

# Quartic Gauge Couplings from $K3$ Geometry

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## Abstract

We show how certain  $F^4$  couplings in eight dimensions can be computed using the mirror map and  $K3$  data. They perfectly match with the corresponding heterotic one-loop couplings, and therefore this amounts to a successful test of the conjectured duality between the heterotic string on  $T^2$  and  $F$ -theory on  $K3$ . The underlying quantum geometry appears to be a 5-fold, consisting of a hyperkähler 4-fold fibered over a  $\mathbb{P}^1$  base. The natural candidate for this fiber is the symmetric product  $\text{Sym}^2(K3)$ . We are lead to this structure by analyzing the implications of higher powers of  $E_2$  in the relevant Borchers counting functions, and in particular the appropriate generalizations of the Picard-Fuchs equations for the  $K3$ .

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# 1 Introduction

We consider certain threshold corrections  $\Delta(T, U)$  to  $F^4$  couplings in eight dimensional string compactifications with  $N = 1$  supersymmetry. Such theories are obtained from the heterotic string compactified on  $T^2$  (with moduli  $T, U$  plus 16 Wilson lines that we will suppress), or dually, from  $F$ -theory [1] compactified on elliptic fibered  $K3$ 's. Threshold corrections of this kind have been considered by various authors, either from the heterotic string point of view or from the dual Type I string perspective [14], [15], [25], [43]. Furthermore, an attempt was made in [25] to compute these couplings from  $K3$  geometry in  $F$ -theory; it is the purpose of the present paper to extend and improve upon this approach.

The motivation for studying this subject is, of course, not that eight dimensions would be phenomenologically very important, but rather that we expect to learn more about how to do exact non-perturbative computations in  $D$ -brane physics.<sup>2</sup> Experience suggests that whenever we study BPS-saturated couplings [20], [4], [13] in an effective action, there should be a purely geometrical method for computing them. Indeed, we will argue that there is a beautiful structure behind the 7-brane interactions in eight dimensions: the relevant quantum geometry appears to be a 5-fold, given by a fibration of a hyperkähler 4-fold over a  $\mathbb{P}^1$  base. This 4-fold is nothing but the symmetric product  $\text{Sym}^2(K3) \equiv \frac{K3 \otimes K3}{S_2}$  of the underlying  $K3$ .

For simplicity, we will focus in this paper only on a certain class of couplings for one-parameter families of elliptic  $K3$ 's, and intend to present a more thorough geometrical treatment in a companion paper [30]. We will consider couplings of the form

$$\text{Re}[\Delta_{G_1 G_2}(T)] \text{Tr}[F_{G_1} \wedge F_{G_1}] \wedge \text{Tr}[F_{G_2} \wedge F_{G_2}] , \quad (1.1)$$

where  $G_{1,2}$  are non-abelian gauge groups (e.g.,  $E_8$ ). There is no holomorphic prepotential underlying this kind of coupling. Recall that it is only the  $U(1)$  couplings of the form  $\text{Re}[\Delta_{TTUU}]F_T^2 F_U^2$  etc. that possess an underlying holomorphic prepotential, *i.e.*,  $\Delta_{TTUU} \sim \partial_T^2 \partial_U^2 \mathcal{G}(T, U)$  [25]. The latter class of couplings, and their prepotentials will be discussed in [30].

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<sup>2</sup>Other interesting aspects of  $D = 8$  theories have been recently discussed in [51].

The situation in eight dimensions is analogous to the more familiar  $N = 2$  supersymmetric theories in four dimensions, which are obtained from the heterotic string on  $K3 \times T^2$  and from the type IIA/B strings on Calabi-Yau 3-folds: there is no holomorphic prepotential,  $\mathcal{F}$ , for couplings of the form  $\text{Re}[\Delta_{N=2}^{\frac{d=4}{N=2}}(T, U)] \text{Tr}[F_G \wedge F_G]$ ,<sup>3</sup> whereas there is such a prepotential for the couplings of the  $U(1)$  gauge fields  $F_T$  and  $F_U$ .

More explicitly, the Wilsonian one-loop heterotic string threshold corrections in four dimensions, after performing the modular integrations, can be expressed in terms of Borcherds modular products [10], [47], [49], [5], [9], [48]

$$\begin{aligned} \Delta_{N=2}^{\frac{d=4}{N=2}}(T, U) &= \log[\Psi] , & \text{where} & & (1.2) \\ \Psi &= (q_T)^a (q_U)^b \prod_{(k,l)>0} (1 - q_T^k q_U^l)^{c(kl)} , \end{aligned}$$

for some  $a, b$ . Here,  $q_T = e^{2\pi iT}$ ,  $q_U = e^{2\pi iU}$ , the product runs over  $k > 0, l \in \mathbb{Z} \wedge k = 0, l > 0$  in the chamber  $T_2 \equiv \text{Im}T > U_2 \equiv \text{Im}U$ , and  $c(n)$  are the expansion coefficients of a certain nearly holomorphic and quasi-modular form,<sup>4</sup>  $\mathcal{C}(\tau) = \sum c(n)q^n$ . The precise form of the “counting function”  $\mathcal{C}$ , depends on the model and specific gauge group factor that is considered [46].

In spite of the lack of a prepotential, there is a natural geometric formulation of the four dimensional couplings  $\Delta_{N=2}^{\frac{d=4}{N=2}}(T, U)$ , and this still involves the mirror map, and is closely related to the counting of elliptic curves.<sup>5</sup> More precisely, the four-dimensional couplings are sections of a line bundle, which can be trivialized at large Kähler structures using the mirror map  $t_k(z_l)$  and the fundamental period  $\varpi_0$ . Following an argument given in [40],  $\Psi \det_{kl}(\frac{\partial t_k}{\partial z_l}) \varpi_0^{3+h_{1,1}-\chi/12}$  is an invariant ratio of

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<sup>3</sup>We consider only the *perturbative* one-loop piece in four dimensions, and send the dilaton to weak coupling, i.e.,  $e^{-4\pi S} \rightarrow 0$ . In eight dimensions the heterotic one-loop result is supposed to be exact [14], [15].

<sup>4</sup>A modular function is called *nearly* holomorphic if it is meromorphic with poles only at cusps ( $\tau = i\infty$  for  $SL(2, \mathbb{Z})$ ), and we will call such a form *quasi*-modular if it can be written in the form  $\mathcal{C}(\tau) = P(E_2, E_4, E_6)/\Delta^m$ , where  $P$  is some (quasi-homogeneous) polynomial,  $\Delta = \eta^{24}(\tau)$ , and where  $E_n$  are the familiar Eisenstein functions.

<sup>5</sup>The prepotentials for the couplings of  $F_T$  and  $F_U$  are, of course, related to the counting of rational curves.

sections, whose only singularities can be on the discriminant locus of the CY 3-fold. Thus, denoting the components of the discriminant by  $D_i$  and taking the logarithm, we know from this general reasoning that the couplings can be written in the form:

$$\Delta_{N=2}^{d=4} = \log \left[ \prod_i \left( D_i(z)^{\alpha_i} \right) \det_{kl} \left( \frac{\partial z_l}{\partial t_k} \right) \varpi_0^{\chi/12-3-h_{1,1}} \right] \quad (1.3)$$

This has the same form as the topological partition function  $\mathcal{F}_1$  [31], which counts elliptic curves in the 3-fold. The couplings differ from  $\mathcal{F}_1$  in the values of the discriminant exponents  $\alpha_i$ , but we see here the sense in which the threshold couplings are related to the counting of elliptic curves. In practice, there is no easy way to determine the  $\alpha_i$ , other than by matching the asymptotic behaviour of (1.3) and (1.2) at large Kähler structures.

By performing the relevant heterotic one-loop modular integrals [14]-[25], it turns out that the threshold couplings  $\Delta_{G_1 G_2}(T, U)$  in eight dimensions have a product representation that is completely analogous to the four-dimensional expression in (1.2). One thus may expect that there should be some way to compute these expressions geometrically, similar in spirit to (1.3). It is the purpose of the present paper to show that this expectation bears out, by showing that the  $F_{G_1}{}^2 F_{G_2}{}^2$  threshold corrections can be represented in a way analogous to (1.3) (where again a few parameters  $\alpha_i$  need to be matched against the heterotic one-loop result). Our results make major use of, and indeed generalize the mirror map of the relevant  $K3$  surface, and once again, the threshold corrections are related to counting elliptic curves in  $K3$ .

In the next section, we will first analyze the structure of the relevant Borchers products that underlie the heterotic one-loop couplings, for  $G_{1,2} = E_8$ . The novel feature as compared to the well-known four-dimensional story is the appearance of  $E_2^2$  in the counting functions. In Section 2.2 we translate this into properties of the Picard-Fuchs system that the geometrical ( $F$ -theory) formulation of the problem must provide. In Section 2.3 we generalize this to a whole sequence of models with different gauge symmetries, which have essentially the same structure. In Section 3 we then interpret the inhomogenous Picard-Fuchs equations of Section 2.2 in terms of geometry, and are thereby naturally lead to symmetric products of  $K3$  and their fibrations.

Finally, in appendix A we discuss some properties of quasi-modular Borcherds products, while in appendix B we present a streamlined technique for the computation of the heterotic one-loop couplings. Here we also show that these couplings can be obtained concisely in terms of a generating function that has an intriguing interpretation in terms of  $D$ -strings.

## 2 Borcherds products and mirror map

### 2.1 Building blocks

To simplify the discussion, we focus here on the model with  $E_8 \times E_8$  non-abelian gauge symmetry and fixed modulus  $U = \rho \equiv e^{2\pi i/3}$ ; we will later show how our arguments can easily be generalized to a whole series of one-parameter models.

An algebraic representation of the relevant singular  $K3$  with two  $E_8$  singularities is given by

$$W(x, y, \xi) = y^2 + x^3 + \xi^5(\xi - 1)(\xi - z^*(\tau)) = 0. \quad (2.1)$$

The mirror map, namely the map to the flat coordinate  $T$ , is [36], [25]

$$z^*(T) = \left( \sqrt{-j(T)/1728} + \sqrt{1 - j(T)/1728} \right)^2, \quad (2.2)$$

which is nothing but the hauptmodul for a certain  $\mathbb{Z}_2$  extension of the modular group,  $SL(2, \mathbb{Z})$ . It is more convenient for our work to use  $SL(2, \mathbb{Z})$  modular forms, and so we introduce

$$z(T) \equiv -4 \frac{z^*(T)}{(1 - z^*(T))^2} = \frac{1728}{j(T)}. \quad (2.3)$$

Our task is to represent the  $F^4$  heterotic threshold corrections, as computed [15] and rederived in appendix B.2, in terms of the mirror map pertaining to the  $K3$  surface (2.14). The product form of these looks exactly like (2.1), but with  $U = \rho$ :

$$\begin{aligned} \Delta_{E_8 E'_8}(T) &= -48 \log[\Psi] \Big|_{U \equiv \rho} , & \text{with } a = -2, b = 0 , \\ & & \text{and counting function} \\ \mathcal{C} &= \frac{1}{12} \frac{1}{\eta^{24}} [E_2 E_4 - E_6]^2 \\ \Delta_{E_8 E_8}(T) &= -24 \log[\Psi] \Big|_{U \equiv \rho} , & \text{with } a = 8, b = 12 , \\ & & \text{and counting function} \\ \mathcal{C} &= \frac{1}{12} \frac{E_4}{\eta^{24}} \\ & & [E_2^2 E_4 - 2E_2 E_6 + E_4^2] . \end{aligned} \tag{2.4}$$

We see (due to the finite number of quasi-modular forms with a given degree) that all couplings are composed out of a finite number of building blocks. Most importantly, note that there are two kinds of ingredients:

- (i) those terms that are fully modular and are polynomials in the Eisenstein series  $E_4(\tau)$  and  $E_6(\tau)$
- (ii) those terms that are quasi-modular, because they involve powers of  $E_2(\tau)$ .

The theorems of Borcherds [16] state that the product  $\Psi(T, U)$  in (1.2) has good modular properties essentially if  $\mathcal{C}(\tau)$  is a modular function<sup>6</sup>. However these theorems do not apply if  $\mathcal{C}$  contains  $E_2$ . This means that while the pieces of (2.4) that do not contain  $E_2$  map into the ring of modular functions generated by  $z(T)$ ,  $z(T) - 1$  and  $z(T)'$ , the  $E_2$ -parts cannot map into this ring. However, as we will see in the next section and in appendix A, we can make good use of the fact that the  $E_2$  pieces arise from taking derivatives of true modular forms (and that the  $E_2$  pieces can be removed from the counting function by further judicious differentiation). In this spirit, we parametrize the non-modular pieces in the following way:

$$\mathcal{C}_1 \equiv \frac{1}{2\pi i} \frac{d}{d\tau} \frac{E_4 E_6}{\eta^{24}} = -\frac{1}{\eta^{24}} \left( \frac{1}{2} E_4^3 + \frac{1}{3} E_6^2 + \frac{1}{6} E_2 E_4 E_6 \right) ,$$

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<sup>6</sup>If the counting function,  $\mathcal{C}$ , has weight zero then the constant term of its  $q$ -expansion is required to be divisible by 24.

$$\begin{aligned}
 c_2 &\equiv \left( \frac{1}{2\pi i} \frac{d}{d\tau} \right)^2 \frac{E_4^2}{\eta^{24}} \\
 &= -\frac{1}{\eta^{24}} \left( \frac{13}{36} E_4^3 + \frac{2}{9} E_6^2 + \frac{1}{3} E_2 E_4 E_6 + \frac{1}{12} E_2^2 E_4^2 \right),
 \end{aligned}$$

and define

$$\mu_i(T) = a_i \log \left[ q_T^{-1} \prod_{(k,l)>0} (1 - q_T^k q_U^l)^{c_i(k,l)} \Big|_{U \equiv \rho} \right], \quad i = 1, 2. \tag{2.5}$$

where  $a_i$  are some normalization constants that will be fixed later ( $a_1 = -3$ ,  $a_2 = -9/2$ ). Combining this with the modularly well-behaved pieces, we can now rewrite the threshold couplings in terms of these building blocks in the following way:

$$\begin{aligned}
 \Delta(T) = -48 \Big( \log [z(T)^{\alpha_1} (z'(T))^{\alpha_2} (z(T) - 1)^{\alpha_3}] \\
 + \beta_1 \mu_1(T) + \beta_2 \mu_2(T) \Big). \tag{2.6}
 \end{aligned}$$

Explicitly, comparing with (2.4), we find that<sup>7</sup>

$$\begin{aligned}
 \Delta_{E_8 E'_8} : \alpha_1 = -2, \alpha_2 = 0, \alpha_3 = 0, \beta_1 = -1, \beta_2 = 2/9, \tag{2.7} \\
 \Delta_{E_8 E_8} : \alpha_1 = -16, \alpha_2 = 18, \alpha_3 = -9, \beta_1 = -1/2, \beta_2 = 1/9.
 \end{aligned}$$

Equation (2.6) is the analogue, and in fact the generalization of the threshold formula (1.3) in four dimensions. Indeed the corresponding four-dimensional expression can be written exactly in this form, but with  $\beta_2 = 0$ . Note that the four-dimensional expression (1.3) is modular as a function of *all* the Calabi-Yau moduli, including the dilaton modulus  $z_S \sim e^{-4\pi S}$ . The lack of modularity (due to the  $\mu_1$ ) comes from identifying the perturbative coupling,  $S$ , and extracting the weak coupling limit. The non-modular function  $\mu_1 - \log(z)$  then turns up as the finite residue in  $\lim_{S \rightarrow \infty} (\log(z_S) - S)$ . (The  $\log(z)$  term subtracts the singularity at  $T = U = \rho$ .)

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<sup>7</sup>Note that  $\Delta_{E_8 E'_8} - 2\Delta_{E_8 E_8} = 288 \log[\eta(T)^{24}]$ , which represents the eight dimensional analog of the well-known result [7] about differences of four dimensional threshold couplings.

The vanishing of  $\beta_2$  is a reflection of the fact that for the four-dimensional gauge couplings,  $E_2$  appears only linearly in the counting function, and not quadratically. The new feature in eight dimensions is thus the presence of the function  $\mu_2$ , whose Borchers formula has a counting function containing  $E_2^2$ . This raises the question as to how such functions would naturally appear from the intrinsic geometry of  $K3$ . In fact, counting functions of curves of algebraic genus  $g$  with  $n$ -nodes, passing through  $g$  points on  $K3$ , have been found in [33]:

$$C_g \equiv \sum_{n=0}^{\infty} c_g(n)q^n = \left( \frac{\partial}{\partial q} E_2(q) \right)^g \frac{q}{\eta(q)^{24}} .$$

These involve arbitrary high powers of  $E_2$ , and in particular one has:  $C_1 = \frac{1}{12} \frac{E_2^2 - E_4}{\eta^{24}}$ . This means that the threshold corrections in eight dimensions can formally be related to the counting of nodal elliptic curves in  $K3$ .

## 2.2 Picard-Fuchs equations with sources

We now wish to relate the functions  $\mu_i$  to the geometry of the dual  $F$ -theory: that is, to the geometry of the relevant elliptically fibered  $K3$  (2.1). In practice this means that we want to obtain a generalization of the usual Picard-Fuchs operator. At  $U = \rho$ , this PF operator is of second order, and after transforming to the variable  $z(T)$  in (2.3), it becomes:

$$\mathcal{L}^{(2)} \equiv \frac{1}{z} \left[ \theta_z^2 - z \left( \theta_z + \frac{5}{12} \right) \left( \theta_z + \frac{1}{12} \right) \right] , \tag{2.8}$$

where  $\theta_z \equiv z \frac{d}{dz}$ . The fundamental solutions to  $\mathcal{L}^{(2)}\varpi_i(z) = 0$  are given by the periods

$$\varpi_0(z) = {}_2F_1 \left( \frac{1}{12}, \frac{5}{12}; 1, z \right) = (E_4)^{1/4}, \quad \varpi_1(z) = T \varpi_0 = T (E_4)^{1/4} . \tag{2.9}$$

As was noted in [18], there is a canonical association of (2.8) to the following third-order operator:

$$\mathcal{L}^{(3)} \equiv \frac{1}{z} \left[ \theta_z^3 - z \left( \theta_z + \frac{5}{6} \right) \left( \theta_z + \frac{1}{2} \right) \left( \theta_z + \frac{1}{6} \right) \right] , \tag{2.10}$$



The two operators  $\mathcal{L}^{(2)}$  and  $\mathcal{L}^{(3)}$  are naturally related with one another for a number of reasons. First, their fundamental solutions are quadratically related:

$$\omega_j(z) = \varpi_{j-i} \varpi_i = T^j (E_4)^{1/2}, \quad j = 0, 1, 2, \quad (2.11)$$

where  $\mathcal{L}^{(3)}\omega_i(z) = 0$ . This fact will be important later when we discuss the interpretation of the underlying geometry.

More generally, these two operators satisfy some interesting identities when filtered through the mirror map: for any function  $f(z)$  one has

$$\begin{aligned} z \mathcal{L}^{(2)} (f(z\varpi_0(z))) &= \frac{1}{E_4(q_T)} (\theta_{q_T}^2 f(z(q_T))) \varpi_0, \\ z \mathcal{L}^{(3)} (f(z)\omega_0(z)) &= \frac{1}{E_6(q_T)} (\theta_{q_T}^3 f(z(q_T))) \omega_0. \end{aligned} \quad (2.12)$$

From this and (A.6) it follows that the functions we seek,  $\mu_j(z)$ , satisfy the following inhomogenous, or “source” PF equations:

$$\begin{aligned} \mathcal{L}^{(2)} (\mu_1 \omega_0(z)) &= \varpi_0 \\ \mathcal{L}^{(3)} (\mu_2 \omega_0(z)) &= \omega_0, \end{aligned} \quad (2.13)$$

where we have fixed the normalization constants,  $a_1 = -3$ ,  $a_2 = -9/2$ , in (2.5) by requiring “unit sources” on the right-hand sides of these equations. The solutions of these equations are ambiguous up to additions of the homogeneous solutions, which amount to irrelevant addition of terms linear in  $T$  to  $\mu_1$  and up to quadratic terms in  $T$  to  $\mu_2$ .

Amongst other things, these equations mean that at  $U = \rho$  the Borchers products  $\mu_i$  become solutions to relatively simple linear systems of equations. In particular, note that  $\mathcal{L}^{(2)}(\mathcal{L}^{(2)}(\mu_1 \varpi_0(z))) = 0$  and  $\mathcal{L}^{(3)}(\mathcal{L}^{(3)}(\mu_2 \omega_0(z))) = 0$ . In other words, we find that the ingredients  $\mu_i$  in the threshold corrections (2.6) satisfy generalized hypergeometric equations of fourth and sixth order, respectively.

The question arises as to the physical and geometrical interpretation of the inhomogenous Picard-Fuchs equations (2.13). We derived them by working backwards, *i.e.*, by investigating how to reproduce the threshold corrections originally obtained from the heterotic string. However, before we discuss the physical and geometric interpretation, we first wish to generalize our ideas to a larger class of models.

### 2.3 Generalization to certain one-parameter families of $K3$ 's.

Consider the sequence of models that have been introduced in ref. [25]. They represent certain one-parameter<sup>8</sup> families of singular  $K3$  surfaces, with the special property that the modulus  $\tau_s$  of the elliptic fiber (the type IIB string coupling) remains constant over the base  $\mathbb{P}^1$ . These families can be represented by the following polynomial equations  $W(x, y, \xi) = 0$ :

$$\begin{aligned}
 (E_8^2 H_0^2) & : y^2 + x^3 + \xi^5(\xi - 1)(\xi - z^*(T)) = 0 \\
 (E_7^2 H_1^2) & : y^2 + x^3 + x\xi^3(\xi - 1)(\xi - z^*(T)) = 0 \\
 (E_6^2 H_2^2) & : y^2 + x^3 + \xi^4(\xi - 1)^2(\xi - z^*(T))^2 = 0 \\
 (D_4^4) & : y^2 + x^3 + \xi^3(\xi - 1)^3(\xi - z^*(T))^3 = 0.
 \end{aligned} \tag{2.14}$$

The first model is exactly the model with  $E_8 \times E_8$  gauge symmetry that we discussed above. Each of these models has four singularities in the  $z$ -plane of the indicated types, leading to corresponding gauge symmetries in  $D = 8$  (the Kodaira singularities of type  $H_n$  lead to gauge groups  $A_n$ ). There exist actually further models of the same kind, which we will not discuss in great detail in the following (but which could be treated in a similar way). That is, the list of one-parameter families with constant coupling and four singularities in the  $z$ -plane includes also the models  $(E_8 H_0 D_4^2)$ ,  $(E_7 H_1 D_4^2)$ ,  $(E_6 H_2 D_4^2)$ ,  $(E_8 H_0 E_6 H_2)$ ,  $(E_6^2 D_4 H_0)$  and  $(H_2^2 D_4 E_8)$ .

One feature these models have in common is that their mirror maps are uniformly given by certain Thompson series; this is much in line of the findings of ref. [18]. The abovementioned models indeed match very well with the list of replicable arithmetic triangle functions discussed in [27]. More specifically, explicit computations show that the mirror maps are determined by the Schwarzian equation

$$\frac{z^{*'''}}{z^{*'}} - \frac{3}{2} \left( \frac{z^{*''}}{z^{*'}} \right)^2 = -2Q(z^*) z^{*'/2}, \tag{2.15}$$

where

$$Q(z^*) = \frac{1}{4} \left\{ \frac{1 - \lambda^2}{z^{*2}} + \frac{1 - \mu^2}{(z^* - 1)^2} + \frac{\lambda^2 + \mu^2 - \nu^2 - 1}{z^*(z^* - 1)} \right\}. \tag{2.16}$$

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<sup>8</sup>The  $T^2$  modulus  $U$ , as well as the Wilson lines, are frozen to particular finite values [25].

The solution of (2.15) is given by the Schwarzian triangle function

$$T(z^*) = s(\lambda, \mu, \nu; z^*) , \tag{2.17}$$

where  $(\pi\lambda, \pi\mu, \pi\nu)$  are the angles of the relevant fundamental domain (which depends on the specific model). We list these and other data, partly taken from [27], in Table 1.

Elliptic Singularities	Constant IIB coupling $\tau_s$	Angles $(\lambda, \mu, \nu)$	Hyper-geometric indices $(a, b; c)$	Inverse mirror map = Hauptmodul $z^*(T)$
$E_8^2 H_0^2$	$\rho$	$(0, \frac{2}{3}, 0)$	$(\frac{1}{6}, \frac{1}{6}, 1)$	$(\sqrt{-J(T)} + \sqrt{1 - J(T)})^2$
$E_7^2 H_1^2$	$i$	$(0, \frac{1}{2}, 0)$	$(\frac{1}{4}, \frac{1}{4}, 1)$	$-\frac{1}{64}(\frac{\eta(T)}{\eta(2T)})^{24}$
$E_6^2 H_2^2$	$\rho$	$(0, \frac{1}{3}, 0)$	$(\frac{1}{3}, \frac{1}{3}, 1)$	$-\frac{1}{27}(\frac{\eta(T)}{\eta(3T)})^{12}$
$D_4^4$	any	$(0, 0, 0)$	$(\frac{1}{2}, \frac{1}{2}, 1)$	$-\frac{1}{16}(\frac{\eta(T)}{\eta(4T)})^8$
$E_8 H_0 D_4^2$	$\rho$	$(0, \frac{1}{3}, \frac{1}{3})$	$(\frac{1}{6}, \frac{1}{2}, 1)$	$\frac{\sqrt{3}i(\theta_2^4(2T) - e^{\pi i/3}\theta_3^4(2T))^3}{9\theta_2^4(2T)\theta_3^4(2T)\theta_4^4(2T)} - 1$
$E_7 H_1 D_4^2$	$i$	$(0, \frac{1}{4}, \frac{1}{4})$	$(\frac{1}{4}, \frac{1}{2}, 1)$	$\frac{i(\theta_3^2(2T) + i\theta_4^2(2T))^4}{8\theta_2^4(2T)\theta_3^2(2T)\theta_4^2(2T)} - 1$
$E_6 H_2 D_4^2$	$\rho$	$(0, \frac{1}{6}, \frac{1}{6})$	$(\frac{1}{3}, \frac{1}{2}, 1)$	$\frac{\sqrt{3}i(\eta^6(2T) + 3\sqrt{3}i\eta^6(6T))^2}{36\eta^6(2T)\eta^6(6T)} - 1$
$E_8 H_0 E_6 H_2$	$\rho$	$(0, \frac{1}{2}, \frac{1}{6})$	$(\frac{1}{6}, \frac{1}{3}, 1)$	
$E_6^2 D_4 H_0$	$\rho$	$(\frac{1}{6}, \frac{1}{3}, \frac{1}{6})$	$(\frac{1}{6}, \frac{1}{3}, \frac{5}{6})$	
$H_2^2 D_4 E_8$				

**Table 1.** Complete list of one-parameter families of  $K3$  surfaces with four elliptic singularities and constant coupling. The triple  $(\lambda, \mu, \nu)$  describes the angles of the fundamental region of the relevant triangle group, and  $(a, b; c)$  the indices of the corresponding hypergeometric equation. Every vanishing angle corresponds to a cusp and thus to a decompactification limit  $\text{Im}T \rightarrow \infty$ ; the last two models obviously do not have such a limit ( $J \equiv j/1728$ ).

Note that for these models all monodromies (induced by encircling the four singularities in the  $z$ -plane) are of finite order. As was discussed in [25], this means that the geometry of the singular  $K3$ 's can be described by a finite covering of the  $z$ -plane and thus effectively reduces to the one of Riemann surfaces; the four 7-planes then correspond to the branch points of these curves. More specifically, for the four models in (2.14) one finds the following  $\mathbb{Z}_N$ -symmetric curves

$$\Sigma_N : \quad x^N = \xi^{-1}(\xi - 1)(\xi - z^*) \tag{2.18}$$

of genus  $g = N - 1$ , where  $N = 6, 4, 3, 2$ , respectively. Indeed, the relevant period integrals  $\varpi_i = \int_{\gamma_i} dx d\xi / (\partial_y W(x, y, \xi))$  of the  $K3$  surfaces (2.14) can be directly obtained from the curves (2.18). This can be seen by changing variables in the integral by setting  $x = v \xi^{2(1-1/N)} (\xi - 1)^{2/N} (\xi - z^*)^{2/N}$ , upon which the integral then factorizes into:  $\int \frac{dv}{\sqrt{v^3+1}} \int \frac{d\xi}{\xi^{1-1/N} (\xi - 1)^{1/N} (\xi - z^*(T))^{1/N}}$ . The integral over  $v$  is simply a constant normalization, and we thus reduce the relevant  $K3$  periods to the periods of the  $\mathbb{Z}_N$  curves:

$$\varpi_i = \int \frac{d\xi}{\xi^{1-1/N} (\xi - 1)^{1/N} (\xi - z^*(T))^{1/N}} .$$

This can also be interpreted [25] as integrals over open string metrics [28],  $d\xi \prod_{i=1}^{24} (\xi - \xi_i)^{-1/12}$ . The periods may be written as hypergeometric functions

$$\begin{aligned} \varpi_0 &= (-1)^{-2/N} \pi \csc(\pi/N) {}_2F_1(1/N, 1/N, 1; z^*) \\ \varpi_1 &= z^{*-1/N} (-1)^{-2/N} \pi \csc(\pi/N) {}_2F_1(1/N, 1/N, 1; 1/z^*) . \end{aligned} \tag{2.19}$$

of the corresponding  $(a, b; c)$  type, as indicated in Table 1. The flat coordinate is then alternatively given by  $T = \varpi_1 / \varpi_0$ .

The issue is to compute couplings of the form  $\Delta_{G_1 G_2}(T) F_{G_1}^2 F_{G_2}^2$  (1.1), where  $G_{1,2}$  are the non-abelian gauge groups of any two given 7-planes, out of the total of four. As discussed in [25], the primary, and potentially singular contribution to this coupling comes from integrating out the exchange of the  $RR$  four-form tensor field  $C^{(4)}$  between the two given 7-planes, simply because each of the planes carries a world-volume coupling of the form  $C^{(4)} \wedge F_{G_i} \wedge F_{G_i}$ .

It was proposed in [25] that the coupling should be given by a logarithmic correlation function between the two relevant branch points (7-planes) of  $\Sigma_N$ .<sup>9</sup> This correlator is supposedly nothing but the Green’s function  $\mathcal{G}^{\Sigma_N}$  between appropriate  $1/N$ -period points of a scalar field on  $\Sigma_N$ , ie.,  $\Delta_{G_1 G_2} \sim \mathcal{G}^{\Sigma_N}(\xi_1, \xi_2)$ .

The problem is that a Green’s function is not uniquely defined since there is the freedom of adding a non-singular piece to it,  $\mathcal{G}^{\Sigma_N}(\xi_1, \xi_2, T) \rightarrow \mathcal{G}^{\Sigma_N}(\xi_1, \xi_2, T) + \beta_i \mu_i(T)$ . The canonical choice for it, given by the

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<sup>9</sup>We suspect that this can be naturally expressed in terms of a “logarithmic” conformal field theory, along the lines of ref. [26].

prime form, turns out not to give the complete result in general. More precisely, somewhat tedious explicit computations show that the canonical Green’s function between any two relevant branch points  $z_i$  is composed out of the Hauptmodul  $z^*(T)$  and has the general form

$$\mathcal{G}_{\text{prime form}}^{\Sigma_N}(\xi_1, \xi_2, T) = \log [z^{*\alpha_1}(1 - z^*)^{\alpha_2}(z^*)^{\alpha_3}] \tag{2.20}$$

for an appropriate choice of  $\alpha_i$  (this is essentially a combination of the generalized Halphen functions discussed in ref. [27].) We find that this Green’s function yields the correct result for the couplings (1.1) only for the model with  $D_4^4$  gauge symmetry, as was shown in ref. [25].<sup>10</sup>

The point is that the prime form (2.20) describes only the “modular” part of the threshold correction, but misses the functions  $\mu_i$  in (2.6). Physically, (2.20) describes only the tree-level exchange of  $C$  fields, but misses certain instanton contributions. Namely, loops of  $(p, q)$  strings in the  $\xi$ -plane will be closed in general only on the covering surface  $\Sigma_N$ , so that such strings effectively wrap the Riemann surfaces. Wrapping entire world-sheets of such strings will thus in general generate extra instanton-like contributions. In the  $D_4^4$  model considered in [25] there are no such instanton corrections ( $\beta_i = 0$ ) because  $\Sigma_2$  has genus  $g = 1$ , so that from the point of view of the  $(p, q)$  instantons the situation is like a type IIB compactification on  $T^2$  with maximal supersymmetry: it is known [29] that for this compactification there are no  $(p, q)$  instanton corrections to parity-odd couplings.

The functions  $\mu_i$  to be added to the canonical Greens functions (2.20) can be obtained in exactly the same way as we did before. We first perform a quadratic change of variables,<sup>11</sup>

$$z(T) \equiv -4 \frac{z^*(T)}{(1 - z^*(T))^2}, \tag{2.21}$$

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<sup>10</sup>Note that the heterotic loop computation in [25] missed a term, which slightly modifies the result given in [25]; however, the correlators can be still represented in the form (2.20) with the choice:  $(\alpha_1, \alpha_2, \alpha_3) = (1, -1, 0), (-1, -1, 0), (-1, 1, 0)$  referring to  $\Delta_{12}, \Delta_{13}, \Delta_{14}(T)$ , respectively. The correct computation can be found in a separate erratum.

<sup>11</sup>For  $E_{8-k}H_kD_4^2$  ( $k = 0, 1, 2$ ), the transformation is  $z(T) = -\frac{z^*(T)^2}{4(1-z^*(T))}$ , which maps to the equations (2.22) and (2.24). For  $E_3H_0E_6H_2$ , we have simply  $z(T) = z^*(T)$  which maps to these equations for  $N = 3$ . For the last two entries in Table 1, the transformation (2.21) maps to hypergeometric systems of types  ${}_2F_1(1/12, 1/4; 5/6, z)$  and  ${}_3F_2(1/6, 1/2, 1/3; 2/3, 5/6, z)$ , respectively.

in terms of which the Picard-Fuchs operators are:

$$\mathcal{L}_N^{(2)} \equiv \frac{1}{z} \left[ \theta_z^2 - z \left( \theta_z + \frac{1}{2N} \right) \left( \theta_z + \frac{1}{2} - \frac{1}{2N} \right) \right] \tag{2.22}$$

where  $N = 2, 3, 4, 6$ , respectively. The fundamental solutions to  $\mathcal{L}_N^{(2)} \varpi_i(z) = 0$  are

$$\begin{aligned} \varpi_0(z) &= {}_2F_1 \left( \frac{1}{2N}, \frac{1}{2} - \frac{1}{2N}; 1, z \right) = \sqrt{z'z^{-1}(1-z)^{-1}} \\ \varpi_1(z) &= T \varpi_0. \end{aligned} \tag{2.23}$$

The third-order operators that are associated with (2.22) are simply [18]

$$\mathcal{L}_N^{(3)} \equiv \frac{1}{z} \left[ \theta_z^3 - z \left( \theta_z + 1 - \frac{1}{N} \right) \left( \theta_z + \frac{1}{2} \right) \left( \theta_z + \frac{1}{N} \right) \right], \tag{2.24}$$

whose solutions are again quadratic in terms of  $\varpi_i$ :  $\omega_j(z) = \varpi_{j-i} \varpi_i$ . We can then analogously write down the source equations:

$$\begin{aligned} \mathcal{L}_N^{(2)} (\mu_1 \varpi_0(z)) &= \varpi_0 \\ \mathcal{L}_N^{(3)} (\mu_2 \omega_0(z)) &= \omega_0, \end{aligned} \tag{2.25}$$

which finally determine the extra contributions,  $\mu_i(z(T))$ . Once again, for simplicity we have chosen to normalize the  $\mu_i$  to satisfy these equations with “unit source”.

In order to test our ideas explicitly, we now consider the remaining models in the list (2.14), i.e., the ones with  $(E_6 \times A_2)^2$  and  $(E_7 \times A_1)^2$  gauge symmetry, and compare the geometric data with the heterotic one-loop couplings (these one-loop couplings are computed in appendix B). Since these models have a greater variety of non-abelian group factors than the  $E_8 \times E_8$  and  $D_4^4$  models, there are more couplings to test.

The upshot is that we indeed find that the generic expression (2.6) reproduces the heterotic one-loop results, provided that we choose the coefficients  $\alpha_i, \beta_i$  appropriately (where, of course,  $z^*(T) = -\frac{1}{27}(\eta(T)/\eta(3T))^{12}$  or  $z^*(T) = -\frac{1}{64}(\eta(T)/\eta(2T))^{24}$ , respectively, and where  $\mu_{1,2}$  are the solutions of (2.25) with  $N = 3, 4$ ). Explicitly, by matching the asymptotic  $q$ -expansions of these building blocks with

the heterotic couplings (B.26) at  $U = \rho - 1$  [25], we have for the  $E_6$  model:

$$\begin{aligned}
 \Delta_{E_6 E_6'}(T) &= \log [z^{*-1/3}(z^* - 1)^{2/3}] - \frac{1}{12}\mu_1 + \frac{1}{108}\mu_2 \\
 \Delta_{E_6 A_2}(T) &= \log [z^{*-1/6}(z^* - 1)^{-1/3}] + \frac{1}{108}\mu_2 \\
 \Delta_{E_6 A_2'}(T) &= \log [z^{*-1/3}(z^* - 1)^{-1/3}] + \frac{1}{108}\mu_2 \\
 \Delta_{A_2 A_2'}(T) &= \log [z^{*-1/3}(z^* - 1)^{2/3}] + \frac{1}{12}\mu_1 + \frac{1}{108}\mu_2 .
 \end{aligned}
 \tag{2.26}$$

Quite similarly, for the  $E_7$  model we find that at  $U = 1 + i$ :

$$\begin{aligned}
 \Delta_{E_7 E_7'}(T) &= \log [z^{*-1/12}(z^* - 1)^{1/6}] - \frac{1}{32}\mu_1 + \frac{1}{192}\mu_2 \\
 \Delta_{E_7 A_1}(T) &= \log [z^{*-1/24}(z^* - 1)^{-1/12}] + \frac{1}{192}\mu_2 \\
 \Delta_{E_7 A_1'}(T) &= \log [z^{*-1/12}(z^* - 1)^{-1/12}] + \frac{1}{192}\mu_2 \\
 \Delta_{A_1 A_1'}(T) &= \log [z^{*-1/12}(z^* - 1)^{1/6}] + \frac{1}{32}\mu_1 + \frac{1}{192}\mu_2 .
 \end{aligned}
 \tag{2.27}$$

Thus, including the results of [25] and of Section 2.1, we have verified that for all  $K3$  surfaces in (2.14) we can match the geometric data to the corresponding heterotic one-loop results. This represents, we believe, the most complete quantitative test of the heterotic/ $F$ -theory duality to date.

### 3 Interpretation and Discussion

We have demonstrated that the inhomogenous Picard-Fuchs equations (2.25) carry the relevant information about the  $F^4$  couplings (1.1). We now give two interpretations of these equations.

The first is to note that the structure of the inhomogenous Picard-Fuchs equations is highly reminiscent of the equations of Seiberg and Witten [35]. Indeed, the geometry of the specific families (2.14) of singular elliptic  $K3$ 's effectively reduces to the one of  $SU(N)$  SW curves. More generally, remember that the periods,  $a$  and  $a_D$ , of the Seiberg-Witten differential satisfy a first order system of differential equations:

$$\frac{\partial}{\partial z^*} a_D = \varpi_1 , \quad \frac{\partial}{\partial z^*} a = \varpi_0 , \tag{3.1}$$

where the functions  $\varpi_i$  are the standard periods (2.19) of the  $\mathbb{Z}_N$  curves (2.18).

For our quartic gauge couplings in eight dimensions it is not first order, but second order operators  $\mathcal{L}_N^{(2)}$  whose application yields the standard periods of the curves. This means that  $\mu_1$  may be seen as a period of another meromorphic differential on these Riemann surfaces. Similarly, since the differential operators  $\mathcal{L}_N^{(3)}$  are the PF operators associated [18] with the  $K3$  manifolds  $X_6(1, 1, 1, 3)$ ,  $X_4(1, 1, 1, 1)$ ,  $X_{2,3}(1, 1, 1, 1, 1)$  and  $X_{2,2,2}(1, 1, 1, 1, 1, 1)$ , respectively, this suggests that one could associate  $\mu_2$  with the periods of certain meromorphic differentials on these  $K3$  surfaces.

A second, and more directly useful interpretation can be given for the second order equation in (2.25) for  $\mu_1$ , and this will then help us to get a better understanding of the third-order equation.

As mentioned earlier, the function  $\mu_1$  naturally appears also in the four dimensional,  $N = 2$  supersymmetric theories arising from 3-fold compactifications of type II strings. This function is essentially the difference of  $\log(z_S) - S$  in the large base space limit of the relevant Calabi-Yau 3-fold (in which the non-perturbative contributions to the threshold corrections drop out). The relevant 3-folds are known to be  $K3$  fibrations [3] over a  $\mathbb{P}^1$  base, and this implies that the Picard-Fuchs operators of these Calabi-Yau manifolds must involve, in some way, the differential operators  $\mathcal{L}_N^{(2)}$  in (2.22).

More precisely, the “fibered” PF operators are obtained, to leading order in  $z_S \sim e^{-4\pi S}$ , by the replacement  $\theta_z^2 \rightarrow \theta_z(\theta_z - 2\theta_{z_S})$  in the first term of  $\mathcal{L}_N^{(2)}$ . If one now recalls that  $S\varpi_0 \sim (\log(z_S) + \mu_1 - \log(z))\varpi_0$  is a period of the Calabi-Yau manifold and if one keeps all the finite terms in the Calabi-Yau Picard-Fuchs system in the limit as  $S \rightarrow \infty$ , one finds that  $\theta_{z_S}(\log(z_S)\varpi_0)$  contributes a finite term that may be written as a  $\mathcal{L}^{(2)}((\mu_1 - \log(z))\varpi_0) = 2\theta_z\varpi_0$ . This equation then trivially reduces to (2.25).

In other words, the source term of the inhomogenous second order equation (2.25) is nothing but a remnant of the heterotic dilaton in the large base space, or weak coupling limit.

This suggests a natural interpretation of the third order equation (2.25), which appears only for the eight dimensional, but not for the four dimensional couplings. A crucial insight can be gained by paying attention to the structure of the solutions of the *homogenous* equation,



$\mathcal{L}_N^{(3)}\omega_i(z) = 0$ : the three solutions are nothing but quadratic products of the ordinary  $K3$  periods. We believe that these periods are to be interpreted as those of the symmetric product,  $\text{Sym}^2(K3)$ , of the underlying  $K3$ .

The appearance of  $\text{Sym}^2(K3)$  is indeed quite natural in the context of  $D$ -brane physics. That is, the contribution to the couplings (1.1) we consider comes from pairs of 7-branes, and a system of two branes (or points on  $K3$ ) is thought to be described by a non-linear sigma-model whose target space is  $\text{Sym}^2(K3)$  [34]. Since this is a hyperkähler manifold,<sup>12</sup> and a sigma-model on such a space has  $N = (4, 4)$  supersymmetry, the quantum cohomology is trivial and this is exactly what is reflected by the product structure of the periods.

More generally, any hyperkähler manifold has a holomorphic  $(2, 0)$ -form and a holomorphic  $(4, 0)$ -form (which may be thought of as the square of the  $(2, 0)$ -form). It is the variation of the Hodge structure of the holomorphic  $(2, 0)$ -form and  $(4, 0)$ -form that seems to underly our two functions  $\mu_1$  and  $\mu_2$ . More precisely, what we should have is a fibration of these forms, which –in the large base limit– manifests itself in the source terms of the inhomogenous equation (2.25). The  $\mathbb{P}^1$  fibration yields in total a 5-fold, and indeed it was suggested in [25] a 5-fold should underlie the  $F^4$  couplings in eight dimensions.

We have made extensive, and thus far unsuccessful, attempts to obtain algebraic (hyperkähler-fibered) 5-folds, whose Picard-Fuchs systems would reduce to the source equations presented in this paper. However, it is notoriously difficult to find algebraic descriptions of hyperkähler manifolds [39], and so our lack of success may merely be reflection of this fact.

The question whether the threshold corrections described in this paper can indeed be realized in terms of a fibration of  $\text{Sym}^2(K3)$  or not, has potentially important physical significance. Remember that what we just have been arguing is that the heterotic one-loop couplings are given by the large base space limit of this fibration, just as for the well-known couplings in four dimensions. However, in four dimensions this is not the full story, in that the expansion away from the large base space limit gives the dilaton dependent, non-perturbative corrections

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<sup>12</sup>For a review and references, see [39] (and also [37], [38]).

to the one-loop couplings.

One may thus be tempted to ask for an interpretation of the higher orders of expansion in the base-space parameter,  $z_S$ , of the 5-fold. It has been suggested [14][15], however, that the heterotic one-loop corrections to  $F^4$  are exact in eight dimensions and that there are no further non-perturbative corrections. If this were true, then the source equations discussed in this paper would indeed capture the complete story. However, being related to a singular geometrical limit, this seems a little unnatural; perhaps there is, in fact, a physically meaningful extra dependence on a geometrical modulus which perturbs away from the singular limit. In fact, it is known that  $\text{Sym}^2(K3)$  has an extra modulus that controls the blow up of its  $\mathbb{Z}_2$  singularity,<sup>13</sup>

$$\dim H^{1,1}(\text{Sym}^2(K3)) = \dim H^{1,1}(K3) + 1 = 21, \quad (3.2)$$

and it is a non-trivial fact [38] that this modulus behaves exactly like a string coupling constant. We hope to give a more detailed presentation of these matters elsewhere.

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## A Quasi-modular Borcherds products

It was shown by Borcherds that if the “counting function”  $\mathcal{C}(\tau) \equiv \sum c(n)q^n$  is a true modular form of weight  $-s/2$ , then there is a canonical choice of the exponents  $a, b$  in

$$\Psi = (q_T)^a (q_U)^b \prod_{(k,l)>0} (1 - q_T^k q_U^l)^{c(kl)}, \quad (\text{A.1})$$

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<sup>13</sup>It deforms  $\text{Sym}^2(K3)$  to a smooth Hilbert scheme [37].

such that  $\Psi$  is a meromorphic modular form of  $(T, U)$ -weights  $(c(0)/2, c(0)/2)$ . Moreover, the zeroes and poles of  $\Psi$  are given precisely by the vanishing of the various factors in the product. Perhaps the most familiar example of these Borcherds formulae is:  $\mathcal{C}(\tau) = E_4^3/\Delta - 744$  then for  $T_2 > U_2$  one has  $a = -1, b = 0$ , and

$$\Psi_0 = j(T) - j(U) . \tag{A.2}$$

We want to find some form of generalized Borcherds formulae for simplifying modular products involving  $E_2$ . Counting functions involving  $E_2$  can be obtained by differentiating modular polylogarithms. That is, consider

$$\chi(T, U) = \left( \frac{1}{2\pi i} \right)^{2n+1} \sum_{(k,l)>0} c(kl) \mathcal{L}i_{2n+1}[q_T^k q_U^l] , \tag{A.3}$$

where the polylogarithm is defined by  $(a \geq 1)$ :

$$\mathcal{L}i_a(z) = \sum_{p>0} \frac{z^p}{p^a} , \quad \text{with} \left( z \frac{\partial}{\partial z} \right)^{a-1} \mathcal{L}i_a(z) = -\log(1 - z) , \tag{A.4}$$

and, as usual, the sum in (A.3) runs over the positive roots  $k > 0, l \in \mathbb{Z} \wedge k = 0, l > 0$ . It then follows that if one defines  $\Psi$  by taking  $\log(\Psi) = (-\frac{1}{4\pi^2} \partial_U \partial_T)^n \chi$ , then  $\Psi$  has counting function  $(\frac{1}{2\pi i} \partial_\tau)^n \mathcal{C}(\tau)$ .

The issue is that the obvious modular quantity (A.3) has polylogarithmic singularities, while the natural meromorphic object,  $\Psi$ , is not modular. However one can find a meromorphic, modular object by further differentiating  $\log(\Psi)$ . It is elementary to show that if  $F^{(-2m)}(\tau)$  is a modular form of weight  $-2m$ , then  $G^{(2m+2)} = \frac{d^{2m+1}}{d\tau^{2m+1}} F^{(-2m)}$  is a modular form of weight  $2m + 2$ . That is,  $G^{(2m+2)}$  contains *no*  $E_2$ 's. Moreover  $\frac{d^m}{d\tau^m} F^{(-2m)}$  is an quasi-modular function that contains a factor of  $E_2^m F^{(-2m)}$ . Thus not only is it most natural to think of any  $E_2$  in  $\mathcal{C}(\tau)$  as coming from derivatives of other modular forms, but one can render such functions modular once again by taking a suitable number of derivatives. For example, define:

$$\Psi_i = q_T^{-1} \prod_{(k,l)>0} (1 - q_T^k q_U^l)^{c_i(kl)} , \quad i = 1, 2$$

$$\mathcal{C}_1 \equiv \frac{1}{2\pi i} \frac{d}{d\tau} \frac{E_4 E_6}{\eta(\tau)^{24}}$$

$$\begin{aligned}
 &= -\frac{1}{\eta(\tau)^{24}} \left( \frac{1}{2}E_4^3 + \frac{1}{3}E_6^2 + \frac{1}{6}E_2E_4E_6 \right); \tag{A.5} \\
 \mathcal{C}_2 &\equiv \left( \frac{1}{2\pi i} \frac{d}{d\tau} \right)^2 \frac{E_4^2}{\eta(\tau)^{24}} \\
 &= \frac{-1}{\eta(\tau)^{24}} \left( \frac{13}{36}E_4^3 + \frac{2}{9}E_6^2 + \frac{1}{3}E_2E_4E_6 + \frac{1}{12}E_2^2E_4^2 \right).
 \end{aligned}$$

One can easily check that  $(\frac{1}{2\pi i}\partial_\tau)^2\mathcal{C}_1 = (984 - j(\tau))E_4$  and  $(\frac{1}{2\pi i}\partial_\tau)^3\mathcal{C}_2 = (240 - j(\tau))E_6$ , which are modular forms of weight 4 and 6 respectively.

Now consider  $\chi_1 = \partial_U^2\partial_T^2\log(\Psi_1)$  and  $\chi_2 = \partial_U^3\partial_T^3\log(\Psi_2)$ . These may be viewed as modular “polylogarithms” of the form (A.3) with  $n = -2$  and  $n = -3$ . The functions  $\mathcal{L}i_a$  for  $a \leq 0$  are rational, and indeed the corresponding modular “polylogarithms” are the positive weight automorphic forms generated via the Hecke transformations, and are thus nearly holomorphic modular forms [16]. The weights of these modular “polylogarithms” is the same as the weight of the counting function, and so the functions  $\chi_1$  and  $\chi_2$  have  $(T, U)$  weight  $(4, 4)$  and  $(6, 6)$  respectively. One can use this, and the manifest zeroes and poles of  $\Psi_i$  to uniquely identify the  $\chi_i$ . We will not do this here, but instead focus on the special point  $U = \rho \equiv e^{2\pi i/3}$ .

Since we are taking  $U = \rho$ , we will only be interested in the modular and holomorphic properties as a function of  $T$ . We therefore consider the  $\mu_j = a_j \log(\Psi_j)|_{U=\rho}$  with the constants  $a_j$  as in (2.5), and define  $\Phi_j = (\frac{1}{2\pi i}\partial_T)^{j+1} \mu_j$ ,  $j = 1, 2$ . The function  $\Phi_j$  is thus a modular form of weight  $2(j + 1)$ . From the product formula (A.5), and the fact that  $\mathcal{C}_i \sim -\frac{1}{q} + const + \dots$ , one sees that the the functions  $\log(\Psi_i)$  are *only* singular at  $T = U = \rho$ , and moreover, at this point  $\Phi_1$  and  $\Phi_2$  have double and triple poles respectively. One can also easily see that  $\Phi_1$  and  $\Phi_2$  both vanish at  $T = i\infty$ . This determines the functions  $\Phi_i$  up to overall normalizations, and the latter can be fixed by using the fact that  $E_4(\tau)/E_6(\tau) \sim -\frac{2\pi i}{3}(\tau - \rho)$  as  $\tau \rightarrow \rho$  and using (A.5) to obtain the coefficient of the pole in  $\Phi_j$ . One needs to be a little careful in that the product in (A.5) has a simple zero at  $T = U$ , but in the limit  $U \rightarrow \rho$  this becomes a triple zero because  $U = \rho$  is a  $\mathbb{Z}_3$ -orbifold point of the fundamental domain. One finds:

$$\Phi_1 = 1728 \frac{E_4(q_T)}{j(q_T)}, \quad \Phi_2 = 1728 \frac{E_6(q_T)}{j(q_T)}. \tag{A.6}$$

## B Elliptic genera and heterotic $F^4$ -corrections

In this appendix we compute the one-loop threshold corrections in the heterotic string picture. They are needed in Section 2 for the comparison with the geometric  $K3$  data. In subsection B.1 we will first write a compact generating functional, from which these couplings can be obtained by differentiation and which has an interesting  $D$ -string interpretation. In B.2 we consider the model with  $E_8^2$  gauge symmetry, and in B.3 we extend this to the remaining models with  $[E_7 \times SU(2)]^2$  and  $[E_6 \times SU(3)]^2$  gauge symmetry. Finally, in B.4 we collect some data on Jacobi forms.

### B.1 1/2 BPS-saturated $F^n$ -amplitudes

We will present here a formal expression for heterotic one-loop corrections to  $\text{Tr}F^n, (\text{Tr}F^{n/2})^2, \dots$  (in general  $n$ -derivative) gauge couplings ( $n$ =even), where the gauge fields originate from  $E_8 \times E_8'$  and where the trace is taken in the adjoint representation. Furthermore, we restrict to  $T^2 \times X$  heterotic string compactifications and 1/2-BPS saturated amplitudes. The latter restriction guarantees that the whole left-moving fermionic part of the partition function (supplemented with  $2n$  fermionic zero modes) cancels against the left-moving bosonic oscillator contribution. This leads to a world-sheet torus integral whose integrand is essentially the product of the torus partition function  $Z_{2,2}(T, U)$  and the holomorphic genus  $\Phi_{-n}(q, y)$ . More precisely, we have

$$\Delta_{(\text{Tr}F_{E_8}^{n/2})^2} = \frac{1}{(2\pi i)^n} \frac{\partial^n}{\partial \bar{z}^n} \int \frac{d^2\tau}{\tau_2} [Z_{2,2}(q, \bar{q}) \tilde{\Phi}_{-n}(\bar{q}, \bar{y}) - c_{(n/2)}(0)] \Big|_{\bar{z}=0}, \tag{B.1}$$

where  $\tilde{\Phi}_{-n}(q, y) = e^{m\pi \frac{z^2}{\tau_2}} \Phi_{-n}(q, y)$  with  $y = e^{2\pi iz}$  and  $q = e^{2\pi i\tau}$ . As usual, the non-harmonic pieces are needed for modular invariance and come from the coincidence of external gauge legs. The parameter,  $y$ , represents one of the skew eigenvalues of the background gauge field,  $F$ . Here we simplify our calculations (without loss of generality) by

restricting attention to a single such parameter. The constant  $c_{(n/2)}(0)$  in (B.1) is defined to be  $E_2^{n/2}(q)\Phi_{-n}(q, 1)|_{\text{coeff}(q^0)}$  and is needed to keep the integral IR-finite. The  $\Phi_{-n}(q, y)$  are Jacobi functions with weight  $-n$  and index  $m = 4$ , and we define their expansion coefficients  $c(k, b)$  by:

$$\Phi_{-n}(q, y) = \sum_{k \geq 0} \sum_{b^2 \leq 4mk} c(k, b)y^b q^k. \tag{B.2}$$

In contrast to (B.28), the function  $\tilde{\Phi}_{-n}(q, y)$  has a well-behaved transformation behaviour:

$$\tilde{\Phi}_{-n}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^{-n} \tilde{\Phi}_{-n}(\tau, z). \tag{B.3}$$

It is this property that allows to use in (B.1) the orbit decomposition method of [7], and after some work to eventually arrive at (for the chamber  $T_2 > U_2$  and regularization  $\epsilon \rightarrow \infty$ )

$$\begin{aligned} \Delta_{(\text{Tr}F_{E_8}^{n/2})^2} &= \frac{1}{(2\pi i)^n} \frac{\partial^n}{\partial z^n} \times \\ &\left\{ \left[ \sum_b \sum_{(k,l) > 0} \sum_{p > 0} \frac{2}{\sqrt{p^2 - \frac{mz^2}{T_2 U_2}}} \right. \right. \\ &\quad \left. \left. e^{-2\pi(kT_2 + lU_2)} \sqrt{p^2 - \frac{mz^2}{T_2 U_2}} e^{2\pi ip(kT_1 + lU_1)} c(kl, b)y^b + hc. \right] \right. \\ &+ \sum_b \left[ \frac{U_2}{\pi} \sum_{j > 0} \frac{2}{j^2 - mz^2 \frac{U_2}{T_2}} \right. \\ &\quad \left. + \sum_{j > 0} \left( \frac{2}{\sqrt{j^2 - \frac{mz^2}{T_2 U_2}}} - \frac{2}{\sqrt{j^2 - \frac{mz^2}{T_2 U_2} + \frac{\epsilon}{\pi T_2 U_2}} \right) \right] c(0, b)y^b \Big\} \\ &+ \frac{\pi}{3} T_2 \sum_{s=0}^{n/2} \frac{1}{s+1} E_2^{s+1} F_s \Big|_{\text{coeff}(q^0)} \\ &- c_{(n/2)}(0) \left[ \ln \epsilon + \gamma_E + 1 + \ln \left( \frac{2}{3\sqrt{3}} \right) \right] \Big|_{z=0}, \end{aligned} \tag{B.4}$$

with  $\frac{3}{2n/2} (y \frac{\partial}{\partial y})^n \tilde{\Phi}(q, y) \Big|_{z=0} =: \sum_{s=0}^{n/2} \widehat{E}_2^s F_s$ . Note that the last four terms give simply polynomials in  $T_2$  and  $U_2$ .

The formula (B.4) can be easily generalized to combinations  $\text{Tr}F_1^{n/2}\text{Tr}F_2^{n/2}$  of different gauge groups, by including further Wilson lines  $z_i \equiv \text{Tr}F_i$  and differentiating with respect to them ( $z^2 \rightarrow \sum_i z_i^2$ ,  $y^b \rightarrow \prod_i y_i^{b_i}$ ).

The complicated formula (B.4) has an intriguing physical interpretation in term of the dual Type I string picture of the heterotic string, by recognizing the exponentiated square root as a Born-Infeld action (this generalizes the observations of [14]). Specifically, in eight dimensions where  $n = 4$ , (B.4) can be rewritten in terms of the Born-Infeld action of a  $D$ -string, which reads [50]:

$$S_{BI}[\mathcal{G}, \mathcal{B}, \mathcal{F}, \mathcal{C}_2] = \int d^2\sigma e^{-\phi} \sqrt{\det(\mathcal{G} + \mathcal{B} + \mathcal{F})} - i \int \mathcal{C}_2, \tag{B.5}$$

where  $\mathcal{F} = \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix}$  is the open string world-volume  $U(1)$  gauge background field. Moreover, in (B.5) we also have the induced moduli fields  $\mathcal{G}_{\alpha\beta} = G_{ij}\partial_\alpha X^i\partial_\beta X^j$ ,  $\mathcal{B}_{\alpha\beta} = B_{ij}\partial_\alpha X^i\partial_\beta X^j$  (in what follows  $\mathcal{B} = 0$ ) and the  $RR$  2-form  $\mathcal{C}_2$  on the world volume. The sum  $k > 0$ ,  $l \in \mathbb{Z}$  in (B.4) over the heterotic winding states thus can be seen as the  $D$ -instanton sum, so that

$$\Delta_{(\text{Tr}F^2)^2} = \frac{\partial^4}{\partial f^4} \sum \frac{1}{\sqrt{\det(\mathcal{G} + \mathcal{F})}} e^{-S_{BI}[\mathcal{G}, \mathcal{F}, \mathcal{C}_2]} \Phi_{-4}(\mathcal{U}, \sqrt{\det\mathcal{F}}) \Big|_{\mathcal{F}=0}, \tag{B.6}$$

with the  $D$ -brane complex structure  $\mathcal{U} = \frac{j+pU_1}{k} + \frac{p}{k}U_2\sqrt{\frac{\det(\mathcal{G}+\mathcal{F})}{\det\mathcal{G}}}$ , gauge field  $e^{-\phi}f = izk\sqrt{m\frac{T_2}{U_2}}$ ,  $e^{-\phi}\sqrt{\det\mathcal{G}} = kpT_2$  and  $\mathcal{C}_2 = kpT_1$ . On the other hand, the part of (B.4) that does not involve winding states ( $k = 0$ ) gives the perturbative contributions in Type I language [14].

We now apply the generating function in (B.4) to the three physical models that we discuss in the present paper.

## B.2 Gauge group $E_8 \times E_8$

Literally taken, the expression for  $\Delta_{\text{Tr}F_{E_8}^n}$  in (B.1) directly applies to heterotic compactifications on: (i)  $K3 \times T^2$  (for  $n = 2$ ), or (ii)  $T^2$  (for

$n = 4$ ). Indeed, using

$$\begin{aligned} \left(y \frac{\partial}{\partial y}\right)^2 J_{E_8}(q, y) \Big|_{z=0} &= 4 \left(y \frac{\partial}{\partial y}\right)^2 E_{4,1}(q, y) \Big|_{z=0} & (B.7) \\ &= \frac{2}{3}(E_2 E_4 - E_6), \left(y \frac{\partial}{\partial y}\right)^4 J_{E_8}(q, y) \Big|_{z=0} \\ &= \frac{4}{3}(E_2^2 E_4 - 2E_2 E_6 + E_4^2). \end{aligned}$$

we can immediately rederive from (B.4) the results of [4] and [15]:

(i)  $F^2$  in  $d = 4$ , with  $\Phi_{-2}(q, y) = \frac{E_6 J_{E_8}(q, y)}{\eta^{24}}$ :

$$\begin{aligned} \Delta_{\text{Tr}F_{E_8}^2} &= 4\text{Re} \left\{ \sum_{(k,l)>0} c_{(1)}(kl) \mathcal{L}i_1(x) \right. \\ &\quad \left. - \frac{3}{\pi T_2 U_2} c(kl) \left[ (kT_2 + lU_2) \mathcal{L}i_2(x) + \frac{1}{2\pi} \mathcal{L}i_3(x) \right] \right\} \\ &\quad - c_{(1)}(0) \ln(\mathcal{K}T_2 U_2) - \frac{\pi c(0) U_2^2}{15 T_2} & (B.8) \\ &\quad - \frac{3c(0)\zeta(3)}{\pi^2 T_2 U_2} + \frac{\pi}{3} c_{(1)}(0) U_2 + 288\pi T_2, \end{aligned}$$

with  $\frac{3}{2^{s/2}} \left(y \frac{\partial}{\partial y}\right)^s \Phi_{-n}(q, y) \Big|_{z=0} = \sum_m c_{(s/2)}(m) q^m$ ,  $s \neq 0$ ,  $\Phi_{-n}(q, 1) := \sum_m c(m) q^m$  and  $\mathcal{K} = \frac{8\pi}{3\sqrt{3}} e^{1-\gamma_E}$ . This gives precisely the integrals  $\tilde{\mathcal{I}}, \mathcal{I}$  given in eq. (A.31) and (A.47) of [4].

(ii)  $F^4$  in  $d = 8$ , with  $\Phi_{-4}(q, y) = \frac{E_4 J_{E_8}(q, y)}{\eta^{24}}$ :

$$\begin{aligned} \Delta_{(\text{Tr}F_{E_8}^2)^2} &= -c_{(2)}(0) \ln(\mathcal{K}T_2 U_2) \\ &\quad + 4\text{Re} \left\{ \sum_{(k,l)>0} c_{(2)}(kl) \mathcal{L}i_1(x) \right. \\ &\quad \left. - \frac{6}{\pi T_2 U_2} c_{(1)}(kl) \left[ (kT_2 + lU_2) \mathcal{L}i_2(x) + \frac{1}{2\pi} \mathcal{L}i_3(x) \right] \right. \\ &\quad \left. + \frac{9}{\pi^2 T_2^2 U_2^2} c(kl) \left[ (kT_2 + lU_2)^2 \mathcal{L}i_3(x) \right. \right. & (B.9) \\ &\quad \left. \left. + \frac{3}{2\pi} (kT_2 + lU_2) \mathcal{L}i_4(x) + \frac{3}{4\pi^2} \mathcal{L}i_5(x) \right] \right\} \\ &\quad - \frac{2\pi c_{(1)}(0) U_2^2}{15 T_2} - \frac{6c_{(1)}(0)\zeta(3)}{\pi^2 T_2 U_2} + \frac{4\pi c(0) U_2^3}{105 T_2^2} \end{aligned}$$



$$+ \frac{27c(0)\zeta(5)}{2\pi^4 T_2^2 U_2^2} + \frac{\pi}{3} c_{(2)}(0) U_2 + 384\pi T_2 .$$

This gives precisely the integrals given in eq. (E.27) of [15], which we need in section 2.1.

The correction  $\Delta_{\text{Tr}F_{E_8}^2 \text{Tr}F_{E_8'}^2}$ , which we also need in section 2.1, is easily obtained from (B.9) by replacing the coefficients  $c_{(s/2)}(n)$  with<sup>14</sup>:

$$\begin{aligned} \sum c(m)q^m &= \frac{J_{E_8}(q, 1)^2}{\eta^{24}} \\ \sum c_{(1)}(m)q^m &= \frac{3}{4} \left( y_1 \frac{\partial}{py_1} \right)^2 \frac{J_{E_8}(q, y_1) J_{E_8}(q, y_2)}{\eta^{24}} \\ &+ \frac{3}{4} \left( y_2 \frac{\partial}{py_2} \right)^2 \frac{J_{E_8}(q, y_1) J_{E_8}(q, y_2)}{\eta^{24}} \Big|_{z_i=0} \sum c_{(2)}(m)q^m \\ &= \frac{9}{4} \left( y_1 \frac{\partial}{py_1} \right)^2 \left( y_2 \frac{\partial}{py_2} \right)^2 \frac{J_{E_8}(q, y_1) J_{E_8}(q, y_2)}{\eta^{24}} \Big|_{z_i=0} . \end{aligned} \tag{B.10}$$

We see that the (harmonic)  $\mathcal{L}i_1$ -term arises from maximally differentiating the Jacobi function  $\Phi_{-n}(q, y)$ , i.e., its coefficients  $c_{(n/2)}(kl)$  involve powers of  $E_2^{n/2}$ . On the other hand, for the maximally non-harmonic terms (proportional to  $\frac{1}{(T_2 U_2)^{n/2}}$ ) the coefficients  $c(kl)$  of  $\Phi_{-n}(q, 0)$  appear. In fact, the expressions in the brackets [ ] are precisely the Bloch-Ramakrishnan-Wigner polylogarithms [5], [11].

### B.3 Gauge groups $G \times G' \subset E_8 \times E_8'$

Threshold corrections for gauge groups  $G \times G' \subset E_8 \times E_8'$  are obtained by introducing Wilson lines. We consider two cases: (I)  $[SU(2) \times E_7]^2$  and (II)  $[SU(3) \times E_6]^2$ , for which appropriate (discrete) Wilson lines are:

$$\begin{aligned} (I) \quad a_1^I &= \frac{1}{2}(1, -1, 0, 0, 0, 0, 0, 0), \frac{1}{2}(1, -1, 0, 0, 0, 0, 0, 0) , \tag{B.11} \\ (II) \quad a_1^I &= \frac{1}{3}(1, 1, -2, 0, 0, 0, 0, 0), \frac{1}{3}(1, 1, -2, 0, 0, 0, 0, 0) . \end{aligned}$$

The internal part of the partition function  $Z_{(18,2)}(q, \bar{q})$  [6] becomes a  $\mathbb{Z}_M$  orbifold with Kähler modulus  $\tilde{T} = MT$  and complex structure modulus

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<sup>14</sup>The last term becomes  $-192\pi T_2$ .

$\tilde{U} = U/M$  (where  $M = 2$  and  $M = 3$ , respectively):

$$Z_{(18,2)}^{G \times G'}(q, \bar{q}) = \sum_{(h,g)} \sum_{\substack{m_1, m_2 \\ n_1 + \frac{h}{M}, n_2}} e^{\frac{2\pi i}{M} g m_1} q^{\frac{1}{2} \bar{P}_L^2} \bar{q}^{\frac{1}{2} \bar{P}_R^2} \mathcal{C}_{(h,g)}(q), \tag{B.12}$$

It is shifted by  $\theta = \frac{1}{M}(0, 0, 1, 0)$  in  $(P_L, P_R) \in \mathcal{N}_{2,2}$  and  $\Theta = a_1$  in  $E_8 \times E_8$  with:

$$\mathcal{C}_0 := \mathcal{C}_{(0,0)}(q) = \frac{E_4^2}{\eta^{24}} = \eta^{-24} (Z_{E_7^0} Z_{A_1^0} + Z_{E_7^1} Z_{A_1^1})^2, \tag{B.13}$$

$$\mathcal{C}_1 := \mathcal{C}_{(0,1)}(q) = \frac{1}{4} \eta^{-24} (\theta_4^2 \theta_3^6 + \theta_3^2 \theta_4^6)^2 = \eta^{-24} (Z_{E_7^0} Z_{A_1^0} - Z_{E_7^1} Z_{A_1^1})^2.$$

We have introduced here the lattice partition functions for  $E_7$  [41] and  $A_1$ :

$$\begin{aligned} Z_{E_7^0} &= \theta_3^7(2\tau) + 7\theta_3^3(2\tau)\theta_4^4(2\tau) \\ Z_{E_7^1} &= \theta_2^7(2\tau) + 7\theta_2^3(2\tau)\theta_3^4(2\tau) \\ Z_{A_1^0} &= \theta_3(2\tau) \\ Z_{A_1^1} &= \theta_2(2\tau). \end{aligned} \tag{B.14}$$

The twisted sector functions follow from modular invariance. Similarly, for the  $E_6$  model we get:

$$\mathcal{C}_0 := \mathcal{C}_{(1,1)}(q) = \frac{E_4^2}{\eta^{24}} = \eta^{-24} (Z_{E_6^0} Z_{A_2^0} + 2Z_{E_6^1} Z_{A_2^1})^2 \tag{B.15}$$

$$\mathcal{C}_1 := \mathcal{C}_{(1,\theta)}(q) = \mathcal{C}_{(1,\theta^2)}(q) = \eta^{-24} (Z_{E_6^0} Z_{A_2^0} - Z_{E_6^1} Z_{A_2^1})^2,$$

where we have introduced the following  $E_6$  [41] and  $A_2$ -characters:

$$\begin{aligned} Z_{E_6^0} &= \frac{1}{2} \left\{ \theta_3(3\tau)\theta_3(\tau)^5 + \theta_4(3\tau)\theta_4(\tau)^5 + \theta_2(3\tau)\theta_2(\tau)^5 \right\} \\ Z_{E_6^1} &= \frac{1}{2} \left\{ \theta \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} (3\tau)\theta_2(\tau)^5 + \theta \begin{bmatrix} 4/3 \\ 0 \end{bmatrix} (3\tau)\theta_3(\tau)^5 \right. \\ &\quad \left. - \rho^{\frac{1}{2}} \theta \begin{bmatrix} 4/3 \\ 1 \end{bmatrix} (3\tau)\theta_4(\tau)^5 \right\} \\ Z_{E_6^{\bar{1}}} &= \frac{1}{2} \left\{ \theta \begin{bmatrix} 5/3 \\ 0 \end{bmatrix} (3\tau)\theta_2(\tau)^5 + \theta \begin{bmatrix} 2/3 \\ 0 \end{bmatrix} (3\tau)\theta_3(\tau)^5 \right. \\ &\quad \left. - \rho \theta \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} (3\tau)\theta_4(\tau)^5 \right\} \\ Z_{A_2^0} &= \theta_3(2\tau)\theta_3(6\tau) + \theta_2(2\tau)\theta_2(6\tau) \end{aligned} \tag{B.16}$$

$$Z_{A_2^1} = \theta_3(2\tau)\theta\left[\begin{smallmatrix} 4/3 \\ 0 \end{smallmatrix}\right](6\tau) + \theta_2(2\tau)\theta\left[\begin{smallmatrix} 1/3 \\ 0 \end{smallmatrix}\right](6\tau).$$

Again, the twisted sector functions follow from modular invariance. The dependence on the skew eigenvalues of  $F$  may be easily introduced for each sector by replacing the  $\theta$ -functions with Jacobi functions (B.29):

$$(I) \quad J_{E_7,i}(q, y_1, y_2) = Z_{E_7^0}(q, y_1)Z_{A_1^0}(q, y_2) - Z_{E_7^1}(q, y_1)Z_{A_1^1}(q, y_2) \tag{B.17}$$

$$(II) \quad J_{E_6,i}(q, y_1, y_2) = Z_{E_6^0}(q, y_1)Z_{A_2^0}(q, y_2) - Z_{E_6^1}(q, y_1)Z_{A_2^1}(q, y_2) ,$$

for the coset  $i = 1$ . For the subsequent world-sheet  $\tau$ -integration, it is convenient to express the orbifold sector sum in (B.12) as sum over the cosets [8]

$$Z(q, \bar{q}, \tilde{T}, \tilde{U})_i = \nu_i \sum_{A_i} q^{\frac{1}{2}|\tilde{P}_L|^2} \bar{q}^{\frac{1}{2}|\tilde{P}_L|^2} , \quad i = 1, \dots, M + 1 , \tag{B.18}$$

with the  $A_1 = \{m_1 \in M\mathbb{Z}; m_2, n_1, n_2 \in \mathbb{Z}\}$ ,  $A_2 = \{n_1 \in \frac{\mathbb{Z}}{M}; m_1, m_2, n_2 \in \mathbb{Z}\}$  etc. and  $\nu_i = vol(\mathcal{N}_{2,2,i}) = \{1, \frac{1}{M}, \dots, \frac{1}{M}\}$ . The function  $\tau_2 Z(q, \bar{q}, \tilde{T}, \tilde{U})_1$  is invariant under  $\Gamma_0(M)_\tau \times \Gamma^0(M)_{\tilde{T}} \times \Gamma_0(M)_{\tilde{U}}$ .

After expressing the  $G \times G'$  currents as  $E_8 \times E_8$  currents, we follow [42] to extract the relevant gauge contractions:

$$\begin{aligned} \Delta_{\text{Tr}F_\alpha^2 \text{Tr}F_\beta^2} &= \int \frac{d^2\tau}{\tau_2} \left\{ a[Z(q, \bar{q}, \tilde{T}, \tilde{U})_0 - 1] + \frac{1}{(2\pi i)^4} \frac{\partial^4}{\partial \bar{z}_\alpha^2 \partial \bar{z}_\beta^2} \right. \\ &\quad \times \int \frac{d^2\tau}{\tau_2} \sum_{i=1}^{M+1} b \left[ Z(q, \bar{q}, \tilde{T}, \tilde{U})_i \right. \\ &\quad \left. \left. \frac{\tilde{J}_{G,i}(\bar{q}, \bar{y}_1, \bar{y}_2) \tilde{J}_{G',i}(\bar{q}, \bar{y}_3, \bar{y}_4)}{\bar{\eta}^{24}} - \nu_i b_i(0) \right] \right|_{z_i=0} , \end{aligned} \tag{B.19}$$

with  $b_i = \frac{\partial^4}{\partial y_\alpha^2 \partial y_\beta^2} \frac{J_{G,i} J_{G',i}}{\eta^{24}} \Big|_{z_i=0} = \sum_k b(k)_i q^k$ . This expression is the generalization of (B.1) to subgroups  $G \times G' \subset E_8 \times E'_8$ . We have displayed the coefficients  $(\alpha, \beta, b_1, b_2)$  in the following tables, next to two additional numbers  $c, \bar{b}$ , which will prove to be useful later to write down the final result in a closed form:

$\text{Tr}F_\alpha^2 \text{Tr}F_\beta^2$	$a$	$b$	$b_1$	$b_2$	$c$	$\tilde{b}$
$\text{Tr}F_{E_7}^2 \text{Tr}F_{E_7'}^2$	0	$\frac{2}{49}$	0	0	0	$\frac{2}{49}$
$\text{Tr}F_{A_1}^2 \text{Tr}F_{A_1'}^2$	0	2	0	0	0	2
$\text{Tr}F_{A_1}^2 \text{Tr}F_{E_7'}^2$	-6	$\frac{2}{7}$	0	21	0	$\frac{2}{7}$
$\text{Tr}F_{A_1}^2 \text{Tr}F_{E_7}^2$	6	$\frac{2}{7}$	-21	0	-6	$\frac{2}{7}$
$\text{Tr}F_{E_7}^2 \text{Tr}F_{E_7}^2$	0	$\frac{2}{147}$	63	63	2	$\frac{2}{49}$
$\text{Tr}F_{A_1}^2 \text{Tr}F_{A_1}^2$	0	2	-5	7		

**Table 2.** Coefficients  $(a, b, b_i)$  in (B.19) for heterotic  $F^4$  corrections with gauge group  $[E_7 \times SU(2)]^2$ . In addition,  $\Delta_{\text{Tr}F_{A_1}^4} = -\frac{1}{3}\Delta_{(\text{Tr}F_{A_1}^2)^2}$  and  $\Delta_{\text{Tr}F_{E_7}^4} = 0$ .

$\text{Tr}F_\alpha^2 \text{Tr}F_\beta^2$	$a$	$b$	$b_1$	$b_2$	$c$	$\tilde{b}$
$\text{Tr}F_{E_6}^2 \text{Tr}F_{E_6'}^2$	0	$\frac{1}{8}$	0	0	0	$\frac{1}{18}$
$\text{Tr}F_{A_2}^2 \text{Tr}F_{A_2'}^2$	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$
$\text{Tr}F_{A_2}^2 \text{Tr}F_{E_6'}^2$	-3	$\frac{1}{6}$	0	18	0	$\frac{1}{6}$
$\text{Tr}F_{A_2}^2 \text{Tr}F_{E_6}^2$	3	$\frac{1}{6}$	-18	0	-3	$\frac{1}{6}$
$\text{Tr}F_{E_6}^2 \text{Tr}F_{E_6}^2$	0	$\frac{1}{24}$	36	36	$\frac{3}{2}$	$\frac{1}{18}$
$\text{Tr}F_{A_2}^2 \text{Tr}F_{A_2}^2$	0	$\frac{3}{4}$	-3	9		

**Table 3.** Coefficients  $(a, b, b_i)$  in (B.19) for heterotic  $F^4$  corrections with gauge group  $[E_6 \times SU(3)]^2$ . In addition,  $\Delta_{\text{Tr}F_{A_2}^4} = -\frac{1}{3}\Delta_{(\text{Tr}F_{A_2}^2)^2}$  and  $\Delta_{\text{Tr}F_{E_6}^4} = 0$ .

The techniques to perform world-sheet torus integrals over Narain coset sums

$$\Delta = \frac{1}{(2\pi i)^n} \frac{\partial^n}{\partial \bar{z}^n} \int \frac{d^2\tau}{\tau_2} \sum_{i=1}^{M+1} [Z(q, \bar{q}, T, U)_i \tilde{\Phi}_{-n,i}(\bar{q}, \bar{y}) - \nu_i c_{(n/2),i}(0)] \Big|_{z=0} \tag{B.20}$$

have been developed in [8] and extended<sup>15</sup> in [25]. Essentially,  $\Delta$  inte-

<sup>15</sup>Later, in [44] also integrals over coset sums have been calculated by an independent method. These results completely agree with our findings in [25].

grates to a sum over two sectors:

$$\begin{aligned}
 \Delta = & \frac{1}{(2\pi i)^n} \frac{\partial^n}{\partial z^n} \times \\
 & \left\{ \left[ \sum_b \sum_{\substack{k>0 \\ l \in \mathbb{Z}}} \sum_{p>0} \frac{2}{\sqrt{p^2 - \frac{mz^2}{MT_2U_2}}} \right. \right. \\
 & \quad e^{-2\pi(kT_2+lMU_2)\sqrt{p^2 - \frac{mz^2}{MT_2U_2}}} e^{2\pi ip(kT_1+lMU_1)} c_1(Mkl, b)y^b \\
 & \quad + \frac{2}{\sqrt{p^2 - \frac{mMz^2}{T_2U_2}}} e^{-2\pi(k\frac{T_2}{M}+lU_2)\sqrt{p^2 - \frac{mMz^2}{T_2U_2}}} e^{2\pi ip(k\frac{T_1}{M}+lU_1)} c_2\left(\frac{kl}{M}, b\right)y^b \\
 & \quad + \sum_{l>0} \sum_{p>0} \frac{2}{\sqrt{p^2 - \frac{mz^2}{T_2U_2}}} e^{-2\pi lMU_2\sqrt{p^2 - \frac{mz^2}{T_2U_2}}} e^{2\pi iplMU_1} c_1(0, b)y^b \\
 & \quad \left. \left. + \frac{2}{\sqrt{p^2 - \frac{mz^2}{T_2U_2}}} e^{-2\pi lU_2\sqrt{p^2 - \frac{mz^2}{T_2U_2}}} e^{2\pi iplU_1} c_2(0, b)y^b + hc. \right] \right. \\
 & \quad + \sum_b \left[ \frac{MU_2}{\pi} \sum_{j>0} \frac{2}{\sqrt{j^2 - mz^2 M^2 \frac{U_2}{T_2}}} \right. \\
 & \quad \left. + \left( \frac{2}{\sqrt{j^2 - \frac{mz^2}{T_2U_2}}} - \frac{2}{\sqrt{j^2 - \frac{mz^2}{T_2U_2} + \frac{\epsilon}{\pi T_2U_2}}} \right) \right] c_1(0, b)y^b \\
 & \quad + \sum_b \left[ \frac{U_2}{\pi} \sum_{j>0} \frac{2}{\sqrt{j^2 - mz^2 \frac{U_2}{T_2}}} \right. \\
 & \quad \left. + \left( \frac{2}{\sqrt{j^2 - \frac{mz^2}{T_2U_2}}} - \frac{2}{\sqrt{j^2 - \frac{mz^2}{T_2U_2} + \frac{\epsilon}{\pi T_2U_2}}} \right) \right] c_2(0, b)y^b \Big\} \quad (\text{B.21}) \\
 & + \frac{\pi T_2}{3M} \sum_{i=1}^{M+1} \sum_{s=0}^{n/2} \frac{1}{s+1} E_2^{s+1} F_{i,s} \Big|_{\text{coeff}(q^0)} \\
 & - (c_{(n/2),1}(0) + c_{(n/2),2}(0)) \left[ \ln \epsilon + \gamma_E + 1 + \ln\left(\frac{2}{3\sqrt{3}}\right) \right] \Big|_{z=0},
 \end{aligned}$$

with  $\frac{3}{2^{n/2}} (y \frac{\partial}{\partial y})^n \tilde{\Phi}_i(q, y) \Big|_{z=0} =: \sum_{s=0}^{n/2} \widehat{E}_2^s F_{i,s}$ . Similar as for the  $E_8$  model

(B.7), we need for case (i):

$$\begin{aligned}
 & \left( y \frac{\partial}{\partial y_1} \right)^2 J_{E_7,1}(q, y_1, y_2) \Big|_{z_i=0} \\
 &= \frac{7}{48} \theta_3^2 \theta_4^2 [E_2(\theta_3^4 + \theta_4^4) + \theta_2^8 - 2\theta_3^4 \theta_4^4], \left( y \frac{\partial}{\partial y_2} \right)^2 J_{E_7,1}(q, y_1, y_2) \Big|_{z_i=0} \\
 &= \frac{1}{48} \theta_3^2 \theta_4^2 [E_2(\theta_3^4 + \theta_4^4) - 3\theta_2^8 - 2\theta_3^8 - 2\theta_4^8 + 2\theta_3^4 \theta_4^4],
 \end{aligned} \tag{B.22}$$

and for case (ii):

$$\begin{aligned}
 & \left( y \frac{\partial}{\partial y_1} \right)^2 J_{E_6,1}(q, y_1, y_2) \Big|_{z_i=0} \\
 &= \frac{1}{4} f_0 (f_0 - 3f_3 - E_2 f_1), \left( y \frac{\partial}{\partial y_2} \right)^2 J_{E_6,1}(q, y_1, y_2) \Big|_{z_i=0} \\
 &= \frac{1}{12} f_0 (f_0 + 9f_3 - E_2 f_1)
 \end{aligned} \tag{B.23}$$

with  $f_1 = Z_{A_2^0}$ ,  $f_0 = \left( \frac{\eta^3(\tau)}{\eta(3\tau)} \right)^3$  and  $f_3 = \left( \frac{3\eta^3(3\tau)}{\eta(\tau)} \right)^3$ .

While we need in the present paper only the harmonic pieces of the threshold corrections, it may nevertheless be instructive to the reader to note how easily also the non-harmonic terms derive from our generating formulae (B.4) and (B.21). Evaluating the harmonic part of (B.19), we then arrive at our final result (after dropping the pieces linear in  $T_2$ , which may be easily derived from (B.22) and (B.23)):

$$\begin{aligned}
 \Delta_{\text{Tr}F_\alpha^2 \text{Tr}F_\beta^2}^{\text{harmonic}} &= 4\text{Re} \left\{ -a \ln \eta(\tilde{T}) \eta(\tilde{U}) \right. \\
 &\quad \left. - c\tilde{b} \ln \eta\left(\frac{\tilde{T}}{M}\right) \eta(\tilde{U}) - c\tilde{b} \ln \eta(\tilde{T}) \eta(M\tilde{U}) \right. \\
 &\quad \left. + \tilde{b} \sum_{(k,l)>0} b_1^{\alpha\beta}(Mkl) \mathcal{L}i_1(e^{2\pi i(k\tilde{T}+lM\tilde{U})}) \right. \\
 &\quad \left. + \tilde{b} \sum_{(k,l)>0} b_2^{\alpha\beta}\left(\frac{kl}{M}\right) \mathcal{L}i_1(e^{2\pi i(k\frac{\tilde{T}}{M}+l\tilde{U})}) \right\},
 \end{aligned} \tag{B.24}$$

with the coefficients  $b_i^{\alpha\beta}$  defined by:

$$\frac{1}{\eta(q)^{24}} \left( y \frac{\partial}{\partial y_\alpha} \right)^2 J_{G,i}(q, y_1, y_2) \left( y \frac{\partial}{\partial y_\beta} \right)^2 J_{G',i}(q, y_1, y_2) \Big|_{z_i=0} = \sum_m b_i^{\alpha\beta}(m) q^m. \quad (\text{B.25})$$

analogous to (B.10).

In order to facilitate the comparison with the geometrical formulae of section 2.3, we present here the first terms of the asymptotic  $q$ -series of the corrections (B.24) for the  $E_6$  model (which indeed coincide with the  $q$  expansions of (2.26)). In fact, the geometrical couplings were defined at a fixed value of  $U$ , and it is not entirely trivial to evaluate (B.24) at this value. Explicitly, for the  $E_6$  model, where  $U = \mathbf{T}^{-1} \cdot \rho = \rho - 1$  [25], we find the following expansions:

$$\begin{aligned} \Delta_{E_6 E_6'}(T) &= -\frac{1}{3} \log(q) + 6q + 14q^3 - \frac{33q^4}{2} + \mathcal{O}(q)^5 \\ \Delta_{E_6 A_2}(T) &= \frac{1}{2} \log(q) - 2q + 15q^2 - \frac{110q^3}{3} + \frac{263q^4}{2} + \mathcal{O}(q)^5 \quad (\text{B.26}) \\ \Delta_{E_6 A_2'}(T) &= \frac{2}{3} \log(q) + 18q^2 - 36q^3 + 135q^4 + \mathcal{O}(q)^5 \\ \Delta_{A_2 A_2'}(T) &= -\frac{1}{3} \log(q) + 24q - 81q^2 + 392q^3 - 1848q^4 + \mathcal{O}(q)^5. \end{aligned}$$

Moreover, the solutions of the inhomogenous PF equation (2.25) for  $N = 3$  look:

$$\begin{aligned} \mu_1(T) &= 108q - 486q^2 + 2268q^3 - 10989q^4 + \mathcal{O}(q)^5 \mu_2(T) \\ &= 108q - 810q^2 + 4572q^3 - 24597q^4 + \mathcal{O}(q)^5. \quad (\text{B.27}) \end{aligned}$$

### B.4 Jacobi functions

A Jacobi form (for more details see [45])  $f_{s,m}$  of weight  $s$  and index  $m$  enjoys

$$\begin{aligned} f_{s,m} \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) &= (c\tau + d)^s e^{2\pi i \frac{m c z^2}{c\tau + d}} f_{s,m}(\tau, z), \\ f_{s,m}(\tau, z + \lambda\tau + \mu) &= e^{-2\pi i m(\lambda^2 \tau + 2\lambda z)} f_{s,m}(\tau, z), \quad (\text{B.28}) \end{aligned}$$

for  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$  and  $\lambda, \mu \in \mathbb{Z}$ . With

$$\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (q, y) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2}\alpha)^2} e^{\pi i(n+\frac{1}{2}\alpha)\beta} y^{n+\frac{1}{2}\alpha} \quad (\text{B.29})$$

the function

$$\begin{aligned} J_{E_8}(q, y) &:= \frac{1}{2} \sum_{(\alpha, \beta)} \theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (q, y)^8 \\ &= 1 + q(126 + 56y^{-2} + 56y^2 + y^{-4} + y^4) + \dots \end{aligned} \quad (\text{B.30})$$

is a Jacobi function of weight 4 and index  $m = 4$ , whereas

$$E_{4,1}(q, y) = \frac{1}{2} [\theta_2(q, y)^2 \theta_2^6 + \theta_3(q, y)^2 \theta_3^6 + \theta_4(q, y)^2 \theta_4^6], \quad (\text{B.31})$$

has index  $m = 1$  ( $J_{E_8}(q, y) = E_{4,1}(q, y^2)$ ). We use the notation  $\theta_1 = \theta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $\theta_2 = \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\theta_3 = \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  and  $\theta_4 = \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

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