Adv. Theor. Math. Phys. **5** (1999) 1289–1324

BF Description of Higher-Dimensional Gravity Theories

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Abstract

In the first-order formalism, pure three-dimensional gravity is just the BF theory. Similarly, four-dimensional general relativity can be formulated as BF theory with an additional constraint term added to the Lagrangian. In this paper we show

e-print archive: http://xxx.lanl.gov/hep-th/9901069

that the same is true also for higher-dimensional Einstein gravity: in any dimension gravity can be described as a constrained BF theory. Moreover, in any dimension these constraints are quadratic in the B field. After describing in detail the structure of these constraints, we scketch the "spin foam" quantization of these theories, which proves to be quite similar to the spin foam quantization of general relativity in three and four dimensions. In particular, in any dimension, we solve the quantum constraints and find the so-called simple representations and intertwiners. These exhibit a simple and beautiful structure that is common to all dimensions.

1 Introduction

In three spacetime dimensions Einstein's general relativity becomes a beautiful and simple theory. There are no local degrees of freedom, and gravity is an example of topological field theory. Owing to this fact, a variety of techniques from TQFT can be used, and a great deal is known about quantization of the theory. More precisely, when written in the first order formalism, three-dimensional gravity is just the BF theory, whose action is given by:

$$S_{\mathrm{BF}} = \int_{\mathcal{M}} \mathrm{Tr}(B \wedge F). \tag{1}$$

Here $\mathcal{M}$ is the spacetime manifold, F is the curvature of the spin connection, and B is the frame field one-form. The trace is taken in the Lie algebra of the relevant gauge group, which in the case of 3D is given by $\mathrm{SO}(2,1)$ for Lorentzian spacetimes and by $\mathrm{SO}(3)$ in the Euclidean case. The quantization of BF theory is well-understood, both canonically and by the path integral method, at least in the Euclidean case. This is one of the possible ways to construct quantum gravity in three spacetime dimensions: it exists as a topological field theory.

It is tempting to apply the beautiful quantization methods from TQFT to other, more complicated theories, including those with local degrees of freedom. An interesting proposal along these lines was made in a series of papers by Martellini and collaborators [9], who proposed to treat Yang-Mills theory as a certain deformation of the BF theory.

This gives an interesting picture of the confining phase of Yang-Mills theory.

Recently, a proposal was made suggesting a way to apply the ideas and methods from TQFT to four-dimensional gravity. The new approach to quantum gravity, for which the name "spin foam approach" was proposed in [2], lies on the intersection between TQFT and loop quantum gravity (see [18] for a recent review on "loop" gravity). As it was advocated by Rovelli and Reisenberger [17], the results of the "loop" approach suggest a possibility of constructing the partition function of 4D gravity as a "spin foam" model. The first "spin foam" model of 4D gravity was constructed by Reisenberger [14], and was intimately related to the self-dual canonical (loop) quantum gravity. Later, Barrett, Crane [4] and Baez [2] proposed another model based on the study of the geometry of a 4-simplex. Both spin foam models deeply use the fact that Einstein's theory in four dimensions can be rewritten as a BF theory with additional quadratic constraints. It is a constrained SU(2) BF theory [13, 5] in the self-dual case, and a constrained SO(4) BF theory [6] for the Barrett, Baez and Crane model. In both cases, the resulting quantum model is given by a certain deformation of the topological BF theory.

While the approach of Martellini et al. [9], which treats Yang-Mills theory as a deformation of the BF theory, is clearly not limited to any spacetime dimension, one might suspect that the similar strategy in the case of gravity works only in three and four dimensions. Indeed, it is believed that in order to quantize a theory in a way similar to the one used in TQFT, the theory must at least have the property that its phase space consists of pairs connection – conjugate electric field. However, already in the case of four dimensions, the fact that the gravitational phase space can be brought to the Yang-Mills form is quite non-trivial. In order to arrive to such a formulation one uses crucially the self-duality available in four dimensions [1]. Thus, one might suspect that the quantization techniques from TQFT that use the connection field as the main variable are limited only to gravity in three and four dimensions.

There is, however, one case that seems to contradict this negative conclusion: the case of the usual SO(4) first-order formulation of gravity in four dimensions. As we have mentioned above, this model can be written as a SO(4) BF theory with additional constraints guaranteeing that the B field comes from the frame field. This formulation serves as the starting point for the quantum model of Barrett, Baez and Crane, which does treat this theory as a deformation of the BF theory. On the other hand, the canonical formulation of this theory is known to contain second class constraints, solving which one does not seem to arrive to a phase space of the Yang-Mills type [3]. Thus, this theory provides us with a puzzle: on one hand, treating it covariantly as BF theory with constraints one can quantize it as a deformation of the topological BF theory, on the other hand one does not expect the methods from TQFT to work because the phase space of this theory is not that simple as that of Yang-Mills theory. While we do not know any simple resolution of this puzzle, it seems from details of the quantum theory that it uses the self-duality in some clever way and thus goes around the problem with the second class constraints of the canonical formulation.

This fact, yet to be understood in full details, opens door to a possibility of applying the "topological" quantization procedure to gravity theories in higher dimensions, hoping that the covariant quantization will be able to go around the problem with second class constraints that are known to be present also in this case. The first step that one has to take towards this goal is to reformulate a higher dimensional gravity theory as a BF theory with constraints. The main aim of this paper is to show that such a formulation is indeed possible. In the second part of the paper we shall study in some details the corresponding "spin foam" quantum theory.

Our results can be summarized as follows. First, in section 2, we show that in any dimension gravity can be written as an SO(D), or SO($D-1,1$), BF theory subjects to quadratic, non-derivative constraints on the B field. Namely we prove that gravity in D dimensions can be described by the following action functional:

$$S[A,B,\Phi] = \int_{\mathcal{M}} \mathrm{Tr}(B\wedge F) + \frac{1}{2}\mathrm{Tr}(B\wedge\Phi(B)). \tag{2}$$

Here A is an SO(D) -for spacetime of euclidean signature, or SO($D-1,1$) for spacetime of Minkowskian signature- connection field. The B field is a Lie algebra valued $(D-2)$-form and Φ is a Lagrange multiplier field that can be contracted in a special way (see below) with the $(D-$

2)-form B to produce a Lie-algebra valued 2-form, denoted by $\Phi(B)$ in (2). Let us emphasize that the Lagrange multiplier term of the action is quadratic in B in any dimension. The precise form of this term will be given below. As we show in the next section, varying this action with respect to the Lagrange multipliers, one obtains equations that guarantee that the B field comes from a frame field e:

$$B = *(e \wedge e).$$

For such B, the action (2) is just the usual action for gravity in the first-order formulation. This means that the theory described by (2) is indeed equivalent to gravity, in the sense that all solutions of Einstein's theory are also solutions of (2).

The second part of our paper is devoted to quantum theory. We study a quantization of the theory described by (2) along the lines of Refs. [4, 2, 8]. This quantization procedure, which can be called a "spin foam" quantization, will be summarized in some details in section 3. For now, let us just note that in this quantization the B field is promoted to a derivative operator acting on the so-called spin networks. The quadratic constraints become constraints on representations and intertwiners labelling the spin networks. Representations satisfying these constraints will be called, following [2], *simple*. In four dimensions simple representation of $\mathrm{SO}(4) \approx \mathrm{SU}(2) \times \mathrm{SU}(2)$ were found [4, 2] to be the ones of the type (j, j), that is, the ones carrying the same spin under the left and right copies of $\mathrm{SU}(2)$. In section 3 we find all possible simple representations in any dimension. Surprisingly, it turns out that the simple representations are in certain precise sense the simplest possible representations of the gauge group. We find that in any dimension these representations are labeled just by a single parameter. Also, in any dimension, we construct an intertwiner satisfying the intersection constraints.

2 Classical theory

This section is devoted to an analysis of the classical theory. We will first present the action in several equivalent formulations and then prove that it is equivalent to the standard Einstein-Hilbert action. In subsection 2.2 we discuss in detail the issue of dependence between the constraints. In this section the gauge group can be taken to be either that corresponding to Euclidean signature or to Lorentzian: all our proofs are independent of this. For definiteness, we will work with the Euclidean version, in which the gauge group is SO(D), for this is what is used in the quantum part of our paper.

2.1 The action

The action for gravity in the BF formulation is a functional of the B field, the connection form A, and Lagrange multipliers Φ. There are two equivalent formulations, which are both worth mentioning. In the first formulation, which is more customary in the context of BF theories, the B field is thought of as a Lie algebra valued $(D-2)$-form. In the second formulation one uses the metric-independent Levi-Civita density to construct from this $(D-2)$-form a densitized rank two antisymmetric covariant tensor, which we will call a bivector. We first present the action in this second formulation, for it looks exactly the same in any dimension $D \geq 4$. Thus, we start by writing B as a bivector $\tilde{B}^{\mu\nu}_{ij}$, where Greek characters are the spacetime indices, latin letters are the internal indices, and a single tilde over the symbol of B represents the fact that its density weight is one.

The action of the theory is then given by:

$$S[A,\tilde{B},\underset{\sim}{\Phi}] = \int d^D x \; \tilde{B}^{\mu\nu}_{ij} F^{ij}_{\mu\nu} + \frac{1}{2} \underset{\sim}{\Phi}^{ijkl}_{\mu\nu\rho\sigma} \tilde{B}^{\mu\nu}_{ij} \tilde{B}^{\rho\sigma}_{kl} . \tag{3}$$

The action is a functional of an SO(D) gauge field A^{ij}_μ, bivector fields $\tilde{B}^{\mu\nu}_{ij}$, and Lagrange multiplier fields $\underset{\sim}{\Phi}^{ijkl}_{\mu\nu\rho\sigma}$. This action is generally covariant: the bivector fields scale as tensor densities of weight one, while the multipliers scale as densities of weight minus one, which is represented by a single tilde below the symbol 'Φ'.

In order to ensure the relation to gravity, the multiplier field $\underset{\sim}{\Phi}^{ijkl}_{\mu\nu\rho\sigma}$ must be such that it is completely anti-symmetric in one set of indices, and its anti-symmetrization on the other set of indices vanishes. There is a freedom, however, on which set of indices the anti-symmetrization is taken to vanish. It turns out to be more convenient for the quantum theory to choose the anti-symmetrization on the spacetime indices to vanish. This is the choice we make. Let us emphasize, however, that from the point of view of the classical theory the two possibilities are completely equivalent in the sense that they are both enough to guarantee the simplicity of the B field (for a generic, non-degenerate field B).

The postulated properties of the Lagrange multiplier field Φ imply that it is of the form:

$$\underset{\sim}{\Phi}^{ijkl}_{\mu\nu\rho\sigma} = \epsilon^{[m]ijkl}\underset{\sim}{\Phi}_{[m]\mu\nu\rho\sigma},$$

where $\epsilon^{[m]ijkl}$ is the totally anti-symmetric form on the Lie algebra, $[m]$ is a completely anti-symmetric cumulative index of length $D-4$, and $\underset{\sim}{\Phi}_{[m]\mu\nu\rho\sigma}$ is a new Lagrange multiplier field, which we, by abuse of notation, also call Φ. This new Lagrange multiplier field also has density weight minus one. The field $\underset{\sim}{\Phi}_{[m]\mu\nu\rho\sigma}$ has a property that its anti-symmetrization on the spacetime indices vanishes:

$$\underset{\sim}{\Phi}_{[m][\mu\nu\rho\sigma]} = 0. \tag{4}$$

Using this new set of Lagrange multipliers the action can be written as:

$$S[A,B,\Phi] = \int d^Dx\ \tilde{B}^{\mu\nu}_{ij}F^{ij}_{\mu\nu} + \frac{1}{2}\underset{\sim}{\Phi}_{[m]\mu\nu\rho\sigma}\epsilon^{[m]ijkl}\tilde{B}^{\mu\nu}_{ij}\tilde{B}^{\rho\sigma}_{kl}. \tag{5}$$

Let us now give another way the action (5) can be written, using the representation of the B field as a $(D-2)$-form. This is more standard in the context of BF theories. Using the definition of the bivector $\tilde{B}^{\mu\nu}$,

$$\tilde{B}^{\mu\nu}_{ij} = \frac{1}{2!(D-2)!}\tilde{\epsilon}^{\mu\nu\beta_1\ldots\beta_{D-2}}B_{\beta_1\ldots\beta_{D-2}\,ij}, \tag{6}$$

one can easily check that the action (5) can be rewritten as

$$\begin{aligned} S[A,B,\Phi] = \frac{1}{2!(D-2)!}\int d^Dx\ & B_{\beta_1\ldots\beta_{D-2}\,ij}F^{ij}_{\mu\nu}\tilde{\epsilon}^{\beta_1\ldots\beta_{D-2}\mu\nu} \\ & + \frac{1}{2}B_{\beta_1\ldots\beta_{D-2}\,ij}\Phi^{ij}_{\mu\nu}(B)\tilde{\epsilon}^{\beta_1\ldots\beta_{D-2}\mu\nu}, \end{aligned} \tag{7}$$

where we have introduced a new two-form field $\Phi(B)$ with values in the Lie algebra. In the index notation it is given by:

$$\Phi^{ij}_{\mu\nu}(B) := \underset{\sim}{\Phi}^{ijkl}_{\mu\nu\rho\sigma}\tilde{B}^{\rho\sigma}_{kl}. \tag{8}$$

Thus, in the abstract notations, one can write the action as

$$\int_{\mathcal{M}} \mathrm{Tr}(B \wedge F) + \frac{1}{2}\mathrm{Tr}(B \wedge \Phi(B)). \tag{9}$$

Thus, there are two equivalent formulations of the theory. One can use the formulation in terms of forms, given by (9), or the formulation in terms of bivectors, given by (3). In what follows, we will mostly use the formulation in terms of bivectors.

Variation of the action (5) with respect to Φ gives the following equations:

$$\epsilon^{[m]ijkl}\tilde{B}^{\mu\nu}_{ij}\tilde{B}^{\rho\sigma}_{kl} = \tilde{\epsilon}^{[\alpha]\mu\nu\rho\sigma}\tilde{c}^{[m]}_{[\alpha]} \tag{10}$$

for some coefficients $\tilde{c}^{[m]}_{[\alpha]}$. Here $[m], [\alpha]$ are cumulative anti-symmetric indices of length $D-4$, Lie algebra and spacetime ones correspondingly. As one can see, when equations (10) are satisfied, the coefficients $\tilde{c}^{[m]}_{[\alpha]}$ are given by:

$$\tilde{c}^{[m]}_{[\alpha]} = \frac{1}{(D-4)!4!}\epsilon^{[m]ijkl}\tilde{B}^{\mu\nu}_{ij}\tilde{B}^{\rho\sigma}_{kl}\,\underset{\sim}{\epsilon}_{[\alpha]\mu\nu\rho\sigma}. \tag{11}$$

The bivector field $\tilde{B}$ can be viewed as a linear map from the space of spacetime two-forms to the space of densitized internal two-forms: $\tilde{B}_{ij}(\theta) \equiv \tilde{B}^{\mu\nu}_{ij}\theta_{\mu\nu}$. We will say that B is generic (or non-degenerate) if this map is invertible.

It is clear that when B comes from a frame field e, B identically satisfies (10). The following theorem states that the reverse is true.

Theorem 1. *In dimension $D > 4$ a generic B field satisfies the constraints (10) if and only if it comes from a frame field. In other words, a non-degenerate B satisfies the constraints (10) if and only if there exist e^{μ}_{i} such that:*

$$\tilde{B}^{\mu\nu}_{ij} = \pm|e|e^{[\mu}_{i}e^{\nu]}_{j}, \tag{12}$$

where $|e|$ is the absolute value of the determinant of the matrix e^{μ}_{i}.

The condition $D > 4$ is there because in four dimensions, under the same assumptions, there is another solution (see [6]) given by:

$$\tilde{B}^{\mu\nu}_{ij} = \pm|e|\epsilon_{ij}{}^{kl}e^{[\mu}_{k}e^{\nu]}_{l}. \tag{13}$$

Thus, our theorem, in particular, claims that this other solution appears only in four dimensions.

Proof. The constraints (10) can be conveniently subdivided into the following categories:

$$\begin{aligned}
&\text{simplicity:} && \tilde{B}^{\mu\nu}_{[ij}\tilde{B}^{\mu\nu}_{kl]} = 0 && \mu, \nu \,\text{distinct} && (14)\\
&\text{intersection:} && \tilde{B}^{\mu\nu}_{[ij}\tilde{B}^{\nu\rho}_{kl]} = 0 && \mu, \nu, \rho \,\text{distinct} && (15)\\
&\text{normalization:} && \tilde{B}^{\mu\nu}_{[ij}\tilde{B}^{\rho\sigma}_{kl]} = \tilde{B}^{\mu\rho}_{[ij}\tilde{B}^{\sigma\nu}_{kl]} && \mu, \nu, \rho, \sigma \,\text{distinct} && (16)
\end{aligned}$$

The reason for this terminology has to do with the conditions imposed by the various constraints.

In appendix A we prove the following two propositions. The first proposition states that imposing the simplicity condition on a non-zero two-form B_{ij} is equivalent to demanding that the two-form is simple, or, in other words, that it factors as the outer product of one-forms:

$$B_{[ij}B_{kl]} = 0 \quad \Leftrightarrow \quad B_{ij} = u_{[i}v_{j]}. \tag{17}$$

Note that we have omitted the density weight of B_{ij} in the above expression. For the discussion that follows, where we treat B_{ij} as a Lie-algebra two-form, the density weight of B is irrelevant.

The second proposition states that the intersection condition on a pair of simple two-forms ensures that they share a common one-form factorizing both of them:

$$B_{[ij}B'_{kl]} = 0 \quad \Leftrightarrow \quad B_{ij} = u_{[i}v_{j]} \;\text{ and }\; B'_{ij} = v_{[i}w_{j]}. \tag{18}$$

Moreover, the common factor v_i is uniquely determined up to scaling when B and B' are not proportional to each other. In case B and B' are proportional to each other, the above statement trivially holds, but the common form v_i is not determined uniquely: any linear combination of it with the other one-form is also a common form.

Let us now discuss the meaning of the normalization condition. Imposing the normalization condition on two pairs of simple two-forms, each pair of which is constructed by taking different outer products of the same 4 one-forms, fixes the relative normalization of the two two-forms. In other words, given four simple two-forms

$$\begin{aligned} B_{ij} &= N u_{[i} v_{j]} \\ B'_{ij} &= N' w_{[i} z_{j]} \\ B''_{ij} &= N'' u_{[i} w_{j]} \\ B'''_{ij} &= N''' z_{[i} v_{j]}, \end{aligned} \tag{19}$$

the conditions

$$B_{[ij} B'_{kl]} = B''_{[ij} B'''_{kl]}$$

imply that $NN' = N''N'''$ as long as the four vectors are linearly independent.

Let us now see what these assertions imply for our theory. First, consider a set of two-forms $B^{12}_{ij}, B^{13}_{ij}, \dots B^{1D}_{ij}$. According to the simplicity relations, each of these two-forms factors into one-forms, and according to the intersection relations each pair shares a unique common factor. Note that our assumption that B is generic implies that all two-forms $B^{\mu\nu}_{ij}$ are non-zero and that B's are not proportional to each other. Let v_i be the non-zero one-form shared by B^{12}_{ij} and B^{13}_{ij}; w_i be the one-form shared by B^{13}_{ij} and B^{14}_{ij}; u_i be the one-form shared by B^{14}_{ij} and B^{12}_{ij}. Then there are three possibilities: the one-forms u_i, v_i, w_i span a linear space of rank (i) 3; (ii) 2; (iii) 1. Let us consider each case separately.

Case (i). Since u_i and v_i are distinct one-forms that both divide B^{12}_{ij}, this two-form is given by a product of u_i, v_i: $B^{12}_{ij} = c u_{[i} v_{j]}$. Likewise, we can express the remaining two bivectors completely in terms of our three vectors. Thus, we have:

$$\begin{aligned} B^{12}_{ij} &= c u_{[i} v_{j]} \\ B^{13}_{ij} &= c' v_{[i} w_{j]} \\ B^{14}_{ij} &= c'' w_{[i} u_{j]}. \end{aligned} \tag{20}$$

Case (ii). Let us assume, without loss of generality, that $v_i = w_i$. Then this vector divides all B^{12}, B^{13} and B^{14}, but u_i divides both B^{12} and B^{14}. So B^{12} and B^{14} are proportional, which is excluded for a generic B.

Case (iii). Let us assume, without loss of generality, that $u_i = v_i = w_i$. Then, from the definitions, we see that this vector must divide all three bivectors. Thus, we can write

$$\begin{aligned} B^{12}_{ij} &= u_{[i}p_{j]} \\ B^{13}_{ij} &= u_{[i}q_{j]} \\ B^{14}_{ij} &= u_{[i}r_{j]} \end{aligned} \tag{21}$$

for some suitable vectors p_i, q_i, r_i.

In four dimensions, the case (i) was associated with the so-called topological sector (13) (see [6] for a discussion on this sector), while the case (iii) was associated with the gravity sector. In dimension higher than four, however, the case (i) cannot occur since we have, for instance, the two-form B^{15}_{ij} to reckon with. This two-form must have a factor in common with the three two-forms considered previously. In the case (i) , there is no factor in common between the three two-forms. So the only possibility is that B^{15} is proportional to for instance B^{12} which is not possible for a generic B. Thus, we are forced to the case (iii), in which, if we assume that the three bivectors are distinct, the only common factor is u_i. We then conclude that u_i divides B^{15}_{ij} as well. Continuing this reasoning, we see that u_i must divide all two-forms $B^{1\nu}_{ij}$.

Repeating the above arguments with different values for spacetime indices, we conclude that there exist one-forms $e'^1_i \dots e'^D_i$ such that e'^μ_i divides $\tilde{B}^{\mu\nu}_{ij}$ for any ν. If we are in the generic case, where these vectors are pairwise distinct, this then implies that $\tilde{B}^{\mu\nu}_{ij} = (e')k^{\mu\nu}e'^{[\mu}_i e'^{\nu]}_j$, where $k^{\mu\nu}$ are some coefficients symmetric in $\mu\nu$, and (e') is the determinant of the matrix e'^μ_i, which is included to give the right density weight to B. To find the coefficients $k^{\mu\nu}$ we have to use the normalization constraints. From these constraints, we conclude that $k^{\mu\nu}k^{\rho\sigma} = k^{\mu\rho}k^{\sigma\nu}$. This relation implies that $k^{\mu\nu} = \pm c^\mu c^\nu$, for some vectors c^μ. Indeed, for $k^{\mu\nu}$ not equal to zero, there exists one-form n_μ such that $k^{\mu\nu}n_\mu n_\nu = \pm 1$. Multiplying the above relation by $n_\mu n_\nu$ we get $k^{\mu\nu} = \pm c^\mu c^\nu$, where $c^\mu = k^{\mu\nu}n_\nu$. Thus, if we rescale our vectors e'^μ_i as

$$e^\mu_i = |c^1 \cdots c^D|^{-\frac{1}{D+2}} c^\mu e'^\mu_i,$$

then we have $\tilde{B}^{\mu\nu}_{ij} = \pm e e^{[\mu}_i e^{\nu]}_j$. In odd dimensions we can always absorb the minus sign by redefining the frame e. But in even dimensions we have $\tilde{B}^{\mu\nu}_{ij} = \pm |e| e^{[\mu}_i e^{\nu]}_j$. □

Substituting this solution of the constraints back into the action, we find

$$S[A, \tilde{B}(e)] = \pm \int d^D x \ |e| e_i^{[\mu} e_j^{\nu]} F_{\mu\nu}^{ij} \tag{22}$$

which is simply the standard Palatini action in terms of the frame field e. Thus, our classical theory is indeed a reformulation of general relativity.

Note that our theorem deals only with the case the B field is non-degenerate. It would be interesting to see what the constraints (10) imply in the case B is degenerate. This is of no relevance for the classical theory, where one does not allow degenerate metrics. However, the case of degenerate B field may be quite relevant in the quantum theory, where, as the example of 2+1 gravity suggests, the degenerate metrics play an important role. Thus, it would be quite interesting to study the degenerate sectors and to analyze their quantization. We do not address this important problem in the present paper, hoping to return to it in the future. For an analysis of degenerate sectors in the case of four dimensions see [15, 10, 12]

2.2 Gauge transformations

This subsection deals with the issue of dependence between the constraints (10). We show that the fact that the constraints are not independent implies the presence of an additional gauge symmetry in the theory. We also discuss the problem of finding an independent subset of the constraints.

The action functional (2) is invariant under three different sets of gauge transformations. Two of these, spacetime diffeomorphisms and frame rotations, are familiar so we need not discuss them here, while the third is specific to our new formulation and arises from the fact that the constraints (10) are not independent in more than four spacetime dimensions.

To understand this redundancy, let us find the number of constraints that need to be imposed to guarantee that the B field comes from a frame, and compare this number with the number of constraints in

(10). In D dimensions, there are D^2 components of e_i^μ and $(C_D^2)^2 = D^2(D-1)^2/4$ components of $\tilde{B}_{ij}^{\mu\nu}$, which means that we need the number of independent constraints equal to the difference between the above two numbers, that is, $D^2(D^2-2D-3)/4$. The number of constraints we have can be calculated by looking at the equations (10) that one obtains by varying the action with respect to the Lagrange multipliers Φ. The free indices in this equation are $[m]$ and anti-symmetric pairs (μ,ν), (ρ,σ). The equations are symmetric in these pairs. Thus, the number of equations in (10) is equal to the product of C_D^4, which is the dimension of the index $[m]$, with the number of independent entrees in a symmetric $C_D^2 \times C_D^2$ matrix. This gives $C_D^4 \frac{C_D^2(C_D^2+1)}{2}$ equations. However, some of these equations are simply definitions of the coefficients $\tilde{c}_{[\alpha]}^{[m]}$, see (11). Thus, to get the number of constraints imposed by (10), we have to subtract from the above number the number of components in $\tilde{c}_{[\alpha]}^{[m]}$. This, finally, gives

$$C_D^4 \frac{C_D^2(C_D^2+1)}{2} - (C_D^4)^2$$

constraints. In the case of four dimensions, this number equals to 20, which is exactly the number of constraints needed to go from the B field to the frame. However, already in five dimensions this number is much larger than the number of independent constraints that are needed: we have 250 constraints in (10) with only 75 independent constraints necessary. The bottom line is that we have more constraints than needed for $D > 4$. Since, as we proved, there is a solution to this set of constraints, this simply means that they are highly redundant for $D > 4$.

Thus, the fact that not all the constraints that follow from our action principle are independent does not cause any problems classically. However, this may lead to problems in the quantum theory, for example, with the definition of the partition function. Indeed, in the partition function we have to integrate over the set of Lagrange multipliers $\underset{\sim}{\Phi}{}^{ijkl}_{\mu\nu\rho\sigma}$. This integration leads formally to a delta distribution of the constraint

$$C_{ijkl}^{\mu\nu\rho\sigma} \equiv K_{ijkl}^{\mu\nu\rho\sigma} - K_{ijkl}^{[\mu\nu\rho\sigma]}, \tag{23}$$

where

$$K_{ijkl}^{\mu\nu\rho\sigma} \equiv \tilde{B}_{[ij}^{\mu\nu} \tilde{B}_{kl]}^{\rho\sigma}. \tag{24}$$

If the constraints are not independent, the integration over all Lagrange multipliers Φ leads to products of delta functions that are ill-defined. Thus, one may worry that the partition function of the theory is not well-defined. The standard strategy to deal with this problem is that of gauge fixing. As we show below, the fact that the constraints are not independent implies that there is an additional "gauge" symmetry in the theory. Gauge fixing this symmetry amounts to finding an independent set of constraints. Below we exhibit such an independent set.

The additional "gauge" symmetry present because of the redundancy of the constraints is given by the following transformation of the multiplier fields:

$$\delta \underset{\sim}{\Phi}^{ijkl}_{\mu\nu\rho\sigma} = \Lambda^{ijklmn}_{\mu\nu\rho\sigma\gamma\delta} \tilde{B}^{\gamma\delta}_{mn}. \tag{25}$$

As we shall illustrate below, this transformation leaves the action invariant. Here $\Lambda^{ijklmn}_{\mu\nu\rho\sigma\gamma\delta}$ is the gauge parameter, which must be symmetric under an interchange of any two of the three antisymmetric index pairs $(\mu,\nu), (\rho,\sigma), (\gamma,\delta)$, anti-symmetric in the internal indices $jklm$, and its anti-symmetrization in spacetime indices $\mu\nu\rho\sigma$ must vanish. In addition, it must be symmetric in the indices i, n. Note also that the density weight of Λ must be -2.

In order to prove that the transformation (25) leaves the action invariant, we first have to show that the following relations hold:

$$\begin{aligned} B^{\mu\nu}_{i[j} K^{\rho\sigma\gamma\delta}_{klm]n} + B^{\mu\nu}_{n[j} K^{\rho\sigma\gamma\delta}_{klm]i} + B^{\rho\sigma}_{i[j} K^{\gamma\delta\mu\nu}_{klm]n} + B^{\rho\sigma}_{n[j} K^{\gamma\delta\mu\nu}_{klm]i} \\ + B^{\gamma\delta}_{i[j} K^{\mu\nu\rho\sigma}_{klm]n} + B^{\gamma\delta}_{n[j} K^{\mu\nu\rho\sigma}_{klm]i} \equiv 0, \end{aligned} \tag{26}$$

where $K^{\mu\nu\rho\sigma}_{ijkl}$ is given by (24). The proof of this fact is as follows. Let us pick an arbitrary vector v^i and set $u_j := v^i B_{ij}$. We then have $v^i K_{ijkl} = u_{[j} B_{kl]}$. Using this relation, we can obtain the following identity:

$$0 = u_{[j} u_k B_{lm]} = u_{[j} K_{klm]n} v^n = v^i B_{i[j} K_{klm]n} v^n, \tag{27}$$

which implies that $B_{i[j} K_{klm]n} + B_{n[j} K_{klm]i} \equiv 0$. This is almost the above relation (26). More precisely, to obtain (26) we set $B_{ij} = x B^{\mu\nu}_{ij} + y B^{\rho\sigma}_{ij} + z B^{\gamma\delta}_{ij}$, use the relation just proved, expand, and equate the coefficient of xyz to zero. Using the relation (26) one can easily convince oneself that the transformation (25), with the postulated symmetry properties of the gauge parameter, leaves the action invariant.

It turns out that the set of gauge transformations (25) is complete, i.e., these are all gauge symmetries appearing because of the redundancy of the constraints. In other words, first, gauge fixing this symmetry amounts to choosing an independent subset of constraints; second, all other constraints follow from this independent subset and the relations (26). Let us now present an independent subset of constraints (10) that is enough to guarantee that B comes from a frame. As we have said above, these independent constraints, plus the relations (26) imply the rest of the constraints. The following proposition is a statement to this effect.

Proposition 1. *The following subset of the constraints*

$$
\begin{aligned}
&\text{(i)} \quad K_{ij12}^{\mu\,\mu+1\,\mu\,\mu+1} = 0\,;\\
&\text{(ii)} \quad K_{123i}^{\mu-1\,\mu\,\mu\,\mu+1} = 0\,;\\
&\text{(iii)} \quad B_{[ij}^{\mu\nu} K_{1]234}^{\mu\,\mu-1\,\mu\,\mu+1} - B_{[ij}^{\mu\,\mu-1} K_{1]234}^{\mu\,\mu+1\,\mu\nu} + B_{[ij}^{\mu\,\mu+1} K_{1]234}^{\mu\nu\mu\,\mu-1} = 0\,,\\
&\qquad\qquad \mu+1<\nu\,; \qquad\qquad (28)\\
&\text{(iv)} \quad B_{[12}^{\mu\nu} K_{i]345}^{\nu\,\nu-1\,\nu\,\nu+1} - B_{[12}^{\nu\,\nu-1} K_{i]345}^{\nu\,\nu+1\,\mu\nu} + B_{[12}^{\mu\,\mu+1} K_{i]345}^{\mu\nu\nu\,\nu-1} = 0\,,\\
&\qquad\qquad \mu+1<\nu\,;\\
&\text{(v)} \quad K_{1234}^{12\mu\nu} = K_{1234}^{1\mu2\nu}\,, \qquad \mu+1<\nu
\end{aligned}
$$

is independent. Moreover, the above constraints, plus the relations (26) imply all the constraints (10).

A proof of this proposition consists of two parts. First, as it is easy to see, the number of constraints in (28) is just the right number of constraints needed to go from the B field to a frame. Indeed, there are $D(C_{D-2}^2)$ constraints in (i), $D(D-3)$ in (ii), $(C_D^2 - D)C_{D-1}^2$ in (iii), $(C_D^2 - D)(D-2)$ in (iv), and, finally, $C_D^2 - D$ constraints in (v). Adding all these numbers together one obtains exactly $(C_D^2)^2 - D^2$, which is the number of DOF in the B field minus the number of DOF in the frame. Thus, the counting shows that the number of constraints (28) is not larger than the number of constraints that is needed. The second part of the proof is to show that the constraints (28) are enough to guarantee that the B field comes from a frame. This is done by expressing all the other constraints as constraints (28) modulo the relations (26). We will not give a proof of this fact here, for it involves a rather lengthy manipulation with the relations (26). □

3 Quantum models

This section is devoted to quantum theory. More precisely, we apply the so-called "spin foam" quantization procedure to our theory. We first review the main steps of this procedure, and then find results analogous to ones available in the case of four dimensions.

3.1 Spin foam quantization

In the paper [8] it was advocated that the knowledge of the generating functional $Z[J]$

$$Z[J] = \int \mathcal{D}A\mathcal{D}B\, e^{i\int_{\mathcal{M}} \mathrm{Tr}[B\wedge F + B\wedge J]} \tag{29}$$

of the BF theory, hence the ability to compute all correlation functions of the B field in the BF theory, opens a way towards understanding of Yang-Mills theories in any dimension and of gravity in three and four dimensions. The key point is to consider these theories as deformations of the BF theory. The knowledge of $Z[J]$ leads to an understanding of these theories in the same way as the knowledge of the generating functional of the free scalar field leads to understanding of an interacting quantum field theory. Similar ideas were also put forward by Martellini and collaborators for the case of Yang-Mills theory [9]. As far as gravity is concerned, a strength of this proposal is in the fact that the BF theory incorporates gauge invariance and needs no background metric for its definition, which are two features desired for a non-perturbative treatment of gravity.

Moreover, in [8] a "spin foam" computation of the generating functional was performed. The resulting "spin foam" version of the generating functional immediately gives the "spin foam" quantization of any theory considered as a constrained BF theory. In this quantization the field B is promoted to a derivative operator (in J) acting on the generating functional $Z(J)$.

In the previous section we showed that gravity in higher dimensions can be written as BF theory with constraints. This means that gravity in any dimension falls within the scope of applicability of the method [8]. Thus, there exists a spin foam model of higher-dimensional gravity, some aspects of which we study below. The spin foam model we obtain is a generalization of the 4D spin foam model proposed in [14, 4, 2] to higher dimensions.

In this paper we describe only the main steps of the spin foam quantization procedure. For details the reader may consult Refs. [4, 2, 8]. One starts with a decomposition of $\mathcal{M}$ into piecewice-linear cells. For simplicity this decomposition is usually taken to be a triangulation which we shall denote by Δ; it will be fixed in what follows. Having a fixed triangulation, one can compute the spin foam "approximation" to the generating functional $Z[J]$ of the BF theory. This is an approximation because it takes into account only special distributional configurations of the B field. However, as it was shown in [8], this approximation is exact for TQFT.

The result of calculation of $Z[J]$ can be described as follows. Up to fine details related to the way the simplex amplitudes are glued together, $Z[J]$ can be thought of as given by a sum over product of amplitudes – one for each D-simplex. These amplitudes depend on the current through a collection of group elements. The current, being a two-form, can be integrated over the special portions of the dual faces of Δ that are called in [14, 8] *wedges*. Each wedge is in one-to-one correspondence with a pair (D-simplex, $(D-2)$-simplex lying in it). Integrating J over all wedges of Δ, one gets a collection of Lie algebra elements. Exponentiating the later one obtains a collection of group elements. The generating functional $Z[J]$ depends on the current through these group elements.

The simplex amplitude is obtained as follows. First, one has to construct a special graph. The boundary of each D-simplex is a $(D-1)$-dimensional manifold triangulated by $(D-1)$-simplices. One can construct a graph dual to this triangulation. In D spacetime dimensions, this graph will have $D+1$ vertex and $D(D+1)/2$ edges. Each vertex will have exactly D edges coming to it, or, in other words, its valency will be D. For each edge, let us take the usual space $L^2(G)$ of square integrable functions on the group. Edges of these graph are in one-to-one correspondence with wedges introduced above, and, thus, with the group elements constructed above from the current J. Thus, we can think of elements of $L^2(G)$ for each edge of our graph as functions of the group elements constructed from the current. To construct the vertex amplitude, which is a function of all $D(D+1)/2$ group elements coming from J, one has to choose the so-called *intertwiner* for each vertex. Intertwiners give a way to construct a function of $D(D+1)/2$ group elements that is invariant under the action of the group. This function is the simplex amplitude. In practice, the simplex amplitudes are given by the so-called *spin networks*, which are constructed by taking a basis in $L^2(G)$ consisting of the matrix elements of the irreducible representations.

Given the generating functional $Z(J)$, the computation of the expectation values of products of B field is given by derivative operators acting on $Z(J)$. Thus, amplitudes for gravity theory can be obtained from the above simplex amplitudes by imposing on them certain differential equations with respect to the current. This procedure can be justified as a projection on the kernel of constraints arising when one takes the path integral over the Lagrange multipliers Φ:

$$\int \mathcal{D}\Phi e^{i\int_{\mathcal{M}} \frac{1}{2}\mathrm{Tr}[B\wedge\Phi(B)]} = \delta(C), \tag{30}$$

where C are the constraints (23). After finding solutions to the differential equations corresponding to these constraints, one evaluates them on $J=0$ to obtain amplitudes for gravity.

There are several types of constraints that one has to impose. First, there are the so-called closure constraints. These arise because one is considering a set of Lie algebra two-forms B_{ij}, one for each $(D-2)$-simplex, that are obtained by integrating the B field, which is a $(D-2)$-form, over these $(D-2)$-simplices, and there are linear dependences between B_{ij} obtained this way. It is straightforward to solve the differential equations corresponding to these constraints for they simply require the simplex amplitude to be gauge invariant.

Second, there are simplicity constraints for each $(D-2)$-simplex, or for each two-form B_{ij}, which require this two-form to be simple. This constraints can also be solved in quantum theory. They imply that only a part of the space $L^2(G)$ is relevant. This relevant part can be written as a direct sum over special representations that can be called *simple representations*. We find and study some properties of these representations in the following subsection.

Third, there are analogs of intersection constraints. In the quantum theory these constraints appear as constraints on intertwiners. We find a solution to these constraints in subsection 3.3.

Finally, there is a problem of imposing analogs of normalization constraints. However, these are non-trivial already in the case of four dimensions. Already in that case, there exists a lot of confusion in the literature as to this problem. We will not discuss it in this paper.

In this paper we will not discuss the spin foam model itself; instead, we would like to understand what are the implications, from the point of view of representation theory, of the simplicity and intersection constraints.

3.2 Simple representations

In this section we use notations and general results on representation theory of $\mathrm{SO}(D)$ that are described in Appendices B and C. We refer the reader to the books [11, 19] for a deeper exposition of the results stated in these two appendices.

Let us denote the basis of the Lie algebra of $\mathrm{SO}(D)$ by X_{ij}, $i, j \in \{1, \ldots, D\}$. The commutation relations are given by (49). As we have discussed in the previous section, two-forms B_{ij} are promoted in the "spin foam" quantization to derivative operators acting on the generating functional. Since Lie algebra is generated by derivatives (vector fields) on the group, this means that B_{ij} is promoted in the quantum theory to an element X_{ij} of the Lie algebra of $\mathrm{SO}(D)$.

The quantum analog of the Pluecker relation (17) is given by:

$$X_{[ij}X_{kl]} = 0, \; \forall i, j, k, l \in \{1, \ldots, D\}, \tag{31}$$

where $[ijkl]$ means that we consider the total anti-symmetrization on these indices.

Given a linear representation V of SO(D) we say will that V is a *simple* representation if the quantum Pluecker relation (31) is identically satisfied on V.

It is clear that if V is simple, it decomposes into a sum of irreducible simple representations of SO(D); so it is enough to concentrate on irreducible simple representations. The purpose of this subsection is to give a complete classification of the space of simple representations of SO(D) for all values of D. We find that the simple representations in any dimension are labelled by only one positive integer. Thus, there is a remarkable similarity between simple representations in any dimension D starting from $D = 3$.

There is a natural representation of SO(D) in the space $\mathcal{L}^2(S^{D-1})$ of square integrable functions on the $(D-1)$-sphere. The group action is given by:

$$g \cdot \phi(x) = \phi(g^{-1}x), \tag{32}$$

where $x = (x_1, \cdots, x_D)$ is a unit vector from R^D. This representation is reducible: any $\mathcal{L}^2$ function on the sphere can be decomposed into spherical harmonics

$$\mathcal{L}^2(S^{D-1}) = \oplus_{N=0}^{\infty} H_N^{(D)}, \tag{33}$$

where $H_N^{(D)}$ represents the space of harmonic homogeneous polynomial of degree N (see Appendix C). The action of the Lie algebra elements X_{ij} in the space $\mathcal{L}^2(S^{D-1})$ is given by:

$$X_{ij} \cdot \phi(x) = x_i \frac{\partial \phi}{\partial x_j}(x) - x_j \frac{\partial \phi}{\partial x_i}(x). \tag{34}$$

It is now obvious to see that the space $\mathcal{L}^2(S^{D-1})$, and, therefore, $\mathcal{H}_N^{(D)}$ gives a simple representation.

$$X_{[ij}X_{kl]}\phi = x_{[i}\partial_j x_k \partial_{l]}\phi = x_{[i}\delta_{jk}\partial_{l]}\phi + x_{[i}x_k\partial_j\partial_{l]}\phi = 0 \tag{35}$$

The first equality is the definition of the representation, the second is obtained by commuting x and ∂, and the third by taking into account the anti-symmetrization on the indices.

A remarkable fact is that the spherical harmonics representations $\mathcal{H}_N^{(D)}$ are the only simple representations of SO(D). The following theorem is a statement to this effect:

Theorem 2. *V is an irreducible simple representation of $SO(D)$, $D \geq 4$, if and only if V is equivalent to one of the representations $\mathcal{H}_N^{(D)}$.*

We would like to present two proofs of this fact, one by recurrence and the other more direct. Both these proofs use the fact that an irreducible representation is uniquely characterized by its highest weight. Moreover, the highest weight characterizing $\mathcal{H}_N^{(D)}$ is Ne_1, in the notation of appendices B and C.

Before we give the proofs, let us make several comments. First, for $D = 3$ there is no Pluecker relation; one can say that all representation of SO(3) are simple. In that case the above theorem is still valid, because the representations $\mathcal{H}_N^{(D)}$ exhaust all representations of SO(3). They correspond, of course, to the integer spin representations of Spin(3) $\equiv$ SU(2).

The second comment is that in dimension $D = 4$, the theorem has already been proved in [4]. However, these authors did not realize the crucial fact that the simple representations are related to spherical harmonics. This is this fact that allows us to find the generalization of simple representations to higher dimensions. Let us see how the simple representations of [4] are related to the ones described in the above theorem. In $D = 4$, $X_{[ij}X_{kl]}$ is an invariant tensor (there is only one such tensor because in dimension four there is a unique totally anti-symmetric tensor of rank four). The value of this tensor on the representation $\Lambda(n_1, n_2) = e_1(n_1 + n_2)/2 + e_2(n_1 - n_2)/2$ (see appendix B) is given by $n_1(n_1 + 2) - n_2(n_2 + 2)$. In that case the quantum Pluecker relation reads $n_1 = n_2 = N$, so simple representations of SO(4) are given by the highest weight $\Lambda = Ne_1$, N being a positive integer. This is precisely the highest weight of the representation $\mathcal{H}_N^{(4)}$. Let us now give the proofs for a general dimension D.

Proof 1. As we have just discussed, the theorem holds for dimensions $D = 3, 4$. Thus, to prove the theorem in any dimension, it is enough to show that from the fact that it holds in dimension D it follows that it holds in $D + 1$. Thus, we assume that representations $\mathcal{H}_N^{(D)}$ are the only simple irreducible representations of $\mathrm{SO}(D)$ and show that from this assumption it follows that $\mathcal{H}_N^{(D+1)}$ are the only simple irreducible representations of $\mathrm{SO}(D+1)$. We prove this by constructing an embedding of $\mathrm{SO}(D)$ into $\mathrm{SO}(D+1)$ and then showing that the pullback of simple representations of $\mathrm{SO}(D+1)$ under this embedding is a simple representation of $\mathrm{SO}(D)$. As the last step of the proof we show that the only irreducible representations of $\mathrm{SO}(D+1)$ that have the property that their pullback contains the representations $\mathcal{H}_N^{(D)}$ are the representations $\mathcal{H}_N^{(D+1)}$.

Let us construct an embedding of $\mathrm{SO}(D)$ into $\mathrm{SO}(D+1)$. Since $\mathrm{SO}(D+1)$ is the group of rotation of $D+1$ dimensional vectors, $\mathrm{SO}(D)$ can be obtained as the subgroup fixing the vector $(0,\ldots,0,1)$. This gives an embedding of $\mathrm{SO}(D)$ into $\mathrm{SO}(D+1)$, which we denote by

$$\phi : \mathrm{SO}(D) \to \mathrm{SO}(D+1).$$

If X_{ij}, $i,j \in \{1,\ldots,D\}$ is a basis of $so(D)$, then the action of the embedding ϕ on the Lie algebra is given by $\phi(X_{ij}) = X_{ij}$ for $i,j \in \{1,\ldots,D\}$. If V is a representation of $\mathrm{SO}(D+1)$ we can define its pullback $\phi^*(V)$ using the embedding ϕ. In other words, when $V_\Lambda^{(D+1)}$ is a representation of $\mathrm{SO}(D+1)$ of highest weight Λ, $g \in \mathrm{SO}(D)$ and $v \in V_\Lambda^{(D+1)}$, then $g \cdot \phi^*(v) = \phi(g) \cdot v$. Thus, $\phi^*(V_\Lambda^{(D+1)})$ is a representation of $\mathrm{SO}(D)$. However, this representation is not necessarily irreducible, but it certainly contains, when decomposed into a sum of irreducible representations, the irreducible subrepresentation of highest weight $\phi^*(\Lambda)$. This is because the embedding ϕ maps positive roots onto positive roots and $\phi^*(v_\Lambda)$ is the highest weight for $\mathrm{SO}(D)$.

Let us now take as $V_\Lambda^{(D+1)}$ a simple representation of $\mathrm{SO}(D+1)$. Then the pullback $\phi^*(V_\Lambda^{(D+1)})$ is a simple representation of $\mathrm{SO}(D)$. Indeed, considering a basis of $so(D)$, we have:

$$X_{[ij}X_{kl]} \cdot \phi^*(V_\Lambda^{(D+1)}) \equiv \phi(X_{[ij})\phi(X_{kl]}) \cdot V_\Lambda^{(D+1)} = X_{[ij}X_{kl]} \cdot V_\Lambda^{(D+1)} = 0. \tag{36}$$

Here the first equality is the definition of the pullback of a representation, in the second we used the definition of the embedding, and the third uses the hypothesis that $V_\Lambda^{(D+1)}$ is simple.

This result means that $\phi^*(\Lambda)$ is the highest weight of a simple representation of $\mathrm{SO}(D)$. We have assumed that all highest weight simple representations of SO(D) come from weights Ne_1. In other words, $\phi^*(\Lambda)$ must equal Ne_1 for some N. We would like to show now that the only highest weights Λ of SO($D+1$) satisfying this property are the ones corresponding to the representations $H_N^{(D+1)}$, i.e., $\Lambda = Ne_1$. To see this we have to consider two cases: (i) $D = 2n$; (ii) $D = 2n+1$.

(i) In this case, SO($2n$) and SO($2n+1$) have the same rank n, and ϕ^* is the identity operator, $\phi^*(e_i) = e_i$. Thus, $\phi^*(\Lambda) = Ne_1$ implies that $\Lambda = Ne_1$.

(ii) In this case, SO($2n+1$) has rank n, while SO($2n+2$) has rank $n+1$. Thus, ϕ is the projection operator $\phi^*(e_i) = e_i$, $i \leq n$ and $\phi^*(e_{n+1}) = 0$. Therefore, $\phi^*(\Lambda) = Ne_1$ implies that $\Lambda = Ne_1 + ke_{n+1}$ for some k. But Λ is a highest weight of SO($2n+1$). This means (see appendices B and C) that $\Lambda = N_1e_1 + N_2e_2 + \cdots + N_{n+1}e_{n+1}$ where $N_1 \geq N_2 \geq \cdots \geq N_{n+1} \geq 0$. Thus, if $n \geq 2$, the only highest weight Λ of SO($2n+1$) satisfying $\phi^*(\Lambda) = Ne_1$ is given by $\Lambda = Ne_1$.

Proof 2. Here is a more direct proof of the theorem that uses the correspondence (50) between the Cartan basis and the usual basis of $so(D)$. Let us denote for $1 \leq i < j \leq n$, $n = [D/2]$,

$$C(i,j) = -3X_{[2i-1,2i}X_{2j-1,2j]}. \tag{37}$$

If $V_\Lambda^{(D)}$ is a simple representation of SO(D), then the action of $C(i,j)$ vanishes on $V_\Lambda^{(D)}$. Using (50) and after some algebra we get:

$$C(i,j) = H_j(H_i+1) + E_{-e_i-e_j}E_{e_i+e_j} - E_{-e_i+e_j}E_{e_i-e_j}. \tag{38}$$

Evaluating this expression on the highest weight vector v_Λ we get:

$$C(i,j)v_\Lambda = (\Lambda|e_j)((\Lambda|e_i)+1)v_\Lambda. \tag{39}$$

The simplicity constraint implies that $(\Lambda|e_j)((\Lambda|e_i)+1) = 0$ for all $1 \leq i < j \leq n$, which means $(\Lambda|e_j) = 0$ for all $1 < j \leq n$. Thus, the only highest weights representations that satisfy the simplicity constraint correspond to $\Lambda = Ne_1$. □

3.3 Simple spin networks

In this section we deal with the quantum version of the intersection constraints. As we discussed above, these become equations on intertwiners. In this subsection we show how to solve these constraints in any dimension. We arrive at a notion of *simple* spin networks, whose edges are labelled by simple representations satisfying the simplicity constraints, and whose intertwiners satisfy the intersection relations. The simple spin networks are higher dimensional generalizations of the relativistic spin networks of [4]. However, let us first recall some general facts about spin networks. A more detailed account is given in [2].

An $\mathrm{SO}(D)$ spin network is a triple $(\Gamma, \Lambda, \imath)$, where (i) Γ is an oriented graph, (ii) Λ is a labelling of each edge e by an irreducible representation Λ_e of $\mathrm{SO}(N)$, (iii) $\imath$ is a labelling of each vertex by an intertwiner $\imath_v$ mapping the tensor product of incoming representations at v to the product of outgoing representations at v. Let us denote by $I(\Gamma, \Lambda, v)$ the space of such intertwiners at vertex v. We can then associate to each colored graph (Γ, Λ) a vector space

$$\mathcal{H}(\Gamma, \Lambda) = \otimes_v I(\Gamma, \Lambda, v). \tag{40}$$

When Γ has only one edge and one vertex (circle), this space is one dimensional and is generated by the function on $\mathrm{SO}(N)$ given by the character in the representation Λ. For a general graph this space can be thought of as the space of functionals in the variables $\Lambda_e(g_e)$ which are invariant under gauge transformation acting at each vertex of Γ. This is the space of spin network functionals based on the colored graph (Γ, Λ).

Given an oriented edge e, let us denote by e_- the vertex where e starts and by e_+ the vertex where e ends. For any pair $(e, \pm)$, we can define a group action on $\mathcal{H}(\Gamma, \Lambda)$. This is given by the right or left multiplication:

$$h^{(e,e_+)}\phi(g_{e_1}, \cdots, g_e, \cdots, g_{e_n}) = \phi(g_{e_1}, \cdots, g_e h, \cdots, g_{e_n}),$$
$$h^{(e,e_-)}\phi(g_{e_1}, \cdots, g_e, \cdots, g_{e_n}) = \phi(g_{e_1}, \cdots, h^{-1} g_e, \cdots, g_{e_n}).$$

We denote $X^{(e,e\pm)}$ the corresponding action of the Lie algebra by the derivative operator. We can now introduce the notion of *simple spin networks.*

We say that an $\mathrm{SO}(N)$ spin network $\phi \in \mathcal{H}(\Gamma, \Lambda)$ is *simple* if for all vertices v and all pairs of edges e, e' meeting at v, the following relation is satisfied:

$$X^{(e,v)}_{[ij} X^{(e',v)}_{kl]} \phi = 0. \tag{41}$$

This relation, for a pair (e,e), amounts to the quantum Pluecker relation (31) for the representation Λ_e. This means that the edges of a simple spin network are labelled by simple representations, that is, are colored by one integer N_e which characterize the simple representations $\mathcal{H}^{(D)}_{N_e}$. The remaining conditions (41) for distinct pairs of edges meeting at a vertex v are conditions on the intertwining operator used at the vertex v.

Let $e_1, \dots, e_n$ be the incoming edges and $e'_1, \dots, e'_p$ be the outgoing edges at the vertex v. Then $\mathcal{H}^{(D)}_{N_{e_i}}, \mathcal{H}^{(D)}_{N_{e'_j}}$ are simple representations associated with edges meeting at v. An intertwiner from the tensor product of incoming simple representations to the product of outgoing ones is given by a multi-linear map

$$I(P_1, \cdots, P_n, \bar{Q}_1, \cdots, \bar{Q}_p), \tag{42}$$

where P_i (Q_j respectively) are harmonic homogeneous polynomials of degree N_{e_i} ($N_{e'_j}$ respectively), and $\bar{Q}$ denotes the complex conjugate of Q. The intertwining property reads

$$I(P_1, \cdots, P_n, \bar{Q}_1, \cdots, \bar{Q}_p) = I(g \cdot P_1, \cdots, g \cdot P_n, g \cdot \bar{Q}_1, \cdots, g \cdot \bar{Q}_p), \tag{43}$$

and an example of the relation (41) is given by:

$$I(X_{[ij} \cdot P_1, X_{kl]} \cdot P_2, \cdots, P_n, \bar{Q}_1, \cdots, \bar{Q}_p) = 0 \tag{44}$$

There exists a very simple and beautiful solution of these constraints. This solution was discovered for the case $D = 4$ by [4]. However, in that work, it was written in a rather cumbersome way as a sum over a product of intertwiners of $SU(2)$. Moreover, the proof that the intertwiner satisfies the intersection constraints used heavily the fact that the universal covering of $SO(4)$ can be written as the product $SU(2) \times SU(2)$. This uses the duality available in $D = 4$, which makes this dimension very special. Thus, it was not at all clear that this solution could be generalized to higher dimensions, where there is no notion of duality. The solution we give shows that the central notion, allowing the construction to work, is not self-duality, but the fact that simple representations are realized in the space of polynomials on the sphere S^{D-1}.

Let P_i (Q_j respectively) be harmonic homogeneous polynomial of degree N_{e_i} ($N_{e'_j}$ respectively) and consider the following intertwiner between

$$\otimes_{i=1}^{n} \mathcal{H}^{(D)}_{N_{e_i}} \text{ and } \otimes_{j=1}^{p} (\mathcal{H}^{(D)}_{N_{e'_j}})$$

given by

$$I_{n,p}(P_1, \cdots, P_n, \bar{Q}_1, \cdots, \bar{Q}_p) = \int_{S^{D-1}} d\Omega(x)\, P_1(x) \cdots P_n(x) \bar{Q}_1(x) \cdots \bar{Q}_p(x). \quad (45)$$

Here $d\Omega$ denotes the invariant measure on the unit sphere S^{D-1}. For definiteness we choose the normalization of this measure such that

$$\int_{S^{D-1}} d\Omega(x) = 1$$

The fact that $I_{n,p}$ is an invariant intertwiner can be easily seen using the invariance of the measure, integration by parts and the Leibniz rule. This is, in a sense, the simplest possible intertwiner one can imagine. Note that the above intertwiner in the case $D = 3$ is the usual intertwiner of SO(3) that is constructed from Clebsch-Gordan coefficients. It is remarkable that such a simple entity gives a simple intertwiner:

Theorem 3. *$I_{n,p}$ satisfies the relation (41).*

Proof. Let us consider a vertex with n incoming and p outgoing edges. Let us choose any two of the incoming edges, which we denote by e_1 and e_2. Let us denote by F the product of all polynomials P, Q except P_{e_1}, P_{e_2}: $F = \prod_{i \neq 1,2} P_{e_i} \prod_j Q_{e_j}$. We then have to prove that the following quantity

$$X^{e_1}_{[ij} X^{e_2}_{kl]} I_{n,p} = \int_{S^{D-1}} d\Omega X_{[ij} \cdot P_{e_1} X_{kl]} \cdot P_{e_2} F \quad (46)$$

is zero. We can use the following identity:

$$2X_{[ij} \cdot P_{e_1} X_{kl]} \cdot P_{e_2} = X_{[ij} \cdot X_{kl]} \cdot (P_{e_1} P_{e_2}) - (X_{[ij} X_{kl]} \cdot P_{e_1}) P_{e_2} - P_{e_1} (X_{[ij} X_{kl]} \cdot P_{e_2}), \quad (47)$$

The last two terms are zero because P's are homogeneous polynomials. Similarly, the first term is zero because the product of homogeneous polynomials is also a homogeneous polynomial. The other intersection relations can be proved analogously. □

One question that remains is the question of uniqueness of the intertwiner we constructed. However, this question is non-trivial already in the case of four dimensions. In the case of $D = 4$ there exists an argument [16] that shows that this intertwiner is the only one satisfying all the constraints. Thus, it may be the case that the intertwiner we found is unique in any dimension. It would be interesting to find a proof of this conjecture. We leave this issue to further research.

4 Discussion

We have seen that, in many aspects, the BF formulation of higher-dimensional gravity is analogous to the four-dimensional case. Indeed, as in four dimensions, one must add to the usual BF action constraints that are quadratic in the B field and that guarantee that it comes from the frame field. We also saw that the quantum spin foam models in higher dimensions are quite similar to their four-dimensional cousin. Strikingly, in any dimension representations that appear are labelled by just one parameter, the structure of the intertwiner that is used to built the model is quite similar to that in four dimensions. Let us emphasize that this similarity between the case of four dimensions and higher dimensional theories is by itself an interesting and unexpected result. Indeed, as we discussed in the Introduction, it is tempting to believe that the four-dimensional case is special, for there the self-duality is available. Our results indicate that the case of four dimensions is not that special.

There are, however, several differences between the case of four dimensions and higher-dimensional gravity that are worth mentioning. First, unlike the four-dimensional case, in higher dimensions it is much harder to single out the independent constraints. In four dimensions the number of Lagrange multipliers that appear in the action is equal to the number of independent constraints. In higher dimensions we were not able to find a covariant formulation with this property: the number of Lagrange multipliers appearing in the action (2) is much larger than the number of independent constraints. We were able, however, to find a description of independent constraints, see the subsection of 2 on gauge transformations, but not in any covariant way. Thus, unlike the four-dimensional case, we don't have an action principle with the number of Lagrange multipliers appearing equal to the number of independent constraints. This does not seem, however, to cause any problems, either classically or quantum mechanically. Classically one finds complicated relations (26) between the constraints appearing from varying the action (2), and this is manifested by the appearance of gauge symmetries discussed in section 2. However, all relations together, although not independent, do imply that the B field comes from a frame field. One might worry that the dependence of constraints may cause problems quantum mechanically. However, as we saw, it doesn't seem to be the case, at least in the spin foam context. As we have seen in the last section, it was possible to impose the simplicity and intersection constraints on spin foam by explicitly constructing the intertwiners satisfying these constraints.

The second important difference between $D = 4$ and the higher-dimensional cases is the absence of the topological sector. As we saw in subsection 2.1, in higher dimensions there is only one type of solutions of the simplicity constraints, in contrast to two different types in the case of four dimensions: the case (i), according to the classification of the subsection 2.1, can exist only in four dimensions, where it leads to the topological sector. This is an interesting feature of higher-dimensional gravity, for it means that one does not have to worry about a possible interference between the two sectors when they are both present in the quantum theory. Also, unlike the four-dimensional case, there is no worry that the spin foam quantization gives a quantization of the topological sector, not gravity: simply because there is no topological sector anymore. Of course, one still has to worry about the issue of "two signs", arising in the solution of the constraint equations. But this comes about even in the simplest case of three dimensions, where the two types of solutions can interfere in the quantum theory and make the problem of finding the "gravitational" sector of the theory very difficult, see [7] for a discussion of this problem.

Let us now discuss implications of our results for the problem of quantization of gravity. We have discussed some aspects of the "spin foam" quantum model of gravity for all $D > 4$. As we saw, these models turn out to be quite similar to the four-dimensional model. There are, however, many problems with this model even in the case of $D = 4$, of which the main one is probably that one does not know how to glue the simplex amplitudes together to form the amplitude of the whole triangulated manifold. The other problem is that we do not know yet how to implement the normalization conditions in the quantum theory. Thus, the quantum theory presented in this paper is far from giving a correct quantization of gravity.

Our results, however, have another interesting implication for the problem of quantum gravity. Our results imply that gravity, as well as Yang-Mills theory in any dimension, can be thought of as an "interacting" BF theory. Indeed, the action of this theories can be rewritten as that of BF theory plus an additional term quadratic in the B field, which can be thought of as the "interaction" term. This means that the problem of quantization of both Yang-Mills and gravity theories in any dimension to a large extent reduces to the problem of finding the generating functional $Z[J]$ of the BF theory:

$$Z[J] = \int \mathcal{D}A\mathcal{D}B\, e^{i\int_{\mathcal{M}} \mathrm{Tr}(B\wedge F)+\mathrm{Tr}(B\wedge J)},$$

where J is the current two-form. Indeed, because actions for the both theories can be represented in the form BF action plus quadratic term in B, all correlation functions of these theories (in B field) can be found by appropriately differentiating the generating functional $Z[J]$ with respect to J. Thus, $Z[J]$ is a universal object, a knowledge of which in a particular spacetime dimension to a large extent means the knowledge of both Yang-Mills and gravity theories in that dimension. This way of approaching the problem of four-dimensional quantum gravity was advocated in [8]. The results of our paper mean that this strategy can also be applied to higher dimensional theories. Let us also mention that a partial progress along the lines of finding $Z[J]$ was achieved in [8], where we found a "spin foam" approximation to this generating functional in any dimension.

Let us conclude by pointing out another interesting implication of our results. In four spacetime dimensions, the use spin foam models was to a large extent motivated by results of the canonical approach to quantum gravity [18]: the known four-dimensional spin foam models are intimately related to the loop canonical quantization of gravity. We have found that the spin foam model formulation of quantum gravity is not limited to four dimensions. Thus, our results point towards an interesting possibility that there exists an analog of canonical connection quantization of gravity in any dimension. It would be interesting to find such a formulation.

Acknowledgements

We are grateful to L. Smolin for a discussion. This work was supported in part by the NSF grant PHY95-14240 and by the Eberly research funds of Penn State. K.K. was supported in part by a Braddock fellowship from Penn State.

A Pluecker relations

Relations which enforce that a multivector factors as an anti-symmetrized product of vectors arise in the geometry of subspaces of linear spaces, and are known as Pluecker relations. In this appendix, we shall review and demonstrate some relevant facts of algebraic geometry concerning those relations.

As the first step, we shall consider the case of a single two-form B_{ij}, and show that the necessary and sufficient condition for it to be an anti-symmetrized product of one-forms is the following:

$$B_{[ij}B_{kl]} = 0. \tag{48}$$

Before showing that this condition implies factorization, we will note that it is equivalent to the following weaker condition:

$$\frac{1}{2}B_{i[j}B_{kl]} = B_{ij}B_{kl} + B_{ik}B_{lj} + B_{il}B_{jk} = 0.$$

To see that this is the case, we simply write out the six terms appearing in the complete anti-symmetrization on the indices $ijkl$, and note that each of the three possible ways of choosing a pair of two indices appears twice. Let us now show that the above condition implies that the two-form B_{ij} factors as a product of one-forms. If B_{ij} is not identically zero, then we can find vectors a^i and b^j such that $B_{ij}a^ib^j = 1$. Let us now define one-forms u_i and v_i as $u_i = B_{ij}a^j$ and $v_i = B_{ij}a^j$. Then, using the above identity, we obtain

$$u_iv_j - v_iu_j = (B_{ik}B_{jl} - B_{il}B_{jk})a^kb^l = B_{ij}B_{kl}a^kb^l = B_{ij}.$$

Thus, this proves the simplicity of B_{ij} by explicitly constructing the two one-forms that divide it.

Next, given two simple two-forms, let us find a condition that they have a non zero common factor. First, assuming that this is the case, we have:

$$B_{ij} = u_{[i}v_{j]} \qquad \text{and} \qquad {B'}_{ij} = v_{[i}w_{j]}.$$

We then see that $\lambda B_{ij} + \mu {B'}_{ij}$ must be a non zero simple bivector for any value of the constants λ, μ. Using the simplicity criterion that we have already proved above, this will be the case if the relation

$$B_{[ij}{B'}_{kl]} = 0$$

is satisfied. This is the relation we looked for. Let us now show that this relation is also a sufficient condition for two two-forms to have a common factor. To show this we shall produce this common factor explicitly. Consider the following entity:

$$B_{i[j}{B'}_{kl]}.$$

If the two bivectors appearing in the above expression were proportional, then, because they are simple, this expression would vanish identically. If they are not proportional, it is possible to choose vectors a^i, b^i, and c^i such that the expression

$$v_i := B_{i[j}{B'}_{kl]}a^jb^kc^l$$

differs from zero. The fact that this one-form is a factor of B_{ij} follows immediately from:

$$B_{[ij}v_{k]} = B_{[ij}B_{k][l}{B'}_{mn]}a^lb^mc^n = 0,$$

where we have used the fact that B_{ij} satisfies the simplicity constraint. To see that our vector also divides the other bivector, we note that there is another expression for v_i. The Pluecker relation reads:

$$0 = B_{[ij}{B'}_{kl]} = B_{i[j}{B'}_{kl]} + {B'}_{i[j}B_{kl]}.$$

This means that the roles of the two bivectors in our formula are interchangeable. Hence v_i is a factor of both bivectors. This vector is unique up to rescaling, for the two two-forms are distinct.

B Some facts about SO(N) and its representation theory

We will denote by X_{ij}, $i,j \in \{1,\dots,D\}$ generators of Lie algebra $SO(D)$. They satisfy the following commutation relations:

$$[X_{ij}, X_{kl}] = \delta^{ik} X_{jl} - \delta^{il} X_{jk} - \delta^{jk} X_{il} + \delta^{jl} X_{ik} \tag{49}$$

Let us consider the Cartan representation of this Lie algebra. There are two cases to consider: (i) $D = 2n+1$; (ii) $D = 2n$. In the first case the corresponding Dynkin diagram is B_n; in the second case it is D_n.

The Cartan subalgebra $\mathcal{H}$ of SO$(2n+1)$ is generated by $H_k = iX_{2k-12k}, k = 1,\dots,n$. We denote by e_k the generators of the dual of $\mathcal{H}$, $e_k(H_j) = \delta_{kj}$.

Let us denote by $\Delta \in \mathcal{H}^*$ the root space of $so(D)$. We have

$$\begin{aligned} \Delta &= \{\pm e_i \pm e_j, with\, 1 \le i < j \le n\} \cup \{\pm e_i, 1 \le i \le n\}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \ D = 2n+1. \\ \Delta &= \{\pm e_i \pm e_j, with\, 1 \le i < j \le n\}, \ \text{ for } \ D = 2n. \end{aligned}$$

The Cartan basis $H_i, E_{\pm\alpha}$, $i \in \{1,\dots,n\}$, $\alpha \in \Delta$ is related to the basis X_{ij} by

$$\begin{aligned} H_i &= iX_{2i-12i} && (50) \\ E_{e_i+e_j} &= \frac{1}{2i}[X_{2i-1,2j-1} - iX_{2i-1,2j} - iX_{2i,2j-1} - X_{2i,2j}], && (51) \\ E_{-e_i-e_j} &= \frac{1}{2i}[X_{2i-1,2j-1} + iX_{2i-1,2j} + iX_{2i,2j-1} - X_{2i,2j}], && (52) \\ E_{e_i-e_j} &= \frac{1}{2i}[X_{2i-1,2j-1} + iX_{2i-1,2j} - iX_{2i,2j-1} - X_{2i,2j}], && (53) \\ E_{-e_i+e_j} &= \frac{1}{2i}[X_{2i-1,2j-1} - iX_{2i-1,2j} + iX_{2i,2j-1} - X_{2i,2j}]. && (54) \end{aligned}$$

where $1 \le i < j \le n$.

The simple roots are given by:

$$\alpha_1 = e_1 - e_2,\ \dots,\ \alpha_{n-1} = e_{n-1} - e_n,\ \alpha_n = e_n, \quad \text{for } D = 2n+1, \tag{55}$$

$$\alpha_1 = e_1 - e_2,\ \dots,\ \alpha_{n-1} = e_{n-1} - e_n,\ \alpha_n = e_{n-1} + e_n, \quad \text{for } D = 2n. \tag{56}$$

One of the central theorems in the theory of Lie groups states that the irreducible representations of the covering group of SO(D), i.e. Spin(D), are in one-to-one correspondence with the dominant integral weights, that is, weights of the form:

$$\Lambda = \sum_{i=1}^{n} n_i \lambda_i,$$

where n_i are positive integers, λ_i are the Dynkin weights satisfying $2(\lambda_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$ and $n = [D/2]$ ($[\cdot]$ is the integral part). Λ denotes the highest weight of the corresponding irreducible representation.

When expressed in the basis given by the weights $e_i, 1 \leq i \leq n$, the highest weights labelling representations of Spin(D) are given by:

$$\begin{aligned}
&\Lambda(n_1, \cdots, n_n) \qquad (57)\\
&= (n_1 + \cdots + n_{n-2} + n_{n-1} + \frac{n_n}{2})e_1 + \cdots + (n_{n-2} + n_{n-1} + \frac{n_n}{2})e_{n-2}\\
&\quad + (n_{n-1} + \frac{n_n}{2})e_{n-1} + (\frac{n_n}{2})e_n, \quad \text{for } D = 2n+1;\\
&\Lambda(n_1, \cdots, n_n) \qquad (58)\\
&= (n_1 + \cdots + n_{n-2} + \frac{n_{n-1} + n_n}{2})e_1 + \cdots + (n_{n-2} + \frac{n_{n-1} + n_n}{2})e_{n-2}\\
&\quad + (\frac{n_{n-1} + n_n}{2})e_{n-1} + (\frac{n_n - n_{n-1}}{2})e_n, \quad \text{for } D = 2n.
\end{aligned}$$

The irreducible representations of SO(D) are in one-to-one correspondence with the irreducible representations of Spin(D) that satisfy the restriction: (i) n_n is an even integer for $D = 2n+1$; (ii) $n_{n-1} + n_n$ is an even integer for $D = 2n$.

C Harmonic polynomial representations of SO(N)

Let $V_N^{(D)}$ be the space of complex-valued homogeneous polynomials of degree N on R^D. Then SO(D) acts on this space by $g \cdot P(x) = P(g^{-1}x)$.

This action induces the following action of the Lie algebra SO(D):

$$X_{ij} \cdot P(x) = x^i \frac{\partial P}{\partial x^j}(x) - x^j \frac{\partial P}{\partial x^i}(x). \tag{59}$$

The basis of weight vectors of $V_N^{(D)}$ is given by:

$$(x_1 + ix_2)^{k_1}(x_1 - ix_2)^{l_1} \cdots (x_{2n-1} - ix_{2n})^{l_1}, \text{ for } D = 2n \tag{60}$$

$$(x_1 + ix_2)^{k_1}(x_1 - ix_2)^{l_1} \cdots (x_{2n-1} - ix_{2n})^{l_1} x_{2n+1}^{k_0}, \text{ for } D = 2n+1, \tag{61}$$

where $N = \sum_j k_j + \sum_i l_i$. This basis diagonalizes the action of the Cartan subalgebra generated by $H_i = iX_{2i-12i}$. Hence, the weights of $V_N^{(D)}$ are given by:

$$\sum_{i=1}^{j=n} (k_i - l_i)e_i. \tag{62}$$

The highest weight of $V_N^{(D)}$ is Ne_1 and its dimension is C_N^{N+D-1}.

This representation is, however, not irreducible. To see this, let us consider the Casimir

$$C = \frac{1}{2} \sum_{ij} X_{ij} X_{ij}.$$

Its action on $V_N^{(D)}$ is given by:

$$C \cdot P(x) = N(N + D - 2)P(x) + |x|^2 \Delta P(x), \tag{63}$$

where $\Delta = (\partial_1)^2 + \cdots + (\partial_D)^2$ is the Laplacian. Thus, an invariant subspace of this action is the subspace $\mathcal{H}_N^{(D)}$ of harmonic homogeneous polynomials on R^D. This space is known to be an irreducible representation of SO(D). Thus, $\mathcal{H}_N^{(D)}$ can be equivalently characterized as the irreducible representation of highest weight $\Lambda = Ne_1$. The dimension of $\mathcal{H}_N^{(D)}$ can be deduced from the dimension of $V_N^{(D)}$ using the following relations

$$\dim \mathcal{H}_N^{(D)} = \dim V_N^{(D)} - \dim V_{N-2}^{(D)}, \tag{64}$$

$$\dim \mathcal{H}_N^{(D)} = \sum_{k=1}^{N} \dim \mathcal{H}_k^{(D-1)}. \tag{65}$$

From the first relation we deduce that

$$\dim\mathcal{H}_N^{(D)} = \frac{(N+D-3)!(2N+D-2)}{N!(D-2)!}. \tag{66}$$

The second equality tells us how $\mathcal{H}_N^{(D)}$, viewed as a representation of $SO(D-1)$, decomposes as a sum of irreducible representations.

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