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Angular Quantization of the Sine-Gordon Model at the Free Fermion Point

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Abstract

The goal of this paper is to analyse the method of angular quantization for the Sine-Gordon model at the free fermion point, which is one of the most investigated models of the two-dimensional integrable field theories. The angular quantization method (see hep-th/9707091) is a continuous analog of the Baxter's corner transfer matrix method. Investigating the canonical quantization of the free massive Dirac fermions in one Rindler wedge we identify this quantization with a representation of the infinite-dimensional algebra introduced in the paper q-alg/9702002 and specialized to the free fermion point. We construct further the main ingredients of the SG theory in terms of the representation theory of this algebra following the approach by M. Jimbo, T. Miwa et al.

1 Introduction

The Sine-Gordon (SG) model in two-dimensional Minkowski space-time is described by the action[1]

$$S_{\mathrm{SG}} = \frac{1}{4\pi}\int dt\, dx\, \left(\frac{1}{2}\frac{\partial^2\Phi(x,t)}{\partial t^2} - \frac{1}{2}\frac{\partial^2\Phi(x,t)}{\partial x^2} \right.$$
$$\left. + \frac{m^2}{\beta^2}\,\left(\cos(\beta\Phi(x,t)) - 1\right)\right). \quad (1.1)$$

The quantum SG theory is perhaps the most fundamental of the integrable quantum field theories in two dimensions, and thus plays an important role in the development of new methods. The S-matrix of soliton-antisoliton scattering was obtained in [34]. This S-matrix (see (3.4)) depends on so called renormalized coupling constant ξ and the relation of this parameter to the SG coupling constant β is

$$\xi = \frac{\beta^2}{2-\beta^2}\,. \quad (1.2)$$

[1]We have rescaled $\beta \to \sqrt{4\pi}\beta$ in comparison with the usual convention, so that the free fermion (FF) point occurs at $\beta^2 = 1$.

The quantum SG model is a superrenormalizable theory for the real values of the coupling constant $0 < \beta^2 < 2$ which corresponds to the restriction to the real positive values of ξ, $0 < \xi < \infty$. The regime $1 < \xi < \infty$ is the breatherless one, where solitons and antisolitons do not form bound states. One can see that modulo the overall scalar factor and for appropriate choice of the multiplicative spectral parameter $z = e^{-\theta/\xi}$, where θ is a rapidity of the particles, the soliton-antisoliton S-matrix can be written in the form

$$S(\theta,\xi) = \rho(\theta,\xi) \begin{pmatrix} zq - z^{-1}q^{-1} & 0 & 0 & 0 \\ 0 & z - z^{-1} & q - q^{-1} & 0 \\ 0 & q - q^{-1} & z - z^{-1} & 0 \\ 0 & 0 & 0 & zq - z^{-1}q^{-1} \end{pmatrix},$$

$$q = \exp\left(\pi i \frac{\xi+1}{\xi}\right) \qquad (1.3)$$

which signifies in particular a quantum group symmetry of the Hilbert space of states of the model with respect to the finite dimensional quantum group $U_q(sl_2)$ [32, 22].

The SG model was also one of the first continuous integrable models where the quantum inverse scattering method (QISM) was tested. It was shown in the paper [12] that the quantum monodromy matrices $\mathcal{T}(u)$ satisfy the commutation relation

$$R(u_1 - u_2, \xi)\mathcal{T}_1(u_1)\mathcal{T}_2(u_2) = \mathcal{T}_2(u_2)\mathcal{T}_1(u_1)R(u_1 - u_2, \xi) \,, \qquad (1.4)$$

where the R-matrix has the same structure as in (1.3) (see (3.6) for the exact formula) in terms of additive an spectral parameter u $\left(z = e^{-u/(\xi+1)}\right)$, but with deformation parameter replaced by

$$q' = \exp\left(\pi i \frac{\xi}{\xi+1}\right). \qquad (1.5)$$

The equation (1.4) implies that

$$[\mathrm{tr}\,\mathcal{T}(u_1), \mathrm{tr}\,\mathcal{T}(u_2)] = 0 \qquad (1.6)$$

and signifies that after proper expansion of the quantity $\mathrm{tr}\,\mathcal{T}(u)$ with respect to the spectral parameter u it generates the local integrals of motion and (1.6) shows that they are in involution. Note that even at the FF point where $\beta^2 = 1$ the R-matrix in the commutation relation of monodromy matrices (1.4) is nontrivial since $q' = i$; this can be traced to the fact that the monodromy matrix is constructed from the fields $\exp(i\Phi/2)$, which are non-local in terms of the fermions since the fermion bilinear is $\exp(i\Phi)$.

As we see, the SG model naturally contains two quantum group symmetries, with different deformation parameters related by the duality transformation (3.7). An attempt to explain this phenomena was made in [26] in the framework of the bosonization technique in massive integrable field theories. This approach was generalized then for the lattice integrable models [14]. Following the ideas presented in these papers a screening current algebra was proposed in [20]. The specific coalgebraic properties of this infinite-dimensional algebra allowed to reconstruct the bosonization approach of [26] from algebraical analysis of the representation theory of the screening current algebra.

Essential progress toward understanding quantum integrable models in the infinite volume limit was made in the framework of Baxter's corner transfer matrix (CTM) method [2]. It was observed that the CTM of some lattice integrable models in the infinite volume limit has equidistant spectrum bounded from below and so can be described by the infinite set of oscillators. This fact allows one to develop a new approach to quantum integrable models on the lattice. This was done by the Kyoto group for the XXZ model in the anti-ferroelectric regime [18]. The model was completely solved, namely, the correlation functions of local operators and form-factors of local operators were calculated explicitly, using infinite-dimensional representations of quantum affine algebra $U_q(\widehat{sl}_2)$ with real parameter of deformation satisfying $-1 < q < 0$. One of the main ideas of the construction is to divide the total Hilbert space of the model, which is identified in the infinite volume limit with an infinite product of two-dimensional spaces where local operators act,

$$\mathcal{H}_{\mathrm{XXZ}} \approx \cdots \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \cdots \tag{1.7}$$

into two semi-infinite products of these spaces

$$\begin{aligned}\mathcal{H}_{\mathrm{XXZ}} &\approx (\cdots \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2) \otimes (\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \cdots) \\ &\approx \mathcal{H}^*_{\mathrm{CTM}} \otimes \mathcal{H}_{\mathrm{CTM}} = \mathrm{End}\,(\mathcal{H}_{\mathrm{CTM}})\end{aligned} \tag{1.8}$$

which are denoted by $\mathcal{H}_{\mathrm{CTM}}$ and where the corner transfer matrix acts naturally. Each of these semi-infinite products is identified with level

1 and level -1 integrable modules of $U_q(\widehat{sl}_2)$, where operators of the algebra act naturally. The decomposition (1.8) led in particular to the identification of the states in the Hilbert space $\mathcal{H}_{XXZ}$ with the operators acting in $\mathcal{H}_{\rm CTM}$. The space End $(\mathcal{H}_{\rm CTM})$ is equipped with a natural scalar product $(A, B) = {\rm Tr}_{\mathcal{H}_{\rm CTM}} AB$ and the vacuum vector in $\mathcal{H}_{\rm XXZ}$ is defined as $(-q)^{H_{\rm CTM}}$, where $H_{\rm CTM}$ is a corner transfer matrix hamiltonian.

The representation theory of the quantum affine algebra $U_q(\widehat{sl}_2)$ provides certain operators which intertwine its action in $\mathcal{H}_{\rm CTM}$ (type I and type II intertwining operators). Type II intertwining operators are used for the construction of the basis of asymptotic states in End $(\mathcal{H}_{\rm CTM})$, and type I operators are used for the construction of the transfer matrix and the local hamiltonian in this picture. Moreover, the adjoint action of the elements of the quantum affine algebra in End $(\mathcal{H}_{\rm CTM})$ describe a level 0 $U_q(\widehat{sl}_2)$ symmetry of the model. As a consequence, the form-factors of the local operators and correlation functions of their product are presented in a form of certain multiple integrals, which come as a trace over $\mathcal{H}_{\rm CTM}$ of certain products of the intertwining operators.

In the continuous integrable models an approach to implement Baxter's CTM method was developed in the papers [26, 5] and was based on the method of the angular quantization. The total Hilbert space of the continuous quantum integrable model in infinite volume was supposed to be embedded into a tensor product

$$\mathcal{H} \hookrightarrow \mathcal{H}_L \otimes \mathcal{H}_R\ , \tag{1.9}$$

where $\mathcal{H}_L$ ($\mathcal{H}_R$) are the Hilbert spaces of the quantization in the left (right) wedge. The right Rindler wedge (RRW) in two-dimensional Minkowski space-time is

$$(x^0)^2 - (x^1)^2 < 0, \quad x^1 > 0\ , \tag{1.10}$$

where x^0 is a time and x^1 is coordinate, while the left Rindler wedge (LRW)

$$(x^0)^2 - (x^1)^2 < 0, \quad x^1 < 0\ . \tag{1.11}$$

Let us fix the parametrization of space-time coordinates in RRW

$$x^0 = r\,{\rm sh}\,\alpha, \quad x^1 = r\,{\rm ch}\,\alpha, \quad r \geqslant 0, \quad \alpha \in \mathbb{R}\ . \tag{1.12}$$

With this parametrization, the coordinates x^0, x^1 cover the RRW, since $x^1 > 0$. The LRW is formally obtained by the rotation $\alpha \to \alpha - i\pi$ or by applying the operator $e^{\pi K}$ where K is Lorentz boost generator $K = -i\partial_\alpha$. The space $\mathcal{H}_L$ can be identified with the dual to $\mathcal{H}_R$ and so the states in the total Hilbert space can be realized as the operators in $\mathcal{H}_R$.

It was suggested in [26] to realize $\mathcal{H}_R$ for the SG model as a Fock space with a natural action of the operators satisfying the commutation relations of Zamolodchikov-Faddeev (ZF) algebra. Further, in [20] these operators were identified with intertwining operators of the scaled elliptic algebra $\mathcal{A}(\widehat{sl}_2)$ which can be observed in the bosonization picture [26] by the presentation using screening currents. One of the main arguments in favor of these mathematical constructions was the coincidence of form factors of certain local operators in SG theory with trace calculations in $\mathcal{H}_R$.

In this paper, we try to develop the method of the angular quantization in two directions. First, we analyze the SG model in RRW at the free fermion point, where the canonical quantization can be done explicitly. We see here that the usual conserved charges [23] diverge and the only chance to get a rich algebra of symmetries is to use a certain analytical continuation of the conserved charges, or equivalently, the scattering data. In this case the bosonization [26] naturally appears. We see further that in order to close the algebra, we are forced to use the currents with dual monodromy properties and the algebra of (nonlocal) conserved currents which we find here coincide with specialization of the scaling elliptic algebra $\mathcal{A}(\widehat{sl}_2)$ proposed in [20] and specialized to the free fermion point ($\xi = 1$).

Second, we go into further details of the description of the continuous SG model analogous to the group-theoretical description of the space of states in the XXZ model [18]. We show that starting from level one representation of the scaling elliptic algebra $\mathcal{A}(\widehat{sl}_2)$ we can correctly define the vacuum, the asymptotic states and operators which act on the space of the asymptotic states, namely, the transfer matrix, the hamiltonian, the local integrals of motion. Contrary to the lattice case they are given now via coefficients of the asymptotic expansion of the family of commuting operators. We define the adjoint action of the algebra $\mathcal{A}(\widehat{sl}_2)$ on the space of states and show that known symmetries of this space related to the conserved nonlocal currents [28, 3] and formulated in terms of quantum affine algebras at level zero, can be obtained from this adjoint action by the asymptotical expansion. Let us roughly explain this description.

The total Hilbert space $\mathcal{H}$ is supposed to be divided as in (1.9). The spaces $\mathcal{H}_R$ and $\mathcal{H}_L$ are level 1 and level -1 highest weight modules over the algebra $\mathcal{A}(\widehat{sl}_2)$ so the states in $\mathcal{H}$ can be identified with some operators in $\mathcal{H}_R$. In particular, the physical vacuum state $|\mathrm{vac}\rangle_{\mathrm{ph}}$ is identified with boost operator

$$|\mathrm{vac}\rangle_{\mathrm{ph}} = e^{\pi K} = e^{-i\pi\partial_\alpha}, \tag{1.13}$$

where α is angular time in RRW and the states $|\theta_1, \dots, \theta_n\rangle_{\varepsilon_1,\dots,\varepsilon_n}$ are identified with the product

$$|\theta_1, \dots, \theta_n\rangle_{\varepsilon_1,\dots,\varepsilon_n} = Z^*_{\varepsilon_1}(\theta_1)\dots Z^*_{\varepsilon_n}(\theta_n)e^{\pi K}\ , \tag{1.14}$$

where $Z^*_\varepsilon(\theta)$ are certain 'twisted' intertwining operators of the screening currents algebra $\mathcal{A}(\widehat{sl}_2)$, which also act in $\mathcal{H}_R$. The adjoint action of the algebra $\mathcal{A}(\widehat{sl}_2)$ is not standard because this algebra is not a Hopf algebra. Indeed, the commutation and comultiplication relations of the algebra $\mathcal{A}(\widehat{sl}_2)$ in terms of L-operators can be written in the form

$$\begin{aligned}&\mathcal{R}(u_1-u_2,\xi+c)L_1(u_1,\xi)L_2(u_2,\xi)\\ &\quad = L_2(u_2,\xi)L_1(u_1,\xi)\mathcal{R}(u_1-u_2,\xi)\end{aligned} \tag{1.15}$$

$$\Delta^{\mathrm{op}}L(u,\xi) = L(u-i\pi c^{(2)}/4,\xi+c^{(2)})\ \dot{\otimes}\ L(u+i\pi c^{(1)}/4,\xi) \tag{1.16}$$

where $\mathcal{R}(u,\xi)$ means R-matrix defined by (3.1) and c is a central element of the algebra $\mathcal{A}(\widehat{sl}_2)$. Note that R-matrices in the left and right hand sides of (1.15) differ by the central element of the algebra, which signifies that the algebra under consideration is not coassociative. This algebra is not a usual Hopf algebra. Nevertheless, a coalgebraic structure of this algebra was used in [20] to construct the intertwining operators for highest weight modules over this algebra at the value of the central element $c = 1$. There are also some indications that this screening current algebra is a quasi-Hopf algebra [11] (see papers [17] on the lattice variants of this algebra).

The adjoint action has the different form on the subspaces $\mathcal{H}_i \in \mathcal{H}$, $i = 0, 1$ of even and odd number of particles and includes the involution of the algebra $\mathcal{A}(\widehat{sl}_2)$

$$\iota\left(L(u)\right) = \sigma_z L(u)\sigma_z \ . \tag{1.17}$$

For the state $X_i \in \mathcal{H}_i$, $i = 0, 1$ it is defined as follows

$$\begin{aligned}\mathrm{Ad}_{L(u;\xi)} \cdot X_k \\ = \iota\left(L^{-1}(u + i\pi c/4;\xi)\right) X_k \, \iota^{k+1}\left(L(u - i\pi + i\pi c/4;\xi)\right) \ . \end{aligned}\tag{1.18}$$

We prove that so defined adjoint action realizes the level zero representation of the algebra $\mathcal{A}(\widehat{sl}_2)$ onto the space of states $\mathcal{H}$, such that n-particle states compose n-fold tensor products of two-dimensional representations. The quantum affine symmetry of the Hilbert space $\mathcal{H}$ found in [3] can be realized via the asymptotical expansion of the adjoint action of the currents of the algebra $\mathcal{A}(\widehat{sl}_2)$.

The paper is organized as follows. In the second section we consider the canonical quantization of the SG model at FF point and its specialization to right Rindler wedge. Then we construct the nonlocal integrals of motion and develop bosonization of all the objects in terms of these integrals of motion. The third section is devoted to the description of the screening currents algebra for general value of SG coupling constant satisfying $1 < \beta < \sqrt{2}$. In the last section we develop the angular quantization scheme in full aspect; for example, we construct the monodromy matrix on the total Hilbert space and investigate some of its properties.

2 Canonical quantization

2.1 Sine-Gordon model at free fermion point

It is well known [6] that the SG model model with the action (1.1) is equivalent on the quantum level to the massive Thirring model defined by the action

$$\begin{aligned}S_{\mathrm{Th}} = \int dx^0 dx^1 \left[\frac{1}{2}\left(\overline{\Psi}(x) i\gamma^\mu \partial_\mu \Psi(x) - \partial_\mu \overline{\Psi}(x) i\gamma^\mu \Psi(x)\right)\right. \\ \left. - m\overline{\Psi}(x)\Psi(x) - \frac{g}{2}\left(\overline{\Psi}(x)\gamma^\mu\Psi(x)\right)^2\right]\end{aligned} \tag{2.1}$$

where $g = \frac{\pi(1-\xi)}{2\xi}$. The equivalence is established by the following bosonization rules

$$\frac{\beta}{2\pi}\varepsilon^{\mu\nu}\partial_\nu\Phi(x^0,x^1) = \overline{\Psi}(x^0,x^1)\gamma^\mu\Psi(x^0,x^1),$$
$$\cos(i\beta\Phi(x^0,x^1)) = \overline{\Psi}(x^0,x^1)\Psi(x^0,x^1) \qquad (2.2)$$

where $\varepsilon^{\mu\nu}$ is antisymmetric tensor normalized $\varepsilon^{01} = 1$.

At the FF point ($\xi = 1$) the interaction in the Thirring model vanishes and its lagrangian becomes a lagrangian of free massive Dirac fermions

$$\mathcal{L}(x^0,x^1) = \left[\frac{1}{2}\left(\overline{\Psi}(x)i\gamma^\mu\partial_\mu\Psi(x) - \partial_\mu\overline{\Psi}(x)i\gamma^\mu\Psi(x)\right) - m\overline{\Psi}(x)\Psi(x)\right], \qquad (2.3)$$

where $\overline{\Psi}(x) = \Psi^\dagger(x)\gamma^0$ is a Dirac conjugated spinor. We fix the γ-matrices to be

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The complete set of solutions to the corresponding linear equation of motion can be chosen in the form

$$\Psi_\theta(x^0,x^1) = \sqrt{\frac{m}{2}}\begin{pmatrix} e^{\theta/2} \\ ie^{-\theta/2} \end{pmatrix} e^{-im\mathrm{ch}(\theta)x^0 + im\mathrm{sh}(\theta)x^1} \qquad (2.4)$$

and satisfies the completeness relation with respect to the scalar product

$$(\Psi_\theta, \Psi_{\theta'}) = \int_{-\infty}^{\infty} dx^1 \overline{\Psi}_\theta(x^0,x^1)\gamma^0\Psi_{\theta'}(x^0,x^1) = \delta(\theta-\theta'). \qquad (2.5)$$

Note that the solutions (2.4) are anti-periodic with respect to the shift $\theta \to \theta + 2\pi i$.

The completeness relation (2.5) allows one to quantize the Dirac field $\Psi(x) \equiv \Psi_+(x)$ and its hermitian conjugate $\Psi_+^\dagger(x) \equiv \Psi_-(x)$

$$\Psi_+(x^0,x^1) = \int_{-\infty}^{\infty} d\theta \left[c(\theta)\Psi_\theta(x) + d^\dagger(\theta)\Psi_\theta^*(x)\right], \qquad (2.6)$$

$$\Psi_-(x^0,x^1)=\int_{-\infty}^{\infty}d\theta\left[d(\theta)\Psi_\theta(x)+c^\dagger(\theta)\Psi^*_\theta(x)\right], \tag{2.7}$$

by imposing equal time anticommutation relations

$$\begin{aligned}\{\psi_+(x^0,x),\psi_-(x^0,x')\}&=2\pi\delta(x-x'),\\ \{\overline{\psi}_+(x^0,x),\overline{\psi}_-(x^0,x')\}&=2\pi\delta(x-x'),\end{aligned} \tag{2.8}$$

where $\psi_\pm(x^0,x^1)$ and $\overline{\psi}_\pm(x^0,x^1)$ are components of spinor $\Psi_\pm(x^0,x^1)$. One can verify now that (2.8) and the normalization condition (2.5) imply the standard anticommutation relations

$$\{c(\theta),c^\dagger(\theta')\}=\delta(\theta-\theta'),\quad \{d(\theta),d^\dagger(\theta')\}=\delta(\theta-\theta') \tag{2.9}$$

and all others are trivial.

The Hilbert space $\mathcal{H}$ of this model is defined by application of the creation operators $c^\dagger(\theta)$ and $d^\dagger(\theta)$ to the 'physical' vacuum vector $|\text{vac}\rangle_{\text{ph}}$ annihilated by the operators $c(\theta)$ and $d(\theta)$.

The integrals of motion are constructed from the conserved currents:

$$\frac{\partial J_y}{\partial\overline{y}}-\frac{\partial J_{\overline{y}}}{\partial y}=0$$

and are given by the contour integral

$$Q^J=\int(dy\ J_y+d\overline{y}\ J_{\overline{y}})\ . \tag{2.10}$$

The charge Q^J is conserved along the evolution which is 'orthogonal' to the contour in the definition of Q^J. In the standard quantization picture this contour is chosen to be equal time line $x^0=\text{const}$ in the space-time. The charges which are conserved along the evolution with respect to the time x^0 can be obtained from the currents

$$J_y^\pm = (D\psi_\pm)\psi_\pm,\quad J_{\overline{y}}^\pm=(D\overline{\psi}_\pm)\overline{\psi}_\pm, \tag{2.11}$$
$$J_y^0 = (D\psi_-)\psi_+,\quad J_{\overline{y}}^0=(D\overline{\psi}_-)\overline{\psi}_+, \tag{2.12}$$

or equivalently from

$$\tilde{J}_y^\pm = \psi_\pm(D\psi_\pm),\quad \tilde{J}_{\overline{y}}^\pm=\overline{\psi}_\pm(D\overline{\psi}_\pm), \tag{2.13}$$
$$\tilde{J}_y^0 = \psi_-(D\psi_+),\quad \tilde{J}_{\overline{y}}^0=\overline{\psi}_-(D\overline{\psi}_+), \tag{2.14}$$

for the operator $D = \partial_y^n$ or $\partial_{\overline{y}}^n$, $n \geqslant 0$. Let us denote the neutral charges which correspond to the first operator as I_n and to the second as $\overline{I}_n$. The charged conserved quantities we denote by $I_n^{\pm}$ and $\overline{I}_n^{\pm}$ respectively. They have the explicit expressions in terms of the operators acting in total Hilbert space:

$$I_n = m^n \int_{-\infty}^{\infty} d\theta \; e^{n\theta} \left(c^{\dagger}(\theta) c(\theta) + d^{\dagger}(\theta) d(\theta) \right) ,$$
$$\overline{I}_n = m^n \int_{-\infty}^{\infty} d\theta \; e^{-n\theta} \left(c^{\dagger}(\theta) c(\theta) + d^{\dagger}(\theta) d(\theta) \right) \qquad (2.15)$$

and similar formulas for $I_n^{\pm}$ and $\overline{I}_n^{\pm}$. The Hamiltonian H which describes the evolution of the quantum fields (2.6) and (2.7) with respect to the time x^0 is given by the sum $(I_1 + \overline{I}_1)/2$ and has eigenvalue $m \mathrm{ch}\, \theta$ on the one-particle states generated by $c^{\dagger}(\theta)$ and $d^{\dagger}(\theta)$ from physical vacuum $|\mathrm{vac}\rangle_{\mathrm{ph}}$.

However there is no a direct way to quantize the SG field $\Phi(x^0, x^1)$ at the FF point using the quantization of the Dirac fermion fields $\Psi(x^0, x^1)$. In particular, it is difficult to construct the realization of the commutation relations (1.4) directly in the infinite volume limit using the canonical anticommutation relations (2.8) and without referring to the lattice regularization.

On the other hand the canonical quantization of the free massive Dirac fermions in RRW allows one to construct the operators which are building blocks of the angular quantization method. This will be done in the next subsections with the main goal being to demonstrate the nonabelian symmetry algebra which appears in the angular quantization approach to the SG model.

2.2 Free fermions in Rindler wedge

Let us solve the equation of motion for free massive Dirac field in RRW using the parametrization (1.12). The solution to the Dirac equation of motion normalized with respect to the scalar product

$$(\Psi, \Psi') = \int_0^{\infty} dr \; \left(e^{-\alpha} \overline{\psi} \overline{\psi}' + e^{\alpha} \psi \psi' \right) , \quad \Psi = \begin{pmatrix} \overline{\psi} \\ \psi \end{pmatrix} \qquad (2.16)$$

is given in terms of MacDonald functions

$$\Psi_\nu(r,\alpha) = \frac{\sqrt{m}}{\sqrt{\pi}\Gamma(i\nu+1/2)} \begin{pmatrix} \mathrm{e}^{(2i\nu+1/2)\alpha} K_{i\nu+1/2}(mr) \\ \mathrm{e}^{(2i\nu-1/2)\alpha} K_{i\nu-1/2}(mr) \end{pmatrix}, \quad \nu \in \mathbb{R}\,, \tag{2.17}$$

and has exponentially decreasing asymptotics in RRW when $r \to \infty$. On the other hand we observe that rotation of angular time α by $2\pi i$ which corresponds to the path around origin in euclidean plane multiplies the solution (2.17) by the factor $-e^{-2\pi\nu}$. This signifies the fact that the space of functions used in canonical and angular quantization are completely different. Nevertheless, the completeness relation

$$(\Psi_\nu, \Psi_{\nu'}) = \delta(\nu+\nu') \tag{2.18}$$

allows one to quantize the Dirac fields in RRW

$$\Psi_\pm(r,\alpha) = \begin{pmatrix} \overline{\psi}_\pm(r,\alpha) \\ \psi_\pm(r,\alpha) \end{pmatrix} = \int_{-\infty}^{\infty} d\nu\; b_\pm(\nu)\Psi_\nu(r,\alpha) \tag{2.19}$$

by imposing the equal 'time' ($\alpha = \mathrm{const}$) anticommutation relations

$$\begin{aligned} \{\psi_+(r,\alpha), \psi_-(r',\alpha)\} &= -\mathrm{e}^{-\alpha}\delta(r-r'), \\ \{\overline{\psi}_+(r,\alpha), \overline{\psi}_-(r',\alpha)\} &= -\mathrm{e}^{\alpha}\delta(r-r') \end{aligned} \tag{2.20}$$

which are equivalent to

$$\{b_\pm(\nu), b_\pm(\nu')\} = 0, \quad \{b_+(\nu), b_-(\nu')\} = \delta(\nu+\nu')\,. \tag{2.21}$$

Rindler fermionic Fock space $\mathcal{H}_R^f$ is defined by the vacuum state $|\mathrm{vac}\rangle_f$ which satisfies

$$b_\pm(\nu)|\mathrm{vac}\rangle_f = 0, \quad \nu > 0\,. \tag{2.22}$$

The left vacuum vector ${}_f\langle\mathrm{vac}|$ is correspondingly defined:

$${}_f\langle\mathrm{vac}|b_\pm(\nu) = 0, \quad \nu < 0\,. \tag{2.23}$$

2.3 Scattering transform

For the quantum fermionic fields (2.19) we introduce the scattering transform [27]

$$\Psi(r,\alpha) \to \Lambda_\pm(\theta,\alpha) = \frac{\sqrt{m}}{2\sqrt{\pi}} \int_0^\infty dr\, e^{-mr\mathrm{ch}\theta} \left(\overline{\psi}_\pm(r,\alpha) e^{(\theta-\alpha)/2} + \psi_\pm(r,\alpha) e^{(\alpha-\theta)/2}\right), \quad (2.24)$$

where $\theta \in \mathbb{C}$ is spectral parameter. Using the free fermion equation of motion in RRW we can verify that the dependence of the operators $\Lambda_\pm$ on the angular time α reduces to a simple shift of the spectral parameter

$$\Lambda_\pm(\theta,\alpha) = \Lambda_\pm(\theta+\alpha) \ , \quad (2.25)$$

where $\Lambda_\pm(\theta)$ is the value of the scattering transform at the initial time, say $\alpha = 0$. It is clear that the scattering transform (2.24) is not defined for all values of the spectral parameter θ. For example, if the solutions $\overline{\psi}_\pm(r,\alpha)$ and $\psi_\pm(r,\alpha)$ have the constant asymptotics when $r \to \infty$ then the integral in (2.24) is convergent if $|\mathrm{Im}\ \theta| < \pi/2$, which follows from the inequality $\mathrm{Re}\ \mathrm{ch}\,\theta > 0$. However, the solutions (2.17) of the Dirac equation in RRW have exponentially decreasing asymptotics. Using the fact that the leading term of the asymptotic of the MacDonald function $K_x(z)$ when $z \to \infty$ does not depend on the index x and is proportional to $z^{-1/2}e^{-z}$ we find that the inequality mentioned above is replaced by the more weak inequality

$$\mathrm{Re}\ \mathrm{ch}\,\theta > -1 \ . \quad (2.26)$$

The solution of (2.26) defines a larger domain of existence of the scattering transform than specified above, namely

$$|\mathrm{Im}\ \theta| < \pi/2 + \epsilon, \quad \text{where} \quad \epsilon = \pi/2 - \arccos\left((\mathrm{ch}\,\mathrm{Re}\ \theta)^{-1}\right) \quad (2.27)$$

so the domain of the possible values of the spectral parameter is a strip whose width depends on the value of $\mathrm{Re}\ \theta$.

An important consequence of this observation is the fact that the points $\mathrm{Im}\ \theta = \pm\pi/2$ are *always* in the domain of existence of the scattering transform. This leads to the fact that the vacuum expectation value of the product $\Lambda_\pm(\theta)\Lambda_\mp(\theta')$ is a well defined meromorphic function in the domain $|\mathrm{Im}\,(\theta-\theta')|\leqslant\pi+\varepsilon$ for some positive number ε. Using the expression of the scattering data operators $\Lambda_\pm(\theta)$ in terms of the fermionic operators $b_\pm(\nu)$

$$\Lambda_\pm(\theta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\nu\ b_\pm(\nu)\Gamma\left(\frac{1}{2}-i\nu\right)e^{i\nu\theta} \tag{2.28}$$

we can calculate this function explicitly:

$$\langle\Lambda_\pm(\theta)\Lambda_\mp(\theta')\rangle=\frac{1}{4\pi^2}\beta\left(\frac{\theta-\theta'+\pi i}{2\pi i}\right), \tag{2.29}$$

where the β-function is $\beta(x)=\partial_x\ln\left(\Gamma\left(\frac{x+1}{2}\right)/\Gamma\left(\frac{x}{2}\right)\right)$. Since the function (2.29) has the poles only in the points $\theta=\theta'-\pi i(2k+1)$, $k=0,1,\ldots$ the domain of the 'existence' of this function can be extended to $\mathrm{Im}\,(\theta-\theta')>-3\pi$ with a simple pole at the point $\theta=\theta'-\pi i$. The function $\langle\Lambda_\pm(\theta)\Lambda_\mp(\theta')\rangle$ is given by the meromorphic function (2.29) in this domain. An immediate consequence of this fact is the anticommutation relation

$$\{\Lambda_+(\theta),\Lambda_-(\theta')\}=\frac{\pi}{\mathrm{ch}((\theta-\theta')/2)},\qquad |\mathrm{Im}\,(\theta-\theta')|<3\pi\ . \tag{2.30}$$

The scattering transform (2.24) describes an evolution of the initial data (the quantum fields $\Psi_\pm(r,0)$ at initial value of the angular time $\alpha=0$) with respect to this angular time. Since this evolution reduces to a simple shift in the spectral parameter one can easily restore the quantum fields $\Psi_\pm(r,\alpha)$ at arbitrary time α by solving inverse scattering problem, restoring quantum fields $\Psi_\pm(r,\alpha)$ from the operators $\Lambda_\pm(\theta)$.

This can be done using the operators $\mathcal{Z}_\pm(\theta)$ related to the operators $\Lambda_\pm(\theta)$ by the integral transform

$$\Lambda_\pm(\theta)=\frac{1}{4\pi}\int_{-\infty}^{\infty}d\theta'\,\frac{\mathcal{Z}_\pm(\theta')}{\mathrm{ch}((\theta-\theta')/2)} \tag{2.31}$$

which can be inverted as follows

$$\mathcal{Z}_\pm(\theta)=\Lambda(\theta+\pi i)+\Lambda(\theta-\pi i)\ . \tag{2.32}$$

The inverse scattering problem has a solution in terms of the operators $\mathcal{Z}_\pm(\theta)$:

$$\Psi_\pm(r,\alpha) = \frac{\sqrt{m}}{2\sqrt{\pi}} \int_{-\infty}^{\infty} d\theta\ \mathcal{Z}_\pm(\theta+\alpha) e^{-mr\,\mathrm{ch}\,\theta} \begin{pmatrix} e^{(\theta+\alpha)/2} \\ e^{-(\theta+\alpha)/2} \end{pmatrix}. \qquad (2.33)$$

One can observe that substitution of (2.33) into (2.24) leads to the integral transform (2.31).

Formula (2.32) allows one to calculate the vacuum expectation values of the different operators

$$\begin{aligned} \langle \Lambda_\pm(\theta_1)\mathcal{Z}_\mp(\theta_2)\rangle &= \langle \mathcal{Z}_\pm(\theta_1)\Lambda_\mp(\theta_2)\rangle \\ &= -\frac{1}{2\pi i}\frac{1}{\theta_1-\theta_2}, \quad \mathrm{Im}\,(\theta_1-\theta_2) > -2\pi, \\ \langle \mathcal{Z}_\pm(\theta_1)\mathcal{Z}_\mp(\theta_2)\rangle &= -\frac{1}{2\pi i}\left[\frac{1}{\theta_1-\theta_2-i\pi} + \frac{1}{\theta_1-\theta_2+i\pi}\right], \\ &\qquad \mathrm{Im}\,(\theta_1-\theta_2) > -\pi. \end{aligned} \qquad (2.34)$$

These formulas allow to verify that the canonical anticommutation relations (2.20) follow from the solution of the inverse scattering problem (2.33). The formula (2.34) demonstrates also that the operators $\mathcal{Z}_\pm(\theta)$ anticommute for real values of the spectral parameter θ.

We would like to remark here that the operators $\mathcal{Z}_\pm(\theta)$ being expressed in terms of the fermionic operators $b_\pm(\nu)$

$$\mathcal{Z}_\pm(\theta) = \int_{-\infty}^{\infty} d\nu\, \frac{b_\pm(\nu)}{\Gamma\left(\frac{1}{2}+i\nu\right)} e^{i\nu\theta} \qquad (2.35)$$

should not be understood literally, but rather as a certain normal ordering expression, where the normal ordering is dictated by the prescription (2.34) on the domains of analyticity of the products of the scattering data operators. The naive use of the vacuum expectation value

$$\langle b_\pm(\nu) b_\mp(\nu')\rangle = \delta(\nu+\nu')\Theta(\nu)$$

where $\Theta(\nu)$ is the step-function, in order to calculate (2.34) does not allow to find the domain where the vacuum expectation value $\langle \mathcal{Z}_\pm(\theta_1)\mathcal{Z}_\mp(\theta_2)\rangle$ is defined since this information is encoded in the analytical properties of the scattering transform.

In order to obtain (2.20) from (2.34) it is convenient to introduce the operators $Z^*_\pm(\theta)$ and $Z_\pm(\theta)$ as a shift by $\pm\pi i/2$ of the operators $\mathcal{Z}_\pm(\theta)$ using the freedom to move the contour in the integral representation (2.33):

$$Z^*_\pm(\theta) = \mathcal{Z}_\pm(\theta - \pi i/2), \quad Z_\pm(\theta) = \mathcal{Z}_\mp(\theta + \pi i/2)\ . \tag{2.36}$$

Then the fermion fields (2.33) can be rewritten in the two equivalent form:

$$\begin{aligned}\Psi_\pm(r,\alpha) &= \frac{\sqrt{m}}{2\sqrt{\pi}}\int_{-\infty}^{\infty} d\theta\ Z_\mp(\theta+\alpha)e^{-imr\mathrm{sh}(\theta)}\begin{pmatrix} e^{\pi i/4}e^{(\theta+\alpha)/2} \\ e^{-\pi i/4}e^{-(\theta+\alpha)/2}\end{pmatrix} \\ &= \frac{\sqrt{m}}{2\sqrt{\pi}}\int_{-\infty}^{\infty} d\theta\ Z^*_\pm(\theta+\alpha)e^{imr\mathrm{sh}(\theta)}\begin{pmatrix} e^{-\pi i/4}e^{(\theta+\alpha)/2} \\ e^{\pi i/4}e^{-(\theta+\alpha)/2}\end{pmatrix}.\end{aligned}$$

The operators $Z_\pm(\theta)$ or $Z^*_\pm(\theta)$ acting in angular fermionic or bosonic Hilbert spaces can be associated with the states in total Hilbert space $\mathcal{H}$ of the model. In the next subsections we will identify them with intertwining and dual intertwining operators for the screening current algebra. The fact that the pole at the point $\theta_1 = \theta_2 + i\pi$ does not produce the restriction on the domain of the analyticity can be seen in the general situation, because the origin of this pole is the pinching of the contour in the integral representation of the function $\langle\mathcal{Z}_\pm(\theta_1)\mathcal{Z}_\mp(\theta_2)\rangle$ when $\theta_1 \to \theta_2 + i\pi$.

2.4 Integrals of motion

The operator which describes the evolution with respect to the angular time α is the Lorentz boost operator K. In terms of the fields:

$$\begin{aligned}K = \frac{i}{2}\int_0^\infty dr\, r\Big[&e^{-\alpha}\left(\overline{\psi}_-\left(\partial_r\overline{\psi}_+\right) - \left(\partial_r\overline{\psi}_-\right)\overline{\psi}_+\right) \\ &- e^{\alpha}\left(\psi_-\left(\partial_r\psi_+\right) - \left(\partial_r\psi_-\right)\psi_+\right) + 2m\overline{\psi}_-\psi_+ - 2m\psi_-\overline{\psi}_+\Big].\end{aligned}$$

Using canonical anticommutation relations (2.20) one can find the action of this operator on the components of the Dirac spinor:

$$[K,\overline{\psi}_\pm(r,\alpha)] \ = \ i\frac{\partial\overline{\psi}_\pm(r,\alpha)}{\partial\alpha} - \frac{i}{2}\overline{\psi}_\pm(r,\alpha),$$

$$[K,\psi_\pm(r,\alpha)] = i\frac{\partial\psi_\pm(r,\alpha)}{\partial\alpha} + \frac{i}{2}\psi_\pm(r,\alpha) \ . \tag{2.37}$$

Note that the boost operator acts differently on the different components of spinors $\Psi_\pm(r,\alpha)$. The formulas (2.37) can be rewritten as follows

$$\begin{aligned} e^{-i\eta K}\overline{\psi}_\pm(r,\alpha)e^{i\eta K} &= e^{-\eta/2}\overline{\psi}_\pm(r,\alpha+\eta), \\ e^{-i\eta K}\psi_\pm(r,\alpha)e^{i\eta K} &= e^{\eta/2}\psi_\pm(r,\alpha+\eta) \ . \end{aligned} \tag{2.38}$$

After rotation of the fermions around the origin by $\eta = 2\pi i$ both of the equations (2.38) become

$$e^{2\pi K}\Psi_\pm(r,\alpha)e^{-2\pi K} = -\Psi_\pm(r,\alpha+2\pi i) \ . \tag{2.39}$$

In terms of Rindler fermions $b_\pm(\nu)$ the boost operator has a form

$$K = \int_{-\infty}^{\infty} d\nu \ \nu \ {:}b_-(\nu)b_+(-\nu){:} \equiv \int_{-\infty}^{\infty} d\nu \ \nu \ {:}b_+(\nu)b_-(-\nu){:} \tag{2.40}$$

and yields the value of continuous mode ν

$$[K, b_\pm(\nu)] = \nu b_\pm(\nu), \quad \forall\ \nu \in \mathbb{R} \ . \tag{2.41}$$

A second important operator is the operator of topological charge which can be written in terms of the fields as follows

$$Q = -\int_0^\infty dr \ \left(e^\alpha\psi_-(r,\alpha)\psi_+(r,\alpha) + e^{-\alpha}\overline{\psi}_-(r,\alpha)\overline{\psi}_+(r,\alpha)\right) . \tag{2.42}$$

In terms of the fermionic modes it has the form

$$Q = -\int_{-\infty}^{\infty} d\nu \ {:}b_-(\nu)b_+(-\nu){:} = \int_{-\infty}^{\infty} d\nu \ {:}b_+(\nu)b_-(-\nu){:} \tag{2.43}$$

and is normalized in such a way that the charges of the Rindler fermions $b_\pm(\nu)$ correspond to their indexes

$$[Q, b_\pm(\nu)] = \pm b_\pm(\nu), \quad \forall \nu \in \mathbb{R}. \tag{2.44}$$

In Rindler's parametrization the contours in the definition of the conserved charges (2.10) are the straight rays $\alpha = \text{const}$. So in RRW we have

$$Q^J = \frac{1}{2}\int_0^\infty dr \left(\mathrm{e}^\alpha J_y + \mathrm{e}^{-\alpha} J_{\overline{y}}\right) . \tag{2.45}$$

Let us consider the charges given by this formula and compute them for the conserved currents J^0 (2.12) and $\tilde{J}^0$ (2.14). We obtain the result

$$I_n = (-1)^n \tilde{I}_n = \int_{-\infty}^{\infty} d\theta \; e^{n\theta} {:}\mathcal{Z}_-(\theta)\Lambda_+(\theta){:}. \tag{2.46}$$

Using formulas (2.28) and (2.35) we can observe that arbitrary non-vanishing matrix elements of the integrals I_n or $\tilde{I}_n$ in the Fermionic Fock space are divergent. To avoid this divergence we consider the analytical continuation of discrete index $n \to -i\lambda$ to the imaginary axis, where $\lambda \in \mathbb{R}$. In this case the charges given by the currents (2.13), (2.14) and (2.11), (2.12) will produce well defined quantities. In contrast to (2.46) the charges corresponding to the currents J^0 and $\tilde{J}^0$ do not coincide. We denote the ones corresponding to neutral currents (2.14) and (2.12) as a_λ, $\tilde{a}_\lambda$. In terms of the scattering data operators or in Rindler fermions they have the form

$$\begin{aligned} a_\lambda &= \int_{-\infty}^{\infty} d\theta \; e^{i\lambda\theta} {:}\mathcal{Z}_-(\theta)\Lambda_+(\theta){:} \\ &= \int_{-\infty}^{\infty} d\nu \; \frac{\Gamma\left(\frac{1}{2} - i\nu\right)}{\Gamma\left(\frac{1}{2} + i(\lambda - \nu)\right)} {:}b_-(\lambda - \nu) b_+(\nu){:} \; , \end{aligned} \tag{2.47}$$

$$\begin{aligned} \tilde{a}_\lambda &= \int_{-\infty}^{\infty} d\theta \; e^{i\lambda\theta} {:}\Lambda_-(\theta)\mathcal{Z}_+(\theta){:} \\ &= \int_{-\infty}^{\infty} d\nu \; \frac{\Gamma\left(\frac{1}{2} - i(\lambda - \nu)\right)}{\Gamma\left(\frac{1}{2} + i\nu\right)} {:}b_-(\lambda - \nu) b_+(\nu){:} \; , \end{aligned} \tag{2.48}$$

where normal ordering is defined with respect to fermionic vacuum vectors.

By comparing the formulas (2.48) and (2.47) we conclude that the conserved charges a_λ and $\tilde{a}_\lambda$ are related to each other by some complicated integral transform. This integral transform can be described algebraically by extending the algebra of the operators a_λ and $\tilde{a}_\lambda$. This will be demonstrated in the Appendix A.

Using (2.21) we see that

$$[a_\lambda, a_\mu] = [\tilde{a}_\lambda, \tilde{a}_\mu] = \lambda \delta(\lambda + \mu) \tag{2.49}$$

and this Heisenberg type commutation relation allows us to use these operators for the bosonization.

2.5 The bosonization and the screening currents

Using (2.21) we can obtain the commutation relations:

$$[a_\lambda, \mathcal{Z}_-(\theta)] = e^{-i\lambda\theta}\mathcal{Z}_-(\theta), \quad [a_\lambda, \Lambda_+(\theta)] = -e^{-i\lambda\theta}\Lambda_+(\theta) \tag{2.50}$$

$$[\tilde{a}_\lambda, \Lambda_-(\theta)] = e^{-i\lambda\theta}\Lambda_-(\theta), \quad [\tilde{a}_\lambda, \mathcal{Z}_+(\theta)] = -e^{-i\lambda\theta}\mathcal{Z}_+(\theta) \tag{2.51}$$

These formulas together with (2.49) allow to bosonize the operators $\Lambda_+(\theta)$ and $\mathcal{Z}_-(\theta)$ in terms of a free bosonic field constructed from continuous bosons a_λ and the operators $\Lambda_-(\theta)$ and $\mathcal{Z}_+(\theta)$ from the analogous free field constructed from the bosons $\tilde{a}_\lambda$.

This bosonization should conserve all the properties of the Hilbert space $\mathcal{H}_R^f$ and the action of the operators $\Lambda_\pm(\theta)$, $\mathcal{Z}_\pm(\theta)$ on it. It is clear that it is impossible to do this using only bosonic modes a_λ because they carry the charge 0, while the Hilbert space $\mathcal{H}_R^f$ is naturally graded with respect to topological charge operator:

$$\mathcal{H}_R^f = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_{R,n}^f, \quad \mathcal{H}_{R,n}^f = \left\{ x \in \mathcal{H}_R^f \;\middle|\; Qx = nx \right\} \tag{2.52}$$

Because of the formulas (2.28) and (2.35) the operators $\Lambda_\pm(\theta)$ and $\mathcal{Z}_\pm(\theta)$ change the topological charge

$$\Lambda_\pm(\theta),\ \mathcal{Z}_\pm(\theta)\ :\ \mathcal{H}_{R,n}^f \to \mathcal{H}_{R,n\pm1}^f\ . \tag{2.53}$$

Note that in our normalization the topological charge operator Q coincides with the operators $-a_0 \equiv -\tilde{a}_0$.

To conserve these properties of $\mathcal{H}_R^f$ in the bosonization picture we introduce a pair of zero mode operators $\mathcal{Q}$ and $\mathcal{P}$ which satisfy the commutation relations

$$[\mathcal{P}, \mathcal{Q}] = i \tag{2.54}$$

and bosonic vacuum vectors $|n\rangle_b$, $n \in \mathbb{Z}$ which are annihilated by all nonnegative bosonic modes and are eigenstates of the operator $\mathcal{P}$

$$a_\lambda|n\rangle_b = 0, \qquad \lambda \geqslant 0, \qquad \mathcal{P}|n\rangle_b = n|n\rangle_b\ . \tag{2.55}$$

We identify $\mathcal{H}_{R,n}^f$ with bosonic space $\mathcal{H}_{R,n}^b$ generated from bosonic vacuum vector $|n\rangle_b$

$$\int_{-\infty}^{0} f_n(\lambda_n)a_{\lambda_n}d\lambda_n \ldots \int_{-\infty}^{0} f_1(\lambda_1)a_{\lambda_1}d\lambda_1\ |n\rangle_b\ , \tag{2.56}$$

where the functions $f_i(\lambda)$ are analytical functions in a neighborhood of $\mathbb{R}_-$ except $\lambda = 0$, where they can have a simple pole.

Then due to (2.51) the operators $\Lambda_-(\theta)$ and $\mathcal{Z}_+(\theta)$ can be bosonized as follows:

$$\Lambda_+(\theta) = \exp\left(i\mathcal{Q} + \fint_{-\infty}^{\infty} \frac{d\lambda}{\lambda}\, a_\lambda e^{i\lambda\theta}\right),$$
$$\mathcal{Z}_-(\theta) = \exp\left(-i\mathcal{Q} - \fint_{-\infty}^{\infty} \frac{d\lambda}{\lambda}\, a_\lambda e^{i\lambda\theta}\right). \tag{2.57}$$

The integral under the exponent is understood as principal value integral to exclude the singularity at zero:

$$\fint_{-\infty}^{\infty} f(\lambda)\, d\lambda = \lim_{\epsilon\to+0}\left(\int_{-\infty}^{-\epsilon} f(\lambda)\, d\lambda + \int_{\epsilon}^{\infty} f(\lambda)\, d\lambda\right). \tag{2.58}$$

We define the products of the operators like (2.57) to be ζ-function regularized [19, 16]

$$\exp\left(\int_{-\infty}^{\infty} d\lambda\; g_1(\lambda)\, a_\lambda\right) \cdot \exp\left(\int_{-\infty}^{\infty} d\mu\; g_2(\mu)\, a_\mu\right)$$
$$= \frac{e^\gamma}{2\pi} \exp\left(\int_{\tilde{C}} \frac{d\lambda\, \ln(-\lambda)}{2\pi i} c(\lambda) g_1(\lambda) g_2(-\lambda)\right)$$
$$\cdot \exp\left(\int_{-\infty}^{\infty} d\lambda\, (g_1(\lambda) + g_2(\lambda))\, a_\lambda\right). \tag{2.59}$$

where $c(\lambda) = \lambda$ and γ is Euler constant and the contour $\tilde{C}$ is shown in Fig. 1.

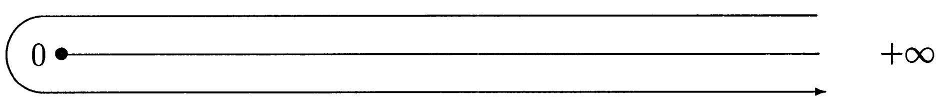

Figure 1.

Naturally there is an alternative way to bosonize the fermionic Fock space $\mathcal{H}_R^f$ using modes $\tilde{a}_\lambda$ and introducing the corresponding zero mode operators $\tilde{\mathcal{Q}}$, $\tilde{\mathcal{P}}$, the bosonic vacuum vectors and bosonic Fock spaces.

It is clear that these vacuum vectors are not a priori the same as for bosons a_λ because of the complicated commutation relations between the bosons a_λ and $\tilde{a}_\lambda$ (see Appendix A). This alternative bosonization looks as follows

$$\Lambda_-(\theta) = \exp\left(-i\tilde{\mathcal{Q}} - \fint_{-\infty}^{\infty} \frac{d\lambda}{\lambda}\, \tilde{a}_\lambda e^{i\lambda\theta}\right),$$
$$\mathcal{Z}_+(\theta) = \exp\left(i\tilde{\mathcal{Q}} + \fint_{-\infty}^{\infty} \frac{d\lambda}{\lambda}\, \tilde{a}_\lambda e^{i\lambda\theta}\right). \tag{2.60}$$

We also define the charged currents in the momentum space which correspond to the charged conserved currents (2.13) and (2.11) by the requirement that they relate the components of the operators $\mathcal{Z}_\pm(\theta)$. Let

$$E(\theta) = :\Lambda_+(\theta + \pi i/2)\Lambda_+(\theta - \pi i/2):,$$
$$\tilde{F}(\theta) = :\Lambda_-(\theta + \pi i/2)\Lambda_-(\theta - \pi i/2): \tag{2.61}$$

be the operator valued currents in the momentum space. Using vacuum expectation values and Wick theorem we can prove the formulas

$$\mathcal{Z}_+(\theta) = -i\int_{C_1} du\; e^{u-\theta}\mathcal{Z}_-(\theta)E(u) + i\int_{C_2} du\; e^{u-\theta}E(u)\mathcal{Z}_-(\theta)$$
$$= \Lambda_+(\theta + \pi i) + \Lambda_+(\theta - \pi i), \tag{2.62}$$
$$\mathcal{Z}_-(\theta) = -i\int_{C_1} du\; e^{u-\theta}\mathcal{Z}_+(\theta)\tilde{F}(u) + i\int_{C_2} du\; e^{u-\theta}\tilde{F}(u)\mathcal{Z}_+(\theta)$$
$$= \Lambda_-(\theta + \pi i) + \Lambda_-(\theta - \pi i), \tag{2.63}$$

where the contour C_1 goes from $-\infty$ to $+\infty$ and is above all the poles in the operator product expansion $\mathcal{Z}_-(\theta)E(u)$ and $\mathcal{Z}_+(\theta)\tilde{F}(u)$ and the contour C_2 is also from $-\infty$ to $+\infty$ and below all the poles in the OPE $E(u)\mathcal{Z}_-(\theta)$ and $\tilde{F}(u)\mathcal{Z}_+(\theta)$. Let us prove (2.62). It follows from the OPE

$$\mathcal{Z}_-(\theta)E(u) = :\mathcal{Z}_-(\theta)E(u): - \frac{1}{2\pi i}\left[\frac{\Lambda_+\left(u - \frac{\pi i}{2}\right)}{\theta - u - \frac{\pi i}{2}} - \frac{\Lambda_+\left(u + \frac{\pi i}{2}\right)}{\theta - u + \frac{\pi i}{2}}\right],$$
$$E(u)\mathcal{Z}_-(\theta) = :\mathcal{Z}_-(\theta)E(u): - \frac{1}{2\pi i}\left[\frac{\Lambda_+\left(u + \frac{\pi i}{2}\right)}{u - \theta - \frac{\pi i}{2}} - \frac{\Lambda_+\left(u - \frac{\pi i}{2}\right)}{u - \theta + \frac{\pi i}{2}}\right].$$

Now the formula (2.62) follows from the trivial calculation of the integrals. The second relation (2.63) is proved analogously.

Using formulas (2.57) and (2.60) we can write down the bosonized expressions for the screening currents

$$E(u) = \exp\left(2iQ + 2\fint_{-\infty}^{\infty}\frac{d\lambda}{\lambda}\, e^{iu\lambda}\mathrm{ch}\,(\pi\lambda/2)\ a_\lambda\right)\,, \tag{2.64}$$

$$\tilde{F}(u) = \exp\left(-2i\tilde{Q} - 2\fint_{-\infty}^{\infty}\frac{d\lambda}{\lambda}\, e^{iu\lambda}\mathrm{ch}\,(\pi\lambda/2)\ \tilde{a}_\lambda\right)\,, \tag{2.65}$$

in terms of the bosons a_λ for the current $E(u)$ and $\tilde{a}_\lambda$ for $\tilde{F}(u)$. Because the operators a_λ and $\tilde{a}_\lambda$ do not form a closed algebra the screening currents $E(u)$ and $\tilde{F}(u)$ also do not form the closed algebra. In the next subsection we will define another pair of the screening currents $\tilde{E}(u)$ and $F(u)$ such that the pairs $E(u)$, $F(u)$ or $\tilde{E}(u)$, $\tilde{F}(u)$ do form the closed algebra both isomorphic to the screening current algebra introduced in [26, 20].

2.6 Another pair of screening currents and quantum Jost functions

The commutation relations (1.4) have a smooth classical limit when $\xi \to 0$ and correspondingly $q' \to 1$. In this limit these commutation relations become Poisson brackets for the elements of monodromy matrices for classical SG model [13]. The elements $Z'_\varepsilon(\alpha)$ of the 'monodromy' matrix associated with the half-line were introduced in [26]. It was shown in [25] that they are the quantum analogs of classical Jost functions. These classical objects can be written explicitly as path-ordered exponents of SG connections in RRW and satisfy the Poisson bracket relation

$$\{\mathcal{Z}'_{\varepsilon_1}(\alpha_1), \mathcal{Z}'_{\varepsilon_2}(\alpha_2)\} = r^{\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2}(\alpha_1-\alpha_2)\mathcal{Z}'_{\varepsilon'_2}(\alpha_2)\mathcal{Z}'_{\varepsilon'_1}(\alpha_1)\,, \tag{2.66}$$

where $r^{\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2}(\alpha)$ is a classical trigonometric r-matrix

$$r^{\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2}(\alpha) = \lim_{\xi\to 0}\frac{R^{\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2}(\alpha,\xi)-1}{\pi i\xi} \tag{2.67}$$

$$= \frac{1}{2}\begin{pmatrix} \mathrm{th}(\alpha/2) & & & \\ & -\mathrm{cth}(\alpha/2) & 2\mathrm{sh}^{-1}(\alpha) & \\ & 2\mathrm{sh}^{-1}(\alpha) & -\mathrm{cth}(\alpha/2) & \\ & & & \mathrm{th}(\alpha/2) \end{pmatrix}$$

obtained from the R-matrices (3.6). Note, that the scalar factor of the R-matrix (3.6) also contribute to the classical r-matrix (2.67).

Unfortunately, a way to obtain the quantum analog of the relations (2.66)

$$\mathcal{Z}'_{\varepsilon_1}(\alpha_1)\mathcal{Z}'_{\varepsilon_2}(\alpha_2) = R^{\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2}(\alpha_1-\alpha_2,\xi)\mathcal{Z}'_{\varepsilon'_2}(\alpha_2)\mathcal{Z}'_{\varepsilon'_1}(\alpha_1) \qquad (2.68)$$

starting from SG Lagrangian is not known. Nevertheless, one can formulate the properties of these operators which allows to reconstruct them uniquely. These properties follow from the interpretation of the operators $\mathcal{Z}_\pm(\theta)$ as the operators in $\mathcal{H}_R$ which correspond to the states in total Hilbert space $\mathcal{H}$ of the model. Since the operators $\mathcal{Z}'_\pm(\alpha)$ are related to the quantum Jost functions and to the integrals of motion it is natural to require their commutativity with the operators $\mathcal{Z}_\varepsilon(\theta)$ up to the phase

$$\mathcal{Z}_\varepsilon(\theta)\mathcal{Z}'_\nu(\alpha) = \varepsilon\nu\ \phi(\theta-\alpha)\ \mathcal{Z}'_\nu(\alpha)\mathcal{Z}_\varepsilon(\theta) \qquad (2.69)$$

where $\phi(\theta)$ is a yet unknown function (see (3.65) below). Because of the relations (2.62) and (2.63) this requirement is equivalent to the following

$$E(u)\mathcal{Z}'_-(\alpha) = -\mathcal{Z}'_-(\alpha)E(u), \quad \tilde{F}(u)\mathcal{Z}'_+(\alpha) = -\mathcal{Z}'_+(\alpha)\tilde{F}(u) \qquad (2.70)$$

These anticommutation relations can be satisfied by the following bosonization of the operators $\mathcal{Z}'_\nu(\alpha)$

$$\mathcal{Z}'_-(\alpha) = \exp\left(iQ/2 + \int_{-\infty}^{\infty}\frac{d\lambda}{\lambda}\,\frac{a_\lambda}{2\mathrm{ch}\,\pi\lambda/2}\,e^{i\lambda\alpha}\right), \qquad (2.71)$$

$$\mathcal{Z}'_+(\alpha) = \exp\left(-i\tilde{Q}/2 - \int_{-\infty}^{\infty}\frac{d\lambda}{\lambda}\,\frac{\tilde{a}_\lambda}{2\mathrm{ch}\,\pi\lambda/2}\,e^{i\lambda\alpha}\right). \qquad (2.72)$$

Using this bosonization and also the rule of the normal ordering (2.59) we can observe that the operators $\mathcal{Z}'_\pm(\alpha)$ are related to the scattering data operators $\Lambda_\pm(\theta)$ as follows

$$\Lambda_{-\varepsilon}(\alpha) = g^{-1}\mathcal{Z}'_\varepsilon\left(\alpha-\frac{i\pi}{2}\right)\mathcal{Z}'_\varepsilon\left(\alpha+\frac{i\pi}{2}\right)$$

$$= :\mathcal{Z}'_\varepsilon\left(\alpha - \frac{i\pi}{2}\right)\mathcal{Z}'_\varepsilon\left(\alpha + \frac{i\pi}{2}\right):. \tag{2.73}$$

The normalization constant is given in terms of double Γ-functions (see Appendix B for the definition of these functions). Note that formula (2.73) allows to identify the scattering data operators $\Lambda_\pm(\theta)$ with the generating functions of the local operators introduced in the paper [26]. From the formulas (2.71) and (2.57) we can easily find the function $\phi(\theta) = \mathrm{ctg}\left(\frac{\pi}{4} + \frac{\theta}{2i}\right)$.

The formulas in (2.73) are equalities in different bosonic Fock spaces generated by the operators a_λ and $\tilde{a}_\lambda$ respectively. We can translate them in equal bosonic spaces introducing another pair of screening currents $F(u)$ and $\tilde{E}(u)$

$$F(u) \equiv \mathcal{Z}_-(u), \quad \tilde{E}(u) \equiv \mathcal{Z}_+(u). \tag{2.74}$$

These relations are given by the integral transforms

$$\begin{aligned} \mathcal{Z}'_+(\alpha) &= 2e^{-2\gamma}\pi^{3/2}\int_{C'} du\, e^{\frac{u-z}{2}}\left[e^{\frac{\pi i}{4}}\mathcal{Z}'_-(\alpha)F(u) + e^{-\frac{\pi i}{4}}F(u)\mathcal{Z}'_-(\alpha)\right] \\ &= \frac{e^{-3\gamma/2}}{\sqrt{2\pi}}\int_{C'} du\, \Gamma\left(\frac{1}{4} + \frac{u-\alpha}{2\pi i}\right)\Gamma\left(\frac{1}{4} - \frac{u-\alpha}{2\pi i}\right) :\mathcal{Z}'_-(\alpha)F(u):, \end{aligned} \tag{2.75}$$

and

$$\begin{aligned} \mathcal{Z}'_-(z) &= 2e^{-2\gamma}\pi^{3/2}\int_{C'} du\, e^{\frac{u-z}{2}}\left[e^{\frac{\pi i}{4}}\mathcal{Z}'_+(\alpha)\tilde{E}(u) + e^{-\frac{\pi i}{4}}\tilde{E}(u)\mathcal{Z}'_+(\alpha)\right] \\ &= \frac{e^{-3\gamma/2}}{\sqrt{2\pi}}\int_{C'} du\, \Gamma\left(\frac{1}{4} + \frac{u-\alpha}{2\pi i}\right)\Gamma\left(\frac{1}{4} - \frac{u-\alpha}{2\pi i}\right) :\mathcal{Z}'_+(\alpha)\tilde{E}(u):, \end{aligned} \tag{2.76}$$

where in both formulas the contour C' goes from $-\infty$ to ∞ along the real axis such that

$$\mathrm{Im}\ \alpha - \pi/2 < \mathrm{Im}\ u < \mathrm{Im}\ \alpha + \pi/2. \tag{2.77}$$

The proof of the fact that the relation (2.75) is equivalent to the relation (2.73) for $\varepsilon = -$ or vice versa (2.76) is equivalent to the relation (2.73) for $\varepsilon = +$ can be found in the Appendix B.

The fact that the second set of the screening currents for the quantum Jost operators $\mathcal{Z}'_\pm(\alpha)$ coincided with scattering data operators

$\mathcal{Z}_{\pm}(\theta)$ is specific to the FF point. But what is true in general is that we have the closed algebra of screening currents either for the pair $E(u)$, $F(u)$ or for the pair $\tilde{E}(u)$, $\tilde{F}(u)$. Both of these algebras can be obtained from the commutation relations (1.15) using different Gauss decompositions of L-operators.

3 Algebra of screening currents

As we already said in the Introduction one of the goals of this paper is to explain the algebraic structures which allows to describe simultaneously two quantum group structures with different parameter of deformations developing ideas of the paper [26]. This algebra will be introduced and explained in this section.

This is a non-abelian algebra of screening currents which can be defined using exact S-matrix of soliton-antisoliton scattering in SG model [34, 20]. We define this algebra for the value of the of the renormalized coupling constant $1 < \xi < \infty$ in so called breatherless regime. We will demonstrate that the representation theory of this algebra has smooth limit when $\xi \to 1$, which corresponds to the FF point of SG model. Using the bosonization we will show that the intertwining operators of the level 1 highest weight modules for the screening current algebra coincide with the operators $\mathcal{Z}_{\pm}(\theta)$ and $\mathcal{Z}'_{\pm}(\theta)$ defined in the previous section from analysis of massive Dirac fermions in RRW.

3.1 R and S Matrices

Consider the following $\mathcal{R}$-matrix.

$$\mathcal{R}^{+}(u,\xi) = \tau^{+}(u)\mathcal{R}(u,\xi), \quad \mathcal{R}(u,\xi) = r(u,\xi)\overline{\mathcal{R}}(u,\xi) \ , \tag{3.1}$$

$$\overline{\mathcal{R}}(u,\xi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u,\xi) & c(u,\xi) & 0 \\ 0 & \tilde{c}(u,\xi) & b(u,\xi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ,$$

$$r(u,\xi)=\frac{\Gamma\left(\frac{1}{\xi}\right)\Gamma\left(1+\frac{iu}{\pi\xi}\right)}{\Gamma\left(\frac{1}{\xi}+\frac{iu}{\pi\xi}\right)}\prod_{p=1}^{\infty}\frac{R_p(u,\eta)R_p(i\pi-u,\eta)}{R_p(0,\xi)R_p(i\pi,\xi)}\,, \tag{3.2}$$

$$R_p(u,\eta)=\frac{\Gamma\left(\frac{2p}{\xi}+\frac{iu}{\pi\xi}\right)\Gamma\left(1+\frac{2p}{\xi}+\frac{iu}{\pi\xi}\right)}{\Gamma\left(\frac{2p+1}{\xi}+\frac{iu}{\pi\xi}\right)\Gamma\left(1+\frac{2p-1}{\xi}+\frac{iu}{\pi\xi}\right)}\,,$$

$$b(u,\xi)=\frac{\mathrm{sh}\,\frac{u}{\xi}}{\mathrm{sh}\,\frac{u-\pi i}{\xi}}\,,\quad c(u,\xi)=-\frac{\mathrm{sh}\,\frac{i\pi}{\xi}}{\mathrm{sh}\,\frac{u-\pi i}{\xi}}\,,\quad \tau^{+}(u)=i\mathrm{cth}\left(\frac{u}{2}\right).$$

The scalar factor $r(u,\xi)$ has an integral representation

$$r(u,\xi)=\exp\left(2i\int_0^{\infty}\frac{d\lambda}{\lambda}\frac{\mathrm{sh}\,\lambda/2}{\mathrm{sh}\,\lambda}\,\frac{\mathrm{sh}\,(\xi-1)\lambda/2}{\mathrm{sh}\,\xi\lambda/2}\,\sin\left(\frac{\lambda u}{\pi}\right)\right),$$
$$-\pi<\mathrm{Im}\,u<\pi\,. \tag{3.3}$$

The R-matrix (3.1) differs from the physical S-matrix which describe the soliton-antisoliton scattering by the transformation

$$S(\theta)=-r(\theta,\xi)\overline{S}(\theta),\quad \overline{S}(\theta)=(\sigma_z\otimes 1)\overline{\mathcal{R}}(\theta,\xi)(1\otimes\sigma_z) \tag{3.4}$$

which change the sign in front of the elements $b(\theta,\xi)$. In the classical limit $\xi\to\infty$ such that u/ξ is fixed the R-matrix $\mathcal{R}(u,\xi)$ goes to identity, while the 'physical' S-matrix goes to diag$(-1,1,1,-1)$. On the other hand at the FF point $\xi\to 1$ S-matrix becomes equal to -1 while R-matrix (3.1) to diag$(1,-1,-1,1)$.

The matrix $\overline{S}(\theta)$ can be written using a multiplicative spectral parameter $z=e^{-\theta/\xi}$ and deformation parameter q introduced by (1.3)

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \frac{z-z^{-1}}{zq-z^{-1}q^{-1}} & \frac{(q-q^{-1})}{zq-z^{-1}q^{-1}} & 0\\ 0 & \frac{(q-q^{-1})}{zq-z^{-1}q^{-1}} & \frac{z-z^{-1}}{zq-z^{-1}q^{-1}} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{3.5}$$

The physical R-matrix which describe the commutation relations of quantum monodromy matrices in SG model (1.4) and quantum Jost functions (2.68) can be similarly written in terms of the matrix $\mathcal{R}(u,\xi)$

$$R(\alpha)=r(\alpha,\xi+1)\overline{R}(\alpha),$$

$$\overline{R}(\alpha) = (\sigma_z \otimes 1)\overline{\mathcal{R}}(-\alpha, -\xi - 1)(1 \otimes \sigma_z) \tag{3.6}$$

Note that matrices $\overline{S}(\theta)$ and $\overline{R}(\alpha)$ are related by the duality transformation

$$\xi \to -\xi - 1, \quad \theta \to -\alpha\ . \tag{3.7}$$

The fact is that the scalar factors of S and R-matrices are related by the same transformation. To see this one should use simple identity

$$\frac{\operatorname{sh}\lambda/2}{\operatorname{sh}\lambda}\left(\frac{\operatorname{sh}(\xi+1)\lambda/2}{\operatorname{sh}\xi\lambda/2} + \frac{\operatorname{sh}(\xi-1)\lambda/2}{\operatorname{sh}\xi\lambda/2}\right) = 1$$

to rewrite the scalar factor $-r(\theta, \xi)$ in the form

$$-r(u,\xi) = \exp\left(-2i\int_0^\infty \frac{d\lambda}{\lambda}\frac{\operatorname{sh}\lambda/2}{\operatorname{sh}\lambda}\frac{\operatorname{sh}(\xi+1)\lambda/2}{\operatorname{sh}\xi\lambda/2}\sin\left(\frac{\lambda u}{\pi}\right)\right). \tag{3.8}$$

Now one can see that functions $r(u, \xi + 1)$ and $-r(u, \xi)$ transform to each other under (3.7), although one should not think about this transformation literally. The point is that the quantization of the SG model is well defined for $0 < \xi < \infty$, so in order to perform the dual transformation (3.7) we should first go to $+\infty$ and then come back to the negative axis from $-\infty$. During this path the properties of the model itself change drastically.

3.2 Algebra of screening currents

Set $1 < \xi < \infty$. Let

$$L(u,\xi) = \begin{pmatrix} L_{++}(u,\xi) & L_{+-}(u,\xi) \\ L_{-+}(u,\xi) & L_{--}(u,\xi) \end{pmatrix} \tag{3.9}$$

be a quantum L-operator whose matrix elements are treated as generating functions for the elements of the algebra given by the commutation relations:

$$R^+(u_1 - u_2, \xi + c)L_1(u_1,\xi)L_2(u_2,\xi)$$
$$= L_2(u_2,\eta)L_1(u_1,\xi)R^+(u_1-u_2,\xi), \tag{3.10}$$
$$\mathrm{qdet}L(u) = L_{++}(u - i\pi)L_{--}(u) - L_{+-}(u-i\pi)L_{-+}(u) = 1. \tag{3.11}$$

Let

$$L(u)=\begin{pmatrix}1 & f(u)\\ 0 & 1\end{pmatrix}\begin{pmatrix}k_1(u) & 0\\ 0 & k_2(u)\end{pmatrix}\begin{pmatrix}1 & 0\\ e(u) & 1\end{pmatrix}, \tag{3.12}$$

be the Gauss decomposition of the L-operator (3.9). This Gauss decomposition corresponds to the algebra of screening currents $E(u)$ and $F(u)$ described in the previous section at the value $\xi=1$. To obtain the algebra related to the currents $\tilde{E}(u)$ and $\tilde{F}(u)$ we should start from another Gauss decomposition

$$\tilde{L}(u)=\begin{pmatrix}1 & 0\\ \tilde{e}(u) & 1\end{pmatrix}\begin{pmatrix}\tilde{k}_1(u) & 0\\ 0 & \tilde{k}_2(u)\end{pmatrix}\begin{pmatrix}1 & \tilde{f}(u)\\ 0 & 1\end{pmatrix}. \tag{3.13}$$

The relation between the Gauss coordinates of both L-operators is complicated enough and can be described on the level of the bosonization of the L-operators by the relations similar to those described in the Appendix A. For the remainder of this paper we will work only with the operator (3.12).

One can deduce from (3.10), (3.11) that

$$k_1(u)=(k_2(u+i\pi))^{-1}.$$

Let

$$h(u)=k_1(u)\,k_2(u)^{-1},\quad h'(u)=k_2(u)^{-1}\,k_1(u)=\frac{\xi\sin(\pi/\xi)}{\xi'\sin(\pi/\xi')}h(u),$$

where by ξ' we denote the combination $\xi+c$ of the parameter ξ and the central element of the algebra $\mathcal{A}(\widehat{sl}_2)$.

The Gauss coordinates $e(u)$, $f(u)$ and $h(u)$ of the L-operator (3.9) satisfy the following commutation relations ($u=u_1-u_2$):

$$e(u_1)f(u_2)-f(u_2)e(u_1)=\frac{\operatorname{sh}(i\pi/\xi')}{\operatorname{sh}(u/\xi')}h'(u_1)-\frac{\operatorname{sh}(i\pi/\xi)}{\operatorname{sh}(u/\xi)}h(u_2), \tag{3.14}$$

$$\begin{aligned}&\operatorname{sh}\left(\frac{u+i\pi}{\xi}\right)h(u_1)e(u_2)-\operatorname{sh}\left(\frac{u-i\pi}{\xi}\right)e(u_2)h(u_1)\\ &=\operatorname{sh}\left(\frac{i\pi}{\xi}\right)\{h(u_1),e(u_1)\},\end{aligned} \tag{3.15}$$

$$\operatorname{sh}\left(\frac{u-i\pi}{\xi'}\right) h(u_1)f(u_2) - \operatorname{sh}\left(\frac{u+i\pi}{\xi'}\right) f(u_2)h(u_1)$$
$$= -\operatorname{sh}\left(\frac{i\pi}{\xi'}\right) \{h(u_1), f(u_1)\}, \tag{3.16}$$
$$\operatorname{sh}\left(\frac{u+i\pi}{\xi}\right) e(u_1)e(u_2) - \operatorname{sh}\left(\frac{u-i\pi}{\xi}\right) e(u_2)e(u_1)$$
$$= \operatorname{sh}\left(\frac{i\pi}{\xi}\right) \left(e(u_1)^2 + e(u_2)^2\right), \tag{3.17}$$
$$\operatorname{sh}\left(\frac{u-i\pi}{\xi'}\right) f(u_1)f(u_2) - \operatorname{sh}\left(\frac{u+i\pi}{\xi'}\right) f(u_2)f(u_1)$$
$$= -\operatorname{sh}\left(\frac{i\pi}{\xi'}\right) \left(f(u_1)^2 + f(u_2)^2\right), \tag{3.18}$$

$$\frac{\operatorname{sh}\left(\frac{u-i\pi}{\xi'}\right)}{\operatorname{sh}\left(\frac{u+i\pi}{\xi'}\right)} h(u_1)h(u_2) = h(u_2)h(u_1)\frac{\operatorname{sh}\left(\frac{u-i\pi}{\xi}\right)}{\operatorname{sh}\left(\frac{u+i\pi}{\xi}\right)} . \tag{3.19}$$

In the next subsections we will describe the finite and infinite dimensional representations of the algbera $\mathcal{A}(\widehat{sl}_2)$. We will consider also the tensor products of the representations where the action of the algebra $\mathcal{A}(\widehat{sl}_2)$ is defined by the following comultiplication structure compatible with the commutation relations (3.10):

$$\begin{aligned} \Delta\, c &= c^{(1)} + c^{(2)} = c\otimes 1 + 1\otimes c\ , \\ \Delta^{\rm op} L(u,\xi) &= L(u - i\pi c^{(2)}/4, \xi + c^{(2)}) \dot{\otimes}\ L(u + i\pi c^{(1)}/4, \xi) \\ \Delta \left(L(u,\xi)\right)^{-1} &= \left(L(u + i\pi c^{(2)}/4, \xi)\right)^{-1} \dot{\otimes} \left(L(u - i\pi c^{(1)}/4, \xi + c^{(1)})\right)^{-1}, \end{aligned} \tag{3.20}$$

where the symbol $\dot{\otimes}$ signifies the matrix tensor product

$$\left(A\dot{\otimes}B\right)_{ij} = \sum_k A_{ik}\otimes B_{kj}\ .$$

The comultiplications of the the Gauss coordinates of L-operators $e(u,\xi)$, $f(u,\xi)$ and $h(u,\xi)$ are

$$\Delta e(u,\xi) = e(u + i\pi c^{(2)}/4, \xi)\otimes 1 + \sum_{p=0}^{\infty}(-1)^p \left(f(u + i\pi c^{(2)}/4 - i\pi, \xi)\right)^p$$

$$\times\, h(u+i\pi c^{(2)}/4,\xi)\otimes\left(e(u-i\pi c^{(1)}/4,\xi+c^{(1)})\right)^{p+1}, \quad (3.21)$$

$$\Delta f(u,\xi) = 1\otimes f(u-i\pi c^{(1)}/4,\xi+c^{(1)}) + \sum_{p=0}^{\infty}(-1)^p\left(f(u+i\pi c^{(2)}/4,\xi)\right)^{p+1} \otimes \tilde h(u-i\pi c^{(1)}/4,\xi+c^{(1)})\left(e(u-i\pi c^{(1)}/4-i\pi,\xi+c^{(1)})\right)^p, \quad (3.22)$$

$$\Delta h(u,\xi) = \sum_{p=0}^{\infty}(-1)^p\frac{\sin(\pi(p+1)/\xi)}{\sin(\pi/\xi)} \times\left(f(u+i\pi c^{(2)}/4-i\pi,\xi)\right)^p h(u+i\pi c^{(2)}/4,\xi) \otimes h(u-i\pi c^{(1)}/4,\xi+c^{(1)})\left(e(u-i\pi c^{(1)}/4-i\pi,\xi+c^{(1)})\right)^p. \quad (3.23)$$

3.3 Finite-dimensional representations and the intertwining operators

Let e, f and h be generators of the algebra $U_{i\pi/\xi}(sl_2)$ with the commutation relations:

$$[h,e]=2e,\qquad [h,f]=-2f,\qquad [e,f]=\frac{\sin(\pi h/\xi)}{\sin(\pi/\xi)}\,. \quad (3.24)$$

The following formulas describe the evaluation homomorphism of the algebra $\mathcal{A}(\widehat{sl}_2)$ at $c=0$ onto the algebra $U_{i\pi/\xi}(sl_2)$:

$$\mathcal{E}v_z\,(e(u)) = -\frac{\mathrm{sh}\,(i\pi/\xi)}{\mathrm{sh}\left(\frac{u-z}{\xi}+\frac{i\pi(h-1)}{2\xi}\right)}\,e = -e\,\frac{\mathrm{sh}\left(i\pi\frac{\xi+1}{\xi}\right)}{\mathrm{sh}\left(\frac{u-z}{\xi}+\frac{i\pi(h+1)}{2\xi}\right)}\,,$$

$$\mathcal{E}v_z\,(f(u)) = -\frac{\mathrm{sh}\,(i\pi/\xi)}{\mathrm{sh}\left(\frac{u-z}{\xi}+\frac{i\pi(h+1)}{2\xi}\right)}\,f = -f\,\frac{\mathrm{sh}\left(i\pi\frac{\xi+1}{\xi}\right)}{\mathrm{sh}\left(\frac{u-z}{\xi}+\frac{i\pi(h-1)}{2\xi}\right)}\,,$$

$$\mathcal{E}v_z\,(h(u)) = \cos\left(\frac{i\pi h}{\xi}\right) - \mathrm{sh}\left(\frac{i\pi}{\xi}\right)\left[\mathrm{cth}\left(\frac{u-z}{\xi}+\frac{i\pi(h-1)}{2\xi}\right)\,ef - \mathrm{cth}\left(\frac{u-z}{\xi}+\frac{i\pi(h+1)}{2\xi}\right)\,fe\right]. \quad (3.25)$$

Let V_n be $(n+1)$-dimensional $U_q(sl_2)$-module with a basis v_k, $k=$

$0, 1, \ldots, n$ where the operators h, e and f act according to the rules

$$h\, v_k = (n-2k)\, v_k,\;\; e\, v_k = \frac{\sin(\pi k\xi)}{\sin(\pi/\xi)}\, v_{k-1},\;\; f\, v_k = \frac{\sin(\pi(n-k)/\xi)}{\sin(\pi/\xi)}\, v_{k+1}. \tag{3.26}$$

Combining these formulas with the evaluation homomorphism we can construct the level zero evaluation representations of the algebra $\mathcal{A}(\widehat{sl}_2)$ in the finite-dimensional space V_n. In particular, in what follows we need the evaluation representation of this algebra into two-dimensional space V_1. It is given by the formulas

$$\pi_z\,(e(u))\, v_+ = 0, \qquad \pi_z\,(f(u))\, v_- = 0, \tag{3.27}$$

$$\begin{aligned}\pi_z\,(e(u))\, v_- &= -\frac{\mathrm{sh}\,(i\pi/\xi)}{\mathrm{sh}\left(\frac{u-z}{\xi}\right)}\, v_+,\\ \pi_z\,(f(u))\, v_+ &= -\frac{\mathrm{sh}\,(i\pi/\xi)}{\mathrm{sh}\left(\frac{u-z}{\xi}\right)}\, v_-,\end{aligned} \tag{3.28}$$

$$\begin{aligned}\pi_z\,(h(u))\, v_\pm &= \cos\left(\frac{\pi}{\xi}\right) \mp \mathrm{sh}\left(\frac{i\pi}{\xi}\right) \mathrm{cth}\left(\frac{u-z}{\xi}\right) v_\pm\\ &= \frac{\mathrm{sh}\left(\frac{u-z\mp i\pi}{\xi}\right)}{\mathrm{sh}\left(\frac{u-z}{\xi}\right)} v_\pm.\end{aligned} \tag{3.29}$$

In these formulas we have identified $v_+ = v_0$ and $v_- = v_1$. Using formulas (3.27)–(3.29) we can define certain intertwining operators between level one highest weight modules over the algebra $\mathcal{A}(\widehat{sl}_2)$.

It was shown in [20] that the algebra $\mathcal{A}(\widehat{sl}_2)$ has the highest weight representations at the value of the central element $c = 1$ which can be bosonized using one free continuous bosonic field. We denote this representation space by the symbol $\mathcal{H}_R$ and will demonstrate in the next subsection that at the FF point it coincides with the bosonized version of the Hilbert space of the free massive Dirac field in the RRW $\mathcal{H}_R^b$. In analogy with the group-theoretical description of the quantum integrable models on the infinite-dimensional lattice [18] we define four types of the *twisted* intertwining operators

$$\begin{aligned} Z'(z) &: \mathcal{H}_R \to \mathcal{H}_R \otimes V_1\,, & Z'^*(z) &: \mathcal{H}_R \otimes V_1 \to \mathcal{H}_R\,,\\ Z^*(z) &: V_1 \otimes \mathcal{H}_R \to \mathcal{H}_R\,, & Z(z) &: \mathcal{H}_R \to V_1 \otimes \mathcal{H}_R\,. \end{aligned} \tag{3.30}$$

The algebra $\mathcal{A}(\widehat{sl}_2)$ acts on the two-dimensional evaluation module V_1 by the formulas (3.27)–(3.29). We require that these operators commute with the action of the algebra $\mathcal{A}(\widehat{sl}_2)$ up to the the involution (1.17)

$$\begin{aligned} Z(z)\iota(x) &= \Delta(x)Z(z)\ , \quad Z'^*(z)\Delta(x) = \iota(x)Z'^*(z)\ , \\ Z^*(\theta)\Delta(x) &= \iota(x)Z^*(z)\ , \quad Z(z)\iota(x) = \Delta(x)Z(z)\ , \end{aligned} \tag{3.31}$$

where $x \in \mathcal{A}(\widehat{sl}_2)$. Due to the dimension of the module V_1 the intertwining operators have two components which are defined as follows:

$$Z'(z)v = Z'_+(z)v \otimes v_+ + Z'_-(z)v \otimes v_-, \quad Z'^*(z)(v \otimes v_\pm) = Z'^*_\pm(z)v,$$
$$Z^*(z)(v_\pm \otimes v) = Z^*_\pm(z)v, \quad Z(z)v = v_+ \otimes Z_+(z)v + v_- \otimes Z_-(z)v,$$

where $v \in \mathcal{H}_R$.

Using the coalgebraic structure of the algebra $\mathcal{A}(\widehat{sl}_2)$ we can rewrite the defining relations (3.31) for the components of the intertwining operators as commutativity with Gauss coordinates of L-operators. For $x = h(u)$ in (3.31) we have

$$h(u)Z^*_-(\theta)h^{-1}(u) = \frac{\operatorname{sh}\left(\frac{u-\theta+5i\pi/4}{\xi}\right)}{\operatorname{sh}\left(\frac{u-\theta+i\pi/4}{\xi}\right)} Z^*_-(\theta)\ , \tag{3.32}$$

$$h(u)Z_+(\theta)h^{-1}(u) = \frac{\operatorname{sh}\left(\frac{u-\theta+i\pi/4}{\xi}\right)}{\operatorname{sh}\left(\frac{u-\theta-3i\pi/4}{\xi}\right)} Z_+(\theta)\ , \tag{3.33}$$

$$h(u)Z'^*_+(\alpha)h^{-1}(u) = \frac{\operatorname{sh}\left(\frac{u-\theta-5i\pi/4}{\xi+1}\right)}{\operatorname{sh}\left(\frac{u-\theta-i\pi/4}{\xi+1}\right)} Z'^*_+(\alpha)\ , \tag{3.34}$$

$$h(u)Z'_-(\alpha)h^{-1}(u) = \frac{\operatorname{sh}\left(\frac{u-\theta-i\pi/4}{\xi+1}\right)}{\operatorname{sh}\left(\frac{u-\theta+3i\pi/4}{\xi+1}\right)} Z'_-(\alpha)\ . \tag{3.35}$$

For $e(u)$ we have

$$\begin{aligned} \operatorname{sh}\left(\frac{i\pi}{\xi}\right) Z^*_+(\theta) = \operatorname{sh}&\left(\frac{u-\theta+i\pi/4}{\xi}\right) e(u)Z^*_-(\theta) \\ &+ \operatorname{sh}\left(\frac{u-\theta+5i\pi/4}{\xi}\right) Z^*_-(\theta)e(u)\ , \end{aligned} \tag{3.36}$$

$$\operatorname{sh}\left(\frac{i\pi}{\xi}\right) Z_-(\theta) = \operatorname{sh}\left(\frac{u-\theta-3i\pi/4}{\xi}\right) e(u) Z_+(\theta) + \operatorname{sh}\left(\frac{u-\theta+i\pi/4}{\xi}\right) Z_+(\theta) e(u) \ , \tag{3.37}$$

$$\{e(u), Z_+^{\prime *}(\theta)\} = \{e(u), Z_-^{\prime}(\theta)\} = 0 \ , \tag{3.38}$$

and finally for $f(u)$

$$\operatorname{sh}\left(\frac{i\pi}{\xi+1}\right) Z_-^{\prime *}(\alpha) = \operatorname{sh}\left(\frac{u-\alpha-i\pi/4}{\xi+1}\right) f(u) Z_+^{\prime *}(\alpha) + \operatorname{sh}\left(\frac{u-\alpha-5i\pi/4}{\xi+1}\right) Z_+^{\prime *}(\alpha) f(u) \ , \tag{3.39}$$

$$\operatorname{sh}\left(\frac{i\pi}{\xi+1}\right) Z_+^{\prime}(\alpha) = \operatorname{sh}\left(\frac{u-\alpha+3\pi/4}{\xi+1}\right) f(u) Z_-^{\prime}(\alpha) + \operatorname{sh}\left(\frac{u-\alpha-i\pi/4}{\xi+1}\right) Z_-^{\prime}(\alpha) f(u) \ , \tag{3.40}$$

$$\{f(u), Z_-^*(\theta)\} = \{f(u), Z_+(\theta)\} = 0 \ . \tag{3.41}$$

Note that in all commutation relations for the operators $Z_\pm(\theta)$ or $Z_\pm^*(\theta)$ appear only the trigonometric functions with the periods $2i\pi/\xi$ while in all those related to the operators $Z_\pm^{\prime}(\alpha)$ or $Z_\pm^{\prime *}(\alpha)$ with the period $2i\pi/(\xi+1)$. This property is encoded into the comultiplication rules (3.21)–(3.23).

We did not write down all the relations following from (3.31) for the components of the intertwining operators but only independent ones. For example, the relation (3.32) is obtained by applying (3.31) to the vector $v_- \otimes v \in V_1 \otimes \mathcal{H}_R$. If we apply it to the vector $v_+ \otimes v$ we obtain the relation

$$\operatorname{sh}\left(\frac{2i\pi}{\xi}\right) Z_-^*(\theta) h(u) e(u-i\pi) = \operatorname{sh}\left(\frac{u-\theta+i\pi/4}{\xi}\right) h(u) Z_+^*(\theta) - \operatorname{sh}\left(\frac{u-\theta-3i\pi/4}{\xi}\right) Z_+^*(\theta) h(u)$$

which is a consequence of (3.36) and (3.15). Nevertheless, the defining relations (3.32)–(3.41) allow one to calculate some properties of the intertwining operators. For example, the commutation relations

$$Z_{\nu_1}^*(\theta_1) Z_{\nu_2}^*(\theta_2) \quad = \quad \rho(\theta_1-\theta_2) \overline{S}_{\nu_1\nu_2}^{\nu_1'\nu_2'}(\theta_1-\theta_2,\xi) Z_{\nu_2'}^*(\theta_2) Z_{\nu_1'}^*(\theta_1) \ ,$$

$$Z'_{\varepsilon_2}(\alpha_2)Z'_{\varepsilon_1}(\alpha_1) \quad = \quad \rho'(\theta_1-\theta_2)\overline{R}^{\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2}(\alpha_1-\alpha_2,\xi)Z'_{\varepsilon'_1}(\alpha_1)Z'_{\varepsilon'_2}(\alpha_2)\ .$$

can be proved using only (3.32)-(3.41) and supposing that the operators $Z^*_-(\theta_1)$, $Z^*_-(\theta_2)$ commute up to some scalar factor and analogously for the operators $Z'_-(\alpha_1)$, $Z'_-(\alpha_2)$.

The defining relations (3.32)–(3.41) allow one to find the bosonization of the intertwining operators from the bosonization of the screening current algebra. Being specialized to the FF point $\xi=1$ the intertwining operators will coincide with the operators $\mathcal{Z}_\pm(\theta)$, and $\mathcal{Z}'_\pm(\alpha)$ constructed in the previous section modulo shifts in the spectral parameters. Also, all the scalar coefficients mentioned above can be fixed using these bosonizations. We will do this in the next subsection.

3.4 Bosonization of the screening operator algebra

The description of the infinite dimensional representations of the algebra $\mathcal{A}(\widehat{sl}_2)$ at non-zero value of the central element C is divided into two steps. The first step is to rewrite the commutation relations (3.14)–(3.19) in terms of the total currents $E(u)$, $F(u)$ and $H(u)$:

$$e\left(u-\frac{ic\pi}{4}\right)+e\left(u-i\pi\xi-\frac{ic\pi}{4}\right)=\xi\sin\left(\pi/\xi\right)E(u)\ , \qquad (3.42)$$

$$f\left(u+\frac{ic\pi}{4}\right)+f\left(u-i\pi\xi'+\frac{ic\pi}{4}\right)=\xi'\sin\left(\pi/\xi'\right)F(u)\ , \qquad (3.43)$$

$$h(u)=2\pi\xi'\ \sin(\pi/\xi')\ H\left(u+\frac{i\pi\xi}{2}+\frac{i\pi c}{4}\right)\ . \qquad (3.44)$$

We write the commutation relations for the total currents in the form adequate for the category of the highest weight representations:

$$[E(u),F(v)]=\left[\delta\left(u-v-\frac{i\pi c}{2}\right)H\left(v+\frac{i\pi(\xi+c)}{2}\right)\right.$$
$$\left.-\delta\left(u-v+\frac{i\pi c}{2}\right)H\left(v-\frac{i\pi(\xi+c)}{2}\right)\right], \qquad (3.45)$$

$$\frac{\Gamma\left(\frac12+\frac1\xi+\frac{i(u-v)}{\pi\xi}\right)}{\Gamma\left(\frac12-\frac1\xi+\frac{i(u-v)}{\pi\xi}\right)}H(u)E(v)=E(v)H(u)\frac{\Gamma\left(\frac12+\frac1\xi-\frac{i(u-v)}{\pi\xi}\right)}{\Gamma\left(\frac12-\frac1\xi-\frac{i(u-v)}{\pi\xi}\right)}, \qquad (3.46)$$

$$\frac{\Gamma\left(\frac{1}{2}-\frac{1}{\xi'}+\frac{i(u-v)}{\pi\xi'}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{\xi'}+\frac{i(u-v)}{\pi\xi'}\right)}H(u)F(v)=F(v)H(u)\frac{\Gamma\left(\frac{1}{2}-\frac{1}{\xi'}-\frac{i(u-v)}{\pi\xi'}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{\xi'}-\frac{i(u-v)}{\pi\xi'}\right)},\tag{3.47}$$

$$\frac{\Gamma\left(1+\frac{1}{\xi}+\frac{i(u-v)}{\pi\xi}\right)}{\Gamma\left(-\frac{1}{\xi}+\frac{i(u-v)}{\pi\xi}\right)}E(u)E(v)=-E(v)E(u)\frac{\Gamma\left(1+\frac{1}{\xi}-\frac{i(u-v)}{\pi\xi}\right)}{\Gamma\left(-\frac{1}{\xi}-\frac{i(u-v)}{\pi\xi}\right)},\tag{3.48}$$

$$\frac{\Gamma\left(1-\frac{1}{\xi'}+\frac{i(u-v)}{\pi\xi'}\right)}{\Gamma\left(\frac{1}{\xi'}+\frac{i(u-v)}{\pi\xi'}\right)}F(u)F(v)=-F(v)F(u)\frac{\Gamma\left(1-\frac{1}{\xi'}-\frac{i(u-v)}{\pi\xi'}\right)}{\Gamma\left(\frac{1}{\xi'}-\frac{i(u-v)}{\pi\xi'}\right)},\tag{3.49}$$

$$\begin{aligned}&\frac{\Gamma\left(1+\frac{1}{\xi}+\frac{i(u-v)}{\pi\xi}\right)\Gamma\left(1-\frac{1}{\xi'}+\frac{i(u-v)}{\pi\xi'}\right)}{\Gamma\left(1-\frac{1}{\xi}+\frac{i(u-v)}{\pi\xi}\right)\Gamma\left(1+\frac{1}{\xi'}+\frac{i(u-v)}{\pi\xi'}\right)}H(u)H(v)\\&=H(v)H(u)\frac{\Gamma\left(1+\frac{1}{\xi}-\frac{i(u-v)}{\pi\xi}\right)\Gamma\left(1-\frac{1}{\xi'}-\frac{i(u-v)}{\pi\xi'}\right)}{\Gamma\left(1-\frac{1}{\xi}-\frac{i(u-v)}{\pi\xi}\right)\Gamma\left(1+\frac{1}{\xi'}-\frac{i(u-v)}{\pi\xi'}\right)}.\end{aligned}\tag{3.50}$$

The commutation relations for the total currents (3.46)–(3.50) are written in the form of equalities of the meromorphic functions without poles and zeros [10, 9]. This means that the product of the currents has the structure of poles and zeros defined by the zeros and poles of the function which is in front of this product in the commutation relations (3.46)–(3.50). For example, the product $E(u)E(v)$ has poles at the points $u=v-i\pi+i\pi\xi k$ and zeros at the points $u=v+i\pi+i\pi\xi(k+1)$, $k\geqslant 0$.

The Frenkel-Ding [8] formulas (3.42) and (3.43) for the total currents can be inverted solving the Riemann-Hilbert problem associated with the strips of the widths $\pi\xi$ and $\pi\xi'=\pi(\xi+c)$ for the currents $E(u)$ and $F(u)$ respectively [20]

$$e(u) = \sin\pi/\xi \int_C \frac{dv}{2\pi i}\,\frac{E(v)}{\mathrm{sh}\,\frac{u-v+ic\pi/4}{\xi}}\,,\tag{3.51}$$

$$f(u) = \sin\pi/\xi' \int_{C'} \frac{dv}{2\pi i}\,\frac{F(v)}{\mathrm{sh}\,\frac{u-v-ic\pi/4}{\xi'}}\,,\tag{3.52}$$

where the contour C' goes from $-\infty$ to $+\infty$, the points $u+ic\pi/4+ik\pi\xi'$ $(k \geqslant 0)$ are above the contour and the points $u-ic\pi/4-ik\pi\xi'$ $(k \geqslant 0)$ are below the contour. The contour C also goes from $-\infty$ to $+\infty$ but the points $u-ic\pi/4+ik\pi\xi$ $(k \geqslant 0)$ are above the contour and the points $u+ic\pi/4-ik\pi\xi$ $(k \geqslant 0)$ are below the contour.

The second step is the bosonization of the currents $E(u)$, $F(u)$ and $H(u)$. To describe the symmetries of the SG model we need the bosonization of the algebra $\mathcal{A}(\widehat{sl_2})$ at the value of the central element $c=1$. To construct this bosonization we define the continuous Heisenberg operators a_λ which satisfy the commutation relations

$$[a_\lambda, a_\mu] = \lambda \frac{\operatorname{sh}\frac{\pi\lambda}{2}}{\operatorname{sh}\pi\lambda} \frac{\operatorname{sh}\frac{\pi\lambda(\xi+1)}{2}}{\operatorname{sh}\frac{\pi\lambda\xi}{2}} \delta(\lambda+\mu) = c(\lambda)\delta(\lambda+\mu) \ . \tag{3.53}$$

The commutation relations of the currents $E(u)$, $F(u)$ and $H(u)$ are satisfied by the operators

$$E(u) = \exp\left(2i\mathcal{Q} + 2 \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda} e^{iu\lambda} \operatorname{ch}(\pi\lambda/2)\, a_\lambda\right), \tag{3.54}$$

$$F(u) = \exp\left(-\frac{2i\xi}{\xi+1}\mathcal{Q} - 2 \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda} e^{iu\lambda} \frac{\operatorname{ch}(\pi\lambda/2)\operatorname{sh}(\pi\lambda\xi/2)}{\operatorname{sh}(\pi\lambda(\xi+1)/2)} a_\lambda\right), \tag{3.55}$$

$$H(u) = \exp\left(\frac{2i}{\xi+1}\mathcal{Q} + \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda} e^{iu\lambda} \frac{\operatorname{sh}(\pi\lambda)}{\operatorname{sh}(\pi\lambda(\xi+1)/2)} a_\lambda\right). \tag{3.56}$$

To verify this statement we should use the normal ordering rule given by (2.59) with the function $c(\lambda)$ specified in (3.53) and formulas given in the Appendix B.

Note that at the FF point the Heisenberg operators a_λ become the same as (2.49) of the nonlocal integrals of motion (2.47) so the bosonization of the current $E(u)$ coincides with the bosonization (2.64), the bosonization of the current $F(u)$ coincides with the bosonization of the scattering operator $\mathcal{Z}_-(u)$ and $H(u)$ with $\Lambda_+(u)$ (cf. (2.57)). The commutation relations (3.46)–(3.50) become in this case

$$\begin{aligned}[H(u), E(v)] = [E(u), E(v)] &= \{H(u), F(v)\} \\ &= \{F(u), F(v)\} = \{H(u), H(v)\} = 0 \ .\end{aligned}$$

The commutation relation (3.45) being multiplied by e^{u-v} and integrated over the parameter u becomes the relation (2.62) which relates the components of the scattering data operators $\mathcal{Z}_\pm(\theta)$.

The formulas (3.32)–(3.41) allow to bosonize the components of the intertwining operators. It is given by the following formulas [26, 20]

$$Z_+(\theta) = \exp\left(-i\mathcal{Q} - \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda}\, \mathrm{e}^{-i\lambda(\theta+\pi i/2)}\, a_\lambda\right), \tag{3.57}$$

$$Z_-(\theta) = \int_C \frac{du}{2\pi}\, e^{(u-\theta)/\xi} \left[(q)^{1/2} E(u) Z_+(\theta) - (q)^{-1/2} Z_+(\theta) E(u)\right], \tag{3.58}$$

$$Z_\pm^*(\theta) = Z_\mp(\theta - i\pi), \tag{3.59}$$

$$Z'_-(\alpha) = \exp\left(\frac{i\xi}{\xi+1}\,\mathcal{Q} + \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda}\, e^{-i\lambda(\alpha-\pi i/2)} \frac{\mathrm{sh}\,(\pi\lambda\xi/2)}{\mathrm{sh}\,(\pi\lambda(\xi+1)/2)}\, a_\lambda\right), \tag{3.60}$$

$$Z'_+(\alpha) = \int_{C'} \frac{du}{2\pi}\, e^{(u-\alpha)/(\xi+1)} \left[(q')^{1/2} Z'_+(\alpha) F(u) - (q')^{-1/2} F(u) Z'_+(\alpha)\right], \tag{3.61}$$

$$Z'^*_\pm(\alpha) = Z'_\mp(\alpha + i\pi), \tag{3.62}$$

where q and q' are given by (1.3) and (1.5) respectively and the contour C' goes from $-\infty$ to $+\infty$ along the real axis leaving the points $z+i\pi/2+ik\pi(\xi+1)$ $(k \geqslant 0)$ above the contour and the points $z-i\pi/2-ik\pi(\xi+1)$ $(k \geqslant 0)$ below the contour. The contour C also goes from $-\infty$ to $+\infty$ but the points $z - i\pi/2 + ik\pi\xi$ $(k \geqslant 0)$ are above the contour and the points $z + i\pi/2 - ik\pi\xi$ $(k \geqslant 0)$ are below the contour.

The formulas (3.57), (3.58) and (3.60), (3.61) demonstrate that at the FF point the operators $Z_\pm(\theta)$ coincide with the operators $\mathcal{Z}_\mp(\theta + \pi i/2)$ modulo certain normalization constants. The same is true for the relation between operators $Z'_\pm(\alpha)$ and $\mathcal{Z}'_\pm(\alpha - \pi i/2)$.

The second remark concerns the form of the contour C in the relation (3.58). The form of this contour is shown on the Fig. 2.

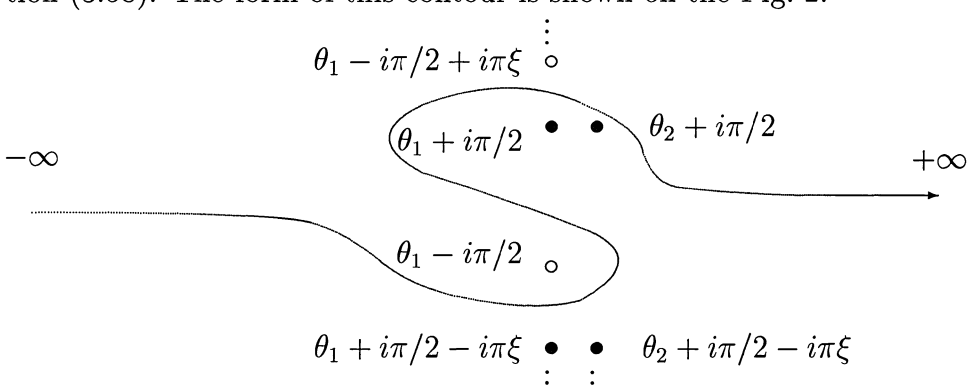

Figure 2.

We can see from this picture that in the limit to the FF point ($\xi \to 1$) there is a double pinching of the integral which leads to the relation (2.62) where the integrals are calculated as the residues in the points $u = \theta \pm \pi i/2$. Moreover, this figure demonstrates that the product $Z_+(\theta_1)Z_-(\theta_2)$ has the pole when $\theta_1 \to \theta_2 + \pi i$ because of the pinching of the contour in the integral representation of this quantity. The origin of this pole due to pinching does not yield the restriction on the domain of the definition of this product and $Z_+(\theta_1)Z_-(\theta_2)$ is an meromorphic function of the variable $\theta_1 - \theta_2$ in the domain $\mathrm{Im}\,(\theta_1 - \theta_2) > -\pi i/2$ with a simple pole in the point $\theta_1 \to \theta_2 + \pi i$.

Using standard techniques [26, 20] we can find the properties of the intertwining operators:

$$
\begin{aligned}
Z_{\nu_1}(\theta_1)Z_{\nu_2}(\theta_2) &= S_{\nu_1\nu_2}^{\nu_1'\nu_2'}(\theta_1-\theta_2,\xi)Z_{\nu_2'}(\theta_2)Z_{\nu_1'}(\theta_1)\ , && (3.63)\\
Z'_{\varepsilon_2}(\alpha_2)Z'_{\varepsilon_1}(\alpha_1) &= R_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'}(\alpha_1-\alpha_2,\xi)Z'_{\varepsilon_1'}(\alpha_1)Z'_{\varepsilon_2'}(\alpha_2)\ , && (3.64)\\
Z_\nu(\theta)Z'_\varepsilon(\alpha) &= \nu\varepsilon\,\mathrm{tg}\left(\frac{i(\theta-\alpha)}{2}-\frac{\pi}{4}\right)Z'_\varepsilon(\alpha)Z_\nu(\theta)\ , && (3.65)\\
\sum_{\varepsilon=\pm} Z'^{*}_\varepsilon(\alpha)Z'_\varepsilon(\alpha) &= g'(\xi)\ \mathrm{id}\ , && (3.66)\\
Z'_{\varepsilon_1}(\alpha)Z'^{*}_{\varepsilon_2}(\alpha) &= g'(\xi)\delta_{\varepsilon_1\varepsilon_2}\ \mathrm{id}\ , && (3.67)\\
Z_{\varepsilon_1}(\theta_1)Z^{*}_{\varepsilon_2}(\theta_2) &= \frac{g(\xi)\delta_{\varepsilon_1\varepsilon_2}\,\mathrm{id}}{\theta_1-\theta_2}+o(z_1-z_2)\ , && (3.68)
\end{aligned}
$$

where the S and R-matrices is given by (3.4) and (3.6) respectively and the normalization constants $g(\xi)$, $g'(\xi)$ can be expressed through double Γ-functions using the formulas given in the Appendix B [20].

4 Angular quantization

Before starting this section we would like to fix the terminology and explain what we mean by the angular quantization in the context of integrable quantum field theory. By this term we mean the possibility to represent the states and operators in the total Hilbert space of the model associated with total space-time as some operators acting in the Hilbert space associated with RRW. So, considering the free fermion in RRW in the second section we did not really consider the angular quantization but did only some preliminary work. The angular quantization of SG model will be considered in this section. But first we would like to recall the angular quantization in lattice integrable models inspired by Baxter's corner transfer matrix method.

4.1 Angular quantization on the lattice

In a series of papers, see e.g. [7, 18] the precise mathematical description of anti-ferroelectric XXZ model in thermodynamic limit was developed in terms of representation theory of quantum affine Lie algebra $U_q(\widehat{sl}_2)$ with the real deformation parameter $-1 < q < 0$. This description, based on Baxter's corner transfer matrix method, looks as follows.

The total Hilbert space of the theory is identified with the space of endomorphisms $\mathrm{End}(\Lambda_0 \oplus \Lambda_1)$ of direct sum of the level one irreducible $U_q(\widehat{sl}_2)$ modules with (complex linear) scalar product given by the natural prescription

$$(A, B) = \mathrm{Tr}_{\Lambda_0 \oplus \Lambda_1} AB \ . \tag{4.1}$$

Two components of degenerated vacuum are identified, up to the constant, with $(-q)^{D^{(i)}}$, where $D^{(i)}$ is principal gradation operator for quantum affine algebra, multiplied by the projection to Λ_i.

The representation theory of $U_q(\widehat{sl}_2)$ provides two types of operators

$$\Phi(\zeta) : \Lambda_i \to \Lambda_{1-i} \otimes V_\zeta, \quad \Psi^*(\zeta) : V_\zeta \otimes \Lambda_i \to \Lambda_{1-i}, \quad i = 0, 1 \tag{4.2}$$

which commute with the action of $U_q(\widehat{sl}_2)$. Here V_ζ is a two-dimensional representation of $U_q(\widehat{sl}_2)$ with basis $v_\pm$ evaluated at the point ζ^2.

The transfer matrix $T(\zeta)$ of the theory acts on the state $A \in \mathrm{End}(\Lambda_0 + \Lambda_1)$ as

$$T(\zeta) \cdot A = \sum_{\varepsilon = \pm} \Phi_\varepsilon(\zeta)\ A\ \Phi_{-\varepsilon}(\zeta) \tag{4.3}$$

and the eigenvectors of the transfer matrix are described in terms of the second type intertwining operators:

$$|\xi_n, \dots, \xi_1\rangle_{\varepsilon_n, \dots, \varepsilon_1; (i)} = c(n) \Psi^*_{\varepsilon_n}(\xi_n) \dots \Psi^*_{\varepsilon_1}(\xi_1) (-q)^{D^{(i)}} \ . \tag{4.4}$$

The local spin operators $\sigma_n^\pm$ acting on the nth cite of the lattice can be described in terms of operators $\Phi(\zeta)$. Due to the definition of scalar product it gives the expressions of the correlation functions of finite products of operators $\sigma_n^\pm$ and of the form-factors of a local operator in terms of traces of products operators $\Phi(\zeta)$ and $\Psi^*(\xi)$ in the Fock space $\Lambda_0 \oplus \Lambda_1$ [18]. Moreover, the adjoint (in a sense of Hopf algebra) action of $U_q(\widehat{sl}_2)$ equips the space of states with a structure of level 0 $U_q(\widehat{sl}_2)$-module, such that n-particle states form n-fold tensor products of the two-dimensional representations of $U_q(\widehat{sl}_2)$.

4.2 Angular quantization in the 2d field theory

A counterpart of the CTM ideology in the integrable models of the 2d quantum field theory in the infinite volume looks as follows [5].

Let $\mathcal{H}_R$ be a Hilbert space of canonical quantization of a theory in the RRW, where boost operator $K = -i\partial_\alpha$ is considered as Hamiltonian. Here ∂_α is the differentiation with respect to the angular 'time' or, what is the same, with respect of the spectral parameters (see (2.25)).

The total Hilbert space $\mathcal{H}$ of the model is supposed to be a properly defined subspace of End $\mathcal{H}_R$ with the scalar product $(A, B) = \mathrm{Tr}_{\mathcal{H}_R} A \cdot B$. The vacuum state in $\mathcal{H}$ is identified with the operator $e^{\pi K}$ in $\mathcal{H}_R$ and the definition of the transfer matrix refers to certain quantum version of Jost functions [26], $Z'_\pm(\alpha)$ (here α is the spectral parameter) whose precise construction on the quantum level is not known. On the classical level these objects in SG theory were introduced by S.Lukyanov in [25] using zero curvature representation of SG equation in RRW: $[\partial_r - A_r, \partial_\alpha - A_\alpha] = 0$.

The asymptotic states $|\theta_1, \dots, \theta_n\rangle_{\varepsilon_1,\dots,\varepsilon_n}$ are presented by the products of the operators

$$|\theta_1, \dots, \theta_n\rangle_{\varepsilon_1,\dots,\varepsilon_n} = Z^*_{\varepsilon_1}(\theta_1) \dots Z^*_{\varepsilon_n}(\theta_n) e^{\pi K}, \tag{4.5}$$

analytically continued to the real line, where $Z^*_\pm(\theta)$ are certain operators acting in the RRW Hilbert space $\mathcal{H}_R$. They can be represented by the bosonized expressions (3.57)-(3.59). The conjugated states are given by the product of the operators $Z_\pm(\theta) = Z^*_\mp(\theta + \pi i)$:

$${}_{\varepsilon_1,\dots,\varepsilon_n}\langle\theta_1, \dots, \theta_n| = e^{\pi K} Z_{\varepsilon_1}(\theta_1) \dots Z_{\varepsilon_n}(\theta_n). \tag{4.6}$$

Analogously to the lattice case, one can assume that any local operator in the theory can be presented in this language in terms of left and right multiplications of certain combinations of the operators $Z'_\varepsilon(\alpha)$ and thus form-factor of operator O can be given by some trace formula

$$ {}_{\mathrm{ph}}\langle \mathrm{vac}|O|\theta_1,\ldots,\theta_n\rangle_{\varepsilon_1,\ldots,\varepsilon_n} = \mathrm{Tr}_{\mathcal{H}_R}\left(e^{2\pi K}\tilde{O}Z^*_{\varepsilon_1}(\theta_1)\ldots Z^*_{\varepsilon_n}(\theta_n)\right), \quad (4.7) $$

where $\tilde{O}$ is some operator acting in $\mathcal{H}_R$ and corresponding to the original operator O. The problem to find an expression for the operator $\tilde{O}$ in terms of the quantum Jost operators $Z'_\pm(\alpha)$ is a complicated problem and has no general solution for the arbitrary operator O although for some simple operators it can be solved by comparing the form factors obtained in the framework of the bootstrap program with those obtained by means of the formula (4.7) (see [26, 29] for the simplest examples in case of the $SU(2)$-invariant Thirring model). We understand the trace in (4.7) as properly regularized to produce the known form factor formulas in SG theory given in [33] (see the paper [30] for the alternative formulation of a continuum analogue of the Baxter corner matrix method).

The possibility to present the matrix element ${}_{\mathrm{ph}}\langle \mathrm{vac}|O|\theta_1,\ldots,\theta_n\rangle_{\varepsilon_1,\ldots,\varepsilon_n}$ as a trace (4.7), the relation (3.59) and the fact that the operators $Z_\pm(\theta)$ commute with the operators $\tilde{O}$ up to numbers related to the locality index [26] allows to demonstrate easily the crossing symmetry of these matrix elements. We have

$$ \begin{aligned} &{}_{\varepsilon'_1,\ldots,\varepsilon'_{n'}}\langle\theta'_1,\ldots,\theta'_{n'}|O|\theta_1,\ldots,\theta_n\rangle_{\varepsilon_1,\ldots,\varepsilon_n} \\ &= \mathrm{Tr}_{\mathcal{H}_R}\left(e^{2\pi K}Z_{\varepsilon'_1}(\theta'_1)\ldots Z_{\varepsilon'_{n'}}(\theta'_{n'})\tilde{O}Z^*_{\varepsilon_1}(\theta_1)\ldots Z^*_{\varepsilon_n}(\theta_n)\right) \\ &={}_{\mathrm{ph}}\langle \mathrm{vac}|O|\theta_1,\ldots,\theta_n,\theta'_1-i\pi,\ldots,\theta'_{n'}-i\pi\rangle_{\varepsilon_1,\ldots,\varepsilon_n,-\varepsilon'_1,\ldots,-\varepsilon'_{n'}}\,. \end{aligned} \quad (4.8) $$

Using the trace formulas we can also verify the completeness of the space of states (4.5) and (4.6) with respect to the scalar product given by the trace over RRW Hilbert space $\mathcal{H}_R$. First of all we observe that the matrix element (4.7) of the unity operator vanishes identically because after substitution of the integral representations of the operators $Z^*_+(\theta)$ (3.58) in (4.7) we obtain the integral with the integrand being the total difference which leads to the vanishing of the integral [29]. On the other hand the pairing of the states (4.5) and (4.6) does not vanish identically but is proportional to some combinations of the δ-functions.

In particular, the simplest pairing of the one-particle states is equal to

$$_{\varepsilon'}\langle\theta'|\theta\rangle_{\varepsilon} = \delta_{\varepsilon\varepsilon'}\delta(\theta-\theta').$$

The delta-function in this formula appears because the trace $\mathrm{Tr}_{\mathcal{H}_R}\left(e^{\sigma K}Z_{\varepsilon'}(\theta')Z^*_{\varepsilon}(\theta)\right)$ has two simple poles in the points $\theta'=\theta$ due to (3.68) and in the point $\theta=\theta'+\sigma+2\pi i$ due to the trace properties. When the parameter σ tends to the value $-2\pi i$ these two poles form the δ-function (see [18] for the detailed description of this mechanism in a case of lattice integrable models). We would like to note here that the same mechanism is responsible for the fact that form factors of the local operators satisfy the annihilation axiom [33]. For XXZ model this fact was established in [31].

These are the general features of the angular quantization approach in the 2d integrable field theory. In order for the angular quantization approach be the self-consistent, in particular the traces (4.7) satisfy all the axioms in the form-factor approach [33], the operators $Z^*_{\pm}(\theta)$ and $Z'_{\pm}(\alpha)$ should satisfy the properties (3.63)–(3.68). Since the representation theory of the algebra $\mathcal{A}(\widehat{sl}_2)$ contains the operators which satisfy such properties we claim that this algebra is the dynamical symmetry algebra of the SG model in the sense claimed in [7] for the quantum XXZ model.

Using the properties of the operators $Z'_{\pm}(\alpha)$ we can find representations of the commutation relations (1.4) for the quantum monodromy matrices and interpret its trace as the generating function of the local integrals of motion through the asymptotical expansion. This will be done in the next subsection. Moreover, we can define appropriate adjoint action of the algebra $\mathcal{A}(\widehat{sl}_2)$ onto the Hilbert space of the SG model $\mathcal{H}$ which describes the known symmetries of this space of states related to the quantum affine algebra $U_q(\widehat{sl}_2)$ [3] and interpret these symmetries as level zero action of the algebra $\mathcal{A}(\widehat{sl}_2)$ in the Hilbert space of states. In the last subsection we will demonstrate that these symmetries being specialized to the FF point become the symmetries governed by the classical affine algebra at level zero and associated with the strip [21].

4.3 Properties of the monodromy matrix in SG model

In this and next subsections we will understand by the operators $Z_\pm^*(\theta)$, $Z_\pm(\theta)$, $Z'_\pm(\alpha)$ and $Z'^*_\pm(\alpha)$ the intertwining operators of the algebra $\mathcal{A}(\widehat{sl}_2)$ which satisfy the properties (3.63)–(3.68).

A monodromy matrix of the model acting on any state $X_k \in \mathcal{H}_k$ of the total Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, where $k = 0, 1$ corresponds to the subspaces of $\mathcal{H}$ of the even and odd number of particles respectively, is defined as follows

$$\mathcal{T}_{\varepsilon\varepsilon'}(\alpha) \cdot X_k = (g'(\xi))^{-1}\ \varepsilon^k Z'_\varepsilon(\alpha) \cdot X_k \cdot Z'_{-\varepsilon'}(\alpha), \quad k = 0, 1\ . \tag{4.9}$$

The commutation relations (3.64) allow to find the commutation relation for this matrices:

$$R(\alpha_1 - \alpha_2, \xi)\mathcal{T}_1(\alpha_1)\mathcal{T}_2(\alpha_2) = \mathcal{T}_2(\alpha_2)\mathcal{T}_1(\alpha_1)R(\alpha_1 - \alpha_2, \xi) \tag{4.10}$$

which coincides with (1.4).

The trace of the monodromy matrix or the transfer matrix $T(\alpha)$ is

$$T(\alpha) \cdot X_k = (g'(\xi))^{-1} \sum_{\varepsilon=\pm} \varepsilon^k Z'_\varepsilon(\alpha) \cdot X_k \cdot Z'_{-\varepsilon}(\alpha)\ . \tag{4.11}$$

The inverse transfer matrix is given in terms of the operators $Z'^*_\pm(\alpha)$:

$$T^{-1}(\alpha) \cdot X_k = (g'(\xi))^{-1} \sum_{\varepsilon=\pm} \varepsilon^k Z'^*_\varepsilon(\alpha) \cdot X_k \cdot Z'^*_{-\varepsilon}(\alpha)\ . \tag{4.12}$$

The fact that operators (4.11) and (4.12) are inverse to each other is a direct consequence of the properties (3.66) and (3.67). The same properties allow to prove that the physical vacuum vector $|\text{vac}\rangle_{\text{ph}} \in \mathcal{H}$ is stable under the action of the operators $T(\alpha)$ and $T^{-1}(\alpha)$:

$$\begin{aligned} T(\alpha)|\text{vac}\rangle_{\text{ph}} &= (g'(\xi))^{-1} \sum_{\varepsilon=\pm} Z'_\varepsilon(\alpha) \cdot e^{\pi K} \cdot Z'_{-\varepsilon}(\alpha) \\ &\overset{(3.62)}{=} e^{\pi K} \sum_{\varepsilon=\pm} Z'^*_{-\varepsilon}(\alpha) Z'_{-\varepsilon}(\alpha) \overset{(3.66)}{=} |\text{vac}\rangle_{\text{ph}}\ . \end{aligned} \tag{4.13}$$

Here and below we will often use the formulas

$$Z_\pm^*(\theta)e^{\pi K} = e^{\pi K} Z_\mp(\theta), \quad Z'_\pm(\alpha)e^{\pi K} = e^{\pi K} Z'^*_\mp(\theta) \tag{4.14}$$

which are consequences of the definition of the boost operator and (3.59), (3.62).

The commutation relations (4.10) imply the commutativity

$$[T(\alpha_1), T(\alpha_2)] = 0 \tag{4.15}$$

which signifies that the operator $T(\alpha)$ can be considered as the generating function of the local integrals of motion. Using the property (3.65) we can calculate the action of the generating function $T(\alpha)$ onto n-particle state $|\theta_1, \dots, \theta_n\rangle_{\varepsilon_1,\dots,\varepsilon_n}$:

$$T(\alpha)\cdot|\theta_1, \dots, \theta_n\rangle_{\varepsilon_1,\dots,\varepsilon_n} = \prod_{j=1}^{n} \varepsilon_j \operatorname{ctg}\left(\frac{\pi}{4} + \frac{\alpha - \theta_j}{2i}\right) |\theta_1, \dots, \theta_n\rangle_{\varepsilon_1,\dots,\varepsilon_n} . \tag{4.16}$$

Using this equality we can see that the quantity

$$I(\alpha) = \sum_s I_s e^{s\alpha} = \frac{1}{2i}\, T^{-1}(\alpha)\frac{\partial T(\alpha)}{\partial \alpha} = \frac{1}{2i}\, \frac{\partial \ln T(\alpha)}{\partial \alpha}, \tag{4.17}$$

has an eigenvalue on the states $|\theta_N, \dots, \theta_1\rangle_{\varepsilon_N,\dots,\varepsilon_1}$

$$I(\alpha)|\theta_N, \dots, \theta_1\rangle_{\varepsilon_N,\dots,\varepsilon_1} = \sum_{j=1}^{N} \frac{2}{\operatorname{ch}(\alpha - \theta_j)} |\theta_N, \dots, \theta_1\rangle_{\varepsilon_N,\dots,\varepsilon_1}$$

and is a generating function of the local integrals of motion (2.15) I_n and $\overline{I}_n$ for odd indeces n:

$$I(\alpha)|\theta_N, \dots, \theta_1\rangle_{\varepsilon_N,\dots,\varepsilon_1} = \begin{cases} \sum_{s\geqslant 0}(-1)^s e^{-(2s+1)\alpha} I_{2s+1}|\theta_N, \dots, \theta_1\rangle_{\varepsilon_N,\dots,\varepsilon_1}, & \alpha \to +\infty \\ \sum_{s\geqslant 0}(-1)^s e^{(2s+1)\alpha} \overline{I}_{2s+1}|\theta_N, \dots, \theta_1\rangle_{\varepsilon_N,\dots,\varepsilon_1}, & \alpha \to -\infty \end{cases} \tag{4.18}$$

where

$$I_{2s+1}|\theta_N, \dots, \theta_1\rangle_{\varepsilon_N,\dots,\varepsilon_1} = \sum_{j=1}^{N} e^{(2s+1)\theta_j} |\theta_N, \dots, \theta_1\rangle_{\varepsilon_N,\dots,\varepsilon_1},$$

$$\overline{I}_{2s+1}|\theta_N, \dots, \theta_1\rangle_{\varepsilon_N,\dots,\varepsilon_1} = \sum_{j=1}^{N} e^{-(2s+1)\theta_j} |\theta_N, \dots, \theta_1\rangle_{\varepsilon_N,\dots,\varepsilon_1}, \quad s \geqslant 0. \tag{4.19}$$

It follows from (4.13) that

$$I_{2s+1}|\mathrm{vac}\rangle_{\mathrm{ph}} = \overline{I}_{2s+1}|\mathrm{vac}\rangle_{\mathrm{ph}} = 0\ . \tag{4.20}$$

It is clear that the form factors of the quantum integrals I_{2s+1} and $\overline{I}_{2s+1}$ vanish identically, but using these quantities we can partially solve the problem of the reconstructing the map $O \to \tilde{O}$ of the local operators into the operators acting in the Hilbert space of the angular quantization. Suppose we know this identification for some particular local operator $\mathcal{O} \to \tilde{\mathcal{O}}$. Then we can immediately find this identification for arbitrary descendant of the operator $\mathcal{O}$ with respect to all integrals of motion I_{2s+1} and $\overline{I}_{2s+1}$: $\mathcal{O}(\alpha) = [\mathcal{O}, I(\alpha)]$ [26]. The answer is

$$\mathcal{O}(\alpha) \mapsto \tilde{\mathcal{O}}(\alpha) = \tilde{\mathcal{O}}\,\tilde{I}(\alpha) - \tilde{I}(\alpha + 2\pi i)\,\tilde{\mathcal{O}}\ , \tag{4.21}$$

where

$$\begin{aligned}\tilde{I}(\alpha) &= \frac{1}{2ig'(\xi)}\sum_{\varepsilon=\pm} Z'_{\varepsilon}(\alpha + i\pi)\partial_{\alpha} Z'_{-\varepsilon}(\alpha) \\ &= \sum_{s>0} \tilde{I}_{\pm(2s+1)} e^{\mp(2s+1)\alpha} \quad \text{when} \quad \alpha \to \pm\infty.\end{aligned} \tag{4.22}$$

The prove of this statement is based on the cyclic property of the trace (4.7) and looks as follows:

$$\begin{aligned}
&{}_{\mathrm{ph}}\langle \mathrm{vac}|\mathcal{O}(\alpha)|\theta_1, \dots, \theta_n\rangle_{\varepsilon_1,\dots,\varepsilon_n} \\
&={}_{\mathrm{ph}}\ \langle \mathrm{vac}|[\mathcal{O}, I(\alpha)]|\theta_1, \dots, \theta_n\rangle_{\varepsilon_1,\dots,\varepsilon_n} \\
&\overset{(4.20)}{=}{}_{\mathrm{ph}}\ \langle \mathrm{vac}|\mathcal{O}I(\alpha)|\theta_1, \dots, \theta_n\rangle_{\varepsilon_1,\dots,\varepsilon_n} \\
&= \mathrm{Tr}_{\mathcal{H}_R}\left(e^{2\pi K}\tilde{\mathcal{O}}\tilde{I}(\alpha) Z^*_{\varepsilon_1}(\theta_1)\dots Z^*_{\varepsilon_n}(\theta_n)\right) - \frac{1}{2ig'(\xi)^2}\sum_{\varepsilon,\mu=\pm}(\varepsilon\mu)^n \mathrm{Tr}_{\mathcal{H}_R} \\
&\quad\cdot\left(e^{\pi K}\tilde{\mathcal{O}} Z'^*_{\varepsilon}(\alpha) Z'_{\mu}(\alpha) Z^*_{\varepsilon_1}(\theta_1)\dots Z^*_{\varepsilon_n}(\theta_n) e^{\pi K} Z'_{-\mu}(\alpha)\partial_{\alpha} Z'^*_{-\varepsilon}(\alpha)\right).
\end{aligned}$$

The last line in the previous calculation can be transformed as follows:

$$\begin{aligned}
&\sum_{\varepsilon,\mu=\pm}(\varepsilon\mu)^n \mathrm{Tr}_{\mathcal{H}_R}\left(e^{\pi K}\tilde{\mathcal{O}} Z'^*_{\varepsilon}(\alpha) Z'_{\mu}(\alpha) Z^*_{\varepsilon_1}(\theta_1)\dots Z^*_{\varepsilon_n}(\theta_n) e^{\pi K} Z'_{-\mu}(\alpha)\partial_{\alpha} Z'^*_{-\varepsilon}(\alpha)\right) \\
&\overset{(3.65)}{=} \sum_{\varepsilon,\mu=\pm} \mathrm{Tr}_{\mathcal{H}_R}\left(e^{\pi K}\tilde{\mathcal{O}} Z^*_{\varepsilon_1}(\theta_1)\dots Z^*_{\varepsilon_n}(\theta_n) Z'^*_{\varepsilon}(\alpha) e^{\pi K} Z'^*_{-\mu}(\alpha) Z'_{-\mu}(\alpha)\partial_{\alpha} Z'^*_{-\varepsilon}(\alpha)\right) \\
&\overset{(3.66)}{=} g'(\xi)\sum_{\varepsilon=\pm} \mathrm{Tr}_{\mathcal{H}_R}\left(Z'^*_{\varepsilon}(\alpha) e^{\pi K}\partial_{\alpha} Z'^*_{-\varepsilon}(\alpha) e^{\pi K}\tilde{\mathcal{O}} Z^*_{\varepsilon_1}(\theta_1)\dots Z^*_{\varepsilon_n}(\theta_n)\right)
\end{aligned}$$

$$\stackrel{(3.62)}{=} g'(\xi)\mathrm{Tr}_{\mathcal{H}_R}\left(e^{2\pi K}\tilde{I}(\alpha+2\pi i)\tilde{O}Z^*_{\varepsilon_1}(\theta_1)\dots Z^*_{\varepsilon_n}(\theta_n)\right) \tag{4.23}$$

so (4.21) is proved.

The generating function (4.22) has also another meaning. Namely, it was proved in [29] that after substitution to the trace (4.7) the coefficients $\tilde{I}_1$ and $\tilde{I}_{-1}$ of the asymptotical expansion (4.22) one obtains the known form factors [33] of the stress energy tensor in $SU(2)$-invariant Thirring model (this model can be obtained from SG model in the limit $\xi \to +\infty$). This allows to conjecture that for the finite ξ the corresponding coefficients of the quantity $\tilde{I}(\alpha)$ will also generate the form factors of stress-energy tensor in the SG model.

4.4 Symmetries of the model

In this subsection we will prove the following three statements.

(ι) The adjoint action of the algebra $\mathcal{A}(\widehat{sl}_2)$ (1.18) on the total Hilbert space is given by the level zero action of this algebra

$$\overline{\mathcal{R}}(u_1-u_2,\xi)\mathrm{Ad}_{L_1(u_1;\xi)}\mathrm{Ad}_{L_2(u_2;\xi)} = \mathrm{Ad}_{L_2(u_2;\xi)}\mathrm{Ad}_{L_1(u_1;\xi)}\overline{\mathcal{R}}(u_1-u_2,\xi)\,. \tag{4.24}$$

($\iota\iota$) The subspace of the n-particle states carries the finite-dimensional representation of the algebra $\mathcal{A}(\widehat{sl}_2)$ given by the formulas

$$\left(\mathrm{id}\otimes\iota\otimes\mathrm{id}\otimes\iota\otimes\dots\iota^{n-1}\right)\Delta^{(n-1)}(x)|\theta_1,\dots,\theta_1\rangle_{\varepsilon_1,\dots,\varepsilon_n}, \qquad x=e(u),\ f(u),\ h(u)\,, \tag{4.25}$$

where $\Delta^{(n)}(x)$ is nth power of the comultiplication maps (3.21)–(3.23) defined inductively

$$\Delta^{(1)}\equiv\Delta,\quad \Delta^{(n)}(x)=(\Delta\otimes\mathrm{id})\,\Delta^{(n-1)}\,,$$

where the action of the Gauss coordinates $e(u)$, $f(u)$ and $h(u)$ on the one-particle states is defined by the formulas (4.30) and (4.31).

($\iota\iota\iota$) The commutation relations of the algebra $\mathcal{A}(\widehat{sl}_2)$ in the form (3.14)–(3.19) allow to define certain asymptotical operators $\mathbf{e}_i$, $\mathbf{f}_i$ and

h such that their commutation and comultiplication relations correspond to those of the Chevalley generators of the quantum affine algebra $U_q(\widehat{sl}_2)$ at level zero with the parameter of deformation $q = \exp\left(\pi i \frac{\xi+1}{\xi}\right)$.

The first statement is a simple consequence of the commutation relation (1.15) and the fact of commutativity $[\mathcal{R}(u,\xi), \sigma_z \otimes \sigma_z] = 0$.

The second statement is a direct consequence of the defining relations (3.32), (3.36) and (3.41) for the operators $Z^*_{\pm}(\theta)$. We start from one particle states $|\theta + \pi i/2\rangle_{\pm}$ and prove that they realize the spin 1/2 representation (3.27)–(3.29). From the definition of the adjoint action (1.18) we have

$$\begin{aligned}
&\mathrm{Ad}_{k_1(u)^{-1}} \cdot Z^*_{\pm}(\tilde\theta) \\
&= k_1(\tilde u)^{-1} Z^*_{\pm}(\tilde\theta) k_1(\tilde u) + k_1(\tilde u)^{-1}\{Z^*_{\pm}(\tilde\theta), f(\tilde u)\} k_2(\tilde u) e(\tilde u), \qquad (4.26)\\
&\mathrm{Ad}_{k_2(u)^{-1}} \cdot Z^*_{\pm}(\tilde\theta) + \mathrm{Ad}_{e(u)k_1(u)^{-1}f(u)} \cdot Z^*_{\pm}(\tilde\theta) \\
&= k_2(\tilde u)^{-1} Z^*_{\pm}(\tilde\theta) k_2(\tilde u) + e(\tilde u) k_1(\tilde u)^{-1}\{Z^*_{\pm}(\tilde\theta), f(\tilde u)\} k_2(\tilde u), \qquad (4.27)\\
&- \mathrm{Ad}_{k_1(u)^{-1}f(u)} \cdot Z^*_{\pm}(\tilde\theta) = k_1(\tilde u)^{-1}\{Z^*_{\pm}(\tilde\theta), f(\tilde u)\} k_2(\tilde u), \qquad (4.28)\\
&- \mathrm{Ad}_{e(u)k_1(u)^{-1}} \cdot Z^*_{\pm}(\tilde\theta) \\
&= e(\tilde u) k_1(\tilde u)^{-1} Z^*_{\pm}(\tilde\theta) k_1(\tilde u) + k_2(\tilde u)^{-1} Z^*_{\pm}(\tilde\theta) k_2(\tilde u) e(\tilde u) \\
&\quad + e(\tilde u) k_1(\tilde u)^{-1}\{Z^*_{\pm}(\tilde\theta), f(\tilde u)\} k_2(\tilde u) e(\tilde u), \qquad (4.29)
\end{aligned}$$

where we denote $\tilde u = u + \pi i/4$ and $\tilde\theta = \theta + \pi i/2$.

The calculation of the adjoint action of the Gauss coordinates of L-operator onto the state $|\tilde\theta\rangle_-$ is an easy part. Indeed, using formulas (3.41) we observe first that the anticommutator $\{Z^*_-(\tilde\theta), f(\tilde u)\}$ vanishes in (4.26)–(4.27) and using then (3.32), (3.36) we obtain

$$\begin{aligned}
\mathrm{Ad}_{f(u)}|\tilde\theta\rangle_- &= 0,\\
\mathrm{Ad}_{e(u)}|\tilde\theta\rangle_- &= -\frac{\mathrm{sh}\, i\pi/\xi}{\mathrm{sh}\left(\frac{u-\theta}{\xi}\right)}|\tilde\theta\rangle_+,\\
\mathrm{Ad}_{h(u)}|\tilde\theta\rangle_- &= \frac{\mathrm{sh}\left(\frac{u-\theta+i\pi}{\xi}\right)}{\mathrm{sh}\left(\frac{u-\theta}{\xi}\right)}|\tilde\theta\rangle_- \qquad (4.30)
\end{aligned}$$

which obviously coincide with the analogous formulas from (3.27)–(3.29). Let us demonstrate how the second formula in (4.30) is obtained. Combining (4.26) and (4.29) and taking into account (3.38) we obtain

$$\begin{aligned}\mathrm{Ad}_{e(u)} \cdot Z^*_-(\tilde\theta) &= -e(\tilde u)Z^*_-(\tilde\theta) - h(\tilde u)Z^*_-(\tilde\theta)h(\tilde u)^{-1}e(\tilde u) \\ &\overset{(3.32)}{=} -e(\tilde u)Z^*_-(\tilde\theta) - \frac{\mathrm{sh}\frac{u-\theta+i\pi}{\xi}}{\mathrm{sh}\frac{u-\theta}{\xi}} Z^*_-(\tilde\theta)e(\tilde u) \\ &\overset{(3.36)}{=} -\frac{\mathrm{sh}\frac{i\pi}{\xi}}{\mathrm{sh}\frac{u-\theta}{\xi}} Z^*_+(\tilde\theta) \,.\end{aligned}$$

The calculation of the adjoint action of the Gauss coordinates onto the state $|\tilde\theta\rangle_+$ is more complicated but straightforward. The main trick is to use the formula (3.36) to replace the operator $Z^*_+(\tilde\theta)$ by the combination of the products $e(v)Z^*_-(\tilde\theta)$ and $Z^*_-(\tilde\theta)e(v)$. Using then the commutation relations of the algebra $\mathcal{A}(\widehat{sl}_2)$ in terms of the Gauss coordinates (3.14)–(3.19) we will find that the dependence on the spectral parameter v is canceled out and we obtain

$$\begin{aligned}&\mathrm{Ad}_{e(u)}|\tilde\theta\rangle_+ = 0, \\ &\mathrm{Ad}_{f(u)}|\tilde\theta\rangle_+ = -\frac{\mathrm{sh}\, i\pi/\xi}{\mathrm{sh}\left(\frac{u-\theta}{\xi}\right)}|\tilde\theta\rangle_-, \\ &\mathrm{Ad}_{h(u)}|\tilde\theta\rangle_+ = \frac{\mathrm{sh}\left(\frac{u-\theta-i\pi}{\xi}\right)}{\mathrm{sh}\left(\frac{u-\theta}{\xi}\right)}|\tilde\theta\rangle_+\end{aligned} \tag{4.31}$$

which coincide with the rest of the formulas (3.27)–(3.29).

To find the action of the Gauss coordinates $e(u)$, $f(u)$ and $h(u)$ on the n-particle states we use the same formulas (4.26)–(4.29) with $Z^*_\pm(\tilde\theta)$ replaced by the n-fold product of these operators. For example, the adjoint action of the Gauss coordinate $e(u)$ on the two-particle state is given by the formula

$$\begin{aligned}&\mathrm{Ad}_{e(u)} \cdot |\theta_1,\theta_2\rangle_{-,-} \\ &= -\frac{\mathrm{sh}\left(\frac{i\pi}{\xi}\right)}{\mathrm{sh}\left(\frac{u-\theta_1}{\xi}\right)}|\theta_1,\theta_2\rangle_{+,-} + \frac{\mathrm{sh}\left(\frac{u-\theta_1+i\pi}{\xi}\right)}{\mathrm{sh}\left(\frac{u-\theta_1}{\xi}\right)}\frac{\mathrm{sh}\left(\frac{i\pi}{\xi}\right)}{\mathrm{sh}\left(\frac{u-\theta_2}{\xi}\right)}|\theta_1,\theta_2\rangle_{-,+} \\ &= \hat\Delta(e(u))\ |\theta_1,\theta_2\rangle_{-,-}\ ,\end{aligned} \tag{4.32}$$

where we denoted by $\hat{\Delta} = (\mathrm{id} \otimes \iota)\,\Delta$ the composition of the comultiplication of the algebra $\mathcal{A}(\widehat{sl}_2)$ (3.21)–(3.23) and the involution (1.17). Repeating these arguments inductively we prove the formula (4.25) where the action of the Gauss coordinates $e(u)$, $f(u)$ and $h(u)$ on the one-particle states are given by the formulas (4.30) and (4.31).

The commutation relations of the algebra $\mathcal{A}(\widehat{sl}_2)$ (3.14)–(3.19) at the zero central element demonstrate that the Gauss coordinates of L-operators have following asymptotics when $\mathrm{Re}\, u \to \pm\infty$:

$$e(u) \sim \exp\left(-\frac{|u|}{\xi}\right), \quad f(u) \sim \exp\left(-\frac{|u|}{\xi}\right), \quad h(u) \sim h(\pm\infty) \equiv h_\pm. \tag{4.33}$$

It follows from (3.15) and (3.16) that Cartan asymptotical generators $h_\pm$ have the following commutation relations with Gauss coordinates $e(u)$ and $f(u)$:

$$h_\pm e(u) h_\pm^{-1} = \exp\left(\pm\frac{2\pi i}{\xi}\right) e(u), \quad h_\pm f(u) h_\pm^{-1} = \exp\left(\mp\frac{2\pi i}{\xi}\right) f(u)\ . \tag{4.34}$$

The comultiplication rule (3.23) yields that the asympotical Cartan elements are primitive and group-like: $\Delta h_\pm = h_\pm \otimes h_\pm$. The commutation relations (4.34) yields that the product h_+h_- is central and also group-like primitive. Due to this we can put this central element to be equal to one so the asymptotical Cartan operators are inverse to each other: $h_+ = h_-^{-1}$.

Let us define the logarithmic Cartan operator $\mathbf{h}$ as follows:

$$h_\pm = \exp\left(\pm i\pi\frac{\xi+1}{\xi}\,\mathbf{h}\right) \tag{4.35}$$

where the operator $\mathbf{h}$ has standard commutation relation with the Gauss coordinates $[\mathbf{h}, e(u)] = 2e(u)$ and $[\mathbf{h}, f(u)] = -2f(u)$. Define also the asymptotical operators

$$\begin{aligned} \mathbf{e}_\pm &= \frac{1}{2}\,\mathrm{sh}\left(i\pi\frac{\xi+1}{\xi}\right)^{-1} \lim_{\mathrm{Re}\, u\to\pm\infty} e^{\pm u/\xi} e(u)\ , \\ \mathbf{f}_\pm &= \frac{1}{2}\,\mathrm{sh}\left(i\pi\frac{\xi+1}{\xi}\right)^{-1} \lim_{\mathrm{Re}\, u\to\pm\infty} e^{\pm u/\xi} f(u)\ . \end{aligned} \tag{4.36}$$

From the commutation relations (3.14)–(3.16) we can obtain the commutation relations of these operators:

$$
\begin{aligned}
&[\mathbf{h},\mathbf{e}_\pm] = \pm 2\mathbf{e}_\pm, \qquad [\mathbf{h},\mathbf{f}_\pm] = \mp 2\mathbf{f}_\pm,\\
&[\mathbf{e}_\pm,\mathbf{f}_\mp] = \pm\frac{\sin(\pi\mathbf{h}(\xi+1)/\xi)}{\sin(\pi(\xi+1)/\xi)} = \pm\frac{q^{\mathbf{h}}-q^{-\mathbf{h}}}{q-q^{-1}}\ ,\\
&q\mathbf{e}_+\mathbf{e}_- = q^{-1}\mathbf{e}_-\mathbf{e}_+, \quad q^{-1}\mathbf{f}_+\mathbf{f}_- = q\mathbf{f}_-\mathbf{f}_+\ ,\\
&q^{\mp 3}\mathbf{e}_\pm^3\mathbf{f}_\pm - \frac{q^3-q^{-3}}{q-q^{-1}}\left(q^{\mp 1}\mathbf{e}_\pm^2\mathbf{f}_\pm\mathbf{e}_\pm - q^{\pm 1}\mathbf{e}_\pm\mathbf{f}_\pm\mathbf{e}_\pm^2\right) - q^{\pm 3}\mathbf{f}_\pm\mathbf{e}_\pm^3 = 0\ ,\\
&q^{\pm 3}\mathbf{f}_\pm^3\mathbf{e}_\pm - \frac{q^3-q^{-3}}{q-q^{-1}}\left(q^{\pm 1}\mathbf{f}_\pm^2\mathbf{e}_\pm\mathbf{f}_\pm - q^{\mp 1}\mathbf{f}_\pm\mathbf{e}_\pm\mathbf{f}_\pm^2\right) - q^{\mp 3}\mathbf{e}_\pm\mathbf{f}_\pm^3 = 0\ , \quad (4.37)
\end{aligned}
$$

where $q = \exp\left(i\pi\frac{\xi+1}{\xi}\right)$. These commutation relations allows to identify the asymptotical operators with the Chevalley generators of the affine quantum algebra $U_q(\widehat{sl}_2)$ at level zero.

Using formulas (4.30), (4.31) and the rule of the Gauss coordinates actions onto multi-particle states (4.25) we can obtain the action of the asymptotical operators $\mathbf{e}_\pm$, $\mathbf{f}_\pm$ and $\mathbf{h}$ onto multiparticle states and prove that it is given by the comultiplication of the algebra $U_q(\widehat{sl}_2)$. To do this we first slightly modify the action of these generators following [32] when they act on the one-particle states $|\theta\rangle_\pm$:

$$
\mathbf{e}_\pm \mapsto \exp\left(\mp\frac{\theta}{\xi}\right)\mathbf{e}_\pm, \quad \mathbf{f}_\pm \mapsto \exp\left(\pm\frac{\theta}{\xi}\right)\mathbf{f}_\pm, \quad \mathbf{h}\mapsto\mathbf{h}. \qquad (4.38)
$$

By the straightforward calculation using the definition of the adjoint action on the multiple-particle states (1.18) and the formulas (3.32)–(3.41) we obtain that this action can be formulated through the comultiplication

$$
\begin{aligned}
\Delta_0\mathbf{e}_\pm &= \mathbf{e}_\pm\otimes 1 + q^{\mp\mathbf{h}}\otimes\mathbf{e}_\pm,\\
\Delta_0\mathbf{f}_\pm &= 1\otimes\mathbf{f}_\pm + \mathbf{f}_\pm\otimes q^{\pm\mathbf{h}},\\
\Delta_0\mathbf{h} &= \mathbf{h}\otimes 1 + 1\otimes\mathbf{h}\ , \qquad (4.39)
\end{aligned}
$$

which can be formally obtained from the comultiplication formulas for the Gauss coordinates (3.21)–(3.23) using (4.33). The action of the asymptotical operators on the one particle states are defined as follows

$$
\mathbf{e}_\pm|\theta\rangle_+ = \mathbf{f}_\pm|\theta\rangle_- = 0,\ \ \mathbf{e}_\pm|\theta\rangle_- = |\theta\rangle_+,\ \ \mathbf{f}_\pm|\theta\rangle_+ = |\theta\rangle_-,\ \ \mathbf{h}|\theta\rangle_\pm = \pm|\theta\rangle_\pm. \qquad (4.40)
$$

For example, let us demonstrate the origin of this comultiplication on the two-particle state. From (4.32) we have

$$\begin{aligned}
&\mathrm{Ad}_{\mathbf{e}_+}\cdot|\theta_1,\theta_2\rangle_{-,-}\\
&=\lim_{\mathrm{Re}\,u\to+\infty}\left(\frac{\exp\left(\frac{u-\theta_1}{\xi}\right)}{2\mathrm{sh}\left(\frac{u-\theta_1}{\xi}\right)}|\theta_1,\theta_2\rangle_{+,-}-\frac{\mathrm{sh}\left(\frac{u-\theta_1+i\pi}{\xi}\right)}{\mathrm{sh}\left(\frac{u-\theta_1}{\xi}\right)}\frac{\exp\left(\frac{u-\theta_2}{\xi}\right)}{2\mathrm{sh}\left(\frac{u-\theta_2}{\xi}\right)}|\theta_1,\theta_2\rangle_{-,+}\right)\\
&=|\theta_1,\theta_2\rangle_{+,-}-\exp\left(\frac{i\pi}{\xi}\right)|\theta_1,\theta_2\rangle_{-,+}\\
&\overset{(4.40)}{=}\left(\mathbf{e}_+\otimes 1+q^{-\mathbf{h}}\otimes\mathbf{e}_+\right)|\theta_1,\theta_2\rangle_{-,-}=\Delta_0(\mathbf{e}_+)\;|\theta_1,\theta_2\rangle_{-,-}\,.
\end{aligned}$$

We would like to note here that the set of the asymptotical generators $\mathbf{e}_+$, $\mathbf{f}_-$ and $\mathbf{h}$ or $\mathbf{e}_-$, $\mathbf{f}_+$ and $\mathbf{h}$ cannot be identified with the set e, f and h used in the construction of the evaluation homomorphism from the algebra $\mathcal{A}(\widehat{sl}_2)$ onto $U_{i\pi/\xi}(sl_2)$, because the first ones are the subalgebras while the second one is factor subalgebra. In particular, the action (4.40) cannot be obtained from the adjoint action onto one-particle states (4.30) and (4.31).

The consideration presented above prove that the adjoint action of the finite-dimensional subalgebra of $\mathcal{A}(\widehat{sl}_2)$ onto the total Hilbert space of the SG model describe the symmetries of this space investigated in [32, 22, 3].

4.5 Symmetries of the model at the FF point

Now we would like to demonstrate how the quantum symmetries of the Hilbert space $\mathcal{H}$ of the SG model become the classical ones (i.e. correspond to undeformed current algebra) at the FF point.

It is clear that the finite-dimensional representations of the algebra $\mathcal{A}(\widehat{sl}_2)$ at the value $\xi=1$ degenerate. Moreover, the operator h_+ becomes the central element of the algebra (cf. (4.34)) and takes the value $(-1)^{k+1}$, $k=0,1$ on the subspace $\mathcal{H}_k$ of the even and odd number of particles of the total Hilbert space $\mathcal{H}$. In order to obtain the nontrivial action of the algebra $\mathcal{A}(\widehat{sl}_2)$ at the FF point on the Hilbert space of states we introduce the rescaled operators

$$\hat{e}(u)=-\left.\frac{e(u)}{\mathrm{sh}(i\pi/\xi)}\right|_{\xi=1},$$

$$\hat{f}(u) = -h_+ \frac{f(u)}{\mathrm{sh}(i\pi/\xi)}\bigg|_{\xi=1},$$
$$\hat{h}(u) = -\frac{h(u)h_+^{-1} - 1}{\mathrm{sh}(i\pi/\xi)}\bigg|_{\xi=1}. \qquad (4.41)$$

The nontrivial commutation relations of the algebra $\mathcal{A}(\widehat{sl}_2)$ reads as follows:

$$[\hat{h}(u), \hat{e}(v)] = 2\mathrm{cth}\,(u-v)\hat{e}(v) - 2\frac{\hat{e}(u)}{\mathrm{sh}\,(u-v)}, \qquad (4.42)$$
$$[\hat{h}(u), \hat{f}(v)] = -2\mathrm{cth}\,(u-v)\hat{f}(v) + 2\frac{\hat{f}(u)}{\mathrm{sh}\,(u-v)}, \qquad (4.43)$$
$$[\hat{e}(u), \hat{f}(v)] = \frac{\hat{h}(u) - \hat{h}(v)}{\mathrm{sh}\,(u-v)}. \qquad (4.44)$$

The algebra (4.42)–(4.44) coincides with the classical current algebra $\widehat{sl}_2$ on the line [21].

Formulas (4.30) and (4.31) of the adjoint action of the operators (4.41) becomes

$$\mathrm{Ad}_{\hat{f}(u)}|\tilde{\theta}\rangle_- = \mathrm{Ad}_{\hat{e}(u)}|\tilde{\theta}\rangle_+ = 0, \quad \mathrm{Ad}_{\hat{h}(u)}|\tilde{\theta}\rangle_\pm = \pm\mathrm{cth}(u-\theta)\theta\rangle_\pm\,,$$
$$\mathrm{Ad}_{\hat{e}(u)}|\tilde{\theta}\rangle_- = \frac{1}{\mathrm{sh}\;(u-\theta)}|\tilde{\theta}\rangle_+, \quad \mathrm{Ad}_{\hat{f}(u)}|\tilde{\theta}\rangle_+ = -\frac{1}{\mathrm{sh}\;(u-\theta)}|\tilde{\theta}\rangle_-\,, \qquad (4.45)$$

and on the multi-particle states are

$$\hat{\Delta}(x) = x\otimes 1 + 1\otimes x, \quad x = \hat{e}(u),\ \hat{f}(u),\ \hat{h}(u)\,, \qquad (4.46)$$

where in order to obtain (4.45) and (4.46) we used the fact that operator h_+ equal to -1 on the one-particle state.

The phenomena that quantum symmetries of the Hilbert space of state for the SG model becomes the classical ones at the FF point is a consequence of the fact that S-matrix in this limit yields the classical r-matrix (2.67):

$$r(u) = \lim_{\xi\to 1} \frac{S(u;\xi) + 1}{\pi i(1-\xi)}. \qquad (4.47)$$

This phenomena was observed in reflectionless SG theory [24] and was used to investigate the space of the local operators in SG model at FF point [23].

5 Discussion

In this paper we further developed the method of angular quantization for the Sine-Gordon model. Technically the application of this method splits into two parts. First, one should explicitly describe canonical quantization $\mathcal{H}_R$ of the model in right Rindler wedge, where the boost plays the role of hamiltonian. Then the space of states and local operators of the theory on the line are described in terms of certain operators acting in $\mathcal{H}_R$.

We studied the SG theory at the free fermion point where the canonical quantization in RRW can be done explicitly. We investigated the integrals of motion and found that the usual local integrals of motion diverge. This forced us to consider nonlocal integrals of motion which are a certain analytical continuation (in the space of eigenvalues of Lorentz boost) of the usual charges and the only possibility to close them into a quadratic current algebra is to use charges with different monodromy properties. They form the specialization of the scaling elliptic algebra $\mathcal{A}(\widehat{sl}_2)$ [20] into free fermion point. The bosonization [26] naturally appears in terms of scattering data.

This indicates that angular quantization of SG model can be done in terms of the representation theory of the algebra $\mathcal{A}(\widehat{sl}_2)$. Starting from level one representations of this algebra in the bosonic Fock space we managed to construct the space of asymptotical states of SG model and some local operators acting into this space of states, in particular, the transfer matrix and the commuting set of the integrals of motion, and demonstrate the mechanism of trace calculations of the form factors of local operators. This approach is an extension of the ideas presented in [18] for XXZ model. The algebra $\mathcal{A}(\widehat{sl}_2)$ is not a Hopf algebra, but we were able to define the adjoint action of this algebra on the space of states, such that n-particle states with given rapidities form n-fold tensor product of two-dimensional representations of the algebra $\mathcal{A}(\widehat{sl}_2)$.

Contrary to the integrable models on the lattice local integrals and local operators of the SG theory appear as coefficients of the asymptotical expansions of certain currents which are constructed explicitly. In particular, the asymptotical expansion of the level zero adjoint action of the algebra $\mathcal{A}(\widehat{sl}_2)$ on the space of states produce the action of Chevelley generators of quantum affine algebra, which was known before. At the free fermion point we get in this way the action of the classical affine algebra $\widehat{sl}_2$ which was constructed in the framework of the radial quantization in the paper [23].

Nevertheless, the understanding of the angular quantization method of SG model for generic value of the renormalized coupling constant ξ is far from being complete. In particular, there is no rigorous construction of the quantum analogs of the Jost functions introduced in [26, 25] without referring to bosonization. SG model admits also natural analog of 'new level zero action' (see [15] and references therein) which is given in terms of L-operators as follows

$$\mathrm{Ad}'_{L(u)} \cdot X = L(u)\, X\, L(u)^{-1} \tag{5.1}$$

and depends on the dual deformation parameter $q' = \exp\left(i\pi\frac{\xi}{\xi+1}\right)$. It will be interesting to extend the results on the spinon bases in conformal field theories investigated in [4] to the massive integrable models. As we mentioned already that the algebra $\mathcal{A}(\widehat{sl}_2)$ is quasi-Hopf algebra, but belonging to a family of dynamical elliptic algebra. The definition of adjoint action, used in this paper, did not refer to the axiomatics of this family. It would be interesting to fill this gap.

Finally, it would also be interesting to further explore the role of the duality transformation (3.7). In terms of the SG coupling β, this is an electric/magnetic duality $\beta \to 2/\beta$ familiar from the conformal field theory of a compactified free boson. As described in the paper this duality relates the q'-deformation parameter of the algebra of the monodromy matrix with the q-deformation parameter of the physical S-matrix. Though we did not present this here, one can define a dual monodromy matrix by the formal replacement $\beta \to 2/\beta$ in the usual monondromy matrix, and show that formally this dual monodromy matrix commutes with the original monodromy matrix. This would imply that the dual monodromy matrix generates additional integrals of motion, presumably related to the quantum affine symmetry described in [3].

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Appendix A. The algebra of bosons a_λ and $\tilde{a}_\lambda$

The normal ordering with respect to the vacuum vectors (2.22) and (2.23)

$$\begin{aligned} b_\pm(\nu)b_\mp(\nu') &= {:}b_\pm(\nu)b_\mp(\nu'){:} + \langle b_\pm(\nu)b_\mp(\nu')\rangle, \\ \langle b_\pm(\nu)b_\mp(\nu')\rangle &= \delta(\nu+\nu')\Theta(-\nu'), \end{aligned} \tag{A.1}$$

where $\Theta(\nu)$ is a 'continuous' step function

$$\Theta(\nu) = \begin{cases} 1, & \text{for} \quad \nu > 0 \\ 1/2, & \text{for} \quad \nu = 0 \\ 0, & \text{for} \quad \nu < 0 \end{cases}, \qquad \Theta(\nu) + \Theta(-\nu) \equiv 1 \tag{A.2}$$

allows to observe that the commutation relations between operators a_λ and $\tilde{a}_\lambda$ is not closed in a sense that the commutator $[a_\lambda, \tilde{a}_\mu]$ cannot be presented as a linear combinations of the same operators with $\mathbb{C}$-number coefficients and $\mathbb{C}$-valued functions. The idea is to consider this commutator as a new bosonic operator and try to close the extended by this operator algebra. Fortunately this extended algebra is closed. To describe its commutation relations we introduce new operators:

$$t_{\lambda,\mu} = \int_{-\infty}^{\infty} d\nu \, \frac{\Gamma\left(\frac{1}{2} - i(\lambda+\nu)\right)}{\Gamma\left(\frac{1}{2} + i(\mu-\nu)\right)} \operatorname{th}\pi\nu \, :b_-(\mu-\nu)b_+(\lambda+\nu): \,, \tag{A.3}$$

$$\tilde{t}_{\lambda,\mu} = \int_{-\infty}^{\infty} d\nu \, \frac{\Gamma\left(\frac{1}{2} - i(\mu-\nu)\right)}{\Gamma\left(\frac{1}{2} + i(\lambda+\nu)\right)} \operatorname{th}\pi\nu \, :b_-(\mu-\nu)b_+(\lambda+\nu): \,. \tag{A.4}$$

The bosonic operators a_λ and $\tilde{a}_\lambda$ are related to the new operators $t_{\lambda,\mu}$ and $\tilde{t}_{\lambda,\mu}$ by the linear transformation:

$$\begin{aligned} \tilde{a}_\lambda &= \operatorname{cth}\pi\lambda \, \tilde{t}_{\lambda,0} - \frac{1}{\operatorname{sh}\pi\lambda} \, t_{0,\lambda}, \\ a_\lambda &= \frac{1}{\operatorname{sh}\pi\lambda} \, \tilde{t}_{\lambda,0} - \operatorname{cth}\pi\lambda \, t_{0,\lambda}. \end{aligned} \tag{A.5}$$

Using simple trigonometric algebra we conclude that the set of the operators $t_{\lambda,\mu}$ and $\tilde{t}_{\lambda,\mu}$ is not independent. For example, the following relation is valid:

$$\operatorname{sh}\pi\lambda \, (t_{\lambda,\mu} - t_{\lambda+\mu,0}) = \operatorname{sh}\pi\mu \, (\tilde{t}_{0,\lambda+\mu} - \tilde{t}_{\lambda,\mu}) \tag{A.6}$$

therefore we can conclude that complete algebra of the bosonic operators reads as follows:

$$[a_\lambda, a_\mu] = \lambda\delta(\lambda+\mu) \tag{A.7}$$

$$[a_\lambda, t_{\mu,\rho}] = t_{\mu,\rho+\lambda} - t_{\mu+\lambda,\rho} + \delta(\lambda+\mu+\rho) \int_0^\lambda d\nu \, \operatorname{th}\pi(\nu+\rho) \tag{A.8}$$

$$\begin{aligned} [t_{\lambda,\mu}, t_{\lambda',\mu'}] &= \operatorname{cth}\pi(\lambda+\mu')(t_{\lambda+\lambda'+\mu',\mu} - t_{\lambda',\lambda+\mu'+\mu}) \\ &\quad + \operatorname{cth}\pi(\lambda'+\mu)(t_{\lambda,\lambda'+\mu'+\mu} - t_{\lambda'+\lambda+\mu,\mu'}) \\ &\quad + \delta(\lambda+\lambda'+\mu+\mu') \int_0^{\lambda+\mu} d\nu \, \operatorname{th}\pi(\nu-\lambda) \, \operatorname{th}\pi(\nu+\mu'). \end{aligned} \tag{A.9}$$

This algebra can be understood as an algebraic realization of the complicated integral transform which relate the operators $\tilde{a}_\lambda$ and a_λ. Indeed, using these commutation relations we can verify that the combination

$$\tilde{a}_\lambda = \operatorname{ch}\pi\lambda \, a_\lambda - \operatorname{sh}\pi\lambda \, t_{0,\lambda} \tag{A.10}$$

also has the commutation relations of the Heisenberg algebra as a_λ do (cf. (A.7)).

Appendix B. Quantum Jost functions at the FF point

We will prove the equivalence of (2.75) to (2.73). The second case can be treated analogously.

First, we write down explicitly all the normal ordering rules which follows from (2.59)

$$\begin{aligned}
Z'_-(z)F(u) &= \frac{e^{\gamma/2}}{(2\pi)^{3/2}} \frac{\Gamma\left(\frac{1}{4}+\frac{u-z}{2\pi i}\right)}{\Gamma\left(\frac{3}{4}+\frac{u-z}{2\pi i}\right)} :Z'_-(z)F(u): \\
F(u)Z'_-(z) &= \frac{e^{\gamma/2}}{(2\pi)^{3/2}} \frac{\Gamma\left(\frac{1}{4}-\frac{u-z}{2\pi i}\right)}{\Gamma\left(\frac{3}{4}-\frac{u-z}{2\pi i}\right)} :Z'_-(z)F(u): \\
F(u)F(v) &= -\frac{e^{2\gamma}}{2\pi i}(u-v):F(u)F(v):
\end{aligned}$$

$$\begin{aligned}
&Z'_+\left(\alpha_1-\frac{i\pi}{2}\right)Z'_+\left(\alpha_2+\frac{i\pi}{2}\right) \\
&= g(\alpha_1-\alpha_2):Z'_+\left(\alpha_1-\frac{i\pi}{2}\right)Z'_+\left(\alpha_2+\frac{i\pi}{2}\right):
\end{aligned}$$

where in the last formula the function $g(\alpha)$ is given in terms of double Γ-functions

$$g(\alpha) = \frac{e^{5\gamma/4}}{2\pi}\frac{\Gamma_2^2(3\pi+i\alpha)}{\Gamma_2(2\pi+i\alpha)\Gamma_2(4\pi+i\alpha)}.$$

For the double and usual Γ-functions we use the integral representations [1, 19]

$$\int_{\tilde{C}} \frac{d\lambda\,\ln(-\lambda)}{2\pi i\lambda}\,\frac{e^{-x\lambda}}{1-e^{-\lambda/\eta}} = \ln\Gamma(\eta x) + \left(\eta x - \frac{1}{2}\right)(\gamma - \ln\eta) - \frac{1}{2}\ln 2\pi\ ,$$

$$\begin{aligned}
&\int_{\tilde{C}} \frac{d\lambda\,\ln(-\lambda)}{2\pi i\lambda}\,\frac{e^{-x\lambda}}{(1-e^{-\lambda\omega_1})(1-e^{-\lambda\omega_2})} \\
&= \ln\Gamma_2(x\mid\omega_1,\omega_2) - \frac{\gamma}{2}B_{2,2}(x\mid\omega_1;\omega_2)\ ,
\end{aligned}$$

where $B_{2,2}(x \mid \omega_1;\omega_2)$ is the double Bernulli polynomial of the second order

$$B_{2,2}(x \mid \omega_1;\omega_2) = \frac{1}{\omega_1\omega_2}\left[x^2 - x(\omega_1+\omega_2) + \frac{\omega_1^2+3\omega_1\omega_2+\omega_2^2}{6}\right].$$

The constant g in the relation (2.73) is the value of the function $g(\alpha)$ at the point $\alpha = 0$.

We have

$$\begin{aligned}
&g^{-1}Z'_+\left(\alpha-\frac{i\pi}{2}\right)Z'_+\left(\alpha+\frac{i\pi}{2}\right)\\
&= \frac{i}{16\pi^6}\int_{C_1}\int_{C_2} du_1du_2 :Z'_-\left(\alpha-\frac{i\pi}{2}\right)Z'_-\left(\alpha+\frac{i\pi}{2}\right)F(u_1)F(u_2):\\
&\quad\cdot\Gamma\left(\frac{1}{2}+\frac{u_1-\alpha}{2\pi i}\right)\Gamma\left(-\frac{u_1-\alpha}{2\pi i}\right)\Gamma\left(\frac{1}{2}-\frac{u_2-\alpha}{2\pi i}\right)\Gamma\left(\frac{u_2-\alpha}{2\pi i}\right)\\
&\quad\cdot(u_1-u_2)\frac{\Gamma\left(\frac{1}{2}+\frac{u_2-\alpha}{2\pi i}\right)\Gamma\left(\frac{1}{2}-\frac{u_1-\alpha}{2\pi i}\right)}{\Gamma\left(1+\frac{u_2-\alpha}{2\pi i}\right)\Gamma\left(1-\frac{u_1-\alpha}{2\pi i}\right)}.
\end{aligned} \tag{B.1}$$

The contours C_1 and C_2 in (B.1) go from $-\infty$ to $+\infty$ and

$$\text{Im }\alpha-\pi < \text{Im }u_1 < \text{Im }\alpha, \quad \text{Im }\alpha < \text{Im }u_2 < \text{Im }\alpha+\pi \tag{B.2}$$

Using the elementary properties of the Γ-functions we can rewrite the integrand in (B.1) in the form

$$\begin{aligned}
&\frac{i}{8\pi^2}\int_{C_1}\int_{C_2} du_1du_2\left[\frac{1}{u_2-\alpha}-\frac{1}{u_1-\alpha}\right]\\
&\qquad\cdot\frac{:Z'_-\left(\alpha-\frac{i\pi}{2}\right)Z'_-\left(\alpha+\frac{i\pi}{2}\right)F(u_1)F(u_2):}{\text{ch}\left(\frac{u_1-z}{2}\right)\text{ch}\left(\frac{u_2-z}{2}\right)}
\end{aligned} \tag{B.3}$$

where contours C_1 and C_2 are specified in (B.2).

Using the fact that integrand in (B.3) is antisymmetric function with respect to variables u_1 and u_2 we conclude:

$$g^{-1}Z'_+\left(\alpha-\frac{i\pi}{2}\right)Z'_+\left(\alpha+\frac{i\pi}{2}\right) = \frac{1}{4\pi}\int_{-\infty}^{\infty} du\frac{F(u)}{\text{ch}\left(\frac{u-\alpha}{2}\right)} = \Lambda_-(\alpha) \tag{B.4}$$

since the current $F(u)$ coincide with the scattering data operator $\mathcal{Z}_-(u)$.

References

[1] E.W. Barnes, *On the theory of the multiple gamma functions*, Trans. Cambridge Philos. Soc., **19** (1904), 374-425.

[2] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic, London, 1982.

[3] D. Bernard and A. LeClair, *Quantum group symmetries and nonlocal currents in 2-d qft*, Commun. Math. Phys., **142** (1991), 99-138.

[4] D. Bernard, V. Pasquier, and D. Serban, *Spinons in conformal field theory*, Nucl. Phys., **B428** (1994), 612;
P. Bouwknegt, A.W.W. Ludwig, and K. Schoutens, *Spinon bases for higher level SU(2) WZW models*, Phys. Lett. B, **359** (1995), 304; *Spinon bases, Yangian symmetry and fermion representations of Virasoro characters in conformal field theory*, Phys. Lett. B, **338** (1994), 448.

[5] V. Brazhnikov and S. Lukyanov, *Angular quantization and form factors in massive integrable models*, Nucl. Phys. B [FS], **512** (1998), 616-636.

[6] S. Coleman, *Quantum Sine-Gordon equation as the massive Thirring model*, Physical Rev. D, **11** (1975), 2088-2097.

[7] B. Davies, O. Foda, M. Jimbo, T. Miwa, and A. Nakayashiki, *Diagonalization of the XXZ Hamiltonian by vertex operators*, Commun. Math. Phys., **151** (1993), 89-153;
M. Jimbo, T. Miwa, and A. Nakayashiki, *Correlation functions of the XXZ model for $\Delta < -1$*, Phys. Lett. A, **168** (1992), 256-263.

[8] J. Ding and I.B. Frenkel, *Isomorphism of two realizations of quantum affine algebras $U_q(gl(n))$*, Commun. Math. Phys., **156** (1993), 277-300.

[9] J. Ding, S. Kharchev, S. Khoroshkin, and S. Pakuliak, *Analytical properties of the elliptic currents*, preprint MPI-1999-8; ITEP-TH-74/98.

[10] J. Ding and S. Khoroshkin, *Weyl group extension of quantized current algebras* preprint `math.QA/9804139`.

[11] V.G. Drinfeld, *Quasi-Hopf algebras*, Leningrad Math. J., **1** (1990), 1419-1457.

[12] L. Faddeev, E. Sklyanin, and L. Takhtadjan, *Quantum inverse problem method*, I, Teor. Mat. Fiz., **40** (1979), 194-220.

[13] L.D. Faddeev and L.A. Takhtadjan, *Hamiltonian methods in the theory of solitons*, Springer, New York, 1987.

[14] O. Foda, K. Iohara, M. Jimbo, R. Kedem, T. Miwa, and H. Yan, *An elliptic quantum algebra for* $\widehat{sl}_2$, Lett. Math. Phys., **32** (1994), 259-268.

[15] M. Jimbo, R. Kedem, H. Konno, T. Miwa, and J.-U.H. Petersen, *Level 0 structure of level 1* $U_q(sl(2))$ *modules and MacDonald polynomials*, J. Phys. A, **28** (1995), 5589-5606.

[16] M. Jimbo, H. Konno, and T. Miwa, *Massless XXZ model and degeneration of the elliptic algebra* $A_{(q,p)}(sl(2))$, in "Deformation theory and symplectic geometry", Ascona, 1996, 117-138.

[17] M. Jimbo, H. Konno, S. Odake, and J. Shiraishi, *Quasi-Hopf twistors for elliptic quantum groups*, preprint `q-alg/9712029`; *Elliptic algebra* $U_{q,p}(\widehat{sl}_2)$: *Drinfeld currents and vertex operators*, preprint `math.QA/9802002`.

[18] M. Jimbo and T. Miwa, *Algebraic Analysis of Solvable Lattice Models*, Conference Board of the Math. Sci., Regional Conference Series in Mathematics, **85** (1995).

[19] M. Jimbo and T. Miwa, *Quantum KZ equation with* $|q| = 1$ *and correlation functions of the XXZ model in the gapless regime*, J. Phys., **A29** (1996), 2923-2958.

[20] S. Khoroshkin, D. Lebedev, and S. Pakuliak, *Elliptic algebra* $\mathcal{A}_{p,q}(\widehat{sl}_2)$ *in the scaling limit*, Commun. Math. Phys., **190** (1998), 597-627.

[21] S. Khoroshkin, D. Lebedev, S. Pakuliak, A. Stolin, and V. Tolstoy, *Classical limit of the scaled elliptic algebra* $\mathcal{A}(\widehat{sl}_2)$, Compositio Math., **115** (1999), 205-230.

[22] A. LeClair, *Restricted Sine-Gordon theory and the minimal conformal series*, Phys. Lett., **230B** (1989), 103.

[23] A. Leclair, *Spectrum generating affine Lie algebras and correlation functions in massive field theory*, Nucl. Phys. B, **415** (1994), 734-780.

[24] A. LeClair and D. Nemeschansky, *Affine lie algebra symmetry of Sine-Gordon theory at reflectionless points*, Mod. Phys. Lett. A, **11** (1996), 139-146.

[25] S. Lukyanov, *Correlators of the Jost functions in the Sine-Gordon model*, Phys. Lett. B, **325** (1994), 409-417.

[26] S. Lukyanov, *Free field representation for massive integrable models*, Commun. Math. Phys., **167** (1995), 183-226.

[27] S. Lukyanov, *Private communication*, 1998.

[28] M. Lüscher, *Quantum non-local charges and absence of particle production in the two-dimensional non-linear σ-model*, Nucl. Phys., **B135** (1978), 1-19.

[29] A. Nakayashiki, S. Pakuliak, and V. Tarasov, *On solutions of KZ and qKZ equations at level zero*, Ann. Inst. Henri Poincaré, **71**(**4**) (1999), 459-496.

[30] M.R. Niedermaier, *Form factors, thermal states and modular structures*, Nucl. Phys. B, **519** (1998), 517.

[31] S. Pakuliak, *Annihilation poles for form factors in the XXZ model*, Inter. J. Modern Phys. A, **9** (1994), 2087-2102.

[32] N. Reshetikhin and F. Smirnov, *Hidden quantum group symmetry and integrable perturbations of conformal field theories*, Comm. Math. Phys., **131** (1990), 157-177.

[33] F. Smirnov, *Form Factors in Completely Integrable Field Theories*, World Scientific, Singapore, 1992.

[34] A.B. Zamolodchikov and Al.B. Zamolodchikov, *Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models*, Ann. Phys. (N.Y.), **120** (1979), 253-291.