Adv. Theor. Math. Phys. **5** (1999) 1201–1225

The Vafa-Witten Theory for Gauge Group $SU(N)$

J.M.F. Labastida and Carlos Lozano[1]

Departamento de Física de Partículas
Universidade de Santiago de Compostela
E-15706 Santiago de Compostela, Spain
lozano@fpaxp1.usc.es

Abstract

We derive the partition function for the Vafa-Witten twist of the $\mathcal{N} = 4$ supersymmetric gauge theory with gauge group $SU(N)$ (for prime N) and arbitrary values of the 't Hooft fluxes $v \in H^2(X, \mathbb{Z}_N)$ on Kähler four-manifolds with $b_2^+ > 1$.

1 Introduction

The study of topological quantum field theories (TQFTs) originated from the twist of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory has been

[1]Present address: Department of Physics, Brandeis University, Waltham, MA 02454, USA
e-print archive: http://xxx.lanl.gov/hep-th/9903172

pursued during the last few years. These studies have led to the full solution for some of the models involved, and have provided important tests of our ideas on duality for Yang-Mills theories in four dimensions. In this paper we generalize previous results obtained by Vafa and Witten [23] for one of the twisted theories.

As in the $\mathcal{N}=2$ case, the $\mathcal{R}$-symmetry group of the $\mathcal{N}=4$ algebra can be twisted to obtain a topological model. But since the $\mathcal{R}$-symmetry group of the $\mathcal{N}=4$ theory is $SU(4)$, this topological twist can be performed in three inequivalent ways, so one ends up with three different TQFTs [23], [28], [13]. The twisted theories are topological in the sense that the partition function as well as a selected set of correlation functions are independent of the metric which defines the background geometry. In the short distance regime, computations in the twisted theory are given exactly by a saddle-point calculation around a certain bosonic background or moduli space, and in fact the correlation functions can be reinterpreted as describing intersection theory on this moduli space. This correspondence can be made more precise through the Mathai-Quillen construction [13]. Unfortunately, it is not possible to perform explicit computations from this viewpoint: the moduli spaces one ends up with are generically non-compact, and no precise recipe is known to properly compactify them.

While for the TQFTs related to asymptotically free $\mathcal{N}=2$ theories the interest lies in their ability to define topological invariants for four-manifolds, for the twisted $\mathcal{N}=4$ theories the topological character is used as a tool for performing explicit computations which might shed light on the structure of the physical $\mathcal{N}=4$ theory. This theory is finite and conformally invariant, and is conjectured to have a symmetry exchanging strong and weak coupling and exchanging electric and magnetic fields, which extends to a full $SL(2,\mathbb{Z})$ symmetry acting on the microscopic complexified coupling τ [21]. In addition to this, since all the fields in the theory take values in the adjoint representation of the gauge group, it is possible to consider non-trivial gauge configurations in $G/Center(G)$ and compute the partition function for fixed values of the 't Hooft flux $v \in H^2(X,\pi_1(G))$ which should behave under $SL(2,\mathbb{Z})$ duality in a well-defined fashion [8]. This has been checked for the physical $\mathcal{N}=4$ theory on T^4 in [6]. It is natural to expect that this property should be shared by the twisted theories on arbitrary four-manifolds. This was checked by Vafa and Witten for one of the twisted theories and for gauge group $SU(2)$ [23], and it was clearly mostly interesting to extend their computation to higher rank groups. Similar results have been recently derived for another twisted version of the $\mathcal{N}= 4$ theory within the u-plane approach [12].

In this paper we will consider the Vafa-Witten theory for gauge group $SU(N)$. The twisted theory does not contain spinors, so it is well-defined on any compact, oriented four-manifold. The ghost-number symmetry of this theory is anomaly-free, and therefore the only non-trivial topological observable is the partition function itself. As we mentioned above, it is possible to consider non-trivial gauge configurations in $G/Center(G)$ and compute the partition function for a fixed value of the 't Hooft flux $v \in H^2(X, \pi_1(G))$. In this case, however, the Seiberg-Witten approach is not available, but, as conjectured by Vafa and Witten, one can nevertheless compute in terms of the vacuum degrees of freedom of the $\mathcal{N} = 1$ theory which results from giving bare masses to all the three chiral multiplets of the $\mathcal{N} = 4$ theory[2]. The partition functions on $K3$ for gauge group $SU(N)$ and trivial 't Hooft fluxes have been computed by Vafa and collaborators in [20]. We will extend their results to arbitrary 't Hooft fluxes and compute the partition function on more general Kähler four-manifolds. A brief account of these results has already appeared in [14].

The paper is organized as follows. In sect. 2 we review the structure of the $\mathcal{N} = 4$ supersymmetric gauge theory in four dimensions and its topological twisting. In sect. 3 we review the Vafa-Witten theory, which arises as a twisted version of the $\mathcal{N} = 4$ theory, and analyze the vacuum structure of the $\mathcal{N} = 1$ theory which arises by giving masses to all the three chiral multiplets of the $\mathcal{N} = 4$ theory. In sect. 4 we derive the partition function on $K3$ for $G = SU(N)$ with prime N and arbitrary values of the 't Hooft fluxes. In sect. 5 we generalize the partition function to more general Kähler manifolds and study the properties of the resulting formulas under duality and under blow-ups. Finally, in sect. 6 we state our conclusions. An appendix deals with a set of useful identities and definitions used in the paper.

[2]A similar approach was introduced by Witten in [24] to obtain the first explicit results for the Donaldson-Witten theory just before the far more powerful Seiberg-Witten approach was available.

2 Twisting $\mathcal{N} = 4$ supersymmetric gauge theory on four-manifolds

In this section we review some aspects of the four-dimensional $\mathcal{N}$=4 gauge theory and its topological twisting.

2.1 The $\mathcal{N} = 4$ supersymmetric gauge theory

We begin with several well-known remarks concerning the $\mathcal{N} = 4$ supersymmetric gauge theory on flat $\mathbb{R}^4$. The $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is unique once the gauge group G and the microscopic coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi^2 i}{e^2}$ are fixed. The microscopic theory contains a gauge or gluon field, four chiral spinors (the gluinos) and six real scalars. All the above fields are massless and take values in the adjoint representation of the gauge group G. From the point of view of $\mathcal{N} = 1$ superspace, the theory contains one $\mathcal{N} = 1$ vector multiplet and three $\mathcal{N} = 1$ chiral multiplets. These supermultiplets are represented in $\mathcal{N} = 1$ superspace by the superfields V and Φ_s $(s = 1, 2, 3)$, which satisfy the constraints $V = V^\dagger$ and $\bar{D}_{\dot{\alpha}}\Phi_s = 0$, $\bar{D}_{\dot{\alpha}}$ being a superspace covariant derivative[3]. The $\mathcal{N} = 4$ supersymmetry algebra has the automorphism group $SU(4)_I$, under which the gauge bosons are singlets, the gauginos transform in the $\mathbf{4} \oplus \bar{\mathbf{4}}$, and the scalars transform as a self-conjugate antisymmetric tensor ϕ_{uv} in the $\mathbf{6}$.

The action takes the following form in $\mathcal{N} = 1$ superspace:

$$\begin{aligned} \mathcal{S} = & -\frac{i}{4\pi}\tau \int d^4x d^2\theta \,\mathrm{Tr}(W^2) + \frac{i}{4\pi}\bar{\tau} \int d^4x d^2\bar{\theta} \,\mathrm{Tr}(W^{\dagger 2}) \\ & + \frac{1}{e^2}\sum_{s=1}^{3} \int d^4x d^2\theta d^2\bar{\theta} \,\mathrm{Tr}(\Phi^{\dagger s} \mathrm{e}^V \Phi_s \mathrm{e}^{-V}) \\ & + \frac{i\sqrt{2}}{e^2} \int d^4x d^2\theta \,\mathrm{Tr}\{\Phi_1[\Phi_2, \Phi_3]\} \\ & + \frac{i\sqrt{2}}{e^2} \int d^4x d^2\bar{\theta} \,\mathrm{Tr}\{\Phi^{\dagger 1}[\Phi^{\dagger 2}, \Phi^{\dagger 3}]\}, \end{aligned} \tag{2.1}$$

[3]We follow the same conventions as in [13].

where $W_\alpha = -\frac{1}{16}\bar{D}^2 e^{-V} D_\alpha e^V$ and Tr denotes the trace in the fundamental representation.

The theory is invariant under four independent supersymmetries which transform under $SU(4)_I$, but only one of these is manifest in the $\mathcal{N} = 1$ superspace formulation (2.1). The global symmetry group of $\mathcal{N} = 4$ supersymmetric theories in $\mathbb{R}^4$ is $\mathcal{H} = SU(2)_L \otimes SU(2)_R \otimes SU(4)_I$, where $\mathcal{K} = SU(2)_L \otimes SU(2)_R$ is the rotation group $SO(4)$. The fermionic generators of the four supersymmetries are $Q^u{}_\alpha$ and $\bar{Q}_{u\dot{\alpha}}$. They transform as $(\mathbf{2}, \mathbf{1}, \bar{\mathbf{4}}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{4})$ under $\mathcal{H}$.

The massless $\mathcal{N} = 4$ supersymmetric theory has zero beta function, and it is believed to be exactly finite and conformally invariant, even non-perturbatively. It is in fact the most promising candidate for the explicit realization of the strong-weak coupling duality symmetry conjectured some twenty years ago by Montonen and Olive [21].

2.2 Twists of the $\mathcal{N} = 4$ supersymmetric theory

The twist in the context of four-dimensional supersymmetric gauge theories was introduced by Witten in [25], where it was shown that a twisted version of the $\mathcal{N} = 2$ supersymmetric gauge theory with gauge group $SU(2)$ is a relativistic field-theory representation of the Donaldson theory of four-manifolds. In four dimensions, the global symmetry group of the extended supersymmetric gauge theories is of the form $SU(2)_L \otimes SU(2)_R \otimes \mathcal{I}$, where $\mathcal{K} = SU(2)_L \otimes SU(2)_R$ is the rotation group, and $\mathcal{I}$ is the chiral $\mathcal{R}$-symmetry group. The twist can be thought of either as an exotic realization of the global symmetry group of the theory, or as the gauging (with the spin connection) of a certain subgroup of the global $\mathcal{R}$-current of the theory.

While in $\mathcal{N} = 2$ supersymmetric gauge theories the $\mathcal{R}$-symmetry group is at most $U(2)$ and thus the twist is essentially unique (up to an exchange of left and right), in the $\mathcal{N} = 4$ supersymmetric gauge theory the $\mathcal{R}$-symmetry group is $SU(4)$ and there are three different possibilities, each corresponding to a different non-equivalent embedding of the rotation group into the $\mathcal{R}$-symmetry group [23], [28], [13]. Two of these possibilities give rise to topological field theories with two independent BRST-like topological symmetries. One of these was considered by

Vafa and Witten in [23]. The second possibility was first addressed by Marcus [18], and his analysis was extended in [13], [2]. The remaining possibility leads to the half-twisted theory, a topological theory with only one BRST symmetry [28], [13]. The generating function of topological correlation functions of this theory has been recently computed for gauge group $SU(2)$ and arbitrary values of the 't Hooft flux in [12] within the u-plane framework [22].

3 The Vafa-Witten theory

The Vafa-Witten theory can be obtained by twisting the $\mathcal{N}$=4 supersymmetric gauge theory as described in [23], [28], [13]. The twisted theory has an anomaly free Abelian ghost-number symmetry which is a subgroup of the $SU(4)_I$ $\mathcal{R}$-symmetry of the $\mathcal{N}$=4 theory. Therefore, the partition function is the only non-trivial topological observable of the theory [23].

The theory has 2 independent BRST charges $Q^{\pm}$ of opposite ghost number. The field content consists of 3 scalar fields $\{\phi^{+2}, \bar{\phi}^{-2}, C^0\}$, 2 one-forms $\{A^0_{\alpha\dot{\alpha}}, \tilde{H}^0_{\alpha\dot{\alpha}}\}$ and 2 self-dual two-forms $\{(B^+_{\alpha\beta})^0, (H^+_{\alpha\beta})^0\}$ on the bosonic (commuting) side; and 2 scalar fields $\{\zeta^{+1}, \eta^{-1}\}$, 2 one-forms $\{\psi^1_{\alpha\dot{\alpha}}, \tilde{\chi}^{-1}_{\alpha\dot{\alpha}}\}$ and 2 self-dual two-forms $\{(\tilde{\psi}^+_{\alpha\beta})^{+1}, (\chi^+_{\alpha\beta})^{-1}\}$ on the fermionic (anticommuting) side. The superscript stands for the ghost number carried by each of the fields.

The twisted $\mathcal{N} = 4$ supersymmetric action breaks up into a Q^+-exact piece (that is, a piece which can be written as $\{Q^+, \mathcal{T}\}$, where $\mathcal{T}$ is a functional of the fields of the theory), plus a topological term proportional to the instanton number of the gauge configuration,

$$\mathcal{S}_{\text{twisted}} = \{Q^+, \mathcal{T}\} - 2\pi i h_v \tau, \tag{3.2}$$

with h_v the instanton number of a gauge bundle with 't Hooft flux v. This is an integer for $SU(N)$ bundles ($v = 0$), but for non-trivial $SU(N)/\mathbb{Z}_N$ bundles with $v \neq 0$ one has

$$h_v = -\frac{N-1}{2N} v \cdot v \mod \mathbb{Z}, \tag{3.3}$$

where $v \cdot v$ stands for $\int_X v \wedge v$. Therefore, as pointed out in [23], one would expect the $SU(N)$ partition function to be invariant under

$\tau \to \tau + 1$, while the $SU(N)/\mathbb{Z}_N$ theory should be invariant under $\tau \to \tau + 2N$ on arbitrary four-manifolds, and under $\tau \to \tau + N$ on spin four-manifolds (where $v \cdot v$ is even). In any case, for odd N, we have invariance under $\tau \to \tau + N$ on any four-manifold. Notice that, owing to (3.2), the partition function depends on the microscopic couplings e and θ only through the combination $2\pi i h_v \tau$, and in particular this dependence is a priori holomorphic (were the orientation of the manifold X reversed, the partition function would depend anti-holomorphically on τ). However there could be situations in which, because of some sort of holomorphic anomaly, the partition function would acquire an explicit anomalous dependence on $\bar{\tau}$. This seems to be the case, for example, for the theory defined on $\mathbb{CP}^2$ [23] and, more generally, on manifolds with $b_2^+ = 1$ [20]. Somewhat related results have been derived for the Donaldson-Witten theory in the context of the u-plane formalism [22].

3.1 Mass perturbations and reduction to $\mathcal{N}$=1

It is a well-known fact that on complex manifolds the exterior differential d splits into the Dolbeaut operators ∂ and $\bar{\partial}$. In a similar way, as pointed out in [24], on a Kähler manifold the number of BRST charges of a twisted supersymmetric theory is doubled, in such a way that, for example, the Donaldson-Witten theory has an enhanced $\mathcal{N}_T = 2$ topological symmetry on Kähler manifolds, while the Vafa-Witten theory has $\mathcal{N}_T = 4$ topological symmetry. In each case, one of the BRST charges comes from the underlying $\mathcal{N} = 1$ subalgebra which corresponds to the formulation of the physical theory in $\mathcal{N} = 1$ superspace. By suitably adding mass terms for some of the chiral superfields in the theory, one can break the extended ($\mathcal{N} = 2$ or $\mathcal{N} = 4$) supersymmetry of the physical theory down to $\mathcal{N} = 1$. For the reason sketched above, the corresponding twisted massive theory on Kähler manifolds should still retain at least one topological symmetry. One now exploits the metric independence of the topological theory. By scaling up the metric in the topological theory, $g_{\mu\nu} \to t g_{\mu\nu}$, one can take the limit $t \to \infty$. In this limit, the metric on X becomes nearly flat, and it is reasonable that the computations in the topological field theory can be performed in terms of the vacuum structure of the $\mathcal{N} = 1$ theory.

One could wonder as to what the effect of the perturbation may be. The introduction of a mass perturbation may (and in general will) distort the original topological field theory. This poses no problem in the case of the Donaldson-Witten theory, as Witten was able to prove that the perturbation is topologically trivial, in the sense that it affects the theory in an important but controllable way [24]. As for the Vafa-Witten theory [23], [15], [4], the twisted massive theory is topological on Kähler four-manifolds with $h^{2,0} \neq 0$, and the partition function is actually invariant under the perturbation. The constraint $h^{2,0} \neq 0$ comes about as follows. In the twisted theory the chiral superfields of the $\mathcal{N} = 4$ theory are no longer scalars, so the mass terms can not be invariant under the holonomy group of the manifold unless one of the mass parameters be a holomorphic two-form ω.

The massive $\mathcal{N} = 1$ theory has the tree level superpotential

$$\begin{aligned}\mathcal{W} = \frac{i\sqrt{2}}{e^2} \int d^4x d^2\theta \, \mathrm{Tr}\{\Phi_1[\Phi_2, \Phi_3]\} + m \int_X d^2\theta \mathrm{Tr}(\Phi_1 \Phi_2) \\ + \int_X d^2\theta \omega \mathrm{Tr}(\Phi_3)^2 + \mathrm{h.c.} \end{aligned} \tag{3.4}$$

Up to a constant rescaling the equations for a critical point of $\mathcal{W}$ are

$$\begin{aligned} [\Phi_3, \Phi_1] &= -m\Phi_3, \\ [\Phi_3, \Phi_2] &= m\Phi_2, \\ [\Phi_1, \Phi_2] &= 2\omega\Phi_3. \end{aligned} \tag{3.5}$$

As noted in [23], [5], these equations are the commutation relations of the Lie algebra of $SU(2)$, and therefore the classical vacua of the resulting $\mathcal{N} = 1$ theory can be classified by the complex conjugacy classes of homomorphisms of the $SU(2)$ Lie algebra to that of $G = SU(N)$.

Eqs. (3.5) admit a trivial solution $\Phi_1 = \Phi_2 = \Phi_3 = 0$ where the gauge group is unbroken and which reduces at low energy to the $\mathcal{N} = 1$ pure $SU(N)$ gauge theory (which has N discrete vacua), and a non-trivial solution (the irreducible embedding in [23]) where the gauge group is completely broken. This corresponds to $\{\Phi_1, \Phi_2, \Phi_3\}$ defining the representation N of $SU(2)$. All these vacua have a mass gap: the irreducible embedding is a Higgs vacuum, while the presence of a mass gap in the trivial vacua is a well-known feature of the $\mathcal{N} = 1$ theory.

When N is prime, these are the only relevant vacua of the $\mathcal{N} = 1$ theory. There are other, more general, solutions to (3.5) which leave different subgroups of G unbroken. However, in all these solutions the unbroken gauge group contains $U(1)$ factors and one expects on general grounds that they should not contribute to the partition function [23]. On the other hand, when N is not prime, there are additional contributions coming from embeddings for which the unbroken gauge group is $SU(d)$, where d is a positive divisor of d. The low-energy theory is again an $\mathcal{N} = 1$ $SU(d)$ gauge theory wihout matter with d massive discrete vacua.

In the long-distance limit, the partition function is given as a finite sum over the contributions of the discrete massive vacua of the resulting $\mathcal{N} = 1$ theory. For $G = SU(N)$ the number of such vacua is given by the sum of the positive divisors of N [5]. The contribution of each vacuum is universal (because of the mass gap), and can be fixed by comparing to known mathematical results [23].

4 The partition function on $K3$

As a first step towards the derivation of the formula for the partition function we will consider the theory on $K3$, where some explicit results are already available. For X a $K3$ surface the canonical divisor is trivial, so there exists a nowhere vanishing section of the bundle of $(2, 0)$ forms. Therefore, the mass perturbation ω does not vanish anywhere and the above analysis of the vacuum structure of the $\mathcal{N}$=1 theory carries over without change.

The structure of the partition function for trivial 't Hooft flux was conjectured in [23]. This conjecture has been confirmed in [20] by studying the effective theory on N coincident $M5$-branes wrapping around $K3 \times T^2$. The partition function for zero 't Hooft flux is *almost* a Hecke transformation of order N [1] of $G(\tau) = \eta(\tau)^{-24}$, with $\eta(\tau)$ the Dedekind function – see eq. (3.7) in [20]:

$$Z_{v=0} \equiv Z_N = \frac{1}{N^2} \sum_{\substack{0 \leq a,b,d \in \mathbb{Z} \\ ad=N,\ b<d}} d\, G\left(\frac{a\tau + b}{d}\right). \tag{4.6}$$

Notice that the number of terms in (4.6) equals the sum of the positive divisors of N as we mentioned above. When N is prime the formula is considerably simpler

$$Z_{v=0} = \frac{1}{N^2} G(N\tau) + \frac{1}{N} \sum_{m=0}^{N-1} G\left(\frac{\tau+m}{N}\right). \tag{4.7}$$

There are $N+1$ terms, the first one corresponding to the irreducible embedding, and the other N to the vacua of the $\mathcal{N}$=1 $SU(N)$ SYM theory.

The $SU(N)$ partition function is defined from (4.6) as $Z_{SU(N)} = \frac{1}{N} Z_{v=0}$. From it, the $SU(N)/\mathbb{Z}_N$ partition function $Z_{SU(N)/\mathbb{Z}_N} = \sum_v Z_v$ can be obtained via a modular transformation [23] (see the appendix for details)

$$\begin{aligned} Z_{SU(N)/\mathbb{Z}_N}(\tau) &= N^{\chi/2} \left(\frac{\tau}{i}\right)^{\chi/2} Z_{SU(N)}(-1/\tau) \\ &= \frac{1}{N^2} \sum_{\substack{a,b,d \\ p=\gcd(b,d)}} d^{12} p^{11} G\left(\frac{a\tau+b}{d}\right). \end{aligned} \tag{4.8}$$

Notice the first equality in (4.8), which is, up to some correction factors which vanish in flat space, the original Montonen-Olive conjecture.

To generalize (4.7) for gauge configurations with arbitrary 't Hooft flux we proceed as in [23]. The N contributions coming from the $\mathcal{N}=1$ pure gauge theory vacua are related by an anomalous chiral symmetry which takes $\tau \to \tau+1$. The anomaly is $2Nh_v - (N^2-1)\left(\frac{\chi+\sigma}{4}\right) = -(N-1)v\cdot v + \cdots$, which is half the anomaly in Donaldson-Witten theory. Hence, the contributions from each vacuum pick anomalous phases $\mathrm{e}^{-i\pi m h_v} = \mathrm{e}^{i\pi \frac{N-1}{N} m v^2}$. As for the contribution coming from the irreducible embedding, modular invariance requires that it vanishes unless $v=0$. Hence,

$$Z_v = \frac{1}{N^2} G(N\tau)\delta_{v,0} + \frac{1}{N} \sum_{m=0}^{N-1} \mathrm{e}^{i\pi \frac{N-1}{N} m v^2} G\left(\frac{\tau+m}{N}\right). \tag{4.9}$$

The Z_v transform into each other under the modular group as predicted in [23]

$$\begin{aligned} Z_v(\tau+1) &= \mathrm{e}^{-i\pi \frac{N-1}{N} v^2} Z_v(\tau), \\ [5pt] Z_v(-1/\tau) &= N^{-11} \left(\frac{\tau}{i}\right)^{-12} \sum_u \mathrm{e}^{\frac{2i\pi u\cdot v}{N}} Z_u(\tau). \end{aligned} \tag{4.10}$$

To evaluate the sum over u we use formulas (A.16) and (A.17) in the appendix[4].

By summing over v in (4.9) we can check (4.8)

$$\begin{aligned} Z_{SU(N)/\mathbb{Z}_N} &= \sum_v Z_v \\ &= \frac{1}{N^2} G(N\tau) + N^{21} G(\tau/N) + N^{10} \sum_{m=1}^{N-1} G\left(\frac{\tau+m}{N}\right). \end{aligned} \tag{4.11}$$

The above results only hold for prime N. The appropriate generalization for arbitrary N should be also investigated.

5 More general Kähler manifolds

On more general Kähler manifolds the spatially dependent mass term vanishes where ω does, and we will assume as in [23], [24] that ω vanishes with multiplicity one on a union of disjoint, smooth complex curves C_j, $j = 1, \ldots n$ of genus g_j which represent the canonical divisor K of X. The vanishing of ω introduces corrections involving K and additional modular functions whose precise form is not known a priori. In the $G = SU(2)$ case, each of the $\mathcal{N} = 1$ vacua bifurcates along each of the components C_j of the canonical divisor into two strongly coupled massive vacua. This vacuum degeneracy is believed to stem [23], [24] from the spontaneous breaking of a $\mathbb{Z}_2$ chiral symmetry which is unbroken in bulk. This is exactly the same pattern that arises in all known examples of twisted $\mathcal{N} = 2$ theories with gauge group $SU(2)$ as the Donaldson-Witten theory and its generalizations [24], [22], [16]. This in turn seems to be related to the possibility of rewritting the corrections near the canonical divisor in terms of the Seiberg-Witten invariants [26]. In fact, it is known that the Vafa-Witten partition function for $G = SU(2)$ can be rewritten in terms of the Seiberg-Witten invariants [4].

The form of the corrections for $G = SU(N)$ is more involved. From related results on Donaldson-Witten theory [19] we know that the higher-rank case presents some new features. We have not been

[4]Note that $K3$ has $\chi = 24$, $\sigma = -16$, $b_1 = 0$ and $b_2 = 22$.

able to disentangle the structure of the vacua near the canonical divisor from first principles. Instead, we will exploit the expected behaviour of the partition function under blow-ups of X. This, together with the modular invariance of the partition function will suffice to completely determine the unknown functions.

5.1 Behaviour under blow-ups

Blowing up a point on a Kähler manifold X replaces it with a new Kähler manifold $\widehat{X}$ whose second cohomology lattice is $H^2(\widehat{X}, \mathbb{Z}) = H^2(X, \mathbb{Z}) \oplus I^-$, where I^- is the one-dimensional lattice spanned by the Poincaré dual of the exceptional divisor B created by the blow-up. Any allowed $\mathbb{Z}_N$ flux $\widehat{v}$ on $\widehat{X}$ is of the form $\widehat{v} = v \oplus r$, where v is a flux in X and $r = \lambda B$, $\lambda = 0, 1, \ldots N-1$. The main result concerning the $SU(2)$ partition function in [23] is that under blowing up a point on a Kähler four-manifold with canonical divisor as above, the partition functions for fixed 't Hooft fluxes $\widehat{Z}_{\widehat{X},\widehat{v}}$ factorize as $Z_{X,v}$ times a level 1 character of the $SU(2)$ WZW model. It would be natural to expect that the same factorization holds for $G = SU(N)$, but now with the level 1 $SU(N)$ characters. In fact, the same behaviour under blow-ups has been proved by Yoshioka [29] for the generating function of Euler characteristics of $SU(N)$ instanton moduli space on Kähler manifolds. This should not come out as a surprise since it is known that, on certain four-manifolds, the partition function of Vafa-Witten theory computes Euler characteristics of instanton moduli spaces [23], [20]. This can be confirmed by realizing the Vafa-Witten theory as the low-energy theory of $M5$-branes wrapped on $X \times T^2$ [3]. It seems therefore natural to assume that the same factorization holds for the partition function with $G = SU(N)$. Explicitly, given a 't Hooft flux $\widehat{v} = v \oplus \lambda B$, $\lambda = 0, 1, \ldots N-1$, on $\widehat{X}$, we assume the factorization [29]

$$Z_{\widehat{X},\widehat{v}}(\tau) = Z_{X,v}(\tau)\, \frac{\chi_\lambda(\tau)}{\eta(\tau)}, \tag{5.12}$$

where $\chi_\lambda(\tau)$ is the appropriate level 1 character of $SU(N)$ – see Appendix A.3 for details. This assumption fixes almost completely the form of the partition functions. Some loose ends can be tied up by demanding modular invariance of the resulting expression.

5.2 The formula for the partition function

Given the assumptions above, and taking into account the structure of the partition function on $K3$, we are in a position to write down the formula for Kähler four-folds X with $h^{(2,0)} \neq 0$. We will first assume that the canonical divisor K is connected and with genus $g - 1 = 2\chi + 3\sigma$. The formula is then

$$Z_v = \left(\sum_{\lambda=0}^{N-1} \left(\frac{\chi_\lambda}{\eta} \right)^{1-g} \delta_{v,\lambda[K]_N} \right) \left(\frac{1}{N^2} G(N\tau) \right)^{\nu/2} + N^{1-b_1} \sum_{m=0}^{N-1} \left(\sum_{\lambda=0}^{N-1} \left(\frac{\chi_{m,\lambda}}{\eta} \right)^{1-g} e^{\frac{2i\pi}{N} \lambda v \cdot [K]_N} \right) \cdot e^{i\pi \frac{N-1}{N} m v^2} \left(\frac{1}{N^2} G\left(\frac{\tau + m}{N} \right) \right)^{\nu/2}, \tag{5.13}$$

where $\nu = \frac{\chi+\sigma}{4}$, $G(\tau) = \eta(\tau)^{-24}$ (with η the Dedekind function) and $[K]_N$ is the reduction modulo N of the Poincaré dual of K. In (5.13) χ_λ are the $SU(N)$ characters at level 1 (see Appendix A.3) and $\chi_{m,\lambda}$ are certain linear combinations thereof

$$\chi_{m,\lambda}(\tau) = \frac{1}{N} \sum_{\lambda'=0}^{N-1} e^{-\frac{2i\pi}{N} \lambda\lambda'} e^{i\pi \frac{N-1}{N} m (\lambda')^2} \chi_{\lambda'}(\tau), \quad 0 \leq m, \ \lambda \leq N - 1. \tag{5.14}$$

The structure of the corrections near the canonical divisor in (5.13) suggests that the mechanism at work in this case is not chiral symmetry breaking. Indeed, near K there is an N-fold bifurcation of the vacuum, and the functions χ_λ, $\chi_{m,\lambda}$ (with m fixed) are not related by a shift in τ as it would be the case were chiral symmetry breaking responsible for the bifurcation. A plausible explanation for this bifurcation could be found in the spontaneous breaking of the center of the gauge group (which for $G = SU(N)$ is precisely $\mathbb{Z}_N$.) This could come about as follows. Let us focus on the irreducible embedding. For trivial canonical divisor the gauge group is almost but not completely Higgsed in this vacuum. In fact, since the scalar fields transform in the adjoint representation of $SU(N)$, the center $\mathbb{Z}_N \subset SU(N)$ remains unbroken. The $SU(N)$ gauge threory has $\mathbb{Z}_N$ string-like solitons [8] which carry non-trivial $\mathbb{Z}_N$-valued electric and magnetic quantum numbers. If these solitons condense, the center $\mathbb{Z}_N$ is completely broken giving rise to an N-fold degeneracy of the vacuum. Each vacuum is singled out by a different value of the $\mathbb{Z}_N$-valued flux. Now for non-trivial canonical divisor K as above, the irreducible vacuum separates into N vacua with magnetic fluxes $\lambda[K]_N$! One could be tempted to speculate further and identify the surface K (or the C_j below) with the world-sheet of the condensed string soliton.

As in [23] we can generalize the above formula for the case that the canonical divisor consists of n disjoint smooth components C_j, $j = 1, \ldots, n$ of genus g_j on which ω vanishes with multiplicity one. The resulting expression is:

$$Z_v = \left(\sum_{\vec{\varepsilon}} \delta_{v, w_N(\vec{\varepsilon})} \prod_{j=1}^{n} \prod_{\lambda=0}^{N-1} \left(\frac{\chi_\lambda}{\eta} \right)^{(1-g_j)\delta_{\varepsilon_j,\lambda}} \right) \left(\frac{1}{N^2} G(N\tau) \right)^{\nu/2}$$
$$+ N^{1-b_1} \sum_{m=0}^{N-1} \left[\prod_{j=1}^{n} \left(\sum_{\lambda=0}^{N-1} \left(\frac{\chi_{m,\lambda}}{\eta} \right)^{1-g_j} \mathrm{e}^{\frac{2i\pi}{N}\lambda v \cdot [C_j]_N} \right) \right]$$
$$\cdot\, \mathrm{e}^{i\pi \frac{N-1}{N} m v^2} \left(\frac{1}{N^2} G\left(\frac{\tau + m}{N} \right) \right)^{\nu/2}, \qquad (5.15)$$

where $[C_j]_N$ is the reduction modulo N of the Poincaré dual of C_j, and

$$w_N(\vec{\varepsilon}) = \sum_j \varepsilon_j [C_j]_N, \qquad (5.16)$$

where $\varepsilon_j = 0, 1, \ldots N-1$ are chosen independently. Notice that (5.15) reduces to (5.13) when $n = 1$.

The formulas for the partition function do not apply directly to the $N = 2$ case. For $N = 2$ there are some extra relative phases t_i – see equations (5.45) and (5.46) in [23] – which are absent for $N > 2$ and prime. Modulo these extra phases, (5.13) and (5.15) are a direct generalization of Vafa and Witten's results. They reduce on $K3$ to the formula of Minahan, Nemeschansky, Vafa and Warner [20] and generalize their results to non-zero 't Hooft flux.

5.2.1 Blow-ups

Given (5.15), we can see explicitly how the factorization property (5.12) works. Let X be a Kähler four-fold with Euler characteristic $\chi = 2(1-$

$b_1) + b_2$, signature $\sigma = b_2^+ - b_2^-$ and canonical divisor $K = \cup_{j=1}^n C_j$, and let $\widehat{X}$ be its one blow-up at a smooth point. Then $\widehat{b}_1 = b_1$, $\widehat{b}_2 = b_2 + 1$, $\widehat{\chi} = \chi + 1$, $\widehat{\sigma} = \sigma - 1$ and $\widehat{K} = K \cup B$, where B is the exceptional divisor, which satisfies $B \cdot C_j = 0$ and $B^2 = -1 = g_B - 1$. Consider a 't Hooft flux $\widehat{v} = v \oplus \widehat{\lambda} B$ in $\widehat{X}$, where v is a flux in X and $\widehat{\lambda}$ is an integer defined modulo N. Now $\widehat{\nu} = \nu$, $\widehat{v}^2 = v^2 - \widehat{\lambda}^2$, $\widehat{v} \cdot C_j = v \cdot C_j$, $\widehat{v} \cdot B = \widehat{\lambda} B^2 = -\widehat{\lambda}$ and $\widehat{w}_N(\vec{\epsilon}) = \sum_{j=1}^n \varepsilon_j [C_j]_N + \widehat{\varepsilon} B$. Thus, the partition function (5.15) takes the form

$$\widehat{Z}_{\widehat{X},\widehat{v}} = \left(\sum_{\vec{\varepsilon},\widehat{\varepsilon}} \delta_{v,w_N(\vec{\varepsilon})} \delta_{\widehat{\lambda},\widehat{\varepsilon}} \prod_{j=1}^{n} \prod_{\lambda=0}^{N-1} \left(\frac{\chi_\lambda}{\eta} \right)^{(1-g_j)\delta_{\varepsilon_j,\lambda}} \left(\frac{\chi_\lambda}{\eta} \right)^{(1-g_B)\delta_{\widehat{\varepsilon},\lambda}} \right) \cdot \left(\frac{1}{N^2} G(q^N) \right)^{\nu/2}$$
$$+ N^{1-b_1} \sum_{m=0}^{N-1} \left[\prod_{j=1}^{n} \left(\sum_{\lambda=0}^{N-1} \left(\frac{\chi_{m,\lambda}}{\eta} \right)^{1-g_j} e^{\frac{2i\pi}{N} \lambda v \cdot [C_j]_N} \right) \cdot \left(\sum_{\lambda=0}^{N-1} \left(\frac{\chi_{m,\lambda}}{\eta} \right)^{1-g_B} e^{-\frac{2i\pi}{N} \lambda \widehat{\lambda}} \right) \right]$$
$$\cdot e^{i\pi \frac{N-1}{N} m v^2} e^{-i\pi \frac{N-1}{N} m \widehat{\lambda}^2} \left(\frac{1}{N^2} G(\alpha^m q^{1/N}) \right)^{\nu/2}, \quad (5.17)$$

where $q = \exp(2\pi i \tau)$, $\alpha = \exp(2\pi i/N)$, and therefore,

$$\widehat{Z}_{\widehat{X},\widehat{v}} = \left(\frac{\chi_{\widehat{\lambda}}}{\eta} \right) \left(\sum_{\vec{\varepsilon}} \delta_{v,w_N(\vec{\varepsilon})} \prod_{j=1}^{n} \prod_{\lambda=0}^{N-1} \left(\frac{\chi_\lambda}{\eta} \right)^{(1-g_j)\delta_{\varepsilon_j,\lambda}} \right) \left(\frac{1}{N^2} G(q^N) \right)^{\nu/2}$$
$$+ N^{1-b_1} \sum_{m=0}^{N-1} \left(\sum_{\lambda=0}^{N-1} \left(\frac{\chi_{m,\lambda}}{\eta} \right) e^{-\frac{2i\pi}{N} \lambda \widehat{\lambda}} e^{-i\pi \frac{N-1}{N} m \widehat{\lambda}^2} \right) \cdot \left[\prod_{j=1}^{n} \left(\sum_{\lambda=0}^{N-1} \left(\frac{\chi_{m,\lambda}}{\eta} \right)^{1-g_j} e^{\frac{2i\pi}{N} \lambda v \cdot [C_j]_N} \right) \right]$$
$$\cdot e^{i\pi \frac{N-1}{N} m v^2} \left(\frac{1}{N^2} G(\alpha^m q^{1/N}) \right)^{\nu/2}. \quad (5.18)$$

Now, from (5.14) it follows that

$$\sum_{\lambda=0}^{N-1}\left(\frac{\chi_{m,\lambda}}{\eta}\right) e^{-\frac{2i\pi}{N}\lambda\widehat{\lambda}}e^{-i\pi\frac{N-1}{N}m\widehat{\lambda}^2}$$
$$=\frac{1}{N}\sum_{\lambda,\lambda'} e^{-\frac{2i\pi}{N}\lambda(\lambda'+\widehat{\lambda})}e^{i\pi\frac{N-1}{N}m((\lambda')^2-\widehat{\lambda}^2)}\left(\frac{\chi_{\lambda'}}{\eta}\right). \tag{5.19}$$

Summing over λ and using (A.12) we get

$$\frac{1}{N}\sum_{\lambda,\lambda'} e^{-\frac{2i\pi}{N}\lambda(\lambda'+\widehat{\lambda})}e^{i\pi\frac{N-1}{N}m((\lambda')^2-\widehat{\lambda}^2)}\left(\frac{\chi_{\lambda'}}{\eta}\right)$$
$$=\sum_{\lambda'}\delta_{\lambda'+\widehat{\lambda},0}\, e^{i\pi\frac{N-1}{N}m((\lambda')^2-\widehat{\lambda}^2)}\left(\frac{\chi_{\lambda'}}{\eta}\right)$$
$$=\frac{\chi_{-\widehat{\lambda}}}{\eta}=\frac{\chi_{N-\widehat{\lambda}}}{\eta}=\frac{\chi_{\widehat{\lambda}}}{\eta}. \tag{5.20}$$

Hence,

$$\widehat{Z}_{\widehat{X},\widehat{v}}=\left(\frac{\chi_{\widehat{\lambda}}}{\eta}\right)\left(\sum_{\vec{\varepsilon}}\delta_{v,w_N(\vec{\varepsilon})}\prod_{j=1}^{n}\prod_{\lambda=0}^{N-1}\left(\frac{\chi_\lambda}{\eta}\right)^{(1-g_j)\delta_{\varepsilon_j,\lambda}}\right)\left(\frac{1}{N^2}G(q^N)\right)^{\nu/2}$$
$$+N^{1-b_1}\sum_{m=0}^{N-1}\left(\frac{\chi_{\widehat{\lambda}}}{\eta}\right)\left[\prod_{j=1}^{n}\left(\sum_{\lambda=0}^{N-1}\left(\frac{\chi_{m,\lambda}}{\eta}\right)^{1-g_j}e^{\frac{2i\pi}{N}\lambda v\cdot[C_j]_N}\right)\right]$$
$$\cdot\, e^{i\pi\frac{N-1}{N}mv^2}\left(\frac{1}{N^2}G(\alpha^m q^{1/N})\right)^{\nu/2}=\left(\frac{\chi_{\widehat{\lambda}}}{\eta}\right)Z_{X,v}, \tag{5.21}$$

as expected.

5.2.2 Modular transformations

We will now study the modular properties of the partition functions (5.13) and (5.15). With the formulas in the appendix one can check that they have the expected modular behaviour[5]

$$Z_v(\tau+1)=e^{\frac{i\pi}{12}N(2\chi+3\sigma)}e^{-i\pi\frac{N-1}{N}v^2}Z_v(\tau),$$
$$Z_v(-1/\tau)=N^{-b_2/2}\left(\frac{\tau}{i}\right)^{-\chi/2}\sum_u e^{\frac{2i\pi u\cdot v}{N}}Z_u(\tau), \tag{5.22}$$

[5]We assume as in [23] that there is no torsion in $H_2(X,\mathbb{Z})$. Were this not case, Eqs. (5.24) and (5.15) above should be modified along the lines explained in [27].

and also, with $Z_{SU(N)} = N^{b_1-1} Z_0$ and $Z_{SU(N)/\mathbb{Z}_N} = \sum_v Z_v$,

$$\begin{aligned} Z_{SU(N)}(\tau+1) &= \mathrm{e}^{\frac{i\pi}{12}N(2\chi+3\sigma)} Z_{SU(N)}(\tau), \\ Z_{SU(N)/\mathbb{Z}_N}(\tau+N) &= \mathrm{e}^{\frac{i\pi}{12}N^2(2\chi+3\sigma)} Z_{SU(N)/\mathbb{Z}_N}(\tau), \end{aligned} \tag{5.23}$$

and

$$Z_{SU(N)}(-1/\tau) = N^{-\chi/2} \left(\frac{\tau}{i}\right)^{-\chi/2} Z_{SU(N)/\mathbb{Z}_N}(\tau), \tag{5.24}$$

which is the Montonen-Olive relation. Notice that since N is odd, the $SU(N)$ (or $SU(N)/\mathbb{Z}_N$) partition function is modular (up to a phase) for $\Gamma_0(N)$, on any four-manifold. On the other hand, for *even* N one would expect on general grounds [23] modularity for $\Gamma_0(2N)$, or at most $\Gamma_0(N)$ on spin manifolds.

5.2.3 The partition function on T^4

We will finish by considering the twisted theory on T^4, where an unexpected result emerges. As $K3$, T^4 is a compact hyper-Kähler manifold (hence with trivial canonical divisor). It has $b_1 = 4$, $b_2 = 6$ and $\chi = 0 = \sigma$. On T^4 the partition function (5.13) reduces to its bare bones

$$Z_v = \delta_{v,0} + \frac{1}{N^3} \sum_{m=0}^{N-1} \mathrm{e}^{i\pi \frac{N-1}{N} m v^2}, \tag{5.25}$$

and does not depend on τ! This should be compared with the formulas in [6]. The Z_v are self-dual in the following sense

$$Z_v = \frac{1}{N^3} \sum_u \mathrm{e}^{\frac{2i\pi u\cdot v}{N}} Z_u. \tag{5.26}$$

Notice that since T^4 is a spin manifold, $v^2 \in 2\mathbb{Z}$, and therefore the sum over m in (5.25) vanishes unless $v^2 = 0$ (modulo N), so Z_v reduces to the rather simple form

$$Z_{T^4,v} = \delta_{v,0} + \frac{1}{N^2} \delta_{v^2,0},$$

which gives the partition function for the physical $\mathcal{N} = 4 SU(N)$ theory in the sector of 't Hooft flux v and in the limit $\bar{\tau} \to \infty$ [23].

6 Conclusions

In this paper we have obtained the partition function of the Vafa-Witten theory for gauge group $SU(N)$ (with prime N) on Kähler four-manifolds with $b_2^+ > 1$. The resulting formulas (5.13) and (5.15) turn out to transform as expected under the modular group, and they can be seen as predictions for the Euler numbers of instanton moduli spaces on those four-manifolds.

It could be interesting to investigate whether (5.13) and (5.15) can be rewritten in terms of the Seiberg-Witten invariants. We believe that this is not the case for the following reason. Let us suppose that it is actually possible to do so. Then one would expect, by analogy with the result for $SU(2)$ [4], that the Donaldson-Witten partition function for $SU(N)$ [19] should be recovered from the Vafa-Witten $SU(N)$ partition function in the decoupling limit $q \to 0$, $m \to \infty$ with $m^4 q$ fixed. In particular, one would expect that the structure of the corrections involving the canonical divisor should be preserved in this limit. Now in the DW partition function in [19], these corrections are written in terms of the Seiberg-Witten classes x [26]. For $G = SU(N)$ these basic classes appear in the generic form $\sum_{x_1,\dots,x_{N-1}} n_{x_1} \cdots n_{x_{N-1}}$ (n_{x_l} are the Seiberg-Witten invariants [26]). Therefore, for $G = SU(N)$ there are $N-1$ independent basic classes contributing to the above sum. On a Kähler manifold with canonical divisor $K = C_1 \cup C_2 \cup \cdots \cup C_n$, with the C_j disjoint and with multiplicity one, each of these basic classes can be written as

$$x_l = \sum_{\rho_l^j} \rho_l^j C_j,$$

with each $\rho_l^j = \pm 1$ [26], and the sum over the basic classes can be traded for a sum over the ρ_l^j. This is analogous to the sum over the ε_j in (5.15), and both sums should contain the same number of terms were it possible to rewrite (5.15) in terms of the basic classes. However, while in the sum over the ρ_l^j there are $2^{n(N-1)}$ terms, the sum over the ε_j contains N^n terms. Notice that these two numbers do coincide when $N = 2$, as it should be, but for $N \neq 2$ this is no longer the case.

It would certainly be mostly interesting to extend these results to all N (not necessarily prime), and to investigate what the large N limit of (5.13) and (5.15) correspond to on the gravity side in the light of the AdS/CFT correspondence [17]. Although there are already some indications of how this correspondence should work [7], [9], a clear understanding is still lacking. We expect to address some of these issues in the near future.

Acknowledgments

We would like to thank M. Mariño, A.V. Ramallo and K. Yoshioka for helpful discussions. C.L. would like to thank the organizers of the "Lisbon School on Superstrings," where some of these results were announced. This work was supported in part by DGICYT under grant PB96-0960, and by the EU Commission under TMR grant FMAX-CT96-0012.

A Appendix

Here we collect some useful formulas which should help the reader follow the computations in the paper.

A.1 Modular forms

The function G is defined as

$$G(\tau) = \frac{1}{\eta(\tau)^{24}}, \tag{A.1}$$

and is a modular form of weight -12

$$G(\tau) \stackrel{\tau\to\tau+1}{\longrightarrow} G(\tau), \qquad G(\tau) \stackrel{\tau\to-1/\tau}{\longrightarrow} \tau^{-12} G(\tau), \tag{A.2}$$

From (A.2) we can determine the modular behaviour of the different modular forms in the $K3$ partition function

$$G(N\tau) \stackrel{\tau\to-1/\tau}{\longrightarrow} \tau^{-12} N^{12} G(\tau/N),$$

$$G\left(\frac{\tau+m}{N}\right) \stackrel{\tau\to-1/\tau}{\longrightarrow} \tau^{-12} G\left(\frac{\tau+h}{N}\right), \tag{A.3}$$

where $1 \le h \le N-1$, $mh = -1 \bmod N$ and N prime.

For arbitrary N one has to consider the modular forms $G\left(\frac{a\tau+b}{d}\right)$, where $ad = N$ and $b < d$ [20]. These functions transform as follows

$$G\left(\frac{a\tau+b}{d}\right) \overset{\tau\to-1/\tau}{\longrightarrow} \tau^{-12}\left(\frac{a}{p}\right)^{12} G\left(\frac{p\tau+ab'}{a\tilde{d}}\right), \tag{A.4}$$

where $p = \gcd\ (b,d)$, $\tilde{d} = d/p$, $\tilde{b} = b/p$, $b'\tilde{b} = -1 \bmod \tilde{d}$. If $b = 0$, then $p = d$ and $b' = 0 = \tilde{b}$. Notice that for prime N (A.4) reduces to (A.3).

A.2 Flux sums

The basic sums we have to consider are of the form

$$I(m,N) = \sum_{\lambda=0}^{N-1} \mathrm{e}^{\frac{i\pi m}{N}\lambda(N-\lambda)} = \sum_{\lambda=0}^{N-1} \mathrm{e}^{i\pi\frac{N-1}{N}m\lambda^2}, \tag{A.5}$$

for $1 \le m \le N-1$, and discrete Fourier transformations thereof

$$\sum_{\lambda=0}^{N-1} \mathrm{e}^{\pm\frac{2i\pi}{N}\lambda\lambda'} \mathrm{e}^{i\pi\frac{N-1}{N}m\lambda^2}, \tag{A.6}$$

from which the sums over fluxes can be easily computed. The basic sum (A.5) is related to a standard Gauss sum $G(m,N) = \sum_{r \bmod N} \mathrm{e}^{2i\pi m r^2/N}$ [11]. In fact, $I(m,N) = I(m+N,N)$ and, since N is odd, it suffices to consider the case where m is even. But in this case

$$I(2a,N) = \sum_{\lambda=0}^{N-1} \mathrm{e}^{i\pi\frac{N-1}{N}2a\lambda^2} = \sum_{\lambda} \mathrm{e}^{-2i\pi a\lambda^2/N} = \overline{G(a,N)}. \tag{A.7}$$

Now, when $a = 1$,

$$G(1,N) = \frac{\sqrt{N}}{2}(1+i)\left(1+\mathrm{e}^{-\frac{i\pi N}{2}}\right), \tag{A.8}$$

([11, p. 165]). Moreover, for $a > 1$ and N an *odd prime*,

$$G(a,N) = \left(\frac{a}{N}\right) G(1,N), \tag{A.9}$$

where $\left(\frac{a}{N}\right)$ is the Legendre symbol [11], which is +1 if a is a perfect square (mod N) and -1 otherwise. Hence, taking (A.7)-(A.9) into account we have the result

$$\sum_{\lambda=0}^{N-1} \mathrm{e}^{i\pi\frac{N-1}{N}m\lambda^2} = \epsilon(m)\sqrt{N}\mathrm{e}^{-\frac{i\pi}{8}(N-1)^2}, \tag{A.10}$$

where

$$\epsilon(m) = \begin{cases} \left(\frac{m/2}{N}\right), & m \text{ even}, \\ \\ \left(\frac{(m+N)/2}{N}\right), & m \text{ odd}, \end{cases} \tag{A.11}$$

If $kh = -1 \bmod N$, $\epsilon(k) = \epsilon(h)$ for $N = 5 \bmod 4$, and $\epsilon(k) = -\epsilon(h)$ for $N = 3 \bmod 4$. This property is essential in proving the second relation in (5.22).

We also have the identity

$$\sum_{\lambda=0}^{N-1} \mathrm{e}^{\pm\frac{2i\pi}{N}\lambda\lambda'} = N\delta_{\lambda',0}, \tag{A.12}$$

and the fundamental result

$$\sum_{\lambda=0}^{N-1} \mathrm{e}^{\pm\frac{2i\pi}{N}\lambda\lambda'}\mathrm{e}^{i\pi\frac{N-1}{N}m\lambda^2} = \epsilon(m)\sqrt{N}\mathrm{e}^{-\frac{i\pi}{8}(N-1)^2}\mathrm{e}^{i\pi\frac{N-1}{N}h(\lambda')^2}, \tag{A.13}$$

with $mh = -1 \bmod N$ and N an *odd prime*.

Now, given (A.10), the basic sum over fluxes $\sum_v \mathrm{e}^{i\pi\frac{N-1}{N}mv^2}$ can be computed in terms of (A.5) as follows – see [23], eq. (3.21)-(3.22):

$$\sum_{v\in H^2(X,\mathbb{Z}_N)} \mathrm{e}^{i\pi\frac{N-1}{N}mv^2} = I(m,N)^{b_2^+}\,\overline{I(m,N)}^{\,b_2^-}, \tag{A.14}$$

so one has (for *prime* N)

$$\sum_{v\in H^2(X,\mathbb{Z}_N)} \mathrm{e}^{i\pi\frac{N-1}{N}mv^2} = \big(\epsilon(m)\big)^{b_2}N^{b_2/2}\mathrm{e}^{-\frac{i\pi}{8}(N-1)^2\sigma}, \tag{A.15}$$

and also, from (A.12) and (A.13)

$$\sum_{v\in H^2(X,\mathbb{Z}_N)} \mathrm{e}^{\frac{2i\pi}{N}u\cdot v} = N^{b_2}\delta_{u,0}, \tag{A.16}$$

$$\sum_{v\in H^2(X,\mathbb{Z}_N)} \mathrm{e}^{\frac{2i\pi}{N}u\cdot v}\mathrm{e}^{i\pi\frac{N-1}{N}mv^2} = \big(\epsilon(m)\big)^{b_2}N^{b_2/2}\mathrm{e}^{-\frac{i\pi}{8}(N-1)^2\sigma}\mathrm{e}^{i\pi\frac{N-1}{N}hu^2}, \tag{A.17}$$

with $mh=-1 \bmod N$ as above.

A.3 $SU(N)$ characters

We have seen above that the corrections to the $SU(N)$ partition function near the canonical divisor of the four-manifold X are given in terms of the level one characters χ_λ of the $SU(N)$ WZW model. These are defined as [10]

$$\chi_\lambda(\tau) = \frac{1}{\eta(\tau)^{N-1}}\sum_{\vec{w}\in[\lambda]} \mathrm{e}^{i\pi\tau\vec{w}^2}, \qquad \lambda\in\mathbb{Z} \bmod N, \tag{A.18}$$

where $[\lambda]$ is the λ-th conjugacy class of $SU(N)$, and the identification $\chi_\lambda(\tau) = \chi_{\lambda+N}(\tau)$ is understood. Also, from the symmetry properties of the inverse Cartan matrix (A.19) it follows that $\chi_\lambda = \chi_{N-\lambda}$. $\lambda=0 \bmod N$ corresponds to $\vec{w}$ in the root lattice, while for $1\leq\lambda\leq N-1$, $[\lambda]=\{\vec{w}\in\Lambda_{\text{weight}} : \vec{w}=\vec{\alpha}^\lambda+\sum_{n^{\lambda'}\in\mathbb{Z}} n^{\lambda'}\vec{\alpha}_{\lambda'}\}$. $\vec{\alpha}_\lambda$ are the simple roots and $\vec{\alpha}^\lambda$ the fundamental weights of $SU(N)$, normalized in such a way that the inverse Cartan matrix $A^{\lambda\lambda'}$ has the standard form

$$A^{\lambda\lambda'} = \vec{\alpha}^\lambda\cdot\vec{\alpha}^{\lambda'} = \mathrm{Inf}\,\{\lambda,\lambda'\} - \frac{\lambda\lambda'}{N}, \qquad 1\leq\lambda,\lambda'\leq N-1. \tag{A.19}$$

The characters (A.18) have the following properties under the modular group [10]

$$\begin{aligned}\chi_\lambda(\tau+1) &= \mathrm{e}^{-\frac{i\pi}{12}(N-1)}\mathrm{e}^{i\pi\frac{N-1}{N}\lambda^2}\chi_\lambda(\tau),\\ \chi_\lambda(-1/\tau) &= \frac{1}{\sqrt{N}}\sum_{\lambda'=0}^{N-1}\mathrm{e}^{-\frac{2i\pi}{N}\lambda\lambda'}\chi_{\lambda'}(\tau).\end{aligned} \tag{A.20}$$

From the characters χ_λ we introduce the linear combinations ($N > 2$ and prime)

$$\chi_{m,\lambda}(\tau) = \frac{1}{N}\sum_{\lambda'=0}^{N-1} \mathrm{e}^{-\frac{2i\pi}{N}\lambda\lambda'}\mathrm{e}^{i\pi\frac{N-1}{N}m(\lambda')^2}\chi_{\lambda'}(\tau), \quad 0 \le m,\lambda \le N-1, \tag{A.21}$$

which have the ciclicity property $\chi_{m+N,\lambda} = \chi_{m,\lambda} = \chi_{m,\lambda+N}$ since N is odd. Under the modular group one has

$$\begin{aligned} \chi_{m,\lambda}(\tau+1) &= \mathrm{e}^{-\frac{i\pi}{12}(N-1)}\chi_{m+1,\lambda}(\tau), \\ \chi_{0,\lambda}(-1/\tau) &= \frac{1}{\sqrt{N}}\chi_\lambda(\tau), \\ \chi_{m,\lambda}(-1/\tau) &= \epsilon(m)\,\mathrm{e}^{-\frac{i\pi}{8}(N-1)^2}\mathrm{e}^{i\pi\frac{N-1}{N}h\lambda^2}\chi_{m,h\lambda}(\tau), \quad m > 0, \end{aligned} \tag{A.22}$$

with $mh = -1 \bmod N$.

References

[1] T.M. Apostol, *Modular functions and Dirichlet series in number theory*, Springer-Verlag, New York, 1976.

[2] M. Blau and G. Thompson, *Aspects of $N_T \geq 2$ topological gauge theories and D-branes*, Nucl. Phys., **B492** (1997), 545; hep-th/9612143.

[3] R. Dijkgraaf, *The mathematics of fivebranes*, hep-th/9810157.

[4] R. Dijkgraaf, J.-S. Park, and B.J. Schroers, *$\mathcal{N} = 4$ supersymmetric Yang-Mills theory on a Kähler surface*, hep-th/9801066.

[5] R. Donagi and E. Witten, *Supersymmetric Yang-Mills theory and integrable systems*, Nucl. Phys., **B460** (1996), 299; hep-th/9510101.

[6] L. Girardello, A. Giveon, M. Porrati, and A. Zaffaroni, *S-duality in $\mathcal{N}$=4 Yang-Mills theories with general gauge groups*, Nucl.Phys., **B448** (1995), 127; hep-th/9502057.

[7] R. Gopakumar and C. Vafa, *Topological gravity as large N topological gauge theory*, Adv. Theor. Math. Phys., **2** (1998), 413, hep-th/9802016; *On the gauge theory/geometry correspondence*, hep-th/9811131.

[8] G.'t Hooft, *On the phase transition towards permanent quark confinement*, Nucl. Phys., **B138** (1978), 1; *A property of electric and magnetic flux in non-Abelian gauge theories*, Nucl. Phys., **B153** (1979), 141.

[9] C.M. Hull, *Timelike T-duality, de Sitter space, large N gauge theories and topological field theory*, J. High Energy Phys., **9807** (1998), 021; hep-th/9806146.

[10] C. Itzykson, *Level one Kac-Moody characters and modular invariants*, Nucl. Phys. B (Proc. Suppl.), **5** (1988) 150.

[11] Hua Loo Keng, *Introduction to number theory*, Springer-Verlag, Berlin, 1982.

[12] J.M.F. Labastida and Carlos Lozano, *Duality in twisted $\mathcal{N} = 4$ supersymmetric gauge theories in four dimensions*, Nucl. Phys., **B537** (1999), 203; hep-th/9806032.

[13] J.M.F. Labastida and Carlos Lozano, *Mathai-Quillen formulation of twisted $\mathcal{N} = 4$ supersymmetric gauge theories in four dimensions*, Nucl. Phys., **B502** (1997), 741; hep-th/9702106.

[14] J.M.F. Labastida and Carlos Lozano, *Duality in the context of topological quantum field theory*, hep-th/9901161.

[15] J.M.F. Labastida and Carlos Lozano, *Mass perturbations in twisted $\mathcal{N} = 4$ supersymmetric gauge theories*, Nucl. Phys., **B518** (1998), 37; hep-th/9711132.

[16] J.M.F. Labastida and M. Mariño, *Polynomial invariants for $SU(2)$ monopoles*, Nucl. Phys., **B456** (1995), 633; hep-th/9507140.

[17] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys., **2** (1998), 231, hep-th/9711200;
S.S. Gubser, I. Klebanov, and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys. Lett., **B428** (1998),

105, hep-th/9802109;
E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys., **2** (1998), 253; hep-th/9802150.

[18] N. Marcus, *The other topological twisting of $\mathcal{N} = 4$ Yang-Mills*, Nucl. Phys., **B452** (1995), 331; hep-th/9506002.

[19] M. Mariño and G. Moore, *The Donaldson-Witten function for gauge groups of rank larger than one*, Commun. Math. Phys., **199** (1998), 25; hep-th/9802185.

[20] J.A. Minahan, D. Nemeschansky, C. Vafa, and N.P. Warner, *E-strings and $\mathcal{N} = 4$ topological Yang-Mills theories*, Nucl. Phys., **B527** (1998), 581; hep-th/9802168.

[21] C. Montonen and D. Olive, *Magnetic monopoles as gauge particles?* Phys. Lett., **B72** (1977), 117.

[22] G. Moore and E. Witten, *Integration over the u-plane in Donaldson theory*, Adv. Theor. Math. Phys., **1** (1998), 298; hep-th/9709193.

[23] C. Vafa and E. Witten, *A strong coupling test of S-duality*, hep-th/9408074; Nucl. Phys., **B431** (1994), 3.

[24] E. Witten, *Supersymmetric Yang-Mills theory on a four-manifold*, J. Math. Phys., **35** (1994), 5101; hep-th/9403193.

[25] E. Witten, *Topological quantum field theory*, Commun. Math. Phys., **117** (1988), 353.

[26] E. Witten, *Monopoles and four-manifolds*, hep-th/9411102; Math. Res. Letters, **1** (1994), 769.

[27] E. Witten, *AdS/CFT correspondence and topological field theory*, J. High Energy Phys., **9812** (1998) 012; hep-th/9812012.

[28] J.P. Yamron, *Topological actions in twisted supersymmetric theories*, Phys. Lett., **B213** (1988), 325.

[29] K. Yoshioka, *Betti numbers of moduli of stable sheaves on some surfaces*, Nucl. Phys. B (Proc. Suppl.), **46** (1996), 263; *Chamber structure of polarizations and the moduli of stable sheaves on a ruled surface*, Int. J. Math., **7** (1996), 411.