On discrete Twist and

Four-Flux in $N = 1$

heterotic/F-theory

compactifications

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Abstract

We give an indirect argument for the matching $G^2 = -\pi_* \gamma^2$ of four-flux and discrete twist in the duality between $N = 1$ heterotic string and F-theory. This treats in detail the Euler number computation for the physically relevant case of a Calabi-Yau fourfold with singularities.

1   Introduction and Summary

Compactification on an elliptic Calabi-Yau three-fold Z with vector bundle V embedded in $E_8 \times E_8$ gives a four-dimensional heterotic string model of $N = 1$ supersymmetry. Originally the case of V the tangent bundle was considered which lead to an unbroken gauge group E_6 times a hidden E_8. The generalisation to an $SU(n)$ bundle V_1 gives unbroken GUT groups like $SO(10)$ and $SU(5)$ (we will in the following focus on the visible sector and therefore assume an E_8 bundle V_2 embedded in the second E_8).

Especially interesting is the case where Z admits an elliptic fibration $\pi : Z \to B_2$ which has a section σ. This makes possible an explicit description of the bundle by using the spectral cover[1] C of B_2. In this description the $SU(n)$ bundle is encoded in two data: a class[2] $\eta_1 = 6c_1 - t$ in $H^{1,1}(B_2)$ and a class γ in $H^{1,1}(C)$ (the latter is connected to the possible existence of chiral matter in these models [14], [18]). In this case it is also possible to give a dual description by F-theory on a Calabi-Yau four-fold X^4 which is $K3$ fibered over B_2 and elliptically fibered over B_3 which in turn is a $\mathbf{P}^1$ fibration described by the class t over B_2. Having an unbroken gauge group G on the heterotic side corresponds then to having a section of G singularities along B_2 in X^4.

It was shown [26] that an anomaly mismatch in the heterotic model causes the occurence of a number n_5 of five-branes wrapping the elliptic

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[1]The details of the spectral cover method and the corresponding F-theory description are reviewed in the appendix.

[2]unspecified cohomology classes refer to B_2; below B_2 will often simply denoted by B

fiber[3] F

$$c_2(Z) = c_2(V_1) + c_2(V_2) + n_5 F \tag{1.1}$$

where the Chern classes were given by (we assume B_2 to be rational)

$$c_2(Z) = 12\sigma c_1 + 12 + 10c_1^2$$
$$c_2(V_1) = \eta_1\sigma - \frac{n_3 - n}{24}c_1^2 - \frac{n}{8}\eta_1(\eta_1 - nc_1) - \frac{1}{2}\pi^*\gamma^2$$
$$= c_2(V_{1;\gamma=0}) - \frac{1}{2}\pi^*\gamma^2$$
$$c_2(V_2) = \eta_2\sigma - 40c_1^2 - 45c_1 t - 15t^2 \tag{1.2}$$

Consistent F-theory compactification on X^4 requires a number of space-time filling threebranes which are localized at points in the base B of the elliptic four-fold. The number of such threebranes was determined in [49] for the case of a *smooth* Weierstrass model for the fourfold by observing that the SUGRA equations have a solution only for a precise number of such threebranes, proportional to the Euler characteristic of the four-fold.

In the case of an $E_8 \times E_8$ vector bundle V, leaving no unbroken gauge group and corresponding to a smooth Weierstrass model for the fourfold it was also shown that the number of five-branes matches the number n_3 of three-branes on the F-theory side whose number is given by

$$\frac{e(X^4)}{24} = n_3 + \frac{1}{2}G^2 \tag{1.3}$$

where $G \in H^{2,2}(X^4)$ is the four-flux [16] (cf. appendix).

For various reasons one can expect G to be associated with γ; *a very condensed version of this argument can be found in the introduction to section C of the appendix.* Part of this association is the following relation

$$G^2 = -\pi^*\gamma^2 \tag{1.4}$$

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[3]We will stick in the following to the ansatz $\eta_1 + \eta_2 = 12\sigma c_1$ which leads only to five-branes wrapping the elliptic fiber. In general other curves would be wrapped as well [22], [19]. Note that in the cases of A_5 and D_6 we take for V_1 a product bundle $SU(n_1^{(1)}) \times SU(n_1^{(2)})$ with $\eta_1^{(1)} = 2c_1$ and $\eta_1^{(2)} = 4c_1 - t$.

which in view of the assumed equality $n_5 = n_3$ and

$$n_5 = c_2(Z) - c_2(V_{1;\gamma=0}) + \frac{1}{2}\pi^*\gamma^2 - c_2(V_2) \qquad (1.5)$$

$$n_3 = \frac{e(X^4)}{24} - \frac{1}{2}G^2 \qquad (1.6)$$

amounts to

$$\begin{aligned}
e(X^4) &= 24(c_2(Z) - c_2(V_{1;\gamma=0}) - c_2(V_2)) \\
&= 288 + (1200 + 107n - 18n^2 + n^3)c_1^2 \\
&\quad + (1080 - 36n + 3n^2)c_1 t + (360 + 3n)t^2
\end{aligned}$$

One would like now to see this equation directly on the F-theory side thereby proving (1.4).

This matching was extended [3] to the general case of heterotic string compactification on an elliptic Calabi-Yau threefolds together with a $SU(n_1) \times SU(n_2)$ vector bundle leading to an unbroken heterotic gauge group which corresponds to a certain locus of degenerated elliptic fibers in the fourfold. This concerned the case of a pure gauge group, corresponding to having singularities of only codimension 1 for X^4. For this the Euler number of the fourfold was expressed directly in their Hodge numbers, which were matched via a direct spectrum comparison with the data of the dual heterotic model; there essential use was made of an index-formula, computing the number of even minus odd vector bundle moduli.

Here we will adopt a different approach. We will express $\frac{e(X_4)}{24}$ in pour Calabi-Yau fourfold data without making any use of dual heterotic data. Now in general one will have also singularities of codimension 2 and even 3. The former arise on the F-theory side from intersection curves of the surface components of the discriminant [4] (the compact parts of the seven-branes in the type IIB interpretation of F-theory) : the I_1 surface and the G surface B_2. They are interpreted as matter [11], [37]. The idea is that for example the collision $E_6 + I_1$ leads to an E_7 which by the adjoint decomposition should correspond to a matter hypermultiplet in the **27** of E_6. On the heterotic side they arise from a similar condition on the cohomology of the bundle which should lead to

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[4]as in the end we want to make a comparison with a dual perturbative (up to the five-branes) heterotic model we consider no more general discriminant configurations

matter and is non-trivial along certain curves where the spectral cover intersects the base (for one class of matter curves). In certain cases (given in the main part of the paper below) of G one can tune the class t resp. the bundle so that such an intersection does not occur, i.e., so that one has only singularities in codimension one and only the case of a pure gauge group. In connection with the appropriate conditions one is also lead to a certain lower bound for the "instanton number" of the vector bundle conjectured in [48]. This is described in *section 2*.

In general one will have the matter curve in $B = B_2$ and for the A and D groups G even two of them, called h and P below, which intersect each other in a codimension three locus, a number of points in B. The corresponding stratification of the discriminant will allow us to compute the Euler number of the fourfold by adding up the parts with singular fibres. The corresponding computation in 6D, i.e., for Calabi-Yau threefolds is described in *section 3*. Essential is the consideration of the cusp locus $C = (f_1 = 0 = g_1)$ inside the I_1 surface component D_1 (of the discriminant surface $D = (4f^3 + 27g^2 = 0)$) which is *approximately* given by $4f_1^3 + 27g_1^2 = 0$ where in f_1, g_1 are split off the parts of f, g causing the G singularity. This is exact for the E_k series; for the $D_{4+n} = I_n^*$ and I_n series further n powers of z, the coordinate transversal to B_1 in the Hirzebruch surface B_2, can be extracted out of $4f_1^3 + 27g_1^2$ and one has actually the equation of divisors $D_1 + nr = (4f_1^3 + 27g_1^2 = 0)$ where r is the class of B_1 in B_2. In those cases one finds that the naive cusp set $C_{old} = (f_1 = 0 = g_1)$ (zero dimensional in the 6D case) contains actually a number x of points of the B_1 line (lying on one of the matter loci given by a divisor h) which are not cusp points[5] and have to be taken out of the cusp set so that the true cusp set is[6] $C = C_{old} - xhr$. This x is evaluated as the intersection multiplicity of f_1 and g_1 along h and computed via their resultant. Moreover not only elliptic fibers with cusp singularities $y^2 = x^3$ lie in the fibers over C but C is also a locus of 'intrinsic' cusp singularities for the D_1 locus. So in 6D one has then to apply the usual Plücker formulas to D_1.

Our general approach in 4D is described in *section 4*. Here we also

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[5]In some cases (A_3, A_4, A_5) other singularities arise at these points (tacnodes and even higher double points).

[6]note that throughout the paper cohomology classes like $h \in H^{1,1}(B_2)$ are identified with their pullbacks under π so that here for example is meant $C = C_{old} - x(\pi^* h \cdot r)$

give the heterotic expectation for $24n_5$ (in the cases of $G = A_1$ and A_2 we give the formula for $c_2(V)$ for $V = E_7$ or E_6 bundle in the appendix). In *section 5* we develop new "Plücker formulas" for the now relevant case of a surface D_1 having a curve of cusp (or higher) point singularities. These formulas are *not just adiabatic extensions* of the usual formulas for singular points on a curve. Actually the story is somewhat more complicated as one also has to treat the case of curves of tacnode point singularities (where a second blow up is needed) which occur in some cases at the collision of the D_1 surface with the G surface B_2 along the h curve. Then we go on to the codimension three loci in *section 6*. There are two types of codimension three loci: the ones related to enhancements of the fiber at the intersection of the matter (=enhancement) curves and the intersection of the cusp curve C with B_2, (because of the precise evaluation of x it turns out that these are actually proportional as cohomology classes) and the ones related to point singularities of D_1. In the final *section 7* we use the techniques accumulated so far to actually compute the Euler number of X^4 and to show that (with suitable assumptions) it equals $24n_5$ from the heterotic side where $\gamma = 0$ is assumed, thereby proving 1.4.

The *appendix* contains the explanation why one is led to expect equ. (1.4) in a general framework and reference material pertaining to the relevant facts about heterotic and F-theory $N = 1$ models.

Acknowledgements

2   The lower bound on η

This is a necessary bound on "how much instanton number has to be turned on to generate/fill out a certain $SU(n)$ bundle", or speaking in terms of the unbroken gauge group G (the commutator of $SU(n)$ in E_8) "to have no greater unbroken gauge group than a certain G". It

is treated here as a warm up because it is related to the consideration of singularities along just B_2 and so in codimension one only[7] (the case that D_1 and B_2 are disjoint) versus singularities in even higher codimension (like the matter curves from the intersection of D_1 and B_2). We will assume that G is an ADE group.[8]

2.1 F-theory arguments

Let us recall the situation in six dimensions. There the easiest duality set-up is given by the duality of the heterotic string on $K3$ with instanton numbers $(12 - m, 12 + m)$ (and no five-branes) with F-theory on the Hirzebruch surface F_m [47]. The gauge group there is described by the singularities of the fibration and a perturbative heterotic gauge group corresponds to a certain degeneration over the zero-section C_0 (of self-intersection $-m$): for example to get an $SU(3)$ one needs a certain A_2 degeneration over C_0 available first for $m = 3$; in general this means that the discriminant divisor $\Delta = 12c_1(F_m)$ has a component $\delta(G)C_0$ where $\delta(G)$ is the vanishing order of the discriminant (equivalently the Euler number of the affine resolution tree of the singularity), giving also the relation $m \leq \frac{24}{12-\delta(G)}$ for the realization over a F_m to have no singularity worse than G. The last relation follows (cf. [48]): from the fact that after taking the C_0 component with its full multiplicity $\delta(G)$ out of Δ the resulting $\Delta' = \Delta - \delta(G)C_0$ has transversal intersection with C_0 and so $\Delta' \cdot C_0 \geq 0$, leading with $c_1(F_m) = 2C_0 + (2 + m)f$ to the mentioned result.

So the instanton number $12 - m$ to give a G gauge group has to be $12 - m \geq 12 - \frac{24}{12-\delta(G)} = (6 - \frac{12}{12-\delta(G)})c_1(B_1)$ with B_1 the common P^1 base of the heterotic $K3$ resp. the F_m. From this it was conjectured in [48] that a similar bound could in four dimensions look like the generalizations of both sides of the six-dimensional bound, i.e., in view of the fact that the $(12 - m, 12 + m)$ structure generalizes in four dimensions to $\eta_1 = 6c_1 - t, \eta_2 = 6c_1 + t$ (for this cf. the anomaly cancellation condition $c_2(V_1) + c_2(V_2) + a_f F = c_2(Z)$ and its component $\eta_1\sigma + \eta_2\sigma = 12c_1\sigma$ concerning the classes not pull-backed from $H^4(B)$

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[7]up to the cusp curve in C_1

[8]The η bound is treated in a toric framework in [10].

for the case of an A model with $W_B = 0$)

$$\eta_1 \geq (6 - \frac{12}{12 - \delta(G)})c_1 \qquad (2.7)$$

Let us now first prove this in the A model with $W_B = 0$ and then include a non-zero W_B. For this recall that the association in six dimensions of the heterotic $(12-m, 12+m)$ with F_m on the F-theory side generalizes [26] to the association of the heterotic $\eta_1 = 6c_1 - t, \eta_2 = 6c_1 + t$ with the following structure of the F-theory base B_3 as a $\mathbf{P}^1$ bundle over the common (with the heterotic side) base $B = B_2$. Look at the $\mathbf{P}^1$ bundle as projectivization of a vector bundle $\mathcal{O} \oplus \mathcal{T}$ with $\mathcal{T}$ a line bundle over B of $c_1(\mathcal{T}) = t$ (this generalizes the twisting condition in the Hirzebruch surface). To make actual computations let us introduce homogeneous coordinates a, b which are sections of $\mathcal{O}(1)$ and $\mathcal{O}(1) \otimes \mathcal{T}$ respectively, where $\mathcal{O}(1)$ is the $\mathcal{O}(1)$ bundle on the $\mathbf{P}^1$ fibers of $c_1(\mathcal{O}(1)) = r$, say, and $r(r + t) = 0$ as a, b have no common zeroes (the disjointness of the zero-section and the section at infinity in the Hirzebruch surface case). Adjunction gives then

$$c_1(B_3) = c_1 + 2r + t \qquad (2.8)$$

and the condition

$$(12c_1(B_3) - \delta(G)B_2)B_2 \geq 0 \qquad (2.9)$$

gives with $B_2 = r$ that

$$12c_1 + (\delta(G) - 12)t \geq 0 \qquad (2.10)$$

resp. formulated in $\eta_1 = 6c_1 - t$ the bound to be proved.

Now let us include the effect of a non-zero W_B. From six dimensions one knows that a heterotic five-brane corresponds to a blow-up in the F-theory base. So here we have to consider the impact of the ruled surface S (in the thereby modified $\tilde{B}_3$) over W_B in $B = B_2$. Its contribution is

$$c_1(\tilde{B}_3) = c_1(B_3) - S \qquad (2.11)$$

leading after intersection with B in the inequality above to a term $-W_B$ on the left hand side or $+W_B$ on the right hand side. On the other hand one has now that $\eta_1 + W_B = 6c_1 - t$ (we think already of the case $\eta_2 = 0$) so that the final bound is unchanged

$$\eta_1 + W_B \geq (6 - \frac{12}{12 - \delta(G)})c_1 + W_B \qquad (2.12)$$

2.2   Heterotic arguments

As the $4D$ bound was guessed from a $6D$ expression, let us point out that it is also possible to see the bound from an adiabatic argument. Namely, assume that B_2 is the adiabatic extension of a $6D$ base B_1, i.e., that B_2 is a Hirzebruch surface F_n. Now as remarked in [48] one has from the investigation of [27] that one of the inequalities $\eta \geq i \cdot c_1$ holds for some i in $2, \ldots n$. Restricting to a fibre $f = \mathbf{P}^1$ of F_n, i.e., restricting to a $K3$ fibre of Z, one gets from $f \cdot c_1(F_n) = 2$ and the $6D$ bound that the relevant i is the same as occurring in $6D$ if one writes there the bound as proportionality to $c_1(B_1) = 2$.

Yet another argument starts from the observation that the class $n\sigma + \eta$ of the spectral cover should be effective, so its image $-nc_1 + \eta$ in the base should be effective too. Now n itself is equal to the $6 - \frac{12}{12 - \delta(G)}$, where $\delta(G) = 11 - n$, for $n = 2, 3$, respectively equals the next bigger integer for $n = 4$ and is even greater (and so implying the lower bound from the stronger spectral cover effectiveness argument $\eta \geq nc_1$) for higher n.

3   The three-dimensional case

Before coming to the actual computation of $e(X)$ let us briefly review the three-dimensional situation, i.e having a Calabi-Yau three-fold Z which is elliptically fibered over a two-dimensional base B_2. For that we reconsider first the case of having a smooth Z, i.e the elliptic fiber does not degenerate worse than I_1 (resp. II at the cusp points) over the discriminant D. Then we proceed and consider the case where the elliptic fiber has a G-singularity (G will be always one of the ADE groups and we will always be in the *split* case of [11]) localised over a codimension one locus in B_2. In our set-up B_2 will be a Hirzebruch surface F_n, i.e., a P^1 fibration over $B_1 = P^1$. Note that apart from $G = E_8$ where $n = 12$ (so we are in the first column of table $A.1$ of [13]) we restrict ourselves to $n = 0, 1, 2$ (the first three rows of the table mentioned). (Because of our interchange (compared to the usual convention) of the bundles one has strictly speaking to put $n = 0, -1, -2$ in the formulae below.)

The fiber enhancement is given as follows: the matter loci are specified in [11] and [37] relates matter and fiber enhancement (intuitively one may think for example of the **27** matter locus of E_6 as located at the E_7 fiber enhancement points given by the collision of the E_6 line B and the I_1 curve D_1).

For background and notation of the elliptically fibered geometry cf. appendix.

3.1   Smooth case

In case that B is two-dimensional, D is a curve in B_2 of class $D = 12c_1(B_2)$. The three-dimensional Calabi-Yau Z over B_2 is described by a smooth Weierstrass model, so one has only type I_1 (and II) singular fibers over D which contribute to $\chi(Z)$. The idea of an Euler number computation from the elliptic fibration data is of course $e(\tilde{Z}) = e(\text{sing. fiber})e(D)$. Since D is a curve we have $-D(D - c_1(B_2)) = -132c_1^2$ (where $c_i = c_i(B_2)$). But D itself will be singular at those points where the divisors associated to the classes $F = 4c_1(B_2)$ and $G = 6c_1(B_2)$ collide, i.e., at $F \cdot G = 24c_1^2$ points. At these points D develops a cusp and the elliptic fiber will be of type II. Using the standard Plücker formula, which takes the double points and cusps into account (cf. [30]), one gets $e(\tilde{D}) = -132c_1^2 + 2(24c_1^2)$, and so we get

$$e(Z) = e(I_1)(-84c_1^2 - 24c_1^2) + e(II)(24c_1^2) = -60c_1^2. \qquad (3.1)$$

3.2   Singular case

Now assume that Z has a section of G-singularities localised over the base curve in the Hirzebruch surface $F_n = B_2$. Consider in F_n the two rational curves given of self-intersection $-n$ resp. $+n$ given by the zero section S_0 and the section at infinity $S_\infty = S_0 + nf$ of the $\mathbf{P}^1$ bundle. Let us localize the G fibers along S_0, where we have an eye on a dual perturbative heterotic[9] description. Now, we can decompose

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[9]we will have $12 - n$ resp. $12 + n$ instantons on the heterotic side corresponding to S_0 resp. S_∞; we put the greater number into the *second* bundle where we want to span an E_8 bundle

the discriminant D into two components: $D = D_1 + D_2$, where D_1 denotes the component with generic I_1 fibers and D_2 has G fibers. Each component is characterized by the order of vanishing of some polynomials as D itself. Denote the class of D_2 by $D_2 = cS_0$, resp. $F_2 = aS_0$ and $G_2 = bS_0$. With the canonical bundle of the Hirzebruch surface $K_{F_n} = -2S_0 - (2+n)f$ we get $D_1 = (24-c)S_0 + (24+12n)f$, resp. $F_1 = (8-a)S_0 + (8+4n)f$ and $G_1 = (12-b)S_0 + (12+6n)f$, so describing the locus of I_1 singularities.

Since only singular fibers contribute to $\chi(Z)$, we get $e(Z) = e(D_1)e(\mathrm{I}_1) + e(S_0)e(G)$. Now D_1 is a curve in the base, which has cusp singularities at $F_1 G_1 = 192 + (6n-12)a + (4n-8)b - abn$ points, so applying the standard Plücker formula, we find $e(D_1) = -D_1(D_1 + K_{F_n}) + 2F_1 G_1 = -1056 + (46-23n)c + c^2n + 2F_1 G_1$. The cusps contribute with $e(II)F_1 G_1$ to $e(Z)$; also we have to take into account that the D_1 branch will intersect the branch of G-singularities S_0 in a number of points (which will modify the cusp set $F_1 G_1$ for $G = I_n, I_n^*$).

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\begin{aligned}
e(Z) = {}& e(\mathrm{I}_1)\left(e(D_1) - e(D_1 \cap S_0) - F_1 G_1 \right) \\
& + e(II) F_1 G_1 \\
& + e(G)\left((e(S_0) - e(D_1 \cap S_0)) \right) \\
& + \sum_{i \in \mathcal{M}} e(G_i^{enh}) e(i) \\
= {}& -480 + (18n-36)a + (12n-24)b + (48-23n)c \\
& + (c^2 - 3ab)n + \sum_{i \in \mathcal{M}} e(i)\left(e(G_i^{enh}) - e(G) - 1 \right) \qquad (3.2)
\end{aligned}
$$

where $\mathcal{M}$ is the set of components of the intersection of D_1 and S_0. For example, for $A_4 = I_5$ one has $D_1 S_0 = 4h_{c_1-t} + P_{8c_1-3t}$, so $\mathcal{M}$ consists of h and P; further $e(h) = 2 - n, e(P) = 16 - 3n$ and $e(G) = 5$, $e(G_h^{enh}) = 6$ and $e(G_P^{enh}) = 7$ corresponding to the generic I_5 fibre, the I_6 enhancement fibre and the D_5 enhancement fiber.

Let us now give a number of cases which illustrate the above formula. To do so we proceed as follows: first, we read off the necessary information about the base geometry from the discriminant, then we

compute the the Euler characteristic of Z and compare our results with the heterotic string side.

3.2.1 $E_8(II^*)$ singularity

From the discriminant of Z

$$\Delta = z_1^{10}(g_{(12-n)}^2(z_2) + \mathcal{O}(z_1^2)) \tag{3.3}$$

we learn that the D_1 locus has $(12-n)$ double points which contribute to $e(D_1)$ and which lying on the intersection points of D_1 and S_0, so we have to apply the Plücker formula for curves which leads to a $2(12-n)$ contribution to the Euler number of D_1. Further $F_1 S_0 = 8$ and $G_1 S_0 = (12-n)$ and the Euler number of Z is given by

$$e(Z) = -240 - 60n \tag{3.4}$$

Let us compare this result with the heterotic string side. There we find $\dim \mathcal{M}_{12+n}(E_8) + \dim \mathcal{M}_{12-n}(SU(1)) + h^{1,1}(K3) = 144 + 29n$ which contribute to the number of hypermultiplets further we have 248 vectors and $13 - n = 1 + 12 - n$ tensors satisfying the gravitational anomaly equation $273 - 144 - 29n + 248 = 29n_T$. This leads to the prediction for $h^{2,1}$ resp. $h^{1,1}$ of the corresponding F-theory model $h^{2,1}(Z) = 152 + 28n$, $h^{1,1}(Z) = 8 + 2 + 1 + 12 - n = 23 - n$ giving $\chi(Z) = -240 - 60n$ which is in agreement with our computation above.

3.2.2 $E_7^s(III^*)$ singularity

Here the discriminant is given by

$$\Delta = z_1^9(4f_{8-n}^3(z_2) + \mathcal{O}(z_1)) \tag{3.5}$$

telling us $D_1 S_0 = 3(8-n)$ so that we should expect an enhancement at $(8-n)$ points, i.e., $e(D_1 \cap S_0)(e(G_{enh}) - e(G) - 1) = (8-n)(e(II^*) - e(III^*) - 1) = 0$. But actually[10] the fibre over these points is not[11]

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[10]as pointed out by Aspinwall [7]

[11]although a generic slice through the singularity might one lead to believe it looks like E_8; but the resolution of the threefold will not give the full E_8 when one does the blow-up explicitly (cf. [43])

of Kodaira type II^* (whose affine diagram is of Euler number 10) but consists of a chain of 8 P^1's (which is not a Kodaira fibre) and has Euler number 9 giving an $-1(8-n)$ contribution to the total Euler number of Z. Thus we find (note also that $F_1 S_0 = (8-n)$, $G_1 S_0 = (12-n)$)

$$e(Z) = -284 - 56n \tag{3.6}$$

From the heterotic side we get

$$\dim \mathcal{M}_{12+n}(E_8) + \dim \mathcal{M}_{12-n}(SU(2)) + h^{1,1}(K3)$$
$$= 153 + 28n = n_H \tag{3.7}$$

so that $n_H = 153 + 28n$ and $n_V = 133$ satisfying the anomaly equation $244 + 133 = 153 + 28n + \frac{56}{2}(8-n)$ and giving $h^{2,1}(Z) = 152 + 28n$ and $h^{1,1}(Z) = 10$, thus equ. (3.6).

3.2.3 $E_6^s(IV^*)$ singularity

$$\Delta = z_1^8 (27 q_{6-n}^4(z_2) + \mathcal{O}(z_1)) \tag{3.8}$$

so we expect $e(D_1 \cap S_0) = D_1 S_0/4 = (6-n)$ collisions between the IV^* and I_1 fiber, further we have $F_1 S_0 = (8-n)$ and $G_1 S_0 = 2(6-n)$. So *Katz/Vafa collision rules* give $e(D_1 \cap S_0)(e(G_{enh}) - e(G) - 1) = (6-n)(e(III^*) - e(IV^*) - 1) = 0$ and we find

$$e(Z) = -300 - 54n \tag{3.9}$$

which can be checked on the heterotic side: $\dim \mathcal{M}_{12+n}(E_8)$ $+ \dim \mathcal{M}_{12-n}(SU(3)) + h^{1,1}(K3) = 160 + 27n = n_H$ and $n_V = 78$, satisfying $244 + 78 = 160 + 27n + 27(6-n)$ and leading to $h^{2,1}(Z) = 159 + 27n$ and $h^{1,1}(Z) = 9$.

3.3   A subtlety concerning the cusp set

Before we will proceed and consider some I_n^* and I_n examples, we have to make a digression concerning the cusp set in these examples.

The reason for that is that, contrary to the case of the E_k series, now the Kodaira values a and b in $f = f_1 + ar$, $g = g_1 + br$ do not

lead by themselves to the value c; instead they would always lead to I_0^* and I_0. To get actually a higher n one has to tune the occurring expressions f_1, g_1 so that in the discriminant n more powers of z (the local coordinate transversal to B_1) than naively expected (i.e., 6 for the D case and 0 for the A case) can be extracted.

Recall that we have (z_1 is the base variable of divisor r)

$$f(z_1, z_2) = \sum_{i=a}^{I} z_1^i f_{8-n(4-i)}(z_2)$$

$$g(z_1, z_2) = \sum_{j=b}^{J} z_1^j g_{12-n(6-j)}(z_2) \tag{3.10}$$

(cf. [11]).

Let us consider this in the example of D_5 where $(a, b, c) = (2, 3, 7)$ and

$$D \sim z^6(4f_1^3 + 27g_1^2) \tag{3.11}$$

But as we have to force a z^7 the coefficients of z^6 have to cancel which leads to the conditions $f_{4c_1-2t} \sim h_{2c_1-t}^2$ and $g_{6c_1-3t} \sim h_{2c_1-t}^3$; furthermore from the split condition (to get really E_6 and not F_4) one gets $g_{6c_1-2t} + f_{4c_1-t}h_{2c_1-t} = q_{3c_1-t}^2$. Altogether this leads to an equation for D

$$D \sim z^7[h_{2c_1-t}^3 q_{3c_1-t}^2 + \mathcal{O}(z)] \tag{3.12}$$

Similarly for I_5, say, one has $(a, b, c) = (0, 0, 5)$ and again the cancellation of the leading terms (there are now higher cancellation conditions as well) leads to $f_{4c_1-4t} \sim h_{c_1-t}^4$ and $g_{6c_1-6t} \sim h_{c_1-t}^6$. All the conditions including the split condition lead to a description by four further relevant sections besides h_{c_1-t}, namely H_{2c_1-t}, q_{3c_1-t}, f_{4c_1-t}, g_{6c_1-t} and a discriminant

$$D \sim z^5 h_{c_1-t}^4 P_{8c_1-3t} \tag{3.13}$$

So the fact that we have to enforce a higher power of z to be extractable leads to the cancellation conditions which come down to $f_1 r = 2h_{2c_1-t}$, $g_1 r = 3h_{2c_1-t}$ for the I_n^* series (for $n > 0$) and to

$f_1 r = 4h_{c_1-t}$, $g_1 r = 6h_{c_1-t}$ for the I_n series (for $n > 0$). This fact then, that f_1 and g_1 have a component h in common, changes the actual cusp set from $f_1 g_1$ to $f_1 g_1 - x \cdot h$ where x is the intersection multiplicity of f_1 and g_1 at h (as computed by the vanishing order of the resultant) which counts the number of times h lies in the intersection product. So to compute x we have to determine the f_1 and g_1 polynomials, i.e., express them in local data near the collision point $D_1 r$. This can be done by using the more general Weierstrass equation

$$y^2 + a_1 xy + a_3 y^3 = x^3 + a_2 x^2 + a_4 x + a_6 \qquad (3.14)$$

where the a_i's are locally defined polynomial functions on the base as f and g (for details see [11]). Further one can express the f and g polynomial in terms of the a_i's

$$f = -\frac{1}{48}((a_1^2 + 4a_2)^2 - 24(a_1 a_3 + 2a_4))$$

$$g = -\frac{1}{864}(-(a_1^2 + 4a_2)^3 + 36(a_1^2 + 4a_2)(a_1 a_3 + 2a_4) - 216(a_3^2 + 4a_6))$$

$$(3.15)$$

The local structure (orders in z) of the a_i's is given by [11] (we are always in the split case)

G	a_1	a_2	a_3	a_4	a_6
I_2	0	0	1	1	2
I_{2k+1}	0	1	k	$k+1$	$2k+1$
I_{2k}	0	1	k	k	$2k$
I_0^*	1	1	2	2	4
I_1^*	1	1	2	3	5
I_2^*	1	1	3	3	5

The h-locus, related to the relevant enhancement, is given for the I series by a_1 (with the exception of I_2 where not h_{c_1-t} but H_{2c_1-2t} is relevant and where the corresponding enhancement locus is given by $b_2 = a_1^2 + 4a_2$) and for the I^* series by a_2 for $n > 0$ and by a_4 for $n = 0$. This is actually refined with a corresponding z power according to $t = a_{2,1} = a_2/z$ for example for D_5 (cf. [11]).

Concerning the value of x one finds that for the I_n series (with two exception: $x = 3$ for $n = 2$, but this would give $x = 6$ if H were h^2 thus fitting the pattern of the I series, and $x = 8$ for $n = 3$) with $n = 4, 5, 6$ it is given by $x = 3n$ and for I_0^*, I_1^*, I_2^* it is given by $x = 0, 2, 3$.

Furthermore we have to note that there is another twist in the story which comes from the fact that D_1 will have a tacnode or an even higher double point when colliding with $r = S_0$ for the cases I_4, I_5, I_6 at the points on $D_1 = r = S_0$ of h, that is a singular point of the form $t^m + z^2 = 0$ with $m = 4$ (tacnode, for I_4) resp. $m = 6$ for I_5, I_6 (after suitable coordinate change).

Group	f_1, g_1	sing. at t	x
I_2	$-\frac{1}{48}(t^2 - 72z)$ $-\frac{1}{864}(-t^3 + 108tz - 1080z^2)$	$-$	3
I_3	$-\frac{1}{48}((t^2 + 4z)^2 - 24(tz + 2z^2)) - \frac{1}{864}(-(t^2 + 4z)^3$ $+36(t^2 + 4z)(tz + 2z^2) - 216(z^2 + 4z^3))$	$-$	8
I_4	$-\frac{1}{48}((t^2 + 4z)^2 - 24(tz^2 + 2z^2)) - \frac{1}{864}(-(t^2 + 4z)^3$ $+36(t^2 + 4z)(tz^2 + 2z^2) - 216(5z^4))$	$t^4 + v^2$	12
I_5	$-\frac{1}{48}((t^2 + 4z)^2 - 24(tz^2 + 2z^3)) - \frac{1}{864}(-(t^2 + 4z)^3$ $+36(t^2 + 4z)(tz^2 + 2z^3) - 216(z^4 + 4z^5))$	$t^6 + v^2$	15
I_6	$-\frac{1}{48}((t^2 + 4z)^2 - 24(tz^3 + 2z^3)) - \frac{1}{864}(-(t^2 + 4z)^3$ $+36(t^2 + 4z)(tz^3 + 2z^3) - 216(5z^6))$	$t^6 + v^2$	18
I_0^*	$-\frac{1}{48}((z^2 + 4z)^2 - 24(z^3 + 2tz^2))/z^2 - \frac{1}{864}(-(z^2 + 4z)^3$ $+36(z^2 + 4z)(z^3 + 2tz^2) - 1080z^4)/z^3$	$-$	0
I_1^*	$-\frac{1}{48}((z^2 + 4tz)^2 - 72z^3)/z^2 - \frac{1}{864}(-(z^2 + 4tz)^3$ $+108(z^2 + 4tz)z^3 - 216(z^4 + 4z^5))/z^3$		2
I_2^*	$-\frac{1}{48}((z^2 + 4tz)^2 - 24(z^4 + 2z^3))/z^2 - \frac{1}{864}(-(z^2 + 4tz)^3$ $+36(z^2 + 4tz)(z^4 + 2z^3) - 216(z^6 + 4z^5))/z^3$	$-$	3

3.3.1 $D_4^s(I_0^*)$ singularity

$$\Delta = z_1^6((h_{4-n}^2 + q_{4-n}^2)(h_{4-n}^2 + \omega q_{4-n}^2)(h_{4-n}^2 + \omega^2 q_{4-n}^2) + \mathcal{O}(z)). \quad (3.16)$$

Here we find $D_1 S_0/6 = (4 - n)$ intersection points between the I_0^* and the I_1 locus. Further we have $F_1 S_0 = 2(4 - n)$ and $G_1 S_0 = 3(4 - n)$. Since we have no additional corrections from the cusps ($x = 0$) and assuming the *Katz/Vafa collision rules* applies in the form $e(G_i^{enh}) = e(G) + 1$ we find

$$e(Z) = -336 - 48n. \quad (3.17)$$

Now let us see which prediction comes from the heterotic side. There we find for the number of hyper-multiplets $3 \cdot 8(4 + n) - 28 = 68 + 24n$ and additional ones coming from $\dim_Q(\mathcal{M}_{inst}^{(n_1+n_2)}) + h^{1,1}(K3) = 88 + 20$ giving a total $n_H = 176 + 24n$ and with $n_V = 28$ vectors we find that the anomaly equation $244 + 28 = 176 + 24n + 3 \cdot 8(4 - n)$ is satisfied. Thus we find $h^{2,1}(Z) = 175 + 24n$ and $h^{1,1}(Z) = 7$ and so $e(Z) = -336 - 48n$ in agreement with our F-theory computation.

3.3.2 $D_5^s(I_1^*)$ singularity

$$\Delta = z_1^7(h_{4-n}^3 q_{6-n}^2 + \mathcal{O}(z_1)). \quad (3.18)$$

We have $D_1 S_0 = 3(4-n) + 2(6-n)$ and $F_1 S_0 = 2(4-n)$, $G_1 S_0 = 3(4-n)$ and further we have to take into account the change of the cusp set since $x = 2$, i.e., we find $C = F_1 G_1 - 2(4 - n)$. Assuming the *Katz/Vafa collision rules* we get

$$e(Z) = -312 - 52n. \quad (3.19)$$

Now the heterotic side gives $\dim_Q(\mathcal{M}_{inst}^{(n_1+n_2)}) + h^{1,1}(K3) = 66 + 20$ hypers resp. $(4 + n)16 + (6 + n)10 - 45 = 79 + 26n$ hypers so $n_H = 165 + 26n$ and $n_V = 45$ satisfying the anomaly cancellation $244 + 45 = 165 + 26n + 16(4 - n) + 10(6 - n)$ and leading to $h^{2,1}(Z) = 164 + 26n$ resp. $h^{1,1}(Z) = 8$ thus $e(Z) = -312 - 52n$.

Let us give for later reference the equation for the D_1 part of the discriminant; here we will already use the 4D notation so that for example the degree $4 - n$ becomes the class $2c_1 - t$.

Now D_1 is given by $h^3_{2c_1-t}q^2_{3c_1-t}+\mathcal{O}(z)=0$. Let us make the accompanying[12] term of the power z explicit in $h^3_{2c_1-t}q^2_{3c_1-t}+S_{12c_1-4t}z+\mathcal{O}(z^2)$. Explicitely one finds with (using the notation $f_i := f_{4c_1-it}$, $g_i := g_{6c_1-it}$ and denote our former f_1,g_1 by F_1,G_1 if there is change of confusion)

$$
\begin{aligned}
F_1 &= \frac{1}{48}(-h^2+zf_1+z^2f_0) \\
G_1 &= \frac{1}{864}(h^3+z(q^2-\frac{3}{2}f_1h)+z^2g_1+z^3g_0)
\end{aligned}
\tag{3.20}
$$

that

$$
\begin{aligned}
& 32\cdot 864\left(4F_1^3+27G_1^2\right) \\
&= z\left(2h^3q^2+\left(-\frac{3}{4}f_1^2h^2+2g_1h^3+3f_0h^4-3f_1hq^2+q^4\right)z\right. \\
&\quad +(f_1^3-3f_1g_1h-6f_1f_0h^2+2g_0h^3+2g_1q^2)z^2 \\
&\quad +(3f_1^2f_0+g_1^2-3f_1g_0h-3f_0^2h^2+2g_0q^2)z^3 \\
&\quad \left.+(3f_1f_0^2+2g_1g_0)z^4+(f_0^3+g_0^2)z^5\right)
\end{aligned}
\tag{3.21}
$$

so that

$$
S_{12c_1-4t}=h^2\left(-\frac{3}{4}f_1^2+2g_1h+3f_0h^2\right)-3f_1hq^2+q^4.
\tag{3.22}
$$

3.3.3 $D_6^s(I_2^*)$ singularity

As explained below one finds the following discriminant structure (with $\Delta = 32\cdot 864(4f_1^3+27g_1^2)$)

$$
\Delta = z_1^8(h_{4-n}^2P_{8-n}^2+\mathcal{O}(z_1)).
\tag{3.23}
$$

Note that in this case we take for V_1 a product bundle $SU(2^{(1)})\times SU(2^{(2)})$ with $\eta_1^{(1)}=2c_1$ and $\eta_1^{(2)}=4c_1-t$ (this is the case $r=0$ of [11]).

We have $D_1S_0=2(4-n)+2(8-n)$ and $F_1S_0=2(4-n)$, $G_1S_0=3(4-n)$ and further we have to take into account the change of the

---

[12]one sees from $r^2=-rt$ that for ascending powers i of z also the t coefficient rises and so the degree of the accompanying term changes as $12c_1-(12-i)t$, keeping always (including the overall z^7) $12c_1-(12-i)t-it=12c_1-12t=Dr$

cusp set since $x = 3$, i.e., we find $C = F_1 G_1 - 3(4 - n)$. Assuming the *Katz/Vafa collision rules* we get

$$e(Z) = -276 - 57n. \tag{3.24}$$

Now the heterotic side gives $\dim_Q(\mathcal{M}_{inst}^{(n_1+n_2)}) + h^{1,1}(K3) = 36 + 20$ hypers resp. $(4 + n)16 + (8 + n)12 - 66 = 94 + 28n$ hypers so $n_H = 150 + 28n$ and $n_V = 45$ satisfying the anomaly cancellation $244 + 28 = 150 + 28n + 32\frac{1}{2}(4 - n) + 12(8 - n)$ and leading to $h^{2,1}(Z) = 149 + 28n$ resp. $h^{1,1}(Z) = 9$ thus predicting $e(Z) = -280 - 56n$.

The further necessary contribution $-1(4 - n)$ is explainable as follows: the enhanced fibre over these points is not (although a generic slice through the singularity might one lead to believe it looks like E_7; but the resolution of the threefold will not give the full E_7 when one does the blow-up explicitly a la Miranda) of Kodaira type III^* (whose affine diagram is of Euler number 9) but consists of a chain of 7 P^1's (which is not a Kodaira fibre) and has Euler number 8 giving an $-1(4 - n)$.

One finds with

$$\begin{aligned}
F_1 &= \frac{1}{48}(-h^2 + zf_1 + z^2 f_0) \\
G_1 &= \frac{1}{864}(h^3 + zg_2 + z^2 g_1 + z^3 g_0)
\end{aligned} \tag{3.25}$$

that the condition to have $c = 8$ leads to

$$g_2 = -\frac{3}{2}hf_1 \tag{3.26}$$

and to get the 'split' $SO(12)$ situation (with parameter $r = 0$) one has [11] to introduce $q = q_{4c_1-t}$ and $u = u_{2c_1}$ and to impose the conditions

$$\begin{aligned}
f_1 &= q + hu \\
g_1 &= \frac{3}{4}qu
\end{aligned} \tag{3.27}$$

(note that [11] have $a_{4,3} \sim F_1$, $a_{6,5} \sim G_1$). This gives

$$\Delta \sim z^8 \left(-\frac{3}{4}h^2 \left(f_1^2 - \frac{8}{3}g_1 h - 4f_0 h^2 \right) + \mathcal{O}(z) \right). \tag{3.28}$$

The identification $a_{4,3}^2 - 4a_{2,1}a_{6,5}$ of the second enhancement locus given in [11] shows that for $f_0 = \frac{1}{4}u^2$ one has $f_1^2 - \frac{8}{3}hg_1 - 4f_0 h^2 = (q + hu)^2 - 2hqu - u^2 h^2 = q^2 =: P^2$.

3.3.4 $A_{k-1}^s(I_k)$ singularity

We consider now the I_k series for $k = 2, 3, 4, 5, 6$ where the leading term of the discriminant is given by $z^2 H_{4-2n}^2 P_{16-6n}$ for I_2, by $z^3 h_{2-n}^3 P_{18-6n}$ for I_3 and by $z^k h_{2-n}^4 P_{16-(8-k)n}$ for I_k with $k = 4, 5, 6$. One has always $FS_0 = 4(2-n)$, $GS_0 = 6(2-n)$ and $D_1 S_0 = 4(2-n) + (16 - (8-k)n)$ and the cusp set is given by $C = F_1 G_1 - x(2-n)$ with x the intersection multiplicity of f_1 and g_1 at $h = 0$. Including the additional singularity contributions (tacnode, etc.) at the h-points of $D_1 \cap r$ $(r = B_1 = S_0)$ for I_4, I_5, I_6 considered below one gets

G	$\chi(Z)$
I_2	$-420 - 24n$
I_3	$-384 - 36n$
I_4	$-352 - 44n$
I_5	$-320 - 50n$
I_6	$-288 - 54n$

Note that the Euler numbers (cf. [13]) match with the heterotic expectations for the spectrum.

Now let us look at the tacnode (and higher double point) singularities of D_1 at h mentioned above (cf. in the following the explicit discriminant forms given in the appendix).

I₄

Note that on h the equation of D_1 is given by $(e := f_2 + H^2)$

$$-\frac{3}{4}e^2\left(h^4 - 2h^2 Hz + \left(H^2 - \frac{4}{3}e\right)z^2\right) + \mathcal{O}(z^3) = 0. \qquad (3.29)$$

So the fact that we do not have a complete square structure in the leading terms (because the expression $\frac{4}{3}e$) shows that we have a generic tacnode structure at h, in contrast to the cases I_5 and I_6. So the $6D$ Euler number contribution will be $+4 - 1 = 3$ (taking into account that one has to go back to the singular model) at each of the $(2-n)$

intersection points of the h component of the intersection of D_1 with the B_1 line.

I$_5$

Note that near h the equation of D_1 is given by (with $e := f_1 H + q^2$)

$$-3Hq^2(h^4 - 2h^2 Hz + H^2 z^2) + \left(-H^2(2g_1 H + 3f_1^2) + \frac{9}{4}e^2\right) z^3$$
$$+ (f_1^3 + 3g_1 e)z^4 + g_1^2 z^5 = 0.$$

So the complete square structure of the leading terms shows that we do not have a generic tacnode structure, just as for I_6 but in contrast to the case I_4. If we replace the variable z by $w := Hz - h^2$ the terms up to third order become (everything up to coefficients) $w^2 + z^3 \rightarrow h^6 + 3h^4 w + w^2 + w^3 \sim h^6 + 3h^4 w + w^2$ near $(h, w) = (0, 0)$ which goes with $w := v - \frac{3}{2}h^4$ to the normal form $h^6 + v^2$. So the $6D$ Euler number contribution will be $+6 - 1 = 5$ (taking into account that one has to go back to the singular model) at each of the $(2 - n)$ intersection points of the h component of $D_1 \cap r$. One finds that one has to adopt a refined analysis to get a missing contribution $+1(2 - n)$.

I$_6$

Note that on h the D_1 equation is given by ($e := \frac{1}{2}f_1 + \mathcal{F}H$ $e' := f_1 + \mathcal{F}H$)

$$-3e^2(h^4 - 2h^2 Hz + H^2 z^2) + f_1\left(\frac{9}{2}\mathcal{F}He' + f_1^2\right) z^3 + \frac{9}{2}\mathcal{F}^2 e'^2 z^4.$$

$$(3.30)$$

So the complete square structure of the leading terms shows that we do not have a generic tacnode structure, just as for I_5 but in contrast to the case I_4. With $w := Hz - h^2$ the terms up to third order become (everything up to coefficients) $w^2 + z^3 \rightarrow h^6 + 3h^4 w + w^2 + w^3 \sim h^6 + 3h^4 w + w^2$ near $(h, w) = (0, 0)$ which goes with $w := v - \frac{3}{2}h^4$ to the normal form $h^6 + v^2$. So the $6D$ Euler number contribution will be $+6 - 1 = 5$ (taking into account that one has to go back to the singular model) at each of the $(2 - n)$ intersection points of the h component of the intersection of D_1 with the B_1 line.

4   The four-dimensional case

In this section we start after the foregoing introductory sections with the Euler number computation in the four-dimensional case. Here we give the smooth case and in the general case the relation to the heterotic situation. In general we will have to consider two types of contributions: singular fibers (corresponding in codimension one to G over B_2 and I_1 over D_1, this is the generic situation in the discriminant surface inside B_3; this is enhanced at the matter curves in B_2 and at the cusp curve C of D_1 in codimension two, and finally further enhanced at the intersection of the matter curves and the intersection of the cusp curve (i.e., the curve of *cuspidal type II fibers above*) with B_2 in codimension three) on the one hand and 'intrinsic' contributions to $e(D_1)$ from its various singularities (the curve of intrinsic cusp *singularities of D_1*, which will always be present, and a curve (actually one of the matter curves, the h curve) of tacnode resp. higher double point singularities in codimension one inside D_1 and various complicated point singularities at the points mentioned above as well as at further points detected by an analysis of the discriminant equation).

In *section 5* we will derive the contributions of the intrinsic singularity curves. In *section 6* we begin the discussion of the codimension loci. In *section 7* we present in a case by case discussion the various examples and further refine investigation of the singularity contributions.

4.1   Smooth case

Now if B is three-dimensional, the discriminant D is a surface in B_3 whose class is given by $D = 12c_1(B_3)$ resp. $G = 4c_1(B_3)$ and $F = 6c_1(B_3)$. In analogy to the Calabi-Yau threefold case we will compute $e(X)$ from $e(sing.fiber)e(D)$. For a smooth D we can obtain from the exact sequence $0 \to T_D \to T_{B_3|D} \to N_{D|B_3} \to 0$ the adjunction formulas (note that $N_{D|B_3} = \mathcal{O}(D)|_D$)

$$\begin{aligned} c_1(B_3)|_D &= c_1(D) + D|_D \\ c_2(B_3)|_D &= c_2(D) + c_1(D)D|_D \end{aligned} \qquad (4.1)$$

which leads to the Euler characteristic of a non-singular D

$$e(D) = c_2(B_3)D - c_1(B_3)D^2 + D^3. \qquad (4.2)$$

But D will be singular along a curve $C = FG$ and we expect a Plücker correction to $e(D)$. For $C = FG$ we can derive the Euler characteristic of C from the above exact sequence by restricting to C and, with the normal bundle of C in B_3 given by $N_C|_{B_3} = (\mathcal{O}(F) \oplus \mathcal{O}(G))|_C$, we get

$$
\begin{aligned}
e(C) &= c_1(B_3)FG - (F+G)FG \\
&= -216c_1^3(B_3) = -1296c_1^2 - 432t^2.
\end{aligned}
\tag{4.3}
$$

Using the *generalised Plücker formulas* derived in the next section we finally get the corrected Euler characteristic of D

$$
\begin{aligned}
e(D) &= c_2(B_3)D - c_1(B_3)D^2 + D^3 + 2(e(C) - DC) \\
&= 288 + 576c_1^3(B_3) = 288 + 3456c_1^2 + 1152t^2
\end{aligned}
\tag{4.4}
$$

and so

$$
\begin{aligned}
e(X) &= e(I_1)(e(D) - e(C)) \\
&\quad + e(II)e(C) \\
&= 288 + 360c_1(B_3)^3 = 288 + 2160c_1^2 + 720t^2.
\end{aligned}
\tag{4.5}
$$

4.2   Singular case

After reproducing the Euler characteristic for smooth X, we will consider the case of having a section of G-singularities located over a surface D_2 in B_3. Let us localize the G fibers along the zero section of the $\mathbf{P}^1$ bundle B_3 over B_2, whose class we denoted above by r. Following the procedure from above, we decompose the discriminant D into $D = D_1 + D_2$ where again D denotes the component with I_1 fibers and D_2 carrys G fibers. With $D_2 = cr$, $F_2 = ar$ resp. $G_2 = br$ and the canonical bundle of the base $K_{B_3} = -c_1 - 2r - t$, we get the classes $D_1 = 12c_1 + (24 - c)r + 12t$, $F_1 = 4c_1 + (8 - a)r + 4t$ and $G_1 = 6c_1 + (12 - b)r + 6t$ which describe our I_1 locus.

As we want to check our results on the Euler number of the F-theory four-fold via $n_3 = n_5$ with a corresponding heterotic computation let us now assume, as we want to use the computations of n_5 from the the spectral cover method for $SU(n)$ bundles [26], that heterotically an $SU(n) \times E_8$ bundle (V_1, V_2) is given and let us look for the first few non-trivial (the case "$n = 1$" of $G = E_8$ is treated below also; furthermore

some other cases of G, mainly in the I_n series, will be discussed; this requires in the case of I_2, I_3 the use of E_7, E_6 bundle V, whose second Chern class is computed in appendix (B) from parabolic methods) cases where the gauge group G is simple (let now $V := V_1$).

V $\quad$ G	a b c	matter curve(s)	fib_{enh} $\quad$ matter	het	het. loc.
$SU(2)$ E_7	3 5 9	$f_{1;4c_1-t}$	"E_8" $\quad$ $(\frac{1}{2})\mathbf{56}$	$H^1(Z,V)$	a_2
$SU(3)$ E_6	3 4 8	q_{3c_1-t}	E_7 $\quad$ $\mathbf{27}$	$H^1(Z,V)$	a_3
$SU(4)$ D_5	2 3 7	h_{2c_1-t}	E_6 $\quad$ $\mathbf{16}$	$H^1(Z,V)$	a_4
		q_{3c_1-t}	D_6 $\quad$ $\mathbf{10}$	$H^1(Z,\Lambda^2V)$	a_3
$SU(5)$ I_5	0 0 5	h_{c_1-t}	D_5 $\quad$ $\mathbf{10}$	$H^1(Z,V)$	a_5
		P_{8c_1-3t}	I_6 $\quad$ $\mathbf{5}$	$H^1(Z,\Lambda^2V)$	$R(a_i)$

with $R(a_i) := a_0 a_5^2 - a_2 a_3 a_5 + a_3^2 a_4$. Here the matter was read off from the Tate formalism [11], and then the enhancement pattern from [37]. Note that, as remarked already in [26], this matches precisely with the heterotic expectations.

$$\begin{aligned}\mathbf{248} &= (\mathbf{2},\mathbf{56}) \oplus (\mathbf{1},\mathbf{133}) \oplus (\mathbf{3},\mathbf{1}) \\ &= (\mathbf{3},\mathbf{27}) \oplus (\mathbf{\bar{3}},\mathbf{\overline{27}}) \oplus (\mathbf{1},\mathbf{78}) \oplus (\mathbf{8},\mathbf{1}) \\ &= (\mathbf{4},\mathbf{16}) \oplus (\mathbf{\bar{4}},\mathbf{\overline{16}}) \oplus (\mathbf{6},\mathbf{10}) \oplus (\mathbf{1},\mathbf{45}) \oplus (\mathbf{15},\mathbf{1}) \\ &= (\mathbf{5},\mathbf{10}) \oplus (\mathbf{\bar{5}},\mathbf{\overline{10}}) \oplus (\mathbf{10},\mathbf{\bar{5}}) \oplus (\mathbf{\overline{10}},\mathbf{5}) \oplus (\mathbf{1},\mathbf{24}) \oplus (\mathbf{24},\mathbf{1}).\end{aligned}$$

There $H^1(Z,V)$ was localized on the curve $a_n = 0$ (meaning $x = \infty$, the zero point in the group law; cf. sect. (B.1), note that a_i is of class $\eta - ic_1$) and $H^1(Z,\Lambda^2V)$ on the common zeroes of P and Q in the representation $w = P(x)+yQ(x) = 0$ of the spectral equation (meaning that y and $-y$ (the inverse bundle) are spectral points in E_b). For example for $G = A_4 = I_5$ one has $P = a_0 + a_2 x + a_4 x^2$, $Q = a_3 + a_5 x$.

So, after we will have computed $e(X^4)$ in the following sections, we will compare with the heterotic expectation derived in the introduction.

$$\begin{aligned}24n_5 = 288 &+ (1200 + 107n - 18n^2 + n^3)c_1^2 \\ &+ (1080 - 36n + 3n^2)c_1 t + (360 + 3n)t^2\end{aligned}$$

with $n = 0, 2, 3, 4, 5$ for $G = E_8, E_7, E_6, D_5, I_5$.

5 "Plücker formulas" for surfaces with curves of singularities

As we will have to compute a number of times the Euler number of a surface component of the discriminant surface in B_3 we give here the general computation. So let D be a surface in B_3 with a curve of singularities along the curve C. Our applications below will include a curve of cusps resp. tacnodes and higher double points. The cusp curve will always be present, while the tacnodes and higher double points occur for $G = A_3 = I_4$ resp. $G = I_5, I_6$. In the actual applications the singularity will be worse at special points on the curve, a possibility which we exclude here.

In *subsection one* we derive the contribution to the Euler number in the general case of a curve of singularities of multiplicity k, resolved by one blow-up. In *subsection two* we prove some main formulae used in this derivation and give an outlook on a closely related application of this technical set-up. In *subsection three* we specialise to the case of the *cusp curve*; here we find the contribution for the smooth case already used in section 4. In *subsection four* we proceed to the case of a *tacnode curve* which makes two successive blow-up's necessary. *Subsection five* treats the case of an even higher double point with three blow-up's.

5.1 The general case of a curve of singularities of multiplicity k, resolved by one blow-up

Let k denote the multiplicity of a surface D (which will be our surface D_1 in the applications) along C (being 2 in the two mentioned examples). We assume at first that the singularity is resolved after one blow-up. So one first blows up C in B_3, producing a three-fold $\pi : \tilde{B}_3 \to B_3$. Then one has for the total transform $\tilde{D}$

$$\tilde{D} = \bar{D} + kE \qquad (5.1)$$

with the proper transform $\bar{D}$ and the exceptional divisor E, a ruled surface over C. One has the relations (which are proved below)

$$c_1(\tilde{B}_3) \;=\; \pi^* c_1(B_3) - E$$

$$c_2(\tilde{B}_3) \;=\; \pi^*(c_2(B_3) + C) - \pi^* c_1(B_3) \cdot E. \tag{5.2}$$

Furthermore the fact that, after blowing up a (itself nonsingular) point on a surface, the self-intersection of the exceptional P^1 is -1 generalizes essentially to a relation $E^2 = -\pi^* C$ up to a correction term aF where F is the fibre of the ruled surface E over C (so $E \cdot F = -1$) and a a number determined by the exterior geometry of C in B_3 (again these relations are proved below)

$$
\begin{aligned}
E^2 &= -\pi^* C - E^3 \, F \\
E^3 &= -\int_C c_1(N_{B_3} C) = -(c_1(B_3)C - e(C)).
\end{aligned}
\tag{5.3}
$$

With these formulas one gets from the usual formula for a smooth surface $c_2(\bar{D}) = c_2(\tilde{B}_3)\bar{D} - c_1(\tilde{B}_3)\bar{D}^2 + \bar{D}^3$ that

$$c_2(\bar{D}) = c_2(B_3)D - c_1(B_3)D^2 + D^3 + \Delta_k \tag{5.4}$$

with the correction term (of course $\Delta_1 = 0$ and one has a factor $(k-1)$)

$$
\begin{aligned}
\Delta_k &= (k-1)[-(3k+1)CD + kc_1(B_3)C - k^2 E^3] \\
&= (k-1)[-(3k+1)CD + k(k+1)c_1(B_3)C - k^2 e(C)]. \tag{5.5}
\end{aligned}
$$

This is seen as follows

$$
\begin{aligned}
c_2\bar{D} &= c_2(\tilde{B}_3)\bar{D} - c_1(\tilde{B}_3)\bar{D}^2 + \bar{D}^3 \\
&= c_2(B_3)D + CD - kc_1(B_3)C \\
&\quad -c_1(B_3)D^2 + k^2 c_1(B_3)C + 2kCD + k^2 E^3 \\
&\quad +D^3 - 3k^2 DC - k^3 E^3. \tag{5.6}
\end{aligned}
$$

In particular for our case of interest $k = 2$ one has

$$\Delta_2 = -7DC + 2c_1(B_3)C - 4E^3. \tag{5.7}$$

5.2   The Chern classes of Blow-ups

Let us now prove the relations used above

$$
\begin{aligned}
(*) \quad c_2(\tilde{B}_3) &= f^*(c_2(B_3) + C) - f^* c_1(B_3) E \\
(**) \quad\quad E^2 &= -\pi^* C - E^3 F.
\end{aligned}
$$

Consider the following blow-up diagram for X a non-singular variety in Y

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\ j\ } & \tilde{Y} \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
X & \xrightarrow{\ i\ } & Y
\end{array}
$$

Further let N be the normal bundle to X in Y with $\mathrm{rank} N = d$, the codimension of X in Y, and identify $\tilde{X}$ with $P(N)$, so $N_{\tilde{X}}\tilde{Y} = \mathcal{O}(-1)$. Then from the above diagram one derives [29] the relation

$$
c_2(\tilde{Y}) - f^* c_2(Y) = -j_*((d-1)g^* c_1(X)
$$
$$
+ \frac{d(d-3)}{2}\mathcal{O}(1) + (d-2)g^* c_1(N)). \qquad (5.8)
$$

5.2.1   The case $d = 2$

For $d = 2$ one has

$$
\begin{aligned}
c_2(\tilde{Y}) - f^* c_2(Y) &= -j_* g^* c_1(X) - [\tilde{X}][\tilde{X}] \\
&= -j_* g^* c_1(X) + f^* i_*[X] - j_* g^* c_1(N_X Y) \\
&= f^* i_*[X] - j_* g^*(c_1(X) + c_1(N_X Y)) \\
&= f^* i_*[X] - j_* g^*(c_1(Y)|_X) \\
&= f^* i_*[X] - f^* c_1(Y)[\tilde{X}] \qquad\qquad (5.9)
\end{aligned}
$$

note in the second line we made use of

$$
\begin{aligned}
f^* i_*[X] &= j_* c_1(F) \\
&= j_* g^* c_1(N_X Y) - [\tilde{X}][\tilde{X}] \qquad\qquad (5.10)
\end{aligned}
$$

with $F = g^* N / N_{\tilde{X}}\tilde{Y}$ and in for the last line one has $j_* g^* i^* u = f^* u[\tilde{X}]$. So identifying $\tilde{Y} = \tilde{B}_3$ and $[\tilde{X}] = E$ resp. $i_*[X] = C$ we arrive at our expression ($*$).

For the second relation let us consider again

$$
\begin{aligned}
f^* i_*[X] &= j_* c_1(F) \\
&= j_* g^* c_1(N_X Y) - [\tilde{X}][\tilde{X}] \\
&= \deg N j_* g^*[pt] - [\tilde{X}][\tilde{X}] \\
&= \deg N\, l - [\tilde{X}][\tilde{X}] \qquad\qquad (5.11)
\end{aligned}
$$

where $l = g^*[pt]$ denotes the class of a fiber in $g : P(N) \to X$. Then from $[\tilde{X}]^2 - [\tilde{X}]g^*c_1(N))|_{[\tilde{X}]} = 0$ we get

$$
\begin{aligned}
[\tilde{X}]^3 &= [\tilde{X}]|_{[\tilde{X}]}g^*c_1(N) \\
&= [\tilde{X}]|_{[\tilde{X}]}\deg N g^*[pt] \\
&= \deg N\, l \cdot [\tilde{X}]|_{[\tilde{X}]} \\
&= -\deg N
\end{aligned}
\tag{5.12}
$$

so that $f^*i_*[X] = -[\tilde{X}]^3\, l - [\tilde{X}]^2$ which is what we were looking for in (**).

5.2.2 The case $d = 3$

Let us give an outlook on a further application of this technique. In connection with our main theme it is also of interest to compute $c_2(X^4)$. First this gives in principle an alternative way to compute the Euler number of X by making use of the relation $c_2^2(X^4) = 480 + e(X^4)/3$ (cf. [49]). Secondly $c_2(X)$ is of interest because of the congruence relation between the four-flux and $c_2(X)/2$ (cf. [53] and appendix (C)).

Consider now a Calabi-Yau 4fold Z embedded (via its Weierstrass representation) into an 5 dimensional ambient space Y, then it follows from adjunction (since Z is a smooth divisor in Y) that

$$
\begin{aligned}
c_1(Y)|_Z &= c_1(Z) + Z|_Z \\
c_2(Y)|_Z &= c_2(Z) + c_1(Z)Z|_Z
\end{aligned}
\tag{5.13}
$$

and thus

$$
c_2(Z) = c_2(Y)|_Z
\tag{5.14}
$$

further recall that $c(Y) = c(B)(1 + r)(1 + r + 2c_1)(1 + r + 3c_1)$ from which we get

$$
\begin{aligned}
c_1(Y) &= 6c_1 + 3r \\
c_2(Y) &= 11c_1^2 + c_2 + 13rc_1 + 3r^2
\end{aligned}
\tag{5.15}
$$

and setting $r^2 = -3rc_1$ (i.e., restricting to Z) then leads to the expression in the smooth case

$$
c_2(Z) = c_2(Y)Z = 11c_1^2 + c_2 + 4rc_1.
\tag{5.16}
$$

Now let us consider the simplest more complicated case, that of an singularity of codimension one which is A_1. In order to do so let us first analyse the change of c_2 of the ambient space. This is computed as follows

$$
\begin{aligned}
c_2(\tilde{Y}) - f^*c_2(Y) &= -j_*g^*(2c_1(X) + c_1(N_XY)) \\
&= -j_*g^*(c_1(X) + c_1(Y)|_X) \\
&= -j_*g^*c_1(X) - j_*g^*i^*c_1(Y) \\
&= -j_*g^*c_1(X) - f^*c_1(Y)[\tilde{X}]. \quad (5.17)
\end{aligned}
$$

Now we have to compute using $\bar{Z} = \tilde{Z} - 2[\tilde{X}]$

$$
\begin{aligned}
c_2(\bar{Z}) = c_2(\tilde{Y})|_{\bar{z}} &= f^*c_2(Y)\bar{Z} - f^*c_1(Y)[\tilde{X}]\bar{Z} - j_*g^*c_1(X)\bar{Z} \\
&= f^*c_2(Y)\tilde{Z} - 2f^*c_2(Y)[\tilde{X}] - f^*c_1(Y)[\tilde{X}]\tilde{Z} \\
&\quad + 2f^*c_1(Y)[\tilde{X}] - j_*g^*c_1(X)\bar{Z} \\
&= c_2(Y)Z - 2f^*c_1(Y)[\tilde{X}]^2 - j_*g^*c_1(X)\bar{Z} \\
&= c_2(Y)Z - 2c_1(Y)r - j_*g^*c_1(X)\bar{Z} \\
&= 11c_1^2 + c_2 + 4rc_1 + 2(6rc_1 - 9rc_1) - j_*g^*c_1(X)\bar{Z} \\
&= 11c_1^2 + c_2 - 2rc_1 - j_*g^*c_1(X)\bar{Z} \quad (5.18)
\end{aligned}
$$

showing the crucial deviation term $-j_*g^*c_1(X)\bar{Z}$ relative to the smooth case.

5.3   Cusp curve

So for example for the cusp curve case (where also $c_2(\bar{D}) = c_2(D)$ as, in contrast to the double point case, no points are identified in blowing down $\bar{D}$ back to its singular version D) one gets that

$$
\Delta_{cusp} = -7CD + 6c_1(B_3)C - 4c_1(C). \quad (5.19)
$$

Note also that for the cases where the cusp curve is given by the uncorrected F_1G_1 (so this includes the smooth case, the pure gauge group case of singularities only in codimension 1, where still D_1 and therefore C is separated from B_2, and furthermore the E series in general)

$$
- 7CD + 6c_1(B_3)C - 4c_1(C)
$$

$$
\begin{aligned}
&= -2CD + 2c_1(C) + 6c_1(B_3)C - 5CD - 6c_1(C) \\
&= -2CD + 2c_1(C) + 6c_1(NC) - 5CD \\
&= -2CD + 2c_1(C) + (6(F_1 + G_1) - 5D_1)C \\
&= -2CD + 2c_1(C) + ((6(F + G) - 5D) - (6(a + b) - 5c)r)C \\
&= -2CD + 2c_1(C) + dCr
\end{aligned}
\tag{5.20}
$$

(note that this D is D_1 in our application) where the term $d := 5c - 6(a + b)$ equals $-4, -3, -2$ for E_8, E_7, E_6 and of course zero for the smooth case. This shows explicitly the deviation $-2CD + dCr$ used above in the smooth case to the naive adiabatic extension $2c_1(C)$ of the one-dimensional Plücker formula.

5.4   Tacnode curve

Now we come to the more complicated case of the tacnode, where we need a second blow-up, as the first blow-up just brings one to the case of an ordinary double point (having distinct tangents as opposed to the tacnode). This second blow-up is along the well-defined (as the two tangent directions of the tacnode points of D along C coincide) proper transform $\bar{C}$ of C under the first blow-up. Note that $\bar{C} = E_{(1)}\bar{D} = E_{(1)}(\tilde{D} - 2E_{(1)})$. Note also that at the end of the procedure we have to go back to the singular model D and to get its Euler number we still have to subtract $e(C)$ as in the second resolution step the double points became separated, i.e., (with $c_2(D)^{ord} = c_2(B_3)D - c_1(B_3)D^2 + D^3$)

$$
\begin{aligned}
c_2(\bar{\bar{D}}) &= c_2(D)^{ord} + \Delta_{tacn} \\
c_2(D) &= c_2(D)^{ord} + \Delta_{tacn} - e(C).
\end{aligned}
\tag{5.21}
$$

In other words the corrections $\Delta_{cusp}, \Delta_{tacn}$ refer in our conventions to the desingularized model (just as in the ordinary Plücker formulas).

Here one gets (up to codimension 3 contributions)

$$
\Delta_{tacn} = -21CD + 26c_1(B_3)C - 20e(C).
\tag{5.22}
$$

To prove this let us follow the two steps of the resolution. Clearly in the second resolution step we are again back in the case of a curve of ordinary double points.

$$
c_2(\bar{\bar{D}}) = c_2(\tilde{\tilde{B}}_3)\bar{\bar{D}} - c_1(\tilde{\tilde{B}}_3)\bar{\bar{D}}^2 + \bar{\bar{D}}^3
$$

$$
\begin{aligned}
&= c_2(\tilde{B}_3)\bar{D} - c_1(\tilde{B}_3)\bar{D}^2 + \bar{D}^3 \\
&\quad -7\bar{D}\bar{C} + 2c_1(\tilde{B}_3)\bar{C} - 4E_{(2)}^3 \\
&= c_2(B_3)D - c_1(B_3)D^2 + D^3 - 7DC + 2c_1(B_3)C - 4E_{(1)}^3 \\
&\quad -7(\tilde{D} - 2E_{(1)})\bar{C} + 2(\pi^* c_1(B_3) - E_{(1)})\bar{C} - 4E_{(2)}^3 \quad (5.23)
\end{aligned}
$$

giving this time

$$
\begin{aligned}
\Delta_{tacn} &= -7DC + 2c_1(B_3)C - 4E_{(1)}^3 \\
&\quad -7\tilde{D}\bar{C} + 2\pi^* c_1(B_3)\bar{C} + 12 E_{(1)}\bar{C} - 4E_{(2)}^3. \quad (5.24)
\end{aligned}
$$

Now, concerning the four new terms in the second line, note that one has, concerning the first three of them, that

$$
\begin{aligned}
\tilde{D}\bar{C} &= \tilde{D}E_{(1)}(\tilde{D} - 2E_{(1)}) = 2DC \\
\pi^* c_1 B_3 \bar{C} &= \pi^* c_1 B_3 E(\tilde{D} - 2E_{(1)}) = 2c_1(B_3)C \\
E_{(1)}\bar{C} &= E_{(1)}^2(\tilde{D} - 2E_{(1)}) = -CD - 2E_{(1)}^3. \quad (5.25)
\end{aligned}
$$

On the other hand concerning the last new term $E_{(2)}^3$ one has again that $E_{(2)}^3 = -c_1(N_{\bar{C}|\tilde{B}_3})$, whereas $E_{(1)}^3 = -c_1(N_{C|B_3})$. Now, to express the former in terms of the latter, note that the short exact sequence

$$
0 \to N_{\bar{C}|E_{(1)}} \to N_{\bar{C}|\tilde{B}_3} \to N_{E_{(1)}|\tilde{B}_3} \to 0 \quad (5.26)
$$

gives

$$
c_1(N_{\bar{C}|\tilde{B}_3}) = c_1(N_{\bar{C}|E_{(1)}}) + c_1(N_{E_{(1)}|\tilde{B}_3}) \quad (5.27)
$$

where the first term on the right hand side is evaluated as $\bar{C}^2$ in $E_{(1)}$, i.e., as

$$
c_1(N_{\bar{C}|E_{(1)}}) = \bar{D}^2 E_{(1)} = (\tilde{D} - 2E_{(1)})^2 E_{(1)} = 4DC + 4E_{(1)}^3. \quad (5.28)
$$

Similarly the second term is $c_1(T) = E_{(1)}|_{E_{(1)}}$ of the tautological bundle T over $E_{(1)}$, restricted to $\bar{C} = \bar{D}|_{E_{(1)}}$, i.e.,

$$
c_1(N_{E_{(1)}|B_3}) = \bar{D}E_{(1)}^2 = (\tilde{D} - 2E_{(1)})E_{(1)}^2 = -DC - 2E_{(1)}^3. \quad (5.29)
$$

So that one gets

$$
E_{(2)}^3 = -c_1(N_{\bar{C}|\tilde{B}_3})
$$

$$
\begin{aligned}
&= -\left(c_1(N_{\tilde{C}|E_{(1)}}) + c_1(N_{E_{(1)}|\tilde{B}_3})\right) \\
&= -\left(4DC + 4E_{(1)}^3 - DC - 2E_{(1)}^3\right) \\
&= -3DC - 2E_{(1)}^3.
\end{aligned}
\tag{5.30}
$$

So finally

$$
\begin{aligned}
\Delta_{tacn} &= -7DC + 2c_1(B_3)C - 4E_{(1)}^3 \\
&\quad -14DC + 4c_1(B_3)C - 12CD - 24E^3 - 4E_{(2)}^3 \\
&= -33CD + 6c_1(B_3)C - 28E_{(1)}^3 - 4(-3DC - 2E_{(1)}^3) \\
&= -21CD + 6c_1(B_3)C - 20E_{(1)}^3 \\
&= -21CD + 26c_1(B_3)C - 20e(C).
\end{aligned}
\tag{5.31}
$$

5.5   Curve of higher double points

If a third blow-up is necessary like for the case of a curve of singularities of type $t^6 + v^2$ one gets

$$
\begin{aligned}
c_2(\bar{\bar{\bar{D}}}) &= c_2(\tilde{B}_3)\bar{D} - c_1(\tilde{B}_3)\bar{D}^2 + \bar{D}^3 \\
&\quad -21\bar{D}\bar{C} + 6c_1(\tilde{B}_3)\bar{C} - 20E_1^3 \\
&= c_2(B_3)D - c_1(B_3)D^2 + D^3 - 7DC + 2c_1(B_3)C - 4E_1^3 \\
&\quad -21(\tilde{D} - 2E_1) + 6\pi^*c_1(B_3)\bar{C} - 6E_1\bar{C} - 20E_1^3 \\
&= c_2(B_3)D - c_1(B_3)D^2 + D^3 \\
&\quad -25DC + 12c_1(B_3)C - 32E_1^3.
\end{aligned}
\tag{5.32}
$$

In the concrete application in the I series discriminant one has along the h curve an equation for D_1 of the form $(x^2 - z)^2 + z^3$ where we have written x for h. One finds in the explicit resolution process further contributions at codimension three loci inside the h curve which we will not need to write down.

6   On the codimension 3 loci

Consider the cases $D_5 = I_1^*$ and I_5. There are two new features compared to the E_k series: first that without further tuning the I_n^* and the

I_n series would remain at $n = 0$ (cf. sect. (3.3)), and secondly the existence of *two* matter curves.

What we want to see in the following is that actually the cohomology classes of the two codimension 3 loci, i.e., of Cr and the intersection of the matter curves h and P, are proportional; more precisely that Cr is a multiple of hP.

Now one has

$$(4f_1^3 + 27g_1^2 = 0) = D_{old} = D_1 + nr \tag{6.1}$$

where n is the subscript in the I_n^* and the I_n series, i.e., the number of powers of z one can extract from the left hand side.

Furthermore one has the decomposition of D_1r into the matter (=enhancement) curves (P means here our q in the I_1^* case)

$$D_1 r = \pi h + \rho P \tag{6.2}$$

where $D_1 r$ is also given by

$$D_{old} r = D_1 r - nt. \tag{6.3}$$

One has a corresponding decomposition

$$(f_1 = 0 = g_1) = C_{old} = C + xhr \tag{6.4}$$

where $x = ord_h res(f_1, g_1)$, so that one also gets (with $\alpha = ord_h f_1, \beta = ord_h g_1$)

$$Cr = \alpha h \cdot \beta h + xht. \tag{6.5}$$

6.1   The I_n series

For the cases $n = 4, 5, 6$ which show the general I series pattern one finds the following. There is $f_1 r = 4h, g_1 r = 6h$ and one finds

$$x = 3n \tag{6.6}$$
$$Cr = 3h(8h + nt) = 3hP_{8c_1-(8-n)t}. \tag{6.7}$$

Let us now understand why Cr is indeed a multiple of hP, considered as cohomology classes, i.e., why the used cohomological relation $8h + nt =$

P is not accidental. For I_0 one has that $D_{old}r = 12c_1 - 12t = 12h_{c_1-t}$. For I_5 one has $D_{old}r = D_1r - 5t = 4h_{c_1-t} + P_{8c_1-3t} - 5t$ and therefore $12h = 4h + P - 5t$ or $8h + 5t = P$. Similarly for I_n.

For the case $n = 2$ one has $f_1r = 2H$, $g_1r = 3H$ and with $x = 3$ one finds $Cr = 2H \cdot 3H + 3Ht = 3H(2H + t) = 3H(4c_1 - 3t) = \frac{3}{2}HP$. For $n = 3$ with $f_1r = 4h$, $g_1r = 6h$ and $x = 8$ one finds $Cr = 8h(3h + t) = 8h(3c_1 - 2t) = \frac{8}{3}h(9c_1 - 6t)$.

6.2 The I_n^* series

The case $D_4 = I_0^*$ is somewhat exceptional as here one has $f_1r = 2h_{2c_1-t}$, $g_1r = 3P_{2c_1-t}$ with different polynomials of the same degree (whereas for $n > 0$ one has $g_1r = 3h$ and P will represent a different cohomology class giving the other matter curve) and a $\mathbf{Z_3}$ symmetry related to D_4 triality. This manifests itself in the discriminant as follows

$$D_1r = 12c_1 - 6t = A^{(0)}_{4c_1-2t} + A^{(1)}_{4c_1-2t} + A^{(2)}_{4c_1-2t} \tag{6.8}$$

where $A^{(i)}_{4c_1-2t} = (h^2_{2c_1-t} + \omega^i P^2_{2c_1-t} = 0)$ with $i = 0, 1, 2$ and $\omega^3 = 1$. Then one has with $x = 0$

$$Cr = 2h \cdot 3P. \tag{6.9}$$

Now note that the locus of simultaneous vanishing of h and P is also the locus of intersection of the $A^{(i)}$.

For I_n^* with $n > 0$ is $f_1r = 2h_{2c_1-t}$, $g_1r = 3h_{2c_1-t}$. For $D_5 = I_1^*$ one has

$$D_1r = 12c_1 - 5t = 3h_{2c_1-t} + 2q_{3c_1-t} \tag{6.10}$$

and with $x = 2$ one finds

$$Cr = 2h \cdot 3h + 2ht = 2h(3h + t) = 2h \cdot 2q \tag{6.11}$$

where $2h$ occurs in Cr actually on the level of divisors.

For $D_6 = II_2^*$ one has (for the parameter $r = 0, 1, 2, 3, 4$ being 0 (cf. the discussion of D_6 in the six-dimensional case and [11])) that

$$D_1r = (12c_1 + 16r + 12t)r = 12c_1 - 4t = 2h_{2c_1-t} + 2P_{4c_1-t} \tag{6.12}$$

and with $x = 3$ one has

$$Cr = 2h \cdot 3h + 3ht = 3h(2h + t) = 3hP_{4c_1 - t}. \qquad (6.13)$$

So one has that for the I_n^* cases with $n = 0, 1, 2$

$$x = 6\frac{n}{2+n} \qquad (6.14)$$

$$Cr = h(6h + xt) = \frac{12}{2+n}h \cdot ((2+n)c_1 - t)$$

$$= \frac{12}{2+n}hP_{(2+n)c_1 - t}. \qquad (6.15)$$

Let us again see why Cr arises as a multiple of hP. For I_0^* one has $D_{old}r = 12c_1 - 6t = 6(2c_1 - t)$. For I_1^* one has $D_{old}r = D_1r - t = 3h + 2q - t$ and therefore $6h = 3h + 2q - t$ or $3h + t = 2q$ as we wanted to prove. Similarly for I_2^* one has $D_{old}r = D_1r - 2t = 2h_{2c_1 - t} + 2P_{4c_1 - t} - 2t$ and so $6h = 2h + 2P - 2t$ or $4h + 2t = 2P$ resp. $2h + t = P$.

Let us finally, for example in the case of D_5, come to the question whether actually the two *sets* $C \cap r$ and $h \cap q$ coincide. From equ. (3.20) it follows that $48F_1h + 864G_1 = z(q^2 - \frac{1}{2}f_1h) + \mathcal{O}(z^2)$ so if we approach $C \cap r$ coming *from the outside* of $B_2 = r = (z = 0)$ we find $0 = q^2 - \frac{1}{2}f_1h + \mathcal{O}(z)$ which goes in the limit $z \to 0$ to the condition $0 = q^2 - \frac{1}{2}f_1h$ resp. $0 = q^2$ for $C \cap r$ as it will lie in the divisor h anyway.

7 The explicit computation of $e(X^4)$

Now, finally, we come to our main computation announced in the introduction. This Euler number computation for the various cases is treated in *subsection (7.2)*. In *subsection (7.1)* we make contact with a formula given in [39] (for the case of singularities in codimension one only) which was guessed there from a list of values based on a computer analysis in a toric framework. The case of pure codimension one is also of interest because in this case the expression $\pi_*(\gamma^2) = -\lambda^2 N\eta(\eta - Nc_1)$ for an $SU(N)$ bundle will vanish as $\eta - Nc_1 = (6 - N)c_1 - t = 0$ for[13] $G = E_8, E_7, E_6, A_4$ and $t = (6 - N)c_1$ with $N = 0, 2, 3, 5$. Finally in

---

[13]The case $G = A_4$ is only a 'pseudo-separation case' between B_2 and D_1 as only one (h) of the two matter curves is turned off cohomologically, but note that over the other matter curve P the enhancement is additive, leading from I_5 to I_6, so

subsection (7.3) we note an observation relating Euler number values in neighbouring cases of certain Higgs chains.

7.1   Euler number formula for codimension one

A byproduct of our analysis is the proof of an Euler number formula for elliptic Calabi-Yau fourfolds for which the elliptic fiber degenerates over the generic codimension one locus B_2 in the Calabi-Yau base B_3. This formula was first written down in [39] based on a toric computer analysis. The formula suitably rewritten reads

$$e(X_4) = 288 + 360 \int_{B_3} c_1^3(B_3) - r(G)c(G)(c(G)+1) \int_{B_2} c_1^2(B_2) \quad (7.1)$$

where $r(G)$ and $c(G)$ are the rank resp. Coxeter number of the gauge group G. Now (with $B_3 = F_{k,m,n}$ the generalized Hirzebruch surface of base $B_2 = F_k$; below we consider yet another example) using the fact that $c_1^3(B_3) = 6c_1^2 + 2t^2$ and that we can express t in data of $F_{k,m,n}$ so that $t = m[b] + n[f]$ with $t^2 = 2mn - m^2k$ where $[b]^2 = -k$ and also noting that $t^2 = 2n^2 = \frac{n^2}{4}c_1^2$ from implementing the codimension one condition $m = n$ and $k = 0$, we can rewrite the above formula as

$$e(X_4) = 288 + (180(12 + n^2) - r(G)c(G)(c(G)+1))c_1^2(B'). \quad (7.2)$$

The case of purely codimension one (fiber) singularity (i.e., especially without matter curves; so this is a 'separation case' what concerns the relative position of the two discriminant components B_2 and D_1) is realizable for $G = E_8, E_7, E_6, D_4, A_2$ over $B_2 = F_0$ with $n = m = 12, 8, 6, 4, 3$. In [39] the authors were restricted to reflexive polyhedra and thus excluding the E_7 case. However, using naively the above formula leads to a prediction for E_7 which will be vindicated by our computation which therefore gives an independent check of this formula. Moreover, we also find agreement for the cases indicated in the following table (always assuming the (pseudo-)separation case)

---

that the Euler number computation is not effectively disturbed, cf. section (7.1); note that by contrast in the case $G = D_5$ the choice $N = 4$ and so $t = 2c_1$ turns off again the h_{2c_1-t} matter curve and again one has the additivity of the enhancement over the other matter curve q but this time there is an intrinsic codimension three locus left over (see below).

G	$e(X_4)$	G	$e(X_4)$
A_1	$288 + 2874c_1^2$		
A_2	$288 + 2856c_1^2$	D_4	$288 + 4872c_1^2$
A_3	$288 + 2820c_1^2$	E_6	$288 + 7704c_1^2$
A_4	$288 + 2760c_1^2$	E_7	$288 + 11286c_1^2$
A_5	$288 + 2670c_1^2$	E_8	$288 + 20640c_1^2$

Let us make some remarks:

A series

The codimension one condition is established by setting $t = c_1$ or equivalently $n = 2$ (*pseudo*-separation, cf. the last footnote). Note that in [39] the A_2 singularity was specified by $n = 3$, i.e., $t = 3/2c_1$ and therefore one has $e(X_4) = 288 + 3756c_1^2$ which matches our computation too.

D series

In the D series we find only for $G = D_4$ a codimension one condition which is $t = 2c_1$ resp. $n = 4$.

E series

The codimension one condition is here established by setting $t = 3c_1$, $4c_1, 6c_1$ resp. $n = 6, 8, 12$ for E_6, E_7, E_8.

As a last point we remark that for the choice of $B_2 = P^2$ of table (6.3) of [39] we find also agreement with our formulae given below.

7.2 The cases

7.2.1 $E_8(II^*)$ singularity

Now, from our above analysis we see that $e(D_1)^{ord} = c_2(B_3)D_1 - c_1(B_3)D_1^2 + D_1^3$ receives two corrections (in later cases more from codimension three contributions). The first one is coming from the fact that D has a cusp curve $C = F_1G_1$ but we have also to take into account that D is "double" along $T = D_1 \cap r$ (of class $6c_1 - t$ related to $g_1(z = 0)$) and has to be resolved (note that this is a 'real' resolution in contrast to the case of the cusp curve where the resolution is only an intermediate computational step to get the contribution of the singular geometry).

Let us first compute the contributions from *fiber* singularities (for the general set-up of the computation involving fiber and intrinsic singularities cf. the introduction to section 4)

$$
\begin{aligned}
e(X) \;=\; & 1(e(D_1) - e(C)) \\
& +2e(C) \\
& +10e(B_2).
\end{aligned}
\tag{7.3}
$$

Then, the intrinsic singularities of D_1 are computed as (where $C \cap r$ is a special codimension three locus)

$$
e(D_1) = c_2(B_3)D_1 - c_1(B_3)D_1^2 + D_1^3 + \Delta_{\text{double}} + \Delta_{\text{cusp}} + \Delta_{C\cap r}
\tag{7.4}
$$

which can be derived as follows. One has

$$
\begin{aligned}
c_2(\bar{\bar{D}}_1) &= c_2(\tilde{\bar{B}}_3)\bar{\bar{D}}_1 - c_1(\tilde{\bar{B}}_3)\bar{\bar{D}}_1^{\,2} + \bar{\bar{D}}_1^{\,3} \\
&= c_2(\tilde{B}_3)\bar{D}_1 - c_1(\tilde{B}_3)\bar{D}_1^{\,2} + \bar{D}_1^{\,3} - 7\bar{D}_1\bar{C} + 2c_1\tilde{B}_3\bar{C} - 4E_{(2)}^3 \\
&= (\pi^*(c_2(B_3) + T) - \pi^*c_1(B_3)E_{(1)})(\tilde{D}_1 - 2E_{(1)}) \\
&\quad - (\pi^*c_1(B_3) - E_{(1)})(\tilde{D}_1 - 2E_{(1)})^2 \\
&\quad + (\tilde{D}_1 - 2E_{(1)})^3 \\
&\quad - 7(\tilde{D}_1 - 2E_{(1)})\bar{C} + 2(\pi^*c_1(B_3) - E_{(1)})\bar{C} - 4E_{(2)}^3 \\
&= c_2(B_3)D_1 - c_1(B_3)D_1^2 + D_1^3 - 7D_1T + 2c_1(B_3)T - 4E_{(1)}^3 \\
&\quad - 7\tilde{D}_1\bar{C} + 14E_{(1)}\bar{C} + 2\pi^*c_1(B_3)\bar{C} - 2E_{(1)}\bar{C} - 4E_{(2)}^3
\end{aligned}
\tag{7.5}
$$

using $\bar{C} = \tilde{C} - \#(C \cap r)\, F$ with F the fiber of $E_{(1)}$ we find

$$
\begin{aligned}
\tilde{D}_1\bar{C} &= \tilde{D}_1(\tilde{C} - \#(C \cap r)\, F) = \tilde{D}_1\tilde{C} = D_1C \\
E_{(1)}\bar{C} &= E_{(1)}(\tilde{C} - \#(C \cap r)\, F) = \#(C \cap r) \\
\pi^*c_1(B_3)\bar{C} &= \pi^*c_1(B_3)(\tilde{C} - \#(C \cap r)\, F) = c_1(B_3)C \\
E_{(2)}^3 &= -(c_1(\tilde{B}_3)C - e(C)) = -(c_1(B_3)C - e(C)) + \#(C \cap r)
\end{aligned}
\tag{7.6}
$$

which then leads to

$$
c_2(\bar{\bar{D}}_1) = c_2(B_3)D_1 - c_1(B_3)D_1^2 + D_1^3 + \Delta_{\text{double}} + \Delta_{cusp} + \Delta_{C\cap r}.
\tag{7.7}
$$

So altogether

$$
c_2(\tilde{\tilde{D}}_1) = 168 + 1702c_1^2 + 1760c_1t + 576t^2 + (12 - 4)\#(C \cap r)
\tag{7.8}
$$

and using

$$
\begin{aligned}
c_2(B_3)D_1 - c_1(B_3)D_1^2 + D_1^3 &= 168 + 5434c_1^2 + 4670c_1t + 1588t^2 \\
e(C) &= -684c_1^2 - 648c_1t - 216t^2 \\
10e(B_2) &= 120 - 10c_1^2 \\
\Delta_{\text{double}} &= -2(29c_1 - 2t)(6c_1 - t) \\
\Delta_{\text{cusp}} &= -3384c_1^2 - 2992c_1t - 1008t^2 \\
\Delta_{C\cap r} &= (12 - 4)\#(C \cap r) \\
\#(C \cap r) &= Cr = 4c_1(6c_1 - t)
\end{aligned} \tag{7.9}
$$

we find

$$
e(X) = 288 + 1008c_1^2 + 1112c_1t + 360t^2 + (12 - 4)Cr \tag{7.10}
$$

which leads to agreement when comparing with the heterotic side where one has

$$
24n_5 = 288 + 1200c_1^2 + 1080c_1t + 360t^2. \tag{7.11}
$$

Finally we remark that this formula also reproduces the computation for the case $B_2 = F_0$ and $t = 0[b] + 12[f]$ in table (6.4) of [39].

7.2.2 $E_7(III^*)$ singularity

Here we have again to take into account the subtlety concerning the fibre enhancement along the matter (=enhancement) curve T of class $4c_1 - t$ related to $f_1(z = 0)$ mentioned in the six-dimensional analysis.

$$
\begin{aligned}
e(X) = 1&((e(D_1) - e(C) - e(T) + \#(C \cap r)) \\
&+ 2(e(C) - Cr) \\
&+ 9(e(B_2) - e(T)) \\
&+ 9(e(T) - \#(C \cap r)) \\
&+ A\#(C \cap r) \\
= e(D_1) &+ e(C) + 9e(B_2) - e(T) + (A - 10)\#(C \cap r) \tag{7.12}
\end{aligned}
$$

with

$$
e(D_1) = c_1(B_3)D_1 - c_1(B_3)D_1^2 + D_1^3 + \Delta_{\text{cusp}}
$$

$$
\begin{aligned}
&= 180 + 2133c_1^2 + 1646c_1 t + 583t^2 \\
e(C) &= -762c_1^2 - 629c_1 t - 217t^2 \\
9e(B_2) &= +108 - 9c_1^2 \\
e(T) &= -(3c_1 - t)(4c_1 - t) \\
C \cap r &= (6c_1 - t)(4c_1 - t) \quad Cr = \#(C \cap r)
\end{aligned} \tag{7.13}
$$

so we find

$$
e(X) = 288 + 1134c_1^2 + 1110c_1 t + 357t^2 + A\#(C \cap r). \tag{7.14}
$$

Now comparing with the heterotic side where one has

$$
24n_5 = 288 + 1350c_1^2 + 1020c_1 t + 366t^2 \tag{7.15}
$$

one is lead from $\Delta_e = 24n_5 - e(X)$ where

$$
\Delta_e = (9 - A)(6c_1 - t)(4c_1 - t) = (9 - A)\#(C \cap r) \tag{7.16}
$$

to a prediction $A = 9$ for the Euler number A of the fiber configuration over the codimension three locus $C \cap r$.

7.2.3 $E_6(IV^*)$ singularity

Now one has with the matter (=enhancement curve) T of class $q_{3c_1 - t}$ related to $g_1(z = 0) = q^2$

$$
\begin{aligned}
e(X) &= 1((e(D_1) - e(C) - e(T) + \#(C \cap r)) \\
&\quad + 2(e(C) - \#(C \cap r)) \\
&\quad + 8(e(B_2) - e(T)) \\
&\quad + 9(e(T) - \#(C \cap r)) \\
&\quad + A\#(C \cap r) \\
&= e(D_1) + e(C) + 8e(B_2) + (A - 10)\#(C \cap r)
\end{aligned} \tag{7.17}
$$

with

$$
\begin{aligned}
e(D_1) &= c_1(B_3)D_1 - c_1(B_3)D_1^2 + D_1^3 + \Delta_{\text{cusp}} \\
&= 192 + 2300c_1^2 + 1552c_1 t + 596t^2 \\
e(C) &= -822c_1^2 - 602c_1 t - 220t^2 \\
8e(B_2) &= +96 - 8c_1^2
\end{aligned}
$$

$$e(T) = -(2c_1 - t)(3c_1 - t)$$
$$\#(C \cap r) = (3c_1 - t)(4c_1 - t), \quad Cr = 2\#(C \cap r) \quad (7.18)$$

(where in the last line one has to take into account that $G_1 r = 2q$ from the split condition $g_1(z = 0) = q^2$) so we find

$$e(X) = 288 + 1470c_1^2 + 950c_1 t + 376t^2 + A\#(C \cap r) \quad (7.19)$$

if we compare this with

$$24n_5 = 288 + 1386c_1^2 + 999c_1 t + 369t^2 \quad (7.20)$$

we find a prediction for A from the vanishing of

$$\Delta_e = (7 - A)(3c_1 - t)(4c_1 - t) = (7 - A)\#(C \cap r). \quad (7.21)$$

7.2.4 $D_4(I_0^*)$ singularity

Let us start with the relevant cohomology classes resp. divisors

$$
\begin{aligned}
F_1 &= 4c_1 + 6r + 4t \Rightarrow F_1 r = 2h \\
G_1 &= 6c_1 + 9r + 6t \Rightarrow G_1 r = 3P \\
D_1 &= 12c_1 + 18r + 12t \Rightarrow D_1 r = A^{(0)} + A^{(1)} + A^{(2)} \quad (7.22)
\end{aligned}
$$

where $h = 2c_1 - t$ and $P = 2c_1 - t$ and $A_{4c_1 - 2t}^{(i)} = \left(h_{2c_1-t}^2 + \omega^i P_{2c_1-t}^2 = 0\right)$ with $i = 0, 1, 2$ and $\omega^3 = 1$ and the last decomposition holds not only on the level of cohomology classes but *actually on the level of divisors* as seen from the equation $D_1 = (A^{(0)} A^{(1)} A^{(2)} + \mathcal{O}(z) = 0)$.

Now we have

$$F_1 G_1 = C_{old} = C_{new} \quad (7.23)$$

So

$$
\begin{aligned}
C &= 24(c_1 + t)(c_1 + 2t) + 6t(3r + 4t) \\
Cr &= 6hP = 6(2c_1 - t)^2 \quad (7.24)
\end{aligned}
$$

further we have

$$e(D_1) = c_2(B_3)D_1 - c_1(B_3)D_1^2 + D_1^3 + \Delta_{cusp}$$

$$= 216 + 7062c_1^2 + 3642c_1 t + 1764t^2 + \Delta_{cusp}. \quad (7.25)$$

One has $\Delta_{cusp} = -4512c_1^2 - 2292c_1 t - 1128t^2$ where we used $e(C) = (c_1 - F_1 - G_1)F_1 G_1 = -960c_1^2 - 498c_1 t - 240t^2$ and altogether $e(D_1) = 216 + 1584c_1^2 + 852c_1 t + 396t^2$ and so

$$\begin{aligned} e(X_4) &= e(D_1) + e(C) + 6(12 - c_1^2) \\ &= 288 + 1584c_1^2 + 852c_1 t + 396t^2. \quad (7.26) \end{aligned}$$

This is in agreement with the computation $e(X_4) = 39264$ of [39] for $B_3 = F_{0,4,4}$ which means $B_2 = F_0$ and so $c_1 = (2,2)$ and $t = (4,4)$. The other choice $t = (0,4)$ and $c_1 = (2,2)$ with $e(X_4) = 19680$ given there leads if one includes in the above formula an fiber enhancement k over the matter curves $e(X_4) = 19776 + 3 \cdot 16(k - 6)$ to a prediction $k = 4$, i.e., an effect similar to the cases of E_7 and D_6 where one did not have the naive additivity of the collision rules.

7.2.5 $D_5(I_1^*)$ singularity

Now one has

$$\begin{aligned} F_1 &= 4c_1 + 6r + 4t \Rightarrow F_1 r = 2h \\ G_1 &= 6c_1 + 9r + 6t \Rightarrow G_1 r = 3h \\ D_1 &= 12c_1 + 17r + 12t \Rightarrow D_1 r = 3h + 2q \quad (7.27) \end{aligned}$$

where $h = 2c_1 - t$ and $q = 3c_1 - t$ and the last decomposition holds not only on the level of cohomology classes but *actually on the level of divisors* as seen from the equation $D_1 = (h^3 q^2 + \mathcal{O}(z) = 0)$.

Now just as

$$(4f_1^3 + 27g_1^2 = 0) = D_1 + r \quad (7.28)$$

we will have a decomposition, again *actually on the level of divisors*,

$$F_1 G_1 = C_{old} = C_{new} + 2hr \quad (7.29)$$

where C_{new} (which we denote simply by C in the following) is the true cusp curve of D_1 we are interested in.

So

$$C_{old} = 6(4(c_1 + t)^2 + 3r(4c_1 + t)) = 24(c_1 + t)^2 + 18r(4c_1 + t)$$

$$C = f_1 g_1 - 2(2c_1 - t)r = 24(c_1 + t)^2 + (68c_1 + 20t)r$$
$$Cr = 2h \cdot 3h + 2ht = 2h(3h + t) = 2h \cdot 2q = 4(2c_1 - t)(3c_1 - t)$$
$$\#(C \cap r) = \#(h \cap q) = hq \qquad (7.30)$$

where in the third line both factors occur not only as cohomology classes but even as divisors: for the part $2h$ this follows from the construction of C and for the part $2q$ we saw this in section 6. Note also that not only $Ch \subset Cr$ but that they are actually equal as sets. The precise multiplicity is given by $C2h = Cr$ as the part $2h$ occurs in the Cr not only as cohomology class but even as divisor. So $Ch = 2pq$.

We proceed now in two steps: first we compute the intrinsic singularity corrections in $e(D_1)$, then we collect the fiber enhancements.

Now $e(D_1)$ is given by (up to contributions from point singularities of D_1 considered below)

$$e(D_1)^{ord} + \Delta_{cusp} = c_2(B_3)D_1 - c_1(B_3)D_1^2 + D_1^3 + \Delta_{cusp}$$
$$= 204 + 6655c_1^2 + 4004c_1 t + 1684t^2 + \Delta_{cusp}. \qquad (7.31)$$

One has $\Delta_{cusp} = -7CD_1 + 6c_1(B_3)C - 4e(C)$ where the first two terms are $-7CD_1 + 6c_1(B_3)C = -7872c_1^2 - 4724c_1 t - 1988t^2$. Now $F_1 G_1 = C_{old} = 2hr + C$ so

$$2\chi_{ar}(C_{old}) = 2\chi_{ar}(2h) + e(C) - 2\chi_{ar}(2hC) \qquad (7.32)$$

Now $2\chi_{ar}(C_{old}) = (c_1 + 2r + t)C_{old} - (F_1 + G_1)C_{old} = -(960c_1^2 + 498c_1 t + 240t^2)$ giving with $2\chi_{ar}(2h) = -2h(2h - c_1) = -(12c_1^2 - 14c_1 t + 4t^2)$ that

$$\begin{aligned} e(C) &= 2\chi_{ar}(C_{old}) - 2\chi_{ar}(2h) + 2(C2h) \\ &= 2\chi_{ar}(C_{old}) - 2\chi_{ar}(2h) + 8hq \\ &= -(900c_1^2 + 552c_1 t + 228t^2) \end{aligned} \qquad (7.33)$$

and so

$$\Delta_{cusp} = -4272c_1^2 - 2516c_1 t - 1076t^2 \qquad (7.34)$$

and altogether

$$e(D_1)^{ord} + \Delta_{cusp} = 204 + 2383c_1^2 + 1488c_1 t + 608t^2. \qquad (7.35)$$

Let us now come to the discussion of the codim 3 contributions. D_1 is given by $h_{2c_1-t}^3 q_{3c_1-t}^2 + S_{12c_1-4t}z + \mathcal{O}(z^2)$ where

$$S_{12c_1-4t} = h^2(-\frac{3}{4}f_1^2 + 2g_1 h + 3f_0 h^2) - 3f_1 h q^2 + q^4. \qquad (7.36)$$

Let us now investigate what happens if, when we are lying on one of the two matter curves h or q, in addition S vanishes, so that the leading terms for the equation of D_1 becomes there $h^3 + Sz + z^2$ resp. $q^2 + Sz + z^2$ and we can expect a singularity at $h \cap S$ resp. $q \cap S$.

Now, on closer inspection, one notices that $h \cap q \subset S$ and $h \cap S \subset q$; therefore $h \cap S = h \cap q$ and $h \cap S \subset q \cap S$. On the other hand $q \cap S$ implies only $h^2(-\frac{3}{4}f_1^2 + 2g_1 h + 3f_0 h^2) = 0$, i.e., we do not necessarily come to lie on h, there is still another divisor R_{8c_1-2t} with $qS = q(2h + R)$ relevant. So one has a disjoint decomposition $q \cap S = (q \cap h) \cup (q \cap R)$ and we will actually consider the loci $h \cap q = h \cap S$ and $q \cap R = (q \cap S) \backslash (h \cap q)$ where the local forms of the singularities of D_1 are respectively

$$h_x \cap q_y \rightarrow x^3 y^2 + (x^2 + xy^2 + y^4)z + z^2$$
$$q_y \cap R_x \rightarrow y^2 + z(x + y^2 + y^4) + z^2 \sim y^2 + zx + z^2 \sim y^2 + x^2 + z^2.$$
$$(7.37)$$

So at $\# h \cap q = hq$ resp. $\#((q \cap S) \backslash (h \cap q)) = \# q \cap R = qR = q(S - 2h) = 4q^2 - 2hq$ points one has an singularity of weighted homogeneous standard form $x^4 + y^8 + z^2$ (as the defining equation is of weights $(2, 1, 4)$ in (x, y, z)) resp. an $\mathbf{A_1}$ singularity which lead to corrections α resp. β to the Euler number of the *singular* surface D_1 or in general to a correction $-\mu$ where μ is the colength (which is finite as the singularity is isolated) of the Jacobian ideal (being also the Euler number of the Milnor fibre minus 1). So $\alpha = 1 \cdot 3 \cdot 7 = -21$ and $\beta = -1$.

Now let us come to the fiber enhancements. Now one has $C \cap r = h \cap q$ and

$$\begin{aligned} \#(C \cap r) &= hq \\ e(C \cup h \cup q) &= e(C) + e(h) + e(q) - hq \end{aligned} \qquad (7.38)$$

giving (l and k parametrize the fiber enhancements at the codimension three loci)

$$e(X) = 1(e(D_1) - e(C) - e(h) - e(q) + hq)$$

$$+2(e(C) - hq)$$
$$+7(e(B_2) - e(h) - e(q) + hq)$$
$$+8(e(h) - hq + e(q) - hq - Rq)$$
$$+lhq + kRq$$
$$= e(D_1) + e(C) + 7e(B_2)$$
$$+(l - 10)hq + (k - 8)Rq$$
$$= e(D_1) + e(C) + 7e(B_2)$$
$$+(l - 2k + 6)hq + (k - 8)4q^2. \qquad (7.39)$$

With

$$
\begin{aligned}
e(D_1) &= e(D_1)^{ord} + \Delta_{cusp} - 2hq + \alpha hq + \beta Rq \\
&= e(D_1)^{ord} + \Delta_{cusp} + (\alpha - 2 - 2\beta)hq + \beta 4q^2 \\
&= e(D_1)^{ord} + \Delta_{cusp} - 21hq - 4q^2 \qquad (7.40)
\end{aligned}
$$

one gets

$$
\begin{aligned}
e(X) &= e(D_1)^{ord} + \Delta_{cusp} + e(C) + 7e(B_2) \\
&\quad +(l - 31)hq + (k - 9)4q^2 \\
&= 288 + 1476c_1^2 + 936c_1 t + 380t^2 \\
&\quad +(l - 31)hq + (k - 9)4q^2 \\
&= 288 + 1476c_1^2 + 936c_1 t + 380t^2 - 8q^2 \\
&\quad +(l - 31)hq + (k - 9 + 2)4q^2
\end{aligned}
$$

$$(7.41)$$

where in the last equation the first line matches with the heterotic expectation

$$24n_5 = 288 + 1404c_1^2 + 984c_1 t + 372t^2 \qquad (7.42)$$

giving corresponding predictions[14] from the vanishing of the other terms.

7.2.6 $D_6(I_2^*)$ singularity

This time one has

$$F_1 = 4c_1 + 6r + 4t \Rightarrow F_1 r = 2h$$

---

[14]Note that the form of deviation of the F-theory result from the heterotic result is already a non-trivial check since we have to tune only two coefficients to match a quadratic expression in c_1 and t with three coefficients.

$$G_1 = 6c_1 + 9r + 6t \Rightarrow G_1r = 3h$$
$$D_1 = 12c_1 + 16r + 12t \Rightarrow D_1r = 12c_1 - 4t = 2h + 2P_{4c_1-t} \quad (7.43)$$

where $h = 2c_1 - t$ and $P = 4c_1 - t$ and the last decomposition holds not only on the level of cohomology classes but *actually on the level of divisors* as seen from the equation $D_1 = (h^2P^2 + \mathcal{O}(z) = 0)$.

$$
\begin{aligned}
C_{old} &= 24(c_1 + t)^2 + 18r(4c_1 + t) \\
C &= f_1g_1 - 3(2c_1 - t)r = 24(c_1 + t)^2 + (66c_1 + 21t)r \\
Cr &= 3(2c_1 - t)(4c_1 - t) = 3hP \\
\#(C \cap r) &= hP. \quad (7.44)
\end{aligned}
$$

The cohomology classes of the two codimension 3 terms Cr and hP are again proportional. Note also that not only $Ch \subset Cr$ but that they are actually equal as sets. The precise multiplicity is given by $C3h = Cr$ as the part $3h$ occurs in the Cr not only as cohomology class but even as divisor.

Now $e(D_1)$ is besides contributions from point singularities (considered below) computed as

$$
\begin{aligned}
e(D_1)^{ord} + \Delta_{cusp} &= c_2(B_3)D_1 - c_1(B_3)D_1^2 + D_1^3 + \Delta_{cusp} \\
&= 192 + 6248c_1^2 + 4296c_1t + 1632t^2 + \Delta_{cusp}. \quad (7.45)
\end{aligned}
$$

Now $F_1G_1 = C_{old} = 3hr + C$ so $2\chi_{ar}(C_{old}) = 2\chi_{ar}(3h) + e(C) - 2(C3h)$ and

$$2\chi_{ar}(C_{old}) = -(960c_1^2 + 498c_1t + 240t^2) \quad (7.46)$$

giving with $2\chi_{ar}(3h) = -3h(3h - c_1) = -30c_1^2 + 33c_1t - 9t^2$

$$e(C) = -882c_1^2 - 567c_1t - 225t^2 \quad (7.47)$$

and

$$
\begin{aligned}
\Delta_{cusp} &= -7CD_1 + 6c_1(B_3)C - 4e(C) \\
&= -4020c_1^2 - 2718c_1t - 1038t^2 \quad (7.48)
\end{aligned}
$$

and altogether

$$e(D_1^{ord}) + \Delta_{cusp} = 192 + 2228c_1^2 + 1578c_1t + 594t^2. \quad (7.49)$$

Let us now come to point singularities of the discriminant of equation $0 = h^2 P^2 + \frac{1}{4} S_{12c_1 - 3t} z + \mathcal{O}(z^2)$ where

$$S = h^3 R - 3h^2 u^2 P + 3hu P^2 + 4P^3 \tag{7.50}$$

where $R = -2u^3 + 8g_0$. Now a similar inspection as for D_5 shows that one has a disjoint decomposition $P \cap S = (P \cap h) \cup (P \cap R)$ and we will actually consider the loci $h \cap P = h \cap S$ and $P \cap R = (P \cap S) \backslash (h \cap P)$ where the local forms of the singularities of D_1 are respectively (we neglect here some coefficients)

$$
\begin{aligned}
h_x \cap q_y \;\; &\rightarrow \;\; -\frac{3}{4} x^2 y^2 + \frac{1}{4}(x^3 R - 3u^2 x^2 y + 3uxy^2 + 4y^3)z + z^2 \\
P_y \cap R_x \;\; &\rightarrow \;\; y^2 + (x + y + y^2 + y^3)z + z^2. \tag{7.51}
\end{aligned}
$$

So at $\# h \cap P = hP$ resp. $\# P \cap R = PR = P(S - 3h) = 3P^2 - 3hP$ points one has singularities which lead to corrections α resp. β to the Euler number of the *singular* surface D_1.

So altogether one finds

$$
\begin{aligned}
e(D_1) &= e(D_1^{ord} + \Delta_{cusp}) + \alpha hP + \beta(3P^2 - 3hP) \\
&= 192 + 2228 c_1^2 + 1578 c_1 t + 594 t^2 + (\alpha - 3\beta)hP + 3\beta P^2.
\end{aligned} \tag{7.52}
$$

Now for $SO(12)$ one has the matter/enhancement scheme: $P - - >$ $\mathbf{12} - - > D_7$ of Euler $9 = 8 + 1$ and $h - - > \mathbf{32} - - > "E_7"$ of Euler 8 (as in the $G = E_7$ case here occurred a collision which is not effectively additive; cf. the six-dimensional discussion).

So now one has for the contributions from the fiber enhancement

$$
\begin{aligned}
e(X) \;\; = \;\; & 1(e(D_1) - e(C) - e(h) - e(P) + hP + \#(C \cap r)) \\
& + 2(e(C) - \#(C \cap r)) \\
& + 8(e(B_2) - e(h) - e(P) + hP) \\
& + 8(e(h) - hP - \#(C \cap r)) + 9(e(P) - hP) \\
& + l\#(C \cap r) + mhP \\
= \;\; & e(D_1) + e(C) + 8e(B_2) - e(h) \\
& + (l - 9)\#(C \cap r) + (m - 8)hP \\
= \;\; & e(D_1) + e(C) + 8e(B_2) - e(h)
\end{aligned}
$$

$$+(l + m - 17)hP \qquad (7.53)$$

giving altogether

$$
\begin{aligned}
e(X) &= 288 + 1340c_1^2 + 1008c_1 t + 370t^2 \\
&\quad + (l + m - 17 + \alpha - 3\beta)hP + 3\beta P^2 \\
&= 288 + 1260c_1^2 + 1044c_1 t + 366t^2 \\
&\quad + (-2 + (l + m - 17 + \alpha - 3\beta))hP + (6 + 3\beta)P^2. \qquad (7.54)
\end{aligned}
$$

Comparing this with the heterotic value

$$24n_5 = 288 + 1260c_1^2 + 1044c_1 t + 366t^2 \qquad (7.55)$$

gives the prediction[15] $\beta = -2$ and $l + m = 13 - \alpha$.

7.2.7 $A_1(I_2)$ singularity

$$
\begin{aligned}
F_1 &= 4c_1 + 8r + 4t \Rightarrow F_1 r = 2H \\
G_1 &= 6c_1 + 12r + 6t \Rightarrow G_1 r = 3H \\
D_1 &= 12c_1 + 22r + 12t \Rightarrow D_1 r = 2H + P \qquad (7.56)
\end{aligned}
$$

where $H = 2c_1 - 2t$ and $P = 8c_1 - 6t$ and the last decomposition holds not only on the level of cohomology classes but *actually on the level of divisors* as seen from the equation $D_1 = (H^2 P + \mathcal{O}(z) = 0)$.

Now $F_1 G_1 = C_{old} = 3Hr + C$ (on the level of divisors) so

$$
\begin{aligned}
C_{old} &= 24(c_1 + t)^2 + 96c_1 r \\
C &= f_1 g_1 - 6(c_1 - t)r = 24(c_1 + t)^2 + 90c_1 r + 6rt \\
Cr &= \frac{3}{2}HP \\
\#(C \cap r) &= H \cdot \frac{1}{2}P = (2c_1 - 2t)(4c_1 - 3t) \qquad (7.57)
\end{aligned}
$$

where in the last line we used the fact that the term $3H$ occurs in Cr as divisor and not just as a cohomology class.

With

$$\Delta_{cusp} = -5748c_1^2 - 588c_1 t - 1728t^2$$

---

[15] For the siginificance of this procedure compare the last footnote.

$$e(C) = -1170c_1^2 - 234c_1t - 324t^2 \qquad (7.58)$$

$$e(D_1) = c_2(B_3)D_1 - c_1(B_3)D_1^2 + D_1^3 + \Delta_{cusp} \qquad (7.59)$$
$$= 264 + 2942c_1^2 + 906c_1t + 756t^2 \qquad (7.60)$$

one finds

$$
\begin{aligned}
e(X_4) &= 1(e(D_1) - [e(C) - \#(C \cap r) + e(H) - HP + e(P)]) \\
&\quad + 2(e(C) - \#(C \cap r)) \\
&\quad + 2(e(B_2) - [e(H) + e(P) - HP]) \\
&\quad + 3(e(H) - HP - \#(C \cap r) + e(P) - HP) \\
&\quad + l\#(C \cap r) + mHP \\
&= e(D_1) + e(C) + 2e(B_2) \\
&\quad + (l - 4)\#(C \cap r) + (m - 3)HP \\
&= 288 + 1770c_1^2 + 672c_1t + 430t^2 \\
&\quad + (\tfrac{1}{2}l + m - 5)HP \\
&= 288 + 1866c_1^2 + 504c_1t + 504t^2 \\
&\quad + (\tfrac{1}{2}l + m - 17)HP. \qquad (7.61)
\end{aligned}
$$

Comparing this with the heterotic side

$$24n_5 = 288 + 1866c_1^2 + 504c_1t + 504t^2 \qquad (7.62)$$

leads again to a corresponding prediction.

7.2.8 $A_2(I_3)$ singularity

$$
\begin{aligned}
F_1 &= 4c_1 + 8r + 4t \Rightarrow F_1 r = 4h \\
G_1 &= 6c_1 + 12r + 6t \Rightarrow G_1 r = 6h \\
D_1 &= 12c_1 + 21r + 12t \Rightarrow D_1 r = 4h + P \qquad (7.63)
\end{aligned}
$$

where $h = c_1 - t$ and $P = 8c_1 - 5t$ and the last decomposition holds not only on the level of cohomology classes but *actually on the level of divisors* as seen from the equation $D_1 = (h^4 P + \mathcal{O}(z) = 0)$.

Now $F_1 G_1 = C_{old} = 8hr + C$ (on the level of divisors) so

$$
\begin{aligned}
C_{old} &= 24(c_1 + t)^2 + 96c_1 r \\
C &= f_1 g_1 - 8(c_1 - t)r = 24(c_1 + t)^2 + 88c_1 r + 8rt \\
Cr &= 8(c_1 - t)(3c_1 - 2t) \\
\#(C \cap r) &= (c_1 - t)(3c_1 - 2t)
\end{aligned}
\tag{7.64}
$$

where in the last line we used the fact that the term $8h$ occurs in Cr as divisor and not just as a cohomology class.

With (note that the discriminant (D.35) shows an intrinsic singularity of D_1 at $h \cap Q_{3c_1 - 2t}$) where

$$
\begin{aligned}
\Delta_{cusp} &= -5656c_1^2 - 776c_1 t - 1632t^2 \\
e(C) &= -1112c_1^2 - 328c_1 t - 288t^2 \\
e(D_1) &= c_2(B_3)D_1 - c_1(B_3)D_1^2 + D_1^3 + \Delta_{cusp} + \alpha h Q \\
&= 252 + 2627c_1^2 + 1360c_1 t + 600t^2 + \alpha h Q
\end{aligned}
\tag{7.65}
$$

one finds

$$
\begin{aligned}
e(X_4) &= 1(e(D_1) - [e(C) - \#(C \cap r) + e(h) - hP + e(P)]) \\
&\quad + 2(e(C) - \#(C \cap r)) \\
&\quad + 3(e(B_2) - [e(h) + e(P) - hP]) \\
&\quad + 4(e(h) - hP - \#(C \cap r) + e(P) - hP) \\
&\quad + l\#(C \cap r) + mhP \\
&= e(D_1) + e(C) + 3e(B_2) \\
&\quad + (l - 5)h(3c_1 - 2t) + (m - 4)hP \\
&= 288 + 1512c_1^2 + 1032c_1 t + 312t^2 \\
&\quad + (l - 5 + \alpha)h(3c_1 - 2t) + (m - 4)hP \\
&= 288 + 1704c_1^2 + 720c_1 t + 432t^2 \\
&\quad + (l - 5 + \alpha)h(3c_1 - 2t) + (m - 4)hP - 24hP \\
&= 288 + 1704c_1^2 + 720c_1 t + 432t^2 \\
&\quad + (l - 5 + \alpha)h(3c_1 - 2t) + (m - 28)hP.
\end{aligned}
\tag{7.66}
$$

Comparing with the heterotic side

$$
24n_5 = 288 + 1704c_1^2 + 720c_1 t + 432t^2
\tag{7.67}
$$

leads to a corresponding prediction.

Finally we remark that the heterotic prediction also matches with the computation for the case $B_2 = F_0$ and $t = 0[b] + 3[f]$ given in table (6.4) of [39].

7.2.9 $A_3(I_4)$ singularity

$$
\begin{aligned}
F_1 &= 4c_1 + 8r + 4t \Rightarrow F_1 r = 4h \\
G_1 &= 6c_1 + 12r + 6t \Rightarrow G_1 r = 6h \\
D_1 &= 12c_1 + 20r + 12t \Rightarrow D_1 r = 4h + P
\end{aligned}
\tag{7.68}
$$

where $h = c_1 - t$ and $P = 8c_1 - 4t$ and the last decomposition holds not only on the level of cohomology classes but *actually on the level of divisors* as seen from the equation $D_1 = (h^4 P + \mathcal{O}(z) = 0)$.

Now $F_1 G_1 = C_{old} = 12hr + C$ (on the level of divisors) so

$$
\begin{aligned}
C_{old} &= 24(c_1 + t)^2 + 96c_1 r \\
C &= f_1 g_1 - 12(c_1 - t)r = 24(c_1 + t)^2 + 84c_1 r + 12rt \\
Cr &= 3hP \\
\#(C \cap r) &= hP
\end{aligned}
\tag{7.69}
$$

where in the last line we used the fact that the term $3h$ occurs in Cr as divisor and not just as a cohomology class.

Furthermore

$$
\begin{aligned}
\Delta_{cusp} &= -5736c_1^2 - 624c_1 t - 1704t^2 \\
e(C) &= -972c_1^2 - 564c_1 t - 192t^2 \\
\Delta_{tacn} &= -2268c_1^2 + 348c_1 t - 122t^2 \\
e(D_1) &= c_2(B_3)D_1 - c_1(B_3)D_1^2 + D_1^3 + \Delta_{cusp} + \Delta_{tacn} - e(h) \\
&= 240 + 2140c_1^2 + 2084c_1 t + 328t^2 + \Delta_{tacn} - e(h) \\
&= 240 + 1914c_1^2 + 2431c_1 t + 207t^2.
\end{aligned}
\tag{7.70}
$$

Now for $SU(4)$ one has the matter/enhancement schemes: $P -- > \mathbf{4} -- > SU(5)$ of Euler $5 = 4 + 1$ and $h -- > \mathbf{6} -- > SO(8)$ of Euler $6 = 4 + 1 + 1$ so that we need to add an $e(h)$ in $e(X^4)$. So one gets for the fiber enhancements

$$
e(X_4) = 1(e(D_1) - [e(C) - \#(C \cap r) + e(h) - hP + e(P)])
$$

$$
\begin{aligned}
&+2(e(C) - \#(C \cap r)) \\
&+4(e(B_2) - [e(h) + e(P) - hP]) \\
&+6(e(h) - hP - \#(C \cap r)) + 5(e(P) - hP)) \\
&+l\#(C \cap r) + mhP \\
= \quad &e(D_1) + e(C) + 4e(B_2) + e(h) \\
&+(l - 7)\#(C \cap r) + (m - 6)hP \\
= \quad &288 + 938c_1^2 + 1868c_1 t + 14t^2 \\
&+(l + m - 13)hP
\end{aligned}
\tag{7.71}
$$

which matches with the codimension one expectation from [39] for $t = c_1$ (cf. section (7.1)) giving $e(X_4) = 288 + 2820c_1^2$.

7.2.10 $A_4(I_5)$ singularity

$$
\begin{aligned}
F_1 &= 4c_1 + 8r + 4t \Rightarrow F_1 r = 4h \\
G_1 &= 6c_1 + 12r + 6t \Rightarrow G_1 r = 6h \\
D_1 &= 12c_1 + 19r + 12t \Rightarrow D_1 r = 4h + P
\end{aligned}
\tag{7.72}
$$

where $h = c_1 - t$ and $P = 8c_1 - 3t$ and the last decomposition holds not only on the level of cohomology classes but *actually on the level of divisors* as seen from the equation $D_1 = (h^4 P + \mathcal{O}(z) = 0)$.

Now $F_1 G_1 = C_{old} = 15hr + C$ (on the level of divisors) so

$$
\begin{aligned}
C_{old} &= 24(c_1 + t)^2 + 96c_1 r \\
C &= f_1 g_1 - 15(c_1 - t)r = 24(c_1 + t)^2 + 81c_1 r + 15rt \\
Cr &= 4h \cdot 6h + 15ht = 3h(8h + 5t) = 3hP \\
\#(C \cap r) &= hP
\end{aligned}
\tag{7.73}
$$

by the similar arguments as in the earlier cases. Note also that not only $Ch \subset Cr$ but that they are actually equal as sets. The precise multiplicity is given by $C3h = Cr$ as the part $3h$ occurs in the Cr not only as cohomology class but even as divisor.

Now (up to a codimension two correction from the higher double point curve along h and codimension three contributions from corrections from point singularities)

$$
e(D_1) = c_2(B_3)D_1 - c_1(B_3)D_1^2 + D_1^3 + \Delta_{cusp}
$$

$$= \ 228 + 7469c_1^2 + 3210c_1t + 1878t^2 + \Delta_{cusp}. \quad (7.74)$$

One has $-7CD_1 + 6c_1(B_3)C = -9222c_1^2 - 3495c_1t - 2259t^2$. Now $F_1G_1 = C_{old} = 15hr + C$ so $2\chi_{ar}(C_{old}) = 2\chi_{ar}(15h) + e(C) - 2(C15h)$ and $2\chi_{ar}(C_{old}) = -1296c_1^2 - 432t^2$ giving with $2\chi_{ar}(15h) = -15h(15h-c_1) = -210c_1^2 + 435c_1t - 225t^2$ and $15Ch = 5Cr = 15hP$ that

$$e(C) = -846c_1^2 - 765c_1t - 117t^2 \quad (7.75)$$

and

$$\Delta_{cusp} = -5838c_1^2 - 435c_1t - 1791t^2 \quad (7.76)$$

and altogether (up to the limitations mentioned above, i.e., the codim 2 and codim 3 contributions from corrections along the higher double point curve h and at their intersection Ch)

$$\begin{aligned} e(D_1) &= 228 + 7469c_1^2 + 3210c_1t + 1878t^2 + \Delta_{cusp} \\ &= 228 + 1631c_1^2 + 2775c_1t + 87t^2. \end{aligned} \quad (7.77)$$

Let us now come to the discussion of the codim 3 contributions. The discriminant equation Δ for D is given by equ. (D.43) so that one has inside the r-plane the equation $h^4P = 0$ with $P = 2h^2g_1 - 3f_1qh - 3Hq^2$. Now note that near h the equation of D_1 is given by (with $e := f_1H + q^2$)

$$-3Hq^2(h^4 - 2h^2Hz + H^2z^2) + \left(-H^2(2g_1H + 3f_1^2) + \frac{9}{4}e^2\right)z^3$$
$$+ (f_1^3 + 3g_1e)z^4 + g_1^2z^5 = 0. \quad (7.78)$$

So the complete square structure of the leading terms showed that we do not have a generic tacnode structure, just as for I_6 but in contrast to the case I_4. With $w := Hz - h^2$ the terms up to third order became (everything up to coefficients) $w^2 + z^3 \to h^6 + 3h^4w + w^2 + w^3$ near $(h,w) = (0,0)$ which goes with $w := v - \frac{3}{2}h^4$ to the normal form $h^6 + v^2$.

Now let us compare with the heterotic side where one has the spectral cover equation $a_0 + a_2x + a_3y + a_4x^2 + a_5xy = 0$ with a_i in the class $\eta_1 - ic_1 = (6-i)c_1 - t$, so (up to inessential factors)

$$\begin{aligned} h_{c_1-t} &= a_5 \\ H_{2c_1-t} &= a_4 \end{aligned}$$

$$q_{3c_1-t} = a_3$$
$$f_1 = f_{4c_1-t} = a_2$$
$$g_1 = g_{6c_1-t} = a_0. \tag{7.79}$$

Further one finds coincidence[16] of the heterotic expression $P \sim a_0 a_5^2 - a_2 a_3 a_5 + a_3^2 a_4$ for this matter curve with the F-theoretic P (up to inessential factors).

The singularity structure, i.e., the overall higher tacnode structure along h, will change if at special loci coefficient functions vanish (say R at $R \cap h = (h \cap q) \cup (h \cap H)$) so that one gets degenerations of the structure equ. (7.78). Now the relevant loci are $h \cap q$ and $h \cap H$ and one has

$$hP = h(2q + H)$$
$$hs = h(2q + 2H)$$
$$hR = h(2q + 3H). \tag{7.80}$$

Both, $h \cap R$ and $h \cap P$, lie in (and are actually equal to) $h \cap (q \cup H)$. In the following we will divide $h \cap P$ and $R \cap h$ into $h \cap q$ and $h \cap H$.

This leads to the consideration of the following loci and singularities (up to coefficients):

$$h_x \cap q_y \Rightarrow x^4(x^2 + xy + y^2) + x^2(x^2 + xy + xy^3 + y^2)z$$
$$+ (x^3y + x^2 + x^2y^2 + xy^3 + xy + y^2)z^2 + z^3 \tag{7.81}$$

respectively

$$h_x \cap H_y \Rightarrow x^4(x^2 + x + y) + x^2(x^2 + x^2y + xy + x + y^2)z$$
$$+ (x^3 + x^2 + x^2y + x^2y^2 + xy + xy^2 + y^3)z^2 + z^3. \tag{7.82}$$

So we expect further corrections (denoted below by α and β) to $e(D_1)$ at these hq resp. hH points.

There are 'B_3-intrinsic' corrections hidden in

$$e(D_1) = c_2(B_3)D_1 - c_1(B_3)D_1^2 + D_1^3 + \Delta_{\text{cusp}} + \Delta_{\text{hightac}}$$
$$- e(h) + \Delta_{C \cap r} + \Delta_{hq} + \Delta_{hH}$$

---

[16]The coincidence of weights for the P's and a_i was already noticed in [26]; here we give the F-theoretic P in h, H, q, f_1, g_1.

$$
\begin{aligned}
&= e(D_1)^{ord} + \Delta_{\text{cusp}} + \Delta_{\text{hightac}} - e(h) + (\gamma - 2)\#(C \cap r) \\
&\quad + (\alpha - 4)\#(h \cap q) + (\beta - 4)\#(h \cap H) \\
&= e(D_1)^{ord} + \Delta_{\text{cusp}} + \Delta_{\text{hightac}} - e(h) \\
&\quad + (2\gamma + \alpha - 8)hq + (\gamma + \beta - 6)hH \\
&=: e(D_1)^{ord} + \Delta_{\text{cusp}} + \Delta_h
\end{aligned}
\tag{7.83}
$$

where this time $\Delta_{C \cap r}$ comprises two effects: first, that the points of C which lie in r are no longer cusp points, and secondly that, as the cusp curve C and the tacnode curve hr intersect, the corrections Δ_{cusp_C} and Δ_{tac_h} get a third term $\Delta_{\text{cusp} \cap \text{hightac}} = \gamma\#(C \cap h) = \gamma\#(C \cap r)$ describing the influence of the intersection locus. Also at the loci $h \cap q$ and $h \cap H$ we have the point singularities of D_1 measured by α and β, where we then have to subtract their pointwise tacnode contribution " $+ 4$", just as we did for the points of $C \cap r$ which were not cusp points where we subtracted their pointwise cusp contribution " $+ 2$".

Now again we now that $Cr = 3hP$ as cohomology classes and $C \cap r \subset h$. One has $C \cap r \not\subset h \cap P$, i.e., that $C \cap r$ lies neither in $h \cap q$ nor in $h \cap H$. Then one has with the disjoint decomposition $h \cap P = (h \cap q) \cup (h \cap H)$ that[17]

$$
\begin{aligned}
\#(C \cap r) &= hP = 2hq + hH \\
\#(h \cap P) &= \#(h \cap q) + \#(h \cap H) = hq + hH.
\end{aligned}
\tag{7.84}
$$

Now for $SU(5)$ one has the matter/enhancement schemes: $P - - > \mathbf{5} - - > SU(6)$ of Euler $6 = 5 + 1$ and $h - - > \mathbf{10} - - > SO(10)$ of Euler $7 = 5 + 1 + 1$ so that we need to add an $e(h)$ in $e(X^4)$. So one finds

$$
\begin{aligned}
e(X) &= 1(e(D_1) - e(C) - e(h) - e(P) + \#(C \cap r) + \#(h \cap P)) \\
&\quad + 2(e(C) - \#(C \cap r)) \\
&\quad + 5(e(B_2) - e(h) - e(P) + \#(h \cap P)) \\
&\quad + 7(e(h) - \#(h \cap P) - \#(C \cap r)) + 6(e(P) - \#(h \cap P)) \\
&\quad + l\#(C \cap r) + m\#(h \cap q) + k\#(h \cap H) \\
&= e(D_1) + e(C) + 5e(B_2) + e(h)
\end{aligned}
$$

---

[17]the left hand side of the first equation is the intersection of h with a divisor of class P but *not with the divisor P itself* by assumption; note that the second equation is the intersection of h with *the divisor P*

$$+(l-8)\#(C \cap r) + \#(m-7)(h \cap q) + (k-7)\#(h \cap H)$$
$$= e(D_1) + e(C) + 5e(B_2) + e(h)$$
$$+(2l+m-23)hq + (l+k-15)hH$$
$$= 288 + 780c_1^2 + 2011c_1t - 31t^2 + \Delta_h$$
$$+(2l+m-23)hq + (l+k-15)hH. \tag{7.85}$$

Comparing this with the heterotic value

$$24n_5 = 288 + 1410c_1^2 + 975c_1t + 375t^2 \tag{7.86}$$

gives the condition (note that Δ_h was proportional to h too)

$$\Delta_h + (2l+m-23)hq + (l+k-15)hH = h(630c_1 - 406t) \tag{7.87}$$

7.2.11 $A_5(I_6)$ singularity

$$F_1 = 4c_1 + 8r + 4t \Rightarrow F_1 r = 4h$$
$$G_1 = 6c_1 + 12r + 6t \Rightarrow G_1 r = 6h$$
$$D_1 = 12c_1 + 18r + 12t \Rightarrow D_1 r = 12c_1 - 6t = 4h + P_{8c_1-2t} \tag{7.88}$$

where $h = c_1 - t$ and $P = 8c_1 - 2t$ and the last decomposition holds not only on the level of cohomology classes but *actually on the level of divisors* as seen from the equation $D_1 = (h^4 P + \mathcal{O}(z) = 0)$.

Now $F_1 G_1 = C_{old} = 18hr + C$ (on the level of divisors) so

$$C_{old} = 24(c_1+t)^2 + 96c_1 r$$
$$C = f_1 g_1 - 18(c_1-t)r = 24(c_1+t)^2 + 78c_1 r + 18rt$$
$$Cr = 4h \cdot 6h + 18ht = 3h(8h+6t) = 3hP. \tag{7.89}$$

Note again that not only $Ch \subset Cr$ but that they are actually equal as sets. The precise multiplicity is given by $C3h = Cr$ as the part $3h$ occurs in the Cr not only as cohomology class but even as divisor.

Now

$$e(D_1) = c_2(B_3)D_1 - c_1(B_3)D_1^2 + D_1^3 + \Delta_{cusp} + \Delta_h$$
$$= 216 + 7062c_1^2 + 3642c_1t + 1764t^2 + \Delta_{cusp} + \Delta_h. \tag{7.90}$$

One has $-7CD_1 + 6c_1(B_3)C = -8820c_1^2 - 4068c_1 t - 2088t^2$. Now $F_1 G_1 = C_{old} = 18hr + C$ so $2\chi_{ar}(C_{old}) = 2\chi_{ar}(18h) + e(C) - 2(C18h)$ and $2\chi_{ar}(C_{old}) = -1296c_1^2 - 432t^2$ giving with $2\chi_{ar}(18h) = -18h(18h - c_1) = -18(c_1 - t)(17c_1 - 18t) = -18(17c_1^2 - 35c_1 t + 18t^2) = -306c_1^2 + 630c_1 t - 324t^2$ and $18Ch = 6Cr = 18hP = 18(c_1 - t)(8c_1 - 2t)$ that

$$
\begin{aligned}
e(C) &= -702c_1^2 - 990c_1 t - 36t^2 \\
\Delta_{cusp} &= -6012c_1^2 - 108c_1 t - 1944t^2
\end{aligned}
\tag{7.91}
$$

and thus

$$
e(D_1) = 216 + 1050c_1^2 + 3534c_1 t - 180t^2 + \Delta_h.
\tag{7.92}
$$

Now for $SU(6)$ one has the matter/enhancement schemes: $P -- > $ $\mathbf{6} -- > SU(7)$ of Euler $7 = 6 + 1$ and $h -- > \mathbf{15} -- > SO(12)$ of Euler $8 = 6 + 1 + \mathbf{1}$ so that, like for $SU(5)$ we need to add an $e(h)$ in $e(X^4)$. So now one has (let us include in Δ_h' as opposed to Δ_h also the fiber enhancements at the codimension three loci, which are also in h)

$$
\begin{aligned}
e(X^4)^{-->cod2} &= e(D_1) + e(C) + e(h) + 6e(B_2) + \Delta_h' \\
&= 288 + 342c_1^2 + 2545c_1 t - 217t^2 + \Delta_h'.
\end{aligned}
\tag{7.93}
$$

Comparing this with the heterotic value

$$
24n_5 = 288 + 1266c_1^2 + 1035c_1 t + 369t^2
\tag{7.94}
$$

gives as prediction $\Delta_h' = h(1150c_1 - 665t)$.

7.3 An observation on the 4D Higgs chains

Let us close with an observation relating the Euler numbers in neighbouring cases in certain Higgs-chains.

7.3.1 6D case

For the chain $\mathbf{E_7} \to \mathbf{E_6} \to \mathbf{D_5} \to \mathbf{I_5}$ with the Euler numbers

$$
e(E_7) = -284 - 56n
$$

$$e(E_6) = -300 - 54n$$
$$e(D_5) = -312 - 52n$$
$$e(I_5) = -320 - 50n \tag{7.95}$$

one gets

$$e(higher) - 2e(enh.locus) = e(lower) \tag{7.96}$$

where the enhancement locus is in this chain always the h locus of the *higher* group, i.e., $f_1 = 4c_1 - t = 8 - n$, $g_1 = 3c_1 - t = 6 - n$, $h = 2c_1 - t = 4 - n$ and $h = c_1 - t = 2 - n$ for E_7, E_6, D_5 and I_5 respectively.

The reason for these relations is of course easy to see. Take for example the case of E_7 and E_6. One has $h^{1,1}(E_7) - h^{1,1}(E_6) = 1$ and from $h^{2,1} = n_H - 1 = \dim \mathcal{M}_{12+n}(E_8) + \dim \mathcal{M}_{12-n}(H_k) + h^{1,1}(K3) - 1 = 112 + 30n + k(12-n) - (k^2-1) + 19$ where $H_k = SU(k)$ is the commutant with $k = 2$ and 3 one gets that $h^{2,1}(E_7) - h^{2,1}(E_6) = [k(12-n) - (k^2 - 1)] - [(k+1)(12-n) - ((k+1)^2 - 1)] = -(12-n) + 2k + 1$ for $k = 2$ and so $e(E_7) - e(E_6) = 2(12 - n - 2k) = 2(2(6 - k) - n)$ where $2(6 - k) - n$ is just the degree of the relevant enhancement curve.

Similarly for the I_k series one gets with

$$e(I_6) = -288 - 54n$$
$$e(I_5) = -320 - 50n$$
$$e(I_4) = -352 - 44n$$
$$e(I_3) = -384 - 36n$$
$$e(I_2) = -420 - 24n \tag{7.97}$$

that again

$$e(higher) - 2e(enh.locus) = e(lower) \tag{7.98}$$

where the relevant curve is in this chain always the P locus of the *higher* group, i.e., $8c_1 - (8 - k)t = 16 - (8 - k)n$ for I_k the higher group (note that in our set-up we made the switch $n \to -n$).

7.3.2   4D case

For the chain $\mathbf{E_7} \to \mathbf{E_6} \to \mathbf{D_5} \to \mathbf{I_5}$ with the euler numbers

$$e(E_7) = 288 + 1350c_1^2 + 1020c_1 t + 366t^2$$

$$e(E_6) = 288 + 1386c_1^2 + 999c_1 t + 369t^2$$
$$e(D_5) = 288 + 1404c_1^2 + 984c_1 t + 372t^2$$
$$e(I_5) = 288 + 1410c_1^2 + 975c_1 t + 375t^2 \tag{7.99}$$

one gets

$$e(higher) - 3e(enh.locus) = e(lower) \tag{7.100}$$

where the enhancement locus is in this chain always the h locus of the *higher* group, i.e., $f_1 = 4c_1 - t$, $g_1 = 3c_1 - t$, $h = 2c_1 - t$ and $h = c_1 - t$ for E_7, E_6, D_5 and I_5 respectively.

Again let us see from a heterotic point of view how this structure emerges. From

$$c_2(SU(n)) = \eta\sigma - \frac{n^3 - n}{24}c_1^2 - \frac{n}{8}\eta(\eta - nc_1) \tag{7.101}$$

one sees that

$$c_2(SU(n+1))$$
$$= c_2(SU(n)) - \frac{1}{8}[(n^2 + n)c_1^2 + \eta(-nc_1 + \eta - c_1 - nc_1)] \tag{7.102}$$

where the correction term is

$$-\frac{1}{8}[(n^2 + n)c_1^2 + (6c_1 - t)((5 - 2n)c_1 - t)] = \frac{1}{8}e(h_{\eta - nc_1}) \tag{7.103}$$

with $\eta - nc_1 = (6 - n)c_1 - t$ the class of the h matter curve ($a_n = 0$). This leads in $24n_5$ to the searched for $3e(h)$.

Furthermore inside the I_k series one has the same relation with

$$e(I_6) = 288 + 1266c_1^2 + 1035c_1 t + 369t^2$$
$$e(I_5) = 288 + 1410c_1^2 + 975c_1 t + 375t^2 \tag{7.104}$$

where the relevant curve is in this chain the P locus of the *higher* group, i.e., $8c_1 - (8 - k)t$ for I_k the higher group.

Appendix

In *section A* we give some background and notation concerning four-dimensional F-theory models, i.e., Calabi-Yau four-folds and recall some facts related to the four-flux.

In *section B* the different constructions of the vector bundles of the dual heterotic models are described and the computation of the second Chern class of E_k bundles in the parabolic framework is derived.

In *section C* some known resp. expected connections between the moduli spaces of compactifications in the two dual pictures of the heterotic string and F-theory are described, especially the analogy between the gamma class of the bundle description and the four-flux on the F-theory side is recalled.

Section D has a somewhat different flavour in that it lists some computational details concerning the explicit discriminant equations and spectrum computations for the case of Calabi-Yau *three*-folds.

A   4d F-theory models

A.1   The geometry of the four-fold

We will consider Calabi-Yau fourfolds X which are elliptically fibered $\pi : X \to B_3$ over a complex three-dimensional base $B_3 = B$, let σ be a section. X can be described by a Weierstrass equation $y^2 z = x^3 + g_2 x z^2 + g_3 z^3$ which embeds X in a $\mathbf{P}^2$ bundle $W \to B$ which is the projectivization of a vector bundle $\mathbf{P}(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}_B)$ with $\mathcal{L}$ being a line bundle over B. Since the canonical bundle of X has to be trivial we get from $K_X = \pi^*(K_B + \mathcal{L})$ the condition $\mathcal{L} = -K_B$. Further we can think of x, y and z as homogeneous coordinates on the $\mathbf{P}^2$ fibers, i.e., they are sections of $\mathcal{O}(1) \otimes K_B^{-2}$, $\mathcal{O}(1) \otimes K_B^{-3}$ and $\mathcal{O}(1)$, whereas g_2 and g_3 are sections of $H^0(B, K_B^{-4})$ and $H^0(B, K_B^{-6})$ respectively and the Weierstrass equation is a section of $\mathcal{O}(1)^3 \otimes K_B^{-6}$. The section σ can be thought of as the point at infinity $x = z = 0, y = 1$. The descriminant of the elliptic fibration is given by $\Delta = 4g_2^3 - 27g_3^2$ which is a section of K_B^{-12}. If $\Delta = 0$ at a point $p \in B$ the type of singular fiber is determined by the orders of vanishing $\mathrm{ord}(g_2) = \mathrm{a}$, $\mathrm{ord}(g_3) = \mathrm{b}$ and $\mathrm{ord}(\Delta) = \mathrm{c}$, and given by Kodaira's classification of elliptic fiber singularities:

a	b	c	fiber	singularity
≥ 0	≥ 0	0	smooth	none
0	0	n	I_n	A_{n-1}
≥ 1	1	2	II	none
≥ 1	≥ 2	3	III	A_1
≥ 2	2	4	IV	A_2
2	≥ 3	$n+6$	I_n^*	D_{n+4}
≥ 2	3	$n+6$	I_n^*	D_{n+4}
≥ 3	4	8	IV^*	E_6
3	≥ 5	9	III^*	E_7
≥ 4	5	10	II^*	E_8

Let us denote by $F = -4K_B$, $G = -6K_B$ and $D = -12K_B$ the classes of the divisors associated to the vanishing of g_2, g_3 and Δ respectively.

We will assume B_3 to be a $\mathbf{P}^1$ bundle $\sigma : B_3 \to B_2$ which is the projectivization $\mathbf{P}(Y)$ of a vector bundle $Y = \mathcal{O} \oplus \mathcal{T}$, with $\mathcal{T}$ a line bundle over B_2 and $\mathcal{O}(1)$ a line bundle on the total space of $\mathbf{P}(Y) \to B_2$ which restricts on each $\mathbf{P}^1$ fiber to the corresponding line bundle over $\mathbf{P}^1$. Further let u, v be homogeneous coordinates of the $\mathbf{P}^1$ bundle and think of a and b as sections of $\mathcal{O}(1)$ and $\mathcal{O}(1) \otimes \mathcal{T}$ over B_2. If we set $r = c_1(\mathcal{O}(1))$ and $t = c_1(\mathcal{T})$ and $c_1(\mathcal{O} \otimes \mathcal{T}) = r + t$ then the cohomology ring of B_3 is generated over the cohomology ring of B_2 by the element r with the relation $r(r+t) = 0$, i.e., the divisors $u = 0$ resp. $v = 0$, which are dual to r resp. $r + t$, do not intersect. From adjunction $c(B_3) = (1 + c_1(B_2) + c_2(B_2))(1+r)(1+r+t)$ one finds for the Chern classes[18] $c_1(B_3) = c_1 + 2r + t$, $c_2(B_3) = c_2 + c_1 t + 2c_1 r$. Note that B_2 will be chosen to be rational. For this recall that the arithmetic genus p_a of B_3 has to be equal to one $1 = p_a = \frac{1}{24} \int_{B_3} c_1(B_3)c_2(B_3) = \frac{1}{12} \int_{B_2} c_1^2 + c_2$ (for more details cf. [2]) in order to satisfy the $SU(4)$ holonomy condition for X, otherwise there are non-constant holomorphic differentials on B_3 which would pull back to X.

---

[18]Unspecified Chern classes refer to B_2.

A.2   The four-flux

In order to obtain consistent F-theory compactifications to four-dimensions on Calabi-Yau fourfold X, it is necessary to turn on a number n_3 of space-time filling three-branes for tadpole cancellation [49]. This is related to the fact that compactifications of the type IIA string on X are destabilized at one loop by $\int B \wedge I_8$, where B is the NS-NS two-form which couples to the string and I_8 a linear combination of the Pontryagin classes p_2 and p_1^2 [51]. So, compactifications of the type IIA string to two-dimensions lead to a tadpole term $\int B$ which is proportional to the Euler characteristic of X. Similar, in M-theory compactifications to three-dimensions on X arises a term $\int C \wedge I_8$ with C being the M-theory three-form, and integration over X then leads to the tadpole term which is proportional to $\chi(X)$, and couples to the 2-brane [24],[9]. Now, as M-theory compactified to three-dimensions on X is related to a F-theory compactification on X to four-dimensions [50], one is lead to expect a term $\int A$ with A now being the R-R four form potential, which couples to the three-brane in F-theory [49]. Taking into account the proportionality constant [52], one finds $\frac{\chi(X)}{24} = n_3$ three-branes in F-theory (or membranes in M-theory resp. strings in type IIA theory) [49].

Furthermore the tadpole in M-theory will be corrected by a classical term $C \wedge dC \wedge dC$, which appears if C gets a background value on X and thus leads to a contribution $\int dC \wedge dC$ to the tadpole [9]. Also one has [53] a quantization law for the four-form field strength G of C (the four-flux) the modified integrality condition $G = \frac{dC}{2\pi} = \frac{c_2}{2} + \alpha$ with $\alpha \in H^4(X, \mathbf{Z})$ where α has to satisfy [15] the bound $-120 - \frac{\chi(X)}{12} \leq \alpha^2 + \alpha c_2 \leq -120$, in order to keep the wanted amount of supersymmetry in a consistent compactification. Altogether[19] one finds [16]

$$\frac{\chi(X)}{24} = n_3 + \frac{1}{2} \int G \wedge G \qquad (\text{A.1})$$

for consistent $N = 1$ F-theory compactifications on X to four-dimensions.

---

[19]the presence of a non-trivial instanton background can also contribute to the anomaly [12]; this adds a term $\sum_j \int_{\Delta_j} c_2(E_j)$ to n_3 where $\int_{\Delta_j} c_2(E_j) = k_j$ are the instanton numbers of possible background gauge bundles E_j inside the 7-brane [12] and Δ_j denotes the discriminant components in B_3

From the connection of F-theory to M-theory via S^1 compactification one expects some lifting of the four-flux of M-theory to play a role in F-theory. This is a $(2,2)$ form in integral cohomology (essentially; its precise quantization law leading to half-integrality is reviewed below) and so according to the Hodge conjecture supported on an algebraic surface S in X_4. From the primitivity condition (again further reviewed below) for the self-dual four-flux ($g^{b\bar{b}}$ the Kaehler metric) [9]

$$F_{a\bar{a}b\bar{b}}g^{b\bar{b}} = 0 \tag{A.2}$$

one gets a well-comed constraint $\int_X J \wedge J \wedge F = 0$ on the moduli.

B $4D$ heterotic models

B.1 The spectral cover method for $SU(n)$ bundles

Let us recall the idea of the spectral cover description of an $SU(n)$ bundle V: one considers the bundle first over an elliptic fibre and then pastes together these descriptions using global data in the base B. Now over an elliptic fibre E an $SU(n)$ bundle V over Z (assumed to be fibrewise semistable) decomposes as a direct sum of line bundles of degree zero; this is described as a set of n points which sums to zero. If you now let this vary over the base B this will give you a hypersurface $C \subset Z$ which is a ramified n-fold cover of B. If one denotes the cohomology class in Z of the base surface B (embedded by the zero-section σ) by $\sigma \in H^2(Z)$ one finds that the globalization datum suitable to encode the information about V is given by a class $\eta \in H^{1,1}(B)$ with

$$C = n\sigma + \eta \tag{B.3}$$

as C is given as a locus $w = a_0 + a_2 x + a_3 y + \dots a_n x^{\frac{n}{2}} = 0$, for n even say and x, y the usual elliptic Weierstrass coordinates, with w a section of $\mathcal{O}(\sigma)^n \otimes \mathcal{N}$ with a line bundle $\mathcal{N}$ of class η. Note that a_i is of class $\eta - ic_1$.

The idea is then to trade in the $SU(n)$ bundle V over Z, which is in a sense essentially a datum over B, for a line bundle L over the n-fold (ramified) cover C of B: one has

$$V = p_*(p_C^* L \otimes \mathcal{P}) \tag{B.4}$$

with $p : Z \times_B C \to Z$ and $p_C : Z \times_B C \to C$ the projections and $\mathcal{P}$ the global version of the Poincare line bundle over $E \times E$ (actually one uses a symmetrized version of this), i.e., the universal bundle which realizes the second E in the product as the moduli space of degree zero line bundles over the first factor.

A second parameter in the description of V is given by a half-integral number λ which occurs because one gets From the condition $c_1(V) = \pi_*(c_1(L) + \frac{c_1(C)-c_1}{2}) = 0$ that with $\gamma \in ker\, \pi_* : H^{1,1}(C) \to H^{1,1}(B)$ one has

$$c_1(L) = -\frac{1}{2}(c_1(C) - \pi_*c_1) + \gamma \qquad (B.5)$$

where γ is being given by $(\lambda \in \frac{1}{2}\mathbf{Z})$[20]

$$\gamma = \lambda(n\sigma - \eta + nc_1) \qquad (B.6)$$

as $n\sigma|_C - \eta + nc_1$ is the only generally given class which projects to zero.

B.2 E_k bundles and del Pezzo description

B.2.1   Fibrewise

The del Pezzo surface dP_k $(k = 0, \dots 8)$ is given by blowing up k points in $\mathbf{P}^2$. So the lattice $L = H^2(dP_k, \mathbf{Z})$ has the signature $(+1, {}_1^k)$ in the basis given by the line H from $\mathbf{P}^2$ and the exceptional divisors E_i $(i = 0, \dots 8)$; all of these classes are $(1,1)$. The anti-canonical class, an elliptic curve, is ample and given by $-F$ for $F = 3H - \sum_i E_i$. For $k = 9$ (and the nine points lying on the intersection of two cubics) one gets the (almost del Pezzo) rational elliptic surface dP_9, with F the elliptic fibre class. "Exceptional" or "(-1)" curves are the curves c with $c^2 = -1$ and $c \cdot F = +1$. The orthogonal complement of F is the E_k lattice. The A_{k-1} lattice occurs too, with the basis $E_i - E_{i+1}$ $(i = 1, \dots k-1)$; the additional root, which leads to E_k and does not lie on the line of the A_{k-1} Dynkin diagram, is given by $H - (E_1 + E_2 + E_3)$. For D_{k-1} take the representation of dP_k as Hirzebruch surface F_1 blown

---

[20]actually λ has to be strictly half-integral resp. integral for n odd resp. even

up in $k-1$ points lying on different fibers, denote the two P^1 in each of the $k-1$ special fibers of type A_2 by $L_{\pm i}$ ($i = 1, \ldots k-1$) of classes $l_{\pm i}$ and by f the fibre class ($f = l_i + l_{-i}$); $f^\perp$ is the sublattice generated by the l_i ($\pm i = 1, \ldots k-1$) and $(f+K)^\perp$ is generated by the root system $R = \{(l_{\pm i} - l_{\pm j})\}, \pm i, \pm j = 1, \ldots k-1, i \neq \pm j$ of type D_{k-1}.

Now, as every point $\mathbf{w}$ in the weight lattice $L/F\mathbf{Z}$ of G determines a representation $\rho_{\mathbf{w}}$ of the maximal torus and, from taking a flat connection A in the representation $\rho_{\mathbf{w}}$, a line bundle $\mathcal{L}_{\mathbf{w}}$ on E - thus providing a homomorphism to $Jac(E) = E$, a semistable $G = E_k$ bundle over an elliptic curve E corresponds to a homomorphism from L to E mapping F to zero. This first dictionary is further translated via the Torelli theorem to essentially[21] the space of complex structures of a dP_k surface keeping a divisor D of class F fixed: namely, keeping D fixed, $y \cdot F = 0$ for $y \in L^{\perp F}$ means that $\mathcal{L}_y|_D$ is of degree zero, so defines a point in $Jac(E)$. One may rephrase (cf. [44]) the association, in the case of E_8 say, saying that the flat gauge field on the elliptic curve D is mapped to the set of eight points on D which represent the intersection of D with divisors generating the E_8 piece in the lattice of the del Pezzo surface. One gets the del Pezzo surface writing a second cubic (besides the elliptic curve D) on $\mathbf{P}^2$ and forcing nine points to lie on their intersection, where the nine points are got in flat coordinates on D from the flat E_8 gauge field represented as $\vec{w} = (w_1, \ldots, w_8)$ in a Cartan basis by $w_i = z_i - z_0$ with $z_0 + z_1 + \cdots + z_8 = 0$.

B.2.2   Globally

The root system is describing (cf. for example [45],[23],[41], [17]) a certain part of the $H^{1,1}(dP, \mathbf{Z})$ so that the variation of the fibre of the spectral cover over B describes the variation of certain (-1) curves l in their variation in a family of surfaces over B (expressing the effective replacement of these lines by points, causing the (1,1)-shift). This leads also to the necessary relation between G_i^2 and $l^2 \pi_*^i \gamma^2 = -\pi_*^i \gamma^2$ (for $i = 1, 2$).

Actually we will see the spectral cover as parametrization of exceptional lines in a surface fibration over B. This occurs by taking into account the description of the 'enlarged' root system in surface

---

[21]for a subtlety involving a certain $(9-k)^{th}$ root of $\mathcal{L}_F|_F$ cf. [26]

cohomology. Note that as the same moduli space $\mathcal{W}_G$ parametrizes G bundles over an elliptic curve E and del Pezzo surfaces dP_G (with E of class $-K$ fixed) one gets by adiabatic extension over the base B that to the bundle V over Z corresponds a fibration $W_G^{het} \to B$ of dP_G surfaces via pulling back the universal object (now the universal surface not the universal bundle) along the section $s : B \to \mathcal{W}_G = \mathcal{M}_{Z/B}$.

So for $G = E_8$ both data, the spectral cover and the del Pezzo fibration are equivalent. The obvious analogue works for type E_n: the character lattice Λ of E_n is still isomorphic to the primitive cohomology group $H_0^2(dP_n, \mathbf{Z})$. For type A_n or D_n we use the fact that the corresponding character lattices can be embedded into the E_n lattice as the orthogonal complement of an appropriate fundamental weight (corresponding to one of the ends of the Dynkin diagram). So one can define a del Pezzo fibration of type A_n or D_n to be a del Pezzo fibration $\pi : U \to B$ of type E_n together with a section of the family of E_n lattices $R^2\pi_*\mathbf{Z}$ which, in each fiber, is in the W orbit of that fundamental weight. For A_n, for example, this additional data consists, in each fiber, of specifying the pullback of a line of the original P^2.

B.2.3   Why spectral covers for $SU(n)$ bundles and del Pezzo fibrations for E_k bundles

When one tries to transfer these results to the (D- and especially to the) E-series, one faces the following problem. For the E-series one does not describe [26] the bundles via the spectral cover construction but instead via the associated del Pezzo fibration, giving not a covering of B but[22] a fibration over it by surfaces. This is related to the following (cf. [36]): consider the type IIA string on an elliptic K3 with ADE singularity times T^2; the $N = 1$ content of this 4D $N = 4$ theory includes three adjoint chiral fields X, Y, Z, whose Cartan vevs, parametrizing the Higgs branch, correspond to blowing up respectively deforming the

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[22]This admits also a representation-theoretic explanation. The Weyl group of A_n admits a small permutation representation $\mathbf{n + 1}$ which decomposes into the sum of *two* irreducible representations: the trivial one and the weights, $\mathbf{Z}[W/W_0] \cong \mathbf{1} \oplus \Lambda$. But every permutation representation of W_{E_n} contains at least *three* irreducible constituents, so the cohomology of an associated spectral cover contains additional pieces. To get an object with the right cohomology one must either go up in dimension or restrict attention to classes which transform correctly under some correspondences.

singularity respectively giving Wilson lines to the ADE gauge group on T^2; the R-symmetry induces an equivalence of the corresponding moduli spaces. This gives the main theorem on the structure of the moduli space $\mathcal{M}_G$ of flat G-bundles on an elliptic curve.

Concretely let us take as elliptic curve $E = P_{1,2,3}[6]$ of equation $e :=$ $z^6 + x^3 + y^2 + \mu zxy = 0$ leading (with w of sect. B.1) to the deformation $e + vw$ of the $SU(n)$ singularity showing at the same time Looijenga's moduli space $(a_0, a_2, a_3, \ldots, a_n) \in \mathbf{P}^{n-1}$ of flat $SU(n)$ bundles over E as well as the 0D spectral geometry consisting of n points $(e = 0) \cap (w = 0)$ on E. Note that *in this case of the A_n group* it is possible to effectively replace [23] a 2D geometry of $\mathbf{P}^1$'s by the zero dimensional representatives as v occurs only linear and so in the process of period integral evaluation to describe the variation of Hodge structure relevant here can be integrated out.

By contrast the same decoupling phenomenon *does not* take place[24] for the the E_k case: there one finds instead for the deformation $e + \sum_{i=1}^{6} a_i v^i z^{6-i} + b_2 v^2 x^2 + b_3 v^3 y + b_4 v^4 x$ of zero locus $dP_8 = \mathbf{P}_{1,1,2,3}[6]$ showing the 2D spectral geometry of the del Pezzo surface with $H^{1,1}(dP, \mathbf{Z})^{\perp_E} = E_8$ and moduli space $\mathbf{P}_{1,2,3,4,5,6,2,3,4}$.

B.3   The parabolic approach and the characteristic classes of E_8, E_7, E_6

In order to get a heterotic prediction for the Euler characteristic of X with a section of I_2, I_3 singularities, we have to compute the second Chern class of the corresponding E_7, E_6 bundles on the heterotic side. Also our 'second' bundle will always be an E_8 bundle.

In order to do so we can use the parabolic bundles description which leads one to easily compute their Chern classes [26], [1]. In this ap-

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[23] For the general phenomenon relating even (0D to 2D, of symmetric intersection form) resp. odd (1D to 3D, of antisymmetric intersection form) cohomology cf. [40]; the same relation underlies the extraction [38] of the 1D Seiberg-Witten curve from the 3D periods of a Calabi-Yau and the relation between $K3$ singularities and *ADE* gauge groups.

[24] Correspondingly there occurs a situation involving E groups, where the Coulomb branch of an $N = 2$ system does not reduce to a Riemann surface but is described in terms of 3-form periods [36].

proach one starts with an unstable bundle on a single elliptic curve E. For this one fixes a point p on E with the associated rank 1 line bundle $\mathcal{O}(p) = W_1$. Rank k line bundles W_k are then inductively constructed via the unique non-split extension $0 \to \mathcal{O} \to W_{k+1} \to W_k \to 0$. If one writes the dual of W_k as W_k^* then the unique (up to translations on E) minimal unstable bundle with trivial determinant on E is given by $V = W_k \oplus W_{n-k}^*$. This can be deformed by an element of $H^1(E, W_k^* \otimes W_{n-k}^*)$ to a stable bundle V' which fits then into the exact sequence $0 \to W_{n-k}^* \to V' \to W_k \to 0$. To get a global version of this construction on replaces the W_k by their global versions, i.e., replace $\mathcal{O}(p)$ by $\mathcal{O}(\sigma_1)$. The global versions of W_k are inductively constructed by an exact sequence $0 \to \mathcal{L}^{n-1} \to W_k \to W_{k-1} \to 0$.

B.3.1 E_8 bundle

The starting point for building an E_8 bundle is $G = SU(6) \times SU(3) \times SU(2)$ and let $X_{6,3,2}$ denote the three factors. Then locally one has a description of the X's given by $X_6 = (W_5 \oplus \mathcal{O}) \otimes \mathcal{O}(p)^{-1/6}$ and $X_3 = (W_3 \otimes \mathcal{O}(p)^{-1/3})$ and $X_2 = W_2 \otimes \mathcal{L}$. The global versions are given by [26]

$$
\begin{aligned}
X_6 &= (W_5 \otimes \mathcal{S}^{-1} \oplus \mathcal{S}^5 \otimes \mathcal{L}^{-1}) \otimes \mathcal{O}(\sigma)^{-1/6} \otimes \mathcal{L}^{-3/2} \\
X_3 &= W_3 \otimes \mathcal{O}(\sigma)^{-1/3} \otimes \mathcal{L}^{-1} \\
X_2 &= W_2 \otimes \mathcal{O}(\sigma)^{-1/2} \otimes \mathcal{L}^{-1/2}
\end{aligned}
\tag{B.7}
$$

the fundamental class $\lambda(V)^{25}$ of this bundle is given by

$$
\lambda(V) = \eta\sigma - 15\eta^2 + 135\eta c_1 - 310 c_1^2
\tag{B.8}
$$

with $\eta = 4c_1 + c_1(\mathcal{S})$ and which then leads to the following expression for the $\chi(X)$

$$
24 n_5 = 288 + 1200 c_1^2 + 1080 c_1 t + 360 t^2.
\tag{B.9}
$$

B.3.2 E_7 bundle

Our starting point for E_7 bundle is $G = SU(4) \times SU(4) \times SU(2)$ and let $X_{1,2,3}$ denote again the three factors. Then locally one has a description

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[25]which is $c_2(V)/60$ for E_8 bundle, also note that $\lambda(V) = c_2(V)/36$ and $\lambda(V) = c_2(V)/24$ for E_7 resp. E_6 bundles which we use below

of the X's given by $X_1 = (W_3 \oplus \mathcal{O}) \otimes \mathcal{O}(p)^{-1/4}$ and $X_2 = (W_4 \otimes \mathcal{O}(p)^{-1/4})$ and $X_3 = W_2 \otimes \mathcal{O}(p)^{-1/2}$. As global versions we choose

$$
\begin{aligned}
X_1 &= (W_3 \otimes \mathcal{S}^{-1} \oplus \mathcal{S}^3 \otimes \mathcal{L}^{-1}) \otimes \mathcal{O}(\sigma)^{-1/4} \otimes \mathcal{L}^{-1/2} \\
X_2 &= W_4 \otimes \mathcal{O}(\sigma)^{-1/4} \otimes \mathcal{L}^{-3/2} \\
X_3 &= W_2 \otimes \mathcal{O}(\sigma)^{-1/2} \otimes \mathcal{L}^{-1/2}
\end{aligned}
\tag{B.10}
$$

the fundamental class of this bundle is given by

$$
\lambda(V) = \eta\sigma - 6\eta^2 + 48\eta c_1 - \frac{399}{4}c_1^2
\tag{B.11}
$$

with $\eta = 7/2c_1 + c_1(\mathcal{S})$ and which then leads to the following expression for the $\chi(X)$

$$
24n_5 = 288 + 1866c_1^2 + 504c_1 t + 504t^2.
\tag{B.12}
$$

B.3.3 E_6 bundle

In order to get a E_6 bundle one chooses as the unstable bundle whose structure group reduces to a group locally $G = SU(3) \times SU(3) \times SU(3)$ following [26]. The fundamental characteristic class of an E_6 bundle whose structure group reduces to G is

$$
\lambda(V) = c_2(X_1) + c_2(X_2) + c_2(X_3).
\tag{B.13}
$$

Now, it was shown in [26] that on a single elliptic curve X_1 and X_2 are given by $X_{1,2} = W_3 \otimes \mathcal{O}(p)^{-1/3}$ and for X_3 one has $X_3 = (W_2 \oplus \mathcal{O}) \otimes \mathcal{O}(p)^{-1/3}$. All we have to do now is to give a global description of these bundles and compute their Chern classes. Therefore we want to consider bundles which are isomorphic to the X_i's on each fiber and have trivial determinant. Thus we can take for $X_{1,2}$

$$
X_{1,2} = W_3 \otimes \mathcal{O}(\sigma)^{-1/3} \otimes \mathcal{L}^{-1}
\tag{B.14}
$$

and for X_3 the most general possibility to write down a global version of it is

$$
X_3 = (W_2 \otimes \mathcal{S}^{-1} \oplus \mathcal{S}^2 \otimes \mathcal{L}^{-1}) \otimes \mathcal{O}(\sigma)^{-1/3}
\tag{B.15}
$$

where $\mathcal{S}$ is an arbitrary line bundle on B which is introduced using the fact that one can twist by additional data coming from the base. ¿From

this we can now compute Chern classes. We find for the fundamental characteristic class of the E_6 bundle

$$\lambda(V) = \eta\sigma - 3\eta^2 + 21\eta c_1 - 39c_1^2 \qquad \text{(B.16)}$$

where $\eta = c_1(\mathcal{S}) + 3c_1(\mathcal{L})$ and with the anomaly formula we derive simply the heterotic expectation for the Euler characteristic of X with I_3 singularity which is

$$24n_5 = 288 + 1704c_1^2 + 720c_1 t + 432t^2. \qquad \text{(B.17)}$$

C   Comparison of the moduli spaces

In view of the application indicated in the title of the paper the most important insight of an association of data between the heterotic and the F-theory side to be expected will be the following: the gamma class γ is an element of $H^{1,1}(C)$ where the spectral cover C is an n-fold ramified cover of B; think, in a naive picture, of this as a curve in C or as a curve (null-cohomologous) in B each of whose points is covered by some preimages in C; now, if one switches from the spectral cover representation of the bundle (where over each base point $b \in B$ sits a finite point set) to the representation by a fibration over B of del Pezzo surfaces (of type ADE in general or of type E in the more widely known case), the points on the elliptic fibre over b are traded in for $(1,1)$ cohomology classes on the del Pezzo surface (sitting now over b) which correspond to divisors intersecting the former elliptic curve (which re-occurs here as anti-canonical divisor) in the formerly given points; then the situation is considered to be embedded in an E_8 del Pezzo surface with the new (formerly missing) classes/divisors shrunken to zero; this is embedded in a dP_9 set-up which re-occurs (with a structure representing precisely the corresponding heterotic bundle) in the stable degeneration of the Calabi-Yau four-fold on the dual F-theory side; as the points were thickened to P^1's the $(1,1)$ class becomes a $(2,2)$ class which is the candidate for the four-flux class.

C.1   General comparison of moduli space and spectra

The moduli in a 4D N=1 heterotic compactification on an elliptic CY, as well as in the dual F-theoretic compactification, break into "base" parameters which are even (under the natural involution of the elliptic curves), and "fiber" or twisting parameters; the latter include a continuous part which is odd, as well as a discrete part. In [15] all the heterotic moduli were interpreted in terms of cohomology groups of the spectral covers, and identified with the corresponding F-theoretic moduli in a certain stable degeneration. For this one uses the close connection of the spectral cover and the ADE del Pezzo fibrations. For the continuous part of the twisting moduli, this amounts to an isomorphism between certain abelian varieties: the connected component of the heterotic Prym variety (a modified Jacobian) and the F-theoretic intermediate Jacobian. The comparison of the discrete part generalizes the matching of heterotic 5-branes/F-theoretic 3-branes.

By working with elliptically fibered Z one can adiabatically extend the known results about moduli spaces of G-bundles over an elliptic curve $E = T^2$, of course taking into account that such a fiberwise description of the isomorphism class of a bundle leaves definitely room for *twisting along the base B*. The latter possibility actually involves a two-fold complication: there is a continuous as well as a discrete part of these data. It is quite easy to see this for $G = SU(n)$: in this case V can be constructed via push-forward of the Poincare bundle on the spectral cover $C \times_B Z$, possibly twisted by a line bundle $\mathcal{N}$ over the spectral surface C (an n-fold cover of B (via π) lying in Z), whose first Chern class (projected to B) is known from the condition $c_1(V) = 0$. So $\mathcal{N}$ itself is known up to the following two remaining degrees of freedom: first a class in $H^{1,1}(C)$ which projects to zero in B (the discrete part), and second an element of $Jac(C) := Pic_0(C)$ (the continuous part; the moduli odd under the elliptic involution).

The continuous part is expected [26] to correspond on the F-theory side to the odd moduli, related there to the intermediate Jacobian $J^3(X^4)$ of dimension $h^{2,1}$, so that the following picture emerges. The moduli space $\mathcal{M}$ of the bundles is fibered $\mathcal{M} \to \mathcal{Y}$, with fibre $Jac(C)$. There is a corresponding picture on the F-theory side: ignoring the Kahler classes (on both sides), the moduli space there is again fibered.

The base is the moduli space of those complex deformations which fix a certain complex structure of Z; the fibre is the intermediate Jacobian $J^3(X^4) = H^3(X, \mathbf{R})/H^3(X, \mathbf{Z})$ In total[26], $h^{2,1}(Z) + h^1(Z, adV) + 1 = h^{3,1} + h^{2,1}$.

One expects a general scheme of a duality dictionary beyond the previously considered cases of relating $h^{2,0}(C)$ and $h^{3,1}(X^4)$ respectively elements of $H^{1,0}(C)$ and $H^{2,1}(X^4)$ (cf. [12], [26], section 1 and the introduction). Together with the proposed identification of the discrete moduli one gets a dictionary of elements related by a $(1,1)$ Hodge shift

C	X^4
$H^{2,0}$	$H^{3,1}$
$H^{1,0}$	$H^{2,1}$
$H^{1,1}$	$H^{2,2}$

where in the first line the deformations of X^4 preserving the given type of singularity (corresponding with the unbroken gauge group; actually we will consider the parts in the W_i) are understood, in the second line a part of the relative jacobian (see below) is understood, and in the last line the subspaces $ker\, \pi_*$ respectively $ker\,(J \wedge \cdot)$.

One can also give [15] an interpretation of all the bundle moduli $H^1(Z, adV)$, even or odd under the involution, in terms of even respectively odd cohomology of the spectral surface , including an interpretation of the $\mathbf{Z}_2$ equivariant index of [26] as giving essentially the holomorphic Euler characteristic of the spectral surface.

Now let us recall that the $\mathbf{Z}_2$ equivariant index $I = n_e - n_o$ of [26], counting the bundle moduli which are even respectively odd under the τ-involution, can be interpreted as giving essentially the holomorphic Euler characteristic of the spectral surface [15] which is

$$1 + h^{2,0}(C) - h^{1,0}(C) = \left.\frac{c_2(C) + c_1^2(C)}{12}\right|_C = \frac{c_2(Z)C + 2C^3}{12}$$
$$= n + \frac{n^3 - n}{6}c_1^2 + \frac{n}{2}\eta(\eta - nc_1) + \eta c_1. \quad \text{(C.18)}$$

---

[26]Unspecified Hodge numbers refer to X^4

Now one identifies the number of local complex deformations $h^{2,0}(C)$ of C with n_e respectively the dimension $h^{1,0}(C)$ of $Jac(C) := Pic_0(C)$ with n_o.

In this way one gets from a spectrum comparison the following relations [3], [4] in a pure gauge case (the case referred to as seperation resp. codimension one (what concerns the discriminant) case in the main body of the paper)

$$
\begin{aligned}
h^{1,1} &= h^{1,1}(Z) + 1 + r \\
h^{2,1} &= n_o \\
h^{3,1} &= h^{2,1}(Z) + I + n_o + 1.
\end{aligned}
\tag{C.19}
$$

Now one has to realize explicitly the map providing the $(1,1)$ shift in Hodge classes. In the end this goes of course back to the additional $\mathbf{P}^1$ one has in F-theory over the heterotic side, as visible already in eight dimensions. More precisely we will reinterpret the spectral cover of B which describes the heterotic $SU(n)$ bundle in terms of a fibration of del Pezzo surfaces over B, where what were n points of C lying over $b \in B$ are then 'exceptional' curves[27] in the del Pezzo surface over b.

Note that the the effective replacing of the $\mathbf{P}^1$ classes by points accounts for the missing dimensions causing the mentioned $(1,1)$ shift in cohomology when comparing the dual results. The description in the E_k case is already well adapted to the F-theory picture of having a fibration $W \to B$ (for each bundle) of del Pezzo surfaces over B.

C.2   Brane-impurities

From the relations (C.19) one finds that $n_3 = n_5$ as follows [3]. First one expresses, from the heterotic identification, the Hodge numbers of X^4 purely in data of the common base B_2 (using Noether $12 = c_1^2 + c_2$ and $\chi(Z) = -60c_1^2$)

$$
h^{1,1}(X^4) = 12 - c_1^2 + r
$$

---

[27]i.e., rational curves l of self-intersection -1 which have intersection $+1$ with the ample anti-canonical class; for the (almost del Pezzo) case of dP_9 these are just sections of the elliptic fibration

$$h^{3,1}(X^4) = 12 + 29c_1^2 + I + n_o \tag{C.20}$$

and one next inserts the expression for the index I resulting from (C.18)

$$I = n - 1 + \frac{n^3 - n}{6}c_1^2 + \frac{n}{2}\eta(\eta - nc_1) + \eta c_1. \tag{C.21}$$

Then one re-expresses I with $c_2(V_i)$

$$I = rk_i - 4\left(c_2(V_i) - \eta_i\sigma + \frac{1}{2}\pi_*\gamma_i^2\right) + \eta_i c_1 \tag{C.22}$$

which gives with $rk_1 + rk_2 = rk = 16 - r$ and $\eta_1 + \eta_2 = 12c_1$ for the total index

$$
\begin{aligned}
I &= rk + 48c_1\sigma + 12c_1^2 - 4(c_2(V_1) + c_2(V_2)) - 4\left(\frac{1}{2}\pi_*\gamma_1^2 + \frac{1}{2}\pi_*\gamma_2^2\right) \\
&= rk - 48 - 28c_1^2 + 4n_5 - 4\left(\frac{1}{2}\pi_*\gamma_1^2 + \frac{1}{2}\pi_*\gamma_2^2\right)
\end{aligned}
\tag{C.23}
$$

giving finally

$$
\begin{aligned}
n_3 + \frac{1}{2}G^2 &= \frac{\chi(X^4)}{24} = 2 + \frac{1}{4}(h^{1,1}(X^4) - h^{2,1}(X^4) + h^{3,1}(X^4)) \\
&= n_5 - \left(\frac{1}{2}\pi_*\gamma_1^2 + \frac{1}{2}\pi_*\gamma_2^2\right).
\end{aligned}
\tag{C.24}
$$

C.3 Stable degeneration and the map from the heterotic theory to F-theory

We consider the stable degeneration [26], [8], [6] $X^4 \to X_{deg}^4 = W_1 \cup_Z W_2$ where the W_i are fibered by del Pezzo surfaces over B. The 8D picture involves a $K3$ degenerating into the union of two rational elliptic surfaces (dP_9, almost del Pezzo). The base of the fibration is the union of two projective lines intersecting in a point Q over which a common elliptic curve E is fibered; roughly speaking the two E_8 contributions in the $K3$ are separated. Recall that the chosen $K3$ had Picard number two with section and fiber as the two algebraic cycles (still allowing 18 deformations which match the heterotic side) and the transcendental lattice $E_8 \oplus E_8 \oplus H$, with H the 2-dimensional hyperbolic plane, which leads to the 18-dimensional space $S := SO(2,18)/SO(2) \times SO(18)$

divided by the appropriate discrete group; one specializes then to two E_8 singularities at positions $z = 0, \infty$ in the P^1 base, which after the 'separation' in two surfaces are again re-smoothed; imagine to take (for the dP_9's to arise) the two f_4, g_6 parts at $z = 0, \infty$ of the original Weierstrass data f_8, g_{12} of the $K3$.

This corresponds on the heterotic side to the large area degeneration of a T^2 of the same complex structure parameter as E [26], [8]. Imagine that the H and its counterpart in S above correspond to the degrees of freedom represented by the complex structure modulus τ and the area ($+ B$-field) modulus ρ of E; then in the $\rho \to i\infty$ limit one finds in the corresponding boundary component of the quotient (discrete$\backslash S$) the two spaces $(W_{E_8} \backslash (E_i \otimes \Lambda_c))$, 'glued' together by $\tau(E_1) = \tau(E_2)$, describing the moduli of the two dP_9's (Λ_c the coroot lattice of E_8). The heterotic invariant $n_5 = c_2 Z - c_2 V_1 - c_2 V_2$ is then mirrored on the F-theory side by $n_3 = -\frac{\chi(Z)}{24} + \frac{\chi(W_1)}{24} + \frac{\chi(W_2)}{24}$.

Note that the (even) deformations of V_i correspond to those deformations of W_i which preserve fiberwise the elliptic curve E common with the heterotic side, so preserving in total the Calabi-Yau Z common to the W_i: their number is given by the dimension of $H^1(W_i, T_{W_i} \otimes \mathcal{O}(-Z)) \cong H^{3,1}(W_i)$. These are the deformations in the stable degeneration of X^4 which are relevant to the respective bundle. Further under the stable degeneration $J^3(X)$ splits off the abelian varieties $J^3(W_i)$, which contain the pieces relevant for the comparison. So essentially this construction interprets those elements of $H^{2,2}_{prim}(X^4, \mathbf{Z})$ that are 'captured by' the corresponding parts in the W_i cohomology[28]. concerning complex structure deformations note that the distribution into deformations of Z respectively those deformations $H^1(W_i, T_{W_i} \otimes \mathcal{O}(-Z)) \cong H^{3,1}(W_i)$ of W_i, which preserve Z, reflects the well known distribution of the deforming monomials of the defining F-theory equation for X^4 into "middle-polynomials" and the rest.

In the representation of the bundle via the del Pezzo construction respectively in the stable degeneration on the F-theory side the data are already in a form appropriate for comparison. For example in the case of E_8 bundles one has just to blow down the section of the dP_9 fibre on the F-theory side to get the dP_8 fibre of the heterotic side showing

---

[28]for the relation of the primitiveness condition to the W, Z geometry cf. the discussion in the section on the four-flux

the relation of the cohomologies and the intermediate jacobians (cf. [15]). For a bundle of group $H \neq E_8$ the section $\theta : B \to X^4$ of G-singularities in the F-theory setup corresponds (assuming G to be simply-laced) to having a bundle of unbroken gauge group G, i.e., an H bundle where H is the commutant of G in E_8, over the heterotic Calabi-Yau Z respectively having[29] a section $s : B \to \mathcal{W}_H = \mathcal{M}_{Z/B}$ or, as the fibre of $\mathcal{W}_H$ over $b \in B$ parametrizes the corresponding del Pezzo surfaces, a bundle $W_H^{het} \to B$ of del Pezzo surfaces dP_H fibered over B. So, if[30] one considers heterotically actually a dP_8 fibration with G singularity instead of the dP_H fibration, then locally at θ, i.e., at the singularity along B (local in the dP fibre and global along B), the picture in the $K3$ fibre of $X^4 \to B$ respectively the dP fibre on the heterotic side is the same.

C.4   Comparison of the discrete data

Of course the ultimate goal will be to make the complex two-cycle supporting the four-flux explicit and check the choice with a dual heterotic situation. Note that, comparing the heterotic contribution of γ^2 in eq. (1.5)

$$n_{5,\gamma} = n_{5,\gamma=0} + \frac{1}{2}\pi_*(\gamma^2) \tag{C.25}$$

with the formula [16]

$$n_3 = \frac{\chi(X^4)}{24} - \frac{1}{2}G^2 , \tag{C.26}$$

we are led to expect an association letting γ_i correspond with G_i giving

$$\pi_*^i(\gamma_i^2) = -G_i^2. \tag{C.27}$$

[29]at least locally over the dense open subset of B where fibres correspond to semistable bundles

[30]this can be done as we have an ADE system of rational (-2) curves lying in $H^{1,1}(K3,\mathbf{Z})$ as well as in $H^{1,1}(dP_8,\mathbf{Z})^{\perp_F}$ (in the case of the E-series, say; F the elliptic curve representing the ample anticanonical divisor); note that the complex structures for dP_H are given by homomorphisms $H^{1,1}(dP_H,\mathbf{Z})^{\perp_F} \to F$ and the complex structures for dP_8 keeping the G singularity are similarly given by the corresponding homomorphisms for dP_8 mapping the G system of rational (-2) curves to zero (i.e., they essentially describe a mapping for the H-part)

Note that two other facts fit with this association of γ and G. *First* the shifted integrality (to half-integrality): the analogy in the data concerned with the discrete part of the twisting degrees of freedom (cf. below) is represented in the following juxtaposition: on the heterotic side one has (cf. sect. (B.1))

$$\gamma = \frac{c_1(C) - \pi^* c_1(B)}{2} + c_1(\mathcal{L}), \qquad (C.28)$$

where the last term is an element of integral cohomology whereas the square root $(K_C^{-1} \otimes K_B)^{\frac{1}{2}}$ does not necessarily exist as a line bundle. Similarly one has on the F-theory side [53]

$$G = \frac{c_2}{2} + \alpha \qquad (C.29)$$

where $\alpha \in H^4(X, \mathbf{Z})$, but c_2 is not necessarily even. Strictly speaking one should consider here the individual G_i $(i = 1, 2)$ corresponding to the γ_i by means of the association provided by the stable degeneration (cf. the introduction to this section).

Secondly the restriction to the subspace *ker* π respectively primitiveness: the G admissible in an $N = 1$ supersymmetric compactification are in $ker(J \wedge \cdot)$ [9]. The last condition comes down for the relevant projected classes in $H^{2,2}(W)$ to the following: on the heterotic side the actual spectral cover construction will in the E_8 case involve the corresponding fibration of dP_8 surfaces over B (the section of dP_9 blown down); now, the embeddings of the 8D heterotic elliptic curves in the 8D del Pezzos patch together to an embedding of Z in the W_i, giving a map $H^{p,q}(W) \to H^{p,q}(Z)$; but for the dP_8 the anticanonical class given by the elliptic curve E is ample, so actually the *ker* $(J \wedge \cdot)$ condition leads to a *ker* $\cdot |_Z : (H^{2,2}(W) \to H^{2,2}(Z))$ condition, respectively, if one combines with the integration over the fibre, in a $ker(H^{2,2}(W) \to H^{2,2}(Z) \to H^{1,1}(B))$ condition; one has then to divide out the class dual to S_b, the del Pezzo fibre of $pr : W \to B$, corresponding to a differential form supported on the base, which is mapped to zero in the integration over the $\pi : Z \to B$ fibre. So finally the space we are concerned with is the $(ker : W \to B)/S_b \mathbf{Z}$ part in $H^{2,2}(W)$. So the primitiveness condition is the analogue of the condition *ker* $\pi : H^{1,1}(C, \mathbf{Z}) \to H^{1,1}(B, \mathbf{Z})$ on γ.

This fits in and actually completes the general scheme of a duality dictionary beyond the previously considered cases of relating $h^{2,0}(C)$

and $h^{3,1}(X^4)$ respectively elements of $H^{1,0}(C)$ and $H^{2,1}(X^4)$ (cf. [12], [26] and the appendix).

Now one has to realize explicitly the map providing the $(1,1)$ shift in Hodge classes. A naive[31] way to obtain the association of γ_i with G_i is via the cylinder map [34],[35]. This replaces each point in C lying over $b \in B$ by a complex projective line L lying above in the del Pezzo surface over b. Indeed, $L^2 = -1$, suggesting the desired relation (C.27).

D   6D computations

We list some spectrum and Euler number computations in 6D for the I_n series.

D.1   Heterotic spectra

The Euler numbers (cf. [13]) match with the heterotic expectations for the spectrum (note that the spectra have to fulfil the gravitational anomaly condition $244 + n_V = n_H$ (here occur the fundamental matter and the antisymmetric tensors) and that always $h^{1,1}(Z) = 3 + (k-1)$, $h^{2,1}(Z) = n_H - 1$ and $n_V = k^2 - 1$):

$$\mathbf{I_2} \qquad\qquad (16 + 6n)(2) - 3 = 29 + 12n$$

$$\dim_Q(\mathcal{M}_{inst}^{(n_1;n_2)}) + h^{1,1}(K3) = 166 + 20 = 186$$

$$n_H^0 = 215 + 12n$$

$$244 + 3 = 215 + 12n + 2(16 - 6n)$$

$$\mathbf{I_3} \qquad\qquad (18 + 6n)(3) - 8 = 46 + 18n$$

$$\dim_Q(\mathcal{M}_{inst}^{(n_1;n_2)}) + h^{1,1}(K3) = 132 + 20 = 152$$

$$n_H^0 = 198 + 18n$$

$$244 + 8 = 198 + 18n + 3(2 - n)$$

---

[31]More precisely the right hand side of (C.27) gets contributions also from distinct lines which intersect in the del Pezzo surface. $H^i(C)$ breaks into several isotypic pieces (five of them, for E_8). The values of γ coming from bundles all live in one of these pieces, where the cylinder map changes the intersection numbers by a factor of -60 (for E_8); so the correct association sends γ to $\frac{1}{60}$ times its cylinder.

$$+ 3(16 - 5n)$$

$\mathbf{I_4}$ $\quad (2+n)(6) + (16+4n)(4) - 15 = 61 + 22n$

$$\dim_{Q}(\mathcal{M}_{inst}^{(n_1;n_2)}) + h^{1,1}(K3) = 102 + 20 = 122$$

$$n_H^0 = 183 + 22n$$

$$244 + 15 = 183 + 22n + 6(2 - n)$$

$$+ 4(16 - 4n)$$

$\mathbf{I_5}$ $\quad (16+3n)(5) + (2+n)(10) - 24 = 76 + 25n$

$$\dim_{Q}(\mathcal{M}_{inst}^{(n_1;n_2)}) + h^{1,1}(K3) = 72 + 20 = 92$$

$$n_H^0 = 168 + 25n$$

$$244 + 24 = 168 + 25n + 10(2 - n)$$

$$+ 5(16 - 3n)$$

$\mathbf{I_6}$ $\quad (16+2n)(6) + (2+n)(15) - 35 = 91 + 27n$

$$\dim_{Q}(\mathcal{M}_{inst}^{(n_1^{(1)}, n_1^{(2)};n_2)}) + h^{1,1}(K3) = 42 + 20 = 62$$

$$n_H^0 = 153 + 27n$$

$$244 + 35 = 153 + 27n + 15(2 - n)$$

$$+ 6(16 - 2n)$$

(note that one gets $G = A_5 = I_6$ from an $SU(2) \times SU(3)$ bundle).

D.2 Discriminant equations

We consider now in detail the discriminant equations (using the notation $f_i := f_{4c_1 - it}$, $g_i := g_{6c_1 - it}$ and $c := 32 \cdot 864$).

The $\mathbf{I_2}$ case

The ansatz with $H = H_{2c_1 - 2t}$

$$
\begin{aligned}
f &= \frac{1}{48}(-H^2 + f_3 z) \\
g &= \frac{1}{864}(H^3 + g_5 z + g_4 z^2)
\end{aligned}
\tag{D.30}
$$

gives, because of the z-linear term $H^3(2g_5 + 3f_3 H)$ and the thereby enforced choice $g_5 = -\frac{3}{2}f_3 H$, that $D_1 r = 2H + P_{8c_1 - 6t}$ with $P =$

$-\frac{3}{4}f_3^2 + 2g_4 H$ as

$$c \cdot \left(4f^3 + 27g^2\right) = z^2 \left[\left(H^2\left(-\frac{3}{4}f_3^2 + 2g_4 H\right)\right) + \left(f_3^3 - 3f_3 g_4 H\right) z + \left(g_4^2\right) z^2\right]. \tag{D.31}$$

The I_3 case

The ansatz

$$\begin{aligned}
f &= \frac{1}{48}(-h^4 + f_3 z + f_2 z^2) \\
g &= \frac{1}{864}(h^6 + g_5 z + g_4 z^2 + g_3 z^3)
\end{aligned} \tag{D.32}$$

gives

$$\begin{aligned}
&c \cdot \left(4f^3 + 27g^2\right) \\
&= h^6(2g_5 + 3f_3 h^2)z \\
&\quad + (g_5^2 - 3f_3^2 h^4 + 2g_4 h^6 + 3f_2 h^8)z^2 \\
&\quad + (f_3^3 + 2g_5 g_4 - 6f_3 f_2 h^4 + 2g_3 h^6)z^3 + \mathcal{O}(z^4). \tag{D.33}
\end{aligned}$$

Solving for the I_3 condition gives at first $g_5 = -\frac{3}{2}f_3 h^2$ and then $f_3^2 = \frac{4}{3}h^2(2g_4 + 3f_2 h^2)$ which leads us to introduce $Q_{3c_1 - 2t}$ with $3Q_{3c_1 - 2t}^2 = 2g_4 + 3f_2 h^2$.

$$\begin{aligned}
g_5 &= -\frac{3}{2}f_3 h^2 = -3h^3 Q \\
g_4 &= \frac{3}{2}(Q^2 - f_2 h^2) \\
f_3 &= 2hQ \tag{D.34}
\end{aligned}$$

Then one gets

$$c \cdot \left(4f^3 + 27g^2\right) = z^3 \left[\left(h^3(-Q^3 - 3f_2 h^2 Q + 2g_3 h^3)\right) + \left(h^2\left(-\frac{3}{4}f_2^2 h^2 - 6g_3 hQ + \frac{15}{2}f_2 Q^2\right) + \frac{9}{4}Q^4\right)z\right.$$

$$+\left(h(-3f_2g_3h + 6f_2^2Q) + 3g_3Q^2\right)z^2$$
$$+\left(f_2^3 + g_3^2\right)z^3\Big].$$
(D.35)

The I_4 case

The ansatz

$$f = \frac{1}{48}(-h^4 + f_3z + f_2z^2 + f_1z^3)$$

$$g = \frac{1}{864}(h^6 + g_5z + g_4z^2 + g_3z^3 + g_2z^4 + g_1z^5)$$ (D.36)

gives

$$c \cdot \left(4f^3 + 27g^2\right)$$
$$= h^6(2g_5 + 3f_3h^2)z$$
$$+ (g_5^2 - 3f_3^2h^4 + 2g_4h^6 + 3f_2h^8)z^2$$
$$+ (f_3^3 + 2g_5g_4 - 6f_3f_2h^4 + 2g_3h^6 + 3f_1h^8)z^3$$
$$+ (3f_2f_3^2 + g_4^2 + 2g_3g_5 + h^4(-3f_2^2 - 6f_1f_3 + 2g_2h^2))z^4$$
$$+ (3f_2^2f_3 + 3f_1f_3^2 + 2g_3g_4 + 2g_2g_5 + h^4(-6f_1f_2 + 2g_1h^2))z^5$$
$$+ (f_2^3 + 6f_1f_2f_3 + g_3^2 + 2g_2g_4 + 2g_1g_5 - 3f_1^2h^4)z^6$$
$$+ (3f_1f_2^2 + 3f_1^2f_3 + 2g_2g_3 + 2g_1g_4)z^7$$
$$+ (3f_1^2f_2 + g_2^2 + 2g_1g_3)z^8$$
$$+ (f_1^3 + 2g_1g_2)z^9$$
$$+ g_1^2z^{10}.$$
(D.37)

Solving for the I_4 condition leads to introduction of H_{2c_1-t} with

$$g_5 = -3h^4H$$
$$g_4 = \frac{3}{2}h^2(H^2 - f_2)$$
$$g_3 = \frac{1}{2}H(H^2 + 3f_2) - \frac{3}{2}f_1h^2$$
$$f_3 = 2h^2H$$
(D.38)

thus giving

$$c \cdot \left(4f^3 + 27g^2\right)$$

$$= z^4 \left[h^4 \left(h^2(2g_2 - 3f_1H) - \frac{3}{4}H^4 - \frac{3}{2}f_2H^2 - \frac{3}{4}f_2^2 \right) \right.$$
$$+ h^2 \left(2g_1h^4 + h^2 \left(-\frac{3}{2}f_1f_2 - 6g_2H + \frac{15}{2}f_1H^2 \right) \right.$$
$$\left. + \frac{3}{2}f_2^2H + 3f_2H^3 + \frac{3}{2}H^5 \right) z$$
$$+ \left(-h^4 \left(6g_1H + \frac{3}{4}f_1^2 \right) + h^2 \left(3g_2H^2 - 3f_2g_2 + \frac{15}{2}f_1f_2H - \frac{3}{2}f_1H^3 \right) \right.$$
$$\left. + f_2^3 + \frac{9}{4}f_2^2H^2 + \frac{3}{2}f_2H^4 + \frac{1}{4}H^6 \right) z^2$$
$$+ \left(h^2(3f_1g_2 - 3f_2g_1 + 6f_1^2H + 3g_1H^2) + 3f_1f_2^2 + 3f_2g_2H + g_2H^3 \right) z^3$$
$$+ \left(-3f_1g_1h^2 + 3f_1^2f_2 + g_2^2 + 3f_2g_1H + g_1H^3 \right) z^4$$
$$+ \left(f_1^3 + 2g_1g_2 \right) z^5$$
$$\left. + g_1^2 \, z^6 \right]. \tag{D.39}$$

$\mathbf{I_5}$ case

So starting from the ansatz

$$F = \frac{1}{48}(-h^4 + f_3z + f_2z^2 + f_1z^3)$$
$$G = \frac{1}{864}(h^6 + g_5z + g_4z^2 + g_3z^3 + g_2z^4 + g_1z^5) \tag{D.40}$$

one has made sure that the constant (z-free) term has already cancelled:

$$c \left(4F^3 + 27G^2 \right)$$
$$= h^6(3h^2f_3 + 2g_5)z$$
$$\quad + (g_5^2 + h^4(3h^4f_2 + 2h^2g_4 - 3f_3^2))z^2$$
$$\quad + (f_3^3 + 2g_5g_4 + h^4(3h^4f_1 + 2h^2g_3 - 6f_3f_2))z^3$$
$$\quad + (3f_3^2f_2 + g_4^2 + 2g_5g_3 + h^4(2h^2g_2 - 3f_2^2 - 6f_3f_1))z^4$$
$$\quad + (3f_3f_2^2 + 3f_3^2f_1 + 2g_4g_3 + 2g_5g_2 + 2h^4(h^2g_1 - 3f_2f_1))z^5$$
$$\quad + (f_2^3 + 6f_3f_2f_1 + g_3^2 + 2g_4g_2 + 2g_5g_1 - 3h^4f_1^2)z^6$$
$$\quad + (3f_2^2f_1 + 3f_3f_1^2 + 2g_3g_2 + 2g_4g_1)z^7$$
$$\quad + (3f_2f_1^2 + g_2^2 + 2g_3g_1)z^8$$
$$\quad + (f_1^3 + 2g_2g_1)z^9$$

$$+ g_1^2 z^{10}. \tag{D.41}$$

To get actually I_5 the terms up to z^4 have to cancel. For this one solves for $g_5, g_4, g_3, g_2, f_3, f_2$ in terms of $h_{c_1-t}, H_{2c_1-t}, q_{3c_1-t}, f_1 = f_{4c_1-t}, g_1 = g_{6c_1-t}$ with the terms H and q and thereby gets succesively the following relations (in the expression in front of z^5 *all* terms finally come with an explicit h^4 factor)

$$
\begin{aligned}
g_5 &= -3h^4 H \\
g_4 &= 3h^2(H^2 - hq) \\
g_3 &= \frac{3}{2}h(2Hq - hf_1) - H^3 \\
g_2 &= \frac{3}{2}(f_1 H + q^2) \\
f_3 &= 2h^2 H \\
f_2 &= 2hq - H^2.
\end{aligned} \tag{D.42}
$$

So the discriminant equation Δ presents itself now in the manifest I_5 form

$$
\begin{aligned}
\Delta = z^5 \Bigg[& h^4(2h^2 g_1 - 3f_1 qh - 3Hq^2) \\
& + h^2\left(-3h^2\left(\frac{1}{4}f_1^2 + 2g_1 H\right) + qh(6f_1 H - q^2) + 6H^2 q^2 \right)z \\
& + \left(-6qh^3 g_1 + \frac{3}{2}h^2(Hf_1^2 + 5q^2 f_1 + 4H^2 g_1) \right. \\
& \qquad \left. + 3qHh(3q^2 - Hf_1) - 3q^2 H^3 \right)z^2 \\
& + \left(-3f_1 g_1 h^2 + 6hq(f_1^2 + g_1 H) - 2g_1 H^3 - \frac{3}{4}f_1^2 H^2 \right. \\
& \qquad \left. + \frac{9}{2}f_1 q^2 H + \frac{9}{4}q^4 \right)z^3 \\
& + \left(f_1^3 + 3g_1(f_1 H + q^2) \right)z^4 \\
& + g_1^2 z^5 \Bigg]
\end{aligned} \tag{D.43}
$$

giving

$$P = 2h^2 g_1 - 3f_1 qh - 3Hq^2. \tag{D.44}$$

The I_6 case

The ansatz

$$f = \frac{1}{48}(-h^4 + f_3 z + f_2 z^2 + f_1 z^3)$$

$$g = \frac{1}{864}(h^6 + g_5 z + g_4 z^2 + g_3 z^3 + g_2 z^4 + g_1 z^5) \qquad \text{(D.45)}$$

gives

$$
\begin{aligned}
c \cdot & \left(4f^3 + 27g^2\right) \\
= & \; h^6(2g_5 + 3f_3 h^2)z \\
& + (g_5^2 + h^4(-3f_3^2 + 2g_4 h^2 + 3f_2 h^4))z^2 \\
& + (f_3^3 + 2g_5 g_4 + h^4(-6f_3 f_2 + 2g_3 h^2 + 3f_1 h^4))z^3 \\
& + (3f_2 f_3^2 + g_4^2 + 2g_3 g_5 + h^4(-3f_2^2 - 6f_1 f_3 + 2g_2 h^2))z^4 \\
& + (3f_2^2 f_3 + 3f_1 f_3^2 + 2g_3 g_4 + 2g_2 g_5 + h^4(-6f_1 f_2 + 2g_1 h^2))z^5 \\
& + (f_2^3 + 6f_1 f_2 f_3 + g_3^2 + 2g_2 g_4 + 2g_1 g_5 - 3f_1^2 h^4)z^6 \\
& + (3f_1 f_2^2 + 3f_1^2 f_3 + 2g_2 g_3 + 2g_1 g_4)z^7 \\
& + (3f_1^2 f_2 + g_2^2 + 2g_1 g_3)z^8 \\
& + (f_1^3 + 2g_1 g_2)z^9 \\
& + g_1^2 z^{10}. \qquad \text{(D.46)}
\end{aligned}
$$

Solving for the I_6 condition leads at first to the identifications (D.42) and then with the condition that $P = 0$ to the introduction of $\mathcal{F} = \mathcal{F}_{2c_1}$ whose product with h replaces the old $q_{3c_1 - t}$, thereby leading to

$$
\begin{aligned}
g_5 &= -3h^4 H \\
g_4 &= 3h^2(H^2 - h^2 \mathcal{F}) \\
g_3 &= \frac{3}{2}h^2(2H\mathcal{F} - f_1) - H^3 \\
g_2 &= \frac{3}{2}(f_1 H + h^2 \mathcal{F}^2) \\
g_1 &= \frac{3}{2}\mathcal{F}(H\mathcal{F} + f_1) \\
f_3 &= 2h^2 H \\
f_2 &= (2h^2 \mathcal{F} - H^2) \qquad \text{(D.47)}
\end{aligned}
$$

giving

$$\Delta = z^6 \left[h^4 \left(-\mathcal{F}^3 h^2 - 3\left(\frac{1}{2}f_1 + \mathcal{F}H\right)^2 \right) \right.$$

$$+ 3h^2 \left(-\frac{1}{2}h^2 \mathcal{F}^2 f_1 + 2H \left(\frac{1}{2}f_1 + \mathcal{F}H \right)^2 \right) z$$

$$+ 3 \left(\mathcal{F}h^2 \left(\frac{3}{4}\mathcal{F}^3 h^2 + \frac{1}{2}f_1^2 + 3\mathcal{F}f_1 H + 3\mathcal{F}^2 H^2 \right) \right.$$

$$\left. - H^2 \left(\frac{1}{2}f_1 + \mathcal{F}H \right)^2 \right) z^2$$

$$+ \left(\frac{9}{2}\mathcal{F} \left(\mathcal{F}^2 h^2 (f_1 + \mathcal{F}H) + f_1 H (f_1 + \mathcal{F}H) \right) + f_1^3 \right) z^3$$

$$+ \frac{9}{2}\mathcal{F}^2 (f_1 + \mathcal{F}H)^2 z^4 \Bigg]. \tag{D.48}$$

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