M-Theory, Topological

Strings and

Spinning Black Holes

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Abstract

We consider M-theory compactification on Calabi-Yau threefolds. The recently discovered connection between the BPS states of wrapped M2 branes and the topological string amplitudes on the threefold is used both as a tool to compute topological string amplitudes at higher genera as well as to unravel the degeneracies and quantum numbers of BPS states. Moduli spaces of k-fold symmetric products of the wrapped M2 brane play a crucial role. We also show that the topological string partition function is the Calabi-Yau version of the elliptic genus of the symmetric product of $K3$'s and use the macroscopic entropy of spinning black holes in 5 dimensions to obtain new predictions for the asymptotic growth of the topological string amplitudes at high genera.

1   Introduction

The study of Calabi-Yau threefolds has been a source of many new ideas in string theory. Not only are they useful as building blocks of various string compactifications, but they also provide interesting examples of exactly computable quantities in string theory. In particular they correspond to the "critical dimension" for the $(N = 2)$ topological string.

Topological strings, roughly speeking, count the number of holomorphic curves inside the Calabi-Yau. As such, one would expect that they should correspond to the partition function of M2 (or D2)-branes wrapped around them. The connection at first sight seems somewhat confusing: The topological string amplitudes exist for each genus, whereas the M2 brane (or D2 brane) degeneracies only care about the charge and not the genus of the curve representing it. It turns out, as discovered in [17] , that the genus dependence of the topological string amplitudes captures the $SU(2)_L$ representation content of BPS states corresponding to wrapped M2 branes upon compactifications of M-theory on Calabi-Yau threefolds. Here $SU(2)_L$ denotes a subgroup of the $SO(4)$ rotation group in 5 dimensions. This identification was

based on the target space interpretation of what the topological string computes [66], [75] and the contribution of BPS states to such terms (using a Schwinger 1-loop computation) [76].

Topological string amplitudes at genus 0 can be computed using mirror symmetry [68]. For higher genera, mirror symmetry is still a powerful principle and can be used to compute the amplitudes up to a finite number of undetermined constants at each genus [66] . Fixing the constant is called fixing the 'holomorphic ambiguity', and for the certain cases they were fixed for genus 1 and genus 2 in [65], [65] . The number of unknown constants grows with the genus. In certain cases one can use direct A-model localization [67] to fix these constants and in particular checking the integrality properties of topological string partition functions, anticipated in [17], at higher $g \leq 5$.

In this paper we wish to use the reformulation of topological string amplitudes as a computation of BPS states in M-theory compactifications [17] to make progress in explicit computations of topological strings at higher genera. The reorganization this introduces into topological string amplitudes is to fix the BPS charge and consider *all* allowed genera of the M2 brane at the same time. For a given degree, there typically is a highest genus curve embedded in the 3-fold which realizes that class[1] . One then studies the moduli space of that curve, together with the flat bundle over it. Understanding of the cohomology of this moduli space and the $SU(2)_L$ action on it, will in particular determine the BPS degeneracy and its $SU(2)_L$ quantum numbers. This in particular affects the topological string amplitudes for all F_r with $r \leq g$ in a well defined way. The main aim of our paper is to develop techniques that at least in some cases allows us to extract from the geometry of this moduli space the $SU(2)_L$ action on its cohomology. We relate the degeneracies for a fixed $SU(2)_L$ spin, and in particular its contribution to F_r, to the Euler characteristic of the $\delta = g - r$ fold symmetric product of the holomorphic curves in the Calabi-Yau 3-fold and to higher F_k's $(r < k \leq g)$. For δ sufficiently small this space is smooth and its Euler characteristic can be computed. For δ too big, in general this space is not smooth and the computation of its Euler

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[1]This highest genus is the arithmetic genus which we will often denote by g. The topological string amplitude at genus g usually denoted F_g has contributions from curves of different arithmetic genera. If the distinction is important we use the label r to refer to the worldsheet genus and write F_r etc.

characteristic requires more care. We will consider examples of both types. We also use these results to fix the holomorphic ambiguities for higher genera in some examples (and in particular we push up the computation of topological strings to higher genera).

We will also discuss the connection of topological string amplitudes and the entropy of spinning black holes corresponding to $M2$ branes wrapped over "large" cycles in the Calabi-Yau. In particular we see how in the case of $K3 \times T^2$ the elliptic genus of the symmetric product of $K3$'s predicts complete answers to the $SU(2)_L$ action on the moduli spaces that we study. For a general Calabi-Yau threefold, we see how the black hole entropy predicts new growth properties for the topological string amplitudes at higher genera that would be interesting to verify.

The organization of this paper is as follows: In Section 2 we review the definition and some results related to A-model topological strings. In Section 3 we review the definition of some of the new invariants which allows one to rewrite topological string amplitudes using integral data. In Section 4 we show how the new invariants can be effectively computed in certain cases. In Section 5 we show that the same invariants can also be computed in a different way and be given a related geometric interpretation. In Section 6 we show in the case of $K3 \times T^2$ how the elliptic genus of symmetric products of $K3$ captures the BPS degeneracies of a wrapped M2 brane and show how our methods can predict some of these results. In Section 7 we use predictions of macroscopic entropy of black holes to estimate the growth of topological string amplitudes for high genera. In Section 8 we give some examples involving non-compact Calabi-Yau 3-folds and show how our methods work in those cases. In Section 9 we do the same, but in the context of compact CY 3-folds. In appendix A we discuss some aspects of del Pezzo surfaces and in appendix B we discuss some aspects of B-model topological strings.

2   Topological Strings (A-model)

In topological string theory (A-model) one considers maps from a Riemann surface Σ_g of genus g to a manifold which in the case of interest

in this paper we take to be a Calabi-Yau threefold X. The partition function depends only on the complexified Kähler moduli of X denoted by $(t_i, \bar{t}_i)$. In the limit whereby one fixes t_i and takes the limit $\bar{t}_i \to \infty$, a holomorphic anomaly decouples, and the theory becomes purely topological. In particular, in this limit the $F_g(t_i)$ are obtained by considering *holomorphic* maps from the Riemann surface to X. Roughly speaking one has

$$F_g(t_i) = \sum_{hol.map f:\Sigma_g \to X} \exp\left(-\int_{\Sigma_g} f^*(k) \right)$$

where k is the Kahler class on X and $f^*(k)$ is its pullback to Σ. The above formula is not quite general because often holomorphic maps come in families. In these cases the sum is replaced by an integral over the moduli space of holomorphic maps representing some top characteristic class on the moduli space. More precisely, in the special case of Calabi-Yau threefolds that we are considering the formal dimension of the moduli space of maps is zero and when there is a moduli space of maps there is an equal dimensional space corresponding to a cokernel of a bundle map. Thus the cokernel vector space forms a bundle on the moduli space of curves whose Euler class enters the relevant topological computation which enters in the above formula (for a more precise mathematical definition and a review of the subject see [68]).[2] The result of such integrals for each fixed topological class of the image curve in X is known as the Gromov-Witten invariants. In other words one can write

$$F_g(t_i) = \sum_{d^i} f_{g,d} \exp(-d_i t^i)$$

where d_i denotes the homology class of the image curve in terms of some basis for $H_2(X, \mathbf{Z})$ and $f_{g,d}$ are the Gromov-Witten invariants. Since in most cases of interest the computation of $f_{g,n}$ involves integrals over moduli spaces, there is a priori no reason for them to be integers, as they are not "counting" the number of holomorphic curves. However some surprising integrality properties have already been observed for small genus which we will review below. From the viewpoint of the topological string this integrality is very surprising and has not been explained.

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[2]Strictly speaking, the obstruction spaces need not form a bundle, and there can be a virtual fundamental class in place of an Euler class.

An explanation of the observed integrality and its generalization to all genera has been found in [17] based on M-theory/type IIA duality which recasts Gromov-Witten invariants in terms of some new integral invariants n_d^r. In this section we first review some aspects of topological strings. We then review the results of [17] in Section 3.

2.1   The genus 0 contribution

Let $\mathrm{Li}_r(x) = \sum_{k=1}^{\infty} k^{-r} x^k$, i.e. $\mathrm{Li}_{-r}(x) = -\left(x \frac{\mathrm{d}}{\mathrm{d}x}\right)^r \log(1-x)$ and $\mathrm{Li}_r(x) = -\left(\int \frac{\mathrm{d}x}{x}\right)^r \log(1-x)$ then [29] gives a formal expansion

$$F_0 = \frac{K_0 t^3}{3!} + t \int_X c_2 J + \frac{\chi}{2}\zeta(3) + \sum_{d=1}^{\infty} n_d^0 \, \mathrm{Li}_3(q^d) \, . \qquad (2.1)$$

Here K_0 is the classical triple intersection number on X, which comes from the degree zero maps.

The curve counting function in genus zero is $K_{ttt} = (\partial_t)^3 F_0 = K_0 + \sum_{d=1}^{\infty} K_d q^d$. By 2.1, K_d is related to the n_d^0 by

$$K_d = \sum_{n|d} \frac{n_{d/n}^0}{n^3}. \qquad (2.2)$$

It was observed that in this way of writing the Gromov-Witten invariants the n_d^0 are integers [29] for the case of the quintic in $\mathbf{P}^4$ at least for degrees up to 300. This was later extended to all d which are not multiples of 5. [3] An explanation of this integrality was suggested in [29] as counting the "number of rational holomorphic curves" in Calabi-Yau space. This was further supported by the fact that it was shown in [9] that the n fold covering of an isolated holomorphic curve of degree d gives a contribution of $1/n^3$ to the Gromov-Witten invariant for degree dn in perfect accord with 2.2 . However the interpretation of n_d^0 as counting holomorphic rational curves in X is in general *not* the right interpretation. In particular a counter-example occurs even for isolated

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[3]Lian and Yau also proved integrality of the coefficients of the mirror map for the quintic, and in all the applications to toric hypersurfaces no non-integer n_d^0 ever appeared.

curves in the quintic. In [77] a contribution of $n^0_{5,\ nod} = 17,601,000$ plane curves with six nodes to total number of curves at degree five $n^0_5 = 229,305,888,887,625$ was found. There are three contributions to K^{10}: degree 10 curves, double covers of degree 5 curves, and additional integer contributions for double covers of the 6-nodal curves corresponding to double covers with 2 components. Correspondingly for each double covering of a nodal curve there is a higher dimensional stratum *and* six points in $\mathcal{M}_{0,0}(X,10)$ [68] . The contribution of the higher dimensional strata to the degree 10 curves must therefore be calculated by a virtual fundamental class calculation, which yields the usual $\frac{1}{2^3}$ and so the double covering contribution is $\frac{1}{2^3} \cdot n^0_{5\ smooth}$ for the smooth $d = 5$ curves but $(6 + \frac{1}{2^3}) \cdot n^0_{5\ nod}$ for the nodal ones. That means that the number[4] of degree 10 curves on the quintic is not given by n^0_{10}, but rather by $N_{10} = K^{10} - \frac{n^0_1}{1000} - \frac{n^0_2}{125} - \frac{n^0_5}{8} - 6n^0_{5,\ nod}$, see Example 7.4.4.1 and Theorem 9.2.6 in [68] . In particular n^0_{10} as defined in 2.2 has no interpretation as a "number" of curves, and there is currently no known mathematical reason to expect it to be integer! One aim of this paper is to outline a physically motivated geometrical definition of the n^r_d, which makes the integrality manifest.

More generally, let $C \subset X$ be a sufficiently general smooth curve of genus g satisfying appropriate genericity hypotheses. Then $C_g(h,d)$ denotes the contribution to F_{g+h} of maps whose image is C whose image has class $d[C]$. It is not yet clear if this notion is well-defined for $g \geq 2$.

Extending [9] Faber and Pandharipande [22] prove that the multi-cover contribution $C_0(h,d)$ of a $\mathbf{P}^1$ is described by

$$C_0(g,d) = \chi_g d^{2g-3} = \frac{|B_{2g}|d^{2g-3}}{2g(2g-2)!} \quad \text{with} \quad \chi_0 = 1, \ \chi_1 = \frac{1}{12}. \quad (2.3)$$

Here $\chi(\mathcal{M}_g) = \frac{|B_{2g}|}{2g(2g-2)!}$ is the Harer-Zagier formula for the orbifold Euler characteristic of $\mathcal{M}_g$ in complete accordance with the predictions of M-theory [17] which will be discussed later.

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[4]We assume that there are finitely many curves (Clemens' Conjecture). N_{10} could still contain multiplicity factors for certain curves.

2.2   The genus 1 contribution

For $r = 1$ the situation is more interesting. The localization [22] gives

$$C_1(0, d) = \frac{\sigma_1(d)}{d}, \ C_1(h, d) = 0 \ (h > 0). \tag{2.4}$$

There is no bubbling contributions of genus 1 curves to higher genus curves, i.e., $C_1(h, d) = 0$ for $h > 0$ in accordance with the above and the zero-mode analysis in [65] . This is a feature one finds in the M-theory approach discussed in the next section.

However the form of F_1 discussed in the next section (which is most natural from the M-theory perspective) is (up to the t-terms)

$$F_1 = \frac{t \int c_2 J}{24} + \sum_{d=1}^{\infty} \left(\frac{1}{12} n_d^0 + n_d^1 \right) \mathrm{Li}_1(q^d) \ . \tag{2.5}$$

Here, n_d^1 is an invariant of certain BPS states typically associated to wrapping M2 branes around degree d elliptic curves E. This differs from the geometric substraction scheme 2.4, as it does not subtract all the multicovering maps from the torus to itself in the definition of the n_d^1, but instead subtracts $1/d$ for the class $d[E]$. Substraction of these would yield[5] [65]

$$F_1 = \frac{t \int c_2 J}{24} + \sum_{d=1}^{\infty} \left(\frac{1}{12} n_d^0 \mathrm{Li}_1(q^d) + n_d^{*1} \log \eta(q^d) \right) \tag{2.6}$$

where n_d^{*1} corresponds to elliptic curves rather than BPS states. The reason for adding back in the multicover contributions is discussed from the BPS perspective in the next section.

Comparing 2.5 with 2.6

$$\sum_{d,n} \frac{n_d^1}{n} q^{dn} = \sum_{d,n} n_d^{*1} \frac{\sigma_1(n)}{n} q^{nd}$$

and keeping in mind the definition $\frac{\sigma_1(n)}{n} = \sum_{k|n} \frac{1}{k}$, we see that the number of BPS states of charge $d[E]$ is $n_d^1 = \sum_{m|d} n_m^{*1}$ as expected from adding up all bound states.

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[5]The genus zero contribution follows from 2.3 in both cases.

2.3   The constant map contribution

We can compute for arbitrary genus in the simple case when the holomorphic maps from the Riemann surface to X are just the constant maps. This is already a case where there is a moduli space of such maps. If the degree of the map f is 0 its moduli space splits into

$$\bar{\mathcal{M}}_{g,n}(X,0) \sim \bar{\mathcal{M}}_{g,n} \times X \qquad (2.7)$$

where $\bar{\mathcal{M}}_{g,n}$ in this case corresponds to moduli space of genus g domain curves. The relevant Gromov-Witten invariant in this case is given by $\frac{1}{2}e(X)\int_{\mathcal{M}_g} c_{g-1}^3(H)$ [66] where $e(X)$ denotes the Euler characteristic of X and H denotes the Hodge bundle (coming from the space of holomorphic one forms on the Riemann surface) over the moduli space. For the Hodge integrals involved in the above formula a closed formula was recently proven in [22] following the approach of [35], [24], [15], which yields

$$\langle 1 \rangle_{g,0}^X = (-1)^g \frac{\chi}{2} \int_{\mathcal{M}_g} \lambda_{g-1}^3 = (-1)^g \frac{\chi}{2} \frac{|B_{2g}B_{2g-1}|}{2g(2g-2)(2g-2)!} \ . \qquad (2.8)$$

This is in perfect agreement with the prediction for constant map contribution from the viewpoint of duality of type IIA and M-theory [17] .

3   M-theory/Type IIA interpretation of the F_g

Recently a series of integer invariants n_d^r were defined [17] for each Calabi-Yau threefold X, labeled by a degree $d \in H_2(X, \mathbf{Z})$ and a positive integer r. Their definition was motivated by consideration of M-theory on Calabi-Yau threefolds. The topological significance of these new invariants is that they can be used to rewrite F_r in terms of them. In particular they explain the integrality properties of F_r for all r and reproduce the expected integrality properties for genus $r = 0, 1$.

The definition of these invariants was motivated by consideration of the spectrum of BPS states in M-theory compactification on Calabi-Yau

3-fold. The spectrum in turn (in simple situations) can be computed by considering a certain $SU(2)$ action on the cohomology of the moduli space of holomorphic curves in a Calabi-Yau X together with a flat bundle. We will now review the definition of these integral invariants.

3.1   The new invariants

The invariants defined in [17] are given as follows: Consider M-theory on a Calabi-Yau threefold X. This gives an $N = 2$ theory in $d = 5$. This theory has $b_2(X)$ gauge fields. The states in $d = 5$ are labeled in particular by the charge under these $U(1)$'s, which thus correspond to an integral element $d \in H_2(X, \mathbf{Z})$. Fix the subsector of the Hilbert space with charge d and consider all states in this sector which are BPS states. These arise in M-theory by considering M2 branes in X wrapped around supersymmetric cycles in the class given by d. In particular their mass is fixed by the Kahler class of X and is given by $d_i t_i$. The BPS state in addition is labeled by how it transforms under the spatial rotation group in $4 + 1$ dimensions which is $SO(4)$, or more precisely $SU(2)_L \times SU(2)_R$. In particular we can write the degeneracy of the BPS states together with their $SO(4)$ quantum numbers as

$$\left[\left(\frac{1}{2}, 0 \right) \oplus 2(0, 0) \right] \otimes \bigoplus_{j_L, j_R} N^d_{j_L, j_R} \left[(j_L, j_R) \right]. \qquad (3.1)$$

The numbers $N^d_{j_L, j_R}$ denote the number of BPS states with charge represented by the class d and with $SU(2)_L \times SU(2)_R$ representation given by the rerpresentation (j_L, j_R), where $j_L, j_R \in (1/2)\mathbf{Z}$ and denote the spin of the representations.

The number of BPS states is not an invariant of the theory and it can jump. Two (short) BPS multiplets can join and become a (long) non BPS multiplet. For example, changing the complex structure of the Calabi-Yau X will change the numbers $N^d_{j_L, j_R}$. However, the left index of the representation does not change. In other words, if we consider the degeneracies with respect to $SU(2)_L$ and sum over all $SU(2)_R$ quantum numbers multiplied by $(-1)^{2j_R} = (-1)^{F_R}$, then this weighted sum of left representations does not change. There are well known examples, e.g. [78], where the individual right spin content changes under complex

structure deformation. Consider a $\mathbf{P}^1$ fibered over a genus g curve C_g. The right Lefschetz decomposition of the base $\mathcal{M} = C_g$ is $\left[\frac{1}{2}\right] + 2g[0]$. Vanishing complex volume of the $\mathbf{P}^1$ corresponds to a special value on the Coulomb branch $\phi = 0$ with a $SU(2)$ gauge enhancement and g hypermultiplets in the adjoint representation. Higgsing w.r.t. to the diagonal components of the hypers corresponds to a complex structure deformation and breaks the gauge group to $U(1)$ and $2g - 2$ charged hypers, which geometrically corresponds to a splitting of the $\mathbf{P}^1$ fibration into $2g-2$ isolated $\mathbf{P}^1$'s, whose right spin content is $(2g-2)[0]$. It is more useful for comparison with topological strings to choose a different basis for the $SU(2)_L$ representations. Let

$$I_r = \left[\left(\frac{1}{2}\right) + 2(0)\right]^{\otimes r} .$$

Using this basis, the procedure is

$$N^d_{j_L,j_R}[(j_L, j_R)] \to \sum N^d_{j_L,j_R}(-1)^{2j_R}(2j_R + 1)[(j_L)] = \sum_r n^r_d I_r \quad (3.2)$$

The above equation defines the invariants n^r_d which appear in the partition function of the topological string. According to [17] we have:

$$F = \sum_{r=0}^{\infty} \lambda^{2r-2} F_r = \sum_{r=0}^{\infty} \sum_{\substack{d=0 \\ m=1}}^{\infty} n^r_d \frac{1}{m}\left(2\sin\frac{m\lambda}{2}\right)^{2r-2} q^{dm}. \quad (3.3)$$

The argument leading to this identification is that the topological string can be viewed as computing $\sum F_r R_+^2 F_+^{2r-2} \lambda^{2r-2}$ amplitudes in four dimensions upon considering type IIA compactification on the Calabi-Yau. Here R_+ and F_+ denote self-dual parts of the Riemann tensor and graviphoton field strength, respectively [66], [75] . Then a 1-loop Schwinger computation as in [76] with the BPS states running around the loop relates the BPS content of states in 5-dimensional M-theory to corrections to $R_+^2 F_+^{2r-2}$ amplitudes. The appearance of the extra sum over m in 3.3 is related to the momentum a BPS state in 5 dimensions can have when compactified on a circle down to 4 dimensions. These appear as 'multi-cover' contributions in the topological string context, as first noticed for the case of genus 0 by [79]. The term $(2\sin\frac{m\lambda}{2})^{2r}$ in the above formula arises from computation of $\text{Tr}(-1)^{2j_L+2j_R}\exp(2im\lambda J_L^3)$ in the I_r representation, where J_L^3 is one of the generators of $SU(2)_L$.

3.2   The higher genus contributions

Expanding 3.3 gives back 2.1, i.e. the naive multicovering formula 2.2 for the rational curves, which first was empirically observed in [29]. Of course the physical picture relates the integers n_d^r naturally to the number of BPS states. Note also that with the ζ-function renormalization one gets the well known [29] subleading contribution to the genus zero pre-potential $\chi(X)\zeta(3)/2$ from 2.73.3 .

Using genus $r = 0, 1$ as a model, we can try to recursively define n_d^r by

$$K_d^r = \sum_{\substack{k|d \\ h \leq r}} n_k^h C_h \left(r - h, \frac{d}{k} \right),$$

where K_d^r are the Gromov-Witten invariants defined by $F_r = \sum K_d^r q^d$. This is the approach taken in [68] for $r = 0, 1$; there the numbers n_d^r are called instanton numbers (see [68] for a precise version of the integrality conjecture for $r = 0$).

For the elliptic curve T^2 and any n, we can consider n D2 branes wrapped on T^2. In this case as discussed in [17] to count the number of BPS states we should consider the moduli space of stable rank n bundles on T^2. There are indecomposable (semi)stable $U(n)$ bundles over T^2, which corresponds to a BPS bound state of n D2 branes wrapping T^2 (the corresponding space for genus 0 is empty which is why we do not have bound states of n D2 branes on a genus 0 curve). This explains the scheme used in the previous section in defining the BPS numbers for genus 1.

For the genus 2,3 expansion we have

$$F_2 = \frac{\chi}{5760} + \sum_{d=1}^{\infty} \left(\frac{1}{240} n_d^0 + n_d^2 \right) \text{Li}_{-1}(q^d) \tag{3.4}$$

$$F_3 = -\frac{\chi}{1451520} + \sum_{d=1}^{\infty} \left(\frac{1}{6048} n_d^0 - \frac{1}{12} n_d^2 + n_d^3 \right) \text{Li}_{-3}(q^d) \tag{3.5}$$

and similarly for higher genus

$$F_r = \frac{(-1)^r \chi |B_{2r} B_{2r-2}|}{4r(2r-2)!(2r-2)} \tag{3.6}$$

$$+ \sum_{d=1}^{\infty} \left(\frac{|B_{2r}|n_d^0}{2r(2r-2)!} + \frac{2(-1)^r n_d^2}{(2r-2)!} \pm \ldots - \frac{r-2}{12} n_d^{r-1} + n_d^r \right) \mathrm{Li}_{3-2r}$$

There is a subtlety for genus $r > 2$ in that n D2 brane bound states can deform off the supporting genus r curve. This is briefly discussed in [17] and is a topic for further study.

4   Computation of n_d^r

The identity 3.3 reexpresses topological string amplitudes in terms of integral quantities n_d^r defined in terms of the BPS spectrum of M-theory on Calabi-Yau threefolds according to 3.2 . If one finds a simple way to compute the new invariants n_d^r this would translate to a practical method of computing topological string amplitudes.

In [17] it was shown how one goes about computing n_d^r (at least in certain good cases). The basic idea is to consider the moduli space of $M2$ branes, which gets translated using M-theory/IIA duality to the study of certain aspects of moduli space of $D2$ branes. One considers supersymmetric $D2$ branes whose class in X is given by $[D2] = d$. The moduli space of such configurations is given, in addition to the embedding of the $D2$ brane, by the choice of a flat bundle on the brane. In general if we have N coincident branes, we will have to consider also the moduli of flat $U(N)$ bundles in addition to the moduli of the embeddings of the $D2$ branes. Let us consider the simple case where we have a single $D2$ brane in class d and let us denote by $\widehat{\mathcal{M}}$ the moduli space of holomorphic curves in X in class d, together with the choice of the flat bundle on the Riemann surface. Let $\mathcal{M}$ denote the moduli space of holomorphic curves in class d, without the choice of the flat bundle. Then we have a map

$$\widehat{\mathcal{M}} \to \mathcal{M}.$$

Let us assume that generically the Riemann surface has genus g. Then the above map has generically a fiber which is T^{2g}, i.e., the Jacobian of

the Riemann surface. However generally speaking there are loci where the genus g surface become singular. For example it can develop nodes by having some pinched cycles. Similarly the Jacobian torus becomes singular in this limit.[6] Nevertheless, one expects the total space $\widehat{\mathcal{M}}$ to be smooth (similar to description of elliptic fibration of K3 where the fibration becomes singular at 24 points, but the K3 is smooth). Because of this smoothness, for many questions it is possible to treat the above fibration *as if* there are no degenerate fibers. In particular, consider the integral (1,1) form k corresponding to the fiber T^{2g} (usually denoted on each non-degenerate fiber by $k|_{fiber} = \theta = dz(\mathrm{Im}\Omega^{-1})dz^*$). We will assume, as is the case with smooth Jacobian varieties, that k makes sense as an integral (1,1) class in $\widehat{\mathcal{M}}$.

Consider the cohomology of the manifold $\widehat{\mathcal{M}}$. These will correspond to BPS states in M-theory compactification. Moreover the $SU(2)_L$ quantum numbers get morally identified with the $SL(2)$ Lefschetz decomposition in the fiber direction (i.e. using k as a raising operator) and the $SU(2)_R$ quantum numbers get morally identified with the $SL(2)$ Lefschetz decomposition in the base direction (i.e. using the Kähler form on the base). In other words we have [17] :

$$H^*(\widehat{\mathcal{M}}) = \sum N^d_{j_1,j_2}[j_1^{fiber}, j_2^{base}] \qquad (4.1)$$

from which we can read off n^r_d according to 3.2. There are precise statements that can be made: the usual Lefschetz decomposition of the cohomology $\widehat{\mathcal{M}}$ is identified with the diagonal $SL(2) \subset SL(2)_{fiber} \times SL(2)_{base}$, and the $SU(2)_R$ content of the highest left spin is identified with the Lefschetz decomposition of $H^*(\mathcal{M})$.

There are two particularly easy cases to compute from the above definition, namely:

$$\begin{aligned} n^g_d &= (-1)^{\dim \mathcal{M}} e(\mathcal{M}) \\ n^0_d &= (-1)^{\dim \widehat{\mathcal{M}}} e(\widehat{\mathcal{M}}) \end{aligned} \qquad (4.2)$$

where $e(...)$ denotes the Euler characteristic of the space. The relations follow from the definition of what the double Lefschetz action is. As we will demonstrate in the next section, the other non-vanishing n's, i.e. the n^r_d for $0 < r < g$ can also be related to particular combinations

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[6]More precisely, the compactified Jacobian becomes singular in this limit.

of Euler characteristics of certain subspaces in $\widehat{\mathcal{M}}$. Sometimes we will write $r = g - \delta$ where δ is a positive integer less than or equal to g.

The existence of such a double Lefschetz decomposition is expected from the M-theory description of the BPS states in 5 dimensions and so it should be possible to rigorize the existence of the above double $SL(2)$ decomposition of the cohomology of $\widehat{\mathcal{M}}$. However here we would like to get the new invariants with the minimal amount of assumptions about the properties of $\widehat{\mathcal{M}}$. As we will discuss in the next section all we really need for computation of n_d^r is the existence of a smooth manifold $\widehat{\mathcal{M}}$ and an integral (1,1) class k which on smooth fibers is the canonical $(1, 1)$ class on the Jacobian torus. This will also lead to a simple formulation for the computation of all n_d^r in terms of Euler characteristics of relative Hilbert schemes, which are frequently easy to compute.

4.1   Computational Scheme for n_d^r

As is clear from 4.1 and 3.2 all we need to compute the n_d^r is the Lefschetz action in the fiber direction. We will see that in fact we can compute n_d^r without this assumption in a reasonably general setting.

For each point on the base $\mathcal{M}$ let C denote the corresponding Riemann surface and $\mathcal{J}(C)$ its Jacobian. The Riemann surface together with the choice of p points on it, is what is called the Hilbert scheme of p points on C, and denoted by $\mathrm{Hilb}^p(C)$. [80]:

$$f_p : \mathrm{Hilb}^p(C) \to \mathcal{J}(C) \tag{4.3}$$

whose image is denoted by W_p. We can relate the cohomology of W_p to the cohomologies of both $\mathrm{Hilb}^p(C)$ and $\mathcal{J}(C)$, thereby relating these two cohomologies directly.

We have the map $H_*(W_p) \to H_*(\mathcal{J}(C))$, which by Poincaré duality is identified with a map $i : H^*(W_p) \to H^*(\mathcal{J}(C))$ whose image we wish to compute. Let $\theta \in H^{1,1}(\mathcal{J}(C))$ be the cohomology class of the zero locus of the theta function on $\mathcal{J}(C)$. Since the image W_p of f_p is dual to $\theta^{g-p}/(g - p)!$ [80], the composition of the restriction map $r : H^*(\mathcal{J}(C)) \to H^*(W_p)$ with i is (up to the constant which we ignore)

just the multiplication map

$$\theta^{g-p} : H^*(\mathcal{J}(C)) \to H^*(\mathcal{J}(C)).$$

Since there is also a map $f_p^* : H^*(W_p) \to H^*(\mathrm{Hilb}^p(C))$, we expect to be able to relate $H^*(\mathrm{Hilb}^p(C))$ with the image of θ^{g-p}.

Here is our strategy. Once we understand this relation, we consider varying the point on the base $\mathcal{M}$. In this way the Abel-Jacobi map f_p gets promoted to a map

$$\hat{f}_p : \mathcal{C}^{(p)} \to \widehat{\mathcal{M}}. \tag{4.4}$$

Here $\mathcal{C}^{(p)}$ denotes the moduli space of holomorphic curves of degree d together with the choice of p points on the Riemann surface.[7] Therefore we relate $H^*(\mathcal{C}^p)$ to the image of multiplication by k^{g-p} on $H^*(\widehat{\mathcal{M}})$, where k is the $SU(2)$ raising operator in the fiber direction. This can be used to compute $n_d^{g-\delta}$ according to 4.1 and 3.2.

Before we carry this out, it is first convenient to review some facts about the cohomology of the Hilbert scheme of p points on the Riemann surface C. Let C be a smooth curve of genus g. Then we have for its cohomology, as an $SU(2)$ Lefschetz representation

$$H^*(C) = \left(\frac{\mathbf{1}}{\mathbf{2}}\right) \oplus (2g)(\mathbf{0}). \tag{4.5}$$

For a smooth curve, its Hilbert scheme is the same as its symmetric product. Taking symmetric products, we have for the Lefschetz $SU(2)$ decomposition

$$H^*(\mathrm{Hilb}^k(C)) = \bigoplus_r \mathrm{Sym}^r\left(\frac{\mathbf{1}}{\mathbf{2}}\right) \otimes \wedge^{k-r}(2g)(\mathbf{0}) \tag{4.6}$$

$$= \bigoplus_{r=0}^{k} \binom{2g}{k-r} \left(\frac{\mathbf{r}}{\mathbf{2}}\right)$$

---

[7]More precisely, we choose a length p subscheme Z of the curve C, which means that $\dim \mathcal{O}_Z = p$. For smooth curves, a length p subscheme is the same thing as a subset of p points of C (including multiplicity). If the curve is singular, these notions can differ. In Section 5, we will see how the difference plays a crucial role in relating our methods to geometry.

Note that since the $2g(\mathbf{0})$ represent odd cohomology of C, we must antisymmetrize.

For convenience, we explicitly list the first two cases of 4.6

$$H^*(\mathrm{Hilb}^2(C)) = (\mathbf{1}) \oplus (2g) \left(\frac{\mathbf{1}}{\mathbf{2}}\right) \oplus (2g^2 - g)(\mathbf{0}) \qquad (4.7)$$

$$H^*(\mathrm{Hilb}^3(C)) = \left(\frac{\mathbf{3}}{\mathbf{2}}\right) \oplus (2g)(\mathbf{1}) \oplus (2g^2 - g) \left(\frac{\mathbf{1}}{\mathbf{2}}\right) \oplus$$
$$\left(\frac{4}{3}g^3 - 2g^2 + \frac{2}{3}g\right)(\mathbf{0}).$$

The Jacobian $\mathcal{J}(C)$ is a principally polarized abelian variety [80], so as we have already mentioned has a canonical Kähler class $\theta \in H^{1,1}(\mathcal{J}(C))$, whose corresponding divisor is the zero locus of the theta-function on $\mathcal{J}(C)$. It is straightforward to check that the resulting Lefschetz $SU(2)$ representation content of $H^*(\mathcal{J}(C))$ can be identified with I_g, which is the representation we defined before. Furthermore, the class θ in this context is identified with the $SU(2)$ raising operator k. So $\theta^{g-p}H^*(\mathcal{J}(C))$ is the same as $k^{g-p}I_g$, and we are just dealing with a simple problem in the representation theory of $SU(2)$.

One easily proves by induction that

$$I_g = \bigoplus_{r=0}^{g} \left(\begin{pmatrix} 2g \\ g-r \end{pmatrix} - \begin{pmatrix} 2g \\ g-r-2 \end{pmatrix} \right) \left(\frac{\mathbf{r}}{\mathbf{2}}\right) \qquad (4.8)$$

as an $SU(2)$ representation. It follows immediately that

$$\theta^{g-1}H^*(\mathcal{J}(C)) = \left[\frac{\mathbf{1}}{\mathbf{2}}\right] \oplus (2g)\,[\mathbf{0}],$$

$$\theta^{g-2}H^*(\mathcal{J}(C)) = [\mathbf{1}] \oplus (2g) \left[\frac{\mathbf{1}}{\mathbf{2}}\right] \oplus (2g^2 - g - 1)\,[\mathbf{0}], \qquad (4.9)$$

$$\theta^{g-p}H^*(\mathcal{J}(C)) = \bigoplus_{r=1}^{p} \left(\begin{pmatrix} 2g \\ p-r \end{pmatrix} - \begin{pmatrix} 2g \\ p-r-2 \end{pmatrix} \right) \left[\frac{\mathbf{r}}{\mathbf{2}}\right] \quad \text{in general.}$$

In 4.9 we are being a bit imprecise with notation, since the $k^{g-p}I_g$ are not representations of $SU(2)$. What we mean by $[\mathbf{r}/\mathbf{2}]$ is a collection of $r + 1$ classes of the form $v, kv, k^2 v, \ldots k^r v$. We are not assuming

that v is killed by the $SU(2)$ annihilation operator. Here v can have any $U(1)$ charge $m \geq -r$, so that $[\mathbf{r}/\mathbf{2}]$ has $U(1)$ charges shifted to $m, m+1, \ldots, m+r$.

Now we are ready to relate $H^*(\mathrm{Hilb}^k(C))$ and the image of multiplication by θ^{g-p}. We can write the precise relationship by comparing 4.5 and 4.6 with 4.9 :

$$
\begin{aligned}
H^*(C) &= k^{g-1} I_g \\
H^*(\mathrm{Hilb}^2(C)) &= k^{g-2} I_g \oplus (\mathbf{0}) \\
H^*(\mathrm{Hilb}^3(C)) &= k^{g-3} I_g \oplus H^*(C) \\
H^*(\mathrm{Hilb}^p(C)) &= k^{g-p} I_g \oplus H^*(\mathrm{Hilb}^{p-2}(C)) \quad \text{in general.}
\end{aligned}
\tag{4.10}
$$

Note that i takes $H^i(W_p)$ to $H^{2g-2p+i}(\mathcal{J}(C))$. This shift by $2g - 2p$ is precisely what is needed to match up the $U(1)$ charges in 4.10, which is understood as an identification of $U(1)$ charges.

Again by induction we note from 4.9 that

$$
\mathrm{Tr}(-1)^F k^{g-p} I_g = \frac{2}{p!}(g-p) \prod_{i=1}^{p-1}(2g-i) \equiv a(g,p)
\tag{4.11}
$$

Now we vary over $\mathcal{M}$. We write the representation of the BPS states as $R = \sum_\delta n^{g-\delta} I_{g-\delta}$. Allowing the curve to vary over the moduli space of the curve $\mathcal{M}$, we get from 4.10

$$
\begin{aligned}
H^*(\mathcal{C}^{(1)}) &= k^{g-1} R \\
H^*(\mathcal{C}^{(2)}) &= k^{g-2} R \oplus H^*(\mathcal{C}^{(0)}) \\
H^*(\mathcal{C}^{(3)}) &= k^{g-3} R \oplus H^*(\mathcal{C}^{(1)}) \\
H^*(\mathcal{C}^{(p)}) &= k^{g-p} R \oplus H^*(\mathcal{C}^{(p-2)}) \quad \text{in general}
\end{aligned}
\tag{4.12}
$$

with the definitions $\mathcal{C}^{(0)} \equiv \mathcal{M}$ and $\mathcal{C}^{(1)} \equiv \mathcal{C}$. We now apply $\mathrm{Tr}(-1)^F$ to both sides of 4.12, and get, using 4.11

$$
(-1)^{\dim(\mathcal{M})+\delta}\left(e(\mathcal{C}^{(\delta)}) - e(\mathcal{C}^{(\delta-2)})\right)
$$
$$
= \sum_{p=0}^{\delta} a(g-p, \delta-p)\, n^{g-p}, \quad \text{for} \quad \delta = 1, \ldots \tag{4.13}
$$

where we set $e(\mathcal{C}^{(-1)}) \equiv 0$, $a(g,0) \equiv 1$. In particular, the first two equations read

$$(-1)^{\dim(\mathcal{M})+1} e(\mathcal{C}^{(1)}) = (2g-2)n^g + n^{g-1}$$

$$(-1)^{\dim(\mathcal{M})} \left(e(\mathcal{C}^{(2)}) - e(\mathcal{C}^{(0)}) \right)$$

$$= \frac{1}{2}(2g-2)(2g-1)n^g + (2g-4)n^{g-1} + n^{g-2} \qquad (4.14)$$

$$= \frac{1}{2}(2g-2)(2g-1)n^g + (2g-4)n^{g-1} + n^{g-2}$$

If we solve 4.14 and $n^g = (-1)^{\dim(\mathcal{M})} e(\mathcal{C}^{(0)})$ for n^g, n^{g-1}, n^{g-2} we get

$$n^{g-1} = (-1)^{\dim(\mathcal{M})+1} \left(e(\mathcal{C}^{(1)}) + (2g-2)e(\mathcal{M}) \right) \qquad (4.15)$$

$$n^{g-2} = (-1)^{\dim(\mathcal{M})} \left(e(\mathcal{C}^{(2)}) + (2g-4)e(\mathcal{C}^{(1)}) + \frac{1}{2}(2g-2)(2g-5)e(\mathcal{C}^{(0)}) \right).$$

In general one shows that the solution to 4.13 yields

$$n^r = n^{g-\delta} = (-1)^{\dim(\mathcal{M})+\delta} \sum_{p=0}^{\delta} b(g-p, \delta-p)\, e(\mathcal{C}^{(p)}), \qquad (4.16)$$

with

$$b(g,k) \equiv \frac{2}{k!}(g-1) \prod_{i=1}^{k-1} (2g-(k+2)+i), \quad b(g,0) \equiv 0.$$

Note that we do not require the Lefschetz action on $\widehat{\mathcal{M}}$ to apply these formulas, only the Lefschetz action on the spaces $\mathcal{C}^{(k)}$. These exist whenever the spaces $\mathcal{C}^{(k)}$ are smooth.

5   Considerations of Enumerative Geometry

In this section, we put forward some natural geometric principles which allow us to relate the invariants n_d^r to computations on other, but related geometrical objects. Moreover, this reasoning points us to introduce correction terms for certain families of reducible curves. We

illustrate our formulas by a few examples which yield numbers which can be checked by other methods. More systematic checks are done in the remaining sections of this paper.

In the previous section we have seen that the contribution of a family of genus g curves to F_0 comes from the $e(\widehat{\mathcal{M}})$. Since the general fibers of $\widehat{\mathcal{M}}$ are Jacobian varieties, they have Euler characteristic 0 and hence do not contribute to $e(\widehat{\mathcal{M}})$. This is similar to the situation considered in [27]. There, the calculation of $e(\widehat{\mathcal{M}})$ can be localized to a calculation on the set of nodal curves. For a genus g curve we would like to be able to compute $e(\widehat{\mathcal{M}})$ by localizing on curves with g nodes.

An additional motivation comes from a glance at 4.2. It is natural to expect from 4.2 that there would be subspaces

$$\mathcal{M} = \widehat{\mathcal{M}}_0 \subset \widehat{\mathcal{M}}_1 \subset \cdots \subset \widehat{\mathcal{M}}_\delta \subset \ldots \subset \widehat{\mathcal{M}}_g = \widehat{\mathcal{M}} \qquad (5.1)$$

such that $n_d^{g-\delta} = (-1)^{\dim \widehat{\mathcal{M}}_\delta} e(\widehat{\mathcal{M}}_\delta)$ for some suitable spaces $\widehat{\mathcal{M}}_\delta$.

Quite independently of the existence of such subspaces, we can still ask for a localication type of computation for all these cases, as in [27]. Consider for example the $\delta = g$ case. In this case we are degenerating the curve of genus g to genus zero with g nodes. On the other hand, the genus 0 isolated curves have information only about the I_0 content of BPS states. Since an isolated genus $g - \delta$ curve has information only about the $I_{g-\delta}$ content of BPS states, one would expect that $n_d^{g-\delta}$ which counts BPS states in the representation $I_{g-\delta}$ is localized on curves of genus $g - \delta$. This reasoning would thus lead us to identify

$$n_d^{g-\delta} = (-1)^{\dim \widetilde{\mathcal{M}}_\delta} e(\widetilde{\mathcal{M}}_\delta) \qquad (5.2)$$

where $\widetilde{\mathcal{M}}_\delta \subset \mathcal{M}$ denotes the moduli space of irreducible curves with δ ordinary nodes, i.e. with genus $r = g - \delta$. Note that this proposal also fits with the top genus contribution where $\delta = 0$, namely

$$n^g = (-1)^{\dim \mathcal{M}} e(\mathcal{M}). \qquad (5.3)$$

where we have noted that $\mathcal{M}$ parameterizes generic genus g curves. Regardless of the existence or definition of $\widehat{\mathcal{M}}_\delta$ and the localization of its Euler characteristic to $e(\widetilde{\mathcal{M}}_\delta)$ we would like to explore the potential validity of 5.2 and in particular see if we get a match with the computations done in the previous section.[8]

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[8]We could use the relation of invariants to classes considered in the previous

5.1   Alternative interpretation of the n_d^r

In this section, we undertake the geometric calculation of the desired formulas. Recall that in [27], a formula for n^0 was derived assuming that the singular curves in the relevant family of curves had only nodes as singularities. In our situation, we will make similar simplifying assumptions on the geometry of the singular curves in our family. Our viewpoint is that since we expect to derive formulas which are generally valid, we are free to make extra assumptions in order to derive them. In fact, our assumptions rarely hold, but we will be able to argue that the formulas obtained are sound. In this way, we greatly enhance our ability to calculate geometrically.

Calculation of invariants of $\widetilde{\mathcal{M}}_\delta$ can be tricky due to the irreducibility requirement. It is easier instead to study the spaces $\mathcal{M}_\delta$, the set of curves with δ nodes, dropping the irreducibility hypothesis. It is easier still to consider $\overline{\mathcal{M}}_\delta$, the closure of the $\mathcal{M}_\delta$ in $\mathcal{M}$. The spaces $\overline{\mathcal{M}}_\delta$ parametrize curves with at least δ nodes, and also curves with possibly more complicated singularities or higher multiplicities. We will calculate $e(\overline{\mathcal{M}}_\delta)$ using a simple topological argument in certain good situations.

Let us consider a special case where many of the difficulties are supressed. Suppose that all curves C_i parametrized by the points of $\overline{\mathcal{M}}_i - \overline{\mathcal{M}}_{i+1}$ have exactly i nodes for $0 \leq i \leq \delta$, where we have put $\mathcal{M}_0 = \mathcal{M}$. Then the Euler characteristic of C_i is $2 + i - 2g$.

If in addition $\mathcal{M}_{\delta+1}$ is empty, then $\mathcal{M}_\delta = \overline{\mathcal{M}}_\delta = \overline{\mathcal{M}}_\delta - \overline{\mathcal{M}}_{\delta+1}$. We next set out to calculate $e(\mathcal{M}_\delta)$. Note that if the curves of $e(\mathcal{M}_\delta)$ are irreducible, so that $\mathcal{M}_\delta = \widetilde{\mathcal{M}}_\delta$, then we can calculate $n^r = n^{g-\delta}$ from this Euler characteristic using 5.2. We are continuing to assume that $\mathcal{M}_\delta$ is smooth.

Recall that the Hilbert scheme $\mathrm{Hilb}^k(C)$ of degree k subschemes of a single curve C parametrizes subsets $S \subset C$ of k points. The points of S are allowed to occur with multiplicity. There is a bit more structure placed on the higher multiplicity points which are located at the singularities. We will give examples later. For the moment, we just

---

section to construct spaces formally related to what we want, but that viewpoint does not appear useful in the present context.

observe that this Hilbert scheme has dimension k.

Let $\pi : \mathcal{C} \to \mathcal{M}$ be the universal curve, so that if $m \in \mathcal{M}$ corresponds to a curve $C_m \subset X$, then $\mathcal{C} \subset X \times \mathcal{M}$ is such that $\pi^{-1}(m) = C_m \times \{m\}$. For each k, let $\pi_k : \mathcal{C}^{(k)} \to \mathcal{M}$ be the relative Hilbert scheme of degree k subschemes of the fibers of π. In other words, we build $\mathcal{C}^{(k)}$ from the universal curve by taking the Hilbert scheme of the curves $C_m = \pi^{-1}(m)$ for each $m \in \mathcal{M}$, and $\mathcal{C}^{(k)}$ is constructed as the union of these as m varies in $\mathcal{M}$. Thus the fiber of π_k over m is $\mathrm{Hilb}^k(C_m)$, and $\mathcal{C}^{(k)}$ has dimension $\dim \mathcal{M} + k$.

Our assumptions imply that all fibers of $\mathcal{C}^{(k)}$ over points of $\overline{\mathcal{M}}_i - \overline{\mathcal{M}}_{i+1}$ have the same computable Euler characteristic, which will be an explicit function $f(g, i, k)$ of g, i, and k. We will calculate $f(g, i, k)$ explicitly soon for all g and a few values of i and k.

Then letting $\mathcal{C}_i^{(k)}$ be the preimage under π_k of $\overline{\mathcal{M}}_i$, we get

$$ e(\mathcal{C}_i^{(k)}) - e(\mathcal{C}_{i+1}^{(k)}) = f(g, i, k) \left(e(\overline{\mathcal{M}}_i) - e(\overline{\mathcal{M}}_{i+1}) \right) . $$

Note that $\mathcal{C}_{\delta+1}^{(k)}$ is empty. Summing these equations from i from 0 to δ, we get an equation expressing $e(\mathcal{C}^{(k)})$ in terms of the $e(\overline{\mathcal{M}}_i)$, $0 \leq i \leq \delta$. If we generate δ such equations by taking k from 1 to δ, then these equations can be solved for the δ variables $e(\overline{\mathcal{M}}_i)$, $1 \leq i \leq \delta$. In particular, we can solve for $e(\overline{\mathcal{M}}_\delta) = e(\mathcal{M}_\delta)$ in terms of these $e(\mathcal{C}^{(k)})$ and $e(\mathcal{M})$.

As already stated, this gives the desired formula for n^r when the irreducibility assumption holds.

We now carry out this procedure for small δ. The results are

$$
\begin{aligned}
e(\mathcal{M}_1) &= e(\mathcal{C}) + (2g-2)e(\mathcal{M}) \\
e(\mathcal{M}_2) &= e(\mathcal{C}^{(2)}) + (2g-4)e(\mathcal{C}) + \frac{1}{2}(2g-2)(2g-5)e(\mathcal{M}) \\
e(\mathcal{M}_3) &= e(\mathcal{C}^{(3)}) + (2g-6)e(\mathcal{C}^{(2)}) + \frac{1}{2}(2g-4)(2g-7)e(\mathcal{C}) + \\
&\quad \frac{1}{6}(2g-2)(2g-6)(2g-7)e(\mathcal{M}) \\
e(\mathcal{M}_4) &= e(\mathcal{C}^{(4)}) + (2g-8)e(\mathcal{C}^{(3)}) + \frac{1}{2}(2g-6)(2g-9)e(\mathcal{C}^{(2)}) + \\
&\quad \frac{1}{6}(2g-4)(2g-8)(2g-9)e(\mathcal{C}) + \\
&\quad \frac{1}{24}(2g-2)(2g-7)(2g-8)(2g-9)e(\mathcal{M})
\end{aligned}
\tag{5.4}
$$

These formulas suggest that in general, we have

$$
\begin{aligned}
e(\mathcal{M}_\delta) &\doteq e(\mathcal{C}^{(\delta)}) + (2g-2\delta)e(\mathcal{C}^{(\delta-1)}) + \\
&\quad \sum_{i=2}^{\delta} \frac{1}{i!}(2g-2\delta+2i-2)(2g-2\delta+i-3) \times \\
&\quad \times (2g-2\delta+i-4)\cdots(2g-2\delta-1)e(\mathcal{C}^{(\delta-i)})
\end{aligned}
\tag{5.5}
$$

where we have put $\mathcal{C}^{(1)} = \mathcal{C}$ and $\mathcal{C}^{(0)} = \mathcal{M}$. These are precisely the formulas given in 4.16, as we asserted at the beginning of this section!

Consider for example the case when X is a local $\mathbf{P}^2$. Since homogeneous polynomials of degree d in the three variables have $(d+2)(d+1)/2$ coefficients and scalar multiplication of the equation does not alter the curve, with get $\mathcal{M} = \mathbf{P}^{d(d+3)/2}$. In particular, if $d = 4$, we get that $\mathcal{M} = \mathbf{P}^{14}$, with Euler characteristic 15. To understand $\mathcal{C}$, we consider the projection $\mathcal{C} \to \mathbf{P}^2$. The fiber over $p \in \mathbf{P}^2$ is the set of plane quartic curves which contain p, and this is a $\mathbf{P}^{13}$ for all p, as the equation $f(p) = 0$ imposes one linear equation on the 15 coefficients of f. Thus $\mathcal{C}$ is a $\mathbf{P}^{13}$ bundle over $\mathbf{P}^2$, hence smooth, and $e(\mathcal{C}) = (3)(14) = 42$. We therefore get $e(\widetilde{\mathcal{M}}_1) = 42 + 4(15) = 102$.

Note that $\widetilde{\mathcal{M}}_1 = \mathcal{M}_1$ in this instance. To see this, observe that a reducible curve of degree 4 would have to be the union of a line and a degree 3 curve, a union of two degree 2 curves, or more degenerate

configurations. All such curves must have at least 3 nodes or worse singularities, so are not contained in $\mathcal{M}_1$. But it is not true that $\mathcal{M}_2$ is empty in this case. Nevertheless, we find from table 4 that $n_4^2 = -102$, exactly as we would have found if $\mathcal{M}_2$ were empty!

This situation turns out to be quite common. We derive formulas for the n^r, and they turn out to have greater validity.

A few words are in order now about the assumption we made that $\mathcal{M}_{\delta+1}$ is empty. Recall that we want to derive a formula for $e(\overline{\mathcal{M}}_\delta)$. But this Euler characteristic is only asserted to correctly calculate the appropriate n^r if $\overline{\mathcal{M}}_\delta$ is smooth. This is relevant because at a point of $\mathcal{M}_{\delta+1} \subset \overline{\mathcal{M}}_\delta$, the space $\overline{\mathcal{M}}_\delta$ tends to be singular. Here is the reason. If C has $\delta + 1$ nodes, then choosing any subset of δ of these nodes, we get a branch of $\overline{\mathcal{M}}_\delta$ which parameterizes curves for which the last node is allowed to smooth out while the original subset of δ nodes remains nodal. Since there are $\delta + 1$ choices of subsets of δ nodes, we see that $\overline{\mathcal{M}}_\delta$ has $\delta + 1$ branches at a general point of $\mathcal{M}_{\delta+1}$. In particular, $\overline{\mathcal{M}}_\delta$ is singular.

Said differently, once we assume that $\overline{\mathcal{M}}$ is smooth, then the assumption that $\mathcal{M}_{\delta+1}$ is empty is quite natural. Once we drop the smoothness assumption, then there is no reason that the formula $n^{g-\delta} = e(\overline{\mathcal{M}}_\delta)$ should be valid. The pleasant surprise is that we have discovered that the formula we derive for the $n^{g-\delta}$ are correct more generally. In fact, these formulas do not always compute the $e(\overline{\mathcal{M}}_\delta)$, but that is of no concern to us: the bottom line is that the formulas compute the invariants that we are actually interested in.

We now have to say something about a common situation when $\widetilde{\mathcal{M}}_i \neq \mathcal{M}_i$, namely when there are reducible curves in our family with exactly i nodes. We expect a nice geometric situation when all components parameterizing reducible families are irreducible components of $\overline{\mathcal{M}}_\delta$.

Here is the problem. If a curve C has δ nodes and is irreducible, then its desingularization $\widetilde{C}$ has genus $g - \delta$ and comes with a map $\widetilde{C} \to C$ which gives an explicit geometric contribution to the instanton sums. However, if C has δ nodes but is reducible, then its desingularization at δ nodes can split C into disjoint components. Since the worldsheet must be connected, such a configuration does not contribute to the

instanton sums. However, it does contribute to our calculations which have ignored the issue of irreducibility. We must find a way to correct for this.

We consider such a component which parameterizes reducible curves of the form $C = C_1 \cup C_2 \cup \ldots \cup C_k$. Some of the curves C_i may also have a fixed number of nodes, hence a fixed geometric genus r_i. Explicitly, if C_i has degree d_i and δ_i nodes, then $r_i = (d_i - 1)(d_i - 2)/2 - \delta_i$. Since each degree d_i is strictly less than the degree d of C, we can inductively compute the instanton numbers $n_{d_i}^{r_i}$ for their respective degrees and geometric genus.

Suppose that these curves split into disjoint components after desingularizing at the δ nodes. We propose that this component contributes $\prod_i n_{d_i}^{r_i}$ to the numbers naively computed by multiplying formulas 5.4 and 5.5 by the appropriate sign. In other words, we are proposing the following algorithm for computing the instanton number of genus $r \leq g$ associated to a family of curves of arithmetic genus g.

Supposing that $\mathcal{C}^{(k)}$ is smooth for $0 \leq k \leq g$, we put $\delta = g - r$ and calculate $(-1)^{\dim \mathcal{M}_\delta} e(\overline{\mathcal{M}}_\delta)$, where by $e(\overline{\mathcal{M}}_\delta)$ we mean the value calculated from 5.4 or 5.5. Then identify any components $M_1, \ldots M_r$ of $\overline{\mathcal{M}}_\delta$ which parameterize reducible curves. Each component M_j gives a contribution of the form $\prod_i n_{d_{ij}}^{r_{ij}}$ as explained above. We have introduced a second subscript on d and g to emphasize that we may have to consider several components. Our proposal is then

$$n^r = (-1)^{\dim \overline{\mathcal{M}}_\delta} e(\overline{\mathcal{M}}_\delta) - \sum_{j=1}^r \prod_i n_{d_{ij}}^{r_{ij}}. \qquad (5.6)$$

We illustrate again when X is a local $\mathbf{P}^2$ and again $d = 4$. This time, we will calculate the genus 0 instanton number. We have to impose $\delta = 3$ nodes to get the genus to 0. We have already computed that $e(\mathcal{M}) = 15$ and $e(\mathcal{C}) = 42$. We consider the map $\rho_2 : \mathcal{C}^{(2)} \to \text{Hilb}^2 \mathbf{P}^2$ which takes a multiplicity 2 scheme in a degree 4 curve and views it as a multiplicity 2 scheme in $\mathbf{P}^2$. We can compute the Euler characteristic of $\text{Hilb}^2 \mathbf{P}^2$ either from counting fixed points of a torus action or from

the generating function

$$\sum_{k=0}^{\infty} e\left(\text{Hilb}^k \mathbf{P}^2\right) q^k = \prod_{n=1}^{\infty} (1-q^n)^{-3}. \tag{5.7}$$

Either way, we get 9 for this Euler characteristic. It is not hard to see that the fiber of ρ_2 over any point $Z \in \text{Hilb}^2 \mathbf{P}^2$ has codimension 2 in the space of all degree 4 curves, as this fiber is just the space of all quartics containing Z. Said differently, the condition that $f|_Z = 0$ places 2 independent linear conditions on the 15 coefficients of f. If $Z = \{p, q\}$, these two conditions are just $f(p) = f(q) = 0$. If Z is concentrated at a single point, then after a change of coordinates we can write Z locally as $y = x^2 = 0$. The space of f which contain Z is just the space of (not necessarily homogeneous) degree 4 polynomials in x and y whose constant terms and coefficient of x vanish, again a codimension 2 linear subspace. After projectivizing, This space is therefore a $\mathbf{P}^{12}$, with Euler characteristic 13. We therefore see that $\mathcal{C}^{(2)}$ is smooth, and we compute that $e(\mathcal{C}^{(2)}) = 9 \cdot 13 = 117$.

Similarly, we get that $\mathcal{C}^{(3)}$ is smooth and $e(\mathcal{C}^{(3)}) = 22 \cdot 12 = 264$, since $\text{Hilb}^3 \mathbf{P}^2$ has Euler characteristic 22 by 5.7 and the space of quartic curves containing a fixed multiplicity 3 scheme is a $\mathbf{P}^{11}$.

Now using these numbers and $g = 3$ in 5.4, we get $e(\overline{\mathcal{M}}_3) = 222$.

But this is not the entire story, since there are reducible quartics which are unions of lines and cubic curves which have three nodes. Lines and cubics have respective instanton numbers 3 and -10. Since the space of three nodal curves has dimension 11, we therefore get the corrected number $n_4^0 = (-1)^{11} 222 - (3)(-10) = -192$. This is in agreement with the value we will exhibit from the B-model in Table 4 of Section 8.3.

We think of our calculational method as giving corrections to 5.4 and 5.5. Unfortunately, it does not apply in all cases, since the $\mathcal{C}^{(k)}$ can be singular. The simplest case we are aware of is n_6^2 in local $\mathbf{P}^2$. This case has $\delta = 8$, and for $\delta < 8$ our method applied successfully every time we are able to check it by mirror symmetry or localization [67]. Our proposal is therefore a very powerful check of the M-theory integrality prediction. We presume that the eventual reconciliation with more general cases (including n_6^2) will come from more subtle corrections.

As an interesting aside, we note that our method sometimes applies nevertheless when $\mathcal{C}^{(k)}$ is singular. The simplest case is if $C \subset X$ is a single isolated curve of arithmetic genus 1 with a single node. The identity map $C \to C$ is a genus 1 stable map, and the normalization map $\mathbf{P}^1 \to C$ is a genus 0 stable map. It is clear that these are the only degree 1 stable maps onto C up to isomorphism. We arrive at the conclusion that $n^1 = 1$ and $n^0 = 1.$[9] Since $\mathcal{M}$ is a point, we get from 5.3 that $n^1 = 1$. As for the genus 0 contribution, the first line of 5.4 (or more simply, the method of [27]) gives $n^0 = 1$, since $\mathcal{C} = C$ has Euler characteristic 1.

Continuing with this digression, note that this reconciles the count of BPS states with stable maps of degree 1 for any isolated irreducible curve of arbitrary genus and number of nodes. Suppose that an irreducible curve D has arithmetic genus g and k nodes. By [82], the compactified Jacobian of D is isomorphic to a product of factors. One factor is the Jacobian $\mathcal{J}(\tilde{D})$ of the smooth genus $g-k$ desingularization of D. There are k other factors, one for each node, and each of these are isomorphic to the curve C above with genus 1 and a single node. By [17], $\mathcal{J}(\tilde{D})$ contributes an I_{g-k} representation, and we have just established that each copy of C contributes an $I_1 + I_0$ representation. So the total representation is

$$I_{g-k} \otimes (I_1 + I_0)^k = \sum_{\delta=0}^{k} \binom{k}{\delta} I_{g-\delta}. \tag{5.8}$$

The right hand side of 5.8 predicts $n^{g-\delta} = \binom{k}{\delta}$. This matches the stable maps perfectly, as we get a genus $g - k$ stable map by picking δ of the k nodes and partially normalizing D only at this subset. Since there are $\binom{k}{\delta}$ ways to do this, we have complete agreement.

We now derive 5.4, beginning with $\delta = 1$. Since all smooth curves of genus g have Euler characteristic $2 - 2g$, we get

$$e(\mathcal{C}) - e(\mathcal{C}_1) = (2 - 2g) \left(e(\mathcal{M}) - e(\overline{\mathcal{M}}_1) \right).$$

By our assumption that all curves of $\overline{\mathcal{M}}_1$ have exactly one node, we have

$$e(\mathcal{C}_1) = (3 - 2g)e(\overline{\mathcal{M}}_1),$$

---

[9]The higher degree invariants have recently been computed in [81].

since one nodal curves have Euler characteristic $3 - 2g$. Adding these two equations, we obtain

$$e(\overline{\mathcal{M}}_1) = e(\mathcal{C}) + (2g - 2)e(\mathcal{M}), \qquad (5.9)$$

which is the first equation in 5.4.

The case of $\delta = 2$ requires additional explanation.

We have to calculate $e(\overline{\mathcal{M}}_2)$. We have the equations

$$
\begin{aligned}
e(\mathcal{C}_2) &= (4 - 2g)e(\overline{\mathcal{M}}_2) \\
e(\mathcal{C}_1) - e(\mathcal{C}_2) &= (3 - 2g)(e(\overline{\mathcal{M}}_1) - e(\overline{\mathcal{M}}_2)) \\
e(\mathcal{C}) - e(\mathcal{C}_1) &= (2 - 2g)(e(\mathcal{M}) - e(\overline{\mathcal{M}}_1))
\end{aligned}
$$

obtained as before. Adding these equations gives

$$e(\mathcal{C}) = (2 - 2g)e(\mathcal{M}) + e(\overline{\mathcal{M}}_1) + e(\overline{\mathcal{M}}_2). \qquad (5.10)$$

We next derive another equation to eliminate $e(\overline{\mathcal{M}}_1)$ by considering $\mathcal{C}^{(2)}$. As above, let $\mathcal{C}_i^{(2)}$ be the restriction of the map $\mathcal{C}^{(2)} \to \mathcal{M}$ to the part lying over $\overline{\mathcal{M}}_i \subset \mathcal{M}$ for $i = 1, 2$. We calculate the Euler characteristics of the strata $\mathcal{C}_i^{(2)} - \mathcal{C}_{i+1}^{(2)}$, a new point needing explanation being the role of the nodes. Writing the node locally as $xy = 0$, we see that there is a $\mathbf{P}^1$ moduli space for the schemes of multiplicity 2 at the origin. Recall that locally schemes are the same thing as ideals, so a scheme of multiplicity 2 at the origin is just an ideal I of polynomials in x, y such that the origin has multiplicity 2. It is easy to see that I must be generated by a linear and a quadratic polynomial in x, y, both vanishing at the origin. Given a linear polynomial $ax + by$, there is actually no need to specify a choice of quadratic polynomial $q(x, y)$, since q, taken together with the quadratic polynomials $x(ax + by)$ and $y(ax + by)$ spans a 3 dimensional space, necessarily the entire space of quadratic polynomials vanishing at the origin. Explicitly, these are the schemes $Z_{a,b}$ defined by the ideals $I_{a,b} = (ax + by, x^2, xy, y^2)$, where $(a, b) \in \mathbf{P}^1$. Note that $xy \in I_{a,b}$, so that the $Z_{a,b}$ are indeed contained in the nodal curve.

This is the new ingredient that we need to calculate the Euler characteristics of the strata. To get $\mathrm{Hilb}^2(C)$, where C is a curve with i nodes, we take its second symmetric product, and replace the single

point 2(node) with a $\mathbf{P}^1$, for each node. This says that since C has i nodes, then

$$
\begin{aligned}
e(\text{Hilb}^2 C) &= e(\text{Sym}^2 C) + i \\
&= \binom{i+3-2g}{2} + i.
\end{aligned}
\tag{5.11}
$$

This leads immediately to the equations

$$
e(\mathcal{C}_2^{(2)}) = \left(\binom{5-2g}{2} + 2 \right) e(\overline{\mathcal{M}}_2)
$$

$$
e(\mathcal{C}_1^{(2)}) - e(\mathcal{C}_2^{(2)}) = \left(\binom{4-2g}{2} + 1 \right) (e(\overline{\mathcal{M}}_1) - e(\overline{\mathcal{M}}_2))
$$

$$
e(\mathcal{C}^{(2)}) - e(\mathcal{C}_1^{(2)}) = \left(\binom{3-2g}{2} \right) (e(\mathcal{M}) - e(\overline{\mathcal{M}}_1))
$$

We add these equations and obtain

$$
e(\mathcal{C}^{(2)}) = \binom{3-2g}{2} e(\mathcal{M}) + (4 - 2g)e(\overline{\mathcal{M}}_1) + (5 - 2g)e(\overline{\mathcal{M}}_2).
\tag{5.12}
$$

We now can eliminate $e(\overline{\mathcal{M}}_1)$ from 5.10 and 5.12 and get

$$
e(\overline{\mathcal{M}}_2) = e(\text{Hilb}^2(\mathcal{C}/\mathcal{M})) + (2g - 4)e(\mathcal{C}) + (g - 1)(2g - 5)e(\mathcal{M}),
\tag{5.13}
$$

the second equation in 5.4.

We turn next to $\delta = 3$. The calculation begins as in the previous cases, and we get the equations

$$
\begin{aligned}
e(\mathcal{C}) &= (2 - 2g)e(\mathcal{M}) + e(\overline{\mathcal{M}}_1) + e(\overline{\mathcal{M}}_2) + e(\overline{\mathcal{M}}_3) \\
e(\mathcal{C}^{(2)}) &= \binom{3-2g}{2} e(\mathcal{M}) + (4 - 2g)e(\overline{\mathcal{M}}_1) \\
&\quad + (5 - 2g)e(\overline{\mathcal{M}}_2) + (6 - 2g)e(\overline{\mathcal{M}}_3).
\end{aligned}
\tag{5.14}
$$

We now have to bring in $\mathcal{C}^{(3)}$ to derive one more equation for the purpose of eliminating $e(\overline{\mathcal{M}}_1)$ and $e(\overline{\mathcal{M}}_2)$. We have to explain how to calculate $\mathcal{C}^{(3)}$. So we need to know how to calculate the Euler characteristic of Hilb^3 of a curve C of arithmetic genus g with i nodes for

$i = 1, 2, 3$. We study the map $\mathrm{Hilb}^3 C \to \mathrm{Sym}^3 C$ and see where it fails to be an isomorphism. This is precisely over the points $2p + q$ and $3p$ of $\mathrm{Sym}^3 C$, where $p \in C$ is a node and $q \neq p$ is arbitrary. As in the discussion leading to 5.11, we replace $2p + q$ by $\mathbf{P}^1 \times q$, where $\mathbf{P}^1$ is the $\mathbf{P}^1$ of tangent directions to C at p. So for each node p_i, we replace a subset $\{p_i\} \times C - \{p_i\}$ by $\mathbf{P}^1 \times C - \{p_i\}$, adding $i(i + 1 - 2g)$ to the Euler characteristic. As for $3p$, we write the node locally as $xy = 0$ and look for multiplicity 3 schemes contained in $xy = 0$ and concentrated at $(0, 0)$. It suffices to compute the Euler characteristic, which is just the number of fixed points of a torus action $(x, y) \mapsto (t^a x, t^b y)$ where $a \neq b$ are arbitrary. These are just the points $(x, y^3), (y, x^3)$, and (x^2, xy, y^2), so the Euler characteristic is 3.[10] This gives

$$
\begin{aligned}
e\left(\mathrm{Hilb}^3 C\right) &= \binom{4 + i - 2g}{3} + i(i + 1 - 2g) + 2i \\
&= \binom{4 + i - 2g}{3} + i(i + 3 - 2g).
\end{aligned}
$$

We now immediately get the equations

$$
\begin{aligned}
e(\mathcal{C}_3^{(3)}) &= \binom{4 - 2g}{3} e(\overline{\mathcal{M}}_3) \\
e(\mathcal{C}_2^{(3)}) - e(\mathcal{C}_3^{(3)}) &= \left(\binom{5 - 2g}{3} + 4 - 2g \right) \left(e(\overline{\mathcal{M}}_2) - e(\overline{\mathcal{M}}_3) \right) \\
e(\mathcal{C}_1^{(3)}) - e(\mathcal{C}_2^{(3)}) &= \left(\binom{6 - 2g}{3} + 2(5 - 2g) \right) \left(e(\overline{\mathcal{M}}_1) - e(\overline{\mathcal{M}}_2) \right) \\
e(\mathcal{C}^{(3)}) - e(\mathcal{C}_1^{(3)}) &= \left(\binom{7 - 2g}{3} + 3(6 - 2) g \right) \left(e(\mathcal{M}) - e(\overline{\mathcal{M}}_1) \right)
\end{aligned}
$$

Adding, we get the formula

$$
e(\mathcal{C}^3) = \binom{4 - 2g}{3} e(\mathcal{M}) + \left(\binom{4 - 2g}{2} + 4 - 2g \right) e(\overline{\mathcal{M}}_1) \tag{5.15}
$$
$$
+ \left(\binom{5 - 2g}{2} + 6 - 2g \right) e(\overline{\mathcal{M}}_2) + \left(\binom{6 - 2g}{2} + 8 - 2g \right) e(\overline{\mathcal{M}}_3).
$$

---

[10]It can be seen that the set of all multiplicity 3 schemes is isomorphic to $\mathbf{P}^2$, while those contained in the nodal curve are $\mathbf{P}^1 \cup \mathbf{P}^1$, the first $\mathbf{P}^1$ being $\{(ax + by^2, x^2, xy, y^3)\}$ for $(a, b) \in \mathbf{P}^1$, the other $\mathbf{P}^1$ being obtained by interchanging x and y.

We can now solve our 3 equations in 5.14 and 5.15 for $e(\overline{\mathcal{M}}_3)$, obtaining

$$
\begin{aligned}
e(\overline{\mathcal{M}}_3) \;=\;& e(\mathcal{C}^{(3)}) + (2g-6)e(\mathcal{C}^{(2)}) + (g-2)(2g-7)e(\mathcal{C}) \quad (5.16)\\
& +\frac{2}{3}(g-1)(g-3)(2g-7)e(\mathcal{M}) \;,
\end{aligned}
$$

the third equation in 5.4.

The same method applied to $\delta = 4$ yields the fourth equation in 5.4. We use the general fact that the Hilbert scheme of multiplicity k points concentrated at a node has Euler characteristic k. This fact can be easily verified using fixed points of torus actions. We have already seen this result explicitly for $k \leq 3$.

Note that it is clear that this method of calculation generalizes to arbitrary genus. We do not know how to carry this out in closed form, but do presume that the answer is given by 5.5 since we have derived this in 4.16 and we will offer several checks of these formulas in later sections.

6   Application to Counting M2 branes in $K3 \times T^2$

Before we come to the application of the formalism developed in Sections 4 and 5 to the general case of Calabi-Yau threefolds, we consider the simpler $K3 \times T^2$ case[11]. In this case the topological string amplitudes are rather trivial (except for genus 1 which is 24 times the logarithm of the η function). The reason for this triviality is also easy to explain from the view point of BPS degeneracy of wrapped M2 branes in M-theory compactification on $K3 \times T^2$: The BPS spectrum of states which preserve exactly 1/4 of the supersymmetry (which has the same amount of supersymmetry preservation as for the $M2$ branes wrapped around a generic Calabi-Yau) are longer: They are in general of the form

$$
R \otimes I_1^L \otimes I_1^R \tag{6.1}
$$

---

[11]A similar analysis can be done for $T^4 \times T^2$ with symmetric product of $K3$ playing the same role as symmetric product of $K3$ plays here.

where R is some representation of $SU(2)_L \times SU(2)_R$ and $I_1^L = (1/2, 0) + 2(0,0)$ and $I_1^R = (0, 1/2) + 2(0, 0)$. These states were recently [83] starting from a type II one-loop computation. We observe here that there is an extra factor of I_1^R in the above representation. When we consider the relevant index contributing to topological string amplitudes, by summing over the right representation with a $(-1)^{F_R}$, the I_1^R factor kills the contribution. The geometric explanation of this, in term of the moduli space we have discussed is that the moduli space of M2 brane configurations is a product space with a factor including a T^2. This is because we can use the $U(1) \times U(1)$ symmetry of the T^2 to obtain a new holomorphic curve from any given one. This introduces an extra factor I_1^R in the representation. The only case where this action is trivial, and the I_1^R is absent from the above representation, is if the $M2$ brane lies entirely in the T^2, i.e. it is a point in $K3$ and wraps the T^2. From the point of view of representation theory the fact that the I_1^R does not appear in that case is that this BPS state preserves $1/2$ of the supersymmetry and so it is a shorter multiplet. The moduli space of the $M2$ brane wrapping T^2 and projecting to a point in $K3$ is simply $K3$, whose Euler characteristic is 24. Taking into account the N fold bound state which always exists at genus 1, we reproduce the prediction of the topological string amplitude at genus 1 and its vanishing at all other genera.

However, clearly there is an enormous amount of information in precisely which representations R appear in 6.1 . In particular, for these BPS states, we can omit the factor of I_1^R, and again concentrate on the $SU(2)_L$ action and sum over the right states (with a $(-1)^{F_R}$) and define the degeneracy number n_d^r, just as in the generic Calabi-Yau case. Here d is an integral H_2 class of $K3 \times T^2$ and r labels the $SU(2)_L$ representation content in terms of I_r. We can still ask how to compute n_d^r numbers using the techniques of this paper. For simplicity of notation we consider a "topological string amplitude" F_r using these numbers as input parameters, without worrying whether or not they come from any 2d topological theories[12].

The class d can be viewed as coming from a class $C \in H_2(K3)$ and

---

[12]It is likely that they do. For example it is natural to expect that by some insertion of operators at higher genera one can effectively "cancel" the I_1^R contribution above. For example an insertion of $\int J_L J_R$ on the world sheet, where $J_{L,R}$ denote the left- and right-moving $U(1)$ currents of $N = 2$ algebra may do the job.

a class in $H_2(T^2)$ defined by an integer M times the basic class. By diffeomorphism symmetry of $K3$ the number of BPS states for $d = [C, M]$, as far as the C dependence goes should only depend on $C^2 = 2N - 2$. Thus we can recast the computation in terms of finding the degeneracy associated to the choice of two integers (N, M).

6.1 Zero winding on the T^2

Let us first consider the case where $M = 0$, i.e. that the BPS states correspond to wrapping only the $K3$ space, and to a choice of a point on T^2. The T^2 space enters in a rather trivial way (simply giving the I_1^R factor noted above) and essentially drops out of further consideration for this case. We are then just asking about the BPS spectrum of $M2$ brane wrapped over M-theory compactifications on $K3$. Since M-theory on $K3$ is dual to heterotic string on T^3 [84], the heterotic string dual gives an immediate prediction for the number of BPS states, as well as their $SU(2)_L \times SU(2)_R$ quantum numbers. The answer for the dimension of the representation R (summed over all states weighted with $(-1)^{F_L + F_R}$) gives the degeneracy of the left oscillator of the heterotic string at level N. In other words

$$\sum n_{N,0}^{g=0} q^N = \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^{24}}.$$

This structure follows from the fact that there are 24 left oscillators α_{-n}^i where i runs from 1 to 24 and n runs over all positive integers. For example, if $N = 3$, the BPS states are specified by the symmetrization of the states

$$\alpha_{-1}^i \alpha_{-1}^j \alpha_{-1}^k |0\rangle, \quad \alpha_{-1}^i \alpha_{-2}^j |0\rangle, \quad \alpha_{-3}^k |0\rangle,$$

of the heterotic string. In fact we can also easily read off the $SU(2)_L \times SU(2)_R$ content of these states as well, because the $SU(2)_L \times SU(2)_R = SO(4)$ is identified with its canonical embedding in $SO(24)$. In other words, each oscillator α_{-n}^i corresponds to the representations[13]

$$[\alpha_{-n}^i] = 20(0,0) + \left(\frac{1}{2}, \frac{1}{2}\right)$$

---

[13]The same reasoning applies to the $SU(2)_L \times SU(2)_R$ decomposition of rational elliptic surfaces. This makes it easy to calculate the higher genus invariants in a fixed class $[B] + n[F]$ [5] .

We can thus decompose the BPS states above in terms of the $SU(2)_L \times SU(2)_R$ quantum numbers inherited from each oscillator. For example, the contributions of the three types of states given above are

$$\binom{22}{3}(\mathbf{0}, \mathbf{0}) + \binom{21}{2}\left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}\right) + 20((\mathbf{1}, \mathbf{1}) + (\mathbf{0}, \mathbf{0}))$$

$$+ \left(\frac{\mathbf{3}}{\mathbf{2}}, \frac{\mathbf{3}}{\mathbf{2}}\right) + \left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}\right) + 400(\mathbf{0}, \mathbf{0}) + 40\left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}\right)$$

$$+ (\mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{0}) + (\mathbf{0}, \mathbf{1}) + (\mathbf{0}, \mathbf{0}) + 20(\mathbf{0}, \mathbf{0}) + \left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}\right)$$

$$= 1984(\mathbf{0}) - 504\left(\frac{\mathbf{1}}{\mathbf{2}}\right) + 64(\mathbf{1}) - 4\left(\frac{\mathbf{3}}{\mathbf{2}}\right) ,$$

where we have summed in the last expression over the right representation with $(-1)^{F_R}$. Reexpressed in the I_g the result reads

$$3200 I_0 - 800 I_1 + 88 I_2 - 4 I_3 . \tag{6.2}$$

The above calculation can be easily systematized by writing the oscillator partition function for the oscillators in the representation $20(\mathbf{0}, \mathbf{0}) + \left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}\right)$, one for each integer. In particular one obtains

$$\prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^{20}(1 - yq^n)^2(1 - y^{-1}q^n)^2} \tag{6.3}$$

$$= \sum_{r=0, d=0}^{\infty} (-1)^r n_{N,0}^r (y^{\frac{1}{2}} - y^{-\frac{1}{2}})^{2r} q^d .$$

On the right-hand side $n_{N,0}^r$ is the number of BPS states in the representation I_r^L with charge whose square is $2N - 2$. The identification follows by noting that I_g contains $\binom{2g}{g+i}$ states with J_L^3 eigenvalue $i/2$, or alternatively since I_1 has one state with J_L^3 eigenvalue $\pm 1/2$, while $I_g = I_1^{\otimes g}$. The expression 6.3 contains information about all genus, and with 3.3 one can resum it to write the total free energy as $F = \sum_{m=1}^{\infty} \frac{1}{m} F^{(m)}$, where the last sum is over the multicovering

contributions with[14]

$$F^{(m)} = \left(2 \sin \frac{m\lambda}{2}\right)^{-2} \times$$

$$\prod_{n=1}^{\infty} \frac{1}{(1 - q^{mn})^{20}(1 - e^{i\lambda m}q^{mn})^2(1 - e^{-i\lambda m}q^{mn})^2} \cdot$$

(6.4)

Of course all information about the n_d^r is already in $F^{(1)}$.

Let us summarize for concreteness some of the low degree invariants.

$n_{N,0}^r$	$r = 0$	1	2	3	4	5	6
$N = 0$	1						
1	24	−2					
2	324	−54	3				
3	3200	−800	88	−4			
4	25650	−8550	1401	−126	5		
5	176256	−73440	15960	−2136	168	−6	
6	1073720	−536860	145214	−25670	3017	−214	7

Table 1: The weighted sum of BPS states $n_{N,0}^r$ for classes in the $K3$.

These predictions for the spectrum of BPS states for M-theory compactification on $K3$ is based on duality with heterotic strings. One can ask if one can derive these spectra directly in M-theory context. In particular as discussed before, we would have to study the moduli space of holomorphic curves in $K3$ together with a flat bundle, in the class C whose self-intersection is $2N - 2$. This has been considered [85], and the above result from heterotic string was reproduced. The basic idea is rather simple: With some choice of complex structure, we can assume $K3$ is an elliptic surface over $\mathbf{P}^1$. Moreover, by global diffeomorphisms we can assume the cycle C is represented by the class $[B] + N[F]$ where $[B]$ denotes the class of the base $\mathbf{P}^1$ and $[F]$ denotes the class of the

---

[14]This decomposition into spins gives the higher genus result also for curves in the $K3$ classes of $K3$ fibered threefolds. If we take into account the multiplicity due to base $\mathbf{P}^1$ integration (-2) and the lattice sum, captured already at genus zero, and multiply therefore by $(-2)E_4(q)E_6(q)$ we get the higher genus answer for the $K3$ fibration $X_{24}(1,1,2,8,12)$, which was obtained from a one loop computation in [25] .

elliptic fiber. The moduli space of curves in this class corresponds to degenerate Riemann surfaces of genus N and is simply given by the choice of N points on $\mathbf{P}^1$ to which the N elliptic fibers are attached. In the computation of the BPS states, we are instructed to consider also flat bundles on the Riemann surface. In this degenerate limit, that choice is easy: it simply corresponds to the choice of a flat bundle on each elliptic fiber. That in turn is equivalent to a choice of a point on a dual elliptic fiber. All said and done, the choice of N points on $\mathbf{P}^1$ and a point on the dual elliptic fiber over each point, shows that the moduli space of curves with the flat bundle is equivalent to the choice of N points on the T-dual $K3$. Since the ordering of the points are immaterial, this corresponds to the N fold symmetric product of $K3$, or more precisely, the Hilbert scheme of N points on $K3^{15}$. Thus the moduli space is given by

$$\widehat{\mathcal{M}} = \mathrm{Hilb}^N(K3)$$

The cohomology of this space can be identified in the usual way [86] with the Hilbert space of 24 oscillators at level N, and exactly reproduces the above results for the heterotic string. Moreover the $SU(2)_L \times SU(2)_R$ decomposition can also be deduced from the corresponding decomposition for the cohomology of a single copy of $K3$. With the identification of $SU(2)_L$ with the elliptic fiber direction and $SU(2)_R$ with the base directions, we immediately get the decomposition

$$24 \to 20(0,0) + \left(\frac{1}{2}, \frac{1}{2}\right),$$

as this is the unique representation whose diagonal $SU(2)$ content is $(\mathbf{1}) \oplus 21(\mathbf{0})$, the Lefschetz representation of K3, while the $SU(2)_R$ content of left-spin 1/2 is $(\mathbf{1/2})$, the Lefschetz representation on the base $\mathcal{M} = \mathbf{P}^1$. This reproduces the result based on duality with heterotic strings given above.

In [27], the coefficients c_N of I_0, which as discussed is the Euler characteristic of $\mathrm{Hilb}^N(K3)$, were related to genus zero curves coming from degenerate genus N curves with exactly N nodes. As the N

---

[15]It would be nice to make this argument mathematically more rigorous. What has to be checked is that this correspondence continues to hold when several fibers are allowed to coincide. The details will require a mathematical study of sheaves on non-reduced curves.

continuous parameters of the moduli space $\mathbf{P}^N$ of the genus N curve are completely killed by the imposition of the N nodes, this eventually leads to the counting of points. Here we consider the intermediate cases, the genus $N - \delta$ curves, where we impose $0 \leq \delta \leq N$ nodes. As this leaves a δ dimensional moduli space, an appropriate virtual fundamental class on this space is needed to reduce the dimension to 0^{16}. The formula 5.2 is equivalent to the assumption that the obstruction bundle in the case of a smooth moduli space is the cotangent bundle, since the Euler class of the cotangent bundle is the Euler characteristic of the moduli space up to sign.

For example, the coefficients in 6.2 correspond to invariants n_3^r associated to genus $r = 0, 1, 2, 3$ curves obtained by putting nodes on the degree $d = 3$ genus $g = 3$ curve. The moduli space of such curves has dimension $r = 0, 1, 2, 3$, and the virtual fundamental class has the same codimension. So the n_3^r (and multiple cover/bubbling contributions) can be thought of as computable by taking the virtual class and performing an additional localization on the positive dimensional moduli space $\bar{\mathcal{M}}_\delta$ of curves with δ nodes. By the discussion of Section 4 and 5, we can instead calculate these using the invariants $e(\mathcal{C}_{[N]}^{(k)})$ for $k \leq \delta$. In other words, in this case we have two geometric models for computing $n_{[N,0]}^r$: One is based on the Hilbert scheme of N points on $K3$, which we have already discussed, and it agrees with the predicted answer from heterotic string. Another way to compute these numbers is to follow the strategy developed in previous sections and relate these numbers to $e(\mathcal{C}_{[N]}^{(k)})$. This will be useful, as it will also tell us how in some cases where these spaces are singular, we may nevertheless define unambiguous answers.

Let's check a few cases of these numbers. For any N, we have $\mathcal{M} = \mathbf{P}^N$.

For $N = 0$, the moduli space is a point, and $n_0^0 = 1$.

For $N = 1$, the moduli space is $\mathcal{M} = \mathbf{P}^1$, giving $n_1^1 = -2$ by 5.3. Choosing the complex structure so that the K3 is elliptically fibered and our family of curves is the fiber class, we see that $\mathcal{C}$ is just the K3. So from [27] or from 5.4, we get $n_1^0 = e(\mathcal{C}) = 24$.

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[16]A related problem was considered in [87], where the dimensions of the moduli space was reduced to 0 by forcing the curves to go through $k - \delta$ points.

For $N = 2$, we again get $n_2^2 = 3$ (and more generally, $n_g^g = (-1)^g(g + 1)$). Let us choose the complex structure to be that of $S = \mathbf{P}(1,1,1,3)[6]$. The projection $\pi : S \to \mathbf{P}^2$ onto the first 3 coordinates is a 2-1 cover. The inverse images C via π of the lines in $\mathbf{P}^2$ define the genus two curves. To see this, letting H be the hyperplane class of $\mathbf{P}^2$ we compute

$$C^2 = (\pi^*(H))^2 = \pi^*(\text{point}) = 2,$$

since 2 points of S lie over a point of $\mathbf{P}^2$. Since $C^2 = 2N - 2 = 2$, this verifies that the genus is $N = 2$. To calculate $\mathcal{C}$, as usual we project $\mathcal{C}$ onto S and note that the fiber is always $\mathbf{P}^1$ as follows. Given a point p of a curve C (so that $(p, C) \in \mathcal{C}$), the curves C through p are in 1-1 correspondence with the lines of $\mathbf{P}^1$ through $\pi(p)$, and this is always a $\mathbf{P}^1$. This gives $e(\mathcal{C}) = e(S)e(\mathbf{P}^1) = 48$, and by 5.4, we get $n_2^1 = -(48 + 2 \cdot 3) = -54$.

But now something interesting happens. The space $\mathcal{C}^{(2)}$ is not a projective bundle over $\text{Hilb}^2(S)$. To see this, let's pick a point Z of $\text{Hilb}^2(S)$. This usually projects via π to a point Z' of $\text{Hilb}^2(\mathbf{P}^2)$. When this happens, there is a unique line ℓ connecting the two points of Z' in $\mathbf{P}^2$, hence a unique curve $C = \pi^{-1}(\ell)$ in S passing through the two points of Z. So we might have thought that $\mathcal{C}^{(2)}$ is isomorphic to $\text{Hilb}^2(S)$, with Euler characteristic 324. But this is not true! If the point Z consisted of the 2 points of $\pi^{-1}(p)$ for any point $p \in \mathbf{P}^2$, then it maps via π to just the point p of $\mathbf{P}^2$. There is a $\mathbf{P}^1$ of lines through p, whose inverse images via π are a $\mathbf{P}^1$ of curves in S through Z. In other words, there is an isomorphic copy of $\mathbf{P}^2$ embedded in $\text{Hilb}^2(S)$ via the map $p \mapsto \pi^{-1}(p)$, and each of these points gets replaced by a $\mathbf{P}^1$ in $\mathcal{C}^{(2)}$. It is not difficult to see that $\mathcal{C}^{(2)}$ is the blowup of $\text{Hilb}^2(S)$ along $\mathbf{P}^2$. This tells us that $\mathcal{C}^{(2)}$ is smooth, so that 5.4 applies. It also tells us that $e(\mathcal{C}^{(2)}) = e(\text{Hilb}^2(S)) + e(\mathbf{P}^2) = 324 + 3 = 327$. Now, 5.4 gives $n_2^0 = 324$. We will give a physical way to calculate $e(\mathcal{C}^{(2)})$ using $K3 \times T^2$ soon.

In general, if the map $\mathcal{C}^{(k)} \to \text{Hilb}^k(K3)$ is a projective bundle, then we see that $\mathcal{C}^{(k)}$ is smooth, and its Euler characteristic is $(g - k + 1)e(\text{Hilb}^k(K3))$, where g is the genus N. We can then calculate n_g^{g-k} using 5.5. If it is not a projective bundle, then there are two problems. First, the Euler characteristic is more difficult to calculate. Second, and more seriously, the space $\mathcal{C}^{(k)}$ need not be smooth, so there may be a

virtual fundamental class to calculate instead of the Euler characteristic and our formulas need not be valid.

Let's check $N = 3$. We have $\mathcal{M} = \mathbf{P}^3$ and $n_3^3 = -4$. It is not hard to see (and we will check presently) that $\mathcal{C}$ and $\mathcal{C}^{(2)}$ are projective bundles, so that $e(\mathcal{C}) = 3 \cdot 24 = 72$ and $e(\mathcal{C}^{(2)}) = 2 \cdot 324 = 648$. This gives, by 5.4, $n_3^2 = 72 + 4 \cdot 4 = 88$, and $n_3^1 = -(648 + 2 \cdot 72 + 2 \cdot 4) = -800$.

To compute $e(\mathcal{C}^{(3)})$, we choose our K3 to have the complex structure of a degree 4 hypersurface $S \subset \mathbf{P}^3$, and the genus 3 curves C to be the plane sections of S (which are all degree 4 plane curves). Note that there is a $\mathbf{P}^2$ of planes passing through any point of S, and a $\mathbf{P}^1$ of planes containing any 2 points of S, verifying the projective bundle structure on $\mathcal{C}$ and $\mathcal{C}^{(2)}$ just mentioned. Given three points of S, there is typically a unique plane containing these points (the plane they span), but this fails when the 3 points lie on a line ℓ, in which case we get a $\mathbf{P}^1$ of planes. In other words, the set of triples of collinear points in S forms a subset T of $\mathrm{Hilb}^3(S)$, and we replace T by a $\mathbf{P}^1$ bundle over T to get $\mathcal{C}^{(3)}$. Thus $e(\mathcal{C}^{(3)}) = e(\mathrm{Hilb}^3(S)) + e(T)$.

There is an alternative way to describe T. Let $p_0 \in S$ be arbitrary. Let ℓ be any line containing p_0 (for fixed p_0, there is a $\mathbf{P}^2$ of such lines). Then $\ell \cap S$ consists of 4 points, of which p_0 is one of them. So there are 3 remaining points p_1, p_2, p_3, which collectively give a point of T. Thus the data of T is a point of S and a point of $\mathbf{P}^2$, and so T is smooth and $e(T) = e(S)e(\mathbf{P}^2) = 72$. Furthermore, it is again straightforward to check that $\mathcal{C}^{(3)}$ is the blowup of $\mathrm{Hilb}^3(S)$ along T, so is smooth. Thus $e(\mathcal{C}^{(3)}) = 3272$ (which we will check soon by a different method), and by 5.4, $n_3^0 = 3272 + 0 - 72 + 0 = 3200$.

In principle these calculations can be continued to higher N, at the expense of having to use increasingly difficult projective geometry to complete the calculation. We will return to more of these calculations shortly when we see that many of the moduli spaces $\mathcal{M}$ for $K3 \times T^2$ with $M \neq 0$ are precisely the relative Hilbert schemes $\mathcal{C}^{(k)}$ for $K3$!

6.2   More general H_2 classes in $K3 \times T^2$

We will now relax the assumption that $M = 0$ and also consider the case where the $M2$ brane wraps M times around the T^2. When $M = 0$ we used three ways to compute it in the previous section: One was the duality with heterotic strings. The other was the direct definition in M-theory, leading to Hilbert scheme of $K3$ and the third one, was based on the methods we developed in this paper. The first two approaches gave a complete answer, whereas the third approach was somewhat incomplete (even though in principle one should be able to push that program of computation sketched in some cases in the previous section). For the case when $M \neq 0$ it turns out we only know one method to compute the exact answer and that is based on duality between M-theory on $K3 \times T^2$ and Type IIB string on $K3 \times S^1$. The other two methods, are more difficult and we do not know how to use the direct definition of BPS states of M2 brane to obtain the results predicted by this duality. We will discuss aspects of these computations in special cases below. But first we show what the duality between M-theory and type IIB predicts for all the numbers $n^r_{[N,M]}$ defined before.

M-theory on T^2 is dual to type IIB on S^1. Through this duality the $M2$ brane in 9 dimensions, gets mapped to a $D3$ brane wrapped around S^1. Moreover the momentum quantum number around S^1 of type IIB gets mapped to the quantum number of the $M2$ brane wrapping number over T^2. Now consider compactifying further on a $K3$. Then an M2 brane wrapped in some 2-cycle class of $K3 \times T^2$ given by $[C, M]$, gets mapped via this duality to $D3$ branes wrapped around $C \times S^1$ carrying momentum M around S^1. In fact, in the type IIB setup, these are exactly the class [88]. Let us consider the limit where the $K3$ is small. In this limit we have an effective leftover $1 + 1$ dimensional worldvolume of the $D3$ brane which is a supersymmetric sigma model on $\mathrm{Sym}^N(K3)$. Thus the moduli of D3 branes in the class C of $K3$ with $C^2 = 2N - 2$ is given by $\mathrm{Sym}^N(K3) \times R^4$, where the extra R^4 comes from the position of the $D3$ brane in the rest of the space. Thus the low energy dynamics of the 1+1 dimensional effective theory is a supersymmetric sigma model on this space. We are considering the spatial direction to be wrapped over an S^1 and we are looking for states which preserve 1/4 supersymmetry in the full theory, which correspond to states which preserve 1/2 supersymmetry in the $1 + 1$ dimensional

sigma model. These come from states with purely left-moving oscillator excitations, and restricting to the ground states of the right-moving Hilbert space. Moreover if we are interested in M2 brane BPS states with wrapping number M around T^2, we should look at left-moving oscillator excitation M. The $SU(2)_L \times SU(2)_R$ symmetry is realized in the Hilbert space of the sigma model by a left-moving current algebra which realizes $SU(2)_L$ and a right-moving current algebra [89], as far as the degrees of freedom is concerned on the symmetric product of $K3$'s. However for the R^4 factor, the $SU(2)_L$ current algebra is a right-moving current and the $SU(2)_R$ current is a left-mover[17]. Summing over the R states with $(-1)^{F_R}$ is *exactly* the definition of the elliptic genus of the sigma model on $\mathrm{Sym}^N(K3) \times R^4$. It was in fact the elliptic genus that was used to verify the predictions for black hole entropy [88], [89] ! So in this case we see that the left/right asymmetric treatment of the $SU(2)_L \times SU(2)_R$ is exactly the same as the useful elliptic genus index for supersymmetric sigma models, which has the required stability under deformations which allows one to predict at least a lower bound for the black hole entropy. The spinning quantum number of the black hole gets mapped in this case to the J_L^3 quantum number. We will use this link with black holes in the next section to connect the growth of n_d^r with predictions based on black hole entropy for M-theory compactification on a general Calabi-Yau 3-fold.

Due to interest from black hole physics the elliptic genus of symmetric products of $K3$ has been computed, including the modification due to the J_L^3 quantum number [90]. Let us denote the elliptic genus of a general manifold M

$$\chi(M; q, y) = \mathrm{Tr}_{\mathcal{H}(M)} (-1)^F y^{F_L} q^H = \sum_{M \geq 0, l \in Z} c(M, l) q^M y^l \qquad (6.5)$$

then the orbifold construction of [90] gives for the N-fold symmetric

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[17]The reason for this switch, relative to the $K3$ degrees of freedom is that in the gauge theory language in 6 dimensions (as in the dual $D1$–$D5$ brane dual systems), $K3$ degrees of freedom come from the Higgs branch and the R^4 from the Coulomb branch, and the fermions in these two multiplets have opposite 6-dimensional chiralities. This translates to the statement that the $SO(2)$ and $SO(4)$ chiralities are oppositely correlated in the two cases. This in particular means that the left-moving fermions do not carry any $SU(2)_L$ quantum number for the R^4 sigma models. Of course the bosonic oscillators are vectors and do carry both quantum numbers.

product $\mathrm{Sym}^N(M)$

$$\sum_{N=0}^{\infty} p^N \chi(S^N K3; q, y) = \prod_{\substack{N>0, M\geq 0, l\in Z}} \frac{1}{(1 - p^N q^M y^l)^{c(N\cdot M, l)}} \cdot \quad (6.6)$$

The elliptic genus of the K3 can be easily made concrete from an orbifold representation of the $K3$ (e.g. $K3 = T4/Z_2$) or a Landau-Ginzburg description as

$$\chi(K3; q, y) = 24 \left(\frac{\theta_3(q, y)}{\theta_3(q)}\right)^2 - 2\frac{\theta_4^4(q) - \theta_2^4(q)}{\eta^4} \left(\frac{\theta_1(q, y)}{\eta}\right)^2 .$$

We note the first terms of this expansion

$$q^0 : \frac{2}{y} + 20 + 2y = 2\begin{bmatrix}1\\2\end{bmatrix} + 20\,[0]\,, \quad q^1 : \frac{20}{y^2} - \frac{128}{y} + 216 - 128\,y + 20\,y^2.$$

Clearly the q^0 part of 6.6 gives 6.4 back. So far we ignored the fact that the momentum can be distributed in the space time directions as well. Taking this into account (and noting as discussed before that the $SU(2)_L$ current algebra is not active as far as the left-moving fermions from the R^4 part are concerned) we get for the single cover contribution

$$F^{(1)} = \left(2\sin\frac{\lambda}{2}\right)^{-2} \prod_{\substack{k>0 \, n>0 \\ m\geq 0, l\in Z}} \frac{(1 - q^k)^4}{(1 - e^{i\lambda}q^k)^2(1 - e^{-i\lambda}q^k)^2} \frac{1}{(1 - p^n q^m e^{il\lambda})^{c(n\cdot m, l)}}$$

$$=: \left(2\sin\frac{\lambda}{2}\right)^{-2} \sum_{N\geq 0, M\geq 0, L\in Z} c_{N,M,L}\, p^N q^M e^{iL\lambda}. \quad (6.7)$$

As with 6.4, we can sum over all multicoverings $F^{(m)}$ to obtain the free energy. Moreover we may multiply by a lattice theta functions if we wish to exhibit the sum over classes in $H^2(K3, Z)$.

Let us recapitulate the dictionary which relates this to the counting of $M2$ branes of M-theory in $K3 \times T^2$. N corresponds to the genus g of a smooth curve in the $K3$, which after a suitable complex structure deformation can be always thought as a curve C with degree N with respect to an elliptic K3's fiber class which has degree 1 in the base class. The powers M correspond to the degree with respect to the T^2.

The powers L (note the $y \to 1/y$ symmetry) correspond to the number of nodes δ, and for a given class $[M, N]$ we have $\delta = N + M - L$.

Let us consider the case $\delta = 0$ first. Geometrically the moduli space of these curves is modeled by $\mathcal{C}^{(M)}_{[N]}$, the degree M relative Hilbert scheme of the universal curve. To see this, note that a general curve of this type consists of a sort of "comb", the union of a single curve C on the K3[18] and M copies of T^2. The only moduli is where on C to attach the copies of T^2. This is given by specifying the M points on C, and the total moduli is $\mathcal{C}^{(M)}_{[N]}$, where $\mathcal{C}_{[N]}$ is now the universal curve of genus N on the K3 (i.e. there is no T^2 being considered at all).

By the general formula, $n^g_{[M,N]} = (-1)^{\dim \mathcal{M}} e(\mathcal{M})$, the arithmetic genus g being $M + N$. The $c_{N,M,M+N}$ give the Euler number $e(\mathcal{C}^{(M)}_{[N]})$ at least for the case that $\mathcal{C}^{(M)}_{[N]}$ is smooth and the relevant class for the obstruction theory is in the cotangent bundle of $\mathcal{M}$. If $\mathcal{C}^{(M)}_{[N]}$ is singular then the physical prediction $c_{N,M,M+N}$ should calculate an integral over a suitable, but not yet understood, virtual fundamental class, the result of which we may call in a slight abuse of notion also $e(\mathcal{C}^{(M)}_{[N]})$. The important point here is that we find that the relations between the n^r_d and the so defined $e(\mathcal{C}^{(M)}_{[d]})$ are exactly reproduced by our general formula 4.16 !

Let us list the predictions for the values for the $e(\mathcal{C}^{(M)}_{[N]}) = n^r_{[M,N]} = c_{N,M,M+N}$ and compare with the direct computation in several cases.

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[18]We have adjusted the complex structure of the K3 away from the elliptically fibered structure, so instead of C being the base plus g fibers, the curve C is typically irreducible.

$n_{N,M}^g$ $M=0$	1	2	3	4	5	6
$N=0$ 1	-2	3	-4	5	-6	7
1 -2	24	-48	72	-96	120	-144
2 3	-48	327	-648	972	-1296	1620
3 -4	72	-648	3272	-6404	9600	-12800
4 5	-96	972	-6404	26622	-51396	76955
5 -6	120	-1296	9600	-51396	185856	-353808
6 7	-144	1620	-12800	76955	-353808	1150675

Table 2: The weighted sum of BPS states $n_{N,M}^g$ for classes in $K3 \times T^2$ with $\delta = 0$.

These moduli spaces are the relative Hilbert schemes of the K3. The Euler characteristics of these relative Hilbert schemes can be sometimes computed directly, and are in agreement when smooth. As usual, we have $\mathcal{M} = \mathbf{P}^g$. As for the computation of $\mathcal{C}^{(M)}$, this is our standard method. We consider the map $\rho : \mathcal{C}^{(M)} \to \mathrm{Hilb}^M(K3)$. If this is a projective bundle then we can immediately compute $e(\mathcal{C}^{(M)})$. If not, then we analyze where it fails to be a projective bundle and correct as appropriate.

Letting $\mathcal{C}$ be the universal curve on a general K3 of genus N, we have seen from the comb description that $\mathcal{M} = \mathcal{C}^{(M)}$, leading to $n_{[M,N]}^g = e(\mathcal{C}^{(M)})$ when $\mathcal{C}^{(M)}$ is smooth. In particular, this occurs if N is large, with the asymptotic formula

$$n_{[M,N]}^g = (-1)^{N+M}(N+1-M)e(\mathrm{Hilb}^M(K3)), \qquad N \gg 0$$

which is consistent with the appearance of the columns of the above table. The deviations from this formula occur when there is not a projective bundle structure for $\mathcal{C}^{(M)} \to \mathrm{Hilb}^M(K3)$. We have already explained this for $M = N = 2$ and for $M = N = 3$.

Let's look at $N = 4$ and $M = 3$. We choose the complex structure to be that of $S = \mathbf{P}^4[2,3]$, the intersection of a degree 2 hypersurface Q and a degree 3 hypersurface T. The curves of genus 4 are the hyperplane sections. By dimension reasons, we might expect the fibers of $\mathcal{C}^{(3)} \to \mathrm{Hilb}^3(S)$ to be $\mathbf{P}^1$, but if the 3 points happen to lie on a straight line, there is a $\mathbf{P}^2$ of hyperplanes through the line, giving a fiber of $\mathbf{P}^2$ rather than $\mathbf{P}^1$. Let's call this locus $B \subset \mathrm{Hilb}^3(S)$. Now, if 3 points of S lie

on a line ℓ, then the quadratic equation defining Q has at least 3 zeroes on the line ℓ, hence vanishes identically. Thus $\ell \subset Q$. Conversely, any line contained in Q will meet S in 3 points (the three points where the line meets T). This shows that B is the set of lines contained in Q. So $\mathcal{C}^{(3)}$ is a $\mathbf{P}^1$ bundle over $\mathrm{Hilb}^3(S)$, except over B where it is a $\mathbf{P}^2$ bundle. This gives $e(\mathcal{C}^{(3)}) = 2 \cdot e(\mathrm{Hilb}^3(S)) + e(B)$. We can use standard techniques in algebraic geometry [91] and the Schubert software [92] to compute $e(B) = 4$. This gives the invariant 6404, in agreement with the table.

We can also check the $N = 1$ row. We have by 5.3 $n_{[1,M]}^{M+1} = (-1)^{M+1} e(\mathcal{C}^{(M)})$, where $\mathcal{C}$ is the universal curve of the elliptic fibration. If C is a smooth fiber, then the corresponding fiber of $\mathcal{C}^{(M)}$ is the Mth symmetric product of C, which has Euler characteristic 0. So it is only the 24 singular fibers that contribute to $e(\mathcal{C}^{(M)})$. Each of these singular fibers C has exactly one node, so $e(\mathrm{Hilb}^M(C)) = M$ by the discussion in the paragraph following 5.16. This gives $e(\mathcal{C}^{(M)}) = 24M$, or $n_{[1,M]}^{M+1} = (-1)^{M+1} 24M$, in agreement with the table.

Note that the symmetry of the table is immediate from duality and our calculations, but the geometric content of this symmetry is non-trivial, even in identifying the $N = 1$ row with the $M = 1$ column.

We can also consider the more difficult situation where we put nodes $\delta = N + M - L > 0$. Some invariants for $\delta = 1$ are given in the table below.

$n_{N,M}^{g-1}$ $M = 0$	1	2	3	4	5	6
$N = 0$ 0	0	−6	16	−30	48	−70
1 24	0	216	−432	768	−1200	1728
2 −54	720	−162	5712	−9768	15552	−22680
3 88	−1488	11832	−4368	83496	−135456	204872
4 −126	2376	−23016	138696	−65112	884184	−1398486
5 168	−3360	36000	−258048	1292712	−701856	7546536
6 −214	4440	−50328	396392	−2324790	10160160	−6086258

Table 3: The weighted sum of BPS states $n_{N,M}^{g-1}$ for classes in $K3 \times T^2$.

Note that the $M = 0$ column reproduces the K3 results found earlier.

7 Black Hole Entropy and Topological Strings

As discussed in the previous section in the context of M-theory compactification on $K3 \times T^2$, the spectrum of M2 branes wrapped around various cycles for large enough cycle classes would physically correspond to black holes. Moreover the $SU(2)_L$ content of the BPS state corresponds to the spin of the black hole. In such a case the growth of wrapped M2 branes in a given H_2 class and with given spin is anticipated by the Bekenstein-Hawking entropy of macroscopic black holes, which has been verified. Moreover, there are index-like quantities appearing in topological string theories, computed as a sum over the right-moving states with $(-1)^{F_R}$, which in the type IIB setup is perturbation invariant and is related to the computation of the elliptic genus. These are exactly the kind of computations which yield black hole entropy in the regimes where string perturbation breaks down.

It is thus natural to expect that this relation between BPS states in M-theory compactification on a general Calabi-Yau threefold continue to hold, namely the growth of the left BPS degeneracy with fixed $SU(2)_L$ content but summed over right $SU(2)_R$ quantum numbers with $(-1)^{F_R}$ will also yield the black hole entropy. In what we shall write here we will not worry about numerical constants in the formula for the growth of black hole entropy.

If we consider the macroscopic prediction for black hole entropy coming from a large charge $d \gg 1$ and for a given $J_L^3 = m$ spin one obtains [89], in the regime $d \gg m$ and $d \gg 1$,

$$N_{d,m} \sim \exp\sqrt{d^3 - m^2} \qquad (7.1)$$

This is meant to convey the exponential growth and its dependence in uniform rescaling of d and m, and is valid up to numerical coefficients in the exponent. Also there may be a sub-leading power law correction prefactor in front of the exponent.

One can try to compare this macroscopic prediction with the microscopic prediction. Since the $SU(2)_L$ content of the black hole entropy is captured by the numbers n_d^r, we can compare with the total number of BPS states with charge d and with J_L^3 spin m. All we need to do is

to recall that the full representation of the BPS states with charge d is given by

$$R = [\sum_{r=0}^{g} n_d^r I_r] \otimes I_1 \qquad (7.2)$$

This sum is finite, as for a given degree there is a maximum genus curve which realizes it. Moreover if we consider the I_g content of the state we have

$$Tr_{I_k} y^{J_L^3} = (y^{\frac{1}{2}} + y^{\frac{-1}{2}})^{2k}$$

This in particular means that the number of states in I_k with $J_L^3 = m$ spin is given by

$$\binom{2k}{k+m}$$

Applying this to 7.2 we see that the number $N_{d,m}$ is given by

$$N_{d,m} = \sum_r n_d^r \binom{2r+2}{r+1+m}$$

Comparing this with the growth expected from macroscopic considerations 7.1 , we get the prediction in the limit $d >> 1, d >> m$

$$\sum_r n_d^r \binom{2r+2}{r+1+m} \sim \exp\sqrt{d^3 - m^2} \qquad (7.3)$$

This is a very delicate sum, in that n_d^r are typically very big numbers for d large, which alternate in sign when one changes g by one unit, as we have seen in many examples in this paper. It is known that n_d^r for a fixed r grows with large d as $\exp(d)$. This however, is not in contradiction with the prediction 7.3 as one is summing over all non-vanishing r, and for large d's the allowed range in r is also large. In other words one is considering a different region of parameters and the equation 7.3 is a new prediction of the growth of these numbers in a different direction. It would be interesting to verify them (some special cases of this formula for Calabi-Yau threefold has been verified in [93] in the context of elliptic 3-folds).

8 Computations in local Calabi-Yau geometries

In this section we will apply the techniques developed in sections 4 and 5 to local Calabi-Yau geometries. By a local Calabi-Yau model we mean the total space $\mathcal{O}(K_B) \to B$ of the canonical line bundle fibered over a (two) dimensional Fano[19] variety B. For the base B we discuss here del Pezzo surfaces $\mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$ and E_n, the blow up of $\mathbf{P}^2$ in $n = 1, \ldots, 8$ points[20]. To calculate the invariants for a curve with δ nodes we have to calculate $e(\mathcal{C}^{(k)})$ for $k \leq \delta$, which is easy when there is a bundle structure $\rho_k : \mathcal{C}^{(k)} \to \mathrm{Hilb}^{(k)}(B)$. For a given degree d we find in general a bound $\delta < \delta_{max}(d)$ for which this is true. For instance, on $\mathbf{P}^2$ the first restriction comes from $\delta_{max}(6) = 8$. In general one can show that in this case $\delta_{max}(d) = d + 2$. Another complication arises when the curves are reducible. However we found a systematic procedure to account for the corrections due to reducible curves, which gives the expected answer whenever $e(\mathcal{C}^{(k)})$ could be calculated using the bundle structure. The information from these calculations were sufficient to fix the B-model ambiguity up to genus $g = 4, 5$ in the examples. When there is no bundle structure for $\mathcal{C}^{(k)}$ the modeling of these spaces becomes more complicated. While the bundle structure guarantees smoothness, we find in examples[21] that some of the spaces lacking the bundle structure are singular. Hence the Euler number $e(\mathcal{C}^{(k)})$ has to be replaced by an integral $\int_{[\mathcal{C}^{(k)}]_{vir}} c_{top}$ over a suitable virtual fundamental class. We have not attempted to define this virtual fundamental class. However in $K3 \times T^2$ case the quantity $\int_{[\mathcal{C}^{(k)}]_{vir}} c$ appeared in two places in our calculation fitting exactly our approximation of $H^*(\mathcal{J}(C))$ as expressed in $\mathcal{C}^{(k)}$ by 4.16 . We are therefore optimistic that there is natural definition of $\int_{[\mathcal{C}^{(k)}]_{vir}} c_{top}$ for the non-smooth $\mathcal{C}^{(k)}$ in the general case, which calculates the n_d^r invariants by the formulas in sections 4, 5.

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[19]The condition for B to be Fano can be relaxed, see [74] .

[20]Quantum intersection rings on these surfaces were [94]. In these cases the dimension of the moduli space of the curves is reduced to zero by requiring the curves to go through fixed points. There is no obvious relation between these invariants and the ones we calculate and relate to M-theory BPS states [95].

[21]$\mathcal{C}^{(8)}$ at $d = 6$ and $\mathcal{C}^{(9)}$ at $d = 7$ on $\mathbf{P}^2$ to be discussed below.

8.1 Basic concepts

First we need to calculate the Euler number of the moduli space $e(\mathcal{M})$ for the non-degenerate curves C, which are embedded in the surface B. More precisely, these curves are characterized by the fact that their geometric genus and their arithmetic genus coincide. We obtain the dimension of the deformation space $\mathbf{P}H^0(\mathcal{O}(C))$ from $\chi(\mathcal{O}(C)) = \sum_{i=0}^{n}(-1)^i h^i(\mathcal{O}(C))$ under the assumption that $H^1(\mathcal{O}(C)) = H^2(\mathcal{O}(C)) = 0$. This is true [96]. Then, using adjunction

$$C^2 + KC = (2g - 2)$$

and the Riemann-Roch formula

$$\chi(\mathcal{O}(C)) = \frac{C^2 - KC}{2} + 1,$$

we obtain $\chi(\mathcal{O}(C)) = g + d$, where d is the degree of the curve with respect to the anticanonical class $-K$ of B. Since the moduli space is obtained by projectivizing $H^0(\mathcal{O}(C))$, we get as the result

$$e(\mathcal{M}) = e(\mathbf{P}^{g+d-1}) = (g + d). \tag{8.1}$$

To be concrete, recall that the classes on a E_n del Pezzo are the $\mathbf{P}^2$ hyperplane class H and the exceptional divisors of the blowups e_i in n points $p_1, \ldots, p_n$, with $H^2 = 1$ and $e_i e_j = -\delta_{ij}$ (for $\mathbf{P}^1 \times \mathbf{P}^1$ see the later discussion). A curve of multidegree $(a; b_1, \ldots, b_n)$ refers to the class of curves obtained from blowing up the plane curves of degree a which pass through the point p_i with multiplicity b_i. These curves have class $aH - \sum b_i E_i$. We will typically rearrange the order of points to write the b_i in nonincreasing order. We will also omit the b_i which are 0. And we will use exponential notation b^k to refer to a subsequence of k copies of b. The exceptional divisors e_i do not fit into this classification scheme and will be denoted separately. To follow the subsequent discussion one needs an overview over the low degree classes in the del Pezzos surfaces, which we provide in Appendix A. They are ordered with respect to their arithmetic genus

$$g = \frac{(a-1)(a-2)}{2} - \sum_{i=1}^{n} \frac{b_i(b_i - 1)}{2} \tag{8.2}$$

and their degree with respect to the anticanonical class of the del Pezzos $-K_{E_n} = 3H - \sum_{i=1}^{n} e_i$

$$d = 3a - \sum_{i=1}^{n} b_i \ . \tag{8.3}$$

Let us consider as an easy example the E_1 del Pezzo surface. As it was observed by mirror symmetry in [42] there are invariants[22] $n_{1,a}^0 = 2a + 1$ from curves wrapping 1 times the base H and a times the fibre $H - e_1$ of the Hirzebruch surface $F_1 = E_1$, i.e. since $a = HC$ and $1 = (H - e_1)C$ these genus zero curves are in the class $(a; a - 1)$ and by 8.3, 8.1, 5.3 we immediately obtain the mirror symmetry prediction.

To calculate the invariants for curves with one node $\delta = 1$ we need next to determine the Euler number of the universal curve $e(\mathcal{C})$. We first fix a class $[C]$ with a curve C of arithmetic genus g and degree d in it. Then we calculate the contribution of a nodal curve in $[C]$ to the invariant n_d^{g-1} at genus $g - 1$ by 4.16 or 5.4, 5.3 . In the simplest situation the nodal curve is irreducible. The space $\mathcal{C}$, comes with a fibration structure as follows. Fixing the location of a point on the curve gives a linear constraint in the moduli space $\mathbf{P}^{g+d-1}$ of the genus g curve. As the point is free to vary over B, $\mathcal{C}$ comes with a natural fibration $\mathcal{C} \to B$ which, if it is smooth, gives rise to the Euler number

$$e(\mathcal{C}) = e(\mathbf{P}^{g+d-2})e(B) = (g + d - 1)e(B) \ . \tag{8.4}$$

In general if there is the bundle structure $\rho_k : \mathcal{C}^{(k)} \to \mathrm{Hilb}^{(k)}(B)$ then

$$e(\mathcal{C}^{(k)}) = e(\mathbf{P}^{g+d-1-k})e(\mathrm{Hilb}^{(k)}(B)) \tag{8.5}$$

with $\sum_k e(\mathrm{Hilb}^{(k)}(B))q^k = \prod_n \frac{1}{(1-q^n)^{e(B)}}$. The easiest example for the treatment of reducible curves is the calculation of n_4^0 in $\mathbf{P}^2$ in section 5.2 (after eq. 5.7). In the case of general del Pezzo surfaces we have to sum over various classes as will be explained in Section 8.5 for the case of the E_5 del Pezzo when we calculate n_4^0.

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[22]Which resum to the logarithm capturing the running of the gauge coupling in the N=2 gauge theory at the appropriate locus in the moduli space.

8.2 $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$

The conifold geometry does not support higher genus curves and the only rational curve that exists is the $\mathbf{P}^1$. Therefore the F_g of the type IIA topological theory[23] are completely governed by the multicover and bubbling contribution of this one rational curve. In this sense this simplest Calabi-Yau background to which 2-d gravity can be coupled is completely solved by 2.3 in full accordance to 3.3 . It is interesting that in this case the part of the F_g which follows from the anomaly can be made vanishing in the holomorphic limit by noting that there is a gauge choice in which all propagators of Kodaira-Spencer become identically zero, compare [67] . In this sense all holomorphic information here actually comes from the ambiguity.

8.3   **Local $\mathbf{P}^2$:** $\mathcal{O}(-3) \to \mathbf{P}^2$

This is the next simplest case. This geometry supports an infinite number of curves of different arithmetic genera, making the geometry much more interesting than in the conifold case.

We first present the result for the invariants. The numbers in the Table 4 marked with a diamond were obtained with the technique discussed in sections 4 and 5, as explained in detail below. Once we can fix the holomorphic ambiguity completely for a genus r, which was possible up to $r = 5$ in this case (see sect 8.6), the B-model gives an immediate answer for F_r at all degree d. Using this B-model result integrality of the n_d^r was checked up to $d = 300$. For $r = 6, 7, 8$ we could determine the ambiguity only up to $S = \{2, 4, 6\}$ constants, which can be parameterized by the n_d^r, $d = 8, \ldots 8 + S_i$. Assuming that they are integer we checked that the B-model gives rise to integer n_d^r up to $d = 300$. As a further check we compared with the calculation in [67] , which uses the $c = 1$ KDV hierarchy and direct localization techniques. Numbers marked with a star were calculated this way.

The $r = 1$ numbers listed in Table 4 follow the geometric subtraction scheme 2.4 . The differences in genus 1 between the geometric and the

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[23]Not to be confused with the IIB asymptotic behavior at the locus where an S^3 shrinks, which we discuss in Sections 2,5,6.

physical subtraction schemes $\Delta_d = (n_d^1 - n_d^{*1})$ are $\Delta_6 = -10$, $\Delta_8 = 231$, $\Delta_9 = -10$ and $\Delta_{10} = 4452$ up to given order.

d	$r = 0$	1	2	3	4	5
1	$3^\diamond_*$	0	0	0	0	0
2	$-6^\diamond_*$	0	0	0	0	0
3	$27^\diamond_*$	-10°_*	0	0	0	0
4	-192°_*	231_*	-102°_*	15°_*	0	0
5	1695°_*	-4452°_*	5430°_*	-3672°_*	1386°_*	-270°_*
6	-17064_*	80948_*	-194022_*	290853°_*	-290400°_*	$196857^\diamond$
7	188454	−1438086	5784837	−15363990	29056614	−40492272
8	−2228160	25301295	−155322234	649358826	−2003386626	4741754985
9	27748899	−443384578	3894455457	−23769907110	109496290149	−396521732268
10	−360012150	7760515332	−93050366010	786400843911	−5094944994204	26383404443193

d	$r = 6$	7	8	9	10	11	12	13	14	15
1	0	0	0	0	0	0	0	0	0	0
$\vdots$										
4	0	0	0	0	0	0	0	0	0	0
5	21°	0	0	0	0	0	0	0	0	0
6	-90390°	27538°	-5310°	585°	-28°	0	0	0	0	0
7	42297741°	-33388020°	19956296°	-9001908°	3035271°	-751218°	132210°	-15636°	1113°	-36°

Table 4: The weighted sum of BPS states n_d^r for the local $\mathbf{P}^2$ case. (d is the degree w.r.t. H)

We now turn to a more detailed discussion of our methods outlined in Sections 4 and 5. We will give the results in terms of the degree d to the extent possible.

Since the degree d plane curves have moduli space $\mathcal{M} = \mathbf{P}^{d(d+3)/2}$, we have from 5.3 that

$$n_d^{(d-1)(d-2)/2} = (-1)^{d(d+3)/2}(d+1)(d+2)/2.$$

This is in complete agreement with Table 4.

Note that if $d \geq 3$, then any reducible curves must have at least 2 nodes or worse singularities. This is seen by considering the ways that a degree d curve can split into curves of degrees d_i with $\sum d_i = d$, together with the observation that general plane curves of degrees d_i and d_j meet in $d_i d_j$ points. Therefore we only need the first term on the right hand side of 5.6 to compute instanton numbers with $\delta = 1$.

Note that the fiber of $\mathcal{C} \to \mathbf{P}^2$ is the set of degree d plane curves containing p, which is a $\mathbf{P}^{d(d+3)-1}$. Thus $e(\mathcal{C}) = e(\mathbf{P}^2)e(\mathbf{P}^{d(d+3)/2-1})$,

or $3d(d+3)/2$. Using the first equation of 5.4 and 5.6 together with $g = (d-1)(d-2)/2$, we get a formula

$$n_d^{d(d-3)/2} = (-1)^{d(d+3)/2} \binom{d}{2} (d^2 + d - 3),$$

which agrees with the results found for $d = 3, 4, 5, 6$ in Table 4 above.

As our next check for $\mathbf{P}^2$, we observe as before that for $d \geq 4$, all reducible curves have at least 3 nodes. So as above, the calculation simplifies for $\delta = 2$ and $d \geq 4$ by requiring only the first term on the right hand side of 5.6.

Consider the map $\rho_2 : \mathcal{C}^{(2)} \to \mathrm{Hilb}^2 \mathbf{P}^2$. The fiber of ρ_2 over $Z \in \mathrm{Hilb}^2\mathbf{P}^2$ is the set of plane curves of degree d containing $Z \subset \mathbf{P}^2$. Either $Z = \{p, q\}$ or Z consists of a point $p \in \mathbf{P}^2$ and a tangent direction at p. Either way, we see that it is two conditions on a plane curve to contain Z (to say that C contains $Z = (p, v)$ with $v \in T_p\mathbf{P}^2$ means that $p \in C$ and v is tangent to C at p). Thus the fiber is a $\mathbf{P}^{d(d+3)/2-2}$. This gives

$$
\begin{aligned}
e(\mathcal{C}^{(2)}) &= e(\mathbf{P}^{d(d+3)/2-2}) e(\mathrm{Hilb}^2\mathbf{P}^2) \\
&= 9\left(\frac{d(d+3)}{2} - 1\right).
\end{aligned}
$$

We get from the second equation of 5.4 and 5.6

$$
\begin{aligned}
n_d^{(d^2-3d-2)/2} = (-1)^{d(d+3)/2} \Bigg(& 9\left(\frac{d(d+3)}{2} - 1\right) \\
& + (d^2 - 3d - 2) \cdot 3\frac{d(d+3)}{2} \\
& + \left(\frac{d^2 - 3d}{2}\right)(d^2 - 3d - 3)\binom{d+2}{2}\Bigg).
\end{aligned}
$$

In particular, we verify that $n_4^1 = 231$, as noted earlier, as well as $n_5^4 = 1386$.

Continuing, we have that for $d \geq 5$, all reducible curves have at least 4 nodes, so the calculation for $\delta = 3$ simplifies. We consider the map $\mathcal{C}^{(3)} \to \mathrm{Hilb}^3\mathbf{P}^2$, and check that the fiber is always a $\mathbf{P}^{d(d+3)/2-3}$, yielding $e(\mathcal{C}^{(3)}) = 22((d+4)(d-1)/2)$. Using the third equation from

5.4 and 5.6, we get

$$n_d^{\frac{d^2-3d-4}{2}} = (-1)^{\frac{(d+1)(d+2)}{2}} \left(22\frac{(d+4)(d-1)}{2} \right.$$

$$+ ((d-1)(d-2) - 6)\, 9 \left(\frac{d(d+3)}{2} - 1 \right)$$

$$+ \frac{((d-1)(d-2) - 4)\,((d-1)(d-2) - 7)}{2} \left(\frac{3d(d+3)}{2} \right)$$

$$+ \frac{((d-1)(d-2) - 2)\,((d-1)(d-2) - 6)}{6} \times$$

$$\left. \times \frac{((d-1)(d-2) - 7)}{1} \binom{d+2}{2} \right) \right) \tag{8.6}$$

In particular, substituting $d = 5, 6$ into 8.6, we get $n_5^3 = -3672$ and $n_6^7 = 27538$, see Table 4.

We recall from Section 5 that we can also handle the situation where there are reducible 3 nodal curves, namely $d = 4$. The problem is that we have the locus of quartic curves which are unions of a cubic and a line. We saw how this leads to $n_4 = -192$.

The case of n_5 is even more interesting, in that there are two extra components contained in $\overline{\mathcal{M}}_6$. The first component is the union of degree 2 and degree 3 curves, contributing $n_2^0 n_3^1 = 60$, and the second component is the union of lines and degree 4 curves with 2 nodes, contributing $n_1^0 n_4^1 = 693$. According to 5.6, we substitute $d = 5$ in 5.5 with $\delta = 6$ and $g = 6$ and subtract the contributions of 60 and 693, obtaining $n_5^0 = 1695$, as in Table 4. The $e(\mathcal{C}^{(k)})$ can be computed as above because in these cases the projection $\mathcal{C}^{(k)} \to \mathrm{Hilb}^k \mathbf{P}^2$ is a bundle of projective spaces and $e(\mathrm{Hilb}^k \mathbf{P}^2)$ can be computed by 5.7.

Using 5.5 and 5.6, we can verify n_6^r for $r \geq 3$. The projection $\mathcal{C}^{(k)} \to \mathrm{Hilb}^k \mathbf{P}^2$ is again a bundle of projective spaces for $k \leq 7$. Recall that this bundle structure implies that $\mathcal{C}^{(k)}$ is smooth, so that our method applies. Thus we have a check of our method for each $\delta \leq 7$.

This bundle structure does not occur in general. In the computation of n_6^2, we encounter the map $\mathcal{C}^{(8)} \to \mathrm{Hilb}^8 \mathbf{P}^2$. Over a general point of $\mathrm{Hilb}^8 \mathbf{P}^2$, the fiber is a $\mathbf{P}^{27-8} = \mathbf{P}^{19}$, as a multiplicity 8 scheme usually imposes 8 conditions on curves of a given degree. But now suppose

the 8 points all lie on a line L, and we identify the projective space of degree 6 curves containing these 8 points (some of which may coincide). For such a degree 6 curve, the restriction of its degree 6 equation to L has 8 zeros, hence vanishes identically. Thus the degree 6 equation contains the equation of the line as a factor, and we can multiply it by an arbitrary degree 5 polynomial and obtain a degree 6 polynomial containing the desired 6 points. Since the degree 5 polynomials form a $\mathbf{P}^{20}$, we get a fiber of $\mathbf{P}^{20}$ rather than $\mathbf{P}^{19}$ in this case.

The lack of a bundle structure for $\rho_8 : \mathcal{C}^{(8)} \to \mathrm{Hilb}^8 \mathbf{P}^2$ causes us to ask if our methods apply in this case. We have to see whether or not $\mathcal{C}^{(8)}$ is smooth. Consider the projection $\pi_8 : \mathcal{C}^{(8)} \to \mathbf{P}^{27}$ onto the other factor. That is, given an element (C, Z) of $\mathcal{C}^{(8)}$, so that C is a plane curve of degree 6, and Z is a multiplicity 8 scheme in C, i.e. an element of $\mathrm{Hilb}^8 C$, put $\pi_8(C, Z) = C$, where C is now identified with the corresponding element of $\mathcal{M} = \mathbf{P}^{27}$. Since the fibers of π_8 are all 8 dimensional, we can see that $\mathcal{C}^{(8)}$ has dimension $27 + 8 = 35$. To show that $\mathcal{C}^{(8)}$ is singular, we need only exhibit a single element of $\mathcal{C}^{(8)}$ at which the tangent space of $\mathcal{C}^{(8)}$ has dimension strictly greater than 35. Since the tangent space of $\mathcal{C}^{(8)}$ at (C, Z) is naturally identified with the space of first order deformations of the pair (C, Z), we need only find a (C, Z) for which we can exhibit 36 independent first order deformations.

Towards this end, for C we take a degree 6 curve $l(x_1, x_2, x_3)^2 f(x_1, x_2, x_3) = 0$, where l is linear and f is homogeneous of degree 4 in the homogeneous coordinates (x_1, x_2, x_3) of $\mathbf{P}^2$. For Z we take any 8 points on the line $l(x_1, x_2, x_3) = 0$, which can possibly occur with multiplicity.

Note first that we have 16 independent deformations obtained by moving these 8 points arbitrarily in $\mathbf{P}^2$ while keeping C fixed. This is because of the factor of l^2, which ensures that all motions of the points stay within C to leading order. Then we take the 20 deformations noted above, where we fix Z, but now vary the degree 6 curve to an arbitrary curve of the form $l(x, y, z)g(x, y, z) = 0$, where g is homogeneous of degree 5. These 20 deformations are in fact honest deformations, not just first order deformations. Combining these two classes of first order deformations where we deform C and Z separately, we have all together the needed $16 + 20 = 36$ deformations.

While our methods gave the correct results for singular moduli

spaces arising from isolated nodal curves in Section 4, we are not so lucky this time. The calculation based on 5.5 and 5.6 differs from the value of n_6^2 in Table 4 by 45. We expect that the singular locus of $\mathcal{C}^{(8)}$ provides a correction term to 5.6, much as the second term of 5.6 can be viewed as providing a correction to 5.5. This is a topic for future investigation. If such a correction can be understood, then we can find F_g for all g geometrically!

We close our discussion of the local $\mathbf{P}^2$ with a tantalizing observation about the correction. We see that the image of the singular locus of $\mathcal{C}^{(8)}$ via the projection $\pi_8 : \mathcal{C}^{(8)} \to \mathbf{P}^{27}$ is the set of curves which factor into a degree 1 factor with multiplicity 2 and degree 4 factor. This locus is parameterized by $\mathbf{P}^2 \times \mathbf{P}^{14}$, which has Euler characteristic 45, exactly equal to the desired correction!

But we can't get too excited yet about this observation. The singular locus itself is a $\mathbf{P}^8$ bundle over this space, since we must consider Z itself, and Hilb^8 of a line is just $\mathbf{P}^8$. So the singular locus of $\mathcal{C}^{(8)}$ is parameterized by a space of Euler characteristic $9 \cdot 45$.

Nevertheless, we suspect that this is more than a coincidence. For $d = 7$, we see that $\mathcal{C}^{(9)}$ can be singular when the degree 7 curve factors into the square of a linear factor times a degree 5 factor. This space is parameterized by $\mathbf{P}^2 \times \mathbf{P}^{20}$, which has Euler characteristic 63. Once again, this is exactly the discrepancy between the value of n_7^6 computed using the B-model, and the value computed by 5.6! These examples provide a big hint which needs to be better understood.

8.4   Local $\mathbf{P}^1 \times \mathbf{P}^1$: $\mathcal{O}(K) \to \mathbf{P}^1 \times \mathbf{P}^1$

As expected from the Segre embedding of $P^1 \times P^1$ into P^3 as a degree 2 surface, we have the diagonal perturbation of the local $P^1 \times P^1$ case, i.e. $\sum_{i+j=r} n_{i,j}^{P^1 \times P^1} = n_r^{X_2(\bar{1},1,1,1)}$, which we have checked for genus $0, 1$. Still saying it differently it sums up the instantons in the compact elliptically fibered CY over F_0 which survive the limit where the fibre volume becomes infinite.

d	$r = 0$	1	2	3	4
1	$-4^\diamond$	0	0	0	0
2	$-4^\diamond$	0	0	0	0
3	$-12^\diamond$	0	0	0	0
4	$-48^\diamond$	9	0	0	0
5	$-240^\diamond$	$136^\diamond$	$-24^\diamond$	0	0
6	-1356	1616	-812	$186^\diamond$	$-16^\diamond$
7	-8428	17560	-17340	9712	$-3156^\diamond$
8	-56000	183452	-302160	307996	-206776
9	-392040	1878664	-4688912	7590720	-8583824
10	-2859120	19027840	-67508988	159995520	-274149876

Table 5: The weighted sum of BPS states n_d^r for the local $\mathbf{P}^1 \times \mathbf{P}^1$ case.

We now compute the $\diamond$ numbers in Table 5 geometrically. We recall that a curve of bi-degree (a, b) on $\mathbf{P}^1 \times \mathbf{P}^1$ has total degree $d = a + b$, genus $g = (a - 1)(b - 1)$, and $\mathcal{M} = \mathbf{P}^{(a+1)(b+1)-1}$. By 5.3, this gives a contribution of $(-1)^{(a+1)(b+1)-1}(a + 1)(b + 1))$ to $n_{a+b}^{(a-1)(b-1)}$.

If $d = 2k$ is fixed, then relative to all possibilities for $d = a + b$, the choice $(a, b) = (k, k)$ gives the maximum possible genus $(k - 1)^2$. Thus

$$n_{2k}^{(k-1)^2} = (-1)^{k^2+2k}(k + 1)^2 = (-1)^k(k + 1)^2.$$

This verifies our results for $k = 1, 2,$ and 3.

If $d = 2k + 1$, then the maximal genus $k(k - 1)$ is attained for $(a, b) = (k + 1, k)$ or $(k, k + 1)$. We may as well consider one of these cases and multiply the result by 2. By 5.3 we get

$$n_{2k+1}^{k(k-1)} = -2(k + 1)(k + 2),$$

where the definite sign follows since $(k + 2)(k + 1)$ is always even. This verifies our results for $k = 0, 1, 2$.

We next turn to curves with 1 node, $\delta = 1$. If $d = 2k + 1$, then the only contributions to $n_{2k+1}^{k(k-1)-1}$ come from $(a, b) = (k + 1, k)$ or $(k, k + 1)$. If $d \geq 5$, there are no reducible curves of these bi-degrees with only 1 node. We restrict to the first case and will later multiply

the contribution by 2. The map $\mathcal{C} \to \mathbf{P}^1 \times \mathbf{P}^1$ has fiber over p the space of curves of bi-degree $(k+1, k)$ which contain p, a $\mathbf{P}^{(k+2)(k+1)-2}$. This leads to $e(\mathcal{C}) = 4((k+2)(k+1) - 1) = 4(k^2 + 3k + 1)$. We get

$$n_{2k+1}^{k^2-k-1} = 2\left(4(k^2 + 3k + 1) + (2k(k-1) - 2)(k+2)(k+1)\right),$$

which in particular verifies $n_5^1 = 136$.

If $d = 2k$ and $k \geq 2$, a new interesting possibility arises, which is typical in the sequel. We get contributions to $n_{2k}^{(k-1)^2-1}$ some of which have $\delta = 0$, while others have $\delta = 1$. The former type comes from $(a, b) = (k+1, k-1)$ or $(k-1, k+1)$. This gives a contribution of $2(-1)^{k(k+2)-1}k(k+2)$ by 5.3 . The $\delta = 1$ contribution comes from $(a, b) = (k, k)$, and we get $(-1)^{(k+1)^2-2}(4((k+1)^2 - 1) + (2(k-1)^2 - 2)(k+1)^2)$ from the first equation in 5.4 together with 5.6, as there are no reducible curves of this bi-degree with only 1 node. Combining these two contributions gives

$$n_{2k}^{k^2-2k} = (-1)^{k(k+2)-1}$$
$$\left(2k(k+2) + 4\left((k+1)^2 - 1\right) + \left(2(k-1)^2 - 2\right)(k+1)^2\right),$$

which agrees with Table 5 for $k = 2, 3$.

We next consider the case of $d = 7, g = 4$. Here there are the possible bi-degrees $(4, 3)$ and $(5, 2)$. Since the latter case already has $g = 4$, we get a moduli space $\mathcal{M} = \mathbf{P}^{17}$, which gives a contribution of -18 by 5.3. Curves of bi-degree $(4, 3)$ have genus $g = 6$ and $\mathcal{M} = \mathbf{P}^{19}$, with $e(\mathcal{M}) = 20$. We get as usual $e(\mathcal{C}) = 19 \cdot 4 = 76$ and $e(\mathcal{C}^{(2)}) = 18 \cdot 14 = 252$ (we have used $e(\mathrm{Hilb}^2(\mathbf{P}^1 \times \mathbf{P}^1)) = 14$ here). So the second equation of 5.4 together with 5.6 gives a contribution of $(-1)^{17}(252 + 8 \cdot 76 + 5 \cdot 7 \cdot 20) = -1560$, as there are no reducible curves of this bi-degree with 2 nodes. Combining with the -18 and multiplying by 2 to account for bi-degrees $(3, 4)$ and $(2, 5)$, we get the consistent answer -3156.

More interesting is the geometric calculation of n_5^0. We have to consider bidegrees $(4, 1)$ and $(3, 2)$. The first case is handled by 5.3, and we get -10. In the second case, we have $g = \delta = 2$, $e(\mathcal{M}) = 12$, $e(\mathcal{C}) = 4 \cdot 11 = 44$, $e(\mathcal{C}^{(2)}) = 14 \cdot 10 = 140$. There are also reducible curves of type $(1, 0) \cup (2, 2)$ with 2 nodes. By 5.3 , we have $n_{1,0}^0 = -2$ and $n_{2,2}^1 = 9$. So by the second equation in 5.4 and 5.6, we get $n_{3,2}^0 = -(140 - 12) - (-2) \cdot 9 = -110$. Thus $n_5^0 = 2(n_{3,0}^0 + n_{4,1}^0) = -240$, in agreement with Table 5.

8.5 Other local Del Pezzo geometries: E_5, E_6, E_7, and E_8

The E_5 del Pezzo:

d	$r = 0$	1	2	3	4
1	$16^\diamond$	0	0	0	0
2	$-20^\diamond$	0	0	0	0
3	$48^\diamond$	0	0	0	0
4	$-192^\diamond$	$5^\diamond$	0	0	0
5	$960^\diamond$	$-96^\diamond$	0	0	0
6	-5436	$1280^\diamond$	$-80^\diamond$	0	0
7	33712	-14816	$2512^\diamond$	$-160^\diamond$	0
8	-224000	160784	-51928	$8710^\diamond$	$-680^\diamond$
9	1568160	-1688800	886400	-274240	51040
10	-11436720	17416488	-13552940	6643472	-2167656

Table 6: The weighted sum of BPS states n_d^r for the local E_5 del Pezzo.

As in $\mathbf{P}^1 \times \mathbf{P}^1$, to verify our calculations we need to break up our degrees into subclasses. Verification of the E_n geometries is complicated mainly by the fact that there are many possibilities contributing to a given degree, and we must consider all the possibilities for any fixed d that we want to understand.

As typical examples, let us verify n_4^1 and n_4^0. We look at the $d = 4, g = 1$ data first. We have the class $(3; 1^5)$ with $\mathcal{M} = \mathbf{P}^4$ (by Riemann-Roch, one expects $\mathcal{M} = \mathbf{P}^{d+g-1}$ in general). This verifies $n_4^1 = 5$ by 5.3. But there is also a $\delta = 1$ contribution to n_4^0. We have $\mathcal{C} \to \mathcal{M}$ with fiber $\mathbf{P}^3$, hence

$$e(\mathcal{C}) = e(\mathbf{P}^3)e(E_5) = 4 \cdot 8 = 32.$$

This gives a contribution of -32 to n_4^0 by the first equation of 5.4 and 5.6.

We next look at the $d = 4, g = 0$ data. Each of these classes has $\mathcal{M} = \mathbf{P}^3$. Each $\mathbf{P}^3$ gives a contribution of -4. So we have to count numbers of such families. For $(2; 1^2)$ we have to choose 2 out of the 5 points at which to put the two ones; there are $\binom{5}{2} = 10$ ways of

doing this. For $(3; 2, 1^3)$ we choose 1 point for the two, and 3 of the remaining points for the ones, and there are $5\binom{4}{2} = 20$ ways of doing this. Finally, for $(4; 2^3, 1^2)$ there are 10 choices. All together, there are $10+20+10 = 40$ distinct $d = 4, g = 0$ families. The total contribution is $40(-4) = -160$. Combining with the -32, we have $-160 - 32 = -192$, which agrees with the calculated value of n_4^0.

Now let's calculate n_5^0. We have for $d = 5, g = 1$ the families $(3; 1^4), (4; 2^2, 1^3), (5, 2^5)$ and we want $\delta = 1$. Including permutations, there are 16 such families. In each case, we have reducible curves with 1 node:

$$\begin{aligned}
(3, 1^4) &= (3, 1^5) \cup E_5 \\
(4, 2^2, 1^3) &= (3, 1^5) \cup (1, 1^2) \\
(5, 2^5) &= (3, 1^5) \cup (2, 1^5)
\end{aligned}$$

Since the curves of type $(3, 1^4)$ form a $\mathbf{P}^4$, and the respective curves E_5, $(1, 1^2)$, and $(2, 1^5)$ are all isolated, it follows that the term to be subtracted on the right hand side of 5.6 is -5. Since $e(\mathcal{C}) = 5 \cdot 8 = 40$ in each case, we get for each the contribution $-40 - (-5) = -35$ by the first equation of 5.4 and 5.6. Combining this calculation with the more standard calculations for $\delta = 0$ based on 5.3, we get $n_5 = 80(5) + 16(35) = 960$, in agreement with Table 6.

We can similarly verify all the other cases indicated with a diamond using just these techniques. We can also check some cases requiring curves with $\delta > 1$ nodes, but these become increasingly tedious.

The E_6 del Pezzo:

d	$r = 0$	1	2	3	4
1	27$^\diamond$	0	0	0	0
2	$-54^\diamond$	0	0	0	0
3	243$^\diamond$	$-4^\diamond$	0	0	0
4	$-1728^\diamond$	135$^\diamond$	0	0	0
5	15255	$-3132^\diamond$	189$^\diamond$	0	0
6	-153576	62976	$-10782^\diamond$	789$^\diamond$	$-10^\diamond$
7	1696086	-1187892	397899	-75114	7641
8	-20053440	21731112	-12055770	4188726	-948186
9	249740091	-391298442	326385279	-179998572	69918830
10	-3240109350	6985791864	-8218296072	6602867631	-3896482536

Table 7: The weighted sum of BPS states n_d^r for the local E_6 del Pezzo.

Note that the second equation in 5.4 is needed for n_6^2, and we get from several different families of curves, 5.3, the first two equations in 5.4, and 5.6, the result $n_6^2 = 270(-8) + 108(-72) - 846 = -10782$, again in agreement with Table 7.

The E_7 del Pezzo:

d	$r=0$	1	2	3
1	56$^\diamond$	0	0	
2	$-272^\diamond$	3	0	
3	3240$^\diamond$	$-224^\diamond$	0	
4	-58432	12042	$-844^\diamond$	7$^\diamond$
5	1303840	-574896	116880	$-12112^\diamond$
6	-33255216	26127574	-10554800	2654912
7	930431208	-1163157616	787322120	-358173504
8	-27855628544	51336812456	-52707074296	37805931096
9	878169863088	-2258519658288	3292406219040	-3430645737616
10	-28835628521920	99301270680473	-196037258631040	280764828128124

Table 8: The weighted sum of BPS states n_d^r for the local E_7 del Pezzo.

Again the numbers marked with $\diamond$ have been checked. Here we determine the number n_5^3. First let us calculate the contribution of the smooth genus 3 curves with $d = 5$. From Appendix A with simple combinatorics follows that there are $1 + 35 + 105 + 7 + 140 + 7 + 140 + 105 + 35 + 1 = 576$ curves all with moduli space $\mathbf{P}^7$ by 8.15.3 , hence contributing $(-1)^7 8 \cdot 576 = -4608$ to n_5^3. Also from the table we read off that there are $7 + 21 + 21 + 7 = 56$ genus[24] 4 curves with $d = 5$. In fact we have $n_5^4 = 56(-1)^8 \cdot 9 = 504$ from 8.15.3 as there are no $g = 5$ curves in classes with $d = 5$, which could degenerate to $g = 4$. The universal curve for each of the of the 56 $\delta = 1$ curves has by 8.4 $e(\mathcal{C}) = -8 \cdot 10$. Application of 4.16 gives hence a contribution of $-56 \cdot 80 - (2 \cdot 4 - 2)504$ of the nodal curves to give a total of $n_5^3 = -4608 - 7504 = -12112$.

For the E_8 del Pezzo surface we obtained

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[24]Note the classical fact that for fixed d these curves are in the representation of the Weyl-group of E_7.

d	$r = 0$	1	2	3	4	5	6	7
1	$252°$	$-2°$						
2	$-9252°$	$762°$	$-4°$					
3	848628	-246788	30464	$-1548°$	$7°$			
4	-114265008	76413073	-26631112	5889840	-835236	69587	$-2642°$	$11°$

Table 9: The weighted sum of BPS states n_d^r for the local E_8 del Pezzo.

The case of the E_8 del Pezzo is interesting because it can be easily can be related to the $\frac{1}{2}K_3$, which gives an additional check. Extending [72] it was observed [5] that there is a modular anomaly in this case for higher genus.

More precisely let $F = -K_{\frac{1}{2}K_3}$ the fibre (also the anticanonical class) and $B = e_9$ the base class of the elliptic half K_3 surface. Following [5] we can solve the modular anomaly for this two parameter subspace and write the contribution to F_r for a fixed base class n as a quasimodular form $P_{2g+2n-2}(E_2, E_4, E_6)\frac{q^{\frac{m}{2}}}{\eta^{12n}}$. In particular we can compare the diagonal class $K_{E_8} = F + e_9$ and test the modular anomaly against the holomorphic anomaly calculation. We observed complete accordance up to genus 8 degree 10.

8.6   The topological string perspective

In the local case one has for all genera an explicit virtual fundamental class on the moduli spaces of maps and therefore a direct A-model localization calculation of the topological string amplitudes at higher genera is in principle possible [67] . At genus zero there is a virtual fundamental class also for the global case and equivalence of the A-model and the B-model calculation was proven [73] . This sort of proof was adapted to the local case [74] . Hence as an immediate check we can perform in certain cases A-model computations to compare with. However using the $c = 1$ KDV hierarchy and localization on the toric ambient space becomes extremely difficult for high g, where the M-theory calculation is still very easy, provided that $\delta < \delta_{max}(d)$.

Furthermore we can consider the B-model and use the M-theory calculation to resolve the holomorphic ambiguity. The B-model, which

has some additional simplifications in the local case [67], has the virtue that it comes naturally with the analytic techniques on the complex moduli space of the mirror manifold, which allows us to relate the answers that we get in the infinite volume limit to physical systems at other degenerations by analytic continuation. This yields further consistency checks.

To describe the analytic structure we note that the vectors $\vec{a} = (a_1, a_2)$

$$\mathbf{P}^2: \quad \vec{a} = \left(\frac{1}{2}, \frac{2}{3}\right), \qquad \mathbf{P}^1 \times \mathbf{P}^1: \quad \vec{a} = \left(\frac{1}{2}, \frac{1}{2}\right),$$

$$E_5: \quad \vec{a} = \left(\frac{1}{2}, \frac{1}{2}\right), \qquad E_6: \quad \vec{a} = \left(\frac{1}{3}, \frac{2}{3}\right),$$

$$E_7: \quad \vec{a} = \left(\frac{1}{4}, \frac{3}{4}\right), \qquad E_8: \quad \vec{a} = \left(\frac{1}{6}, \frac{5}{6}\right),$$

determine the Picard-Fuchs differential equation, which governs the complex geometry of the mirror for the local cases

$$\left(\theta^3 - z \prod_{i=1}^{3}(\theta - a_i + 1)\right) \int_{\gamma_i} \Omega = 0 . \tag{8.7}$$

They are solved by Meier's G-function $G(a_1, a_2, 1; x)$, compare with [97]. In particular their Riemann symbol is

$$\mathbf{P} \left\{ \begin{matrix} 0 & \infty & 1 \\ 0 & 0 & 0 \\ 0 & a_1 & 1 \\ 0 & a_2 & 1 \end{matrix} , z \right\} \tag{8.8}$$

and shows that $z = 0$ is the maximally unipotent point, $z = 1$ is the conifold point. We can also read from the Riemann symbol that for the $\mathbf{P}^1 \times \mathbf{P}^1$ and the E_5, $z = \infty$ is not an orbifold point, but has also logarithmic solutions[25].

As in the quintic case of [66], it is convenient for the higher genus calculation to work in the ψ variable as it avoids fractional exponents

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[25]Further properties of the solutions have been made explicit in [58], [71], [67]. In particular the monodromy is related to $\Gamma_0(n)$ for E_{9-n} [67].

in the B-model propagators. In this variables the Yukawa are [26]

$$\psi^{-\beta} \equiv \pm z, \qquad Y_{\psi\psi\psi} = \frac{\alpha(X)\beta^3\psi^{\beta-3}}{(1 \pm \psi^\beta)}, \quad \text{with } \beta = \sum_{i=1}^{l} d_i \qquad (8.9)$$

The mirror map is normalized so that $\alpha(X) = \prod_{i=1}^{k} d_i / \prod_{j=1}^{l} w_l$ is just the classical triple intersection number expressed in terms of the degrees d_i of the complete intersections and the weights of the ambient space. The E_n $n = 5, \dots, 8$ del Pezzo surfaces can be represented as complete intersections of degree $(2, 2)$ in $\mathbf{P}^4$, a degree 3 hypersurface in $\mathbf{P}^3$, a degree 4 hypersurface in weighted projective space $\mathbf{P}^3(1, 1, 1, 2)$ and a degree 6 hypersurface in $\mathbf{P}(1, 1, 2, 3)$.

The B-model multi-loop contributions to the free energy are determined from the holomorphic anomaly equations [66]. In solving these differential equations one is left at the end with a holomorphic ambiguity, which is a holomorphic section of $\mathcal{L}^{2-2r}$ over the complex moduli space[27]. With the right choice of gauge there is no singularity of the holomorphic ambiguity at $z = 0$, but from the singularity structure is clear that we have to generalize the ansatz for the ambiguity [66], [67] to allow beside the singularities at the conifold also for residue terms at $z = \infty$. Hence we make in general the ansatz

$$f_r(\psi) = \sum_{k=0}^{2r-2} \frac{A_k^r}{\mu^k} + \sum_{k=1}^{r-1} \frac{B_k^r}{\rho^{2k}} , \qquad (8.10)$$

where $\mu = c_1(1 \pm \psi^\beta)$ and $\rho = c_2\psi^\beta$.

The finite number of constants A_k^r, B_k^r can be fixed by the direct calculation of the n_d^r, but at least part of it is also encoded in the universal behavior of F_r at singular loci in the moduli space. In particular near a conifold singularity one expects [1] from the duality with the $c = 1$ critical string theory at the selfdual radius an expansion

$$F_{S^3}(\mu) = \frac{1}{2}\mu^2 \log(\mu) - \frac{1}{12}\log(\mu) + \sum_{r=2}^{\infty} \frac{B_{2r}}{(2r-2)(2r)}\mu^{2-2r} \qquad (8.11)$$

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[26]The sign is minus for the local $\mathbf{P}^1 \times \mathbf{P}^1$ case and minus for all other cases. For the somewhat exceptional $\mathbf{P}^2$ and $\mathbf{P}^1 \times \mathbf{P}^1$ case we have $\alpha(X) = 1/3, \beta = 3$ and $\alpha(X) = 1, \beta = -2$.

[27]Here we denote the worldsheet genus of the B-model r, to distinguish it from the arithmetic genus g.

where $B_{2r}/(2r-2)2r = -\frac{1}{240}, \frac{1}{1008}, -\frac{1}{1440}, \ldots$ for genus $r = 2, 3, 4, \ldots$. Relative to the genus expansion of the type II free energy $F(t) = \sum_{r=0}^{\infty} \lambda^{2r-2} F_r(t)$ 8.11 is a double scaling limit in which the distance t to the conifold in the moduli space and the string coupling go to zero while their ratio μ is kept fixed. This selects the corresponding leading term μ^{2r-2} form F_r in 8.11. For the local propagators we use the same gauge as in [67] . In this gauge the singular behavior at the conifold is captured entirely by the ambiguity. The relation between $t = (1 \pm \psi^{\beta})$ and μ can be fixed from the genus zero result by comparing the asymptotic of the Yukawa coupling with 8.11. We found the simple systematics $\mu = \frac{(1 \pm \psi^{\beta})}{\sqrt{\alpha(X)}}$ for the local and $\mu = \frac{i(1 \pm \psi^{\beta})}{\sqrt{\alpha(X)}}$ for the global cases.

It has been speculated that the next to leading order might be related to correlations functions of the $c = 1$ string at the selfdual radius involving discrete states whose $SU(2)_L \times SU(2)_R$ charges could be read off from next to leading order monomial in the perturbations of the local equations [98]. It would be very interesting to make this more concrete. At any rate we will report this behavior at the conifold and other singularities along with our previous results at the large complex structure limit. Beside the verification of the claim in [1] we find also some regularities for the next to leading order residue. For instance, we observed up to $r = 4$ that the A_{2r-3}^r of the quintic Calabi-Yau threefold are exactly $(4/5)$ times smaller as the A_{2r-3}^r for the local $\mathbf{P}^1 \times \mathbf{P}^1$ case, which in turn are identical to the analogous quantities on the E_6 del Pezzo.

For the $\mathbf{P}^2$ case we fixed the constants in the ambiguity 3.4 as follows

r	A_0^r	A_1^r	A_2^r	A_3^r	A_4^r	A_5^r	A_6^r	A_7^r	A_8^r
2	$\frac{3e-220}{17280}$	$\frac{107}{4320\sqrt{3}}$	$-\frac{1}{240}$						
3	$\frac{9940-3e}{4354560}$	$-\frac{2519}{181440\sqrt{3}}$	$\frac{6479}{653184}$	$-\frac{13}{1440\sqrt{3}}$	$\frac{1}{1008}$				
4	$\frac{-1864975+27e}{2351462400}$	$\frac{2117}{230400\sqrt{3}}$	$-\frac{32949197}{2351462400}$	$\frac{14229281}{440899200\sqrt{3}}$	$-\frac{143987}{10886400}$	$\frac{15041}{1814400\sqrt{3}}$	$\frac{-1}{1440}$		
5	$\frac{252764050-243e}{587865600 s_5}$	$\frac{-2607911}{340200\sqrt{3}s_5}$	$\frac{164095927}{8398080 s}$	$\frac{-35952843613}{35952843613\sqrt{3}s_5}$	$\frac{88814851411}{1322697600 s_5}$	$\frac{-616239319}{6123600\sqrt{3}s_5}$	$\frac{348989}{11664 s_5}$	$\frac{-5519}{378\sqrt{3}s_5}$	$\frac{1}{s_5}$

with $\mu = \sqrt{3}(1 + \psi^3)$ and s_g as in 8.11 . This is in perfect agreement with expected behaviour at the conifold.

Because of the additional singularity at $\psi = 0$ the holomorphic ambiguity is more interesting in the $\mathbf{P}^1 \times \mathbf{P}^1$ case. Using the Landau-

Ginzburg orbifold description the mirror manifold can be represented as a hypersurface [1]

$$W = x_1^2 + x_2^2 + x_3^2 + x_4^2 - 2\left(\psi x x_1 x_2 x_3 x_4 + \frac{\psi}{x}\right) \qquad (8.12)$$

in a non-compact $\mathbf{P}^4(-2,1,1,1,1)$. Note that at $x = 1$, $x_i = 1$ for $\psi = 1 + \epsilon$ one has the usual conifold singularity, but $x = 1$, $x_i = 0$ for small ψ does not correspond to a conifold singularity. Because of the Z_2 $(x \to x; x_i \to -x_i)$ identification symmetry in the $\mathbf{P}^4(-2,1,1,1,1)$ it rather is a S^3/Z_2 lens space, called $L(2,1)$, which is shrinking. In the IIB picture this shrinking lens space creates a logarithmic singularity (with a logarithmic shift by 2 at $\psi = 0$ due to the bound state of D-3 branes wrapping $L(2,1)$). It explains why we need here the poles at $\psi = 0$ in the generalized ansatz 3.4 . In fact using the relation $\rho = 2^4 \psi^4$, fixed from the asymptotic behaviour of the Yukawa coupling, we find

r	A_0^r	A_1^r	A_2^r	A_3^r	A_4^r	A_5^r	A_6^r	B_1^r	B_2^r	B_3^r
2	$\frac{e-100}{5760}$	$\frac{83}{5760}$	$-\frac{1}{240}$					$-\frac{2^3}{240}$		
3	$\frac{10045-2e}{2903040}$	$-\frac{413}{51840}$	$\frac{5783}{645120}$	$-\frac{193}{40320}$	$\frac{1}{1008}$			$-\frac{23}{48384}$	$\frac{2^5}{1008}$	
4	$\frac{8e-935025}{696729600}$	$\frac{1001683}{2^{16}5 3^4 7}$	$-\frac{16034329}{2^{15}3^5 5^2 7}$	$\frac{9924889}{2^{14}3^5 5^2 7}$	$-\frac{2871}{27648}$	$\frac{10001}{2^9 3^3 5^2 7}$	$\frac{1}{1440}$	$\frac{150671}{2^{16}3^4 5^2 7}$	$-\frac{3641}{2^{13}3^3 5^2 7}$	$\frac{2^7}{1440}$

Thus the leading behavior at $\psi = 0$, captured in the B_i^r, is exactly as expected[98] for two particles with half the mass ρ leading to $\mathcal{F}_{L(2,1)}(\rho) = 2\mathcal{F}_{S^3}(\frac{\ell}{2})$ with $\mathcal{F}_{S^3}$ as in 8.11 .

For the E_n $n = 5, \ldots , 8$ cases one expect a complicated singularity structure at $\psi = 0$, due to the simultaneous occurence of light electric and magnetic states. For the E_5 case we got

r	A_0^r	A_1^r	A_2^r	A_3^r	A_4^r	A_5^r	A_6^r	B_1^r	B_2^r	B_3^r
2	$\frac{2e-155}{11520}$	$\frac{83}{11520}$	$-\frac{1}{240}$					$-\frac{38}{240}$		
3	$\frac{26425-32e}{46448640}$	$-\frac{3997}{3317760}$	$\frac{6119}{2580480}$	$-\frac{193}{80640}$	$\frac{1}{1008}$			$-\frac{467}{30240}$	$\frac{158}{1008}$	
4	$\frac{2^{11}e-9710925}{2^{22}3^5 5^2 7}$	$\frac{1405589}{2^{15}3^4 7}$	$-\frac{19054729}{2^{19}3^5 5^2 7}$	$\frac{10715449}{2^{17}3^5 5^2 7}$	$-\frac{2057}{2^{12}3^3 7}$	$\frac{10001}{2^{10}3^3 5^2 7}$	$-\frac{1}{1440}$	$-\frac{625063}{2^{15}3^4 5^2 7}$	$-\frac{137563}{2^{10}3^3 5^2 7}$	$-\frac{638}{1440}$

where we normalized $(1 + \psi^2) = \mu/2$ and $\psi^2 = \rho/2^5$. Here we observe the same next to leading order behavior at the conifold as in the $\mathbf{P}^1 \times \mathbf{P}^1$ examples. However the residue at $\psi = 0$ has yet to be interpreted.

Let shortly summarize the remaining cases. For the E_6 we got

r	A_0^r	A_1^r	A_2^r	A_3^r	A_4^r	A_5^r	A_6^r	B_1^r	B_2^r
2	$\frac{-1660+27e}{155520}$	$\frac{107}{12960\sqrt{3}}$	$-\frac{1}{240}$						
3	$\frac{20300-81e}{117573120}$	$-\frac{10301}{14696640\sqrt{3}}$	$\frac{35419}{29393280}$	$-\frac{13}{4320\sqrt{3}}$	$\frac{1}{1008}$			1	
4	$\frac{19683e-11499775}{1714216089600}$	$\frac{6364177}{95234227200\sqrt{3}}$	$-\frac{43056877}{190468454400}$	$\frac{15901121}{11904278400\sqrt{3}}$	$\frac{148787}{97977600}$	$\frac{15041}{5443200\sqrt{3}}$	$-\frac{1}{1440}$	$\frac{1}{54}$	486

with $(1 + \psi^3) = \mu/\sqrt{3}$ and $\psi^3 = \rho/3^9$. Here we note that the next to leading order at the conifold is $\frac{1}{3}$-times the one of the $\mathbf{P}^2$ theory.

The E_7 case:

r	A_0^r	A_1^r	A_2^r	A_3^r	A_4^r	B_1^r
2	$\frac{8e-635}{46080}$	$\frac{143}{23040\sqrt{2}}$	$-\frac{1}{240}$			
3	$\frac{110215-512e}{743178240}$	$-\frac{749}{1658880\sqrt{2}}$	$\frac{68857}{61931520}$	$-\frac{131}{53760\sqrt{2}}$	$\frac{1}{1008}$	1

with $(1 + \psi^4) = \mu/\sqrt{2}$ and $\phi^4 = \rho/2^{15}$.

Finally for the E_8:

r	A_0^r	A_1^r	A_2^r	A_3^r	A_4^r	A_5^r	A_6^r	B_1^r
2	$\frac{9e-1100}{51840}$	$\frac{251}{51840}$	$-\frac{1}{240}$					
3	$\frac{12775-54e}{78382080}$	$-\frac{517}{1399680}$	$\frac{225613}{156764160}$	$-\frac{251}{120960}$	$\frac{1}{1008}$			
4	$\frac{-1547875+5832e}{507915878400}$	$\frac{2590747}{96745881600}$	$-\frac{19158899}{67722117120}$	$\frac{122105573}{101583175680}$	$-\frac{43501}{18662400}$	$\frac{45281}{21772800}$	$-\frac{1}{1440}$	$\frac{2525}{3869835264}$

with $\mu = (1 + \psi^6)$ and $\rho = \psi^6$.

9   Computations in compact Calabi-Yau geometries

9.1   Compact one modulus cases

Here we analyze the higher genus contribution for compact one modulus Calabi-Yau spaces. In this case there is no virtual fundamental class for the higher genus topological string calculation known. In absence of this approach we combine the topological B model calculation and the M-theory description of the invariants to determine the higher genus

F_r. We can carry out the M-theory computation of certain $n_d^{g-\delta}$ for small δ. We expect to be able to make much further progress even in the compact case once we understand how to systematically correct for singularities in the relative Hilbert schemes $\mathcal{C}^{(k)}$ for $k \leq \delta$.

Examples are hypersurfaces or complete intersections in weighted projective space which avoid the singularities of the ambient space. Denoting a complete intersection of degree $(d_1, \dots, d_k)$ in $\mathbf{P}^n(w_1, \dots, w_l)$ by $X_{d_1,\dots,d_k}(w_1, \dots, w_l)^{28}$ we have the following complete list of such examples (compare the second ref. in [51] for the $r = 0, r = 1$ results)

$$X_5(1^5) : \vec{a} = \left(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right), \quad X_6(1^4, 2) : \vec{a} = \left(\frac{1}{6}, \frac{2}{6}, \frac{4}{6}, \frac{5}{6}\right),$$

$$X_8(1^4, 4) : \vec{a} = \left(\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right), \quad X_{10}(1^3, 2, 5) : \vec{a} = \left(\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}\right),$$

$$X_{3,3}(1^6) : \vec{a} = \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right), \quad X_{4,2}(1^6) : \vec{a} = \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}\right),$$

$$X_{3,2,2}(1^7) : \vec{a} = \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{2}, \frac{1}{2}\right), \quad X_{2,2,2,2}(1^8) : \vec{a} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$X_{4,3}(1^5, 2) : \vec{a} = \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}\right), \quad X_{4,4}(1^4, 2^2) : \vec{a} = \left(\frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}\right),$$

$$X_{6,2}(1^5, 3) : \vec{a} = \left(\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{1}{2}\right), \quad X_{6,4}(1^3, 2^2, 3) : \vec{a} = \left(\frac{1}{6}, \frac{5}{6}, \frac{1}{4}, \frac{3}{4}\right),$$

$$X_{6,6}(1^2, 2^2, 3^2) : \vec{a} = \left(\frac{1}{6}, \frac{5}{6}, \frac{1}{6}, \frac{5}{6}\right)$$

The components of the vector $\vec{a}$ specify the Picard-Fuchs operator for the mirror manifold as follows

$$\left(\theta^4 - z \prod_{i=1}^{4}(\theta + a_i)\right) \int_{\gamma_i} \Omega = 0 . \tag{9.1}$$

---

[28] n-times repeating weights will be denoted by w_i^n.

From the Riemann Symbol

$$
\mathbf{P}\left\{
\begin{array}{ccc}
0 & \infty & 1 \\
0 & a_1 & 0 \\
0 & a_2 & 1 \\
0 & a_3 & 2 \\
0 & a_4 & 1
\end{array}
, z \right\}
\tag{9.2}
$$

we conclude that $z = 0$ is the maximally unipotent point with three logarithmic solutions and $z = 1$ is the conifold point with one logarithmic solution. From the Riemann symbol it also clear that the hypersurfaces have a cyclic monodromy of order[29] β (with β as in 8.9) at $z = \infty$, while the complete intersections have degenerate indices and therefore logarithmic degeneration of the periods with shift monodromy and vanishing cycles at $z = \infty$, which justifies the general ansatz for the ambiguity 8.10 . We choose the gauge so that the propagators of the B are regular at the conifold and at $z = \infty$. This amounts, in the notation of sections 7.2 of [66], to a choice of $f(\psi) = \psi$ for quintic and sextic, $f(\psi) = \psi^2$ for the bicubic and $f(\psi) = \psi^4$ for the four conics, while $v(\psi) = 1$ for all cases. The normalization of the Yukawa couplings is as in 8.9 .

9.2   Higher genus results on the quintic

We come now to the simplest compact Calabi-Yau, the zero locus of the quintic

$$
\sum_{i=1}^{5} x_i^5 - 5\psi \prod_{i=1}^{5} x_i = 0.
\tag{9.3}
$$

in $\mathbf{P}^4$. The unique analytic solution at $z = 0$ is $w_0 = \sum_{n=0}^{\infty} \frac{5n!}{(n!)^5} \left(\frac{z}{5^5}\right)^n$, the three-point coupling is as in 8.9. The rational and elliptic curves have been computed in [29], [65] .

---

[29]For the hypersurfaces this was just an orbifold singularity with non-vanishing B-field. In fact the triple intersection corresponds to canonical normalized kinetic terms, precisely to the one calculated in the associated Gepner model [29].

d	$r = 0$	1
1	2875	0
2	609250	0
3	317206375	609250
4	242467530000	3721431625
5	229305888887625	12129909700200
6	248249742118022000	31147299733286500
7	295091050570845659250	71578406022880761750
8	375632160937476603550000	154990541752961568418125

Table 10.a: The weighted sum of BPS states n_d^r for the quintic.

We agree with those results. Note however that the BPS numbers on the genus 1 cases differ from the invariants defined in [65], where the maps from the torus to itself are subtracted as explained in sec /ref-sec:3.3.

Unlike the non-compact cases, the reduction of the holomorphic anomaly involves a global property of the model, the Euler number. The general form of F_2 has been given in [66]. In appendix B we give the complete result of the reduction of F_3[30]. In view of the fast growing number of terms in F_r with the worldsheet genus r one may hope that the ring of modular functions on the moduli space of the concrete Calabi-Yau – here the quintic– transforming in $\mathcal{L}^{2-2r}$ has a much lower dimension, so that there are many relations between the terms in F_r. Restricting the expression given in appendix B for F_3 to the one-moduli case one has 50 terms of different functional form. Somewhat surprisingly we find only one relation between the terms, which is reported in the Appendix B.

With these formulas we obtain the following genus $r = 2, 3$ results.

---

[30]The F_4, F_5 expressions can be made available on request.

d	$r = 2$	3	4
1	0	0	0
2	0	0	0
3	0	0	0
4	534750	8625	0
5	75478987900	−15663750	15520
6	871708139638250	3156446162875	−7845381850
7	5185462556617269625	111468926053022750	243680873841500
8	22516841063105917766750	1303464598408583455000	25509502355913526750

Table 10.b: The weighted sum of BPS states n_d^r for the quintic.

Note that for $d > 7$ the numbers of genus 2 invariants are expected to differ from the ones given in [66] , because of the different definition of the integer invariants related to F_2.

Using the vanishing of $n_1^2 = n_2^2 = n_3^2 = 0$, as explained below, we can fix the genus 2 ambiguity. Comparing 8.9 with 8.11 we learn that the distance to the singularity $t = (1 - z) = i\mu/\sqrt{\alpha}$. Written in terms of μ it reads

$$f_2 = -\frac{571}{36000} + \frac{83i}{7200\sqrt{5}}\mu^{-1} - \frac{1}{240}\mu^{-2} \ .$$

Note that the leading singularity confirms 8.11. Further the constant map contribution is in accordance with [66], [22].

To fix the genus 3 anomaly we used the vanishing of $n_1^2 = n_2^2 = n_3^2 = 0$ and the fact that $n_4^2 = (-1)^2 \cdot 3 \cdot 2875 = 8625$, as explained below. The number n_5^3 turns out to be negative, which is plausible as these curves come in singular families, but the actual value has not been checked. Together with [22] for the constant map piece this determines

$$f_3 = \frac{26857}{126000000} + \frac{356921i}{567000000\sqrt{5}}\mu^{-1} - \frac{5393}{5040000}\mu^{-2}$$
$$- \frac{193i}{50400\sqrt{5}}\mu^{-3} + \frac{1}{1008}\mu^{-4} \ ,$$

i.e. the leading behavior is in perfect agreement with the $c = 1$ interpretation at the conifold singularity. We note that the old definition of the invariant yielding $n_8^2 = 22516841063105918836250$ seems incompatible with the BPS interpretation as it destroys the integrality of the

expansion of F_3. The leading behavior of the ambiguity at genus 4 is

$$f_4 = \ldots + \frac{10001i}{3024000\sqrt{5}}\mu^{-5} - \frac{1}{1440}\mu^{-6}$$

Most notably in order to get an integer expansion we had also to assume that there is also a residue at $\psi^5 = 0$ more precisely $f_4 \sim \frac{11853023518768}{8859375}\psi^{-5}$.

We turn our attention to the application of 5.6 to verify some of the instanton numbers that we have calculated for the quintic. Before we can do this, we have to understand the restrictions on the degree and arithmetic genus of a curve in projective space. This is part of the subject of Castelnuovo theory [99], [100]. Castelnuovo theory gives the maximum arithmetic genus of a nondegenerate irreducible curve C of degree d in $\mathbf{P}^r$. Here, nondegenerate means that C is not contained in any hyperplane. This is not a restriction, since any curve is nondegenerate inside the linear subspace that it spans.

We give a somewhat detailed description for $\mathbf{P}^2$, $\mathbf{P}^3$, and a general formula for $\mathbf{P}^r$.

In $\mathbf{P}^2$, Castelnuovo theory is trivial, since a degree d curve necessarily has arithmetic genus $g = (d-1)(d-2)/2$.

In $\mathbf{P}^3$, the result of [101] says that a nondegenerate curve of degree d either lies on a quadric surface or else there is a number $g(d)$ (to be described presently) such that any curve in $\mathbf{P}^3$ has genus $g \leq g(d)$.

Note that a quadric surface is just $\mathbf{P}^1 \times \mathbf{P}^1$ so all possibilities for (d, g) can be computed from the bidegrees (a, b) and $d = a + b$, $g = (a-1)(b-1)$. Some easy algebra shows that a curve with degree and genus (d, g) can be found on a quadric precisely when $(d-2)^2 - 4g$ is a perfect square.

For $g(d)$, we get

$$\begin{aligned} g(d) &= \tfrac{d^2-3d+6}{6} \quad d \equiv 0 \pmod 3 \\ &= \tfrac{d^2-3d+2}{6} \quad d \not\equiv 0 \pmod 3 \end{aligned} \tag{9.4}$$

For $\mathbf{P}^r$, we give a less complete answer, and just note that the

maximum genus possible is found by writing

$$d - 1 = m(r - 1) + \epsilon$$

with $0 \leq \epsilon < m$. The formula is

$$g \leq \binom{m}{2}(r - 1) + m\epsilon. \tag{9.5}$$

If $r = 3$, it can be checked that the right hand side of 9.5 is the genus of a degree d curve on a quadric, compare with Section 8.4.

Let's apply these formulas to the quintic in low degree.

For $d = 1, 2$, it is clear that the only genus possible is 0. For $d = 3$, we can get $g = 1$ in the plane, and $g = 0$ is possible, even in $\mathbf{P}^3$ (e.g. bidegree $(2, 1)$ in $\mathbf{P}^1 \times \mathbf{P}^1$). Thus $n_1^g = n_2^g = 0$ for $g > 0$ and $n_2^g = 0$ for $g > 1$, consistent with Tables 10.a and 10.b.

Now $d = 4$ is more interesting. This is possible in $\mathbf{P}^2$ only for $g = 3$. In $\mathbf{P}^3$ it is either on a quadric or $g \leq 1$ by Castelnuovo theory. But we check immediately that $g = 0, 1$ are the only possibilities on a quadric as well. Applying Castelnuovo theory to $\mathbf{P}^4$, we again get $g \leq 1$. So $g = 2$ is impossible on a quintic, and $n_4^g = 0$ for $g \geq 4$.

But this discussion does not mean that n_4^2 vanishes. Quite the contrary, we can apply 5.4 and 5.6 to the family of $g = 3$ curves, which are plane quartics. The plane intersects the quintic in a quintic plane curve containing the quartic, leaving a residual line. So we get the moduli space of these $d = 4, g = 3$ curves by taking a line on the quintic, and passing all possible 2-planes through it, leaving a quartic over by reversing the above reasoning. So the moduli space is 2875 copies of $\mathcal{M} = \mathbf{P}^2$. In passing, we note that $n_4^3 = 2875(3) = 8625$ by 5.3. The universal curve $\mathcal{C}$ is a bit subtle. The projection $\mathcal{C} \to X$ (X is the quintic) is 1-1 except over the line (since a point of X not on the line determines a unique 2-plane containing the line). But the fiber of this projection over the line is a $\mathbf{P}^1$, so we must add 2 to the Euler characteristic of the CY (-200) to get the Euler characteristic (-198) of $\mathcal{C}$. Then the first equation in 5.4 together with 5.6 gives for n_4^2 the quantity $2875(-1)(-198 + (4)(3)) = 534750$, as there are no reducible curves of degree 4 in the plane. This is in agreement with Table 10.b.

We can similarly apply Castelnuovo theory to degree 5. Here the result is that plane quintics have genus 6, and otherwise the genus is at most 2. The moduli space of plane quintic curves is the same as the moduli space $G = G(2, 4)$ of $\mathbf{P}^2$s in $\mathbf{P}^4$. By 5.3, we get $n_5^6 = (-1)^6 e(G) = 10$. We can also in principle get n_5^g for $3 \leq g \leq 5$ from 5.4 and 5.6. Unfortunately, we don't know how to compute the Euler characteristic of $\mathcal{C}^{(3)}$, so we can only compute n_5^g for $g = 4, 5$ at present.

We can calculate $e(\mathcal{C})$ as usual by considering the projection $\mathcal{C} \to X$. The fiber over a point $p \in X$ is identified with the set of 2 planes in $\mathbf{P}^4$ which contain p. This in turn is identified with the Grassmannian $G(1, 3)$ of lines in $\mathbf{P}^3$. So $e(\mathcal{C}) = e(X)e(G(1, 4)) = -200 \cdot 10 = -2000$. Then the first equation in 5.4 together with 5.6 gives $n_5^5 = (-1)^5(-2000 + (2 \cdot 3 - 2)10) = 1960$.

The map $\mathcal{C}^{(2)} \to \mathrm{Hilb}^2 X$ has fiber over $Z \in \mathrm{Hilb}^2 X$ the set of all 2 planes in $\mathbf{P}^4$ containing Z. Since Z is either a pair of distinct points or a single point of multiplicity 2 with a distinct tangent direction, we see that this fiber is isomorphic to $\mathbf{P}^3$ in either case. Also, $\mathrm{Hilb}^2 X$ is obtained from the symmetric product $\mathrm{Sym}^2 X$ by blowing up the diagonal, replacing each point of the diagonal with a $\mathbf{P}^2$. Thus

$$e(\mathrm{Hilb}^2) = e(\mathrm{Sym}^2 X) + e(X)(e(\mathbf{P}^2) - 1)$$
$$= \binom{-199}{2} + (-200) \cdot 2 = 19500$$

from which it follows from the second equation in 5.4 together with 5.6 that $n_5^4 = 15520$.

For $0 \leq g \leq 2$, there are both smooth curves of degree 5 and genus g in the quintic as well as a contribution from singular plane quintics. In the case of $g = 0$, there are actually finitely many singular curves, which were enumerated in [77].

9.3   The sextic, the bicubic and four conics

A further typical hypersurface example is the sextic in the weighted projective space $\mathbf{P}^4(1, 1, 1, 1, 2)$. We find the following all integer[31] n_d^r

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[31]In [66] it was claimed that this example has a half-integral invariant for $n_4^{(2)}$, which would be in contradiction with the M-theory interpretation of the $R^2 F^{2g-2}$

invariants for this case

d	$r = 0$	1
1	7884	0
2	6028452	7884
3	11900417220	145114704
4	34600752005688	1773044322885
5	124595034333130080	17144900584158168
6	513797193321737210316	147664736456952923604
7	2326721904320912944749252	1197243574587406496495592
8	11284058913384803271372834984	9381487423491392389034886369

d	$r = 2$	3
1	0	0
2	0	0
3	17496	576
4	10801446444	−14966100
5	571861298748384	1412012838168
6	13753100019804005556	403369763928730938
7	233127389355701229349884	552961951281452536352
8	3246006977306701566424657380	560485610266924061005490676

Table 11: The weighted sum of BPS states n_d^r for the sextic.

The ambiguity can be fixed from the vanishing of $n_1^2 = n_2^2 = 0$ plus the general form of the constant map contribution. Again we find the leading term of f_2 in accordance with 8.11 $f_2 = -\frac{473}{25920} - \frac{463}{51840\sqrt{3}}\mu^{-1} - \frac{1}{240}\mu^{-2}$.

For the genus 3 contribution we can fix a combination of n_3^3, n_4^3 by demanding the expected behavior of the μ^{-4} term in

$$f_3 = \frac{3917}{26127360} + \frac{61447i}{117573120\sqrt{3}}\mu^{-1} - \frac{107945}{94058496}\mu^{-2} - \frac{25i}{8064\sqrt{3}}\mu^{-3} + \frac{1}{1008}\mu^{-4} .$$

amplitude. Luckily we find that the problem relied on a computational error.

This yields $n_4^3 = 36(-865581 + 781n_3^3)$. The fact the μ^{-3} term is a relatively simple fraction for other cases leads to the conjecture that the correct value is $n_3^3 = 576$.

This can be checked by geometry as follows. Note that by projection to the first four coordinates, X admits a $3-1$ cover of $\mathbf{P}^3$. Writing the equation of the CY in the form

$$x_5^3 + f_4(x_1, x_2, x_3, x_4)x_5 + f_6(x_1, x_2, x_3, x_4) = 0,$$

we see that the branch locus $4f_4^3 + 27f_6^2$ has degree 12. A curve of degree 3 must either map isomorphically to a degree 3 curve in $\mathbf{P}^3$ or be a triple cover of a line. Since degree 3 curves have genus at most 1, and we are interested in genus 3, we must have a triple cover of a line branched at 4 points, which has genus 4. Thus $\mathcal{M} = G(1,3)$, the Grassmannian of $\mathbf{P}^1$s in $\mathbf{P}^3$. This has dimension 4 and Euler characteristic 6. So we already see that $n_3^4 = 6$.

We project the universal curve to X as usual; the fiber over $p \in X$ is the set of triple covers of lines which contain p, which is in 1-1 correspondence with the set of lines in $\mathbf{P}^3$ which contain the image of p in $\mathbf{P}^3$. This is isomorphic to $\mathbf{P}^2$. We get $e(\mathcal{C}) = e(\mathbf{P}^2)e(X) = 3(-204) = -612$. Now an application of the first equation in 5.4 and 5.6 gives $(-1)^3(-612 + 6 \cdot 6) = 576$.

We next consider as simplest complete intersection cases two cubics in $\mathbf{P}^5$, i.e. $X_{3,3}(1^6)$ and four conics in $\mathbf{P}^7$, denoted $X_{2,2,2,2}(1^8)$.

d	$r = 0$	1	2	3
1	1053	0	0	0
2	52812	0	0	0
3	6424326	3402	0	0
4	1139448384	5520393	0	0
5	249787892583	4820744484	5520393	0
6	62660964509532	3163476678678	23395810338	6852978
7	17256453900822009	1798399482469092	42200615912499	174007524240
8	5088842568426162960	944929890847710108	50349477671013600	785786604262830

Table 12: The weighted sum of BPS states n_d^r for the compl. intersection $X_{3,3}(1^6)$.

r	A_0^r	A_1^r	A_2^r	A_3^r	A_4^r	B_1^r	B_2^r
2	$-\dfrac{2861}{116640}$	$\dfrac{103i}{15552}$	$-\dfrac{1}{240}$			$-\dfrac{71}{64}$	
3	$\dfrac{1921447}{6348948480}$	$\dfrac{1358671i}{2116316160}$	$-\dfrac{118421}{70543872}$	$\dfrac{377i}{181440}$	$\dfrac{1}{1008}$	$\dfrac{5}{27648}$	$-\dfrac{240501}{35840}$

with $\mu = 3(1 - \psi^6)/i$ $\rho = 3^6\psi^6$.

d	$r = 0$	1	2	3
1	512	0	0	0
2	9728	0	0	0
3	416256	0	0	0
4	25703936	14752	0	0
5	1957983744	8782848	0	0
6	170535923200	2672004608	1427968	0
7	16300354777600	615920502784	2440504320	86016
8	1668063096387072	123699143078400	1628589698304	2403984384

Table 13: The weighted sum of BPS states n_d^r for the compl. intersection $X_{2,2,2,2}(1^8)$.

The indices of the Picard-Fuchs system is 4 fold degenerate at $\psi^8 = 0$ and we find the leading behavior from the ambiguity

r	A_0^r	A_1^r	A_2^r	A_3^r	A_4^r	B_1^r	B_2^r
2	$-\dfrac{133741}{2949120}$	$\dfrac{349i}{46080}$	$-\dfrac{1}{240}$			$-\dfrac{5377}{15}$	
3	$\dfrac{110365853}{1014686023680}$	$\dfrac{1859479i}{1698693120}$	$-\dfrac{30277}{13762560}$	$\dfrac{149i}{64512}$	$\dfrac{1}{1008}$	$-\dfrac{2181905}{221184}$	$\dfrac{23115884}{2835}$

with $\mu = 4(1 - \psi^8)/i$ and $\rho = \psi^8 2^{16}$. It would be very interesting to find the analog of the $c = 1$ model at the $\rho = 0$ singularity.

Acknowledgement

10   Appendix A: Low degree classes on the del Pezzo Surfaces

d	classes
	$g=0$ classes
1	$e_i\ ^1|(1;1^2)\ ^2|\ (2;1^5)\ ^5|\ (3;2,1^6)\ ^7|\ (6;3,2^7)\ (5;2^6,1^2)\ (4;2^3,1^5)\ ^8|$
2	$(1;1)\ ^1|(2;1^4)\ ^4|(3;2,1^5)\ ^6|(5;2^6,1)\ (4;2^3,1^4)\ ^7|$ $(11;4^7,3)\ (10;4^4,3^4)\ (9;4^2,3^5,2)\ (8;4,3^4,2^3)\ (8;3^7,1)\ (7;4,3,2^6)\ (7;3^4,2^3,1)$ $(6;3^2,2^4,1^2)\ (5;3,2^3,1^4)\ (4;3,1^7)\ ^8|$
3	$(1;0)\ ^0|\ (2;1^3)\ ^3|(3;2,1^4)\ ^5|(5;2^6)\ (4;2^3,1^3)\ ^6|$ $(8;3^7)\ (7;3^4,2^3)\ (6;3^2,2^4,1)\ (5;3,2^3,1^3)\ (4;3,1^6)\ ^7|$
4	$(2;1^2)\ ^2|(3;2,1^3)\ ^4|(4;2^3,1^2)\ ^5|(6;3^2,2^4)\ (5;3,2^3,1^2)\ (4;3,1^5)\ ^6|$
5	$(2;1)\ ^1|(3;2,1^2)\ ^3|(4;2^3,1)\ ^4|(5;3,2^3,1)\ (4;3,1^4)\ ^5|$ $(8;4,3^5)\ (7;4,3^2,2^3)\ (7;3^5,1)\ (6;3^3,2,1^2)\ (6;4,2^4,1)\ (5;3^2,1^4)\ ^6|$
6	$(2;0)^0|\ (3;2,1)\ ^2|\ (4;2^3)\ ^3|\ (5;3,2^2)\ (4;3,1^3)\ ^4|(7;3^5)\ (6;3^3,2,1)\ (6;4,2^4)\ (5;3^2,1^3)\ ^5|$
7	$(3;2)\ ^1|(4;3,1^2)\ ^3|(6;3^3,2)\ (5;3^2,1^2)\ ^4|$ $(8;4^2,3^3)\ (7;4,3^3,1)\ (7;4^2,2^3)\ (6;4,3,2,1^2)\ (5;4,1^4)\ ^5|$
8	$(4;3,1)\ ^2|(5;3^2,1)\ ^3|(7;4,3^3)\ (6;4,3,2,1)\ (5;4,1^3)\ ^4|$
	$g=1$ classes
1	$(3;1^8)\ ^8|$
2	$(3;1^7)\ ^7|(9;4,3^7)\ (8;3^6,2^2)\ (7;3^3,2^5)\ (6;3,2^6,1)\ (5;2^5,1^3)\ (4;2^2,1^6)\ ^8|$
3	$(3;1^6)\ ^6|(6;3,2^6)\ (5;2^5,1^2)\ (4;2^2,1^5)\ ^7|$
4	$(3;1^5)\ ^5|\ (5;2^5,1)\ (4;2^2,1^4)\ ^6|$ $(9;4^2,3^5)\ (8;4,3^4,2^2)\ (7;3^4,2^2,1)\ (7;4,3,2^5)\ (6;3^2,2^3,1^2)\ (5;3,2^2,1^4)\ ^7|$
5	$(3;1^4)\ ^4|(5;2^5)\ (4;2^2,1^3)\ ^5|\ (7;3^4,2^2)\ (6;3^2,2^3,1)\ (5;3,2^2,1^3)\ ^6|$
6	$(3;1^3)\ ^3|(4;2^2,1^2)\ ^4|(6;3^2,2^3)\ (5;3,2^2,1^2)\ ^5|$ $(9;4^3,3^3)\ (8;4^2,3^2,2^2)\ (7;4,3^2,2^2,1)\ (6;4,2^3,1^2)\ (6;3^3,1^3)\ ^6|$
7	$(3;1^2)\ ^2|(4;2^2,1)\ ^3|(5;3,2^2,1)\ ^4|(7;4,3^2,2^2)\ (6;4,2^3,1)\ (6;3^3,1^2)\ ^5|$
8	$(3;1)\ ^1|(4;2^2)\ ^2|(5;3,2^2)\ ^3|(6;4,2^3)\ (6;3^3,1)\ ^4|$
	$g=2$ classes
2	$(6;2^8)\ ^8|$
3	$(14;5^7,4)\ (13;5^4,4^4)\ (12;5^2,4^5,3)\ (11;4^7,2)\ (11;5,4^4,3^3)\ (10;4^4,3^3,2)\ (10;5,4,3^6)$ $(9;4^2,3^4,2^2)\ (8;4,3^3,2^4)\ (8;3^6,2,1)\ (7;4,2^7)\ (7;3^3,2^4,1)\ (6;3,2^5,1^2)\ (5;2^4,1^4)$ $(4;2,1^7)\ ^8|$
4	$(8;3^6,2)\ (7;3^3,2^4)\ (6;3,2^5,1)\ (5;2^4,1^3)\ (4;2,1^6)\ ^7|$
5	$(6;3,2^5)\ (5;2^4,1^2)\ (4;2,1^5)\ ^6|$
6	$(5;2^4,1)\ (4;2,1^4)\ ^5|\ (8;4,3^4,2)\ (7;3^4,2,1)\ (7;4,3,2^4)\ (6;3^2,2^2,1^2)\ (5;3,2,1^4)\ ^6|$
7	$(5;2^4)\ (4;2,1^3)\ ^4|(7;3^4,2)\ (6;3^2,2^2,1)\ (5;3,2,1^3)\ ^5|$
8	$(4;2,1^2)\ ^3|(6;3^2,2^2)\ (5;3,2,1^2)\ ^4|(8;4^2,3^2,2)\ (7;4,3^2,2,1)\ (6;4,2^2,1^2)\ ^5|$

d	$g=3$ classes
3	$(12;5,4^7)\ (11;4^6,3^2)\ (10;4^3,3^5)\ (9;4,3^6,2)\ (8;3^5,2^3)\ (7;3^2,2^6)\ (6;2^7,1)\ ^8|$
4	$(6;2^7)\ ^7|$
5	$(11;4^7)\ (10;4^4,3^3)\ (9;4^2,3^4,2)\ (8;3^6,1)\ (8;4,3^3,2^3)\ (7;4,2^6)\ (7;3^3,2^3,1)$ $(6;3,2^4,1^2)\ (5;2^3,1^4)\ (4;1^7)\ ^7|$
6	$(8;3^6)\ (7;3^3,2^3)\ (6;3,2^4,1)\ (5;2^3,1^3)\ (4;1^6)\ ^6|$
7	$(6;3,2^4)\ (5;2^3,1^2)\ (4;1^5)\ ^5|$
8	$(5;2^3,1)\ (4;1^4)\ ^4|\ (8;4,3^4)\ (7;3^4,1)\ (7;4,3,2^3)\ (6;3^2,2,1^2)\ (5;3,1^4)\ ^5|$
	$g=4$ classes
3	$(9;3^8)\ ^8|$
4	$(18;7^2,6^6)\ (17;7,6^5,5^2)...^8|$
5	$(9;4,3^6)\ (8;3^5,2^2)\ (7;3^2,2^5)\ (6;2^6,1)\ ^7|$
6	$(6;2^6)\ ^6|$ $(13;5^5,4^2)\ (12;6,4^6)\ (12;5^3,4^3,3)\ (11;5,4^5,2)\ (11;5^2,4^2,3^3)\ (10;4^5,2^2)\ (10;5,4^2,3^3,2)$ $(9;4^3,3,2^3)\ (9;4^2,3^4,1)\ (9;5,3^4,2^2)\ (8;4,3^3,2^2,1)\ (8;4^2,2^5)\ (7;4,2^5,1)\ (7;3^3,2^2,1^2)$ $(6;3,2^3,1^3)\ (5;2^2,1^5)\ ^7|$
7	$(9;4^2,3^4)\ (8;4,3^3,2^2)\ (7;4,2^5)\ (7;3^3,2^2,1)\ (6;3,2^3,1^2)\ (5;2^2,1^4)\ ^6|$
8	$(7;3^3,2^2)\ (6;3,2^3,1)\ (5;2^2,1^3)\ ^5|$
	$g=5$ classes
4	$(17;6^7,5)\ (16;6^4,5^4)\ (15;6^2,5^5,4)\ (14;6,5^4,4^3)\ (14;5^7,3)\ (13;5^4,4^3,3)\ (13;6,5,4^6)$ $(12;5^2,4^4,3^2)\ (11;4^6,3,2)\ (11;5,4^3,3^4)\ (10;5,3^7)\ (10;4^3,3^4,2)\ (9;4,3^5,2^2)\ (8;3^4,2^4)$ $(7;3,2^7)\ ^8|$
5	$(18;8,7^3,5^4)...^8|$
6	$(12;5^2,4^5)\ (11;5,4^4,3^2)\ (10;4^4,3^2,2)\ (10;5,4,3^5)\ (9;4^2,3^3,2^2)\ (8;3^5,2,1)\ (8;4,3^2,2^4)$ $(7;3^2,2^4,1)\ (6;2^5,1^2)\ ^7|$
7	$(8;3^5,2)\ (7;3^2,2^4)\ (6;2^5,1)\ ^6|$
8	$(6;2^5)\ ^5|$
	$g=6$ classes
4	$(15;6,5^7)\ (14;5^6,4^2)\ (13;5^3,4^5)\ (12;5,4^6,3)\ (11;4^5,3^3)\ (10;4^2,3^6)\ (9;3^7,2)\ ^8|$
5	$(18;8,7,6^5,4)...^8|$
6	$(11;4^6,3)\ (10;4^3,3^4)\ (9;4,3^5,2)\ (8;3^4,2^3)\ (7;3,2^6)\ ^7|$
7	$(15;6^3,5^4)...^7|$
8	$(10;4^4,3^2)\ (9;4^2,3^3,2)\ (8;3^5,1)\ (8;4,3^2,2^3)\ (7;3^2,2^3,1)\ (6;2^4,1^2)\ ^6|$

11 Appendix B: B-model expression for F_g

The F_g can be determined by recursively solving the B-model anomaly equation [66]. As each contribution comes from the boundary of the moduli space of Riemann surfaces the result has a graph interpretation in which each graph corresponds to a possible degeneration of the genus g curve into components with lower genera. In the local case the descendent of the dilaton decouples and only one sort of propagator $S^{i,j}$ occurs. They are in one to one correspondence with the tubes connecting the irreducible components, in this way each contribution of F_g

corresponds precisely to one boundary stratum [67] . In the global case one has three sorts of propagators shown below

Propagors:

$$-S^{ij} \qquad -S^{i\varphi} \qquad -2\,S^{\varphi\varphi}$$

irreducible components

$$g{=}0 \quad g{=}1 \quad g{=}2 \qquad\qquad F^{(1)}_{\varphi} = (\tfrac{\chi}{24} - 1\,)$$
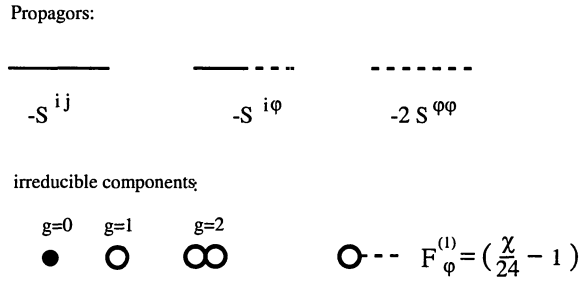
The following Feynman rules hold for $n, m = 0, 1, 2 \ldots$

$$F^{(0)}_{\varphi^n} = 0, \quad F^{(0)}_{i\varphi^n} = 0, \quad F^{(0)}_{ij\varphi^n} = 0,$$
$$F^{(g)}_{i_1, \ldots i_m, \varphi^{n+1}} = (2g - 2 + n + m) = F^{(g)}_{i_1, \ldots i_m, \varphi^n}$$

The graph contribution is divided by the following symmetry factors: $k!$ for k equal (self)links joining the same vertices, 2 for each selflink $S^{\phi,\phi}$, $S^{i,i}$ times the order of the graph automorphism obtained by permuting the vertices. The generation of graphs proceeds along the line described for the the A-model in [67] .

Twelve graphs contribute to F_2, which were derived in [66] . Further we find that 193 graphs contribute to the free energy at genus 3. As it depends in an universal combinatorial way on the lower genus boundary components, which also applies more generally to non-topological string 3 loop calculations, and we have a powerfull check via integrality of the BPS states on the quintic on its expression, we will give it below, despite its complicated nature. Let $\bar{\chi} = \frac{\chi}{24}$ with χ the Euler number of the target space, F the genus 0 prepotential, $G = F_1$, $H = F_2$ and S^{ij},

S^i, S the propagators of Kodaira-Spencer gravity [66] , then we found

$$F_3 = 2(1 + 2\bar{\chi})HS + \bar{\chi}(1 + \bar{\chi} - 2\bar{\chi}^2)S^2 + 2G_i HS^i +$$

$$(2 + \bar{\chi})H_i S^i - 2\bar{\chi}(1 + 2\bar{\chi})G_i S S^i - \frac{1}{2}(2 + 3\bar{\chi})G_i G_j S^i S^j -$$

$$\frac{1}{2}(2 + 3\bar{\chi} - \bar{\chi}^2)G_{ij}S^i S^j + \frac{1}{6}\bar{\chi}(2 + 3\bar{\chi} + \bar{\chi}^2)F_{ijk}S^i S^j S^k +$$

$$G_i H_j S^{ij} + \frac{1}{2}H_{ij}S^{ij} - (1 + 2\bar{\chi})G_i G_j S S^{ij} - (1 + 2\bar{\chi})G_{ij}S S^{ij} -$$

$$G_i G_j G_k S^k S^{ij} - (2 + \bar{\chi})G_i G_{jk}S^k S^{ij} - (1 + \bar{\chi})G_{ijk}S^k S^{ij} +$$

$$\bar{\chi}(1 + 2\bar{\chi})F_{ijk}S S^k S^{ij} + \frac{1}{4}(2 + 3\bar{\chi} + \bar{\chi}^2)F_{ijkl}S^k S^l S^{ij} -$$

$$\frac{1}{2}G_i G_{jkl}S^{il}S^{jk} + \frac{1}{2}(2 + 3\bar{\chi} + \bar{\chi}^2)F_{ijk}G_l S^j S^k S^{il} - G_i G_{jk}S^i S^{jk} -$$

$$\frac{1}{2}G_i G_j G_{kl}S^{il}S^{jk} + (1 + \frac{3}{2}\bar{\chi})F_{ijk}G_l S^k S^l S^{ij} - \frac{1}{2}F_{ijk}H_l S^{il}S^{jk} +$$

$$(1 + 2\bar{\chi})F_{ijk}G_l S S^{il}S^{jk} + F_{ijk}G_l G_m S^m S^{il}S^{jk} +$$

$$(1 + \frac{1}{2}\bar{\chi})F_{ijk}G_{lm}S^m S^{il}S^{jk} + (1 + \frac{1}{2}\bar{\chi})F_{ijkl}G_m S^l S^{im}S^{jk} -$$

$$\frac{1}{4}G_{ij}G_{kl}S^{ik}S^{jl} + (1 + \frac{1}{2}\bar{\chi})F_{ijk}G_l G_m S^k S^{im}S^{jl} - \frac{1}{8}G_{ijkl}S^{ij}S^{kl} +$$

$$(1 + \frac{1}{2}\bar{\chi})F_{ijk}G_{lm}S^k S^{il}S^{jm} - \frac{1}{4}(2 + 3\bar{\chi} + \bar{\chi}^2)F_{ijk}F_{lmn}S^k S^n S^{il}S^{jm} +$$

$$\frac{1}{4}(1 + 2\bar{\chi})F_{ijkl}S S^{ij}S^{kl} + \frac{1}{8}(2 + \bar{\chi})F_{ijklm}S^m S^{ij}S^{kl} +$$

$$\frac{1}{4}F_{ijkl}G_m S^m S^{ij}S^{kl} + \frac{1}{4}(1 + \bar{\chi})F_{ijk}F_{lmn}F_{pqr}S^r S^{il}S^{jp}S^{kn}S^{mq} +$$

$$\frac{1}{8}(2 + \bar{\chi})F_{ijk}F_{lmn}F_{pqr}S^k S^{ip}S^{jn}S^{lm}S^{qr} + \frac{1}{4}F_{ijkl}G_{mn}S^{im}S^{jn}S^{kl} +$$

$$\frac{1}{2}F_{ijk}G_l G_{mn}S^{im}S^{jn}S^{kl} + \frac{1}{6}F_{ijk}G_{lmn}S^{il}S^{jm}S^{kn} -$$

$$\frac{1}{6}(1 + 2\bar{\chi})F_{ijk}F_{lmn}S S^{il}S^{jm}S^{kn} - \frac{1}{6}(1 + \bar{\chi})F_{ijk}F_{lmnp}S^p S^{il}S^{jm}S^{kn} -$$

$$\frac{1}{6}F_{ijk}F_{lmn}G_p S^p S^{il}S^{jm}S^{kn} - \frac{1}{2}(2 + \bar{\chi})F_{ijk}F_{lmn}G_p S^n S^{il}S^{jm}S^{kp} +$$

$$\frac{1}{2}F_{ijk}G_l G_m S^k S^{ij}S^{lm} + \frac{1}{16}F_{ijk}F_{lmn}F_{pqrs}S^{ij}S^{ks}S^{lm}S^{np}S^{qr} -$$

$$\frac{1}{8}(2 + 3\bar{\chi})F_{ijk}F_{lmn}S^k S^n S^{ij}S^{lm} - \frac{1}{2}F_{ijk}F_{lmn}G_p S^n S^{ip}S^{jk}S^{lm} -$$

$$\frac{1}{4}(2 + 3\bar{\chi} + \bar{\chi}^2)F_{ijk}F_{lmn}S^j S^k S^{in}S^{lm} -$$

$$\frac{1}{2}(2 + \bar{\chi})F_{ijk}F_{lmn}G_p S^k S^{ip}S^{jn}S^{lm} + \frac{1}{8}F_{ijklm}G_n S^{ij}S^{kn}S^{lm} -$$

$$\frac{1}{16}F_{ijk}F_{lmn}F_{pqr}F_{stu}S^{is}S^{jt}S^{kn}S^{lp}S^{mq}S^{ru} + \frac{1}{4}F_{ijk}G_{lmn}S^{ij}S^{kn}S^{lm} +$$

$$\frac{1}{48}F_{ijklmn}S^{ij}S^{kl}S^{mn} - \frac{1}{4}F_{ijk}F_{lmn}G_p S^p S^{ij}S^{kn}S^{lm} -$$

$$\frac{1}{4}(2+\bar{\chi})F_{ijk}F_{lmnp}S^p S^{ij}S^{kn}S^{lm} +$$

$$\frac{1}{12}F_{ijk}F_{lmn}F_{pqrs}S^{ip}S^{jq}S^{kr}S^{ls}S^{mn} - \frac{1}{4}F_{ijk}F_{lmn}G_{pq}S^{ip}S^{jq}S^{kn}S^{lm} -$$

$$\frac{1}{48}F_{ijkl}F_{mnpq}S^{im}S^{jn}S^{kp}S^{lq} - \frac{1}{4}(2+\bar{\chi})F_{ijk}F_{lmnp}S^k S^{il}S^{jp}S^{mn} -$$

$$\frac{1}{4}(1+2\bar{\chi})F_{ijk}F_{lmn}S\ S^{ij}S^{kn}S^{lm} - \frac{1}{4}F_{ijk}F_{lmnp}G_q S^{il}S^{jq}S^{kp}S^{mn} -$$

$$\frac{1}{4}F_{ijk}F_{lmn}G_p G_q S^{ip}S^{jq}S^{kn}S^{lm} - \frac{1}{4}F_{ijk}F_{lmn}G_p G_q S^{il}S^{jq}S^{kn}S^{mp} -$$

$$\frac{1}{4}F_{ijk}F_{lmn}G_{pq}S^{il}S^{jp}S^{kn}S^{mq} - \frac{1}{12}F_{ijk}F_{lmnpq}S^{il}S^{jm}S^{kq}S^{np} +$$

$$\frac{1}{4}F_{ijk}F_{lmn}F_{pqr}G_s S^{ip}S^{js}S^{kn}S^{lq}S^{mr} +$$

$$\frac{1}{8}F_{ijk}F_{lmn}F_{pqrs}S^{ip}S^{jq}S^{kn}S^{lr}S^{ms} +$$

$$\frac{1}{4}F_{ijkl}G_m G_n S^{im}S^{jn}S^{kl} + \frac{1}{2}F_{ijk}G_l G_{mn}S^{ij}S^{kn}S^{lm} +$$

$$\frac{1}{16}F_{ijk}F_{lmnpq}S^{ij}S^{kq}S^{lm}S^{np} -$$

$$\frac{1}{48}F_{ijk}F_{lmn}F_{pqr}F_{stu}S^{ip}S^{js}S^{kn}S^{lm}S^{qr}S^{tu} -$$

$$\frac{1}{16}F_{ijkl}F_{mnpq}S^{ij}S^{km}S^{lq}S^{np} +$$

$$\frac{1}{4}(2+\bar{\chi})F_{ijk}F_{lmn}F_{pqr}S^n S^{il}S^{jm}S^{kr}S^{pq} -$$

$$\frac{1}{6}F_{ijk}F_{lmnp}G_q S^{il}S^{jm}S^{kp}S^{nq} + \frac{1}{12}F_{ijk}F_{lmn}F_{pqr}S^r S^{il}S^{jm}S^{kn}S^{pq} -$$

$$\frac{1}{4}F_{ijk}F_{lmnp}G_q S^{ij}S^{kl}S^{mq}S^{np} +$$

$$\frac{1}{8}F_{ijk}F_{lmn}F_{pqr}G_s S^{ip}S^{js}S^{kn}S^{lm}S^{qr} +$$

$$\frac{1}{4}F_{ijk}F_{lmn}F_{pqr}G_s S^{il}S^{jm}S^{kr}S^{ns}S^{pq} +$$

$$\frac{1}{8}F_{ijk}F_{lmn}F_{pqr}S^r S^{ij}S^{kn}S^{lm}S^{pq} + \frac{1}{6}F_{ijk}G_l G_m G_n S^{im}S^{jn}S^{kl} +$$

$$\frac{1}{8}F_{ijk}F_{lmn}F_{pqrs}S^{il}S^{jp}S^{ks}S^{mn}S^{qr} +$$

$$\frac{1}{8}F_{ijk}F_{lmn}F_{pqrs}S^{il}S^{jm}S^{ks}S^{np}S^{qr} + \frac{1}{2}F_{ijk}G_{lm}S^k S^{ij}S^{lm} -$$

$$\frac{1}{8}F_{ijk}F_{lmnp}S^k S^{ij} S^{lm} S^{np} -$$

$$\frac{1}{24}F_{ijk}F_{lmn}F_{pqr}F_{stu}S^{ip} S^{js} S^{kn} S^{lq} S^{mt} S^{ru} -$$

$$\frac{1}{8}F_{ijk}F_{lmn}G_{pq}S^{ij} S^{kq} S^{lm} S^{np} - F_{ijk}HS^k S^{ij} -$$

$$\frac{1}{16}F_{ijk}F_{lmn}F_{pqr}F_{stu}S^{il} S^{jm} S^{kr} S^{ns} S^{pq} S^{tu} -$$

$$\frac{1}{8}F_{ijk}F_{lmn}F_{pqr}F_{stu}S^{il} S^{jp} S^{ku} S^{mq} S^{nr} S^{st}$$

4780 graphs contribute to F_4, which starts with

$$F_4 = \frac{S^3}{248832}(\chi - 24)\chi(1728 + 168\chi + 5\chi^2) + \ldots$$

This first term comes from the graphs
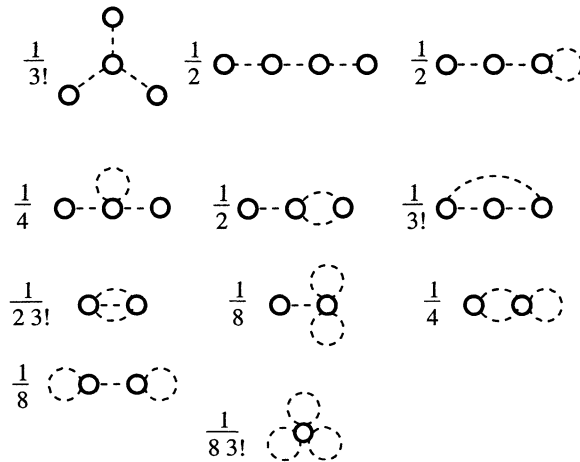
The contributions of the remaining graphs have been calculated and used to evaluate the generating function for the genus 4 curves on the quintic see Table 10. 172631 graphs contribute to F_5. These data are available on request.

We finally report a "Ward-identity" between the correlators at genus 3 on the quintic.

$$
\begin{aligned}
(S^{\psi\psi})^4 (&134560 S^{\Psi} F_{\psi\psi\psi}^3 - 7305(S^{\psi\psi})^2 - 8139 F_{\psi\psi\psi\psi}^2 \\
&+3364 F_{\psi\psi\psi} F_{\psi\psi\psi\psi\psi} - 3372 F_{\psi\psi\psi} F_{\psi\psi\psi\psi} G_{\psi} \\
&+1440 F_{\psi\psi\psi}^2 G_{\psi}^2 - 4S\psi\psi F_{\psi\psi\psi}^2 (2877 F_{\psi\psi\psi} G_{\psi} \\
&-1697 F_{\psi\psi\psi\psi}) + 20184 F_{\psi\psi\psi}^2 G_{\psi\psi}) = 0.
\end{aligned}
$$

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