produce two-point functions. For example, we have

$$\langle \tau_{0,1} \tau_{m_2,\alpha_2} \tau_{m_3,\alpha_3} \rangle_0 \qquad (2.18)$$

$$= d'' \langle \tau_{m_2,\alpha_2} \tau_{m_3,\alpha_3} \rangle_0 + \langle \tau_{m_2-1,\alpha_2+1} \tau_{m_3,\alpha_3} \rangle_0 + \langle \tau_{m_2,\alpha_2} \tau_{m_3-1,\alpha_3+1} \rangle_0 .$$

Hence, using the previous computations of the two-point invariants, we can now also compute arbitrary three-point functions, whose closed-form expressions are possible but not instructive. We thus compute them numerically using a computer program and tabulate some of the invariants in Appendix C.1.

Similarly and unfortunately, we have reduced the problem of computing the higher-point invariants to an exercise in computer programming. We differentiate the TRR (2.8) repeatedly and use the divisor, dilaton, and string equations to reduce the number and degrees of the fields appearing in the right-hand side of equation. Using this simple but tedious approach, we have completed a computer program which computes up to 7-point functions of arbitrary degrees and have included a few examples in Appendix C.

2.3   Genus-One

Because there exists another TRR [38] in genus-1, it is also possible to compute the genus-1 descendant GW-invariants. For $\mathbb{P}^1$, the relation has the following simple form:

$$\langle\!\langle \tau_{n,\alpha} \rangle\!\rangle_1 = \langle\!\langle \tau_{n-1,\alpha}\tau_{0,0} \rangle\!\rangle_0 \langle\!\langle \tau_{0,1} \rangle\!\rangle_1 + \langle\!\langle \tau_{n-1,\alpha}\tau_{0,1} \rangle\!\rangle_0 \langle\!\langle \tau_{0,0} \rangle\!\rangle_1$$
$$+ \frac{1}{12} \langle\!\langle \tau_{n-1,\alpha}\tau_{0,0}\tau_{0,1} \rangle\!\rangle_0 . \tag{2.19}$$

Setting all $t_m^\alpha = 0$, (2.19) yields

$$\langle \tau_{2d+1,0} \rangle_{1,d} = \begin{cases} \frac{1}{12\,d!^2}\left(c_d - 2dc_{d-1} - 1\right) & \text{for } d \neq 0, \\ \frac{1}{12} & \text{for } d = 0, \quad \text{and} \end{cases}$$
$$\langle \tau_{2d,1} \rangle_{1,d} = -\frac{1 - 2d}{24\,d!^2} . \tag{2.20}$$

To compute the two-point functions[7], we differentiate (2.19) with respect to a descendant variable t_m^β and get an equation which is valid in the large phase space:

$$\langle\!\langle \tau_{n,\alpha}\tau_{m,\beta} \rangle\!\rangle_1 = \langle\!\langle \tau_{n-1,\alpha}\tau_{m,\beta}\tau_{0,0} \rangle\!\rangle_0 \langle\!\langle \tau_{0,1} \rangle\!\rangle_1 + \langle\!\langle \tau_{n-1,\alpha}\tau_{0,0} \rangle\!\rangle_0 \langle\!\langle \tau_{0,1}\tau_{m,\beta} \rangle\!\rangle_1$$
$$+ \langle\!\langle \tau_{n-1,\alpha}\tau_{m,\beta}\tau_{0,1} \rangle\!\rangle_0 \langle\!\langle \tau_{0,0} \rangle\!\rangle_1 \tag{2.21}$$
$$+ \langle\!\langle \tau_{n-1,\alpha}\tau_{0,1} \rangle\!\rangle_0 \langle\!\langle \tau_{0,0}\tau_{m,\beta} \rangle\!\rangle_1 + \frac{1}{12} \langle\!\langle \tau_{n-1,\alpha}\tau_{m,\beta}\tau_{0,0}\tau_{0,1} \rangle\!\rangle_0 .$$

At zero couplings, it becomes

$$\langle \tau_{n,\alpha}\tau_{m,\beta} \rangle_1 = -\frac{1}{24} \langle \tau_{n-1,\alpha}\tau_{m,\beta}\tau_{0,0} \rangle_0 + \langle \tau_{n-2,\alpha} \rangle_0 \langle \tau_{0,1}\tau_{m,\beta} \rangle_1 \tag{2.22}$$
$$+ \langle \tau_{n-1,\alpha}\tau_{0,1} \rangle_0 \langle \tau_{m-1,\beta} \rangle_1 + \frac{1}{12} \langle \tau_{n-1,\alpha}\tau_{m,\beta}\tau_{0,0}\tau_{0,1} \rangle_0 ,$$

which contains only genus-0 invariants and genus-1 one-point functions, upon using the divisor axiom. Closed-form answers for these two-point functions are again possible, but they are not perhaps so illuminating. We thus omit the explicit expressions in the paper but list some of their values in Appendix C.2.

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[7]There are of course a few obvious ones that one can compute by using the divisor and string equations. For example, one can show that $\langle \tau_{0,1}\tau_{0,1} \rangle_1 = 0$, $\langle \tau_{1,0}\tau_{0,1} \rangle_1 = -1/24$, $\langle \tau_{1,0}\tau_{1,0} \rangle_1 = 1/12$.

It is clear that the higher-point genus-1 descendant GW-invariants can be similarly computed in terms of the genus-0 and the lower-point genus-1 invariants by repeatedly differentiating (2.19). We have again implemented the algorithm into a computer program which computes up to 5-point functions of arbitrary degrees, and we have also tabulated a collection of our results up to 3-point functions in Appendix C.2.

2.4   Genus-Two

Combined with our previous computations, we now use a genus-2 TRR [17] to compute up to 3-point descendant GW-invariants in arbitrary degrees. Getzler's TRR (A.3) directly leads to the one-point functions, and their derivatives yield the desired two- and three-point functions (See Appendix C.3). One could also obtain the same numbers by using the two-point function TRRs (A.4).

2.5   Hodge Integrals

This section marks the end of our torture with programming.

Let π, Σ and ω be as in §2.1. The Hodge bundle $E = \pi_* \omega$ over $\overline{\mathcal{M}}_{g,n}(V, \beta)$ is a rank-g sheaf of holomorphic sections of $H^0(\Sigma, \omega_\Sigma)$, where ω_Σ is the canonical sheaf of Σ. The λ_i classes are defined to be the i-th Chern classes of the Hodge bundle, and a generalization of the Gromov-Witten integral of the form (2.3) including the λ-classes is called a Hodge integral. In [15], Faber and Pandharipande have found a set of differential operators that annihilate the generating function for Hodge integrals. In principle, their theorem allows one to compute the Hodge integrals on the moduli space of stable maps in terms of the descendant Gromov-Witten invariants. In practice, however, it is difficult to compute the GW-potential in the large phase space, and it is precisely for this reason that some kind of an integrable structure such as the Virasoro constraints is desirable in studying the intersection theory. It is, however, often the case that the Virasoro constraints alone are not strong enough to determine the GW-invariants on non-trivial target spaces. In this paper, we have taken a different approach to computing the invariants, and we have seen that for $\mathbb{P}^1$, the known topological

recursion relations allow one to compute all the GW-invariants up to genus-2. Using these results, the work of Faber and Pandharipande completely determines all the Hodge integrals for $\mathbb{P}^1$ up to genus-2.

The expressions of the differential equations for $\mathbb{P}^1$ are particularly simple:

$$
\langle\!\langle ch_{2\ell-1}(E)\rangle\!\rangle_g = \frac{B_{2\ell}}{(2\ell)!}\left[\langle\!\langle \tau_{2\ell,0}\rangle\!\rangle_g - \sum_m^\infty t_m^0 \langle\!\langle \tau_{m+2\ell-1,0}\rangle\!\rangle_g\right.
$$
$$
- \sum_m^\infty t_m^1 \langle\!\langle \tau_{m+2\ell-1,1}\rangle\!\rangle_g + \sum_{m=0}^{2\ell-2}(-1)^m \langle\!\langle \tau_{m,0}\tau_{2\ell-2-m,1}\rangle\!\rangle_{g-1}
$$
$$
\left.+ \sum_{g'+g''=g}\sum_{m=0}^{2\ell-2}(-1)^m \langle\!\langle \tau_{m,0}\rangle\!\rangle_{g'}\langle\!\langle \tau_{2\ell-2-m,1}\rangle\!\rangle_{g''}\right],
$$

where $B_{2\ell}$ are the Bernoulli numbers.

The first non-trivial Hodge integral is $\langle \tau_{0,0}\,\lambda_1\rangle_{1,0}$, which can be computed explicitly as follows:

$$
\begin{aligned}
\langle \tau_{0,0}\,\lambda_1\rangle_{1,0} &= \int_{\mathbb{P}^1\times\overline{\mathcal{M}}_{1,1}} \lambda_1(c_1(\mathbb{P}^1)-\lambda_1)\\
&= \int_{\mathbb{P}^1\times\overline{\mathcal{M}}_{1,1}} c_1(\mathbb{P}^1)\,\lambda_1\\
&= 2\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1\\
&= \frac{1}{12},
\end{aligned}
\tag{2.23}
$$

where we have used the formula for the Euler class of the obstruction bundle from [29], Mumford's relation $\lambda_g\lambda_g = 0$, and the numerical value of $\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 = 1/24$ from [15]. As simple illustrations, we have implemented our computer program to compute this and other arbitrary Hodge integrals involving up to two-descendants in genus-1. (See Appendix C.2 for a partial list.) The genus-2 cases are similarly treated: The $\ell = 1$ relations in (2.23) lead to λ_1 Hodge integrals, and the Mumford's relation $2\lambda_2 = \lambda_1^2$ yields the λ_2 integrals. Since the algorithm is obvious by now, we do not explicitly carry out the computations.

3   Virasoro Constraints

Let Z be the generating function[8] for GW-invariants:

$$Z = \exp\left[\sum_g \lambda^{2g-2}\left\langle\!\!\left\langle \exp\left(\sum_{m,a} t_m^a \tau_{m,a}\right)\right\rangle\!\!\right\rangle_g\right], \qquad (3.1)$$

and define $z_{n,g}$ to be the genus-g contribution[9] to $Z^{-1}L_n Z$. The Virasoro constraints for $\mathbb{P}^1$ are

$$
\begin{aligned}
z_{n,g} &= 0 \\
&= \sum_{m=0}^{\infty}\Big[\alpha(m,n)\, t_m^0 \left\langle\!\!\left\langle \tau_{n+m,0}\right\rangle\!\!\right\rangle_g \\
&\quad + 2\,\beta(m,n)\, t_m^0 \left\langle\!\!\left\langle \tau_{n+m-1,1}\right\rangle\!\!\right\rangle_g + \gamma(m,n)\, t_m^1 \left\langle\!\!\left\langle \tau_{n+m,1}\right\rangle\!\!\right\rangle_g\Big] \\
&\quad - \alpha(1,n)\left\langle\!\!\left\langle \tau_{n+1,0}\right\rangle\!\!\right\rangle_g - 2\,\beta(1,n)\left\langle\!\!\left\langle \tau_{n,1}\right\rangle\!\!\right\rangle_g \qquad (3.2)\\
&\quad + \sum_{m=0}^{n-2}\delta(m,n)\left[\left\langle\!\!\left\langle \tau_{m,1}\,\tau_{n-m-2,1}\right\rangle\!\!\right\rangle_{g-1} + \sum_{g'+g''=g}\left\langle\!\!\left\langle \tau_{m,1}\right\rangle\!\!\right\rangle_{g'}\left\langle\!\!\left\langle \tau_{n-m-2,1}\right\rangle\!\!\right\rangle_{g''}\right],
\end{aligned}
$$

where we have assumed that $n > 0$ and the constants are given by

$$
\begin{aligned}
\alpha(m,n) &= m\left(\frac{(n+m)!}{m!}\right), \\
\beta(m,n) &= \frac{(n+m)!}{m!}\left[1 + m(c_{n+m} - c_m)\right], \\
\gamma(m,n) &= \frac{(n+m+1)!}{m!}, \\
\delta(m,n) &= (m+1)!\,(n-m-1)!\,. \qquad (3.3)
\end{aligned}
$$

3.1   Genus-Zero

The genus-0 Virasoro constraints are

$$0 = z_{n,0}$$

---

[8] Z is called the partition function in the physics literature.

[9] That is, the coefficient of λ^{2g-2} in $Z^{-1}L_n Z$.

$$= \sum_{m=0}^{\infty} \Big[\alpha(m,n)\, t_m^0 \, \langle\!\langle \tau_{n+m,0} \rangle\!\rangle_0 + 2\,\beta(m,n)\, t_m^0 \, \langle\!\langle \tau_{n+m-1,1} \rangle\!\rangle_0$$

$$+ \gamma(m,n)\, t_m^1 \, \langle\!\langle \tau_{n+m,1} \rangle\!\rangle_0 \Big]$$

$$- \alpha(1,n) \, \langle\!\langle \tau_{n+1,0} \rangle\!\rangle_0 - 2\,\beta(1,n) \langle\!\langle \tau_{n,1} \rangle\!\rangle_0$$

$$+ \sum_{m=0}^{n-2} \delta(m,n) \, \langle\!\langle \tau_{m,1} \rangle\!\rangle_0 \, \langle\!\langle \tau_{n-m-2,1} \rangle\!\rangle_0 \,. \tag{3.4}$$

Taking derivatives of (3.4) with respect to the variables t_m^α yields a set of equations which the GW-invariants must satisfy. Let $\mathcal{I} = \{1, 2, \ldots k\}$ and $\mathcal{J} = \{1, 2, \ldots \ell\}$ be two index sets, and $\mathcal{I}', \mathcal{I}'', \mathcal{J}'$, and $\mathcal{J}''$ their partitions into two complementary subsets. The set $\mathcal{I}$ labels the descendants of the identity and $\mathcal{J}$ those of the hyperplane class. Then, one finds

$$0 = \sum_{i=1}^{k} \alpha(m_i, n) \left\langle \tau_{n+m_i,0} \left(\prod_{j \neq i} \tau_{m_j,0} \right) \tau_{s_1,1} \cdots \tau_{s_\ell,1} \right\rangle_0$$

$$- \alpha(1,n) \langle \tau_{n+1,0} \tau_{m_1,0} \cdots \tau_{m_k,0} \tau_{s_1,1} \cdots \tau_{s_\ell,1} \rangle_0$$

$$+ 2 \sum_{i=1}^{k} \beta(m_i, n) \left\langle \tau_{n-1+m_i,1} \left(\prod_{j \neq i} \tau_{m_j,0} \right) \tau_{s_1,1} \cdots \tau_{s_\ell,1} \right\rangle_0$$

$$- 2\beta(1,n) \langle \tau_{n,1} \tau_{m_1,0} \cdots \tau_{m_k,0} \tau_{s_1,1} \cdots \tau_{s_\ell,1} \rangle_0$$

$$+ \sum_{a=1}^{\ell} \gamma(s_a, n) \left\langle \tau_{m_1,0} \cdots \tau_{m_k,0} \tau_{n+s_a,1} \left(\prod_{b \neq a} \tau_{s_b,1} \right) \right\rangle_0$$

$$\cdot \sum_{q=0}^{n-2} \sum_{\mathcal{I}',\mathcal{I}'',\mathcal{J}',\mathcal{J}''} \delta(q,n) \left\langle \tau_{q,1} \left(\prod_{i \in \mathcal{I}'} \tau_{m_i,0} \right) \left(\prod_{a \in \mathcal{J}'} \tau_{s_a,1} \right) \right\rangle_0$$

$$\cdot \left\langle \tau_{n-q-2,1} \left(\prod_{j \in \mathcal{I}''} \tau_{m_j,0} \right) \left(\prod_{b \in \mathcal{J}''} \tau_{s_b,1} \right) \right\rangle_0 \,. \tag{3.5}$$

The Virasoro constraints are actually proven to hold in genus-0 [32], and we have numerically checked that the constraints (3.5) are indeed satisfied for roughly 5000 cases containing up to four-point functions. This test makes it fairly certain that our computer generated answers of the genus-0 GW-invariants are correct.

3.2   Genus-One

Since $\mathbb{P}^1$ has a semi-simple quantum cohomology, the Virasoro conjecture is true also in genus-1 [7, 33]. In this case, the Virasoro constraints take the form

$$
\begin{aligned}
0 = z_{n,1} \\
= \sum_{m=0}^{\infty} & \Big[\alpha(m,n) \, t_m^0 \, \langle\!\langle \tau_{n+m,0} \rangle\!\rangle_1 \\
& + 2\,\beta(m,n)\, t_m^0 \, \langle\!\langle \tau_{n+m-1,1} \rangle\!\rangle_1 + \gamma(m,n)\, t_m^1 \, \langle\!\langle \tau_{n+m,1} \rangle\!\rangle_1 \Big] \\
& - \alpha(1,n) \, \langle\!\langle \tau_{n+1,0} \rangle\!\rangle_1 - 2\,\beta(1,n) \, \langle\!\langle \tau_{n,1} \rangle\!\rangle_1 \\
& + \sum_{m=0}^{n-2} \delta(m,n) \, \Big[\langle\!\langle \tau_{m,1}\, \tau_{n-m-2,1} \rangle\!\rangle_0 + 2 \langle\!\langle \tau_{m,1} \rangle\!\rangle_0 \langle\!\langle \tau_{n-m-2,1} \rangle\!\rangle_1 \Big] \ , \quad (3.6)
\end{aligned}
$$

and they yield constraints that are similar to (3.5). Using the genus-0 and genus-1 descendant invariants that were computed in §2.2 and §2.3, we have checked that over 7000 Virasoro constraints which involve up to 4-point genus-1 GW-invariants are satisfied.

3.3   Genus-Two

The status of the Virasoro constraints in genus-2 is still conjectural, and it would be interesting to see if the GW-invariants which are obtained from either rigorously derived TRRs or algebraic geometry actually satisfy the Virasoro constraints in this case and in higher genera.

We have checked that our results are indeed consistent with about 1100 Virasoro constraints containing up to 3-point genus-2 invariants. As previously mentioned, one could use either (A.3) or (A.4) to check the consistency with the topological axioms and the Virasoro conjecture.

3.4   Speculations on Higher-Genus Cases: TRRs and Localizations

We find that the Virasoro constraints by themselves do not provide an efficient computational tool unless we already know many of the GW-invariants that are to be used in the constraint equations. In the pure gravity case, the Virasoro constraints relate $\tau_{n+1,0}$ with $\tau_{n,0}$, thus providing an effective recursions among the descendants. In the $\mathbb{P}^1$ case, however, the Virasoro constraints relate $\tau_{n+1,0}$ with $\tau_{n,1}$, but there is no relation between $\tau_{n,1}$ and $\tau_{n-1,0}$. This pattern of recursion explains why the Virasoro constraints generally cannot determine the GW-invariants by themselves.

Motivated by the previous computations, it is tempting to speculate that there may exist higher-genus TRRs that completely determine the GW-invariants. The only higher-genus TRRs that are known to us so far are those found by Eguchi and Xiong in [13]. Unfortunately, their derivation crucially depends on the assumption that the genus-g free energy $\mathcal{F}_g(t) = \left\langle \exp\left(\sum_{m,a} t_m^a \tau_{m,a}\right)\right\rangle_g$ is a function of genus-0 correlation functions in the large phase space. That is, their derivation assumes that

$$\mathcal{F}_g(t) = \mathcal{F}_g(u_{\alpha_1}(t), u_{\alpha_1\alpha_2}(t), \ldots, u_{\alpha_1\alpha_2\cdots\alpha_{3g-1}}(t)) \qquad (3.7)$$

where $u_{\alpha_1\alpha_2\cdots\alpha_k} := \langle\!\langle \tau_{0,0}\tau_{0,\alpha_1}\cdots\tau_{0,\alpha_k}\rangle\!\rangle_0 = \partial^{k+1}\mathcal{F}_0/\partial t_0^0\cdots\partial t_0^{\alpha_k}$. At first sight, it tells us that a genus-g GW-invariant can be expressed in terms of genus-0 invariants; more precisely, it determines the functional dependence of the free energy on the variables t_m^α through the genus-zero quantities $u_{\alpha_1}(t), u_{\alpha_1\alpha_2}(t), \ldots, u_{\alpha_1\alpha_2\cdots\alpha_{3g-1}}(t)$. In the rest of this section, we use the technique of localization[10] to comment on the validity of this assumption for complex projective spaces $\mathbb{P}^r$ admitting torus actions.

We will be very brief and use the results of [22, 28]. Given a compact complex projective variety V and a holomorphic vector bundle

---

[10]We are grateful to Prof. Tian for suggesting this analysis. We are ineluctably led to make it absolutely clear at this point that we do not have a satisfactory understanding of the ideas involving localizations and that the ensuing statements are only speculative. As we do not feel competent enough to present a rigorous proof, we are somewhat reluctant to present our arguments here. Nevertheless, with the hope that our honesty would engender further objectivity and caution from the readers than they would normally require, we proceed.

$E \to V$, equipped with a torus action $T \simeq \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ on (V, E), the Atiyah-Bott fixed points formula reduces the integrals of characteristic classes of E over V to new integrals over fixed loci of the torus action on V [1, 2]. Recall that the GW-invariants are defined to be integrals of certain characteristic classes over the virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(V, \beta)]^{\mathrm{vir}}$. The torus action on V can be naturally lifted to $\overline{\mathcal{M}}_{g,n}(V, \beta)$ by translating the stable maps. The work of Graber and Pandharipande [22] states that for non-singular projective varieties V, there exists a localization formula for the virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(V, \beta)]^{\mathrm{vir}}$, and thus the associated GW-invariants can be defined by integrals over the virtual classes of the fixed loci of the torus action on the moduli space. In particular, the localization formula holds for projective spaces $\mathbb{P}^r$, and the final result which we need is that an arbitrary GW-invariant of $\mathbb{P}^r$ can be expressed as a sum of Hodge integrals over products of the moduli spaces $\overline{\mathcal{M}}_{g',n'}$ of pointed Riemann surfaces; that is, roughly

$$\langle \tau_{m_1,\alpha_1} \cdots \tau_{m_n,\alpha_n} \rangle_{g,d} = \sum_\Gamma \int_{\overline{\mathcal{M}}_\Gamma} \psi_1^{m_1} \cdots \psi_n^{m_n} \frac{\text{weights}}{e(N_\Gamma^{\mathrm{vir}})} \qquad (3.8)$$

where the sum is over all the fixed loci represented by certain "graphs" Γ, $e(N_\Gamma^{\mathrm{vir}})$ is the Euler class of the virtual normal bundle to $\overline{\mathcal{M}}_\Gamma$, ψ_i the pull-back of the first Chern class of the cotangent bundle at i-th marked point on the Riemann surface, and the "weights" are determined by the torus action and on the cohomology classes $\gamma_{\alpha_j} \in H^*(\mathbb{P}^r, \mathbb{C})$. Furthermore, the fixed loci represented by the graph Γ are products of the moduli spaces of pointed stable curves:

$$\overline{\mathcal{M}}_\Gamma = \prod_{\text{vertices}} \overline{\mathcal{M}}_{g(v),\mathrm{val}(v)} \, , \qquad (3.9)$$

with $g(v) \leq g$ representing the arithematic genus of the contracted component of the domain curve. We refer the reader to [22, 28] for the specific definitions of the notations which are actually not so essential for our discussion.

The Euler class $e(N_\Gamma^{\mathrm{vir}})$ of the virtual normal bundle introduces the λ-classes, and the resulting Hodge integrals can be reduced to pure ψ integrals by using Faber's algorithm [14]. Now, we recall the fact that for the intersection theory of the tautological divisors on the moduli space of stable pointed curves, the genus-g free energy, for $g > 0$, is

actually a function of the genus-0 correlators [8]:

$$\mathcal{F}_g(t)_{\mathrm{point}} = \mathcal{F}_g(u^{(1)}(t), \cdots, u^{(3g-2)}(t))_{\mathrm{point}}, \qquad (3.10)$$

where $u := \langle\!\langle \tau_0 \tau_0 \rangle\!\rangle_0 = \partial^2 \mathcal{F}_0/\partial t_0\, \partial t_0$ and $u^{(i)} = \partial^i u/\partial t_0{}^i$. Combined with the localization formula (3.8), the form of (3.10) implies that indeed each GW-invariant can be expressed in terms of $u_{\alpha_1}(0), u_{\alpha_1\alpha_2}(0), \ldots,$ $u_{\alpha_1\alpha_2\cdots\alpha_{3g-1}}(0)$ and the values of their derivatives at the origin of the phase space. This statement is however much weaker than the assumption (3.7). That is, our analysis does not show that the functional dependence of $\mathcal{F}_g(t)$ on t_m^α is only through $u_{\alpha_1}(t), u_{\alpha_1\alpha_2}(t), \ldots,$ $u_{\alpha_1\alpha_2\cdots\alpha_{3g-1}}(t)$. Even though we cannot prove the statement at the moment, we believe that our approach deserves a further consideration.

4   Recursion Relations for Simple Hurwitz Numbers

Hurwitz numbers, whose study had been initiated by Hurwitz more than a century ago [25], count the number of inequivalent ramified coverings of a sphere by Riemann surfaces with specified branching conditions over one point called ∞. The original approach of Hurwitz relates the problem to transitive factorizations of permutations into transpositions. Recently, new insights have been gained from developments in the absolute and relative Gromov-Witten theory [21, 30, 37].

In this section, we take a very modest goal of obtaining new recursion relations for the genus-0 and genus-1 simple Hurwitz numbers which enumerate the coverings with no ramification over ∞. We take two different approaches which yield similar but inequivalent recursion relations. In the notations of the previous sections, the genus-g simple Hurwitz numbers are defined[11] by the descendant GW-invariants of $\mathbb{P}^1$ as

$$H_d^g := \left\langle \tau_{1,1}^{2d+2g-2} \right\rangle_g. \qquad (4.1)$$

We first show that the genus-0 and genus-1 TRRs immediately lead to relations among the simple Hurwitz numbers in these genera. We then use the Virasoro constraints to derive new relations which could be generalized to higher genera.

---

[11]We are grateful to R. Vakil for this definition.

4.1   From Topological Recursion Relations

In this subsection, by using the TRRs (2.8) and (2.19), we derive a new recursion relation for simple Hurwitz numbers in genus-0 and reproduce the known result of Graber and Pandharipande in genus-1.

Claim 4.1. *The genus-0 simple Hurwitz numbers satisfy*

$$H_d^0 = 4 \sum_{k=0}^{d-3} \binom{2d-5}{2k} (d-k-1)(d-k-2)(k+1)^2 H_{k+1}^0 \, H_{d-k-1}^0. \quad (4.2)$$

Proof. We need the following equations which are implied by the string and the divisor equations:

$$\begin{aligned}
\left\langle \tau_{0,1}\tau_{0,0}\tau_{1,1}^{2k-1} \right\rangle_0 &= (2k-1)\,k^2 \left\langle \tau_{1,1}^{2k-2} \right\rangle_0 \\
\left\langle \tau_{0,1}\tau_{0,1}\tau_{1,1}^{2k} \right\rangle_0 &= (k+1)^2 \left\langle \tau_{1,1}^{2k} \right\rangle_0,
\end{aligned} \quad (4.3)$$

and similarly for $\left\langle \tau_{0,0}\tau_{1,1}^{2k-1} \right\rangle_0$ and $\left\langle \tau_{0,1}\tau_{1,1}^{2k} \right\rangle_0$. Differentiating the genus-0 TRR (2.8) yields

$$\begin{aligned}
\left\langle \tau_{1,1}^{2n} \right\rangle_0 &= \sum_{\ell=0}^{2n-3} \binom{2n-3}{\ell} \left[\left\langle \tau_{0,1}\tau_{0,0}\tau_{1,1}^{\ell} \right\rangle_0 \left\langle \tau_{0,1}\tau_{1,1}^{2n-\ell-1} \right\rangle_0 \right. \\
&\qquad \left. + \left\langle \tau_{0,1}\tau_{0,1}\tau_{1,1}^{\ell} \right\rangle_0 \left\langle \tau_{0,0}\tau_{1,1}^{2n-\ell-1} \right\rangle_0 \right] \\
&= \sum_{k=1}^{n-1} \binom{2n-3}{2k-1} \left\langle \tau_{0,1}\tau_{0,0}\tau_{1,1}^{2k-1} \right\rangle_0 \left\langle \tau_{0,1}\tau_{1,1}^{2n-2k} \right\rangle_0 \\
&\qquad + \sum_{k=0}^{n-2} \binom{2n-3}{2k} \left\langle \tau_{0,1}\tau_{0,1}\tau_{1,1}^{2k} \right\rangle_0 \left\langle \tau_{0,0}\tau_{1,1}^{2n-2k-1} \right\rangle_0 \,,
\end{aligned}$$

where we have used the fact that many of the correlation functions vanish for dimensional reasons and we have relabeled indices. Now, using (4.3) and relabeling the summation yields the desired result.   □

Together with the initial conditions $H_1^0 = 1$ and $H_2^0 = 1/2$, these recursion relations easily determine all the simple Hurwitz numbers in genus-zero. The formulae (4.2) are qualitatively similar to those

found by Graber and Pandharipande [16], but they are in fact different recursion relations.

Similarly, we use the genus-1 TRR to derive recursion relations for genus-1 Hurwitz numbers $H_d^1 = \left\langle \tau_{1,1}^{2d} \right\rangle_{1,d}$:

Claim 4.2. *The genus-1 simple Hurwitz numbers satisfy*

$$H_d^1 = 2 \sum_{k=1}^{d-1} k^2(d-k)(2d-2k+1) \binom{2d-1}{2k-2} H_k^0 H_{d-k}^1$$
$$+ \frac{1}{12} d^2(d-1)(2d-1) H_d^0 \quad (4.4)$$

Proof. As in the proof of (4.2), we differentiate the genus-1 TRR and use dimensional arguments to get

$$\left\langle \tau_{1,1}^{2n} \right\rangle_1$$
$$= \sum_{k=1}^{n} \binom{2n-1}{2k-1} \left\langle \tau_{0,1}\tau_{0,0}\tau_{1,1}^{2k-1} \right\rangle_0 \left\langle \tau_{0,1}\tau_{1,1}^{2n-2k} \right\rangle_1$$
$$+ \sum_{k=0}^{n-1} \binom{2n-1}{2k} \left\langle \tau_{0,1}\tau_{0,1}\tau_{1,1}^{2k} \right\rangle_0 \left\langle \tau_{0,0}\tau_{1,1}^{2n-2k-1} \right\rangle_1 + \frac{2n-1}{12} \left\langle \tau_{0,1}^3 \tau_{1,1}^{2n-2} \right\rangle_0$$

Taking caution that $\left\langle \tau_{0,1} \right\rangle_1 = -1/24$, we obtain (4.4) upon using the divisor and the string equations. $\qquad \square$

Unlike the genus-0 case, with a minor rearrangement of terms, it is easy to see that our recursion relation (4.4) is actually equal to that of Graber and Pandharipande.

4.2   From Virasoro Constraints

It is also possible to derive new recursion relations for genus-g Hurwitz numbers by combining the Virasoro constraints with some TRRs. Namely, the L_1 Virasoro constraints yield

$$3H_d^g := 3\left\langle \tau_{1,1}^{2d+2g-2} \right\rangle_g \qquad (4.5)$$

$$= 3(2d + 2g - 3)\left\langle \tau_{2,1}\tau_{1,1}^{2d+2g-4}\right\rangle_g - \left\langle \tau_{2,0}\tau_{1,1}^{2d+2g-3}\right\rangle_g.$$

For example, for genus-0, we deduce

Claim 4.3. *The genus-0 simple Hurwitz numbers H_d^0 satisfy the recursion relations of the form*

$$H_d^0 = (2d - 3)\sum_{k=1}^{d-3}(k + 1)(d - k - 1)\left[\binom{2d - 6}{2k}(2k + 1)\right.$$
$$\left. + \binom{2d - 6}{2k - 1}(2d - 2k - 3)\right]H_{k+1}^0 H_{d-k-1}^0$$
$$- \frac{1}{3}\sum_{k=1}^{d-3}\binom{2d - 5}{2k}(k + 1)(d - k - 1)\left[(2k - 1)(2d - 2k - 3)\right.$$
$$\left. + 2k(2d - 2k - 5)\right]H_{k+1}^0 H_{d-k-1}^0$$
$$+ \frac{4}{3}(d - 1)(2d - 3)H_{d-1}^0. \tag{4.6}$$

Proof. We need the following two recursion formulas which are obtained from the genus-0 TRR:

$$\left\langle \tau_{2,1}\tau_{1,1}^{2m}\right\rangle_0 = (m + 1)\left\langle \tau_{1,1}^{2m}\right\rangle_0$$
$$+ \sum_{k=1}^{m-1}(k + 1)(m - k + 1)\left[\binom{2m - 2}{2k}(2k + 1)\right.$$
$$\left. + \binom{2m - 2}{2k - 1}(2m - 2k + 1)\right]\left\langle \tau_{1,1}^{2k}\right\rangle_0\left\langle \tau_{1,1}^{2m-2k}\right\rangle_0 \tag{4.7}$$

and

$$\left\langle \tau_{2,0}\tau_{1,1}^{2m+1}\right\rangle_0 = -(m + 1)(2m + 1)\left\langle \tau_{1,1}^{2m}\right\rangle_0$$
$$+ \sum_{k=1}^{m-1}\binom{2m - 1}{2k}(k + 1)(m - k + 1)$$
$$\cdot \left[(2k - 1)(2m - 2k + 1)\right.$$
$$\left. + 2k(2m - 2k - 1)\right]\left\langle \tau_{1,1}^{2k}\right\rangle_0\left\langle \tau_{1,1}^{2m-2k}\right\rangle_0. \tag{4.8}$$

The L_1 Virasoro constraint (4.5) now implies our claim.   □

Similarly, after some algebra, one can show

Claim 4.4. *The genus-1 simple Hurwitz numbers satisfy*

$$H_d^1 = \frac{4}{3} \sum_{k=1}^{d-1} \binom{2d-1}{2k-2} k\,(d-k)(2k-1)(2d-2k+1)\,H_k^0 H_{d-k}^1$$
$$+ \frac{d(d-1)(2d-1)^2}{18} H_d^0. \tag{4.9}$$

It can be easily checked that these relations are actually different from (4.2) and (4.4) and from the ones obtained by Graber and Pandharipande.

Remark. It is important to note that since the Virasoro conjecture has been proven to hold in genera-zero and -one, the recursion relations (4.6) and (4.9) are also true and are not mere conjectures. Indeed, we have verified numerically that they lead to the correct simple Hurwitz numbers.

Further investigation is needed to gain a geometric understanding of the recursion relations that we have obtained.

It is also possible to obtain similar relations for higher genus simple Hurwitz numbers from (4.5), but there are two important distinctions from the above two cases. Firstly, the Virasoro constraints are still conjectural in genus-2 and higher, thus the resulting recursions are not rigorous, even though they will provide an interesting check for the conjecture. Secondly, there are no effective TRRs that can be used to express $\left\langle \tau_{2,1} \tau_{1,1}^{2d+2g-4} \right\rangle_g$ and $\left\langle \tau_{2,0} \tau_{1,1}^{2d+2g-3} \right\rangle_g$ in terms of Hurwitz numbers. In principle, Getzler's TRRs (A.3) in genus-2 could be used to express these quantities in terms of lower genus simple Hurwitz numbers and H_k^2, for $k < d$. The TRRs however involve a large number of terms and render computations somewhat intractable. As there already exists a much simpler recursion relation [21], we omit the derivation here. In higher genus, we are not aware of any effective TRRs that can be applied. What seems to be required in this study is a TRR that eliminates the descendant $\tau_{2,\alpha}$ from correlators, just as the string, divisor and dilaton equations eliminate the $\tau_{0,0}$, $\tau_{0,1}$ and $\tau_{1,0}$ insertions, respectively. It would be interesting to see if there exists a geometric reason for such an equation.

5   Conclusion

In principle, one could, with much patience and stamina, extend our program to higher genera. Being novices that we are in computer science, we however stop at genus-2 and would now like to discuss what we have learned from these exercises.

It is instructive to recall how the KdV conjecture for pure gravity, stating that the intersection theory of tautological classes on $\overline{\mathcal{M}}_{g.n}$ is governed by the KdV hierarchy and the string equation, was proven by Witten in genera-zero and -one [38]. First, recall the algebro-geometric way of determining the descendant integrals: In genera-zero and -one, purely dimensional arguments require the non-vanishing descendant integrals to include a certain number of puncture and dilaton operators. Then, the string and dilaton equations are used to reduce the integrals to $\left\langle \tau_{0,0}^3 \right\rangle_0$ in genus-0 and $\left\langle \tau_{1,0} \right\rangle_1$ in genus-1, whose values can be determined from algebraic geometry. Witten's proof is based on the fact that the string and dilaton equations and the initial values of $\left\langle \tau_{0,0}^3 \right\rangle_0$ and $\left\langle \tau_{1,0} \right\rangle_1$, which together determine all the descendant integrals completely, can be derived[12] from his KdV conjecture. Hence, the algebraic geometry and his KdV conjecture yield the precisely same algorithm for computing the descendant integrals in genera-zero and -one. In the case of point target space, there is thus no further need to invoke additional topological recursion relations, which are nevertheless consistent with the KdV structure.

Something similar but crucially different persists in the picture of the Virasoro constraints for $\mathbb{P}^1$ in low genera. One could compute all the GW-invariants in genera-zero, -one and -two by using only the string, divisor and dilaton equations together with the aforementioned topological recursion relations. On the other hand, the Virasoro constraints are not strong enough to determine the GW-invariants by themselves. It thus seems that the Virasoro constraints are *weaker* than the topological recursion relations. As we have checked numerically, the GW-invariants obtained from the TRRs satisfy the Virasoro constraints in genus-zero and -one, as they should according to the rigorous proofs of mathematicians, and even in genus-two, which has yet no direct

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[12]More precisely, the dilaton equation can be derived from the string and the KdV equations.

proof. Thus, as in the pure gravity case, the TRRs are consistent with the conjectured integrable hierarchy manifested by the Virasoro constraints which however, unlike the pure gravity case, do not determine the generating functions completely.

The relations between the TRRs and the Virasoro constraints appear to be quite mysterious. Even in the case of pure gravity, although the TRRs seem redundant, it is not known how to derive them directly from the KdV hierarchy or whether it is possible to do so at all. An analogous question in the case of $\mathbb{P}^1$, in which the TRRs and the Virasoro constraints reverse their roles in some sense, would be: *Do TRRs imply Virasoro constraints in genera-zero, -one, and -two?* Since L_{-1} and L_2 generate the (half branch of) Virasoro algebra and since L_{-1} is just the string equation, which is true for general topological string theories, in order to answer the question, one only needs to prove that the TRRs imply the L_2 condition. We have tried an inductive approach to show that all derivatives of the $L_2 Z$ vanish by the TRRs and the L_{-1} constraint, but it does not seem possible to prove the statement.

The study of Virasoro constraints is presently only at its rudimentary stage, and any subsequent effort to understand their hidden structure would require unraveling their relation with various topological recursion relations and also with the constraints arising from the study of Hodge integrals. In this paper, we have used the L_{-1}, L_0 conditions and TRRs to compute the descendant GW-invariants of $\mathbb{P}^1$ in low genera. The ineffectiveness of the Virasoro constraints suggests that there may exist an enlarged algebra including the Virasoro algebra and giving us a "master" hierarchy encoding the TRR relations in all genera. Since the virtual localization technique expresses all GW-invariants in terms of Hodge integrals over products of the moduli space of stable pointed curves [22], it is also tempting to speculate that the Virasoro conjecture and the TRRs can be translated into a statement of some kind of an integrable hierarchy involving the *very large* phase space of Manin and Zograf [34].

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A   Topological Recursion Relations

We here summarize the topological recursion relations that form the essential bases for our discussion.

Genus-Zero [38]:

$$\langle\langle \tau_{m_1,\alpha_1}\tau_{m_2,\alpha_2}\tau_{m_3,\alpha_3}\rangle\rangle_0 = \langle\langle \tau_{m_1-1,\alpha_1}\tau_{0,0}\rangle\rangle_0 \langle\langle \tau_{0,1}\tau_{m_2,\alpha_2}\tau_{m_3,\alpha_3}\rangle\rangle_0$$
$$+ \langle\langle \tau_{m_1-1,\alpha_1}\tau_{0,1}\rangle\rangle_0 \langle\langle \tau_{0,0}\tau_{m_2,\alpha_2}\tau_{m_3,\alpha_3}\rangle\rangle_0 \ . \quad \text{(A.1)}$$

Genus-One [38]:

$$\langle\langle \tau_{n,\alpha}\rangle\rangle_1 = \langle\langle \tau_{n-1,\alpha}\tau_{0,0}\rangle\rangle_0 \langle\langle \tau_{0,1}\rangle\rangle_1$$
$$+ \langle\langle \tau_{n-1,\alpha}\tau_{0,1}\rangle\rangle_0 \langle\langle \tau_{0,0}\rangle\rangle_1 + \frac{1}{12}\langle\langle \tau_{n-1,\alpha}\tau_{0,0}\tau_{0,1}\rangle\rangle_0 \ . \quad \text{(A.2)}$$

Genus-Two [17]:

$$\langle\langle \tau_{k+2,\alpha}\rangle\rangle_2 = \langle\langle \tau_{k+1,\alpha}\tau_{0,a}\rangle\rangle_0 \eta^{ab}\langle\langle \tau_{0,b}\rangle\rangle_2$$
$$+ \langle\langle \tau_{k,\alpha}\tau_{0,a}\rangle\rangle_0 \eta^{ab}\langle\langle \tau_{1,b}\rangle\rangle_2$$
$$- \langle\langle \tau_{k,\alpha}\tau_{0,a}\rangle\rangle_0 \eta^{ab}\langle\langle \tau_{0,b}\tau_{0,c}\rangle\rangle_0 \eta^{cd}\langle\langle \tau_{0,d}\rangle\rangle_2$$
$$+ \frac{7}{10}\langle\langle \tau_{k,\alpha}\tau_{0,a}\tau_{0,c}\rangle\rangle_0 \eta^{ab}\langle\langle \tau_{0,b}\rangle\rangle_1 \eta^{cd}\langle\langle \tau_{0,d}\rangle\rangle_1$$
$$+ \frac{1}{10}\langle\langle \tau_{k,\alpha}\tau_{0,a}\tau_{0,c}\rangle\rangle_0 \eta^{ab}\eta^{cd}\langle\langle \tau_{0,b}\tau_{0,d}\rangle\rangle_1$$
$$- \frac{1}{240}\langle\langle \tau_{k,\alpha}\tau_{0,a}\rangle\rangle_1 \eta^{ab}\eta^{cd}\langle\langle \tau_{0,b}\tau_{0,c}\tau_{0,d}\rangle\rangle_0$$

$$+ \frac{13}{240} \langle\!\langle \tau_{k,\alpha}\tau_{0,a}\tau_{0,b}\tau_{0,c}\rangle\!\rangle_0 \eta^{ab}\eta^{cd} \langle\!\langle \tau_{0,d}\rangle\!\rangle_1$$

$$+ \frac{1}{960} \langle\!\langle \tau_{k,\alpha}\tau_{0,a}\tau_{0,b}\tau_{0,c}\tau_{0,d}\rangle\!\rangle_0 \eta^{ab}\eta^{cd} ,$$

$$(A.3)$$

where the metric is given by $\eta_{ab} = \delta_{a,1-b}$.

For two-point functions[13] in the large phase space, the relations are

$$\langle\!\langle \tau_{k+1,x}\tau_{l+1,y}\rangle\!\rangle_2$$
$$= \langle\!\langle \tau_{k+1,x}\tau_0^a\rangle\!\rangle_2 \langle\!\langle \tau_{l,y}\tau_{0,a}\rangle\!\rangle_0 + \langle\!\langle \tau_{k,x}\tau_{0,a}\rangle\!\rangle_0 \langle\!\langle \tau_{l+1,y}\tau_0^a\rangle\!\rangle_2$$
$$- \langle\!\langle \tau_{k,x}\tau_{0,a}\rangle\!\rangle_0 \langle\!\langle \tau_{l,y}\tau_{0,b}\rangle\!\rangle_0 \langle\!\langle \tau_0^a\tau_0^b\rangle\!\rangle_2$$
$$+ 3 \langle\!\langle \tau_{k,x}\tau_{l,y}\tau_0^a\rangle\!\rangle_0 \left[\langle\!\langle \tau_{1,a}\rangle\!\rangle_2 - \langle\!\langle \tau_{0,a}\tau_{0,b}\rangle\!\rangle_0 \langle\!\langle \tau_0^b\rangle\!\rangle_2 \right]$$
$$+ \frac{13}{10} \langle\!\langle \tau_{k,x}\tau_{l,y}\tau_{0,a}\tau_{0,b}\rangle\!\rangle_0 \langle\!\langle \tau_0^a\rangle\!\rangle_1 \langle\!\langle \tau_0^b\rangle\!\rangle_1 + \frac{4}{5} \langle\!\langle \tau_{k,x}\tau_{0,a}\rangle\!\rangle_1 \langle\!\langle \tau_{l,y}\tau_0^a\tau_{0,b}\rangle\!\rangle_0 \langle\!\langle \tau_0^b\rangle\!\rangle_1$$
$$+ \frac{4}{5} \langle\!\langle \tau_{k,x}\tau_0^a\tau_{0,b}\rangle\!\rangle_0 \langle\!\langle \tau_{l,y}\tau_{0,a}\rangle\!\rangle_1 \langle\!\langle \tau_0^b\rangle\!\rangle_1 - \frac{4}{5} \langle\!\langle \tau_{k,x}\tau_{l,y}\tau_{0,a}\rangle\!\rangle_0 \langle\!\langle \tau_0^a\tau_{0,b}\rangle\!\rangle_1 \langle\!\langle \tau_0^b\rangle\!\rangle_1$$
$$+ \frac{23}{240} \langle\!\langle \tau_{k,x}\tau_{l,y}\tau_{0,a}\tau_0^a\tau_{0,b}\rangle\!\rangle_0 \langle\!\langle \tau_0^b\rangle\!\rangle_1 + \frac{1}{48} \langle\!\langle \tau_{k,x}\tau_{0,a}\tau_0^a\tau_{0,b}\rangle\!\rangle_0 \langle\!\langle \tau_{l,y}\tau_0^b\rangle\!\rangle_1$$
$$+ \frac{1}{48} \langle\!\langle \tau_{k,x}\tau_0^b\rangle\!\rangle_1 \langle\!\langle \tau_{l,y}\tau_{0,a}\tau_0^a\tau_{0,b}\rangle\!\rangle_0 - \frac{1}{80} \langle\!\langle \tau_{k,x}\tau_{l,y}\tau_0^b\rangle\!\rangle_1 \langle\!\langle \tau_{0,a}\tau_0^a\tau_{0,b}\rangle\!\rangle_0$$
$$+ \frac{7}{30} \langle\!\langle \tau_{k,x}\tau_{l,y}\tau_{0,a}\tau_{0,b}\rangle\!\rangle_0 \langle\!\langle \tau_0^a\tau_0^b\rangle\!\rangle_1 + \frac{1}{30} \langle\!\langle \tau_{k,x}\tau_{0,a}\tau_{0,b}\rangle\!\rangle_0 \langle\!\langle \tau_{l,y}\tau_0^a\tau_0^b\rangle\!\rangle_1$$
$$+ \frac{1}{30} \langle\!\langle \tau_{k,x}\tau_{0,a}\tau_{0,b}\rangle\!\rangle_1 \langle\!\langle \tau_{l,y}\tau_0^a\tau_0^b\rangle\!\rangle_0 - \frac{1}{30} \langle\!\langle \tau_{k,x}\tau_{l,y}\tau_{0,a}\rangle\!\rangle_0 \langle\!\langle \tau_0^a\tau_{0,b}\tau_0^b\rangle\!\rangle_1$$
$$+ \frac{1}{576} \langle\!\langle \tau_{k,x}\tau_{l,y}\tau_{0,a}\tau_0^a\tau_{0,b}\tau_0^b\rangle\!\rangle_0 , \qquad\qquad \text{for } k,l \geq 0, \qquad (A.4)$$

where $\tau_0^a := \eta^{ab}\tau_{0,b}$.

B   Genus-Zero Two-Point Descendants

The GW-invariants of the form $\langle \tau_{m,\alpha}\tau_{0,\beta}\rangle_0$ are easily obtained by using the following TRR for two-point functions [11] which is valid in the

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[13]We thank C.-S. Xiong for providing us with this corrected version of the TRR. We have checked the consistency of these relations against (A.3) up to 3-point functions at the origin of the phase space.

large phase space:

$$\langle\langle \tau_{n,\alpha}\tau_{0,\beta}\rangle\rangle_0 = \frac{1}{n+\alpha+\beta}\left[M_\beta^\gamma \langle\langle \tau_{n-1,\alpha}\tau_{0,\gamma}\rangle\rangle_0 - 2\langle\langle \tau_{n-1,\alpha+1}\tau_{0,\beta}\rangle\rangle_0\right] , \quad (B.1)$$

where the matrix M is given by

$$M_\alpha^\beta = \begin{pmatrix} \langle\langle \tau_{0,0}\tau_{0,1}\rangle\rangle_0 & 2 \\ 2\langle\langle \tau_{0,1}\tau_{0,1}\rangle\rangle_0 & \langle\langle \tau_{0,0}\tau_{0,1}\rangle\rangle_0 \end{pmatrix}. \quad (B.2)$$

When all the couplings are turned off, the matrix M takes the form

$$M_\alpha^\beta = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \quad (B.3)$$

From (B.1), one finds

$$\langle \tau_{2n+1,1}\tau_{0,0}\rangle_{0,n+1} = \langle \tau_{2n,1}\rangle_{0,n+1} = \left[\frac{1}{(n+1)!}\right]^2$$

$$\langle \tau_{2n,1}\tau_{0,1}\rangle_{0,n+1} = \left(\frac{1}{n+1}\right)\frac{1}{(n!)^2}$$

$$\langle \tau_{2n,0}\tau_{0,0}\rangle_{0,n} = \langle \tau_{2n-1,0}\rangle_{0,n} = -\frac{2\,c_n}{(n!)^2}$$

$$\langle \tau_{2n+1,0}\tau_{0,1}\rangle_{0,n+1} = \left(\frac{1}{n+1}\right)\frac{1}{(n!)^2}\left[-2\,c_n - \frac{1}{n+1}\right]. \quad (B.4)$$

For more general invariants, we use the approach discussed in §2.2:

$$\langle \tau_{2m,0}\tau_{2d-2m,0}\rangle_{0,d} = -2\frac{c_d}{d!^2} + \sum_{k=1}^m \Delta_1(k,d)$$

$$\langle \tau_{2m-1,0}\tau_{2d-2m+1,0}\rangle_{0,d} = 2\frac{c_d}{d!^2} + \sum_{k=2}^m \widetilde{\Delta_1}(k,d) \quad (B.5)$$

where

$$\Delta_1(k,d) = A(k,d) - A(d-k+1,d)$$
$$\widetilde{\Delta_1}(k,d) = A(d-k+1,d) - A(k-1,d). \quad (B.6)$$

Similarly, we find

$$\langle \tau_{2m,1}\tau_{2d-2m-2,1}\rangle_{0,d} = \frac{1}{d(d-1)!^2} + \sum_{k=1}^m \Delta_2(k,d)$$

$$\langle \tau_{2m-1,1}\tau_{2d-2m-1,1}\rangle_{0,d} \;=\; \frac{d-1}{d(d-1)!^2} + \sum_{k=2}^{m}\widetilde{\Delta}_2(k,d)\,, \quad \text{(B.7)}$$

where

$$\begin{aligned}
\Delta_2(k,d) &= M(k,d) - M(d-k,d)\\
\widetilde{\Delta}_2(k,d) &= M(d-k,d) - M(k-1,d)\\
M(k,d) &= \frac{1}{(k)!^2(d-k)(d-k-1)!^2}\,.
\end{aligned} \quad \text{(B.8)}$$

Finally, we have

$$\langle \tau_{2m,0}\tau_{2d-2m-1,1}\rangle_{0,d} \;=\; \frac{1}{d!^2} + \sum_{k=1}^{m}\Delta_3(k,d)$$

$$\langle \tau_{2m-1,0}\tau_{2d-2m,1}\rangle_{0,d} \;=\; -\frac{1}{d!^2} + \sum_{k=2}^{m}\widetilde{\Delta}_3(k,d)\,, \quad \text{(B.9)}$$

where

$$\begin{aligned}
\Delta_3(k,d) &= W_1(k,d) + W_2(k,d)\\
\widetilde{\Delta}_3(k,d) &= -W_1(k-1,d) - W_2(k,d)\\
W_1(k,d) &= -\frac{2c_{k-1}+1/k}{k(k-1)!^2(d-k)!^2}\\
W_2(k,d) &= \frac{2\,c_{k-1}}{(k-1)!^2(d-k+1)(d-k)!^2}\,.
\end{aligned} \quad \text{(B.10)}$$

Note that the summations are set to zero whenever the lower limit exceeds the upper limit.

C   Partial Lists of the GW-Invariants

For those who are interested in the numerical values of the GW-invariants and for the sake of completeness, we here present, in fine prints, a few examples of the non-vanishing invariants. In most cases, we omit the ones that can be reduced by using the string, dilaton, or divisor equations.

C.1 Genus-Zero Descendants

1-Point Descendants

n	$\left\langle \tau_{2n+1,0} \right\rangle_{0,\,n+1}$
0	-2
1	$-\dfrac{3}{4}$
2	$-\dfrac{11}{108}$
3	$-\dfrac{25}{3456}$
4	$-\dfrac{137}{432000}$
5	$-\dfrac{49}{5184000}$
6	$-\dfrac{121}{592704000}$
7	$-\dfrac{761}{227598336000}$
8	$-\dfrac{7129}{165919186944000}$
9	$-\dfrac{7381}{16591918694400000}$
10	$-\dfrac{83711}{22083843782246400000}$

n	$\left\langle \tau_{2n,1} \right\rangle_{0,\,n+1}$
0	1
1	$\dfrac{1}{4}$
2	$\dfrac{1}{36}$
3	$\dfrac{1}{576}$
4	$\dfrac{1}{14400}$
5	$\dfrac{1}{518400}$
6	$\dfrac{1}{25401600}$
7	$\dfrac{1}{1625702400}$
8	$\dfrac{1}{131681894400}$
9	$\dfrac{1}{13168189440000}$
10	$\dfrac{1}{1593350922240000}$

2-Point Descendants

$$I = \langle \tau_{2n,0}\tau_{2d-2n,0}\rangle_{0,d}$$

d	$2n$	$2d-2n$	I
1	0	2	-2
2	0	4	$-\frac{3}{4}$
2	2	2	$\frac{5}{4}$
3	0	6	$-\frac{11}{108}$
3	2	4	$\frac{35}{54}$
4	0	8	$-\frac{25}{3456}$
4	2	6	$\frac{109}{1152}$
4	4	4	$\frac{61}{128}$
5	0	10	$-\frac{137}{432000}$
5	2	8	$\frac{83}{12000}$
5	4	6	$\frac{479}{6000}$
6	0	12	$-\frac{49}{5184000}$
6	2	10	$\frac{319}{1036800}$
6	4	8	$\frac{131}{20736}$
6	6	6	$\frac{221}{15552}$

$$I = \langle \tau_{2n-1,0}\tau_{2d-2n+1,0}\rangle_{0,d}$$

d	$2n-1$	$2d-2n+1$	I
1	1	1	2
2	1	3	$\frac{3}{4}$
3	1	5	$\frac{11}{108}$
3	3	3	$\frac{50}{27}$
4	1	7	$\frac{25}{3456}$
4	3	5	$\frac{59}{128}$
5	1	9	$\frac{137}{432000}$
5	3	7	$\frac{641}{13500}$
5	5	5	$\frac{257}{2000}$
6	1	11	$\frac{49}{5184000}$
6	3	9	$\frac{113}{41472}$
6	5	7	$\frac{73}{5184}$

$$I = \langle \tau_{2n,1}\tau_{2d-2n-2,1}\rangle_{0,d}$$

d	$2n$	$2d-2n-2$	I
1	0	0	1
2	0	2	$\frac{1}{2}$
3	0	4	$\frac{1}{12}$
3	2	2	$\frac{1}{3}$
4	0	6	$\frac{1}{144}$
4	2	4	$\frac{1}{16}$
5	0	8	$\frac{1}{2880}$
5	2	6	$\frac{1}{180}$
5	4	4	$\frac{1}{80}$
6	0	10	$\frac{1}{86400}$
6	2	8	$\frac{1}{3456}$
6	4	6	$\frac{1}{864}$

$$I = \langle \tau_{2n-1,1}\tau_{2d-2n-1,1}\rangle_{0,d}$$

d	$2n-1$	$2d-2n-1$	I
2	1	1	$\frac{1}{2}$
3	1	3	$\frac{1}{6}$
4	1	5	$\frac{1}{48}$
4	3	3	$\frac{1}{16}$
5	1	7	$\frac{1}{720}$
5	3	5	$\frac{1}{120}$
6	1	9	$\frac{1}{17280}$
6	3	7	$\frac{1}{1728}$
6	5	5	$\frac{1}{864}$

$$I = \langle \tau_{2n,0}\tau_{2d-2n-1,1}\rangle_{0,d}$$

$$I = \langle \tau_{2n-1,0}\tau_{2d-2n,1}\rangle_{0,d}$$

d	$2n$	$2d-2n-1$	I
1	0	1	1
2	0	3	$\frac{1}{4}$
2	2	1	$-\frac{3}{4}$
3	0	5	$\frac{1}{36}$
3	2	3	$-\frac{2}{9}$
3	4	1	$-\frac{17}{36}$
4	0	7	$\frac{1}{576}$
4	2	5	$-\frac{5}{192}$
4	4	3	$-\frac{11}{64}$
4	6	1	$-\frac{43}{576}$
5	0	9	$\frac{1}{14400}$
5	2	7	$-\frac{1}{600}$
5	4	5	$-\frac{9}{400}$
5	6	3	$-\frac{53}{1800}$
5	8	1	$-\frac{247}{43200}$

d	$2n-1$	$2d-2n$	I
1	1	0	-1
2	1	2	$-\frac{1}{4}$
2	3	0	$-\frac{5}{4}$
3	1	4	$-\frac{1}{36}$
3	3	2	$-\frac{7}{9}$
3	5	0	$-\frac{5}{18}$
4	1	6	$-\frac{1}{576}$
4	3	4	$-\frac{9}{64}$
4	5	2	$-\frac{13}{64}$
4	7	0	$-\frac{47}{1728}$
5	1	8	$-\frac{1}{14400}$
5	3	6	$-\frac{11}{900}$
5	5	4	$-\frac{1}{25}$
5	7	2	$-\frac{29}{1350}$
5	9	0	$-\frac{131}{86400}$

3-Point Descendants

$$I = \langle \tau_{m,0} \tau_{n,0} \tau_{\ell,0} \rangle_0 \qquad\qquad I = \langle \tau_{m,1} \tau_{n,1} \tau_{\ell,1} \rangle_0$$

m	n	ℓ	I
2	2	3	-2
2	2	5	$-\frac{3}{4}$
2	3	4	$-\frac{5}{2}$
2	4	5	$-\frac{15}{16}$
3	3	3	-8
3	3	5	-3
3	4	4	$-\frac{25}{8}$
3	5	5	$-\frac{9}{8}$
4	4	5	$-\frac{75}{64}$
5	5	5	$-\frac{27}{64}$

m	n	ℓ	I
1	1	2	1
1	1	4	$\frac{1}{4}$
1	2	3	$\frac{1}{2}$
1	2	5	$\frac{1}{12}$
1	3	4	$\frac{1}{8}$
1	4	5	$\frac{1}{48}$
2	2	2	1
2	2	4	$\frac{1}{4}$
2	3	3	$\frac{1}{4}$
2	3	5	$\frac{1}{24}$
2	4	4	$\frac{1}{16}$
2	5	5	$\frac{1}{144}$
3	3	4	$\frac{1}{16}$
3	4	5	$\frac{1}{96}$
4	4	4	$\frac{1}{64}$
4	5	5	$\frac{1}{576}$
5	5	4	$\frac{1}{576}$

$$I = \left\langle \tau_{m,1}\tau_{n,0}\tau_{\ell,0} \right\rangle_0 \qquad\qquad I = \left\langle \tau_{m,1}\tau_{n,1}\tau_{\ell,0} \right\rangle_0$$

m	n	ℓ	I
1	2	3	2
1	2	5	$\frac{3}{4}$
1	3	4	$\frac{5}{2}$
1	4	5	$\frac{15}{16}$
2	2	2	1
2	2	4	$\frac{5}{4}$
2	3	3	4
2	3	5	$\frac{3}{2}$
2	4	4	$\frac{25}{16}$
2	5	5	$\frac{9}{16}$
3	2	3	1
3	2	5	$\frac{3}{8}$
3	3	4	$\frac{5}{4}$
3	4	5	$\frac{15}{32}$
4	2	2	$\frac{1}{4}$
4	2	4	$\frac{5}{16}$
4	3	3	1
4	3	5	$\frac{3}{8}$
4	4	4	$\frac{25}{64}$
4	5	5	$\frac{9}{64}$
5	2	3	$\frac{1}{6}$
5	2	5	$\frac{1}{16}$
5	3	4	$\frac{5}{24}$
5	4	5	$\frac{5}{64}$

m	n	ℓ	I
1	1	3	-2
1	1	5	$-\frac{3}{4}$
1	2	2	-1
1	2	4	$-\frac{5}{4}$
1	3	3	-1
1	3	5	$-\frac{3}{8}$
1	4	2	$-\frac{1}{4}$
1	4	4	$-\frac{5}{16}$
1	5	3	$-\frac{1}{6}$
1	5	5	$-\frac{1}{16}$
2	2	3	-2
2	2	5	$-\frac{3}{4}$
2	3	2	$-\frac{1}{2}$
2	3	4	$-\frac{5}{8}$
2	4	3	$-\frac{1}{2}$
2	4	5	$-\frac{3}{16}$
2	5	2	$-\frac{1}{12}$
2	5	4	$-\frac{5}{48}$
3	3	3	$-\frac{1}{2}$
3	3	5	$-\frac{3}{16}$
3	4	2	$-\frac{1}{8}$
3	4	4	$-\frac{5}{32}$
3	5	3	$-\frac{1}{12}$
3	5	5	$-\frac{1}{32}$
4	4	3	$-\frac{1}{8}$
4	4	5	$-\frac{3}{64}$
4	5	2	$-\frac{1}{48}$
4	5	4	$-\frac{5}{192}$
5	5	3	$-\frac{1}{72}$
5	5	5	$-\frac{1}{192}$

C.2   Genus-One Descendants

1-Point Descendants

n	$\langle \tau_{n,0} \rangle_1$
1	$\frac{1}{12}$
5	$-\frac{7}{96}$
7	$-\frac{49}{2592}$
9	$-\frac{163}{82944}$
11	$-\frac{391}{3456000}$
13	$-\frac{173}{41472000}$
15	$-\frac{4579}{42674688000}$
17	$-\frac{2227}{1092472012800}$
19	$-\frac{118673}{3982060486656000}$
21	$-\frac{137719}{398206048665600000}$

n	$\langle \tau_{n,1} \rangle_1$
0	$-\frac{1}{24}$
2	$\frac{1}{24}$
4	$\frac{1}{32}$
6	$\frac{5}{864}$
8	$\frac{7}{13824}$
10	$\frac{1}{38400}$
12	$\frac{11}{12441600}$
14	$\frac{13}{609638400}$
16	$\frac{1}{2601123840}$
18	$\frac{17}{3160365465600}$
20	$\frac{19}{316036546560000}$

1-Point Hodge Integrals

m	$\langle \tau_{m,0} \lambda_1 \rangle_1$
0	$\frac{1}{12}$
2	$-\frac{1}{12}$
4	$-\frac{5}{48}$
6	$-\frac{5}{216}$
8	$-\frac{47}{20736}$
10	$-\frac{131}{1036800}$
12	$-\frac{71}{15552000}$
14	$-\frac{353}{3048192000}$
16	$-\frac{1487}{682795008000}$
18	$-\frac{6989}{221225582592000}$
20	$-\frac{1451}{3982060486656000}$

m	$\langle \tau_{m,1} \lambda_1 \rangle_1$
1	$\frac{1}{12}$
3	$\frac{1}{24}$
5	$\frac{1}{144}$
7	$\frac{1}{1728}$
9	$\frac{1}{34560}$
11	$\frac{1}{1036800}$
13	$\frac{1}{43545600}$
15	$\frac{1}{2438553600}$
17	$\frac{1}{175575859200}$
19	$\frac{1}{15801827328000}$
21	$\frac{1}{1738201006080000}$

2-Point Descendants

m	n	$\langle\tau_{m,0}\tau_{n,0}\rangle_1$
1	1	$\frac{1}{12}$
1	5	$-\frac{7}{96}$
1	7	$-\frac{49}{2592}$
1	9	$-\frac{163}{82944}$
2	2	$-\frac{1}{6}$
2	4	$-\frac{5}{48}$
2	6	$\frac{43}{864}$
2	8	$\frac{115}{6912}$
3	3	$-\frac{1}{12}$
3	5	$\frac{29}{96}$
3	7	$\frac{313}{2592}$
3	9	$\frac{461}{27648}$
4	4	$\frac{1}{3}$
4	6	$\frac{83}{384}$
4	8	$\frac{421}{10368}$
5	5	$\frac{29}{96}$
5	7	$\frac{817}{10368}$
5	9	$\frac{1015}{110592}$
6	6	$\frac{151}{1728}$
6	8	$\frac{20305}{1492992}$
7	7	$\frac{49}{2916}$
7	9	$\frac{7825}{4478976}$
8	8	$\frac{157}{82944}$

m	n	$\langle\tau_{m,1}\tau_{n,0}\rangle_1$
1	2	$\frac{1}{8}$
1	4	$-\frac{11}{96}$
1	6	$-\frac{107}{864}$
1	8	$-\frac{121}{4608}$
2	1	$\frac{1}{24}$
2	3	$-\frac{1}{8}$
2	5	$-\frac{1}{4}$
2	7	$-\frac{203}{2592}$
2	9	$-\frac{823}{82944}$
3	4	$-\frac{7}{32}$
3	6	$-\frac{191}{1728}$
3	8	$-\frac{791}{41472}$
4	1	$\frac{1}{32}$
4	3	$-\frac{5}{32}$
4	5	$-\frac{49}{384}$
4	7	$-\frac{53}{1728}$
4	9	$-\frac{71}{20736}$
5	2	$-\frac{7}{288}$
5	4	$-\frac{31}{384}$
5	6	$-\frac{35}{1152}$
5	8	$-\frac{2263}{497664}$
6	1	$\frac{5}{864}$
6	3	$-\frac{35}{864}$
6	5	$-\frac{43}{1728}$
6	7	$-\frac{53}{10368}$
6	9	$-\frac{1549}{2985984}$
7	2	$-\frac{1}{192}$
7	4	$-\frac{41}{3456}$
7	6	$-\frac{479}{124416}$
7	8	$-\frac{29}{55296}$
8	1	$\frac{7}{13824}$
8	3	$-\frac{7}{1536}$
8	5	$-\frac{5}{2048}$
8	7	$-\frac{85}{186624}$

m	n	$\langle\tau_{m,1}\tau_{n,1}\rangle_1$
1	3	$\frac{5}{48}$
1	5	$\frac{7}{144}$
1	7	$\frac{1}{128}$
1	9	$\frac{11}{17280}$
2	2	$\frac{1}{6}$
2	4	$\frac{11}{96}$
2	6	$\frac{11}{432}$
2	8	$\frac{37}{13824}$
3	3	$\frac{1}{8}$
3	5	$\frac{23}{576}$
3	7	$\frac{19}{3456}$
3	9	$\frac{19}{46080}$
4	4	$\frac{5}{96}$
4	6	$\frac{11}{1152}$
4	8	$\frac{25}{27648}$
4	10	$\frac{71}{1382400}$
5	5	$\frac{1}{96}$
5	7	$\frac{53}{41472}$
5	9	$\frac{37}{414720}$
6	2	$\frac{11}{432}$
6	4	$\frac{11}{1152}$
6	6	$\frac{1}{648}$
6	8	$\frac{67}{497664}$
7	7	$\frac{1}{6912}$
7	9	$\frac{19}{1990656}$
8	8	$\frac{11}{995328}$

2-Point Hodge Integrals

m	n	$\langle \tau_{m,0}\tau_{n,0}\lambda_1\rangle_1$
0	1	$\frac{1}{12}$
0	3	$-\frac{1}{12}$
0	5	$-\frac{5}{48}$
0	7	$-\frac{5}{216}$
0	9	$-\frac{47}{20736}$
1	2	$-\frac{1}{12}$
1	4	$-\frac{5}{48}$
1	6	$-\frac{5}{216}$
1	8	$-\frac{47}{20736}$
1	10	$-\frac{131}{1036800}$
2	3	$\frac{1}{12}$
2	5	$\frac{5}{48}$
2	7	$\frac{5}{216}$
2	9	$\frac{47}{20736}$
3	4	$\frac{7}{16}$
3	6	$\frac{4}{27}$
3	8	$\frac{133}{6912}$
3	10	$\frac{1381}{1036800}$
4	5	$\frac{49}{192}$
4	7	$\frac{119}{2592}$
4	9	$\frac{335}{82944}$
5	6	$\frac{131}{1728}$
5	8	$\frac{763}{82944}$
5	10	$\frac{253}{414720}$
6	7	$\frac{199}{15552}$
6	9	$\frac{1615}{1492992}$
7	8	$\frac{1673}{1119744}$
7	10	$\frac{2161}{22394880}$
8	9	$\frac{4409}{35831808}$
9	10	$\frac{9313}{1194393600}$

m	n	$\langle \tau_{m,1}\tau_{n,1}\lambda_1\rangle_1$
0	1	$\frac{1}{12}$
0	3	$\frac{1}{12}$
0	5	$\frac{1}{48}$
0	7	$\frac{1}{432}$
0	9	$\frac{1}{6912}$
1	2	$\frac{1}{6}$
1	4	$\frac{1}{16}$
1	6	$\frac{1}{108}$
1	8	$\frac{5}{6912}$
1	10	$\frac{1}{28800}$
2	3	$\frac{1}{8}$
2	5	$\frac{1}{36}$
2	7	$\frac{5}{1728}$
2	9	$\frac{1}{5760}$
3	4	$\frac{1}{24}$
3	6	$\frac{5}{864}$
3	8	$\frac{1}{2304}$
3	10	$\frac{7}{345600}$
4	5	$\frac{5}{576}$
4	7	$\frac{1}{1152}$
4	9	$\frac{7}{138240}$
5	6	$\frac{1}{864}$
5	8	$\frac{7}{82944}$
5	10	$\frac{1}{259200}$
6	7	$\frac{7}{62208}$
6	9	$\frac{1}{155520}$
7	8	$\frac{1}{124416}$
7	10	$\frac{1}{2764800}$
8	9	$\frac{1}{2211840}$
9	10	$\frac{1}{49766400}$

m	n	$\langle \tau_{m,0}\tau_{n,1}\lambda_1\rangle_1$
0	2	$\frac{1}{12}$
0	4	$\frac{1}{24}$
0	6	$\frac{1}{144}$
1	1	$\frac{1}{12}$
1	3	$\frac{1}{24}$
1	5	$\frac{1}{144}$
2	2	$-\frac{1}{12}$
2	4	$-\frac{1}{24}$
2	6	$-\frac{1}{144}$
3	1	$-\frac{1}{4}$
3	3	$-\frac{5}{24}$
3	5	$-\frac{7}{144}$
4	0	$-\frac{1}{6}$
4	2	$-\frac{13}{48}$
4	4	$-\frac{3}{32}$
4	6	$-\frac{23}{1728}$
5	1	$-\frac{1}{6}$
5	3	$-\frac{11}{96}$
5	5	$-\frac{7}{288}$
6	0	$-\frac{1}{16}$
6	2	$-\frac{37}{432}$
6	4	$-\frac{47}{1728}$
6	6	$-\frac{19}{5184}$

3-Point Descendants

$I = \langle \tau_{m,0}\tau_{n,0}\tau_{\ell,0}\rangle_1$

m	n	ℓ	I
2	2	3	$\frac{1}{4}$
2	2	5	$\frac{17}{48}$
2	3	4	$\frac{23}{24}$
2	3	6	$-\frac{17}{216}$
2	4	5	$-\frac{21}{128}$
2	5	6	$-\frac{1297}{3456}$
3	3	3	$\frac{4}{3}$
3	3	5	$-\frac{71}{48}$
3	4	4	$-\frac{319}{192}$
3	4	6	$-\frac{7}{4}$
3	5	5	$-\frac{473}{192}$
3	6	6	$-\frac{7159}{7776}$
4	4	5	$-\frac{187}{64}$
4	5	6	$-\frac{21331}{13824}$
5	5	5	$-\frac{1681}{768}$
5	6	6	$-\frac{1105}{1728}$

$I = \langle \tau_{m,1}\tau_{n,1}\tau_{\ell,1}\rangle_1$

m	n	ℓ	I
1	1	2	$\frac{7}{24}$
1	1	4	$\frac{3}{8}$
1	1	6	$\frac{11}{96}$
1	2	3	$\frac{31}{48}$
1	2	5	$\frac{85}{288}$
1	3	4	$\frac{41}{96}$
1	3	6	$\frac{175}{1728}$
1	4	5	$\frac{83}{576}$
1	5	6	$\frac{11}{384}$
2	2	2	$\frac{25}{24}$
2	2	4	$\frac{17}{24}$
2	2	6	$\frac{149}{864}$
2	3	3	$\frac{25}{32}$
2	3	5	$\frac{157}{576}$
2	4	4	$\frac{137}{384}$
2	4	6	$\frac{7}{96}$
2	5	5	$\frac{275}{3456}$
2	6	6	$\frac{409}{31104}$
3	3	4	$\frac{19}{48}$
3	3	6	$\frac{31}{384}$
3	4	5	$\frac{67}{576}$
3	5	6	$\frac{437}{20736}$
4	4	4	$\frac{59}{384}$
4	4	6	$\frac{389}{13824}$
4	5	5	$\frac{53}{1728}$
4	6	6	$\frac{49}{10368}$
5	5	6	$\frac{71}{13824}$
5	6	5	$\frac{71}{13824}$
6	6	6	$\frac{31}{41472}$

$I = \langle \tau_{m,1}\tau_{n,0}\tau_{\ell,0}\rangle_1$

m	n	ℓ	I
1	2	3	$-\frac{7}{12}$
1	2	5	$-\frac{7}{96}$
1	3	4	$\frac{3}{8}$
1	3	6	$\frac{47}{54}$
1	4	5	$\frac{281}{192}$
1	5	6	$\frac{3091}{3456}$
2	2	2	$-\frac{7}{24}$
2	2	4	$-\frac{1}{4}$
2	2	6	$\frac{185}{864}$
2	3	3	$\frac{1}{4}$
2	3	5	$\frac{41}{24}$
2	4	4	$\frac{767}{384}$
2	4	6	$\frac{4631}{3456}$
2	5	5	$\frac{727}{384}$
2	6	6	$\frac{4799}{7776}$
3	2	3	$-\frac{1}{4}$
3	2	5	$\frac{49}{192}$
3	3	4	$\frac{139}{96}$
3	3	6	$\frac{73}{72}$
3	4	5	$\frac{217}{128}$
3	5	6	$\frac{611}{768}$
4	2	2	$-\frac{5}{48}$
4	2	4	$\frac{77}{384}$
4	2	6	$\frac{215}{1152}$
4	3	3	$\frac{47}{48}$
4	3	5	$\frac{37}{32}$
4	4	4	$\frac{527}{384}$
4	4	6	$\frac{995}{1536}$
4	5	5	$\frac{709}{768}$
4	6	6	$\frac{7903}{31104}$
5	2	3	$\frac{13}{144}$
5	2	5	$\frac{187}{1152}$
5	3	4	$\frac{23}{32}$
5	3	6	$\frac{443}{1296}$
5	4	5	$\frac{661}{1152}$
5	5	6	$\frac{3125}{13824}$
6	2	2	$\frac{5}{864}$
6	2	4	$\frac{83}{864}$
6	2	6	$\frac{1501}{31104}$
6	3	3	$\frac{151}{432}$
6	3	5	$\frac{5}{18}$
6	4	4	$\frac{1513}{4608}$
6	4	6	$\frac{15899}{124416}$
6	5	5	$\frac{283}{1536}$
6	6	6	$\frac{1393}{31104}$

$$I = \langle \tau_{m,1} \tau_{n,1} \tau_{\ell,0} \rangle_1$$

m	n	ℓ	I
1	1	3	$\frac{1}{12}$
1	1	5	$-\frac{2}{3}$
1	2	2	$\frac{1}{4}$
1	2	4	$-\frac{27}{32}$
1	2	6	$-\frac{631}{864}$
1	3	3	$-\frac{7}{12}$
1	3	5	$-\frac{11}{12}$
1	4	2	$-\frac{5}{96}$
1	4	4	$-\frac{71}{96}$
1	4	6	$-\frac{1339}{3456}$
1	5	3	$-\frac{55}{144}$
1	5	5	$-\frac{197}{576}$
1	6	2	$-\frac{7}{144}$
1	6	4	$-\frac{227}{1152}$
1	6	6	$-\frac{841}{10368}$
2	2	3	$-\frac{11}{12}$
2	2	5	$-\frac{3}{2}$
2	3	2	$-\frac{1}{24}$
2	3	4	$-\frac{253}{192}$
2	3	6	$-\frac{1255}{1728}$
2	4	3	$-\frac{43}{48}$
2	4	5	$-\frac{107}{128}$
2	5	2	$-\frac{1}{9}$
2	5	4	$-\frac{599}{1152}$
2	5	6	$-\frac{2315}{10368}$
2	6	3	$-\frac{109}{432}$
2	6	5	$-\frac{157}{864}$
3	3	3	$-\frac{47}{48}$
3	3	5	$-\frac{89}{96}$
3	4	2	$-\frac{31}{192}$
3	4	4	$-\frac{289}{384}$
3	4	6	$-\frac{2251}{6912}$
3	5	3	$-\frac{19}{48}$
3	5	5	$-\frac{37}{128}$
3	6	2	$-\frac{47}{864}$
3	6	4	$-\frac{379}{2304}$
3	6	6	$-\frac{3803}{62208}$
4	4	3	$-\frac{33}{64}$
4	4	5	$-\frac{73}{192}$
4	5	2	$-\frac{29}{384}$
4	5	4	$-\frac{181}{768}$
4	5	6	$-\frac{3679}{41472}$
4	6	3	$-\frac{197}{1728}$
4	6	5	$-\frac{1001}{13824}$
5	5	3	$-\frac{215}{1728}$
5	5	5	$-\frac{91}{1152}$
5	6	2	$-\frac{23}{1296}$
5	6	4	$-\frac{1847}{41472}$
5	6	6	$-\frac{631}{41472}$
6	6	3	$-\frac{335}{15552}$
6	6	5	$-\frac{97}{7776}$

C.3   Genus-Two Descendants

1-Point Descendants

n	$\langle \tau_{n,0} \rangle_2$
3	$-\dfrac{1}{240}$
5	$\dfrac{1}{576}$
7	$-\dfrac{61}{23040}$
9	$-\dfrac{977}{622080}$
11	$-\dfrac{551}{2211840}$
13	$-\dfrac{15913}{829440000}$
15	$-\dfrac{26407}{29859840000}$
17	$-\dfrac{39911}{1463132160000}$
19	$-\dfrac{158371}{262193283072000}$
21	$-\dfrac{9635659}{9556945167797440000}$
23	$-\dfrac{4189489}{31856483893248000000}$
25	$-\dfrac{8347673}{6057282865987584000000}$

n	$\langle \tau_{n,1} \rangle_2$
2	$\dfrac{7}{5760}$
4	$\dfrac{1}{1920}$
6	$\dfrac{13}{7680}$
8	$\dfrac{23}{41472}$
10	$\dfrac{77}{1105920}$
12	$\dfrac{43}{9216000}$
14	$\dfrac{583}{2985984000}$
16	$\dfrac{13}{2322432000}$
18	$\dfrac{73}{624269721600}$
20	$\dfrac{1411}{758487711744000}$
22	$\dfrac{589}{25282923724800000}$
24	$\dfrac{103}{437033395814400000}$

2-Point Descendants

m	n	$\langle \tau_{m,0}\tau_{n,0}\rangle_2$
2	2	$-\dfrac{5}{288}$
2	4	$\dfrac{41}{2880}$
2	6	$-\dfrac{59}{2560}$
2	8	$-\dfrac{83}{10368}$
2	10	$-\dfrac{77}{2211840}$
3	3	$\dfrac{29}{1440}$
3	5	$\dfrac{271}{7680}$
3	7	$\dfrac{211}{62208}$
3	9	$\dfrac{75019}{6635520}$
4	4	$-\dfrac{49}{1440}$
4	6	$\dfrac{1589}{69120}$
4	8	$\dfrac{28027}{829440}$
4	10	$\dfrac{4227469}{497664000}$
5	5	$\dfrac{161}{3840}$
5	7	$\dfrac{21173}{311040}$
5	9	$\dfrac{139061}{6635520}$
6	6	$\dfrac{33377}{414720}$
6	8	$\dfrac{54833}{1658880}$
6	10	$\dfrac{19399463}{3583180800}$
7	7	$\dfrac{9899}{248832}$
7	9	$\dfrac{5954869}{716636160}$
8	8	$\dfrac{185363}{19906560}$
8	10	$\dfrac{7319723}{5971968000}$
9	9	$\dfrac{375793}{268738560}$
10	10	$\dfrac{60039767}{429981696000}$

m	n	$\langle \tau_{m,1}\tau_{n,0}\rangle_2$
1	2	$\dfrac{7}{1920}$
1	4	$\dfrac{1}{384}$
1	6	$-\dfrac{9}{2560}$
1	8	$-\dfrac{901}{69120}$
1	10	$-\dfrac{4499}{1105920}$
2	3	$\dfrac{1}{192}$
2	5	$-\dfrac{47}{7680}$
2	7	$-\dfrac{29}{864}$
2	9	$\dfrac{139307}{9953280}$
3	2	$\dfrac{1}{192}$
3	4	$\dfrac{13}{4608}$
3	6	$\dfrac{4931}{103680}$
3	8	$\dfrac{30409}{1105920}$
3	10	$-\dfrac{672173}{124416000}$
4	3	$\dfrac{89}{7680}$
4	5	$-\dfrac{1117}{23040}$
4	7	$\dfrac{34397}{829440}$
4	9	$\dfrac{69811}{6635520}$
5	2	$\dfrac{91}{7680}$
5	4	$-\dfrac{49}{1920}$
5	6	$-\dfrac{423}{10240}$
5	8	$\dfrac{142513}{9953280}$
5	10	$-\dfrac{3204979}{1492992000}$
6	3	$-\dfrac{811}{103680}$
6	5	$-\dfrac{5107}{165888}$
6	7	$-\dfrac{353}{23040}$
6	9	$\dfrac{35275}{11943936}$
7	2	$-\dfrac{23}{11520}$
7	4	$-\dfrac{1303}{92160}$
7	6	$-\dfrac{1613}{138240}$
7	8	$-\dfrac{40423}{13271040}$
7	10	$-\dfrac{126889}{331776000}$
8	3	$-\dfrac{37}{8192}$
8	5	$-\dfrac{1525}{221184}$
8	7	$-\dfrac{302807}{119439360}$
8	9	$-\dfrac{292267}{716636160}$
9	2	$-\dfrac{121}{1105920}$
9	4	$-\dfrac{43769}{16588800}$
9	6	$-\dfrac{186277}{119439360}$
9	8	$-\dfrac{33649}{99532800}$
9	10	$-\dfrac{1071761}{28665446400}$
10	3	$-\dfrac{10163}{13824000}$
10	5	$\dfrac{253273}{331776000}$
10	7	$-\dfrac{86129}{373248000}$
10	9	$-\dfrac{1557227}{47775744000}$

m	n	$\langle \tau_{m,1}\tau_{n,1}\rangle_2$
1	5	$\dfrac{91}{11520}$
1	7	$\dfrac{23}{3840}$
1	9	$\dfrac{121}{92160}$
2	2	$\dfrac{1}{576}$
2	4	$\dfrac{71}{3840}$
2	6	$\dfrac{67}{3840}$
2	8	$\dfrac{809}{165888}$
2	10	$\dfrac{1711}{2764800}$
3	3	$\dfrac{25}{1152}$
3	5	$\dfrac{1001}{34560}$
3	7	$\dfrac{319}{30720}$
3	9	$\dfrac{209}{129600}$
4	4	$\dfrac{421}{11520}$
4	6	$\dfrac{149}{8640}$
4	8	$\dfrac{3697}{1105920}$
4	10	$\dfrac{56371}{165888000}$
5	5	$\dfrac{451}{23040}$
5	7	$\dfrac{4061}{829440}$
5	9	$\dfrac{60523}{99532800}$
6	6	$\dfrac{791}{138240}$
6	8	$\dfrac{17651}{19906560}$
6	10	$\dfrac{38407}{497664000}$
7	7	$\dfrac{1627}{1658880}$
7	9	$\dfrac{3467}{331776000}$
8	8	$\dfrac{7037}{59719680}$
8	10	$\dfrac{1823}{199065600}$
9	9	$\dfrac{2377}{238878720}$
10	10	$\dfrac{15589}{23887872000}$

3-Point Descendants

$$I = \left\langle \tau_{m,0} \tau_{n,0} \tau_{\ell,0} \right\rangle_2 \qquad\qquad I = \left\langle \tau_{m,1} \tau_{n,1} \tau_{\ell,1} \right\rangle_2$$

m	n	ℓ	I
2	2	3	$\frac{11}{96}$
2	2	5	$-\frac{239}{1280}$
2	3	4	$-\frac{1411}{5760}$
2	3	6	$\frac{4211}{25920}$
2	4	5	$\frac{7147}{23040}$
2	5	6	$\frac{162199}{414720}$
2	6	5	$\frac{162199}{414720}$
3	3	3	$-\frac{1}{3}$
3	3	5	$\frac{4}{9}$
3	4	4	$\frac{569}{1152}$
3	4	6	$\frac{1247}{5760}$
3	5	5	$\frac{2017}{11520}$
3	6	6	$-\frac{43471}{69120}$
4	4	5	$\frac{971}{15360}$
4	5	6	$-\frac{1040729}{829440}$
5	5	5	$-\frac{1709}{960}$
5	6	6	$-\frac{1547117}{995328}$

m	n	ℓ	I
1	1	4	$\frac{61}{1920}$
1	1	6	$\frac{113}{1920}$
1	2	3	$\frac{35}{576}$
1	2	5	$\frac{413}{2880}$
1	3	4	$\frac{821}{3840}$
1	3	6	$\frac{13}{90}$
1	4	5	$\frac{589}{2880}$
1	5	6	$\frac{18247}{207360}$
2	2	2	$\frac{19}{192}$
2	2	4	$\frac{1217}{3840}$
2	2	6	$\frac{22547}{103680}$
2	3	3	$\frac{107}{288}$
2	3	5	$\frac{12473}{34560}$
2	4	4	$\frac{5203}{11520}$
2	4	6	$\frac{27499}{138240}$
2	5	5	$\frac{1561}{6912}$
2	6	6	$\frac{9187}{138240}$
3	3	4	$\frac{12193}{23040}$
3	3	6	$\frac{48271}{207360}$
3	4	5	$\frac{22799}{69120}$
3	5	6	$\frac{22859}{207360}$
4	4	4	$\frac{4217}{10240}$
4	4	6	$\frac{38239}{276480}$
4	5	5	$\frac{130099}{829440}$
4	6	6	$\frac{38257}{995328}$
5	5	6	$\frac{36073}{829440}$
6	6	6	$\frac{15431}{1658880}$

$$I = \langle \tau_{m,1}\tau_{n,0}\tau_{\ell,0} \rangle_2 \qquad\qquad I = \langle \tau_{m,1}\tau_{n,1}\tau_{\ell,0} \rangle_2$$

m	n	ℓ	I
1	2	3	$\frac{1}{32}$
1	2	5	$-\frac{317}{5760}$
1	3	4	$-\frac{361}{5760}$
1	3	6	$-\frac{6703}{51840}$
1	4	5	$-\frac{2021}{23040}$
1	5	6	$\frac{48073}{103680}$
2	2	2	$\frac{23}{576}$
2	2	4	$-\frac{47}{1280}$
2	2	6	$-\frac{4259}{17280}$
2	3	3	$-\frac{7}{160}$
2	3	5	$-\frac{35}{144}$
2	4	4	$-\frac{2381}{11520}$
2	4	6	$\frac{7969}{15360}$
2	5	5	$\frac{18077}{23040}$
2	6	6	$\frac{133319}{138240}$
3	2	3	$\frac{11}{576}$
3	2	5	$-\frac{3521}{11520}$
3	3	4	$-\frac{107}{360}$
3	3	6	$\frac{4081}{17280}$
3	4	5	$\frac{27301}{46080}$
3	5	6	$\frac{48989}{41472}$
3	6	5	$\frac{48989}{41472}$
4	2	2	$\frac{53}{768}$
4	2	4	$-\frac{1399}{5760}$
4	2	6	$\frac{20933}{138240}$
4	3	3	$-\frac{383}{1440}$
4	3	5	$\frac{331}{1536}$
4	4	4	$\frac{6605}{18432}$
4	4	6	$\frac{147143}{165888}$
4	5	5	$\frac{37829}{30720}$
4	6	6	$\frac{2502289}{2985984}$
5	2	3	$-\frac{413}{3456}$
5	2	5	$-\frac{9617}{69120}$
5	3	4	$\frac{179}{2560}$
5	3	6	$\frac{1805}{4608}$
5	4	3	$\frac{179}{2560}$
5	4	5	$\frac{101917}{138240}$
5	5	6	$\frac{596827}{829440}$
6	2	2	$-\frac{373}{34560}$
6	2	4	$-\frac{7121}{82944}$
6	2	6	$\frac{101}{23040}$
6	3	3	$-\frac{143}{10368}$
6	3	5	$\frac{9827}{34560}$
6	4	4	$\frac{320921}{829440}$
6	4	6	$\frac{5923543}{14929920}$
6	5	5	$\frac{28171}{51840}$
6	6	6	$\frac{4004861}{14929920}$

m	n	ℓ	I
1	1	5	$\frac{17}{720}$
1	2	2	$\frac{1}{192}$
1	2	4	$\frac{61}{1152}$
1	2	6	$-\frac{397}{2304}$
1	3	3	$\frac{17}{192}$
1	3	5	$-\frac{551}{2880}$
1	4	2	$\frac{151}{1920}$
1	4	4	$-\frac{71}{720}$
1	4	6	$-\frac{25687}{69120}$
1	5	3	$\frac{7}{4320}$
1	5	5	$-\frac{4201}{13824}$
1	6	2	$\frac{167}{3840}$
1	6	4	$-\frac{533}{3456}$
1	6	6	$-\frac{26251}{138240}$
2	2	3	$\frac{11}{96}$
2	2	5	$-\frac{3283}{11520}$
2	3	2	$\frac{17}{128}$
2	3	4	$-\frac{25}{144}$
2	3	6	$-\frac{26813}{41472}$
2	4	3	$-\frac{1}{180}$
2	4	5	$-\frac{1019}{1536}$
2	5	2	$\frac{1267}{11520}$
2	5	4	$-\frac{1733}{4608}$
2	5	6	$-\frac{199573}{414720}$
2	6	3	$\frac{6767}{51840}$
2	6	5	$-\frac{3313}{9216}$
3	3	3	$-\frac{1}{144}$
3	3	5	$-\frac{8957}{11520}$
3	4	2	$\frac{913}{5760}$
3	4	4	$-\frac{3203}{5760}$
3	4	6	$-\frac{29155}{41472}$
3	5	3	$\frac{2501}{11520}$
3	5	5	$-\frac{82409}{138240}$
3	6	2	$\frac{413}{13824}$
3	6	4	$-\frac{44633}{138240}$
3	6	6	$-\frac{646351}{2488320}$
4	4	3	$-\frac{11}{48}$
4	4	5	$-\frac{11419}{15360}$
4	5	2	$\frac{973}{23040}$
4	5	4	$-\frac{126253}{276480}$
4	5	6	$-\frac{458771}{1244160}$
4	6	3	$-\frac{6419}{34560}$
4	6	5	$-\frac{58183}{207360}$
5	5	3	$-\frac{27}{128}$
5	5	5	$-\frac{1223}{3840}$
5	6	2	$-\frac{559}{82944}$
5	6	4	$-\frac{2747}{15552}$
5	6	6	$-\frac{60387}{552960}$
6	6	3	$-\frac{47233}{622080}$
6	6	5	$-\frac{1255591}{14929920}$

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