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Constructing D-Branes from K-Theory

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Abstract

A detailed review of recent developments in the topological classification of D-branes in superstring theory is presented. Beginning with a thorough, self-contained introduction to the techniques and applications of topological K-theory, the relationships between the classic constructions of K-theory and

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the recent realizations of D-branes as tachyonic solitons, coming from bound states of higher dimensional systems of unstable branes, are described. It is shown how the K-theory formalism naturally reproduces the known spectra of BPS and non-BPS D-branes, and how it can be systematically used to predict the existence of new states. The emphasis is placed on the new interpretations of D-branes as conventional topological solitons in other brane worldvolumes, how the mathematical formalism can be used to deduce the gauge field content on both supersymmetric and non-BPS branes, and also how K-theory predicts new relationships between the various superstring theories and their D-brane spectra. The implementations of duality symmetries as natural isomorphisms of K-groups are discussed. The relationship with the standard cohomological classification is presented and used to derive an explicit formula for D-brane charges. Some string theoretical constructions of the K-theory predictions are also briefly described.

1 Introduction and Overview

The second superstring revolution (see [1] for reviews) came with the realization that all five consistent superstring theories in ten dimensions (Type I, Type IIA/B, Heterotic $SO(32)/E_8 \times E_8$) along with 11-dimensional supergravity are merely different perturbation expansions of a single 11-dimensional quantum theory called M-Theory [2]. The evidence for this is provided by the various non-perturbative duality relations that connect the different corners of the moduli space of M-Theory corresponding to the various string theories. The classic examples are the self-duality of the Type IIB superstring [3] and the duality between the Type I and $SO(32)$ heterotic strings [2, 4].

A new impetus into the duality conjectures came with the realization that certain nonperturbative degrees of freedom, known as Dirichlet p-branes (or Dp-branes for short), are charged with respect to the $p+1$-form gauge potentials of the closed string Ramond-Ramond (RR) sector of Type II superstring theory [5]. Dp-branes are supersymmetric extended objects which form $p+1$-dimensional hypersurfaces in spacetime on which the endpoints of open strings can attach (with Dirichlet boundary conditions). They can be thought of as topological defects in spacetime which give explicit realizations of string solitons [6]. The crucial observation [5] was that D-branes have precisely the correct properties to fill out duality multiplets whose other elements are fundamental string states and ordinary field theoretic solitons. D-branes have thereby provided a more complete and detailed dynamical picture of string duality. They have also provided surprising new insights into the quantum mechanics of black holes and into the nature of spacetime at very short distance scales.

The important property of D-branes is that they are examples of BPS states, which may be characterized by the property that their mass is completely determined by their charge with respect to some gauge field. They form ultra-short multiplets of the supersymmetry algebra of the string theory, and are thereby stable and protected from quantum radiative corrections. Their properties can therefore be analysed perturbatively at weak coupling in a given theory and then extrapolated to strong coupling where they can be reinterpreted as non-perturbative configurations of the dual theory. For some time it was thought that this supersymmetry property, which protects the D-brane configurations via non-renormalization theorems, was crucial to ensure their stability and provide the appropriate non-perturbative tests of the duality conjectures.

However, this picture of D-branes has drastically changed in the last year and a half. It may be observed [7] that the spectrum of a superstring theory can contain states which do not have the BPS property, but which are nevertheless stable because they are the lightest states of the theory which carry a given set of conserved quantum numbers which prevent them from decaying. Such stable non-BPS states can be studied using standard string perturbation theory and their properties determined at weak coupling. It has been realized recently [8]–[12] that when these states are extrapolated to strong coupling, the resulting non-perturbative configuration behaves in all respects like an ordinary D-brane (see [13] for recent reviews). This provides a highly non-trivial check of the non-perturbative duality conjectures beyond the level of BPS configurations. For instance, this idea can be applied to heterotic-Type I duality at a non-BPS level [9, 10]. The $SO(32)$ heterotic string contains states which are not supersymmetric, but are stable because they are the lightest states that carry the quantum numbers of the spinor representation of the $SO(32)$ gauge group. It turns out that the corresponding non-perturbative stable configuration which is a spinor of $SO(32)$ is the object that comes from the bound state of a Type I D-string and anti-D-string (wrapped on a circle and with a $\mathbb{Z}_2$-valued Wilson line in the worldvolume). The D-string pair becomes tightly bound, forming a solitonic kink which behaves exactly as a D-particle but which carries a non-additive charge taking values in $\mathbb{Z}_2$ that prevents one from building stacks of non-BPS D-branes.

Generally, this new perspective for understanding D-branes and their conserved charges treats the branes as topological defects in the worldvolumes of higher dimensional unstable systems of branes (such as brane-antibrane pairs). Such systems are unstable because their spectrum contains a tachyonic state that is not removed by the usual GSO projection. However, it is unclear whether these modes are incurable instabilities in the system or if they play a more subtle role in the dynamics. A better understanding of the string theory tachyon has been recently achieved [7]–[9], [14]–[16], with the new belief that the tachyonic mode of an open string stretching between a D-brane and an anti-D-brane (or connecting an unstable brane to itself) is a Higgs-type excitation which develops a stable vacuum expectation value, and the unstable state decays into a stable state. Configurations of unstable D-branes can sometimes carry lower dimensional D-brane charges, so that when the tachyon field rolls down to the minimum of its potential and the state decays, it leaves behind a state which differs from the vacuum configuration by a lower-dimensional D-brane charge. The resulting stable state thereby contains topological defects that correspond to stable D-branes.

In addition to producing new D-brane configurations, the bound state construction of branes through the process of tachyon condensation can be achieved for the known spectrum of supersymmetric branes. This leads to various new connections between different types of D-branes which are known as "descent relations" [11], [17]–[19]. These relations form a remarkable web of mappings between BPS and non-BPS branes that provides various different ways of thinking about the origins of D-branes, and they could lead to a better understanding of the dynamics of different D-branes and their roles in string theory and in M-Theory. The situation in the case of Type II superstring theory is depicted in Fig. 1 [11, 13]. If we consider, say, a Dp-brane anti-Dp-

brane (or D$\overline{p}$-brane for short) bound state pair of Type IIB string theory (p odd), then its open string spectrum contains a tachyonic excitation whose ground state corresponds to the supersymmetric vacuum configuration. However, one can consider instead a tachyonic kink solution on the brane-antibrane pair which describes a non-BPS D$(p-1)$-brane of the IIB theory. This system also contains a tachyonic excitation in its worldvolume field theory, so that one can consider a tachyonic kink solution on the D$(p-1)$-brane which results in a BPS D$(p-2)$-brane of IIB.

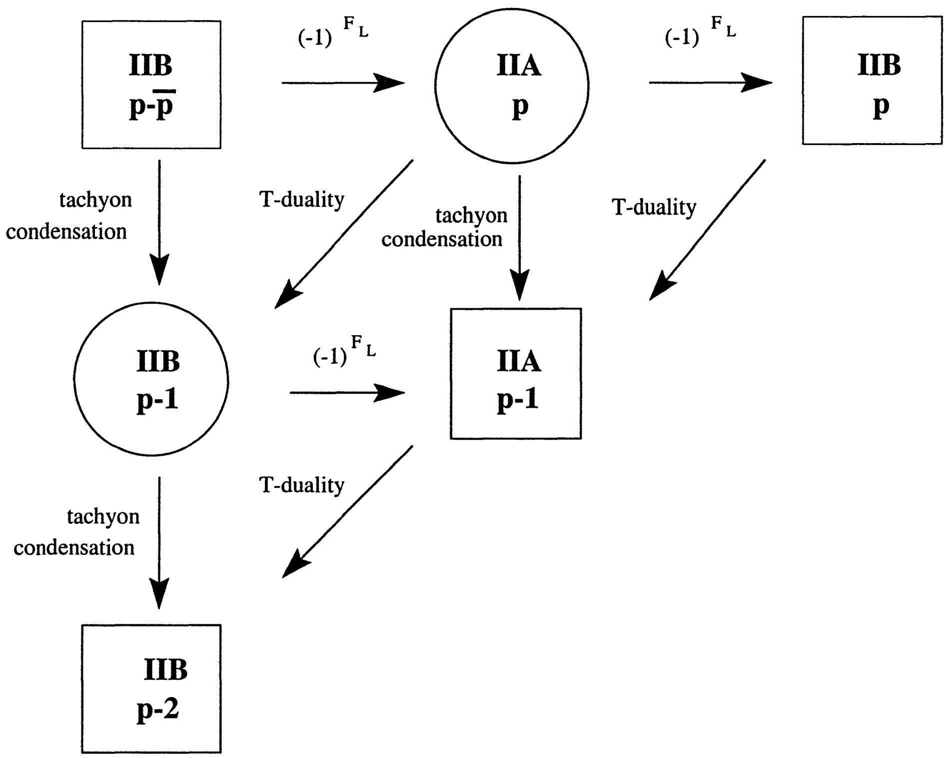

Figure 1: The relationships between different D-branes in Type II superstring theory. The squares represent stable supersymmetric BPS branes or a combination of such a brane with its antibrane, while the circles depict unstable non-BPS configurations. The horizontal arrows represent the result of quotienting the theory by the operator $(-1)^{F_L}$, the vertical arrows the effect of constructing a tachyonic kink solution in the brane worldvolume field theory, and the diagonal arrows the usual T-duality transformations.

Another set of relations comes from modding out the p-$\overline{p}$ brane pair by the operator $(-1)^{F_{\rm L}}$ which acts as -1 on all the Ramond sector states in the left-moving part of the fundamental string worldsheet, and leaves all other sectors unchanged. In particular, it exchanges a D-brane with its antibrane, so that a brane-antibrane pair is invariant under $(-1)^{F_{\rm L}}$ and it makes sense to take the quotient of this configuration. A careful study of the open string spectrum reveals that the result is a non-supersymmetric Dp-brane of IIA, and that a further quotient by $(-1)^{F_{\rm L}}$ yields a supersymmetric p-brane of IIB [11, 20]. When combined with the usual T-duality transformations between the Type IIB and IIA theories [21], we find that any p-brane configuration in Type II superstring theory may be obtained from any higher dimensional brane configuration. In particular, all branes of the Type II theories descend from a bound state of D9-$\overline{\rm D9}$ pairs. Thus all possible stable D-branes appear as topological defects in the worldvolume tachyonic Higgs field on the spacetime filling D9-branes, so that the spacetime filling brane system provides a universal medium in which all stable D-brane charges are carried by conventional topological solitons.

The standard coupling in Type II superstring theory of a BPS Dp-brane to a closed string $p+1$-form RR potential $C^{(p+1)}$ is described by the action [5]

$$S_{(p)} = \mu_{(p)} \int\limits_{\mathcal{M}_p} C^{(p+1)} \tag{1.1}$$

where $\mu_{(p)}$ is the $p+1$-form charge of the p-brane. In addition, the topological charge on the worldvolume manifold $\mathcal{M}_p$ of a Dp-brane couples to the spacetime RR fields through generalized Wess-Zumino type actions [22, 23] (here we work in string units with $2\pi\alpha' = 1$ and suppress the dependence on the Neveu-Schwarz two-form field B as well as on correction terms due to non-vanishing manifold curvature):

$$S^{(p)}_{\rm WZ} = \mu_{(p)} \int\limits_{\mathcal{M}_p} {\rm tr}\left({\rm e}^F\right) \wedge \sum_{p'} C^{(p'+1)} \tag{1.2}$$

where F is the field strength of some gauge field which lives on $\mathcal{M}_p$. The nature of the gauge fields depends on the configurations of D-branes. When N branes are brought infinitesimally close to one another, their generic $U(1)^N$ gauge symmetry is enhanced to $G = U(N)$ [24]. This introduces the possibility of embedding supersymmetric gauge theories of various dimensions into string theory (see [25] for a review). The coupling (1.2) also allows an alternative interpretation of the topological charge as the RR charge due to the presence of lower dimensional branes in the worldvolume of higher dimensional branes [23, 26]. This enables the topological classification of RR charge in terms of worldvolume defects [27] in much the same spirit as that described above.

In fact, the new understanding of the tachyon in an unstable brane configuration as a Higgs type excitation in the spectrum of open string states leads to a topological classification of the resulting brane charges when D-branes are viewed as the tachyonic solitons. Generally, the topological charges of these objects are determined by the homotopy groups of a homogeneous space G/H, where G is a compact Lie group and H is a closed subgroup of G. The fibration

$$H \overset{i}{\hookrightarrow} G \overset{\pi}{\longrightarrow} G/H \tag{1.3}$$

with i the inclusion and π the canonical projection, induces a long exact sequence of homotopy groups,

$$\begin{aligned} \ldots \longrightarrow \pi_{n-1}(H) \overset{i^*}{\longrightarrow} \pi_{n-1}(G) \overset{\pi^*}{\longrightarrow} & \\ & \pi_{n-1}(G/H) \overset{\partial^*}{\longrightarrow} \pi_{n-2}(H) \longrightarrow \ldots. \end{aligned} \tag{1.4}$$

In the present case, G is the worldvolume gauge group of a given configuration of branes and the tachyon scalar field T is a Higgs field for the breaking of the gauge symmetry down to the subgroup H. The tachyonic soliton must be accompanied by a worldvolume gauge field A of corresponding topological charge in the unbroken subgroup of the gauge group, in order that the energy per unit worldvolume of the induced lower dimensional brane be finite. It can be argued [7, 9] that the brane worldvolume field theory admits finite energy, static soliton solutions which have asymptotic pure gauge configurations at infinity,

$$T \simeq T_v\,U, \qquad A \simeq i\,U^{-1}\,dU \tag{1.5}$$

where T_v is a constant, and U is a G/H valued function corresponding to the identity map (of a given winding number) from the asymptotic boundary of the worldvolume soliton to the group manifold of the space G/H of vacua. This leads to topologically distinct sectors in the space of all field configurations, and the charges which distinguish these sectors take values in the appropriate homotopy group of the vacuum manifold. Precisely, if the induced brane configuration has codimension n

in the higher dimensional worldvolume, then the corresponding soliton carries topological charge taking values in $\pi_{n-1}(G/H)$. This homotopy group may be computed using the exact sequence (1.4) [27] (for instance, if the induced boundary homomorphism ∂^* is a trivial mapping, so that $\ker\partial^* = \pi_{n-1}(G/H)$, then $\pi_{n-1}(G/H) = \pi_{n-1}(G)/\pi_{n-1}(H)$).

The coupling (1.1) would seem to imply that, since the massless RR fields $C^{(p+1)}$ are differential forms, the RR charges of D-branes are determined by cohomology classes, i.e., by integrating the $C^{(p+1)}$ over suitable cycles of the spacetime manifold X. However, the new interpretation of D-brane charge as a topological charge actually suggests a different characterization (at least when all spacetime dimensions are much larger than the string scale so that no new stringy phenomena occur). Let us consider Type IIB superstring theory, and go back to the realization of RR charge in terms of a configuration of N 9-branes and M $\overline{9}$-branes. Type II theories have no gauge group, so in order to cancel the tadpole anomaly there must be the same number of 9-branes and $\overline{9}$-branes, $N = M$. The 9-branes and $\overline{9}$-branes fill out the spacetime manifold X. The system of N 9-branes carries a $U(N)$ gauge bundle E and the system of N $\overline{9}$-branes carries a $U(N)$ gauge bundle F. The system of $9-\overline{9}$-branes can therefore be labelled by a pair of $U(N)$ vector bundles (E, F) which characterize the gauge field configurations corresponding to the soliton.

We would now like to determine when two such pairs of bundles (E, F) and (E', F') should be considered equivalent. They should be regarded the same if they determine the same topological class of the soliton. It turns out that the basic equivalence relation is brane-antibrane creation and annihilation. The $9-\overline{9}$ and $\overline{9}-9$ open strings have the opposite GSO projection, so that the massless vector fields are projected out and the tachyonic mode survives [14]. As we have discussed above, it is conjectured that the instability associated with the tachyon represents a flow toward annihilation of the brane-antibrane pair, i.e., by giving the tachyon field a suitable expectation value one can return to the vacuum state without this pair [7]–[9]. Thus if we add an equal number M of 9-branes and $\overline{9}$-branes with the same $U(M)$ gauge bundle H on them, then the tachyon field associated with the open strings stretched between the 9-branes and the $\overline{9}$-branes is a section of a trivial bundle, and hence it can condense to the minimum of its potential everywhere on the $9-\overline{9}$ worldvolume. We suppose that any such collection of brane-antibrane pairs can be created and annihilated, so that the configuration is equivalent to the vacuum which carries no D-brane charges (this is much like the situation in ordinary quantum field theory). We conclude that adding such pairs has no effect on the topological class of the soliton, i.e., the pair (E, F) can be smoothly deformed to the pair $(E \oplus H, F \oplus H)$ for any such bundle H. Thus in terms of the conserved D-brane charges, a property of the system that is invariant under smooth deformations, we conclude that RR-charge is classified topologically by specifying a pair of $U(N)$ vector bundles (E, F) subject to the equivalence relation

$$(E, F) \sim (E \oplus H, F \oplus H) \tag{1.6}$$

for any $U(M)$ vector bundle H. In a manner of speaking (that will soon be made precise), the D-brane charge is determined by the "difference" between the Chan-Paton gauge bundles on the 9-branes and anti-9-branes.

The mathematical conditions described above define the so-called K-theory group $\mathrm{K}(X)$ of the spacetime X. This proposal that D-brane charge takes values in the K-theory of spacetime was made initially in [17], and then extended in [18, 19], [28]–[33]. However, the solitonic description of D-brane states discussed above was not the first evidence that RR-charge should be understood in terms of K-theory rather than cohomology. The strongest prior proposal [34] had been the observation (extending earlier calculations in [22, 23, 26, 35]) that when a D-brane wraps a submanifold Y of spacetime, its RR-charge depends on the geometry of Y, of its normal bundle and on the gauge fields on Y in a manner which suggests that D-brane charges take values in $\mathrm{K}(X)$. Other earlier hints at a connection with K-theory may be found in [36, 37].

The arguments presented above for spacetime filling branes clearly show that when a D-brane wraps a submanifold Y of the spacetime X, its charges are classified by the group $\mathrm{K}(Y)$. One of the profound observations of [17] is that there is a standard K-theory construction, called the Thom isomorphism, which embeds $\mathrm{K}(Y)$ into $\mathrm{K}(X)$ and is equivalent to the bound state construction of D-branes described above, and hence to the representation of all branes in terms of 9-branes and antibranes. In this way, one gets a complete classification of D-brane charges in terms of the topology of the underlying spacetime manifold. The main feature of K-theory which parallels the above soliton constructions is its intimate relationship with homotopy theory. Moreover, another standard K-theory construction, known as the Atiyah-Bott-Shapiro construction, can be used to obtain explicit forms for the classical gauge field configurations which live on a given D-brane. These remarkable facts have been used to reproduce the construction of the Type I non-BPS D-particle discovered in [9], and to predict the existence of new D-branes in the spectrum of the Type I theory and also other superstring theories (the homotopic soliton construction of the Type I D-string was first carried out in [38]). Indeed, the K-groups of a spacetime can be much more general than the corresponding cohomology groups. In many instances the K-groups can have torsion while the cohomology groups are torsion free, lending a natural explanation to the fact that some D-branes (such as the Type I 0-brane) carry torsion charges [34]. The recent string theoretical construction of these new Type I objects [39] illustrates the strong predictions that can follow from the K-theory formalism. In addition, for the spectrum of supersymmetric D-branes (where the RR charge is integer valued) there is a mapping, known as the Chern homomorphism, onto cohomology, thereby making contact with the expectations which follow from the coupling (1.1) to the spacetime RR fields.

1.1 Outline

In this paper we will review the mathematical formalism of topological K-theory and its use as a systematic tool in the topological classification of D-branes in superstring theory. As K-theory now turns out to be at the forefront of mathematical physics as far as its applications to string theory are concerned and, while cohomology and differential geometry are already well-known to most theoretical physicists, K-theory may seem rather obscure, we have attempted to merge the mathematics with the physics in such a way that the naturality of K-theory as a classification tool is evident. The main purpose will be to collect all the relevant mathematical material in one place in a way that should be accessible to a rather general audience of string theorists and mathematicians. The level of this review is geared at string theorists with a relatively good background in algebraic topology and differential geometry (at the level of the books [40] and the review article [41]), and at mathematicians with a rudimentary background in string theory (at the level of the books [42]). More references and background will be given as we proceed.

Before giving a quick outline of the structure of this paper, let us briefly indicate the omissions in our presentation, which can also be taken as directions for further research. Throughout this review we will consider only superstring compactifications for which the curvature of the Neveu-Schwarz B-field is cohomologically trivial. The problems with incorporating this two-form field are discussed in [17], and at present it is not fully understood what the appropriate K-theory should be in these instances. Some steps in this direction have appeared recently in [33, 43, 44]. Related to this problem is how to correctly incorporate S-duality into K-theoretic terms, and in particular the description of the self-duality of Type IIB superstring theory. The analysis of [43] is a first step in this direction. Another related aspect is making contact with the correct construction for M-Theory. The description of M-branes has been discussed in [33, 43] and the appropriate relations in Matrix Theory in [45]. Using the approach of [33], which is based on *algebraic* K-theory, there may be an intimate connection with the gauge bundles for Matrix Theory compactifications used in [46] based on noncommutative geometry. These are all problems that do not as of yet have a natural description in terms of K-theory. It is hoped that the exposition of this paper, in addition to providing the reader with the necessary tools to pursue the subject further, could provoke some detailed investigations of such matters. Finally, we note that the analysis given in the following is meant to serve only as a topological classification of the spectra of branes in the various string theories. The second step, which is omitted in this review, is to actually carry out string theoretical constructions of the D-branes predicted by the K-theory formalism and hence describe their dynamics, especially for the new non-BPS states. This is addressed in [7]–[13], [17, 29, 32, 39]. Indeed, the K-theory classification of D-branes has revealed many interesting new effects and constructions in string theory. We shall only briefly touch upon such matters here, in order to keep the presentation as self-contained as possible.

The structure of this review is as follows. In section 2 we present a thorough, self-contained introduction to the ideas and fundamental constructions of topological K-theory. This section deals with the mathematical highlights of the formalism that will follow in subsequent sections. Many tools for computing K-groups are described which are useful in particular to the various superstring applications that we shall discuss. They will in addition turn out to give many unexpected connections between the various different superstring theories. Here the reader is assumed to have a good background in algebraic topology and the theory of fiber bundles. In section 3 we begin the classification of D-branes using K-theory, starting with the simplest case of Type IIB superstring theory. We start by giving a quick description of the relevant physics of the brane-antibrane pair. For more details, the reader is refered to the original papers [7]–[12] and the recent review articles [13]. We then describe the bound state construction and how it naturally implies the pertinent connection to K-theory, following [17] for the most part. In section 4 we carry out the analogous constructions for Type IIA superstring theory. The relevant K-theory for Type IIA D-branes was suggested in [17] and developed in detail in [18]. In section 5 we then move on to Type I superstring theory, in which new non-BPS D-branes are predicted, again following [17] for a large part. In section 6 we turn our attention to orbifold and orientifold superstring theories. Orbifolds were dealt with in [28] while orientifolds were described in [29, 19]. The various T-dual orientifolds of the Type I theory were discussed in [18, 32, 19]. Finally, we conclude our analysis in section 7 with a description of the modifications of the previous constructs when global topology of the spacetime and of the worldvolume embeddings is taken into account. Here we present the global version of the bound state construction, as described in [17], highlighting the extra structures and care that must be taken into account as compared to previous cases of flat manifolds. We then move on to describe the appropriate K-theory for dealing with general superstring compactifications [32], and introduce a useful connection to the index theory of Dirac operators which has been at the forefront of many applications in theoretical physics (see [41] for a comprehensive review). We then apply these ideas to describe how the celebrated T-duality transformations of D-branes are represented as natural isomorphisms of K-theory groups [30, 32]. We end the review with what can be considered as the origin of the material discussed in this paper, the derivation of the K-theoretic charge formula of [34]. This formula gives the explicit relationship between the K-theoretic and cohomological descriptions of RR charge, and it thereby allows one to explicitly compute D-brane charges in terms of densities integrated over the spacetime manifold.

2 Elements of Topological K-Theory

K-theory was first introduced in the 1950's by Grothendieck in an alternative formulation of the Riemann-Roch theorem (see [47]). It was subsequently developed in the 1960's by Atiyah and Hirzebruch who first introduced the general K-theory group $\mathrm{K}(X)$ of a topological space X [48]. Since then, K-theory has become an indispensable tool in many areas of topology, differential geometry and algebra. Generally speaking, topological K-theory can be regarded as a cohomology theory for vector bundles that emphasizes features which become prominent as the ranks of the vector bundles become large. Actually, it is an example of a *generalized* cohomology theory, in that K-theory does not satisfy all of the Eilenberg-Steenrod axioms [49] of a cohomology theory (it satisfies all axioms except the dimension axiom which defines in advance the cohomology of the topological space consisting of a single point, e.g., $H^n(\mathrm{pt},\mathbb{Z}) = \delta^{n,0}\,\mathbb{Z})$. This extensive section will review the core of the mathematical material that we will need later on in this paper and no mention of physics will be made until section 3. We will not give a complete review of the material, but rather focus only on those aspects that are useful in superstring applications. For more complete expositions of the subject, the reader is refered to the books [50]–[53], where the proofs of the theorems quoted in the following may also be found.

2.1 The Grothendieck Group

In this subsection we shall start with an abstract formulation that will naturally lead to the definition of the group $\mathrm{K}(X)$. Although the formalism is not really required for this definition, it will be of use later on, and moreover it is this definition which allows one to generalize the K-theory of topological spaces to more exotic groups, such as the K-theory of vector spaces, C^*-algebras, etc., which could prove important in future applications of the general formalism of K-theory to string theory and M-theory. Let $\mathcal{A}$ be an abelian monoid, i.e., a set with an addition which satisfies all the axioms of a group except possibly the existence of an inverse. One can naturally associate to $\mathcal{A}$ an abelian group $S(\mathcal{A})$ by the following construction. Consider the equivalence relation $\sim$ on the Cartesian product monoid $\mathcal{A}\times\mathcal{A}$ with $(E,F)\sim(E',F')$

if there exists an element $G \in \mathcal{A}$ such that

$$E + F' + G = F + E' + G. \tag{2.1}$$

The abelian group $S(\mathcal{A})$, called the *symmetrization* of $\mathcal{A}$, is then defined to be the set of equivalence classes of such pairs:

$$S(\mathcal{A}) = \mathcal{A} \times \mathcal{A} / \sim \ . \tag{2.2}$$

The equivalence class of the pair (E,F) is denoted by $[(E,F)]$ and the inverse of such an element in $S(\mathcal{A})$ is $[(F,E)]$. This follows from the fact that for any $E \in \mathcal{A}$, $[(E,E)]$ is a representative of the zero element in $S(\mathcal{A})$. An alternative definition of $S(\mathcal{A})$ is obtained by using in $\mathcal{A} \times \mathcal{A}$ the equivalence relation $(E,F) \sim (E',F')$ if there exist $G, H \in \mathcal{A}$ such that

$$(E,F) + (G,G) = (E',F') + (H,H). \tag{2.3}$$

As a simple example, for $\mathcal{A} = \mathbb{Z}^+$ (the non-negative integers under addition), we have $S(\mathcal{A}) = \mathbb{Z}$. Also, for $\mathcal{A} = \mathbb{Z} - \{0\}$ (an abelian monoid under multiplication), we have $S(\mathcal{A}) = \mathbb{Q} - \{0\}$.

The completion $S(\mathcal{A})$ of the monoid $\mathcal{A}$ can be characterized by the following universal property. For any abelian group $\mathcal{G}$, and any homomorphism $f : \mathcal{A} \to \mathcal{G}$ of the underlying monoids, there exists a unique homomorphism $\tilde{f} : S(\mathcal{A}) \to \mathcal{G}$ such that $\tilde{f} \circ s = f$, where s is the natural map $\mathcal{A} \to S(\mathcal{A})$ defined by $s(E) = [(E,0)]$. This means that $S(\mathcal{A})$ is the "smallest" abelian group that can be built from the abelian monoid $\mathcal{A}$ and it implies, in particular, that if $\mathcal{A}$ is itself a group, then $S(\mathcal{A}) = \mathcal{A}$. In general, this property implies that the map $\mathcal{A} \to S(\mathcal{A})$ is a covariant functor from the category of abelian monoids to the category of abelian groups, i.e., if $\gamma : \mathcal{A} \to \mathcal{B}$ is any homomorphism of monoids, then there is a unique group homomorphism $s(\gamma) : S(\mathcal{A}) \to S(\mathcal{B})$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\gamma} & \mathcal{B} \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ S(\mathcal{A}) & \xrightarrow{s(\gamma)} & S(\mathcal{B}) \end{array} \tag{2.4}$$

and such that $s(\gamma \circ \gamma') = s(\gamma) \circ s(\gamma')$, $s(\mathrm{Id}_{\mathcal{A}}) = \mathrm{Id}_{S(\mathcal{A})}$ (here $\mathrm{Id}_{\mathcal{A}}$ denotes the identity morphism on $\mathcal{A}$).

An important example to which this construction applies is the case that $\mathcal{C}$ is an additive category with $\Phi(\mathcal{C})$ the set of isomorphism classes of elements $E \in \mathcal{C}$, which we denote by $[E]$. $\Phi(\mathcal{C})$ becomes an abelian monoid if we define $[E]+[F] \equiv [E \oplus F]$ (this is well-defined since the isomorphism class of $E \oplus F$ depends only on the isomorphism classes of E and F). The *Grothendieck group* of $\mathcal{C}$ is defined as $\mathrm{K}(\mathcal{C}) = S(\Phi(\mathcal{C}))$. Note that every element of $\mathrm{K}(\mathcal{C})$ can be written as a formal difference $[E]-[F]$ and that $[E]-[F] = [E']-[F']$ in $\mathrm{K}(\mathcal{C})$ if and only if there exists a $G \in \mathcal{C}$ such that $E \oplus F' \oplus G \cong E' \oplus F \oplus G$. Notice also that $[E] = [F]$ if and only if there is a $G \in \mathcal{C}$ such that $E \oplus G \cong F \oplus G$. As a simple example, let $\mathbb{F}$ be an algebraic field (e.g., $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), and let $\mathcal{C}$ be the category of finite dimensional vector spaces over $\mathbb{F}$ whose morphisms are linear transformations. Then, since such finite dimensional vector spaces are characterized uniquely by their dimension, $\Phi(\mathcal{C}) = \mathbb{Z}^+$ implying that $\mathrm{K}(\mathcal{C}) = \mathbb{Z}$.

2.2 The Group $\mathrm{K}(X)$

We will use the construction of the previous subsection for a classification of vector bundles over compact manifolds. Let X be a compact manifold and let $\mathcal{C} = \mathrm{Vect}(X)$ be the additive category of complex vector bundles over X with respect to bundle morphisms and Whitney sum (later on we shall also consider real and quaternionic vector bundles). Define I^k to be the trivial bundle of rank k over X, i.e., $I^k \cong X \times \mathbb{C}^k$. The space of all vector bundles can be partitioned into equivalence classes as follows. Bundle E over X is said to be *stably equivalent* to bundle F, denoted by $E \sim F$, if there exists positive integers j,k such that

$$E \oplus I^j \cong F \oplus I^k \ . \tag{2.5}$$

The corresponding equivalence classes in $\mathrm{Vect}(X)/\sim$ are called stable equivalence classes. It is easily seen that if E, F and G are vector bundles over X then

$$E \oplus G \cong F \oplus G \quad \Rightarrow E \sim F, \tag{2.6}$$

i.e., E and F are stably equivalent. In the proof one uses the fact that there exists a bundle G' such that $G \oplus G'$ is trivial (according to Swan's Theorem this requires X to be a compact Hausdorff manifold).

However, one cannot conclude from the left-hand side of (2.6) that E and F are isomorphic as vector bundles. For example, consider $E = T\mathbf{S}^n$, the tangent bundle of the n-sphere $\mathbf{S}^n$, and $G = N(\mathbf{S}^n, \mathbb{R}^{n+1})$, the normal bundle of $\mathbf{S}^n$ in $\mathbb{R}^{n+1}$. G has a global section given by an outward-pointing unit normal vector, which implies that it is trivial with $N(\mathbf{S}^n, \mathbb{R}^{n+1}) \cong I^1$. Furthermore, we have the usual relations

$$I^{n+1} \cong T\mathbb{R}^{n+1} \cong E \oplus G \cong E \oplus I^1. \tag{2.7}$$

However, $E = T\mathbf{S}^n$ is generally not trivial and therefore not equal to I^n (in fact $T\mathbf{S}^n$ is only trivial for parallelizable spheres corresponding to $n = 1, 3, 7$). So generally, $T\mathbf{S}^n$ is only stably trivial.

This example demonstrates that the space of vector bundles over X is not a group under the Whitney sum of vector bundles, but rather a monoid, as there is no subtraction defined for vector bundles. A group can, using the previous setup, be constructed as follows. The *K-group* of a compact manifold X is defined to be the Grothendieck group of the category $\mathrm{Vect}(X)$, $\mathrm{K}(X) \equiv \mathrm{K}(\mathrm{Vect}(X))$, or

$$\mathrm{K}(X) = \mathrm{Vect}(X) \times \mathrm{Vect}(X)/\sim, \tag{2.8}$$

where we have defined an equivalence relation in $\mathrm{Vect}(X) \times \mathrm{Vect}(X)$ according to $(E, F) \sim (E', F')$ if there exists a vector bundle $G \in \mathrm{Vect}(X)$ such that

$$E \oplus F' \oplus G \cong E' \oplus F \oplus G. \tag{2.9}$$

An equivalent definition of $\mathrm{K}(X)$ is that the pair of bundles (E, F) is taken to be equivalent to $(E \oplus H, F \oplus H)$ for any bundle H. Often the notation $\mathrm{K}^0(X)$ or $\mathrm{KU}(X)$ is also used for this group. An element of $\mathrm{K}(X)$ is written as $[(E, F)]$. In $\mathrm{K}(X)$ the unit (zero) element is $[(E, E)]$ so the inverse of the class $[(E, F)]$ is $[(F, E)]$. Any element $[(E, F)]$ can therefore be identified with $[E] - [F]$ where $[E] = [(E, I^n)]$. Furthermore, $[E] = [F]$ in $\mathrm{K}(X)$ if and only if E and F are stably equivalent. The elements of $\mathrm{K}(X)$ are called *virtual bundles*. The map $X \to \mathrm{K}(X)$ is a contravariant functor from the category of compact topological spaces to the category of abelian groups, i.e., if $f : X \to Y$ is continuous, then it induces the usual pullback map on vector bundles over Y, thus inducing a map $f^* : \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$ and hence a homomorphism $\mathrm{K}(Y) \to \mathrm{K}(X)$.

The K-groups have the following important homotopy invariance property. Consider two homotopic maps $f, g : X \to Y$. Then for any vector bundle $E \to Y$, there is an isomorphism of vector bundles over X:

$$f^*E \cong g^*E. \tag{2.10}$$

From this it follows that the maps induced by f and g on K-groups are the same:

$$s(f) = s(g) : \mathrm{K}(Y) \longrightarrow \mathrm{K}(X). \tag{2.11}$$

For example, if X is a compact manifold which is contractible to a point, then we may deduce that $\mathrm{K}(X) = \mathrm{K}(\mathrm{pt}) = \mathbb{Z}$. Geometrically, this expresses the well-known fact that any vector bundle over a contractible space X is necessarily trivial, so that the corresponding K-theory of X is also trivial.

2.3 Reduced K-Theory

The fact that a vector bundle over a point is just a vector space, so that $\mathrm{K}(\mathrm{pt}) = \mathbb{Z}$, motivates the introduction of a reduced K-theory in which the topological space consisting of a single point has trivial cohomology, $\widetilde{\mathrm{K}}(\mathrm{pt}) = 0$, and therefore also $\widetilde{\mathrm{K}}(X) = 0$ for any contractible space X. Let us fix a basepoint of X and consider the collapsing and inclusion maps:

$$p : X \longrightarrow \mathrm{pt}, \qquad i : \mathrm{pt} \hookrightarrow X. \tag{2.12}$$

These maps induce, respectively, an epimorphism and a monomorphism of the corresponding K-groups:

$$p^* : \mathrm{K}(\mathrm{pt}) = \mathbb{Z} \longrightarrow \mathrm{K}(X), \qquad i^* : \mathrm{K}(X) \longrightarrow \mathrm{K}(\mathrm{pt}) = \mathbb{Z}. \tag{2.13}$$

We then have the exact sequences of groups:

$$\begin{gathered} 0 \longrightarrow \mathbb{Z} \xrightarrow{p^*} \mathrm{K}(X) \longrightarrow \widetilde{\mathrm{K}}(X) \longrightarrow 0 \\ 0 \longrightarrow \widetilde{\mathrm{K}}(X) \longrightarrow \mathrm{K}(X) \xrightarrow{i^*} \mathbb{Z} \longrightarrow 0. \end{gathered} \tag{2.14}$$

These sequences have a canonical splitting so that the homomorphism i^* is a left inverse of p^*. The kernel of the map i^*, or equivalently the cokernel of the map p^*, is called the *reduced K-theory group* and is denoted by $\widetilde{\mathrm{K}}(X)$,

$$\widetilde{\mathrm{K}}(X) = \ker i^* = \operatorname{coker} p^* \tag{2.15}$$

and therefore we have the fundamental decomposition

$$\mathrm{K}(X) = \mathbb{Z} \oplus \widetilde{\mathrm{K}}(X). \tag{2.16}$$

Given a vector bundle $E \to X$, let E_x denote the fiber of E over $x \in X$. We define the *rank function* $\mathrm{rk} : X \to \mathbb{Z}^+$ by $\mathrm{rk}(x) = \dim_{\mathbb{C}} E_x$. Since E is locally trivial, the rank function is locally constant, and the space of all locally constant $\mathbb{Z}^+$-valued functions on X forms an abelian monoid $H^0(X, \mathbb{Z}^+)$ under pointwise addition. The map rk extends naturally to a group homomorphism

$$\begin{aligned} \mathrm{rk} : \mathrm{K}(X) &\longrightarrow H^0(X, \mathbb{Z}) \\ \mathrm{rk}\Big([E] - [F]\Big) &= \mathrm{rk}(E) - \mathrm{rk}(F). \end{aligned} \tag{2.17}$$

The integer (2.17) is called the *virtual dimension* of $[(E, F)] \in \mathrm{K}(X)$. Let $\mathrm{K}'(X) = \ker \mathrm{rk}$. Then the short exact sequence

$$0 \longrightarrow \mathrm{K}'(X) \longrightarrow \mathrm{K}(X) \xrightarrow{\mathrm{rk}} H^0(X, \mathbb{Z}) \longrightarrow 0 \tag{2.18}$$

has a canonical split (i.e., rk has a right inverse), so that if X is connected, then $H^0(X, \mathbb{Z}) = \mathbb{Z}$ and

$$\widetilde{\mathrm{K}}(X) = \mathrm{K}'(X) = \ker \mathrm{rk}. \tag{2.19}$$

In this case $\widetilde{\mathrm{K}}(X)$ is the subgroup of $\mathrm{K}(X)$ whose elements have virtual dimension zero (i.e., consisting of equivalence classes of pairs of vector bundles $[(E, F)]$ of equal rank). The fundamental examples are $\widetilde{\mathrm{K}}(\mathbf{S}^{2n}) = \mathbb{Z}$ and $\widetilde{\mathrm{K}}(\mathbf{S}^{2n+1}) = 0$ for any positive integer n (these groups are computed in section 2.7). Note that the rank function (2.17) naturally gives an assignment $\mathrm{ch}_0(E)$ in the zeroth Čech cohomology group of X which depends only on the stable equivalence class of the vector bundle E in $\mathrm{K}(X)$. This is the first, basic example of the Chern character which will be discussed in section 7.1.

For the physical applications of K-theory, which are presented in the subsequent sections, we shall mostly work in K-theory with compact support. This means that for each class $[(E,F)]$, there is a map $T: E \to F$ which is an isomorphism of vector bundles outside an open set $U \subset X$ whose closure $\overline{U}$ is compact. This condition automatically implies that E and F have the same rank, and hence we shall mostly deal with the reduced K-group $\widetilde{\mathrm{K}}(X)$. The corresponding virtual bundle may then be represented as

$$\Big[(E\,,\,F)\Big] = \Big[(\ker T\,,\,\operatorname{coker} T)\Big] \tag{2.20}$$

When X is not compact, we define $\mathrm{K}(X) = \widetilde{\mathrm{K}}(X^+)$, where X^+ is the one-point compactification of X.

2.4 Higher K-Theory and Bott Periodicity

Starting with $\mathrm{K}(X)$ there is a natural way to define so-called "higher" K-groups. These groups are labelled by a positive integer $n = 0, 1, 2, \ldots$ and are defined according to

$$\mathrm{K}^{-n}(X) = \mathrm{K}(\Sigma^n X), \tag{2.21}$$

where $\Sigma^n X \equiv \mathbf{S}^n \wedge X$ is the n-th reduced suspension of the topological space X. Here $X \wedge Y = X \times Y/(X \vee Y)$ is the smash product of X and Y, and $X \vee Y$ is their reduced join, i.e., their disjoint union with a base point of each space identified, which can be viewed as the subspace $X \times \mathrm{pt} \amalg \mathrm{pt} \times Y$ of the Cartesian product $X \times Y$. For $X = \mathbf{S}^n$, the n-sphere, one has $\Sigma\mathbf{S}^n = \mathbf{S}^1 \wedge \mathbf{S}^n \cong \mathbf{S}^{n+1}$. Alternatively, higher K-groups can be defined through the *suspension isomorphism*:

$$\mathrm{K}^{-n}(X) = \mathrm{K}(X \times \mathbb{R}^n), \tag{2.22}$$

where it is always understood that K-theory with compact support is used. In contrast to conventional cohomology theories, one does not in this way generate an infinite number of higher K-groups because of the fundamental *Bott periodicity theorem*:

$$\mathrm{K}^{-n}(X) = \mathrm{K}^{-n-2}(X), \tag{2.23}$$

which states that the complex K-theory functor K^{-n} is periodic with period two. The same is true for the reduced functor $\widetilde{\mathrm{K}}^{-n}$ since the analogous definition to (2.21) holds for reduced K-theory. However, the higher reduced and unreduced K-groups differ according to

$$\mathrm{K}^{-n}(X) = \widetilde{\mathrm{K}}^{-n}(X) \oplus \mathrm{K}^{-n}(\mathrm{pt}). \tag{2.24}$$

Since $\mathrm{K}(\mathrm{pt}) = \mathbb{Z}$, $\mathrm{K}^{-1}(\mathrm{pt}) = 0$, using Bott periodicity we see that for n even these groups differ by a subgroup $\mathbb{Z}$ (as in (2.16)), while for n odd they are identical, so that $\widetilde{\mathrm{K}}^{-1}(X) = \mathrm{K}^{-1}(X)$. Here the basic examples are $\mathrm{K}^{-1}(\mathbf{S}^{2n}) = 0$ and $\mathrm{K}^{-1}(\mathbf{S}^{2n+1}) = \mathbb{Z}$ for any positive integer n.

Note that for any decomposition $X = X_1 \amalg X_2 \amalg \cdots \amalg X_n$ of X into a disjoint union of open subspaces, the inclusions of the X_i into X induce a decomposition of K-groups as $\mathrm{K}^{-n}(X) = \mathrm{K}^{-n}(X_1) \oplus \mathrm{K}^{-n}(X_2) \oplus \cdots \oplus \mathrm{K}^{-n}(X_n)$ (this follows from the fact that a bundle over X may be characterized by its restriction to X_i). However, this is not true for the reduced K-functor, since for example $\widetilde{\mathrm{K}}(\mathbf{S}^0) = \mathbb{Z}$ but $\widetilde{\mathrm{K}}(\mathrm{pt}) = 0$. More generally, given two *closed* subspaces X_1 and X_2 of a locally compact space X with $X = X_1 \cup X_2$, there is the long exact sequence

$$\begin{aligned} \ldots \longrightarrow \mathrm{K}^{-n-1}(X_1) \oplus \mathrm{K}^{-n-1}(X_2) &\xrightarrow{v} \mathrm{K}^{-n-1}(X_1 \cap X_2) \longrightarrow \\ &\xrightarrow{\zeta} \mathrm{K}^{-n}(X_1 \cup X_2) \xrightarrow{u} \mathrm{K}^{-n}(X_1) \oplus \mathrm{K}^{-n}(X_2) \\ &\xrightarrow{v} \mathrm{K}^{-n}(X_1 \cap X_2) \longrightarrow \ldots , \end{aligned} \tag{2.25}$$

where ζ is the zig-zag homomorphism, and u and v are defined by $u([E]) = ([E|_{X_1}], [E|_{X_2}])$ and $v([E_1], [E_2]) = [E_1|_{X_1 \cap X_2}] - [E_2|_{X_1 \cap X_2}]$. The corresponding long exact sequence for two *open* subspaces U_1 and U_2 of X with $X = U_1 \cup U_2$ is

$$\begin{aligned} \ldots \longrightarrow \mathrm{K}^{-n-1}(U_1) \oplus \mathrm{K}^{-n-1}(U_2) &\longrightarrow \mathrm{K}^{-n-1}(U_1 \cup U_2) \longrightarrow \\ &\longrightarrow \mathrm{K}^{-n}(U_1 \cap U_2) \longrightarrow \mathrm{K}^{-n}(U_1) \oplus \mathrm{K}^{-n}(U_2) \\ &\longrightarrow \mathrm{K}^{-n}(U_1 \cup U_2) \longrightarrow \ldots . \end{aligned} \tag{2.26}$$

These latter two sequences are the analogs of the usual Mayer-Vietoris long exact sequences in cohomology.

2.5 Multiplicative Structures

As in any cohomology theory, $\mathrm{K}(X)$ and $\widetilde{\mathrm{K}}(X)$ are actually rings. In this case the multiplication is induced by the tensor product $E \otimes F$ of vector bundles over $X \times X$:

$$\mathrm{K}(X) \otimes_{\mathbb{Z}} \mathrm{K}(X) \longrightarrow \mathrm{K}(X), \tag{2.27}$$

and is defined by

$$\Big[(E\,,\,F)\Big] \otimes \Big[(E'\,,\,F')\Big] \equiv \Delta^* \Big[(E \otimes E' \oplus F \otimes F'\,,\,E \otimes F' \oplus F \otimes E')\Big] \tag{2.28}$$

where $\Delta : X \to X \times X$ is the diagonal map. This multiplication comes from writing $[(E,F)] = [E] - [F]$ and formally using distributivity of the tensor product acting on virtual bundles. Note that it acts on $[(E,F)]$ as if E's are bosonic and F's are fermionic. It is therefore an example of a $\mathbb{Z}_2$-graded tensor product. There is another product, called the external tensor product or *cup product*, which is a homomorphism

$$\mathrm{K}(X) \otimes_{\mathbb{Z}} \mathrm{K}(Y) \longrightarrow \mathrm{K}(X \times Y) \tag{2.29}$$

defined as follows. Consider the canonical projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$. These projections induce homomorphisms between K-groups according to

$$\pi_X^* : \mathrm{K}(X) \to \mathrm{K}(X \times Y), \quad \pi_Y^* : \mathrm{K}(Y) \to \mathrm{K}(X \times Y). \tag{2.30}$$

Then the cup product of $([E],[F]) \in \mathrm{K}(X) \otimes_{\mathbb{Z}} \mathrm{K}(Y)$ is the class $[E] \otimes [F]$ in $\mathrm{K}(X \times Y)$, with

$$[E] \otimes [F] \equiv \pi_X^*\Big([E]\Big) \otimes \pi_Y^*\Big([F]\Big). \tag{2.31}$$

Consider now the canonical injective inclusion and surjective projection maps:

$$X \vee Y \hookrightarrow X \times Y \longrightarrow X \wedge Y. \tag{2.32}$$

The $\widetilde{\mathrm{K}}^{-n}$ functor is contravariant, and thus, as in any cohomology theory, this induces a split short exact sequence of K-groups

$$0 \longrightarrow \widetilde{\mathrm{K}}^{-n}(X \wedge Y) \longrightarrow \widetilde{\mathrm{K}}^{-n}(X \times Y) \longrightarrow \widetilde{\mathrm{K}}^{-n}(X \vee Y) \longrightarrow 0, \tag{2.33}$$

from which it follows that

$$\begin{aligned} \widetilde{\mathrm{K}}^{-n}(X \times Y) &= \widetilde{\mathrm{K}}^{-n}(X \wedge Y) \oplus \widetilde{\mathrm{K}}^{-n}(X \vee Y) \\ &= \widetilde{\mathrm{K}}^{-n}(X \wedge Y) \oplus \widetilde{\mathrm{K}}^{-n}(X) \oplus \widetilde{\mathrm{K}}^{-n}(Y). \end{aligned} \tag{2.34}$$

The formula (2.34) is particularly useful for computing the K-groups of Cartesian products. As an important example, consider the case that $Y = \mathbf{S}^1$, for which we find

$$\begin{aligned}\widetilde{\mathrm{K}}(X \times \mathbf{S}^1) &= \widetilde{\mathrm{K}}(X \wedge \mathbf{S}^1) \oplus \widetilde{\mathrm{K}}(X) \oplus \widetilde{\mathrm{K}}(\mathbf{S}^1) \\ &= \mathrm{K}^{-1}(X) \oplus \widetilde{\mathrm{K}}(X), \end{aligned} \tag{2.35}$$

since $\widetilde{\mathrm{K}}(\mathbf{S}^1) = 0$ and $\widetilde{\mathrm{K}}(\mathbf{S}^1 \wedge X) = \mathrm{K}^{-1}(X)$. Precisely, the canonical inclusion $i : X \hookrightarrow X \times \mathbf{S}^1$ induces a projection $i^* : \widetilde{\mathrm{K}}(X \times \mathbf{S}^1) \to \widetilde{\mathrm{K}}(X)$ such that $\ker i^* = \mathrm{K}^{-1}(X)$. In other words, $\mathrm{K}^{-1}(X)$ can be identified with the set of K-theory classes in $\widetilde{\mathrm{K}}(X \times \mathbf{S}^1)$ which vanish when restricted to $X \times \mathrm{pt}$. Likewise,

$$\begin{aligned}\mathrm{K}^{-1}(X \times \mathbf{S}^1) &= \mathrm{K}^{-1}(X \wedge \mathbf{S}^1) \oplus \mathrm{K}^{-1}(X) \oplus \mathrm{K}^{-1}(\mathbf{S}^1) \\ &= \widetilde{\mathrm{K}}(X) \oplus \mathrm{K}^{-1}(X) \oplus \mathbb{Z}, \end{aligned} \tag{2.36}$$

where we have used Bott periodicity.

The action of the cup product (2.29) on reduced K-theory can also be deduced using (2.16) and (2.34) to get

$$\left(\widetilde{\mathrm{K}}(X) \otimes_{\mathbb{Z}} \widetilde{\mathrm{K}}(Y)\right) \oplus \mathcal{R} \longrightarrow \widetilde{\mathrm{K}}(X \wedge Y) \oplus \mathcal{R}, \tag{2.37}$$

where $\mathcal{R} = \widetilde{\mathrm{K}}(X) \oplus \widetilde{\mathrm{K}}(Y) \oplus \mathbb{Z}$. Since the group $\mathcal{R}$ appears on both sides of (2.37), we can eliminate it by an appropriate restriction and thereby arrive at the homomorphism

$$\widetilde{\mathrm{K}}(X) \otimes_{\mathbb{Z}} \widetilde{\mathrm{K}}(Y) \longrightarrow \widetilde{\mathrm{K}}(X \wedge Y). \tag{2.38}$$

When either $\mathrm{K}(X)$ or $\mathrm{K}(Y)$ is a free abelian group, the mappings in (2.29) and (2.38) are isomorphisms.

One can also calculate $\mathrm{K}^{-n}(X \times Y)$ in a manner that keeps track of the multiplicative structure of the theory. Define $\mathrm{K}^{\#}(X)$ to be the $\mathbb{Z}_2$-graded ring $\mathrm{K}^{\#}(X) = \mathrm{K}(X) \oplus \mathrm{K}^{-1}(X)$. Then, whenever $\mathrm{K}^{\#}(X)$ or $\mathrm{K}^{\#}(Y)$ is freely generated, we get the K-theory analog of the cohomological Künneth theorem:

$$\mathrm{K}^{\#}(X \times Y) = \mathrm{K}^{\#}(X) \otimes_{\mathbb{Z}} \mathrm{K}^{\#}(Y). \tag{2.39}$$

(In the general case there are correction terms on the right-hand side of (2.39) which take into account the torsion subgroups of the K-groups [54]). Explicitly, (2.39) leads to

$$\begin{aligned} \mathrm{K}(X\times Y) &= \Big(\mathrm{K}(X)\otimes_{\mathbb{Z}}\mathrm{K}(Y)\Big)\oplus\Big(\mathrm{K}^{-1}(X)\otimes_{\mathbb{Z}}\mathrm{K}^{-1}(Y)\Big)\;, \\ \mathrm{K}^{-1}(X\times Y) &= \Big(\mathrm{K}(X)\otimes_{\mathbb{Z}}\mathrm{K}^{-1}(Y)\Big)\oplus\Big(\mathrm{K}^{-1}(X)\otimes_{\mathbb{Z}}\mathrm{K}(Y)\Big). \end{aligned} \tag{2.40}$$

For example, since $Y = \mathbf{S}^1$ has freely generated K-groups, using $\mathrm{K}(\mathbf{S}^1) = \mathrm{K}^{-1}(\mathbf{S}^1) = \mathbb{Z}$, we again arrive at (2.35) and (2.36). Similarly, taking $Y = \mathbf{S}^{2n}$ and $Y = \mathbf{S}^{2n+1}$ in (2.39) gives

$$\mathrm{K}(X\times\mathbf{S}^{2n}) = \mathrm{K}(X)\oplus\mathrm{K}(X), \tag{2.41}$$

$$\mathrm{K}(X\times\mathbf{S}^{2n+1}) = \mathrm{K}(X)\oplus\mathrm{K}^{-1}(X), \tag{2.42}$$

as $\mathrm{K}(X)$-modules.

If we choose $Y = \mathbf{S}^2$, then the maps in (2.29) and (2.38) are actually isomorphisms which can be identified with the Bott periodicity property of the reduced and unreduced K-groups. Replacing X by its n-th reduced suspension in (2.38) gives

$$\widetilde{\mathrm{K}}(\Sigma^n X)\otimes_{\mathbb{Z}}\widetilde{\mathrm{K}}(\mathbf{S}^2) = \widetilde{\mathrm{K}}(\Sigma^n X\wedge\mathbf{S}^2), \tag{2.43}$$

which yields the isomorphism

$$\alpha\,:\,\widetilde{\mathrm{K}}^{-n}(X)\otimes_{\mathbb{Z}}\widetilde{\mathrm{K}}(\mathbf{S}^2)\xrightarrow{\approx}\widetilde{\mathrm{K}}^{-n-2}(X). \tag{2.44}$$

The generator $[\mathcal{N}_{\mathbb{C}}]-[I^1]$ of $\widetilde{\mathrm{K}}(\mathbf{S}^2)=\mathbb{Z}$ may be described by taking $\mathcal{N}_{\mathbb{C}}$ to be the canonical line bundle over the complex projective space $\mathbb{C}P^1$, which is associated with the Hopf fibration $\mathbf{S}^3\to\mathbf{S}^2$ that classifies the Dirac monopole [53, 55]. The isomorphism (2.23) is then given by the mapping

$$\Big[(E\,,\,F)\Big]\longmapsto\alpha\Big[(E\otimes\mathcal{N}_{\mathbb{C}}\,,\,F\otimes\mathcal{N}_{\mathbb{C}})\Big], \tag{2.45}$$

for $[(E,F)]\in\widetilde{\mathrm{K}}^{-n}(X)$.

2.6 Relative K-Theory

We will now define a relative K-group $\mathrm{K}(X,Y)$ which depends on a pair of spaces (X,Y), where Y is a closed submanifold of X, and whose classes can be identified with pairs of bundles over X/Y. If $Y \neq \emptyset$, then the topological coset X/Y is defined to be the space X with Y shrunk to a point. If Y is empty we identify X/Y with the one-point compactification X^+ of X.

First we explain how to describe vector bundles over the quotient space X/Y, given a vector bundle E over X. Let α be a trivialization of E over $Y \subset X$, i.e., an isomorphism $\alpha : E|_Y \xrightarrow{\approx} Y \times V$. Define an equivalence relation on $E|_Y$ by taking $e \in E|_Y$ equivalent to $e' \in E|_Y$ if and only if

$$\pi \circ \alpha(e) = \pi \circ \alpha(e'), \tag{2.46}$$

where $\pi : Y \times V \to V$ is the canonical projection. This equivalence relation identifies points in the restriction of E to Y which are "on the same level" relative to the trivialization α. We then extend this relation trivially to the whole of E. The corresponding set of equivalence classes E_α can be shown to be a vector bundle over X/Y, whose isomorphism class depends only on the homotopy class of the trivialization α of E over $Y \subset X$. In fact, there is a one-to-one correspondence between vector bundles over the quotient space X/Y and vector bundles over X whose restriction to Y is a trivial bundle.

The *relative K-group* is now defined as

$$\mathrm{K}(X,Y) \equiv \widetilde{\mathrm{K}}(X/Y). \tag{2.47}$$

Then $\mathrm{K}(X,Y)$ is a contravariant functor of the pair (X,Y) and, since $\mathrm{K}(X) = \widetilde{\mathrm{K}}(X^+)$ (recall $\mathrm{K}(X) = \widetilde{\mathrm{K}}(X) \oplus \mathrm{K}(\mathrm{pt})$), we have $\mathrm{K}(X,\emptyset) = \mathrm{K}(X)$. The *Excision Theorem* states that the projection $\pi : X \to X/Y$ induces an isomorphism

$$\pi^* : \mathrm{K}(X/Y, \mathrm{pt}) \xrightarrow{\approx} \mathrm{K}(X,Y). \tag{2.48}$$

Likewise, one can define higher relative K-groups by

$$\mathrm{K}^{-n}(X,Y) = \mathrm{K}(X \times \mathbf{B}^n, X \times \mathbf{S}^{n-1} \cup Y \times \mathbf{B}^n), \tag{2.49}$$

where $\mathbf{B}^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ is the unit ball in $\mathbb{R}^n$ and $\mathbf{S}^{n-1} = \partial \mathbf{B}^n$. Alternatively, there are the suspension isomorphisms

$$\mathrm{K}^{-n}(X,Y) = \mathrm{K}\Big((X - Y) \times \mathbb{R}^n\Big). \tag{2.50}$$

The relative K-groups have the usual Bott periodicity:

$$\mathrm{K}^{-n}(X,Y) = \mathrm{K}^{-n-2}(X,Y). \tag{2.51}$$

Let $i : Y \to X$ and $j : (X, \emptyset) \to (X, Y)$ be inclusions. Then there is an exact sequence

$$\mathrm{K}(X,Y) \xrightarrow{j^*} \mathrm{K}(X) \xrightarrow{i^*} \mathrm{K}(Y). \tag{2.52}$$

If Y is further equipped with a base-point, then the sequence

$$\mathrm{K}(X,Y) \longrightarrow \widetilde{\mathrm{K}}(X) \longrightarrow \widetilde{\mathrm{K}}(Y) \tag{2.53}$$

is exact. More generally, one of the most important properties of the K-groups is that they possess the excision property, which means that they satisfy the Barratt-Puppe long exact sequence:

$$\begin{aligned} \ldots \longrightarrow \mathrm{K}^{-n-1}(X) \longrightarrow \mathrm{K}^{-n-1}(Y) \xrightarrow{\partial^*} \mathrm{K}^{-n}(X,Y) \longrightarrow \\ \mathrm{K}^{-n}(X) \longrightarrow \mathrm{K}^{-n}(Y) \xrightarrow{\partial^*} \ldots , \end{aligned} \tag{2.54}$$

where ∂ is the boundary homomorphism. The long exact sequence (2.54) connects the K-groups of X and $Y \subset X$, and it is in precisely this sense that K-theory is similar to a cohomology theory. Using Bott periodicity, this sequence can be amazingly truncated to a six-term exact sequence. If Y is a retract of X (i.e., if the inclusion map $i : Y \to X$ admits a left inverse), then the sequence (2.52) splits giving

$$\mathrm{K}^{-n}(X) = \mathrm{K}^{-n}(X,Y) \oplus \mathrm{K}^{-n}(Y). \tag{2.55}$$

The concept of relative K-theory can be reformulated in a way that will prove useful later on. Let $\Gamma(X, Y)$ be the set of triples $(E, F; \alpha)$, where $E, F \in \mathrm{Vect}(X)$ and $\alpha : E|_Y \xrightarrow{\approx} F|_Y$ is an isomorphism of vector bundles when restricted to Y. Two such triples $(E, F; \alpha)$ and $(E', F'; \alpha')$ are said to be isomorphic if there exist isomorphims $f : E \xrightarrow{\approx} E'$ and $g : F \xrightarrow{\approx} F'$ such that the diagram

$$\begin{array}{ccc} E|_Y & \xrightarrow{\alpha} & F|_Y \\ f|_Y \downarrow & & \downarrow g|_Y \\ E'|_Y & \xrightarrow{\alpha'} & F'|_Y \end{array} \tag{2.56}$$

commutes. A triple $(E, F; \alpha)$ is called *elementary* if $E \cong F$ and α is homotopic to $\mathrm{Id}_{E|_Y}$ within automorphisms of $E|_Y$. The sum of $(E, F; \alpha)$ and $(E', F'; \alpha')$ is defined to be

$$(E, F; \alpha) \oplus (E', F'; \alpha') \equiv (E \oplus E', F \oplus F'; \alpha \oplus \alpha'), \tag{2.57}$$

under which $\Gamma(X, Y)$ becomes an abelian monoid. Now consider the following equivalence relation in $\Gamma(X, Y)$. We take the triple $(E, F; \alpha)$ to be equivalent to $(E', F'; \alpha')$ whenever there exist two elementary triples $(G, H; \beta)$ and $(G', H'; \beta')$ such that

$$(E, F; \alpha) \oplus (G, H; \beta) \cong (E', F'; \alpha') \oplus (G', H'; \beta'). \tag{2.58}$$

The set of equivalence classes of such triples (which we denote by $[E, F; \alpha]$) under the operation $\oplus$ becomes an abelian group which can be identified with the relative K-group, $\mathrm{K}(X, Y) = \Gamma(X, Y)/\sim$. Note that $[E, F; \alpha] = 0$ in $\mathrm{K}(X, Y)$ if and only if there exist vector bundles $G, H \in \mathrm{Vect}(X)$ and bundle isomorphisms $u : E \oplus G \to H$, $v : F \oplus G \to H$ such that $v|_Y \circ (\alpha \oplus \mathrm{Id}_{G|_Y}) \circ u^{-1}|_Y$ is homotopic to $\mathrm{Id}_{H|_Y}$ within automorphisms of $H|_Y$. Moreover, $[E, F; \alpha] + [F, E; \alpha^{-1}] = 0$. Notice also that the group $\mathrm{K}(X)$ can in this formalism be described as the set of triples $[E, F; \alpha]$, where $\alpha : E \overset{\approx}{\to} F$ is a bundle isomorphism defined in a neighbourhood of the infinity of the one-point compactification of X. This is precisely the statement that was made in (2.20).

The basic properties of $\mathrm{K}(X, Y)$ are as follows. First, if α, α' are isomorphisms $E|_Y \overset{\approx}{\to} F|_Y$ as described above and if α and α' are homotopic within isomorphisms from $E|_Y$ to $F|_Y$, then $[E, F; \alpha] = [E, F; \alpha']$. Also, when $[E, F; \alpha]$ and $[F, G; \beta]$ are elements of $\mathrm{K}(X, Y)$ then their sum is given by the relation $[E, F; \alpha] + [F, G; \beta] = [E, G; \beta \circ \alpha]$. Thirdly, two elements of $\mathrm{K}(X, Y)$ determine the same equivalence class, $[E, F; \alpha] = [E', F'; \alpha']$, in $\mathrm{K}(X, Y)$ if and only if there exist triples $(G, G; \mathrm{Id}_{G|_Y})$ and $(G', G'; \mathrm{Id}_{G'|_Y})$, and maps $f : E \oplus G \to E' \oplus G'$, $g : F \oplus G \to F' \oplus G'$ such that the diagram

$$\begin{array}{ccc} (E \oplus G)|_Y & \xrightarrow{\alpha \oplus \mathrm{Id}_{G|_Y}} & (F \oplus G)|_Y \\ f|_Y \downarrow & & \downarrow g|_Y \\ (E' \oplus G')|_Y & \xrightarrow{\alpha' \oplus \mathrm{Id}_{G'|_Y}} & (F' \oplus G')|_Y \end{array} \tag{2.59}$$

commutes. Furthermore, the cup product naturally extends to relative K-theory to give the unique bilinear homomorphism

$$\mathrm{K}(X,Y)\otimes_{\mathbb{Z}}\mathrm{K}(X',Y')\longrightarrow \mathrm{K}(X\times X',X\times Y'\cup Y\times X'). \tag{2.60}$$

This agrees with the cup product introduced earlier when $Y=Y'=\emptyset$. Explicitly, the product of two K-theory classes $[E,F;\alpha]$ and $[E',F';\alpha']$ is obtained using the product (2.28) on the pairs of vector bundles and the product isomorphism

$$\beta=\begin{pmatrix}\alpha\otimes\mathrm{Id} & \mathrm{Id}\otimes\alpha'^{\dagger}\\ \mathrm{Id}\otimes\alpha' & -\alpha^{\dagger}\otimes\mathrm{Id}\end{pmatrix} \tag{2.61}$$

acting on (2.28) (note that this requires the introduction of fiber metrics on the bundles involved).

As a simple example, consider the case that $X=\mathbf{B}^2$ and $Y=\mathbf{S}^1\subset\mathbb{R}^2$. Define $[E,F;\alpha]\in\mathrm{K}(\mathbf{B}^2,\mathbf{S}^1)$ by $E=F=\mathbf{B}^2\times\mathbb{C}$, and $\alpha(x,z)=(x,xz)$ for $x\in\mathbf{S}^1\subset\mathbf{B}^2$. It then turns out that $[E,F;\alpha]$ is a generator of $\mathrm{K}(\mathbf{B}^2,\mathbf{S}^1)=\mathbb{Z}$. This is our first instance of the ABS construction which will be described in section 2.8. As another example, consider the complex projective spaces $X=\mathbb{C}P^2$ and $Y=\mathbb{C}P^1$. As mentioned before, a non-trivial generator of $\widetilde{\mathrm{K}}(\mathbf{S}^2)=\mathbb{Z}$ is given by the canonical line bundle over $\mathbb{C}P^1$ which is the restriction of the canonical line bundle over $\mathbb{C}P^2$. It follows that the map $\mathrm{K}(X)\to\mathrm{K}(Y)$ is surjective. Furthermore, $\mathrm{K}^{-1}(\mathbf{S}^2)=0$ and $\mathrm{K}(X,Y)=\widetilde{\mathrm{K}}(\mathbf{S}^4)=\mathbb{Z}$. From (2.54) we then obtain the split short exact sequence

$$0\longrightarrow\widetilde{\mathrm{K}}(\mathbf{S}^4)\longrightarrow\mathrm{K}(\mathbb{C}P^2)\longrightarrow\mathrm{K}(\mathbb{C}P^1)\longrightarrow 0, \tag{2.62}$$

giving $\mathrm{K}(\mathbb{C}P^2)=\widetilde{\mathrm{K}}(\mathbf{S}^4)\oplus\mathrm{K}(\mathbb{C}P^1)=\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$.

2.7 Computing the K-Groups

In this subsection we will show how the K-groups can be computed as homotopy groups of certain classifying spaces, for which there is often a finite dimensional approximation. The basic case is the reduced K-groups $\widetilde{\mathrm{K}}(X)$, since unreduced K-groups are computed from the decomposition (2.16) and since higher K-groups are given by suspensions as in (2.21). Let $\mathrm{Vect}_k(X)$ be the set of isomorphism classes of complex vector bundles $E_k\to X$ of rank k. Then we have the sequence of inclusions,

$\ldots \subset \mathrm{Vect}_k(X) \subset \mathrm{Vect}_{k+1}(X) \subset \ldots$, via the mapping $E_k \mapsto E_k \oplus I^1$. If $[E_k] \in \mathrm{Vect}_k(X)$, then $[E_k] - [I^k] \in \ker \mathrm{rk} = \mathrm{K}'(X)$. The map $[E_k] \mapsto [E_k] - [I^k]$ is actually an isomorphism $\mathrm{Vect}(X) \to \mathrm{K}'(X)$ of abelian monoids, and hence $\mathrm{Vect}(X) \equiv \bigcup_{k=0}^{\infty} \mathrm{Vect}_k(X)$ is an abelian group.

A complex vector bundle E_k of rank k has structure group $GL(k, \mathbb{C})$ (the fiber automorphism group), which upon choosing a metric on X, and thereby inducing a Hermitian inner product on the fibers of E_k, is reducible to the unitary subgroup $U(k)$. The *classifying space* for E_k is the complex Grassmannian manifold:

$$\mathrm{Gr}(k,m;\mathbb{C}) = \frac{U(m)}{U(m-k) \times U(k)}, \quad m > k+n, \tag{2.63}$$

where $n \equiv \dim X$. According to a standard theorem of differential geometry, there exists a so-called universal bundle $Q(k,m;\mathbb{C})$ over $\mathrm{Gr}(k,m;\mathbb{C})$ of rank k whose pullbacks generate vector bundles such as E_k. This means that $f^*Q(k,m;\mathbb{C}) \cong E_k$ for some continuous map $f : X \to \mathrm{Gr}(k,m;\mathbb{C})$, $m > k+n$. Moreover, this isomorphism depends only on the homotopy class of f. Therefore, bundles E_k are classified according to homotopy classes in $[X, \mathrm{Gr}(k,m;\mathbb{C})]$.

Again, we have natural inclusions $\ldots \subset \mathrm{Gr}(k,m;\mathbb{C}) \subset \mathrm{Gr}(k,m+1;\mathbb{C}) \subset \ldots$, and thus taking the inductive limit we arrive at the classifying space for $U(k)$ bundles:

$$BU(k) \equiv \bigcup_{m=k+n+1}^{\infty} \mathrm{Gr}(k,m;\mathbb{C}) \tag{2.64}$$

such that

$$\mathrm{Vect}_k(X) = \Big[X\,,\,BU(k)\Big] \tag{2.65}$$

and

$$\mathrm{K}'(X) = \mathrm{Vect}(X) = \Big[X\,,\,BU(\infty)\Big], \tag{2.66}$$

with $BU(\infty) = \bigcup_{k=1}^{\infty} BU(k)$. Note that if X is compact, then $\mathrm{K}(X) = H^0(X,\mathbb{Z}) \oplus \mathrm{K}'(X)$ (according to (2.18)) with $H^0(X,\mathbb{Z}) = [X,\mathbb{Z}]$. This implies that

$$\mathrm{K}(X) = \Big[X\,,\,\mathbb{Z}\Big] \oplus \Big[X\,,\,BU(\infty)\Big] = \Big[X\,,\,\mathbb{Z} \times BU(\infty)\Big]. \tag{2.67}$$

It turns out, however, that things simplify somewhat as the rank k is increased. This leads to the notion of *stable range*. Let $k_0 = [(n+1)/2]$. Then for all $k > k_0$ there exists a bundle F_{k_0} of rank k_0 such that $E_k \cong F_{k_0} \oplus I^{k-k_0}$. This means that any vector bundle E_k in the stable range is stably equivalent to some other bundle F_{k_0} of lower rank k_0. Then E_k and F_{k_0} belong to the same stable equivalence class and correspond to exactly the same element of $\mathrm{K}'(X)$, i.e., as far as K-theory is concerned, nothing is gained by considering bundles of very high rank, because once the stable range is reached no new K-theory elements are obtained by increasing the rank k. Notice that, in the stable range, two vector bundles of the same rank are stably equivalent if and only if they are isomorphic. This implies that for all $k > \frac{1}{2}n$, $\mathrm{K}'(X) = \mathrm{Vect}_k(X)$, or $\mathrm{K}'(X) = [X, BU(k)]$. Therefore, whenever X is connected, we have

$$\widetilde{\mathrm{K}}(X) = \Big[X\,,\,BU(k)\Big]. \tag{2.68}$$

Let us consider some simple examples. The case of immediate interest is where $X = \mathbf{S}^n$, for which $\widetilde{\mathrm{K}}(X) = [\mathbf{S}^n, BU(k)] = \pi_n(BU(k))$ for all $k > n/2$. We may cover $\mathbf{S}^n$ with upper and lower hemispheres $\mathbf{S}^n_\pm$. Since the $\mathbf{S}^n_\pm$ are contractible, all bundles $E_k|_{\mathbf{S}^n_\pm}$ are trivial and hence determined by the single $U(k)$-valued transition function g on the overlap $\mathbf{S}^n_+ \cap \mathbf{S}^n_-$. But $\mathbf{S}^n_+ \cap \mathbf{S}^n_- \cong \mathbf{S}^{n-1}$, so g determines a map from $\mathbf{S}^{n-1}$ to $U(k)$, i.e., an element of $\pi_{n-1}(U(k))$. It is this element of $\pi_{n-1}(U(k))$ which determines the bundle $E_k \in \mathrm{Vect}(\mathbf{S}^n)$, and hence an element of $\widetilde{\mathrm{K}}(\mathbf{S}^n)$, so that

$$\widetilde{\mathrm{K}}(\mathbf{S}^n) = \pi_{n-1}\Big(U(k)\Big), \qquad k > n/2. \tag{2.69}$$

In particular, we have

$$\widetilde{\mathrm{K}}(\mathbf{S}^n) = \pi_{n-1}\Big(U(\infty)\Big), \tag{2.70}$$

where $U(\infty) = \bigcup_{k=1}^\infty U(k)$. The homotopy groups of classical Lie groups such as $U(k)$ have been extensively studied. Although $\pi_{n-1}(U(k))$ is not known for all n, k, it is precisely in the stable range $k > n/2$ that we have a complete classification. Note that by (2.68), eq. (2.69) is actually the assertion that $[\mathbf{S}^n, BU(k)] \cong [\mathbf{S}^{n-1}, U(k)]$. This follows from the following facts. First of all,

$$\Big[\mathbf{S}^n\,,\,BU(k)\Big] = \Big[\Sigma\mathbf{S}^{n-1}\,,\,BU(k)\Big] = \Big[\mathbf{S}^{n-1}\,,\,\Omega BU(k)\Big], \tag{2.71}$$

where $\Omega^n Y$ denotes the n-th iterated loop space of the topological space Y. The isomorphism $[\mathbf{S}^{n-1}, U(k)] \cong [\mathbf{S}^{n-1}, \Omega BU(k)]$ now follows from the fact that the space $\Omega BU(k)$ is of the same homotopy type as $U(k)$. This means that the loop space operand Ω may be thought of as a type of homotopic inverse to the classifying space operand B. It is precisely this statement which was the original content of the Bott periodicity theorem for the classical Lie groups [56].

As another example, take $X = \mathbf{B}^n$ and $Y = \mathbf{S}^{n-1} = \partial\mathbf{B}^n$ in $\mathbb{R}^n$. The topological coset $\mathbf{B}^n/\mathbf{S}^{n-1}$ can be identified with $\mathbf{S}^n$, which induces a homeomorphism from $\mathbf{S}^n_+/\mathbf{S}^{n-1} \cong \mathbf{B}^n/\mathbf{S}^{n-1}$ to $\mathbf{S}^n$. It then follows from the excision theorem that

$$\mathrm{K}(\mathbf{B}^n, \mathbf{S}^{n-1}) = \mathrm{K}(\mathbf{B}^n/\mathbf{S}^{n-1}, \mathrm{pt}) = \widetilde{\mathrm{K}}(\mathbf{S}^n) = \pi_{n-1}\Big(U(\infty)\Big). \tag{2.72}$$

For example, $\mathrm{K}(\mathbf{B}^2, \mathbf{S}^1) = \pi_1(U(\infty)) = \pi_1(U(1)) = \mathbb{Z}$.

2.8 Clifford Algebras and the Atiyah-Bott-Shapiro Construction

In this subsection we will discuss the relation of Clifford algebras and spinor representations to K-theory. The analysis of Clifford algebras is very simple and many K-theoretic results become transparent when translated into this algebraic language. Let us start by describing the Clifford algebra $C\ell_{r,s}$ associated with the vector space $V = \mathbb{R}^{r+s}$ and the quadratic form $q(x) = x_1^2 + \ldots + x_r^2 - x_{r+1}^2 - \ldots - x_{r+s}^2$ on V, which is invariant under $O(r,s)$-rotations. There is a natural embedding $V \hookrightarrow C\ell_{r,s}$, and the abstract unital algebra $C\ell_{r,s}$ is generated by any q-orthonormal basis $\Gamma_1, \ldots, \Gamma_{r+s}$ of V subject to the relations

$$\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = \begin{cases} -2\delta_{ij}, & i \leq r \\ +2\delta_{ij}, & i > r. \end{cases} \tag{2.73}$$

The minimal representation of the algebra (2.73) consists of Dirac matrices of dimension $2^{[\frac{r+s}{2}]}$. The reflection map $x \mapsto -x$ for $x \in V$ extends to an automorphism $\eta : C\ell_{r,s} \to C\ell_{r,s}$. Since $\eta^2 = \mathrm{Id}$, this leads to the decomposition

$$C\ell_{r,s} = C\ell_{r,s}^+ \oplus C\ell_{r,s}^-, \tag{2.74}$$

where $C\ell^{\pm}_{r,s} = \{\phi \in C\ell_{r,s} : \eta(\phi) = \pm\phi\}$ are the eigenspaces of η. It follows that

$$C\ell^{\alpha}_{r,s} \cdot C\ell^{\beta}_{r,s} \subset C\ell^{\alpha\beta}_{r,s}, \tag{2.75}$$

where $\alpha, \beta = \pm$. The associated graded algebra of $C\ell_{r,s}$ is then naturally isomorphic to the exterior algebra $\Lambda^* V$, i.e., Clifford multiplication defined by (2.73) is a natural enhancement of exterior multiplication which is determined by the quadratic form q. In fact, there is a canonical vector space isomorphism $\Lambda^* V \xrightarrow{\approx} C\ell_{r,s}$, and hence the natural embeddings $\Lambda^n V \subset C\ell_{r,s}$ for all $n \geq 0$.

The spin group $\mathrm{Spin}\,(r, s) \subset C\ell_{r,s}$, of dimension 2^{r+s}, is obtained from the group of multiplicative units of the Clifford algebra through the embedding $\mathbf{S}^{r+s-1} \subset V \subset C\ell_{r,s}$. It is a double cover of the group $SO(r, s)$, as is expressed by the exact sequence:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}\,(r, s) \longrightarrow SO(r, s) \longrightarrow 1. \tag{2.76}$$

The spin group associated to $C\ell_{n,0}$ is $\mathrm{Spin}\,(n)$ which is a double cover of the isometry group $SO(n)$ of the sphere $\mathbf{S}^{n-1}$. We will use the shorthand notation $C\ell_n \equiv C\ell_{n,0}$ and $C\ell^*_n \equiv C\ell_{0,n}$. As a simple example, $C\ell_1$ is generated by the unit element and an element Γ obeying $\Gamma^2 = -1$, so that $C\ell_1 \cong \mathbb{C}$. Similarly it is easily seen that $C\ell^*_1 \cong \mathbb{R} \oplus \mathbb{R}$.

Under the canonical isomorphism $C\ell_n \cong \Lambda^* \mathbb{R}^n$, Clifford multiplication has a particularly nice form. For $x \in \mathbb{R}^n$, we define the *interior product* $x \neg : \Lambda^p \mathbb{R}^n \to \Lambda^{p-1} \mathbb{R}^n$ by

$$\begin{aligned} &x \neg (x_1 \wedge \cdots \wedge x_p) \\ &= \sum_{m=1}^{p} (-1)^{m+1} \sum_{i=1}^{n} x^i (x_m)^i \, x_1 \wedge \cdots \wedge x_{m-1} \wedge x_{m+1} \wedge \cdots \wedge x_p. \end{aligned} \tag{2.77}$$

This defines a skew-derivation of the algebra since $x \neg (\omega \wedge y) = (x \neg \omega) \wedge y + (-1)^p \omega \wedge (x \neg y)$ for all $\omega \in \Lambda^p \mathbb{R}^n$ and all $y \in \Lambda^q \mathbb{R}^n$. Furthermore, $(x \neg)^2 = 0$ for all $x \in \mathbb{R}^n$, so that the interior product extends universally to a bilinear map $\Lambda^* \mathbb{R}^n \otimes \Lambda^* \mathbb{R}^n \to \Lambda^* \mathbb{R}^n$. It is now elementary to show that the Clifford multiplication between $x \in \mathbb{R}^n$ and $\phi \in C\ell_n$ can be written as

$$x \cdot \phi = x \wedge \phi - x \neg \phi, \tag{2.78}$$

with respect to the canonical isomorphism $C\ell_n \cong \Lambda^*\mathbb{R}^n$.

For any pair of positive integers (r,s) there is an explicit presentation of the algebra $C\ell_{r,s}$ as a matrix algebra over one of the fields $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. The first few examples are easy to construct by hand, for example

$$\begin{aligned} C\ell_{1,0} &= \mathbb{C}, & C\ell_{0,1} &= \mathbb{R}\oplus\mathbb{R} \\ C\ell_{2,0} &= \mathbb{H}, & C\ell_{0,2} &= \mathbb{R}(2) \\ & & C\ell_{1,1} = \mathbb{R}(2), \end{aligned} \tag{2.79}$$

where $\mathbb{F}(m)$ denotes the $\mathbb{R}$-algebra of $m\times m$ matrices with entries in the algebraic field $\mathbb{F}$. The complete classification of Clifford algebras is then obtained by using the periodicity relations (valid for any $n,r,s\geq 0$):

$$C\ell_{n,0}\otimes C\ell_{0,2} = C\ell_{0,n+2}, \tag{2.80}$$
$$C\ell_{0,n}\otimes C\ell_{2,0} = C\ell_{n+2,0}, \tag{2.81}$$
$$C\ell_{r,s}\otimes C\ell_{1,1} = C\ell_{r+1,s+1}, \tag{2.82}$$

and

$$C\ell_{n,0}\otimes C\ell_{8,0} = C\ell_{n+8,0}, \tag{2.83}$$
$$C\ell_{0,n}\otimes C\ell_{0,8} = C\ell_{0,n+8}, \tag{2.84}$$

where

$$C\ell_{8,0} = C\ell_{0,8} = \mathbb{R}(16). \tag{2.85}$$

Using these relations and (2.79) it possible to write down the complete set of Clifford algebras $C\ell_{r,s}$ which are summarized in Table 1. From this table one observes some extra intrinsic symmetries of the Clifford algebras, for example

$$C\ell_{r,s} = C\ell_{r-4,s+4}\ , \tag{2.86}$$
$$C\ell_{r,s+1} = C\ell_{s,r+1}\ , \tag{2.87}$$

which can also be proven directly from the definition of $C\ell_{r,s}$.

We will now describe the complexified Clifford algebras which are related to the K-theory of complex vector bundles over spheres. The complexification of the real Clifford algebra $C\ell_{r,s}$ is the $\mathbb{C}$-algebra $\mathbb{C}\ell_{r,s} =$

s	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$
1	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$
2	$\mathbb{R}(2)$	$\mathbb{R}(2)\oplus\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$
3	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4)\oplus\mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$
5	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)\oplus\mathbb{R}(16)$
6	$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$
7	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8)\oplus\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$
8	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16)\oplus\mathbb{H}(16)$	$\mathbb{H}(32)$

s	$r=5$	$r=6$	$r=7$	$r=8$
0	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$
1	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)\oplus\mathbb{R}(16)$
2	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$
3	$\mathbb{H}(8)$	$\mathbb{H}(8)\oplus\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$
4	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16)\oplus\mathbb{H}(16)$	$\mathbb{H}(32)$
5	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$	$\mathbb{H}(32)\oplus\mathbb{H}(32)$
6	$\mathbb{R}(32)\oplus\mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$	$\mathbb{H}(64)$
7	$\mathbb{R}(64)$	$\mathbb{R}(64)\oplus\mathbb{R}(64)$	$\mathbb{R}(128)$	$\mathbb{C}(128)$
8	$\mathbb{C}(64)$	$\mathbb{R}(128)$	$\mathbb{R}(128)\oplus\mathbb{R}(128)$	$\mathbb{R}(256)$

Table 1: The real Clifford algebras $C\ell_{r,s}$ for $0\leq r,s\leq 8$.

n	1	2	3	4	5	6	7	8
$\mathbb{C}\ell_n$	$\mathbb{C}\oplus\mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2)\oplus\mathbb{C}(2)$	$\mathbb{C}(4)$	$\mathbb{C}(4)\oplus\mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8)\oplus\mathbb{C}(8)$	$\mathbb{C}(16)$

Table 2: The complexified Clifford algebras $\mathbb{C}\ell_n$ for $1 \leq n \leq 8$.

$C\ell_{r,s} \otimes_{\mathbb{R}} \mathbb{C}$, which can also be viewed as the Clifford algebra associated with the vector space $\mathbb{C}^{r+s}$ and the complexification $q \otimes \mathbb{C}$ of the quadratic form q. Since all non-degenerate quadratic forms over $\mathbb{C}$ are equivalent, we have the sequence of isomorphisms

$$\mathbb{C}\ell_n \cong C\ell_{n,0} \otimes_{\mathbb{R}} \mathbb{C} \cong C\ell_{n-1,1} \otimes_{\mathbb{R}} \mathbb{C} \cong \cdots \cong C\ell_{0,n} \otimes_{\mathbb{R}} \mathbb{C} \ , \tag{2.88}$$

which makes the classification of the complexified Clifford algebras much simpler, since it means that $\mathbb{C}\ell_{r,s}$ only depends on the sum of r and s: $C\ell_{r,s} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}\ell_{r+s}$. From this it also follows that the periodicity of $\mathbb{C}\ell_n$ is

$$\mathbb{C}\ell_{n+2} \cong \mathbb{C}\ell_n \otimes_{\mathbb{C}} \mathbb{C}\ell_2 \ , \tag{2.89}$$

where $\mathbb{C}\ell_2 = \mathbb{C}(2)$. (We shall see that this periodicity is related to Bott periodicity of the complex K-theory of spheres.) Using these identities one can easily deduce the list of complexified Clifford algebras in Table 2.

Most of the important applications of Clifford algebras come through a detailed understanding of their representations and, by restriction, of the representation theory of their corresponding spin groups. Such properties follow rather easily from the classification just presented. For any algebraic field $\mathbb{F}$ we define an $\mathbb{F}$-representation of the Clifford algebra to be a homomorphism $\rho : C\ell \longrightarrow \mathrm{End}_{\mathbb{F}}(W)$ into the endomorphism algebra of linear transformations of a finite dimensional vector space W over $\mathbb{F}$ (here $C\ell$ could be either $C\ell_{r,s}$ or $\mathbb{C}\ell_n$). In particular ρ satisfies the property $\rho(\phi\psi) = \rho(\phi) \circ \rho(\psi)$ for all $\phi, \psi \in C\ell$. In this way W becomes a Clifford-module over $\mathbb{F}$. For $\phi \in C\ell$ the action of $\rho(\phi)$ on $w \in W$ is denoted by

$$\rho(\phi)(w) \equiv \phi \cdot w \tag{2.90}$$

and is customarily refered to as Clifford multiplication.

As shown above, the tensor products of irreducible representations of certain Clifford algebras gives another irreducible Clifford module (see e.g., (2.83) and (2.89)). In general, however, $C\ell_n \otimes C\ell_m$ is not a Clifford algebra, and so to find a multiplicative structure in the representations of Clifford algebras it is natural to consider a special class of Clifford modules. For this, we define a $\mathbb{Z}_2$-graded module W for $C\ell_n$ as one with a decomposition $W = W^+ \oplus W^-$ such that

$$C\ell_n^\alpha \cdot W^\beta \subset W^{\alpha\beta} \ , \tag{2.91}$$

where $\alpha, \beta = \pm$. An important grading comes from the chirality grading of the corresponding spin groups. Given a positively oriented, q-orthonormal basis Γ_i of the oriented vector space V, we define an oriented volume element Γ_c of $C\ell_{r,s}$ by the chirality element

$$\Gamma_c = \Gamma_1 \cdots \Gamma_{r+s} \ . \tag{2.92}$$

Setting $n = r + s$, this volume element satisfies

$$\begin{aligned} (\Gamma_c)^2 &= (-1)^{\frac{n(n+1)}{2}+s} \ , \\ x\,\Gamma_c &= (-1)^{n-1}\,\Gamma_c\,x \ , \quad \forall x \in \mathbb{R}^n \ , \end{aligned} \tag{2.93}$$

showing that for n odd, Γ_c lies in the center of $C\ell_{r,s}$, whereas for n even, $\Gamma_i \Gamma_c = \Gamma_c\,\eta(\Gamma_i)$. Therefore, when n is even there is a chirality grading induced by the ± 1 eigenspaces of Γ_c.

Let us start by classifying the representations of the real Clifford algebra $C\ell_{r,s}$. A real representation of this algebra is constructed in the obvious way. A $\mathbb{C}$-representation, on the other hand, is constructed as follows. Recall that a complex vector space is just a real vector space W together with a real linear map $J : W \to W$ such that $J^2 = -\mathrm{Id}$. Then, a complex representation of $C\ell_{r,s}$ is a real representation $\rho : C\ell_{r,s} \to \mathrm{End}_{\mathbb{R}}(W)$ that commutes with the complex structure:

$$\rho(\phi) \circ J = J \circ \rho(\phi) \ . \tag{2.94}$$

Similarly one defines quaternionic representations of $C\ell_{r,s}$. By restriction the representations of the algebras $C\ell_n$ give rise to important representations of the spin group. The real spinor representation of $\mathrm{Spin}(n)$ is defined as a homomorphism

$$\Delta_n : \mathrm{Spin}(n) \longrightarrow GL_{\mathbb{R}}(W) \ , \tag{2.95}$$

given by restricting an irreducible real representation $C\ell_n \to \mathrm{End}_{\mathbb{R}}(W)$ to $\mathrm{Spin}(n) \subset C\ell_n$. It can be shown that when $n \neq 0 \pmod 4$ the representation Δ_n is either irreducible or a direct sum of two equivalent irreducible representations, and that the second possibility occurs exactly when $n \equiv 1$ or $2 \pmod 8$. In the other cases there is a decomposition

$$\Delta_{4m} = \Delta_{4m}^+ \oplus \Delta_{4m}^- \ , \tag{2.96}$$

where $\Delta_{4m}^\pm = \frac{1}{2}(1 \pm \Gamma_c)\Delta_{4m}$ are inequivalent irreducible representations of $\mathrm{Spin}(4m)$. The reality properties of these spinor modules are then easily deduced. The only real spinor modules (or, more precisely, the only ones which are complexifications of real representations) are $\Delta_{8k\pm1}, \Delta_{8k}^\pm$, while the representations $\Delta_{8k+3}, \Delta_{8k+4}^\pm, \Delta_{8k+5}$ are the restrictions of quaternionic Clifford modules. The remaining modules $\Delta_{8k+2}^\pm, \Delta_{8k+6}^\pm$ are complex.

We can similarly classify the representations of the complexified Clifford algebra $\mathbb{C}\ell_n$. We define the complex representation of $\mathrm{Spin}(n)$ to be the homomorphism

$$\Delta_n^{\mathbb{C}} : \mathrm{Spin}(n) \longrightarrow GL_{\mathbb{C}}(W) \ , \tag{2.97}$$

given by restricting an irreducible complex representation $\mathbb{C}\ell_n \to \mathrm{End}_{\mathbb{C}}(W)$ to $\mathrm{Spin}(n) \subset \mathbb{C}\ell_n$. Similarly to the real case, it is possible to show that when n is odd the representation $\Delta_n^{\mathbb{C}}$ is irreducible, whereas when n is even there is a decomposition

$$\Delta_{2m}^{\mathbb{C}} = \Delta_{2m}^{\mathbb{C}+} \oplus \Delta_{2m}^{\mathbb{C}-} \ , \tag{2.98}$$

with $\Delta_{2m}^{\mathbb{C}\pm} = \frac{1}{2}(1 \pm i^m\,\Gamma_c)\Delta_{2m}^{\mathbb{C}}$, into a direct sum of two inequivalent irreducible complex representations of $\mathrm{Spin}(n)$.

We finally come to the connection with K-theory, via the classic Atiyah-Bott-Shapiro (ABS) construction [57] which relates the Grothendieck groups of Clifford modules to the K-theory of spheres. For this we will use the definition introduced in section 2.6 of the relative K-group $\mathrm{K}(X,Y)$ as the group of equivalence classes $[E,F;\alpha]$, where α is an isomorphism of the vector bundles E and F when restricted to Y. Let $\mathrm{R}[\mathrm{Spin}(n)]$ be the complex representation ring of $\mathrm{Spin}(n)$, i.e., the Grothendieck group constructed from the abelian monoid generated by the irreducible complex representations, with respect to the direct sum and tensor product of Spin(n)-modules. (We will describe representation rings in more generality in section 6.1). Let $W = W^+ \oplus W^-$ be a $\mathbb{Z}_2$-graded module over the Clifford algebra $C\ell_n$. We then associate to the graded module W the element

$$\varphi(W) = [E^+, E^-; \mu] \in \mathrm{K}(\mathbf{B}^n, \mathbf{S}^{n-1}) \ , \tag{2.99}$$

where $E^\pm \equiv \mathbf{B}^n \times W^\pm$ is the trivial product bundle, and $\mu : E^+ \to E^-$ is the isomorphism over $\mathbf{S}^{n-1}$ given by Clifford multiplication:

$$\mu(x, w) \equiv (x, x \cdot w) \quad , \quad x \in \mathbf{S}^{n-1} \ . \tag{2.100}$$

Note that, since the ball $\mathbf{B}^n$ is contractible, all bundles over it are trivial and the topology all lies in the winding of the homotopically non-trivial map $\mu : E^+ \to E^-$ over $\mathbf{S}^{n-1}$. It is now straightforward to show that the element $\varphi(W)$ depends only on the isomorphism class of the graded module W, and furthermore that the map $W \mapsto \varphi(W)$ is an additive homomorphism. Thus the map (2.99) gives a homomorphism

$$\varphi : \mathrm{R}\Big[\mathrm{Spin}(n)\Big] \longrightarrow \mathrm{K}(\mathbf{B}^n, \mathbf{S}^{n-1}) \ . \tag{2.101}$$

By restriction, the natural inclusion $i : \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ induces an epimorphism $i^* : \mathrm{R}[\mathrm{Spin}(n+1)] \to \mathrm{R}[\mathrm{Spin}(n)]$. It then follows that the homomorphism (2.101) descends to a homomorphism

$$\varphi_n : \mathrm{R}[\mathrm{Spin}(n)]/i^* \, \mathrm{R}[\mathrm{Spin}(n+1)] \to \mathrm{K}(\mathbf{B}^n, \mathbf{S}^{n-1}),$$

which turns out to be a graded ring isomorphism [57]:

$$\mathrm{R}\Big[\mathrm{Spin}(n)\Big] \, / \, i^* \, \mathrm{R}\Big[\mathrm{Spin}(n+1)\Big] \xrightarrow{\approx} \widetilde{\mathrm{K}}(\mathbf{S}^n). \tag{2.102}$$

The groups in (2.102) are isomorphic to $\mathbb{Z}$ for n even, while they vanish when n is odd according to the above classification of Clifford modules. The isomorphism (2.102) is generated by the principal Spin(n) bundle over $\mathbf{S}^n$:

$$\mathrm{Spin}(n) \hookrightarrow \mathrm{Spin}(n+1) \longrightarrow \mathbf{S}^n. \tag{2.103}$$

This theorem also gives us explicit generators for $\widetilde{\mathrm{K}}(\mathbf{S}^{2n})$ defined via representations of Clifford algebras. For example, let $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ be the fundamental $\mathbb{Z}_2$-graded representation space for $\mathbb{C}\ell_{2n}$. There is an isomorphism $\mathrm{R}[\mathrm{Spin}(2n)] \cong \mathbb{Z} \oplus \mathbb{Z}$ with generators given by $\mathcal{S}$ and its "flip" $\widetilde{\mathcal{S}}$, the same graded module with the factors interchanged (this correponds to a reversal of the orientation in $\mathbb{R}^{2n}$). The generator of $i^* \, \mathrm{R}[\mathrm{Spin}(2n+1)] \cong \mathbb{Z}_{\mathrm{diag}}$ is then $[\mathcal{S}] + [\widetilde{\mathcal{S}}]$. Thus the group $\widetilde{\mathrm{K}}(\mathbf{S}^{2n}) = \mathbb{Z}$ is generated by the element

$$\varphi_{\mathbb{C}}^{(n)} = [\mathcal{S}^+, \mathcal{S}^-; \mu], \tag{2.104}$$

where $\mu_x : \mathcal{S}^+ \to \mathcal{S}^-$ denotes Clifford multiplication by $x \in \mathbb{R}^{2n}$. Denoting the generators of $\mathcal{S}$ by Γ_i, the inclusion $\mathbb{R}^{2n} \hookrightarrow \mathbb{C}\ell_{2n}$ along with the definition (2.90) of Clifford multiplication shows that μ_x can be represented via ordinary matrix multiplication by $x = \sum_i x^i \, \Gamma_i \in \mathbb{R}^{2n}$:

$$\mu_x(w) = \left(\sum_{i=1}^{2n} x^i \, \Gamma_i \right) w \quad , \quad x^i \in \mathbb{R}. \tag{2.105}$$

Moreover, from (2.78) it follows that the square of the isomorphism μ_x is just multiplication by the norm of the vector $x \in \mathbb{R}^{2n}$:

$$\mu_x \circ \mu_x(w) = -|x|^2 \, w. \tag{2.106}$$

Note that the Bott periodicity of spheres, $\widetilde{\mathrm{K}}(\mathbf{S}^n) = \widetilde{\mathrm{K}}(\mathbf{S}^{n+2})$, can now be derived from the periodicity property (2.89) of complexified Clifford algebras. Furthermore, using the structure of Clifford modules it is straightforward to show using the cup product that

$$\varphi_{\mathbb{C}}^{(n)} = \left(\varphi_{\mathbb{C}}^{(1)} \right)^n . \tag{2.107}$$

3 Type IIB D-Branes and $\mathrm{K}(X)$

We will now begin describing the systematic applications of K-theory to the classification of D-brane charges in superstring theory. We start in this section by considering the Type IIB theory, for which the simplest analysis can be carried out. Type II superstrings are oriented and therefore have Chan-Paton bundles with unitary structure groups. Except for the new ways of thinking about and constructing D-branes, the K-theory formalism merely reproduces the known spectrum of stable brane charges. However, the analysis we present in the following easily generalizes to more complicated situations where we will see that K-theory makes genuinely new predictions, and it moreover provides a nice consistency check that the mathematical formalism is indeed the correct one.

We will show in this section that the group $\mathrm{K}(X)$ classifies D-branes in Type IIB superstring theory on the spacetime manifold X [17]. More precisely, the RR-charge of a Type IIB D-brane is measured by the K-theory class of its transverse space, so that $\widetilde{\mathrm{K}}(\mathbf{S}^n)$ classifies $(9-n)$-branes in Type IIB string theory on flat $\mathbb{R}^{10}$, for example. The corresponding K-groups are determined by homotopy theory as described in section 2.7:

$$\widetilde{\mathrm{K}}(\mathbf{S}^n) = \pi_{n-1}\Big(U(k)\Big), \qquad k > n/2. \tag{3.1}$$

Taking the inductive limit one has

$$\widetilde{\mathrm{K}}(\mathbf{S}^n) = \pi_{n-1}\Big(U(\infty)\Big), \tag{3.2}$$

where $U(\infty) = \bigcup_k U(k)$ is the infinite unitary group. Bott periodicity states that the corresponding homotopy groups $\pi_n(U(\infty))$ are periodic with period *two*:

$$\pi_n\Big(U(\infty)\Big) = \pi_{n+2}\Big(U(\infty)\Big), \tag{3.3}$$

or

$$\widetilde{\mathrm{K}}(\mathbf{S}^n) = \widetilde{\mathrm{K}}(\mathbf{S}^{n+2}). \tag{3.4}$$

From this and the fact that $\widetilde{\mathrm{K}}(\mathbf{S}^0) = \mathbb{Z}, \widetilde{\mathrm{K}}(\mathbf{S}^1) = 0$ follows the complete classification of D-branes in Type IIB superstring theory, which is summarized in table 3. This table just reflects the fact that the Type IIB theory has stable Dp-branes only for p odd. In this way one recovers the usual spectrum of IIB BPS brane charges.

3.1 The Brane-Antibrane System

The physics behind the K-theory description of D-brane charges hinges on a new interpretation of branes in terms of higher-dimensional branes

D-brane	D9	D8	D7	D6	D5	D4	D3	D2	D1	D0	D(−1)
Transverse space	$\mathbf{S}^0$	$\mathbf{S}^1$	$\mathbf{S}^2$	$\mathbf{S}^3$	$\mathbf{S}^4$	$\mathbf{S}^5$	$\mathbf{S}^6$	$\mathbf{S}^7$	$\mathbf{S}^8$	$\mathbf{S}^9$	$\mathbf{S}^{10}$
$\widetilde{\mathrm{K}}(\mathbf{S}^n)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

Table 3: D-brane spectrum in Type IIB superstring theory from $\widetilde{\mathrm{K}}(\mathbf{S}^n)$.

and antibranes. We shall therefore start by briefly reviewing the properties of brane-antibrane pairs in superstring theory. This system is unstable due to the presence of a tachyonic mode in the open string excitations that start on the brane (respectively antibrane) and end on the antibrane (respectively brane) [14]. The simplest way to see this property is by appealing to the boundary state formalism (see [58] and references therein). A stable supersymmetric Dp-brane can be represented and described by a boundary state

$$|Dp\rangle = |Dp\rangle_{\rm NS} \pm |Dp\rangle_{\rm R}, \tag{3.5}$$

which is a particular coherent state in the Hilbert space of the *closed* string theory. It represents a source for the closed string modes emitted by a Dp-brane. The boundary state (3.5) consists of a part $|Dp\rangle_{\rm NS}$ which is a source for the closed string states of the NS-NS sector of the fundamental string worldsheet, and a piece $|Dp\rangle_{\rm R}$ for the RR sector. The relative sign in (3.5) distinguishes a brane from its antibrane which has opposite RR charges. Taking into account the closed string GSO projection gives the decompositions

$$\begin{aligned} |Dp\rangle_{\rm NS} &= \tfrac{1}{2}\Big(|Dp,+\rangle_{\rm NS} - |Dp,-\rangle_{\rm NS}\Big), \\ |Dp\rangle_{\rm R} &= \tfrac{1}{2}\Big(|Dp,+\rangle_{\rm R} + |Dp,-\rangle_{\rm R}\Big), \end{aligned} \tag{3.6}$$

where the $\pm$ label the two possible implementations of the boundary conditions appropriate for a Dp-brane. The decompositions (3.6) take into account the sum over the four spin structures on the string worldsheet.

The boundary state formalism allows one to easily compute the spectrum of open strings which begin and end on a Dp-brane. This can be found by computing a tree-level two-point function of the boundary state with itself (the cylinder amplitude) and via a modular transformation re-expressing the result as a one-loop trace over open string states (the annulus amplitude) according to

$$\int_0^\infty d\tau \, \left\langle Dp,\alpha \left| \mathrm{e}^{-\pi\tau(L_0+\tilde{L}_0)} \right| Dp,\beta \right\rangle = V_p \int_0^\infty \frac{dt}{t} \, \mathrm{Tr}_{\rm open} \left(\mathrm{e}^{-2\pi t L_0} \right) , \quad (3.7)$$

where V_p is the (infinite) worldvolume and $\alpha,\beta = \pm$. The open string sectors which appear in (3.7) depend both on the closed string sectors and on the spin structures α,β. Of particular importance are the open string NS and NS$(-1)^F$ even spin structures, which correspond respectively to the closed string NS-NS and RR sectors with $\alpha=\beta$. The NS sector is thereby GSO projected in the usual way as NS-NS+RR=NS+NS$(-1)^F$. This leads to the well-known fact that the sum over the contributions from all even spin structures vanishes, and thus the spectrum of open strings which start and end on a Dp-brane is supersymmetric and free from tachyons.

However, if one considers instead a system composed of one Dp-brane and one anti-Dp-brane (which we will also call a D$\overline{p}$-brane), then the contribution to (3.7) from the RR sector changes sign, and the NS open string sector has the "wrong" GSO projection, NS-NS – RR=NS–NS$(-1)^F$. The open string spectrum therefore exhibits a tachyon, and this fact is responsible for the instability of the brane-antibrane pair. This feature is a consequence of the fact that the system sits at the top of a potential well, and it is precisely the presence of this tachyon field in a $p-\overline{p}$ system that makes the connection to K-theory.

We may choose a suitable basis of the open string Chan-Paton gauge group $U(2)$ of the brane-antibrane pair in which diagonal matrices represent the open string excitations which start and end on the same brane or antibrane, while off-diagonal matrices represent the string states which stretch between the brane and its antibrane of a given orientation. The tachyon vertex operators which create the appropriate $p-\overline{p}$ tachyonic open string states are therefore given by

$$\begin{aligned} V_T(z) &= \mathrm{e}^{ik_aX^a(z)} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \\ V_{T^\dagger}(z) &= \mathrm{e}^{ik_aX^a(z)} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \end{aligned} \quad (3.8)$$

where $X^a(z)$ are worldsheet boson fields and k_a is the momentum along the Dp-brane worldvolume. From the structure of the Chan-Paton matrices in (3.8), it is straightforward to see that the only non-vanishing correlation functions are those involving an equal number of T and $T^\dagger$ vertex operators. If $T(x)$ and $T^\dagger(x)$ denote the complex tachyon fields living on the worldvolume of the Dp-brane anti-Dp-brane system, then there is a tachyon potential of the form

$$V(TT^\dagger) = \sum_{n=2}^{\infty} c_n \left(TT^\dagger\right)^n . \tag{3.9}$$

This implies that the tachyon potential depends only on the modulus of T,

$$V(T) = V(|T|^2). \tag{3.10}$$

The presence of a non-trivial tachyon potential $V(T)$ implies that a stable configuration cannot be reached by simply superimposing a Dp-brane and a D$\overline{p}$-brane, since the system is sitting on top of a tachyon well. The lowest energy configuration (i.e., the stable configuration) of the system is obtained by allowing the tachyon to roll down to the minimum T_0 of its potential. From (3.10) it follows that these points live on a circle described by the equation

$$|T| = T_0. \tag{3.11}$$

Note that in terms of the *real* tachyon field $t = T + T^\dagger$, the tachyon potential is an even function of t,

$$V(t) = V(-t), \tag{3.12}$$

and the corresponding minima always come in pairs $\pm\, t_0$.

Furthermore, one may argue that, when the tachyon condenses into one of its vacuum expectation values, the negative potential energy density of the condensate cancels exactly with the positive energy density associated with the tension of the $p-\overline{p}$ pair:

$$2\mathcal{T}_p + V(T_0) = 0, \tag{3.13}$$

where $\mathcal{T}_p$ is the p-brane tension. This shows that the tachyon ground state is indistinguishable from the supersymmetric vacuum configuration, since it carries neither any charge nor any energy. Thus, under these circumstances, the stable configuration of a brane-antibrane pair which is reached by tachyon condensation is nothing but the vacuum state. However, instead of considering the tachyon ground state, one can also construct tachyonic soliton solutions on the brane-antibrane worldvolume. This will be done in the next subsection, where we will see that one of the astonishing features of the K-theory formalism is that is provides a very explicit form for the classical tachyonic soliton field $T(x)$.

The reversal of the GSO projection described above may be formalised as follows. The endpoints of the open string excitations of the $p-\overline{p}$ pair carry a charge which takes values in a two-dimensional quantum Hilbert space. The first component of such a wavefunction may be regarded as bosonic and representing, say, the open strings which end on the p-brane, while the second component is fermionic and represents the open strings which end on the $\overline{p}$-brane. The $p-\overline{p}$ open strings have off-diagonal Chan-Paton wavefunctions which are odd under the adjoint action of the operator

$$(-1)^F = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.14}$$

and are thereby removed by the GSO projection operator

$$P_{\rm GSO} = \tfrac{1}{2}\Bigl(1 + (-1)^F\Bigr). \tag{3.15}$$

On the other hand, the $p-p$ and $\overline{p}-\overline{p}$ open strings have diagonal Chan-Paton wavefunctions. They are even under $(-1)^F$ and are therefore selected by the GSO projection (3.15). Having one bosonic and one fermionic Chan-Paton state leads to a $p-\overline{p}$ worldvolume gauge symmetry with gauge supergroup $U(1|1)$. However, because of the GSO projection, the off-diagonal fermionic gauge fields of $U(1|1)$ are absent, leading to the usual elimination of the massless vector multiplet. The remaining bosonic fields on the $p-\overline{p}$ worldvolume form instead a structure whose lowest modes correspond to the superconnection [59]

$$\mathcal{A} = \begin{pmatrix} A^+ & T \\ T^\dagger & A^- \end{pmatrix} \tag{3.16}$$

on X, where $A^\pm$ are the gauge fields on the bundles E and F of the bosonic and fermionic Chan-Paton states of the p-brane and $\overline{p}$-brane, respectively. The $p-\overline{p}$ tachyon field T is regarded as a map $T : E \to F$, while its adjoint $T^\dagger$ is a map $T^\dagger : F \to E$. Alternatively, T may be regarded as a section of $E \otimes F^*$ and $T^\dagger$ of $E^* \otimes F$, where $E^* = \mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C})$ is the dual vector bundle to E. The superconnection (3.16) has been used recently in [60] for a generalization, to the brane-antibrane system, of the usual Wess-Zumino couplings of RR fields to worldvolume gauge fields (see also [61]). It will play a crucial role in section 7 when we discuss index theory.

3.2 The Bound State Construction

We will now discuss how to construct tachyonic soliton solutions and show that this construction is equivalent to the ABS homomorphism which maps classes in $\mathrm{K}(Y)$ to classes in $\mathrm{K}(X)$, where the D-branes wrap around a submanifold Y of the spacetime X. Until section 7 we shall deal only with flat spacetimes and topologically trivial worldvolume embeddings $Y \hookrightarrow X$. We will start by constructing a stable p-brane in Type IIB superstring theory as the bound state of a $(p+2)$-brane and a coincident $\overline{(p+2)}$-brane. For this, we shall consider an infinite $(p+2)$ brane-antibrane pair stretching over a submanifold $\mathbb{R}^{p+3} \subset X$. Due to the tachyon, this system tends to annihilate itself unless there is some topological obstruction. This obstruction is measured by the K-theory group $\mathrm{K}(X)$.

On the $(p+2) - \overline{(p+2)}$ pair, there is a $U(1) \times U(1)$ gauge field (A^+, A^-) and a tachyon field T of corresponding charges $(1, -1)$. This means that the kinetic energy term for the tachyon field in the worldvolume field theory is of the form $|(\partial_i - iA_i^+ + iA_i^-)T|^2$. We consider a vortex in which T vanishes on a codimension two submanifold $\mathbb{R}^{p+1} \subset \mathbb{R}^{p+3}$, which we interpret as the p-brane worldvolume. We suppose that $|T(x)|$ approaches its vacuum expectation value T_0 at $|x| \to \infty$ (up to a gauge transformation). T is a complex scalar field, so it can have a winding number around the codimension 2 locus where it vanishes, or equivalently at $|x| = \infty$. The basic case is where the winding number is 1, and T breaks the $U(1) \times U(1)$ gauge symmetry of the brane-antibrane pair down to the diagonal $U(1)$ subgroup. To keep the energy per unit p-brane worldvolume finite (i.e., to have finite tension), there is a unit of magnetic flux in the broken $U(1)$ group, which is achieved by giving the gauge field $A^+ - A^-$ on the worldvolume of the $(p+2) - \overline{(p+2)}$ pair a unit of topological charge at infinity. The non-vanishing asymptotic field configuration therefore takes the form

$$T \simeq T_0 \, \mathrm{e}^{i\theta}, \quad A_\theta^+ - A_\theta^- \simeq 1 \qquad \text{for } \ r \to \infty, \tag{3.17}$$

where (r, θ) are polar coordinates on the two-dimensional transverse space $\mathbb{R}^{p+3} - \mathbb{R}^{p+1}$. Then both the kinetic and potential energy terms in the worldvolume field theory vanish sufficiently fast as $r \to \infty$, leading to a static finite energy vortex configuration for the tachyon field. This system has one unit of p-brane charge, but its $(p+2)$-brane charge is zero between the brane and antibrane. With T approaching its vacuum expectation value everywhere except close to the core $\mathbb{R}^{p+1}$ of the vortex, the system looks like the vacuum everywhere except very close to the locus where T vanishes. This soliton thereby describes a stable, finite energy p-brane in Type IIB string theory. By studying the boundary conformal field theory describing this solution, one can prove that this soliton is indistinguishable from the Dp-brane of Type IIB superstring theory and is simply a different representation of the same topological defect in the spacetime X.

One can easily generalize this construction to a $(p + 2k)$ brane-antibrane pair for $k > 1$. First we construct a p-brane from a $(p + 2)$ brane-antibrane pair, then we construct the $(p+2)$ brane and antibrane each as a bound state of a $(p + 4)$ brane-antibrane pair, and so on. After $k - 2$ more steps, we get a p-brane built from 2^{k-1} pairs of $(p + 2k)$-branes and antibranes. However, such a "stepwise" bound state construction breaks the manifest spacetime symmetries and limits the possible applications of this formalism. A more direct construction exhibiting the full symmetries of the system is desired. This is precisely where the formalism of K-theory plays a central role.

To relate these constructions to K-theory, we recall from section 1 that the D-brane charges of tadpole anomaly cancelling Type IIB $9-\overline{9}$-brane configurations are classified by the reduced K-theory group $\widetilde{\mathrm{K}}(X)$ of the spacetime X. Each class in $\widetilde{\mathrm{K}}(X)$ is represented by an equal number N of 9-branes and $\overline{9}$-branes wrapping X, with the class in $\mathrm{K}(X)$ given by the difference $[E]-[F]$ of the Chan-Paton gauge bundles on the 9-branes and $\overline{9}$-branes. Open strings ending on all possible pairs of these branes give rise to a $U(N) \times U(N)$ gauge field, and a tachyon field T in the bifundamental $\mathbf{N} \otimes \overline{\mathbf{N}}$ representation of the gauge group. Although we don't know the precise form of the tachyon potential, we may argue that at the minima $|T| = T_0$ all eigenvalues of T_0 are equal. This follows from the possibility of separating the brane-antibrane pairs. It then follows that the tachyon condensate T_0 breaks the worldvolume gauge symmetry from $U(N) \times U(N)$ down to the diagonal $U(N)$ subgroup.

We will now construct a stable D-brane of the Type IIB theory as a bound state of a system of N 9-branes and N $\overline{9}$-branes which locally near Y resembles a topologically stable vortex of the tachyon field. The number N will be fixed below by the mathematics of the ABS construction. The stable values of T_0 (i.e., the gauge orbits of the tachyon field with minimum energy) live in the vacuum manifold

$$\mathcal{V}_{\rm IIB}(N) = \frac{U(N) \times U(N)}{U(N)_{\rm diag}} \cong U(N). \tag{3.18}$$

Therefore, when viewed as a Higgs field in this description, T supports stable topological defects in codimension $2k$ which are classified by the non-trivial homotopy groups of the vacuum manifold:

$$\pi_{2k-1}\Big(\mathcal{V}_{\rm IIB}(N)\Big) = \pi_{2k-1}\Big(U(N)\Big) = \mathbb{Z}, \qquad N > k. \tag{3.19}$$

So for a p-brane wrapping a submanifold $\mathbb{R}^{p+1} \subset X$, we take $T(x)$ to vanish in codimension $2k = 9 - p$, and let it approach its vacuum orbit at $|x| \to \infty$, with a non-zero topological twist around the locus $\mathbb{R}^{p+1}$ on which it vanishes, and of a given winding number at infinity. These configurations are classified topologically by the homotopy classes of maps $\mathbf{S}^{2k-1} \to U(N)$, or by the K-theory classes

$$\widetilde{\rm K}(\mathbf{S}^{2k}) = \pi_{2k-1}\Big(U(N)\Big) = \mathbb{Z}, \qquad \forall N > k. \tag{3.20}$$

Note that D-brane charges are labelled by reduced K-groups of the transverse spaces to the worldvolumes (compactified by adding a point at infinity). This result makes manifest the relation between homotopy theory (i.e., the classification of topological defects) and K-theory (i.e., the classification of configurations of spacetime-filling branes up to pair creation and annihilation). As discussed in the previous subsection, the negative energy density corresponding to the vacuum condensate of T is equal in magnitude to the positive energy density due to the non-zero tension of the $9 - \overline{9}$ brane system wrapping X. This implies that the total energy density away from the core of the bound state approaches zero rapidly, and the configuration is very close to the supersymmetric vacuum. Therefore, tachyon condensation leaves behind an object wrapped on $Y = \mathbb{R}^{p+1}$ that carries the charge of a supersymmetric Dp-brane wrapping Y.

The embedding $\mathrm{K}(Y) \hookrightarrow \mathrm{K}(X)$ is realized mathematically by the K-theoretic ABS construction that was described in section 2.8. (This is an example of a push-forward map that we will return to in section 7). It corresponds to the mapping of a non-trivial class describing a D-brane wrapping Y into a class where it corresponds to the bound state of a 9-brane $\overline{9}$-brane configuration wrapping the spacetime X. For Y of codimension $2k$ in X, this construction selects the prefered value $N = 2^{k-1}$ of the number of $9-\overline{9}$-brane pairs (recall that this was precisely the prediction of the previous "stepwise" construction), and it moreover gives a particularly simple, natural and useful representation of the tachyon vortex configuration (i.e., of the generator of $\pi_{2k-1}(U(N))$) via Clifford multiplication. Consider the rotation group $SO(2k)$ of the transverse space, which is the group of orientation-preserving automorphisms of the normal bundle of $Y \subset X$. It has two inequivalent positive and negative chirality complex spinor representations $\Delta_{2k}^{\mathbb{C}\pm}$ of dimension 2^{k-1}. They give rise to two spin bundles $\mathcal{S}^{\pm} \to Y$, which can be extended to a neighbourhood of Y in X (modulo some global obstructions, as we will describe in section 7). They therefore define a K-theory class $[\mathcal{S}^+] - [\mathcal{S}^-] \in \widetilde{\mathrm{K}}(X)$, where $E = \mathcal{S}^+$ is the Chan-Paton bundle carried by the 9-branes and $F = \mathcal{S}^-$ by the $\overline{9}$-branes in the above bound state construction.

The gauge symmetry of the 9-brane worldvolume X is $U(2^{k-1}) \times U(2^{k-1})$, and the tachyon field is a map $T : \mathcal{S}^+ \to \mathcal{S}^-$. Let $\Gamma_1, \dots , \Gamma_{2k}$ be the generators of $\Delta_{2k}^{\mathbb{C}+} \oplus \Delta_{2k}^{\mathbb{C}-}$, which can be regarded as maps $\mathcal{S}^+ \oplus \mathcal{S}^- \to \mathcal{S}^+ \oplus \mathcal{S}^-$. Let $(x^1, \dots , x^{2k}) \in \mathrm{S}^{2k-1} \subset \mathbb{R}^{2k}$. Then, using the construction of section 2.8, we define the tachyon field via Clifford multiplication

$$T(x) = f(x)\, \mu_x = f(x) \sum_{i=1}^{2k} \Gamma_i\, x^i, \tag{3.21}$$

where $f(x)$ is a convergence factor with the asymptotic behaviours

$$\lim_{x \in Y} f(x) = \text{const.}, \qquad \lim_{|x| \to \infty} f(x) = \frac{T_0}{|x|}, \tag{3.22}$$

which ensures that far away from the core of the vortex, $T(x)$ takes values in the IIB vacuum manifold (3.18), whereas the tachyonic soliton is located on the submanifold $x^i = 0, i = 1, \dots, 2k$. In the sequel we shall usually not write such convergence factors explicitly. The field $T(x)$ has winding number 1 [62], and according to the ABS construction, it generates $\pi_{2k-1}(U(2^{k-1})) = \mathbb{Z}$, or equivalently $\mathrm{K}(\mathbf{B}^{2k}, \mathbf{S}^{2k-1}) = \mathbb{Z}$. The precise mapping $\mathrm{K}(Y) \hookrightarrow \mathrm{K}(X)$ of K-theory classes is given by the cup product (2.60):

$$\begin{aligned} \lambda \,:\, \widetilde{\mathrm{K}}(Y) \otimes_{\mathbb{Z}} \mathrm{K}(\mathbf{B}^{2k}, \mathbf{S}^{2k-1}) &\xrightarrow{\approx} \mathrm{K}(Y \times \mathbf{B}^{2k}, Y \times \mathbf{S}^{2k-1}) \\ \Big[(E\,,\,F)\Big] &\longmapsto \lambda\left[(E \otimes \mathcal{S}^+ \oplus F \otimes \mathcal{S}^-\,,\, E \otimes \mathcal{S}^- \oplus F \otimes \mathcal{S}^+)\right] \end{aligned} \tag{3.23}$$

where $[(E, F)] \in \widetilde{\mathrm{K}}(Y)$ and we have used the fact that $\mathrm{K}(\mathbf{S}^m)$ for any m is a free abelian group.

One can verify that this "all at once" construction is equivalent to the previous "stepwise" construction. This fact follows from the periodicity property (2.89) of the complexified Clifford algebras, or equivalently from Bott periodicity of complex K-theory. Namely, the process of tachyon condensation of the bound state of a p-brane $\overline{p}$-brane pair into a $p-2$-brane may be regarded as the Bott periodicity isomorphism on the spacetime K-theory group $\widetilde{\mathrm{K}}(X) \to \widetilde{\mathrm{K}}(X)$, which can in turn be described by the ABS map

$$\begin{aligned} \left[(\mathcal{S}^+_{2k}\,,\,\mathcal{S}^-_{2k})\right] &\longmapsto \left[(\mathcal{S}^+_{2k} \otimes (\mathcal{S}^+_2 \oplus \mathcal{S}^-_2)\,,\, \mathcal{S}^-_{2k} \otimes (\mathcal{S}^+_2 \oplus \mathcal{S}^-_2))\right] \\ T_{2k} &\longmapsto \begin{pmatrix} T_{2k} \otimes \mathbb{I}_2 & \mathbb{I}_{2k} \otimes T_2^\dagger \\ \mathbb{I}_{2k} \otimes T_2 & -T_{2k}^\dagger \otimes \mathbb{I}_2 \end{pmatrix}, \end{aligned} \tag{3.24}$$

where $\mathbb{I}_N$ denotes the $N \times N$ identity matrix, and $[\mathcal{S}^+_{2k}, \mathcal{S}^-_{2k}; T_{2k}]$ is the generator of $\widetilde{\mathrm{K}}(X)$ above. Here

$$T_2(x) = \sigma_1\, x^1 + \sigma_2\, x^2 = \begin{pmatrix} 0 & x^1 + ix^2 \\ x^1 - ix^2 & 0 \end{pmatrix} \tag{3.25}$$

is the codimension 2 tachyon field which generates the stable homotopy groups of the vacuum manifold

$$\mathcal{V}_{\rm IIB}(1) = \frac{U(1) \times U(1)}{U(1)_{\rm diag}} \cong U(1), \tag{3.26}$$

and σ_i will always denote the standard $SU(2)$ Pauli spin matrices. Alternatively, as shown at the end of section 2.5, the $p-2$-brane may be identified with a Dirac magnetic monopole vortex [55] in the $p-\overline{p}$-brane worldvolume. This identification is consistent with the topological stability $\pi_1(U(1)) = \mathbb{Z}$ of the worldvolume soliton. Moreover, it identifies the explicit form of the gauge field configuration in the worldvolume field theory as [63]

$$A_\phi^\pm = 0, \qquad A_\theta^\pm = \pm\,\frac{1 \mp \cos\phi}{\sin\phi} \tag{3.27}$$

where (θ, ϕ) are angular coordinates on the transverse space $\mathbf{S}^2$, and the brane-antibrane indices $\pm$ now label the corresponding upper and lower hemispheres $\mathbf{S}^2_\pm$.

Thus, the p-brane charge of the above configuration equals one, while all higher and lower dimensional charges vanish (this can also be verified by using formulas for brane charges induced by gauge fields). Notice that $T : E \to F$ is trivial at infinity (where the system resembles the vacuum), and is an isomorphism $E \cong F$ in a neighbourhood of infinity. This means that the K-theory class $[(E, F)]$ is assumed to be equivalent to the vacuum at infinity, i.e., that one can relax to the vacuum by tachyon condensation at infinity. Thus the RR charge of an excitation of a given supersymmetric vacuum configuration is best measured by subtracting from its K-theory class the K-theory class of the vacuum. The RR charge of an excitation of the vacuum therefore takes values in K-theory with compact support.

4 Type IIA D-Branes and $\mathrm{K}^{-1}(X)$

In this section we will show that D-brane charges in Type IIA superstring theory are classified by the higher K-group $\mathrm{K}^{-1}(X)$ [17, 18]. Again we shall simply reproduce the well-known spectrum of the Type IIA theory, but we shall gain many new insights into the constructions of D-branes as well as the interrelationships between branes in the Type II theories. Furthermore, we shall uncover some remarkable applications of the bound state construction.

4.1 The Group $\mathrm{K}^{-1}(X)$

There are a number of equivalent definitions of $\mathrm{K}^{-1}(X)$, each of which are useful in different situations. The 11-dimensional "M-theory" definition was given in sections 2.4 and 2.5. In this definition, the higher K-group $\mathrm{K}^{-1}(X)$ is the subgroup of $\widetilde{\mathrm{K}}(X \times \mathbf{S}^1)$ which classifies RR charges in Type IIA string theory on the ten-dimensional spacetime X. It is therefore tempting to interpret the $\mathbf{S}^1$ here as the compactification circle used in relating 11-dimensional M-theory and ten-dimensional Type IIA superstring theory. However, there are no spacetime-filling M-branes, i.e., no known 10-branes, and also no hierarchy of branes in M-theory. So at present it is unclear how to interpret the eleven-dimensional extension of X required to classify D-brane charges of Type IIA string theory. It is for these physical reasons that alternative formulations of the group $\mathrm{K}^{-1}(X)$ are desired.

The "string theory" definition of $\mathrm{K}^{-1}(X)$, i.e., with no reference to an 11-dimensional extension of X, is similar to the definition of relative K-theory introduced in section 2.6 and can be given as follows. Let $E \in \mathrm{Vect}(X)$ and let $\alpha : E \xrightarrow{\approx} E$ be an automorphism of the vector bundle E. Two pairs (E, α) and (F, β) are called isomorphic if there exists an isomorphism of vector bundles $h : E \xrightarrow{\approx} F$ such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{h} & F \\ \alpha \downarrow & & \downarrow \beta \\ E & \xrightarrow{h} & F \end{array} \tag{4.1}$$

i.e., $\beta \circ h = h \circ \alpha$. Define the sum of two pairs (E, α) and (F, β) by $(E \oplus F, \alpha \oplus \beta)$. A pair (E, α) is called elementary if α is homotopic to Id_E within the automorphisms of E. Two pairs (E, α) and (F, β) are equivalent, $(E, \alpha) \sim (F, \beta)$, if there exists two elementary pairs (G, γ) and (H, δ) such that

$$(E \oplus G, \alpha \oplus \gamma) \cong (F \oplus H, \beta \oplus \delta). \tag{4.2}$$

The set of equivalence classes of pairs $[(E, \alpha)]$ defines an abelian group, which is precisely $\mathrm{K}^{-1}(X)$. The inverse of a class $[(E, \alpha)]$ is $-[(E, \alpha)] =$

$[(E,\alpha^{-1})]$. To prove this, we need to show that $(E\oplus E,\alpha\oplus\alpha^{-1})$ is an elementary pair, where we may write

$$\alpha\oplus\alpha^{-1}=\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}=\begin{pmatrix}0 & -\alpha\\ \alpha^{-1} & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{4.3}$$

Using the decomposition

$$\begin{pmatrix}0 & -\alpha\\ \alpha^{-1} & 0\end{pmatrix}=\begin{pmatrix}1 & -\alpha\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \alpha^{-1} & 1\end{pmatrix}\begin{pmatrix}1 & -\alpha\\ 0 & 1\end{pmatrix}, \tag{4.4}$$

we may define a continuous map $\sigma:[0,1]\to\mathrm{Aut}(E\oplus E)$ by

$$\sigma(t)=\begin{pmatrix}1 & -t\alpha\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ t\alpha^{-1} & 1\end{pmatrix}\begin{pmatrix}1 & -t\alpha\\ 0 & 1\end{pmatrix} \tag{4.5}$$

with $\sigma(0)=\mathrm{Id}_{E\oplus E}$ and $\sigma(1)$ coinciding with (4.4). It follows that (4.4) is homotopic to $\mathrm{Id}_{E\oplus E}$ within automorphisms of $E\oplus E$, and hence so is $\alpha\oplus\alpha^{-1}$. More generally, it can be shown that $[(E,\alpha)]+[(E,\beta)]=[(E,\alpha\circ\beta)]=[(E,\beta\circ\alpha)]$. Note that the analogous statement for relative K-theory in section 2.6 can be proven in a similar way.

To show that this abelian group is indeed $\mathrm{K}^{-1}(X)$, we need only prove that two automorphisms α,β determine the same class in $\mathrm{K}^{-1}(X)$, i.e., $[(E,\alpha)]=[(F,\beta)]$, if and only if there exists a vector bundle $G\in\mathrm{Vect}(X)$ such that $\alpha\oplus\mathrm{Id}_F\oplus\mathrm{Id}_G$ and $\mathrm{Id}_E\oplus\beta\oplus\mathrm{Id}_G$ are homotopic within the automorphisms of $E\oplus F\oplus G$. For this, we first demonstrate that $[(E,\alpha)]=0$ if and only if there exists a vector bundle $G\in\mathrm{Vect}(X)$ such that $\alpha\oplus\mathrm{Id}_G$ is homotopic to $\mathrm{Id}_{E\oplus G}$ within the automorphisms of $E\oplus G$. Indeed, if $[(E,\alpha)]=0$ then there exists elementary pairs (G,γ) and (G',γ') and an isomorphism $h:E\oplus G\xrightarrow{\approx}G'$ such that the diagram

$$\begin{array}{ccc} E\oplus G & \xrightarrow{h} & G' \\ {\scriptstyle \alpha\oplus\gamma}\downarrow & & \downarrow{\scriptstyle \gamma'} \\ E\oplus G & \xrightarrow{h} & G' \end{array} \tag{4.6}$$

is commutative. Thus $\alpha\oplus\mathrm{Id}_G$ is homotopic to $\alpha\oplus\gamma=h^{-1}\circ\gamma'\circ h$. This, in turn, is homotopic to $h^{-1}\circ\mathrm{Id}_{G'}\circ h=\mathrm{Id}_{E\oplus G}$, which proves the assertion. The converse statement is obvious. Going back to the original assertion, we consider two classes with $[(E,\alpha)] = [(F,\beta)]$. Then, $[(E,\alpha)] - [(F,\beta)] = [(E\oplus F, \alpha\oplus\beta^{-1})] = 0$ and, as just argued, there exists a vector bundle G such that $\alpha\oplus\beta^{-1}\oplus\mathrm{Id}_G$ is homotopic to $\mathrm{Id}_{E\oplus F\oplus G}$. By composing this homotopy equivalence with $\mathrm{Id}_E\oplus\beta\oplus\mathrm{Id}_G$ one sees that $\alpha\oplus\mathrm{Id}_F\oplus\mathrm{Id}_G$ and $\mathrm{Id}_E\oplus\beta\oplus\mathrm{Id}_G$ are homotopic. The converse statement is again obvious.

This "string theory" definition of $\mathrm{K}^{-1}(X)$, as well as its properties described above, generalize to give the higher Grothendieck group $\mathrm{K}^{-1}(\mathcal{C})$ associated to any category $\mathcal{C}$ which is an abelian monoid. Our third and final definition of $\mathrm{K}^{-1}(X)$ is one that relates the "M-theory" and "string theory" definitions, thereby showing their precise equivalence. Going back again to the definition of $\mathrm{K}^{-1}(X)$ as a subgroup of $\widetilde{\mathrm{K}}(X\times\mathbf{S}^1)$, we identify (E,α) with $(E_\alpha, E_{\mathrm{Id}_E})$, where E_α is the vector bundle over $\mathbf{S}^1\times X$ with total space $[0,1]\times E$ modulo the identification $(0,v)\equiv(1,\alpha(v))$ for all $v\in E$.

To relate $\mathrm{K}^{-1}(X) = \mathrm{K}(\Sigma X)$ to homotopy theory, we use the observation stated after eq. (2.71) in section 2.7. It follows that there is a natural isomorphism

$$\mathrm{K}^{-1}(X) = \Big[X\,,\,U(\infty)\Big], \tag{4.7}$$

where $U(\infty) = \bigcup_{k=1}^\infty U(k)$ is the infinite unitary group. In particular, it is possible to show that

$$\mathrm{K}^{-1}(\mathbf{S}^n) = \pi_{n-1}\Big(\mathrm{Gr}(k,2k;\mathbb{C})\Big), \qquad k>n, \tag{4.8}$$

where

$$\mathrm{Gr}(k,2k;\mathbb{C}) = \frac{U(2k)}{U(k)\times U(k)}, \tag{4.9}$$

and where $k>n$ defines the stable range for $\mathrm{K}^{-1}(X)$.

4.2 Unstable 9-Branes in Type IIA String Theory

To describe supersymmetric p-branes of the Type IIA theory as elements of a K-theory group of the spacetime X, we have to resort to looking at bound states of unstable 9-branes. If we relax the usual requirements that D-branes preserve half of the original supersymmetries and that they carry one unit of the corresponding RR charge, then Type IIA p-branes with p odd are allowed and in particular we have spacetime-filling 9-branes. These states are non-supersymmetric unstable excitations in the superstring theory, as there is always a tachyon in the spectrum of open strings connecting a 9-brane to itself. Thus the 9-branes of Type IIA are highly unstable, and we expect that they should rapidly decay to the supersymmetric vacuum by tachyon condensation on the spacetime-filling worldvolume (there are no RR fields in the corresponding Type IIA supergravity that would couple to any such conserved charges). But as before, the unstable D-brane configurations can carry lower-dimensional D-brane charges, so that when the tachyon rolls down to the minimum of its potential and the state decays, it leaves behind a supersymmetric state that differs from the vacuum by a lower-dimensional D-brane charge, i.e., the state decays into a supersymmetric D-brane configuration and one can represent a supersymmetric D-brane state as the bound state of the original system of unstable branes. Note that a representation in terms of bound states of 8-branes and $\overline{8}$-branes is possible using the constructions of the previous section. However, such a construction is undesirable, as it breaks some of the manifest spacetime symmetries (in the choice of an 8-brane worldvolume submanifold of X), and it limits the kinematics of branes that can be studied in this way. We shall therefore present a string theoretical construction that keeps all spacetime symmetries manifest.

The D9-brane boundary state $|D9\rangle$, as a coherent state in the Type IIA closed string Hilbert space, is of the form

$$|D9\rangle = |D9,+\rangle_{\rm NS} - |D9,-\rangle_{\rm NS}, \tag{4.10}$$

where $|D9,\pm\rangle_{\rm NS}$ are the two possible implementations of Neumann boundary conditions on all spacetime coordinates of X. Since

$$(-1)^{F_{L,R}}|D9,\pm\rangle_{\rm R} = |D9,\mp\rangle_{\rm R}, \tag{4.11}$$

no combination of the states $|D9,\pm\rangle_{\rm R}$ is invariant under the Type IIA GSO projection operator $\frac{1}{4}(1-(-1)^{F_L})(1+(-1)^{F_R})$ and hence there is no RR component in the D9-brane boundary state. But this just means that there is no RR tadpole, and thus no spacetime anomalies related to RR tadpoles can arise for the IIA 9-branes, i.e., unlike the Type IIB case, where the number of 9-branes must equal the number of $\overline{9}$-branes, there is no restriction on the number of Type IIA 9-branes. Moreover, the 9-branes carry no conserved charge, and there is no distinction between 9-branes and $\overline{9}$-branes in IIA.

There is no GSO projection in the open string channel of the torus amplitude $\langle D9|D9\rangle$ and therefore, in the NS sector, the open strings connecting a 9-brane to itself will contain both the $U(1)$ gauge field A_μ that a supersymmetric D-brane would contain, and the tachyon field T which would be otherwise eliminated by the GSO projection for supersymmetric branes. Furthermore, in the Ramond sector of the open string both spacetime chiralities χ, χ' of the ground state spinors are retained. We can generalize this construction to the case of N coincident 9-branes. Then the free open string spectrum of massless and tachyonic states gives rise to the following low-energy field content on the spacetime-filling worldvolume: a $U(N)$ gauge field A_μ, a tachyon field T in the adjoint representation of $U(N)$, and two chiral fermion fields χ, χ' of opposite spacetime chirality in the adjoint representation of $U(N)$. This can be compared with the Type IIB case, where N pairs of 9-branes and $\overline{9}$-branes give rise to a spectrum consisting of a $U(N) \times U(N)$ gauge field and a tachyon field in the bifundamental representation of $U(N) \times U(N)$.

Note that the case $N = 1$ is "degenerate" and will be dealt with separately later on in section 4.4. Notice also that the field content on N 9-branes of Type IIA superstring theory coincides with the ten-dimensional decomposition of an 11-dimensional system, with $A_M = (A_\mu, T)$ an 11-dimensional $U(N)$ gauge field, and $\Psi = (\chi, \chi')$ a 32-component spinor field in the adjoint representation of $U(N)$. This indicates a hidden 11-dimensional symmetry of the lowest lying open string states. It hints at a possible connection to M-theory which is in agreement with the properties of the K-group $\mathrm{K}^{-1}(X)$ described in the previous subsection.

Consider the configurations of N 9-branes in Type IIA superstring theory up to possible creation and annihilation of 9-branes to and from the vacuum. An *elementary* 9-brane configuration is one which rapidly decays to the supersymmetric vacuum, and therefore does not contain any lower-dimensional D-brane charges. Any elementary configuration of N' 9-branes wrapping the spacetime X gives rise to a $U(N')$ bundle F, together with a $U(N')$ gauge field A on F and a tachyon field T in the adjoint representation of $U(N')$. The presence or absence of lower D-brane charges is thereby measured by the tachyon condensate T_0. Thus, as before, we assume that a bundle E with tachyon field T can be deformed by processes involving only creation and annihilation of 9-branes into a bundle isomorphic to $E \oplus F$ where F is the Chan-Paton bundle of an elementary 9-brane configuration.

We therefore consider the set of equivalence classes of 9-branes with tachyon condensate, up to creation and annihilation of elementary 9-brane configurations to and from the vacuum. A 9-brane configuration thereby defines an element $[(E, \alpha)] \in \mathrm{K}^{-1}(X)$ where E is the rank-N Chan-Paton bundle carried by the system of N unstable Type IIA 9-branes. We will see in the following that the automorphism α is given by

$$\alpha = -\exp(\pi i T), \tag{4.12}$$

and it acts by the natural adjoint action (conjugation) on E. Here T is the adjoint $U(N)$ tachyon field on the 9-brane worldvolume. The possible 9-brane configurations up to creation and annihilation of elementary 9-branes are therefore classified by $\mathrm{K}^{-1}(X)$. It is instructive to compare this to the situation in Type IIB, where $\mathrm{K}(X) = \mathbb{Z} \oplus \widetilde{\mathrm{K}}(X)$ and D-brane charges are classified by the reduced K-theory group $\widetilde{\mathrm{K}}(X)$ with tadpole anomaly cancellation requiring that the number of 9-branes equals the number of $\overline{9}$-branes (here the integer in $\mathbb{Z}$ is in general the difference between the number of 9-branes and the number of $\overline{9}$-branes). In the Type IIA theory, we have $\widetilde{\mathrm{K}}^{-1}(X) = \mathrm{K}^{-1}(X)$, with no tadpole restriction on the number of IIA 9-branes. Also, as previously computed, we have

$$\begin{aligned} \widetilde{\mathrm{K}}(\mathbf{S}^{2n}) &= \mathbb{Z}, & \widetilde{\mathrm{K}}(\mathbf{S}^{2n+1}) &= 0 \\ \mathrm{K}^{-1}(\mathbf{S}^{2n+1}) &= \mathbb{Z}, & \mathrm{K}^{-1}(\mathbf{S}^{2n}) &= 0. \end{aligned} \tag{4.13}$$

This represents the fact that Type IIB contains supersymmetric p-branes for p odd, while Type IIA has supersymmetric p-branes for p even. Note that in this language, Bott periodicity is the statement that there are only two Type II superstring theories.

4.3 The Bound State Construction

We shall now present an explicit bound state construction of p-branes with worldvolume Y of odd codimension in the spacetime X as bound states of unstable Type IIA 9-branes. This will show that $\mathrm{K}^{-1}(X)$ indeed does classify D-brane charges in Type IIA superstring theory. This bound state construction is simply the analog of the ABS construction, now mapping classes in $\widetilde{\mathrm{K}}(Y)$ to classes in $\mathrm{K}^{-1}(X)$ in K-theory. This shows that whatever can be done with stable lower-dimensional branes can be done with unstable 9-branes of the Type IIA theory.

Consider a system of N unstable 9-branes. The gauge group is $U(N)$ and the tachyon field lives in the adjoint representation $\mathbf{N}\otimes\overline{\mathbf{N}}$ of $U(N)$ with tachyon potential $V(T) = V(-T)$. If we assume that T condenses into one of its vacuum expectation values $T = T_0$, and that the negative energy density associated with the condensate cancels the positive energy density associated with the 9-brane tension, as in (3.13), then the system of 9-branes completely annihilates into the supersymmetric vacuum and is therefore an elementary configuration. In general, T has the tendency to roll down to the minimum of its potential $V(T)$ and break part of the $U(N)$ gauge symmetry. The precise symmetry breaking pattern depends on the structure of the eigenvalues of T_0, i.e., on the precise form of the tachyon potential. For example, consider the symmetric tachyon potential

$$V(T) = -m^2 \operatorname{tr} T^2 + \lambda^2 \operatorname{tr} T^4 + \dots\ , \tag{4.14}$$

which is anticipated from the structure of the disc amplitudes at tree-level in open string perturbation theory. In this case, $T_0 = T_v \cdot \mathrm{diag}(\pm 1, \pm 1, \dots, \pm 1)$ after diagonalization. Corrections to (4.14) from worldsheets with more than one boundary give terms of the form

$$\delta V(T) = \tilde{\lambda}^2 \left[\operatorname{tr} T^2\right]^2 + \cdots\ . \tag{4.15}$$

It can be shown that if $\lambda^2 \geq 0$ and $\tilde{\lambda}^2 > 0$, then the minimum T_0 of the tachyon potential still has only two distinct eigenvalues $\pm T_v$.

We will henceforth assume that the 9-brane system under consideration has tachyon condensate T_0 with the same number of positive and negative eigenvalues. The number of 9-branes is therefore $2N$, and the gauge group $U(2N)$ is broken down to $U(N)\times U(N)$. The Type IIA vacuum manifold is thus

$$\mathcal{V}_{\rm IIA}(2N)=\frac{U(2N)}{U(N)\times U(N)}, \tag{4.16}$$

and it parametrizes the stable vortex-like configurations of the tachyon field. Away from the core of such a stable vortex (at $|x|\to\infty$), the tachyon field approaches its vacuum expectation values. This defines a map $\mathbf{S}^m\to\mathcal{V}_{\rm IIA}(2N)$, where the sphere $\mathbf{S}^m$ asymptotically surrounds the core of a stable vortex of codimension $m+1$ in the spacetime X. Therefore, the stable tachyon vortices are parametrized by classes in

$$\mathrm{K}^{-1}(\mathbf{S}^{m+1})=\pi_m\Big(\mathcal{V}_{\rm IIA}(2N)\Big)=\begin{cases}\mathbb{Z}, & m=2k\\ 0, & m=2k+1.\end{cases} \tag{4.17}$$

From this we see that the Type IIA system exhibits stable bound states in odd codimension $2k+1$. Note that the Type IIA vacuum manifold (4.16) of the tachyon field on the 9-branes is a finite-dimensional approximation to the classifying space $BU(\infty)$ for complex vector bundles over X.

We shall now explicitly construct the bound state tachyon vortices, which we will interpret as supersymmetric D$(2p)$-branes of the Type IIA theory. As before, K-theory selects a prefered natural value for the number $2N$ of 9-branes used to build the bound state (and is the same number that would arise in a "stepwise" construction). Namely, bound states in codimension $2k+1$ are most efficiently described by $2N=2^k$ 9-branes. Then the stable tachyon vortices are classified topologically by the homotopy groups

$$\pi_{2k}\Big(\mathcal{V}_{\rm IIA}(2^k)\Big)=\mathbb{Z}. \tag{4.18}$$

Again we can explicitly construct the classical tachyon soliton field corresponding to the generator of this homotopy group. The spacetime-filling worldvolume of 2^k 9-branes supports a $U(2^k)$ Chan-Paton bundle, which we identify as the spinor bundle $\mathcal{S}$ of the group $SO(2k+1)$ of rotations in the transverse space whose spinor representation of dimension 2^k is irreducible. The tachyon field is then given by

$$T(x)=\sum_{i=1}^{2k+1}\Gamma_i\,x^i, \tag{4.19}$$

where Γ_i are the Dirac matrices of $SO(2k+1)$ and x^i are local coordinates in the transverse space. The tachyon field is a map $T : \mathcal{S} \to \mathcal{S}$ and it asymptotically takes values in the Type IIA vacuum manifold (4.16). This should be contrasted with the Type IIB case, where the tachyon field asymptotically took values in the IIB vacuum manifold (3.18), because of the different structure of the Clifford algebra representations corresponding to the rotation groups $SO(2k)$ and $SO(2k+1)$.

In the present case we can go even further, and construct explicitly the non-trivial $U(2^k)$ gauge field configuration that lives on the 9-branes and must accompany the tachyon vortex above due to the finite energy conditions imposed on the system as a whole. There is a natural map

$$\pi_{2k-1}\left(U(2^{k-1})\right) \longrightarrow \pi_{2k}\left(\mathcal{V}_1(2^k)\right), \tag{4.20}$$

defined by the transformation of tachyon generators

$$T_{\rm IIB}(x) = \sum_{i=1}^{2k} \Gamma_i\, x^i \quad \longmapsto \quad T_{\rm IIA}(x) = T_{\rm IIB}(x) + x^{2k+1}\, \sigma_3 \otimes \mathbb{I}_{2^{k-1}}. \tag{4.21}$$

Here $T_{\rm IIB}(x)$ is the IIB tachyon field, and it is constructed as the unbroken part of the non-trivial $U(2^k)$ gauge field. Decomposing the sphere as before as $\mathbf{S}^{2k} = \mathbf{S}^{2k}_+ \cup \mathbf{S}^{2k}_-$, with $\mathbf{S}^{2k-1} = \mathbf{S}^{2k}_+ \cap \mathbf{S}^{2k}_-$, gauge fields on $\mathbf{S}^{2k}_\pm$ are topologically trivial and can be patched together to give a global gauge field, with the appropriate magnetic charge on $\mathbf{S}^{2k}$, using the transition function on the equator $\mathbf{S}^{2k-1}$. This large gauge transformation is just $T_{\rm IIB}(x)$, and the unbroken long-ranged gauge field of $U(2^{k-1}) \times U(2^{k-1})$ corresponds to that of a generalized magnetic monopole.

Using $T_{\rm IIA}^2 = |x|^2$, it is possible to show [62, 18] that the bundle automorphism (4.12) is actually the generator of the homotopy group $\pi_{2k+1}(U(2^k))$, and that far away from the core of the vortex, $T_{\rm IIA}(x) \in \mathcal{V}_{\rm IIA}(2^k)$. This induces the natural map

$$\pi_{2k}\left(\mathcal{V}_{\rm IIA}(2^k)\right) \longrightarrow \pi_{2k+1}\left(U(2^k)\right), \tag{4.22}$$

defined by

$$T_{\rm IIA} \quad \longmapsto \quad \alpha = -\exp(\pi i\, T_{\rm IIA}). \tag{4.23}$$

Thus the tachyon condensate represents the generator of the relative K-theory group $\mathrm{K}^{-1}(\mathbf{B}^{2k+1}, \mathbf{S}^{2k}) = \mathbb{Z}$, and the above bound state construction is precisely the analog of the ABS construction, now mapping classes in $\widetilde{\mathrm{K}}(Y) \hookrightarrow \mathrm{K}^{-1}(X)$ for Y of odd codimension in the spacetime manifold X wrapped by the unstable 9-branes of the Type IIA theory. Again the precise embedding is given by the cup product (2.60) as

$$\begin{aligned}\widehat{\lambda}\,:\,\widetilde{\mathrm{K}}(Y)\otimes_{\mathbb{Z}}\mathrm{K}^{-1}(\mathbf{B}^{2k+1},\mathbf{S}^{2k}) &\xrightarrow{\approx}\mathrm{K}^{-1}(Y\times\mathbf{B}^{2k+1},Y\times\mathbf{S}^{2k})\\ \Big[(E\,,\,F)\Big] &\longmapsto \widehat{\lambda}\left[\Big((E\otimes\mathcal{S})_{\mathrm{Id}_E\otimes\alpha}\,,\,(F\otimes\mathcal{S})_{\mathrm{Id}_F\otimes\alpha}\Big)\right]\end{aligned} \tag{4.24}$$

for $[(E,F)]\in\widetilde{\mathrm{K}}(Y)$.

In this way we get a hierarchy of bound state constructions in IIA and IIB, represented by brane systems of increasing dimensions which support worldvolume gauge groups that form a natural hierarchy

$$\begin{aligned}U(1)\subset U(1)\times U(1)\subset U(2)\subset U(2)\times U(2)&\\ \subset U(4)\subset U(4)\times U(4)\subset\dots\,.&\end{aligned} \tag{4.25}$$

This property leads to the usual descent relations among D-branes [11, 19]. In this hierarchy, the bound state construction in terms of pairs of stable branes alternates with the bound state construction in terms of unstable branes. It shows that a supersymmetric Dp-brane of Type II superstring theory can be constructed as the tachyonic kink in the worldvolume of an unstable D$(p+1)$-brane (see the next subsection), or alternatively as a bound state vortex in a $(p+2)$-brane $\overline{(p+2)}$-brane pair, or yet as a bound state of two unstable $(p+3)$-branes, and so on. This procedure continues until one reaches the spacetime filling dimension, thereby ending up with a construction in terms of 9-branes in which all spacetime symmetries are manifest.

As a simple example, consider the case of codimension $2k+1=3$. The K-theory gauge group is then $U(2)$, acting on two unstable 9-branes whose Chan-Paton bundle in the $\mathbf{2}$ representation of $U(2)$ is identified with the spinor bundle $\mathcal{S}$ of the rotation group $SO(3)$ of the transverse space. Using standard Pauli spin matrices σ_i for the Dirac matrices of $SO(3)$ gives

$$T(x)=\sum_{i=1}^{3}\sigma_i\,x^i=\begin{pmatrix}x^3 & x^1+ix^2\\ x^1-ix^2 & -x^3\end{pmatrix}. \tag{4.26}$$

The tachyon field (4.26) represents a vortex of vorticity 1. The finite energy condition ties it to the non-trivial $U(2)$ gauge field

$$\begin{aligned} A_i(x) &= \frac{1}{|x|^2}\left(1-\frac{|x|}{\sinh|x|}\right)\sum_{j=1}^{3}\Gamma_{ij}\,x^j \\ \Gamma_{ij} &\equiv \frac{1}{2}\Big[\sigma_i\,,\,\sigma_j\Big]=\sum_{k=1}^{3}\epsilon_{ijk}\,\sigma^k. \end{aligned} \tag{4.27}$$

Up to the trivial lift from $SU(2)$ to $U(2)$ gauge theory this is nothing but the 't Hooft-Polyakov magnetic monopole in $3+1$ dimensions [64], with the convergence factor in (4.27) the usual BPS solution of the 't Hooft-Polyakov ansatz. This monopole represents a supersymmetric stable D($2p$)-brane of Type IIA superstring theory as a bound state of two unstable D($2p+3$)-branes. Alternatively, the diagonal block in (4.26) represents the construction of one D($2p+2$)-brane and one $\overline{\text{D}(2p+2)}$-brane from a pair of D($2p+3$)-branes (see the next subsection), while the off-diagonal block represents a D($2p$)-brane as the bound state of a D($2p+2$)-brane-antibrane pair (c.f. eq. (3.25)). Again, this is an example of the descent relations in Type II superstring theory [11, 19].

4.4 Domain Walls in Type IIA String Theory

The case of codimension 1 (i.e., $k=0$) is "degenerate", as we shall now discuss. According to the general prescription, this represents a stable 8-brane (or $\overline{8}$-brane) of the Type IIA theory constructed as the tachyonic kink of $2^k=1$ 9-brane. The gauge group is $U(1)$ and the tachyon field is a real scalar field of charge 0 which is given by

$$T(x)=\frac{\pm\,T_0\,x^9}{\sqrt{1+(x^9)^2}}, \tag{4.28}$$

since there is now only one Γ-matrix which can be taken to be the 1×1 identity matrix. Here x^9 is the coordinate transverse to the core of the kink which represents a domain wall in spacetime. The sign in (4.28) distinguishes an 8-brane from an $\overline{8}$-brane and it corresponds to the sign of the difference $T(-\infty)-T(+\infty)$ between the asymptotic values of the tachyon field on the two sides of the domain wall. Note that only one 8-brane or $\overline{8}$-brane may be constructed from one 9-brane. In this case there is no symmetry breaking of the $U(1)$ gauge group, and we are left with a $U(1)$ gauge theory and a tachyon field that can condense into either one of the two vacuum expectation values $\pm T_0$. The relevant homotopy group of the vacuum manifold for one 9-brane is

$$\pi_0\Big(O(1)\Big) = \pi_0\Big(\{\pm T_0\}\Big) = \mathbb{Z}_2, \tag{4.29}$$

and so there is not enough room for the anticipated conserved 8-brane RR charge that should be classified by $\mathbb{Z}$. Therefore, each individual 8-brane and $\overline{8}$-brane requires its own 9-brane, and so the smallest 9-brane system that would accommodate the full K-theory group $\mathrm{K}^{-1}(\mathbf{S}^1) = \mathbb{Z}$ of 8-brane charges has an infinite number of spacetime filling branes.

More generally, if the tachyon potential is arranged so that the tachyon field on the worldvolume of N 9-branes condenses into its vacuum expectation value T_0 with $N-n$ positive eigenvalues and n negative eigenvalues, then

$$T_0 = T_v \begin{pmatrix} \mathbb{I}_{N-n} & 0 \\ 0 & -\mathbb{I}_n \end{pmatrix} \tag{4.30}$$

and the $U(N)$ gauge symmetry is broken to $U(N-n) \times U(n)$. As in the $N=1$ case, for $N>1$ the tachyon field forms kinks of codimension 1. Suppose that all eigenvalues of T correspond to a kink localized at a common domain wall Y of codimension 1 in X. Then locally near Y we can write the tachyon field as

$$T(x) = \begin{pmatrix} \frac{T_v\,x^9}{\sqrt{1+(x^9)^2}}\,\mathbb{I}_{N-n} & 0 \\ 0 & -\frac{T_v\,x^9}{\sqrt{1+(x^9)^2}}\,\mathbb{I}_n \end{pmatrix}, \tag{4.31}$$

which describes $N-n$ 8-branes and n $\overline{8}$-branes with coinciding worldvolumes wrapping Y. More general configurations of separated 8-branes and $\overline{8}$-branes may be constructed by letting each eigenvalue vanish along separate submanifolds of codimension 1 in the spacetime X. Again one cannot represent more than N 8-branes and $\overline{8}$-branes as a bound state of N 9-branes, as one would have to do a K-theoretic "stabilization" by adding other 9-branes in order to keep the relevant homotopy groups in the stable range.

4.5 Application to Matrix Theory

Consider N D0-branes in Type IIA superstring theory on $\mathbb{R}^{10}$ (or some compactification thereof). Each D0-brane can be represented as a bound state of 16 unstable spacetime-filling 9-branes, whose world-volume field theory contains a $U(16)$ gauge field, a tachyon field T in the adjoint representation of $U(16)$, and two chiral fermion fields χ, χ' of opposite spacetime chirality in the adjoint representation of $U(16)$. The tachyon field near the core of each stable point-like soliton can be represented as

$$T(x) = \sum_{i=1}^{9} \Gamma_i \, x^i, \tag{4.32}$$

where Γ_i are the Dirac matrices of the $SO(9)$ group of rotations of the transverse space to the core of the vortex. This field generates the homotopy group

$$\pi_8\Big(\mathcal{V}_{\rm IIA}(16)\Big) = \mathbb{Z}, \qquad \mathcal{V}_{\rm IIA}(16) = \frac{U(16)}{U(8)\times U(8)}. \tag{4.33}$$

Moreover, the non-trivial long-ranged gauge field gives rise to a "magnetic charge" of each D0-brane, in addition to the unit vorticity from $T(x)$.

The 16 9-branes live in the **16** representation of the $U(16)$ gauge group, which, in the background of the generalized magnetic monopole-vortex configuration representing the D0-brane, is identified with the spinor **16** representation of $SO(9)$. (This generalizes the three-dimensional 't Hooft-Polyakov monopole, where the **3** representation of the $SU(2)$ gauge group is identified with the spinor representation of the space rotation group $SO(3)$). K-theory implies that the stable string-theoretical soliton carries one unit of D0-brane charge, and therefore represents a D0-brane as the bound state of 16 unstable 9-branes. K-theory also implies that the trivial topology of the D0-brane worldlines in $\mathbb{R}^{10}$ does not require "stabilization" of the configuration by adding extra 9-branes (this is true even in compactifications of $\mathbb{R}^{10}$). Thus in this bound state construction, we never need to assume that the worldline Y is connected, and the spinor bundle $\mathcal{S}$ in this case is actually trivial along Y, and is thus extendable over X as the trivial bundle. In other words, we do not need to introduce a new set of 16 9-branes for each additional D0-brane and therefore *any* system of N D0-branes can be represented as bound states in a fixed system of 16 unstable spacetime-filling 9-branes.

Thus given a multi-D0-brane state described in terms of just 16 Type IIA 9-branes, we can follow this 9-brane configuration as we take the usual Sen-Seiberg scaling limit that defines Matrix theory [65]. Matrix theory can then be formulated as a theory of stable solitons on the spacetime-filling worldvolume of 16 unstable 9-branes. This interpretation of Matrix theory, in terms of vortices in a gauge theory with fixed gauge group, allows one to change the number N of D0-branes in the system without changing the rank of the gauge group. In the conventional formulation of Matrix theory, whereby the large-N limit requires relating theories with gauge groups of different ranks, it is very difficult to understand how systems with different values of N are related (for instance by some renormalization group approach). But the K-theoretic construction of D0-branes as magnetic vortices keeps the gauge group fixed for arbitrary values of N. In summary, the dynamics of Matrix theory appears to be contained in the physics of magnetic vortices on the worldvolume of 16 unstable 9-branes, described at low energies by a $U(16)$ gauge theory.

5 Type I D-Branes and $\mathrm{KO}(X)$

Having used the Type II theories to become well-acquainted with the bound state constructions and the use of them to describe D-branes in terms of new solitonic objects, we shall now start considering more complicated superstring theories in which the K-theory formalism will make some unexpected predictions. In this section we shall deal with the Type I theory, and thereby make contact with the original constructions of non-BPS states in string theory. Type I superstrings are unoriented and their supersymmetric vacuum configuration has gauge group $SO(32)$ which requires there to be always 32 spacetime filling branes in the vacuum state. The K-theory of the corresponding Chan-Paton gauge bundles therefore requires a refinement of what was described in section 2. It is precisely this difference that will lead to a much richer spectrum of D-brane charges in the Type I theory.

5.1 The Group $\mathrm{KO}(X)$

Consider a system of N 9-branes and M $\overline{9}$-branes in Type I superstring theory. Tadpole anomaly cancellation now requires that $N - M = 32$, and the branes support an $SO(N)$ bundle E and the antibranes an $SO(M)$ bundle F. Because of brane-antibrane creation and annihilation, we identify pairs of bundles (E, F) with $(E \oplus H, F \oplus H)$ for any $SO(K)$ bundle H. Pairs (E, F) with this equivalence relation define the real K-group of the spacetime X, $\mathrm{KO}(X)$. By replacing F with $F \oplus I^{32}$, it follows that the configurations of tadpole anomaly cancelling 9-branes and $\overline{9}$-branes are classified by the reduced real K-theory group $\widetilde{\mathrm{KO}}(X)$. Furthermore, it follows from the bound state construction that D-brane configurations of Type I superstring theory are classified by $\mathrm{KO}(X)$ with compact support [17].

Almost everything we said about $\mathrm{K}(X)$ carries through for real K-theory. The important change, however, is the relation to homotopy theory. Namely, for $k > n = \dim X$ (the stable range for $\mathrm{KO}(X)$), we have

$$\widetilde{\mathrm{KO}}(X) = \Bigl[X\,,\,BO(k)\Bigr], \tag{5.1}$$

where $BO(k) = \bigcup_{m>k+n} \mathrm{Gr}(k, m; \mathbb{R})$ is the classifying space for real vector bundles with structure group $O(k)$, and the real Grassmannian manifold is

$$\mathrm{Gr}(k, m; \mathbb{R}) = \frac{O(m)}{O(m-k) \times O(k)}, \qquad m > k + n. \tag{5.2}$$

For $X = \mathbf{S}^n$ we now have that

$$\widetilde{\mathrm{KO}}(\mathbf{S}^n) = \pi_n\Bigl(BO(k)\Bigr) = \pi_{n-1}\Bigl(O(k)\Bigr), \qquad k > n, \tag{5.3}$$

and this group classifies $(9-n)$-brane charges in Type I string theory on flat $\mathbb{R}^{10}$. The stable homotopy of the orthogonal groups is much different than that of the unitary groups. For example,

$$\begin{aligned}
\pi_0\Bigl(O(k)\Bigr) &= \mathbb{Z}_2, &\quad k \geq 1 \\
\pi_1\Bigl(O(k)\Bigr) &= \mathbb{Z}_2, &\quad k \geq 3 \\
\pi_2\Bigl(O(k)\Bigr) &= 0, &\quad k \geq 4 \\
\pi_3\Bigl(O(k)\Bigr) &= \mathbb{Z}, &\quad k \geq 5.
\end{aligned} \tag{5.4}$$

Note that the identification of 9-brane configurations with $\mathrm{KO}(X)$ does not really require brane-antibrane annihilation. This follows from the fact that as $\dim X = 10$, $SO(32)$ bundles on X are classified by $\pi_n(SO(32))$ for $n \leq 9$. These homotopy groups always lie within the stable range, so that all $SO(32)$ bundles on X are automatically classified by $\widetilde{\mathrm{KO}}(X)$.

The Bott periodicity theorem now states that the homotopy groups of $O(\infty)$ are periodic with period *eight*:

$$\pi_n\Big(O(\infty)\Big) = \pi_{n+8}\Big(O(\infty)\Big), \tag{5.5}$$

and there are accordingly eight higher-degree KO-groups, defined by using suspensions as described in section 2.4, with

$$\widetilde{\mathrm{KO}}^{-n}(X) = \widetilde{\mathrm{KO}}^{-n-8}(X), \tag{5.6}$$

and as usual $\mathrm{KO}^{-n}(X) = \widetilde{\mathrm{KO}}^{-n}(X) \oplus \mathrm{KO}^{-n}(\mathrm{pt})$. In particular, for $X = \mathbf{S}^n$, we have that

$$\widetilde{\mathrm{KO}}(\mathbf{S}^n) = \widetilde{\mathrm{KO}}(\mathbf{S}^{n+8}). \tag{5.7}$$

The periodicity (5.7) can be derived from the ABS construction for KO-theory. Let $\mathrm{RO}[\mathrm{Spin}(n)]$ be the real representation ring of the spin group $\mathrm{Spin}(n)$, which is generated by the irreducible real representations. Then following section 2.8, there is a natural homomorphism

$$\varphi : \mathrm{RO}\Big[\mathrm{Spin}(n)\Big] \longrightarrow \mathrm{KO}(\mathbf{B}^n, \mathbf{S}^{n-1}) \tag{5.8}$$

which descends to a graded ring isomorphism

$$\mathrm{RO}\Big[\mathrm{Spin}(n)\Big] \,/\, i^* \,\mathrm{RO}\Big[\mathrm{Spin}(n+1)\Big] \xrightarrow{\approx} \widetilde{\mathrm{KO}}(\mathbf{S}^n). \tag{5.9}$$

Using the periodicity property (2.83) of real Clifford algebras, eq. (5.7) is immediate.

The isomorphism (5.9) can also be used to show how extra $\mathbb{Z}_2$-valued charges such as those in (5.4) appear in the D-brane spectrum of the Type I theory. For example, consider the case $n = 1$ in (5.9). Since $C\ell_1 = \mathbb{C}$, the $C\ell_1$-modules are just complex vector spaces, and the isomorphism $\mathrm{RO}[\mathrm{Spin}(1)] \xrightarrow{\approx} \mathbb{Z}$ is generated by taking the complex dimension. Similarly, since $C\ell_2 = \mathbb{H}$, the $C\ell_2$-modules are quaternionic vector spaces and $\mathrm{RO}[\mathrm{Spin}(2)] \xrightarrow{\approx} \mathbb{Z}$ comes from taking the quaternionic dimension. The map $i^* : \mathrm{RO}[\mathrm{Spin}(2)] \to \mathrm{RO}[\mathrm{Spin}(1)]$ is realized by regarding a quaternionic vector space as a complex vector space under restriction of scalars. This is just the map $\mathbb{Z} \to \mathbb{Z}$ given by multiplication by 2, since the complex dimension is twice the quaternionic one. This leads to $\widetilde{\mathrm{KO}}(\mathbf{S}^1) = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$. Moreover, the generators of the right-hand side of (5.9) can be conveniently represented in terms of spinor modules and Clifford multiplication maps. For example, if $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ is the fundamental graded module for $C\ell_{4m}$, then

$$\varphi_{\mathbb{R}}^{(4m)} = [\mathcal{S}^+, \mathcal{S}^-; \mu] \tag{5.10}$$

is a generator of the group $\widetilde{\mathrm{KO}}(\mathbf{S}^{4m}) = \mathbb{Z}$, where again $\mu_x : \mathcal{S}^+ \to \mathcal{S}^-$ denotes Clifford multiplication by $x \in \mathbb{R}^{4m}$. Using the Clifford module structure and the cup product we can again easily compute that

$$\varphi_{\mathbb{R}}^{(8n)} = \left(\varphi_{\mathbb{R}}^{(8)}\right)^n, \quad 4\varphi_{\mathbb{R}}^{(8)} = \left(\varphi_{\mathbb{R}}^{(4)}\right)^2. \tag{5.11}$$

The new torsion KO-groups also modify various product relations that were described in section 2. For instance, the Künneth formula (2.39) need not hold in general for spheres, because $\mathrm{KO}(\mathbf{S}^n)$ is not necessarily freely generated. Nevertheless, the analog of (2.35), for example, follows using (2.34) to get

$$\widetilde{\mathrm{KO}}(X \times \mathbf{S}^1) = \widetilde{\mathrm{KO}}^{-1}(X) \oplus \widetilde{\mathrm{KO}}(X) \oplus \mathbb{Z}_2. \tag{5.12}$$

5.2 The Bound State Construction

We need only know the first eight KO-groups to determine the complete spectrum of (BPS and non-BPS) D-branes in Type I superstring theory. This spectrum may be found in table 4. The list contains the well-known stable BPS D9-branes, D5-branes and D-strings of the Type I theory (of integer-valued charges). The D0-brane is the $\mathbb{Z}_2$-charged D-particle, which is stable but non-BPS, originally discovered in [7]. The D8, D7 and D(−1)-branes are new predictions of K-theory [17] which imply that the spectrum of the Type I theory should contain new $\mathbb{Z}_2$-charged stable, but non-BPS, 8-branes, 7-branes and instantons. This new spectrum of the Type I theory has been computed in [39] using

D-brane	D9	D8	D7	D6	D5	D4	D3	D2	D1	D0	D(−1)
Transverse space	$\mathbf{S}^0$	$\mathbf{S}^1$	$\mathbf{S}^2$	$\mathbf{S}^3$	$\mathbf{S}^4$	$\mathbf{S}^5$	$\mathbf{S}^6$	$\mathbf{S}^7$	$\mathbf{S}^8$	$\mathbf{S}^9$	$\mathbf{S}^{10}$
$\widetilde{\mathrm{KO}}(\mathbf{S}^n)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Table 4: Type I spectrum of D-branes from $\widetilde{\mathrm{KO}}(\mathbf{S}^n)$.

the boundary state formalism, thus explicitly confirming the K-theory predictions. Since the D9, D5 and D1 branes all carry RR charge, they are described by boundary states of the form

$$|Dp\rangle = |Dp\rangle_{\mathrm{NS}} \pm |Dp\rangle_{\mathrm{R}}, \quad p = 1, 5, 9\,. \tag{5.13}$$

The other non-BPS D-branes do not carry any RR charge, and so have boundary states

$$|Dp\rangle = |Dp\rangle_{\mathrm{NS}}, \quad p = -1, 0, 7, 8, \tag{5.14}$$

and they are their own antibranes. Note that the explicit boundary state descriptions of the non-BPS D-branes proves that all charges in table 4 are carried by spacetime defects onto which open strings can attach.

The Type I theory can be considered as the orientifold projection of Type IIB superstring theory by the worldsheet parity operator Ω which reverses the orientation of the fundamental string worldsheet. The action of the orientifold group Ω on Chan-Paton bundles is anti-linear, i.e., $E \xrightarrow{\Omega} \overline{E}$, where $\overline{E} \cong E^*$ is the conjugate bundle defined by complex-conjugating the transition functions of E. Thus only real bundles survive the orientifold projection, leading to the KO-theory of real virtual bundles for Type I systems. For $p = 1, 5, 9$, the corresponding Type IIB RR charge is invariant under the Ω-projection, i.e., $\Omega|Dp\rangle_{\mathrm{R}} = |Dp\rangle_{\mathrm{R}}$. The associated Type I bound state constructions are then just the orientifold projections of the Type IIB ones. One can describe the non-BPS branes in terms of bound states of a single BPS brane-antibrane pair of lowest possible dimension. For $p = 0, 8$, there is no IIB RR charge, and the boundary state is automatically even under Ω. The D0 (respectively D8) brane are topologically stable kinks in the tachyon field on the worldvolumes of Type I D1–$\overline{\mathrm{D1}}$

D-brane	D0	D1	D5	D7	D8	D9
Gauge group	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$USp(2)$	$U(1)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Table 5: Worldvolume gauge groups of Type I D-branes.

(respectively D9–D$\overline{9}$) systems, with $\mathbb{Z}_2$-valued Wilson lines (c.f. section 4.4). For $p = -1, 7$, Ω exchanges IIB p-branes with $\overline{p}$-branes, i.e., $\Omega|Dp\rangle_{\rm R} = -|Dp\rangle_{\rm R}$, so that the p-brane-antibrane configuration is Ω-invariant. Thus the Type I D(−1) (respectively D7) brane is the orientifold projection of the D(−1)–D$\overline{(-1)}$ (respectively D7–D$\overline{7}$) system in IIB. In these latter two cases, we may write the corresponding Type I boundary states in terms of those of the IIB theory as

$$|Dp\rangle_{\rm I} = |Dp\rangle_{\rm IIB} + |D\overline{p}\rangle_{\rm IIB} = |Dp\rangle_{\rm NS}, \quad p = -1, 7. \tag{5.15}$$

One may then show that the Type IIB tachyon present in the unstable $p - \overline{p}$ state is eliminated by the Ω orientifold projection [39], leading to a stable solitonic state. From these bound state constructions one may also immediately deduce the worldvolume field theories of the non-BPS D-branes, and in particular the worldvolume gauge groups listed in table 5, as we will demonstrate explicitly in the following. In the remainder of this section we shall describe some aspects of the D-brane spectrum of Type I superstring theory using its KO-theory structure. We will consider each type of soliton separately and discuss the features unique to each dimensionality.

5.3 Type I D-Instantons

The perturbative symmetry group of the Type I superstring should really be considered as $O(32)$, rather than $SO(32)$, because orthogonal transformations $\mathcal{O}$ with $\det \mathcal{O} = -1$ are symmetries of Type I perturbation theory, i.e., the central element -1 of $O(32)$ acts trivially on the perturbative spectrum, so that the corresponding symmetry group is $O(32)/\mathbb{Z}_2$. This fact makes a connection with how the perturbative gauge group of the Type I superstring appears, which is locally isomorphic to $SO(32)$. However, S-duality with the $SO(32)$ heterotic string implies that transformations $\mathcal{O}$ of determinant -1 are actually not symmetries. This then implies that there must exist some non-perturbative effect that breaks the group $O(32)$ to its connected subgroup $SO(32)$, and this is precisely the $\mathbb{Z}_2$-charged gauge instanton associated with $\pi_9(SO(32)) = \mathbb{Z}_2$. This is proven in [17] using index-theoretical arguments, namely the fact that a non-trivial bundle on the sphere $\mathbf{S}^{10}$ is characterized by having an odd number of fermionic zero modes of the corresponding chiral Dirac operator (the relationship between index theory and K-theory will be discussed in section 7.4). The Type I D-instanton comes from a bound state of Type IIB 9-brane-antibrane pairs with Chan-Paton bundles $(\mathcal{S}^+, \mathcal{S}^-)$, while the anti-D-instanton has gauge bundles $(\mathcal{S}^-, \mathcal{S}^+)$. Here $\mathcal{S}^\pm$ are the usual 16-dimensional complex chiral spinor representations of $SO(10)$. The orientifold projection acts by complex conjugation, so it reverses the chiralities $\mathcal{S}^+ \leftrightarrow \mathcal{S}^-$ (equivalently the 9-branes and $\overline{9}$-branes) and thereby identifies the instanton and anti-instanton in the Type I theory. For Type I superstrings and KO-theory, the gauge bundles must be *real*, and so we take the Type I 9-brane Chan-Paton bundles to transform as the spinor module $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ which, by regarding complex representation vector spaces as real ones under restriction of the scalars, becomes the unique irreducible, real 32-dimensional spinor representation of $SO(32)$. The (-1)-brane is therefore described by 32 $9-\overline{9}$ brane pairs with Chan-Paton bundles $(\mathcal{S}, \overline{\mathcal{S}})$ and a tachyon field $T(x) = \sum_i \Gamma_i\, x^i$.

5.4 Type I D-Particles

An element of $\mathrm{KO}(\mathbb{R}^9)$ (or $\widetilde{\mathrm{KO}}(\mathbf{S}^9)$) is described by a pair of trivial $SO(N)$ bundles (E, F) over $\mathbb{R}^9$ with a bundle map $T : E \to F$ that is an isomorphism near infinity and such that the rotation group $SO(9)$ acts on the fibers of E and F in the spinor representation. For KO-theory, we must use real spinor representations, and for $SO(9)$ there is a unique such irreducible representation $\mathcal{S}$ of dimension 16. Thus E and F have rank 16 and transform under $SO(9)$ rotations like $\mathcal{S}$. The tachyon field is then given by (4.32).

We can compare this K-theoretical construction to the original construction of the Type I D-particle in [9]. For this, we make an $8+1$ dimensional split of the coordinates and Γ-matrices. Pick an $SO(8)$ subgroup of $SO(9)$, and let $x=(x^a,x^9)$ under this split, with $a=1,\dots,8$. The spinor representation $\mathcal{S}$ of $SO(9)$ breaks up under this split into $SO(8)$ as $\mathcal{S}=\mathcal{S}^+\oplus\mathcal{S}^-$, with $\mathcal{S}^\pm$ the real eight-dimensional chiral spinor representations of $SO(8)$. Write the $SO(8)$ Dirac matrices as $\Gamma_a:\mathcal{S}^-\to\mathcal{S}^+$ and $(\Gamma_a)^\top:\mathcal{S}^+\to\mathcal{S}^-$. It then follows from (4.32) that the tachyon field decomposes as (see (4.21))

$$T(x)=\begin{pmatrix} x^9\,\mathbb{I}_{16} & \sum_a\Gamma_a\,x^a \\ \sum_a(\Gamma_a)^\top\,x^a & -x^9\,\mathbb{I}_{16}\end{pmatrix}. \tag{5.16}$$

Changing the basis of Chan-Paton factors on the $\overline{9}$-branes by the matrix

$$\sigma_1\otimes\mathbb{I}_{16}=\begin{pmatrix} 0 & \mathbb{I}_{16} \\ \mathbb{I}_{16} & 0\end{pmatrix} \tag{5.17}$$

leads to

$$T(x)\longmapsto(\sigma_1\otimes\mathbb{I}_{16})\,T(x)\,(\sigma_1\otimes\mathbb{I}_{16})=\begin{pmatrix}\sum_a\Gamma_a\,x^a & x^9\,\mathbb{I}_{16} \\ -x^9\,\mathbb{I}_{16} & \sum_a(\Gamma_a)^\top\,x^a\end{pmatrix}. \tag{5.18}$$

The diagonal blocks here represent two decoupled systems each containing eight $9-\overline{9}$ pairs. The first set of eight $9-\overline{9}$ pairs has tachyon field

$$T_s(x^a)=\sum_{a=1}^{8}\Gamma_a\,x^a, \tag{5.19}$$

and the second one has $(T_s)^\top$. But T_s describes a D-string located at $x^1=\ldots=x^8=0$ (see section 5.6 below), and, since $(T_s)^\top$ is made from T_s by exchanging 9-branes with $\overline{9}$-branes, the tachyon field $(T_s)^\top$ describes an anti-D-string located at $x^1=\ldots=x^8=0$. This is just the construction in [7] of the Type I D-particle from a coincident D-string and anti-D-string. The off-diagonal blocks correspond to a codimension one tachyon field which connects the D-string and anti-D-string and is odd under the reflection $x^9\to-x^9$. This is precisely the solitonic configuration of the D1–D$\overline{1}$ tachyon field constructed in [9]. Thus, the K-theory formalism can also be used to produce string theoretical constructions of non-BPS states.

The spinor quantum numbers carried by the D-particle also appear naturally in this framework. As shown in [9], the Type I 0-brane transforms in the spinor representation of $SO(32)$, which agrees with the fact that the non-perturbative gauge group of the Type I superstring is really the spin cover $\mathrm{Spin}(32)/\mathbb{Z}_2$ of $SO(32)$. In the above construction this can be seen from the fact that the $\overline{9}$-brane Chan-Paton factors produce an $SO(32)$ vector of fermionic zero modes, whose quantization gives a spinor representation of $SO(32)$ (again one uses the index theoretical fact that a non-trivial $SO(32)$ bundle on $\mathbf{S}^9$ is characterized by having an odd number of fermionic zero modes of the corresponding Dirac operator) [17]. Furthermore, given N coincident Type I 0-branes, the tachyon vertex operators have the form (in the zero-picture)

$$V(\Lambda) = \psi\!\!\!/ \; \mathrm{e}^{ik_0 X^0(z)} \otimes \Lambda, \tag{5.20}$$

where Λ is an $N \times N$ matrix which acts on the Chan-Paton factors and $\psi\!\!\!/$ is a worldsheet fermion field. The Ω projection maps $\Lambda \to \Lambda^\top$. Thus, if $\Lambda^\top = -\Lambda$, then $V(\Lambda)$ is odd under Ω and the tachyon state survives the Ω-projection. An antisymmetric matrix Λ always has an even number of non-zero eigenvalues, such that each pair describes the flow toward annihilation of a pair of 0-branes. This means that the D-particle number is conserved only modulo 2, in agreement with the fact that $\widetilde{\mathrm{KO}}(\mathbf{S}^9) = \mathbb{Z}_2$. A similar argument applies to the Type I D-instantons.

5.5 Domain Walls in Type I String Theory

The Type I D8-brane is described by $\widetilde{\mathrm{KO}}(\mathbf{S}^1) = \pi_0(O(32)) = \mathbb{Z}_2$, which is represented via trivial bundles $E, F \to \mathbb{R}^1$ and a tachyon field $T : E \to F$ that is invertible at infinity. As in section 4.4, the 8-brane is a domain wall, located at $x^9 = 0$, and constructed from a single $9-\overline{9}$ pair with a tachyon field (4.28) that is positive on one side and negative on the other side of the wall. In contrast to the situation of section 4.4, however, the $\mathbb{Z}_2$-valued charges that arise here from the bound state construction are very natural. One way to see this is by appealing to the Bott periodicity map (5.7) which may be described as follows. Take $[(E_0, F_0)] \in \widetilde{\mathrm{KO}}(\mathbf{S}^n)$ with tachyon map $T_0 : E_0 \to F_0$, and construct $[(E, F)] \in \widetilde{\mathrm{KO}}(\mathbf{S}^{n+8})$ by setting

$$E = E_0 \otimes (\mathcal{S}^+ \oplus \mathcal{S}^-), \qquad F = F_0 \otimes (\mathcal{S}^+ \oplus \mathcal{S}^-), \tag{5.21}$$

where $\mathcal{S}^\pm$ are the chiral spinor representations of $SO(8)$ with Dirac matrices $\Gamma_a : \mathcal{S}^- \to \mathcal{S}^+$. Let x^a denote the last eight coordinates of

$\mathbb{R}^{n+8}$. The tachyon field is then given as before by the cup product:

$$T(x) = \begin{pmatrix} T_0 \otimes \mathbb{I}_{16} & \mathrm{Id} \otimes \sum_a \Gamma_a \, x^a \\ \mathrm{Id} \otimes \sum_a (\Gamma_a)^\top \, x^a & -T_0 \otimes \mathbb{I}_{16} \end{pmatrix} . \tag{5.22}$$

For example, setting $n = 1$ we obtain the tachyon field (5.16) with the diagonal matrix representing 16 8-branes and $\overline{8}$-branes and the off-diagonal ones corresponding to the bound state construction of a D-particle in terms of the $8-\overline{8}$ brane pairs. So the relation (5.16) between the 8-brane and the 0-brane is a typical example of the Bott periodicity map in Type I superstring theory.

Alternatively, the Bott periodicity isomorphism (5.6) of KO-groups comes from taking the cup product of an element of $\widetilde{\mathrm{KO}}^{-n}(X)$ with the generator $[\mathcal{N}_{\mathbb{R}}] - [I^8]$ of $\widetilde{\mathrm{KO}}(\mathbf{S}^8) = \mathbb{Z}$, where $\mathcal{N}_{\mathbb{R}}$ is the rank 7 Hopf bundle over the real projective space $\mathbb{R}P^8$ associated with the real Hopf fibration $\mathbf{S}^{15} \to \mathbf{S}^8$. This shows that the construction of a p-brane in terms of $p+8$-$\overline{p+8}$ brane pairs in Type I superstring theory is determined by a D-string solitonic configuration which gives an explicit physical realization of the Spin(8) instanton. The corresponding eight-dimensional non-trivial gauge connections, and the associated spinor structures, may be found in [66]. This identifies the explicit form of the worldvolume gauge fields living on the $p+8$-$\overline{p+8}$ brane pair, required to ensure that the tachyon field is covariantly constant near infinity and hence to produce the finite energy solitonic p-brane configuration, as [66]

$$A_i^-(x) = 0, \qquad A_i^+(x) = -2i \sum_{j=1}^{8} \Gamma_{ij} \, \frac{x^j}{(1+|x|^2)^2}, \tag{5.23}$$

where Γ_{ij} are the generators of Spin(8). These gauge field configurations are Spin(9) symmetric (thereby preserving the manifest spacetime symmetries) and carry unit topological charge. Similar arguments apply to the non-BPS D7-brane.

5.6 Type I D-Strings

We will now exhibit the Type I D-string as a bound state of 9-branes and $\overline{9}$-branes, located at $x^1 = \ldots = x^8 = 0$ in $\mathbb{R}^{10}$ with worldvolume coordinates (x^0, x^9). The group of rotations keeping the D-string worldsheet fixed is $SO(8)$, which rotates the vector $x = (x^1, \dots, x^8)$. The two spinor representations $\mathcal{S}^{\pm}$ of $SO(8)$ are both eight-dimensional, with Dirac matrices $\Gamma_a : \mathcal{S}^+ \to \mathcal{S}^-$. Thus we consider a configuration of eight 9-branes and eight $\overline{9}$-branes with trivial gauge bundles, but with the rotation group $SO(8)$ acting on the Chan-Paton bundles with the rank eight bundle of the 9-branes transforming as $\mathcal{S}^+$ and of the $\overline{9}$-branes as $\mathcal{S}^-$. The tachyon field is then given by (5.19) and the map $x \mapsto \sum_a \Gamma_a x^a/|x|$ is the generator of $\pi_7(SO(8)) = \mathbb{Z}$. As in section 4.5, there are no global obstructions that occur in this bound state construction, since the Type I spacetime X is a spin-manifold and so is the orientable two-dimensional D-string worldsheet (a global version of the bound state construction will be presented in section 7.3). There is also no need to assume in the above construction that the D-string worldvolume is connected. This implies that any collection of (disjoint) D-strings can be represented by a configuration of eight $9-\overline{9}$ pairs and there is no need to introduce eight more pairs for every D-string. This is in contrast to the Type IIB case, where the spacetime X need not admit a spin structure and in general one would have to carry out a K-theoretic stabilization by adding extra $9-\overline{9}$ pairs.

It is interesting to examine some of the gauge solitons we have described above in light of the S-duality between the Type I theory and Spin$(32)/\mathbb{Z}_2$ heterotic string theory. The Type I D-instanton, D-particle and D-string are all manifest in heterotic string perturbation theory. The D-string is equivalent to the perturbative heterotic string [4], so that the second quantized Fock space of perturbative heterotic strings can be described completely by configurations of eight $9-\overline{9}$ brane pairs. (Note that similar conclusions as those of section 4.5 can also be reached for heterotic Matrix string theory.) The D-particle is a gauge soliton in the spinor representation of $SO(32)$, just like some of the particles in the elementary heterotic string spectrum. Finally, the D-instanton gives a mechanism that breaks the disconnected component of $O(32)$, and this symmetry breaking is manifest in heterotic string perturbation theory. Thus, from the point of view of the heterotic string, these three non-perturbative objects can be continuously connected to ordinary perturbative objects. We note also that all of the above bound state constructions, like those of the previous sections, preserve the manifest symmetries of the transverse spaces to the D-branes. Moreover, the extra 32 9-branes which must be added for anomaly cancellation yields an $SO(32)$ gauge symmetry that plays no role in the above constructions. The bound state construction uses extra brane-antibrane pairs to enlarge the gauge group, so that $SO(32)$ invariance is manifest.

5.7 Type I D5-Branes and the Group $\mathrm{KSp}(X)$

The group $\mathrm{KSp}(X)$ classifies Type I D-branes which are quantized using symplectic gauge bundles. The appearance of symplectic gauge symmetry can be understood from the analysis of [67] (see also [37]) where the requirement of closure of the worldsheet operator product expansion was shown to put stringent restrictions on the actions of discrete gauge symmetries on Chan-Paton bundles. In particular, the square of the worldsheet parity operator Ω acts on Chan-Paton indices as

$$\Omega^2 \,:\, |Dp;ab\rangle \longmapsto \sum_{a',b'} \left(\gamma_\Omega^2\right)_a^{~a'} |Dp;a'b'\rangle \left(\gamma_\Omega^{-2}\right)_b^{~b'} = (\pm i)^{(9-p)/2}\,|Dp;ab\rangle \tag{5.24}$$

where a,b are the open string endpoint Chan-Paton labels of a Dp-brane state of the IIB theory, and γ_Ω denotes the adjoint representation of the orientifold group in the Chan-Paton gauge group. While the 9-branes have the standard orthogonal subgroup projection (as required by tadpole anomaly cancellation), eq. (5.24) shows that $\Omega^2 = -1$ when acting on, for example, 5-branes (and also on the corresponding tachyon vertex operators [17]). The 5-branes must therefore be quantized using pseudo-real gauge bundles, i.e., Chan-Paton bundles with structure group $Sp(2N)$ on the 9-branes and $\overline{9}$-branes. An alternative explanation [17] uses the fact that a Type I 5-brane is equivalent to an instanton on the spacetime filling 9-branes that occupy the vacuum [68]. The tachyon field breaks the $SO(4N)\times SO(4N)$ gauge symmetry of the 9-$\overline{9}$ brane configuration to the diagonal subgroup $SO(4N)_{\rm diag}$, which is then further broken down to $Sp(2N)$ by the instanton field. (Note that for a configuration of unit 5-brane number one needs at least $4N=4$ spacetime filling brane-antibrane pairs). Notice that eq. (5.24) also explains the standard spectrum of stable BPS D-branes in the Type I theory, as well as the worldvolume gauge groups listed in table 5.

D-brane	D9	D8	D7	D6	D5	D4	D3	D2	D1	D0	D(–1)
Transverse space	$\mathbf{S}^0$	$\mathbf{S}^1$	$\mathbf{S}^2$	$\mathbf{S}^3$	$\mathbf{S}^4$	$\mathbf{S}^5$	$\mathbf{S}^6$	$\mathbf{S}^7$	$\mathbf{S}^8$	$\mathbf{S}^9$	$\mathbf{S}^{10}$
$\widetilde{\mathrm{KSp}}(\mathbf{S}^n)$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0

Table 6: D-brane spectrum in Type I string theory with symplectic gauge bundles.

For $\mathrm{KSp}(X)$ the connection with homotopy theory is given by

$$\widetilde{\mathrm{KSp}}(\mathbf{S}^n) = \pi_{n-1}\Big(\mathrm{Sp}(k)\Big), \qquad k > n/4, \tag{5.25}$$

where $k > n/4$ defines the stable range for $\mathrm{KSp}(X)$. As previously,

$$\widetilde{\mathrm{KSp}}(\mathbf{S}^n) = \pi_{n-1}\Big(\mathrm{Sp}(\infty)\Big), \tag{5.26}$$

where $\mathrm{Sp}\,(\infty) = \bigcup_k Sp(k)$ is the infinite symplectic group. In this case Bott periodicity takes the special form

$$\pi_n\Big(\mathrm{Sp}(\infty)\Big) = \pi_{n+4}\Big(O(\infty)\Big), \tag{5.27}$$

so that

$$\widetilde{\mathrm{KSp}}(\mathbf{S}^n) = \widetilde{\mathrm{KO}}(\mathbf{S}^{n+4}). \tag{5.28}$$

Thus, any calculation in symplectic K-theory can be reduced to one in real K-theory. The complete spectrum of corresponding brane charges can be found in table 6, which shows that while the spectrum of supersymmetric D-branes remains unchanged, that of the stable non-BPS states differs from before. The isomorphism (5.28) comes from taking the cup product with the class of the rank 2 instanton bundle $\mathcal{N}_{\mathbb{H}}$ associated with the pseudo-real Hopf fibration $\mathbf{S}^7 \to \mathbf{S}^4$, i.e., the holomorphic vector bundle of rank 2 over $\mathbb{C}P^3$ [55]. Thus the relationships between a BPS p-brane and a BPS $p+4$-brane is a 5-brane soliton which may be identified with an $SU(2)$ Yang-Mills instanton field. For example, consider a Type I D-string in the worldvolume of a $5-\overline{5}$ brane pair [7]. The worldvolume gauge symmetry is $SO(4) = SU(2) \times SU(2)$ and the tachyon field transforms in its $\mathbf{2} \otimes \overline{\mathbf{2}}$ representation. The Ω-projection identifies the vacuum manifold of the 5-brane configuration as $SU(2) = Sp(1)$. The topological stability of the D-string is guaranteed by the homotopy group $\pi_3(SU(2)) = \mathbb{Z}$. A finite energy, static string-like solution in the corresponding 5+1 dimensional worldvolume field theory is possible when one imposes the following asymptotic forms on the fields (analogously to (5.23)):

$$T \simeq T_v\, U, \qquad A^- \simeq 0, \qquad A^+ \simeq i\, dU\, U^{-1}. \tag{5.29}$$

Here U is an $SU(2)$ matrix-valued function corresponding to the identity map (of unit winding number) from the asymptotic boundary $\mathbf{S}^3$, of the string-like configuration in five dimensions, to the $SU(2)$ group manifold $\mathbf{S}^3$. Then the string soliton carries 1 unit of instanton number (living on the 5-brane) which is known to be a source of D-string charge in Type I string theory [23, 68]. Further applications of the K-theory of symplectic gauge bundles will be discussed in section 6.

5.8 Relationships between Type I and Type II Superstring Theory

The K-theory formalism has given us many different relations between D-branes in a given superstring theory. It turns out that it also provides new relationships between the Type I and Type II theories, which we shall now proceed to briefly describe. Let us first note that the codimension 1 cases described in sections 4.4 and 5.5 are actually realizations of the elementary Hopf fibration $\mathbf{S}^1 \to \mathbf{S}^1$ with discrete fiber $\mathbb{Z}_2$ [55]. An example of the construction of a Type I non-BPS brane as a kink of brane-antibrane pairs is of course the original construction [9] of the Type I D-particle from a D-string anti-D-string pair. The double cover of $\mathbf{S}^1$ corresponds to the pair of branes, and the winding number of the tachyon field is labelled by the homotopy group (4.29) of the fiber corresponding to the discrete gauge transformation $T \to -T$ (so that the D-string carries a $\mathbb{Z}_2$-valued Wilson line). The cup product with the generator ω of $\mathrm{KO}(\mathbf{B}^1, \mathbf{S}^0) = \mathbb{Z}_2$ then achieves the desired ABS mapping of $\mathbb{Z}_2$-valued KO-theory classes on $\mathrm{KO}(Y) \to \mathrm{KO}(Y \times \mathbf{B}^1, Y \times \mathbf{S}^0)$.

Generally, the Hopf fibration

$$\mathbf{S}^{n-1} \hookrightarrow \mathbf{S}^{2n-1} \longrightarrow \mathbf{S}^n \tag{5.30}$$

is non-trivial only for $n = 1, 2, 4, 8$ when its fiber $\mathbf{S}^{n-1}$ is a parallelizable sphere [51, 53]. This topological fact is related to the algebraic property that there are only four normed division algebras over the field of real numbers, corresponding respectively to the reals, the complex numbers, the quaternions and the octonions (or Cayley numbers). In that case, the classifying map of the fibration, which determines the corresponding topological soliton field, is determined by the principal $\mathrm{Spin}(n)$ bundle (2.103) and is a generator of

$$\pi_{n-1}\Big(\mathrm{Spin}(n)\Big)\,/\,\pi_{n-1}\Big(\mathrm{Spin}(n-1)\Big) = \begin{cases} \mathbb{Z}_2, & n=1 \\ \mathbb{Z}, & n=2,4,8\,. \end{cases} \tag{5.31}$$

As we have seen, the four Hopf fibrations determine all the fundamental bound state constructions of D-branes in Type I and Type II superstring theory, and hence the complete spectrum of D-brane charges in these theories rests on the fact that there are only four such fibrations. For $n \neq 1$, the topological charge of the corresponding soliton is given by the Pontryagin number density which is proportional to $\mathrm{tr}(F^{n/2})$, where F is the curvature of the associated topologically non-trivial gauge field configuration. For $n = 1$ the charge is determined by a $\mathbb{Z}_2$-valued Wilson line, as in [7]. This feature determines string solitons in terms of magnetic monopoles in the Type II theories, while in the Type I theories we obtain non-BPS branes as kinks, BPS branes as $SU(2)$ instantons, and both BPS and non-BPS branes as $\mathrm{Spin}(8)$ instantons. This topological property realizes all D-branes in terms of more conventional solitons, and it moreover determines the explicit forms of the non-trivial gauge fields living on the brane worldvolumes. Therefore, all fundamental D-brane constructions, and hence the complete spectrum of D-brane charges in Type I and Type II superstring theory, are quite naturally determined by the four non-trivial Hopf fibrations [19] which thereby provide a non-trivial link between the two types of string theories.

Some further connections can be deduced from the relationships that exist between the different types of K-theories. Given a complex vector bundle E, the correspondence $E \mapsto \overline{E}$ induces an involution on the group $\mathrm{K}(X)$. Furthermore, the realification and complexification functors r and c on the categories of real and complex vector bundles induce homomorphisms of the corresponding K-groups. The first one associates to each complex vector bundle its underlying real vector bundle, while the second one associates to each real vector bundle E the complex vector bundle $E \otimes_{\mathbb{R}} \mathbb{C} \cong E \oplus E$. Then there are the natural homomorphisms between the K-groups of the Type I and Type IIB theories

$$\mathrm{K}(X) \xrightarrow{r^*} \mathrm{KO}(X) \xrightarrow{c^*} \mathrm{K}(X). \tag{5.32}$$

Note that the composition $r^* \circ c^*$ is multiplication by 2, while $(c^* \circ r^*)([E]) = [E \oplus E]$. For example, consider the generator of $\widetilde{\mathrm{K}}(\mathbf{S}^4) = \mathbb{Z}$, which is the pseudo-real $SU(2)$ instanton bundle described above. To realize it as a generator of $\widetilde{\mathrm{KO}}(\mathbf{S}^4) = \mathbb{Z}$, which labels Type I 5-brane charge, it must be embedded in the orthogonal structure group as $SO(4) = SU(2) \times SU(2)$ to make it real. The embedding in $SO(4)$ doubles the charge, since the natural map in (5.32) from $\mathrm{KO}(\mathbf{S}^4)$ to $\mathrm{K}(\mathbf{S}^4)$ is multiplication by 2, and so the RR charge of a Type I 5-brane is twice that of a Type IIB 5-brane. These facts may be viewed as special instances of the natural periodicity isomorphisms [51]

$$\begin{aligned} \mathrm{KO}^{-n}(X,Y) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{2}\right] &= \mathrm{KO}^{-n-4}(X,Y) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{2}\right], \\ \mathrm{K}^{-n}(X,Y) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{2}\right] &= \left(\mathrm{KO}^{-n}(X,Y) \oplus \mathrm{KO}^{-n-2}(X,Y)\right) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{2}\right], \end{aligned} \tag{5.33}$$

which can be proven using the cup product with the class of the $SU(2)$ instanton bundle. More generally, the Type I and Type II theories are related by the exact sequence

$$\begin{aligned} \mathrm{KO}^{-n-1}(X,Y) \xrightarrow{c^*} \mathrm{K}^{-n-1}(X,Y) \xrightarrow{r^*\circ\beta} \mathrm{KO}^{-n+1}(X,Y) \\ \xrightarrow{\otimes\omega} \mathrm{KO}^{-n}(X,Y) \xrightarrow{r^*} \mathrm{K}^{-n}(X,Y) \end{aligned} \tag{5.34}$$

where $\beta : \mathrm{K}^{-n-1}(X,Y) \to \mathrm{K}^{-n+1}(X,Y)$ is the Bott periodicity isomorphism.

6 D-Branes on Orbifolds and Orientifolds

In this section we will analyze the properties of D-branes in orbifolds and orientifolds of the Type II and Type I theories. As we shall see, the natural K-theoretic arena for this classification is equivariant K-theory which takes into account of a group action on the spacetime. Equivariant K-theory is of enormous interest in mathematics because it merges cohomology with group representation theory. It is therefore of central importance to both topology and group theory. In the following we will see that it also leads to some non-trivial aspects of the D-brane spectrum in these theories.

6.1 Equivariant K-Theory

Consider Type IIB superstring theory on an orbifold X/G, where G is a finite group of symmetries of X. In this subsection we will show that D-branes on X/G are classified by the so-called G-equivariant K-theory group of X [69]. This group is defined as follows. Let X be a smooth manifold and G a group acting on X (in general G is either a finite group or a compact Lie group). In this situation we say that X is a *G-manifold* and write the G-action $G \times X \to X$ as $(g, x) \mapsto g \cdot x$. A *G-map* $f : X \to Y$ between two G-manifolds is a smooth map which commutes with the action of G on X and Y:

$$f(g \cdot x) = g \cdot f(x). \tag{6.1}$$

In other words, f is G-equivariant. A *G-bundle* $E_G \to X$ is a principal fiber bundle $E \to X$ with E a G-manifold and canonical fiber projection π which is a G-map, i.e., $\pi(g \cdot v) = g \cdot \pi(v)$, for all $v \in E$, $g \in G$. A *G-isomorphism* $E_G \to F_G$ between G-bundles over X is a map which is both a bundle isomorphism and a G-map. These conditions define the category of G-equivariant bundles over the G-space X. The corresponding Grothendieck group is called the G-equivariant K-theory $\mathrm{K}_G(X)$, i.e., $\mathrm{K}_G(X)$ consists of pairs of bundles (E, F) with G-action, modulo the equivalence relation $(E, F) \sim (E \oplus H, F \oplus H)$ for any G-bundle H over X.

D-brane configurations on X/G are understood as G-invariant configurations of D-branes on X [37], i.e., the orbifold spacetime is regarded as a G-space. We assume that X is endowed with an orientation and a spin structure, both of which are preserved by G. Given a D-brane configuration, i.e., a virtual bundle $[(E, F)]$, we can assume that G acts on (E, F), since the gauge bundles can be constructed in a completely G-invariant way. In tachyon condensation, we assume that a pair of bundles (H, H) can be created and annihilated only if G acts on both copies of H in the same way (otherwise the requisite tachyon field would not be G-invariant). Thus, we conclude that for Type IIB superstrings on an orbifold X/G, D-brane charge takes values in $\mathrm{K}_G(X)$. For Type IIA one similarly has $\mathrm{K}_G^{-1}(X)$ and for Type I we get $\mathrm{KO}_G(X)$ (here $\mathrm{K}_G^{-1}(X) \equiv \mathrm{K}_G(\Sigma X) = \mathrm{K}_G(\mathbf{S}^1 \wedge X)$ with G acting trivially on the $\mathbf{S}^1$).

Let V_I denote the irreducible, finite-dimensional complex representation vector spaces of the group G. As in section 2.8, the isomorphism classes $[V_I]$ of the additive category of G-modules with respect to the direct sum of vector spaces, i.e., with $[V_I]+[V_J] = [V_I \oplus V_J]$, generates an abelian monoid. The corresponding Grothendieck group $\mathrm{R}(G)$ is called the *representation ring* of the group G. According to the description of section 2.1, each element of $\mathrm{R}(G)$ can be expressed as a formal difference $[V_I] - [V_J]$, where $[V_I]$ and $[V_J]$ are equivalence classes of finite-dimensional representations of G. Thus we have $[V_I]-[V_J] = [V_I']-[V_J']$ if and only if $V_I \oplus V_J'$ is unitarily equivalent to $V_I' \oplus V_J$. As always, the tensor product of vector spaces $V_I \otimes V_J$ induces a commutative ring structure on $\mathrm{R}(G)$. For example, let $G = \mathbf{S}^1$ and let y^m denote the one-dimensional representation defined by

$$\varphi_m(\sigma)\, z = \mathrm{e}^{im\sigma}\, z, \qquad z \in \mathbf{S}^1, \tag{6.2}$$

with $\sigma \in \mathbb{R}$. Then it is easy to see that the representation ring of the compact group $\mathbf{S}^1 = U(1)$ is the ring of formal Laurent polynomials in the variable y:

$$\mathrm{R}(\mathbf{S}^1) = \mathbb{Z}\left[y, y^{-1}\right]. \tag{6.3}$$

The representation ring of the cyclic subgroup $\mathbb{Z}_n \subset \mathbf{S}^1$ is the direct sum of n integer groups:

$$\mathrm{R}(\mathbb{Z}_n) = \mathbb{Z}^{\oplus n}, \tag{6.4}$$

while the representation ring of the torus group $\mathbf{T}^n = U(1)^n$ is the ring of formal Laurent polynomials in n variables $y_1, \dots, y_n$:

$$\mathrm{R}(\mathbf{T}^n) = \mathbb{Z}\left[y_1, y_2, \dots, y_n, (y_1 y_2 \cdots y_n)^{-1}\right]. \tag{6.5}$$

Generally, for any simply connected Lie group G, $\mathrm{R}(G)$ is a polynomial ring over $\mathbb{Z}$ with $\mathrm{rank}(G)$ generators [70].

A more familiar description of $\mathrm{R}(G)$ is in terms of the space of characters of the group G. The isomorphism class of the G-module V_I is completely determined by its character map $\chi_{V_I} : G \to \mathbb{C}$ defined by $\chi_{V_I}(g) = \mathrm{tr}_{V_I}(g)$. Since the characters enjoy the properties $\chi_{V_I \oplus V_J} = \chi_{V_I} + \chi_{V_J}$, $\chi_{V_I \otimes V_J} = \chi_{V_I}\chi_{V_J}$, and $\chi_{V_I}(hgh^{-1}) = \chi_{V_I}(g)$, it follows that the map $V_I \mapsto \chi_{V_I}$ identifies $\mathrm{R}(G)$ as a subring of the ring of G-invariant complex-valued functions on G. We shall see that the representation ring correctly incorporates the structure of the mirror brane charges induced by the action of G on X.

If G acts trivially on the spacetime X then

$$\mathrm{K}_G(X) = \mathrm{K}(X) \otimes \mathrm{R}(G), \tag{6.6}$$

where $\mathrm{K}(X)$ is the ordinary K-group of X. This follows from the fact that, for trivial G-actions, a G-bundle E may be decomposed as

$$E \cong \bigoplus_I \mathrm{Hom}_G(E_I, E) \otimes E_I, \tag{6.7}$$

where $E_I = X \times V_I$ is the trivial bundle over X with fiber V_I. More generally, for any compact G-space X, the collapsing map $X \to \mathrm{pt}$ gives rise to an $\mathrm{R}(G)$-module structure on $\mathrm{K}_G^{\#}(X)$, such that $\mathrm{R}(G)$ is the coefficient ring in equivariant K-theory (rather than simply $\mathbb{Z}$ as in the ordinary case). The K_G-functor enjoys most of the properties of the ordinary K-functor that we described in the previous sections. In addition, K_G is functorial with respect to group homomorphisms. In this sense, $\mathrm{K}_G(X)$ is a generalization of the two important classification groups $\mathrm{K}(X)$ and $\mathrm{R}(G)$, so that equivariant K-theory unifies K-theory and group representation theory. In fact, the trivial space $X = \mathrm{pt}$ gives $\mathrm{K}_G(X) = \mathrm{R}(G)$ while the trivial group $G = \mathrm{Id}$ gives $\mathrm{K}_G(X) = \mathrm{K}(X)$. A useful "excision type" computational feature is that if H is a closed subgroup of G, then for any H-space X, the inclusion $i : H \hookrightarrow G$ induces an isomorphism $i^* : \mathrm{K}_G(G \times_H X) \xrightarrow{\approx} \mathrm{K}_H(X)$.

If the group G acts freely on X (i.e., without fixed points), then X/G is also a topological space and its G-equivariant K-theory is just

$$\mathrm{K}_G(X) = \mathrm{K}(X/G). \tag{6.8}$$

However, in general X/G is not a topological space (let alone a smooth manifold) and the G-equivariant cohomology is far more intricate. Then, a useful theorem for computing equivariant K-theory is the six-term exact sequence that was introduced in section 2.6:

$$\begin{array}{ccccc}
\mathrm{K}_G^{-1}(X,Y) & \longrightarrow & \mathrm{K}_G^{-1}(X) & \longrightarrow & \mathrm{K}_G^{-1}(Y) \\
\partial^* \uparrow & & & & \downarrow \partial^* \\
\mathrm{K}_G(Y) & \longleftarrow & \mathrm{K}_G(X) & \longleftarrow & \mathrm{K}_G(X,Y)
\end{array} \tag{6.9}$$

where Y is a closed G-subspace of a locally compact G-space X, and the relative K-theory is defined by $\mathrm{K}_G^{-n}(X,Y) = \widetilde{\mathrm{K}}_G^{-n}(X/Y)$ (when the quotient space makes sense). The advantage of using this exact sequence is that one may take Y to be the fixed point set of the group action on X, such that the quotient space X/Y has a free G-action on it and its equivariant cohomology can be computed as the ordinary cohomology of its quotient by G, as in (6.8).

From now on, we will assume that G is a finite discrete group of symmetries of the spacetime manifold X. Away from any orbifold singularities of X/G, D-brane charge is classified according to (6.6) which yields the usual Type II spectrum, taking into account the mirror images connected by the G-action. Therefore, we want to understand how brane-antibrane pairs behave at the singular points. For this we need to know how the orbifold projection is realized on the Chan-Paton factors. From the general theory of D-branes in orbifold singularities it is known [67, 37] that the action of G on Chan-Paton indices is given by

$$g \cdot \mathcal{X}(\Lambda) = \mathcal{X}(\gamma_g \, \Lambda), \tag{6.10}$$

where Λ is the Chan-Paton factor of a field $\mathcal{X}$ and γ_g is the representation of $g \in G$ in the Chan-Paton gauge group. (An example is the action of the GSO projection that we described at the end of section 3.1). In particular, like a vector potential $A_i(x)$, the tachyon field transforms in the adjoint representation under the G-orbifold projection, i.e., it is G-equivariant:

$$g: \quad T(x) \quad \longmapsto \quad \gamma_g \, T\left(g^{-1} \cdot x\right) \, (\gamma_g)^{-1}. \tag{6.11}$$

In this way, the tachyon field can be thought of as either a G-bundle map $T : E_G \to F_G$, or equivalently as a G-section of the G-bundle $(E \otimes F^*)_G$.

In considering brane charges in terms of $9 - \overline{9}$ brane pairs on orbifold singularities, considerations similar to earlier ones apply, but now including the mirror images induced under the action of G, i.e., at an orbifold singularity, each brane pair has $|G|$ mirror pairs. The gauge fields from the vector multiplet of the worldvolume spectrum in X/G define a connection $A_i(x)$ of the corresponding Chan-Paton bundle. The GSO projection cancels tachyonic degrees of freedom leaving only the quiver structure of vector multiplets and hypermultiplets [37]. However, when coincident branes and antibranes wrap a submanifold Y/G of the orbifold spacetime, the tachyon field is still preserved by the GSO projection and the massless vector multiplet is projected out, i.e., the G-action commutes with the GSO projection. The worldvolume field theory of N branes wrapped on $Y \subset X$ is described via Chan-Paton bundles E over Y with structure groups $\prod_I U(Nn_I)$, where n_I is the dimension of the I-th regular representation of G. The vacuum configuration at infinity is re-expressed now in a G-equivariant way in terms of the characters of G. Then the resulting G-invariant vacuum may be reached by tachyon condensation, provided that (6.11) holds. The bound state construction may now be carried out just as before. Some explicit constructions of D-branes on orbifolds using equivariant K-theory may be found in [28].

However, it turns out that for sufficiently "regular" orbifolds, equivariant K-theory does not really provide new information or new states that are not already described according to ordinary cohomology theory or K-theory [29]. For instance, equivariant Bott periodicity $\widetilde{\mathrm{K}}_G(\mathbf{S}^{p+2}) = \widetilde{\mathrm{K}}_G(\mathbf{S}^p)$ implies that $\widetilde{\mathrm{K}}_G(\mathbf{S}^{2k}) = \widetilde{\mathrm{K}}_G(\mathbf{S}^0) = \mathrm{R}(G)$, yielding typically $|G|$ copies of the usual RR charge. In the equivariant cases, the Bott periodicity theorems related different sets of $|G|$ branes to each other, where $|G|$ is the number of mirror images in the orbifold. As an illustration, consider the $\mathbb{Z}_3$ AdS-orbifold for Type IIB supergravity on $\mathbf{AdS}_5 \times \mathbf{S}^5$ which is dual to the $\mathcal{N} = 1$ superconformal field theory on its boundary that is an $SU(N)^3$ gauge theory on the worldvolume of N parallel D3-branes placed at an orbifold singularity [71]. The supergravity horizon is the Lens space

$$\mathcal{H} = \mathrm{L}^2(3) \equiv \mathbf{S}^5/\mathbb{Z}_3. \tag{6.12}$$

Extended objects in the boundary theory are understood as Type IIB branes which wrap cycles in $\mathcal{H}$. The nontrivial homology groups of the horizon are

$$H_1(\mathcal{H}) = H_3(\mathcal{H}) = \mathbb{Z}_3, \tag{6.13}$$

which correspond respectively to D3-branes and D5-branes wrapped on a one-cycle and a three-cycle of $\mathcal{H}$. However, there are also wrapped NS5-branes on the three-cycles, corresponding to the discrete symmetry group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \otimes \mathbb{Z}_3$ of the boundary superconformal field theory. The K-group of the Lens space $\mathcal{H}$ is [72]

$$\mathrm{K}(\mathcal{H}) = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \cong H^{\mathrm{even}}(\mathcal{H}, \mathbb{Z}), \tag{6.14}$$

where H^{even} denotes the subring of elements of even degree in the ordinary cohomology ring. Thus the K-group correctly accounts for the D3-brane and D5-brane torsion charges, but it is missing the non-commuting $\mathbb{Z}_3$-valued charge of the NS5-brane. This is not at all surprising, because topological K-theory always has an underlying *commutative* ring structure, and it does not take into account the Neveu-Schwarz B-field (equivalently the S-duality symmetry of Type IIB superstring theory) [17]. For this particular orbifold example, K-theory completely agrees with ordinary cohomology theory and does not supply us with new objects. This example lies in a particular class whereby the spacetime manifold is birationally equivalent to a smooth toric variety for which the K-groups are torsion-free and thus the Chern character, to be discussed in section 7.1, yields an isomorphism with the corresponding cohomology ring [29]. In light of this feature, we will now turn our attention to orientifolds, whereby the discrete geometrical action of G on X is further accompanied by a worldsheet symmetry action on the superstring theory.

6.2 Real K-Theory

Real KR-theory [73] is a generalized K-theory which merges complex K-theory, real KO-theory (as well as quaternionic KSp-theory and self-conjugate KSC-theory) with equivariant K-theory. We will specialize the orbifold construction above to the case $G = \mathbb{Z}_2$, so that the space X is equipped with an involution, i.e., a homeomorphism $\tau : X \to X$ with $\tau^2 = \mathrm{Id}_X$. In addition to the equivariant cohomology, we shall quotient by the action of the worldsheet parity transformation Ω. Ω reverses the orientation of a string, and it induces an anti-linear involution on gauge bundles E over X that commutes with τ. In making the orientifold projection by Ω (in Type IIB string theory on X), we need to retain K-theory classes that are in effect even under the projection by Ω. Since Ω acts on 9-brane (and $\overline{9}$-brane) Chan-Paton bundles by complex conjugation, we consider an induced anti-linear involution $\tau^* : E_x \to E_{\tau(x)}$ acting on the fibers of gauge bundles, with $(\tau^*)^2 = 1$, that maps E to its complex conjugate bundle $\overline{E}$. Thus $\tau^*(E) \cong \overline{E}$ with isomorphism $\psi : \tau^*(E) \to \overline{E}$ satisfying $(\psi\tau^*)^2 = \mathrm{Id}$. Now we define an equivalence relation on the category of such vector bundles by $(E, F) \sim (E \oplus H, F \oplus H)$ for any bundle H that is similarly mapped by the involution τ^* to its complex conjugate. The Grothendieck group of all virtual bundles with involutions on X is called the Real K-group $\mathrm{KR}(X)$.

As usual, one defines higher groups $\mathrm{KR}^{-m}(X)$ by

$$\widetilde{\mathrm{KR}}^{-m}(X) = \widetilde{\mathrm{KR}}(X \wedge \mathbf{S}^m), \tag{6.15}$$

with the involution τ on X extended to $X \wedge \mathbf{S}^m$ by a trivial action on $\mathbf{S}^m$. More generally, one can extend the definition (6.15) to spheres on which τ acts non-trivially. Let $\mathbb{R}^{p,q}$ be the $p+q$ dimensional real space in which an involution acts as a reflection of the last q coordinates, i.e., given $(x,y) \in \mathbb{R}^p \times \mathbb{R}^q$ we have $\tau : (x,y) \mapsto (x,-y)$. Let $\mathbf{S}^{p,q}$ be the unit sphere of dimension $p+q-1$ in $\mathbb{R}^{p,q}$ with respect to the flat Euclidean metric on $\mathbb{R}^p \times \mathbb{R}^q$. With these definitions, we may define a two-parameter set of higher degree KR-groups according to

$$\widetilde{\mathrm{KR}}^{p,q}(X) = \widetilde{\mathrm{KR}}(X \wedge \mathbb{R}^{p,q}_+), \tag{6.16}$$

or by using the suspension isomorphism:

$$\mathrm{KR}^{p,q}(X) = \mathrm{KR}(X \times \mathbb{R}^{p,q}). \tag{6.17}$$

With these definitions we have

$$\mathrm{KR}^{-n}(X) = \mathrm{KR}^{n,0}(X). \tag{6.18}$$

Bott periodicity in KR-theory takes the form

$$\begin{aligned} \mathrm{KR}^{p,q}(X) &= \mathrm{KR}^{p+1,q+1}(X)\,, & (6.19)\\ \mathrm{KR}^{-m}(X) &= \mathrm{KR}^{-m-8}(X). & (6.20) \end{aligned}$$

The relation (6.19) implies that $\mathrm{KR}^{p,q}(X) = \mathrm{KR}^{q-p}(X)$ so that $\mathrm{KR}^{p,q}(X)$ only depends on the difference $p-q$. The relation (6.20) then states that $\mathrm{KR}^{p,q}(X)$ depends only on this difference modulo 8. This implies that one can define negative-dimensional spheres as those with antipodal involutions in KR-theory, with $\mathbf{S}^{n,0}$ being identified as $\mathbf{S}^{n-1}$ and $\mathbf{S}^{0,n}$ as $\mathbf{S}^{-n-1}$. Note that if we identify $\mathbb{R}^{1,1} = \mathbb{C}$ with the involution τ acting by complex conjugation, then the $(1,1)$ periodicity theorem (6.19) takes the particularly nice form

$$\mathrm{KR}(X) = \mathrm{KR}(X \times \mathbb{C}), \tag{6.21}$$

for any locally compact space X.

KR-theory is a generalization of K-theory and KO-theory because of the following internal symmetries. If the involution τ acts trivially on X, then

$$\begin{aligned} \mathrm{KR}^{-m}(X \times \mathbf{S}^{0,1}) &= \mathrm{K}^{-m}(X), \qquad (6.22)\\ \mathrm{KR}^{-m}(X) &= \mathrm{KO}^{-m}(X). \qquad (6.23) \end{aligned}$$

The relation (6.22) follows from the fact that the space $X \times \mathbf{S}^{0,1}$ can be identified with the double cover $\tilde{X} = X \amalg X$ of X, with τ acting by exchanging the two copies of X in $\tilde{X}$. In particular, if X^τ denotes the set of fixed points of the map $\tau : X \to X$, then

$$\mathrm{KR}^{-n}(X^\tau) = \mathrm{KO}^{-n}(X^\tau), \qquad (6.24)$$

because the involution τ acts trivially on a fixed point. There are many further such internal symmetries in Real K-theory, coming from the usage of negative dimensional spheres. Using the multiplication maps in the fields $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, and (1,1) periodicity, one may establish the isomorphisms [73]

$$\mathrm{KR}(X \times \mathbf{S}^{0,p}) = \mathrm{KR}^{-2p}(X \times \mathbf{S}^{0,p}), \qquad (6.25)$$

for $p = 1, 2$ and 4, respectively. This isomorphism for $p = 1$ gives the complex Bott periodicity theorem, while the real periodicity theorem can be deduced from the case $p = 4$. In fact, there is the usual natural isomorphism

$$\mathrm{KR}^{-n}(X \times \mathbf{S}^{0,p}) = \mathrm{KR}^{-n}(X) \oplus \mathrm{KR}^{p+1-n}(X), \qquad (6.26)$$

for all $p \geq 3$. The case $p = 2$, where there is no splitting into KR-groups of X, is special and will be discussed in section 6.5. Again, most of the properties discussed in section 2 have obvious counterparts in the Real case. In particular, the product formulas derived in section 2 can also be extended to KR-theory (as they did for KO-theory). For example, by repeating the steps which led to (2.36) we may obtain, for a trivial action of τ on X, the product formula

$$\begin{aligned} \widetilde{\mathrm{KR}}^{-1}(X \times \mathbf{S}^{1,1}) &= \widetilde{\mathrm{KR}}^{-1}(X \wedge \mathbf{S}^{1,1}) \oplus \widetilde{\mathrm{KR}}^{-1}(X) \oplus \widetilde{\mathrm{KR}}^{-1}(\mathbf{S}^{1,1})\\ &= \widetilde{\mathrm{KR}}^{1,1}(X) \oplus \widetilde{\mathrm{KO}}^{-1}(X) \oplus \mathbb{Z}\\ &= \widetilde{\mathrm{KO}}(X) \oplus \widetilde{\mathrm{KO}}^{-1}(X) \oplus \mathbb{Z}, \qquad (6.27) \end{aligned}$$

where we have used $(1,1)$ periodicity, eq. (6.23), and the fact that

$$\widetilde{\mathrm{KR}}^{-1}(\mathbf{S}^{1,1}) = \mathrm{KR}^{-1+1}(\mathrm{pt}) = \mathrm{KO}(\mathrm{pt}) = \mathbb{Z}. \tag{6.28}$$

For $X = \mathbf{S}^n$ the periodicity theorem (6.19) can also be deduced from the ABS construction for Real K-theory. For this, we define a two-parameter set of Clifford algebras $C\ell(\mathbb{R}^{n,m})$ of the Real space $\mathbb{R}^{n,m}$ as the usual algebra associated with $\mathbb{R}^{n+m}$ together with an involution generated by the action of τ on $\mathbb{R}^{n,m}$. A Real module over $C\ell(\mathbb{R}^{n,m})$ is then a finite-dimensional representation together with a $\mathbb{C}$-antilinear involution which preserves the Clifford multiplication. The corresponding representation ring $\mathrm{R}[\mathrm{Spin}(n,m)]$ is naturally isomorphic to the Grothendieck group generated by the irreducible $\mathbb{R}$-modules $\Delta_{n,m}$ of the Clifford algebra of the space $\mathbb{R}^n \oplus \mathbb{R}^m$ with quadratic form of Lorentzian signature (n,m), as in section 2.8. The ABS map is now the graded ring isomorphism [52, 73]

$$\mathrm{KR}(\mathbb{R}^{n,m}) \;\cong\; \mathrm{R}\Big[\mathrm{Spin}(n,m)\Big] \,/\, i^*\,\mathrm{R}\Big[\mathrm{Spin}(n+1,m)\Big] \;=\; \mathrm{KO}^{n-m}(\mathbf{S}^0), \tag{6.29}$$

where in the last equality we have used the periodicity relations (6.17), (6.19) and (6.20). This isomorphism relates the groups on the left-hand side of (6.29) to the Clifford algebras $C\ell_{n,m}$, so that the topological $(1,1)$ periodicity (6.19) follows from the algebraic $(1,1)$ periodicity (2.82).

Let us now discuss how Real K-theory can be used to classify D-branes in Ω-orientifolds. Generally, the fixed point set of a G-action on X is a number of $p+1$ dimensional planes called orientifold p-planes, or Op-planes for short. They determine the singular points of the given orbifold. For the present orientifold group action, these objects are non-dynamical but they share many of the properties of D-branes themselves. For instance, they carry RR charge and have light open string states connecting them and the D-branes, which enhances the gauge symmetry of coincident branes over an orientifold plane. Having a non-trivial gauge symmetry means that the supersymmetric vacuum state of these theories must contain 32 Dp-branes in order to render the vacuum neutral (this is again the requirement of tadpole anomaly cancellation). We want to determine the charges of stable (but possibly non-BPS) states localized over an orientifold plane of $X/\Omega\cdot G$. Note that far away from the orientifold planes, we can think of the spacetime manifold X as being represented by a double cover $\tilde{X} \to X$ with the orientifold group $\Omega \cdot G$ mapping the two disconnected components of $\tilde{X}$ into each other. Using (6.22), we see that far away from the Op-planes the theory looks just like ordinary Type II superstring theory. We are therefore interested only in what happens to states which are localized on the singular Op-planes.

From the periodicity relations (6.19) and (6.20), we may show quite generally that

$$\mathrm{KR}(\mathbb{R}^{d-p,9-d}) = \mathrm{KR}(\mathbb{R}^{2d-p-1,0}), \tag{6.30}$$

where we have identified the spacetime X with the Real space $\mathbb{R}^{d+1} \times (\mathbb{R}^{9-d}/\Omega \cdot \mathcal{I}_{9-d})$, with $\mathcal{I}_{9-d}$ the reflection τ acting on $9-d$ coordinates of the transverse space. The KR-group (6.30) classifies Dp-brane charges localized over an orientifold d-plane. On the right-hand side of (6.30) we have a Real space with the KR-involution acting trivially, so that

$$\mathrm{KR}(\mathbb{R}^{d-p,9-d}) = \mathrm{KO}(\mathbf{S}^{2d-p-1}). \tag{6.31}$$

Setting $d = 9$ in (6.31) gives the usual group $\mathrm{KO}(\mathbf{S}^{9-p})$ that classifies Dp-brane charge in ordinary Type I superstring theory. Setting $d = 8$ leads to

$$\mathrm{KR}^{-1}(\mathbb{R}^{d-p-1,1}) = \mathrm{KO}^{-1}(\mathbf{S}^{15-p}) = \mathrm{KO}(\mathbf{S}^{8-p}), \tag{6.32}$$

giving a shift by one of the Type I charge spectrum. In the next subsection it is shown that the T-dual of the Type I theory on a spacetime manifold X is classified by $\mathrm{KR}^{-1}(X)$, in agreement with (6.32). In general, for a given dimensionality d of orientifold planes, one may use (6.31) and table 4 with the appropriate period shift to read off the charges of D-branes located over the d-planes. For example, for $d = 5$ we get the classification of stable D-brane charges localized on an O5-plane. This spectrum resembles that of Type I string theory in that there is a $\mathbb{Z}$-charged D-string, a $\mathbb{Z}_2$-charged gauge soliton, and a $\mathbb{Z}_2$-charged gauge instanton. This spectrum agrees perfectly with the bound state construction of an orientifold p-brane in terms of Type IIB p-brane-antibrane pairs, and the result (5.24) which shows that the tachyonic mode is removed by the Ω-projection only for $p = -1, 7$. A similar analysis can be carried out for Type IIA orientifolds.

A physical interpretation of the $(1,1)$ periodicity of KR-theory may also be given [19]. Consider a p-brane of codimension $n + m$ in a Type

II orientifold by $\Omega \cdot \mathcal{I}_m$. The p-brane charge is induced by the tachyon field which is given by Clifford multiplication on the transverse space $\mathbb{R}^{n,m}$, i.e., $T(x) = \sum_i \Gamma_i x^i$ where Γ_i are the generators of the spinor module $\Delta_{n,m}$, and which generates $\widetilde{\mathrm{KR}}(\mathbb{R}^{n,m})$. Under the ABS isomorphism (6.29), this KR-theory class is multiplied, via the cup product, by the Hopf generator of $\widetilde{\mathrm{KR}}(\mathbb{C}P^1) = \mathbb{Z}$ (with its natural Real structure induced by the antilinear complex conjugation involution), or equivalently by the spin bundles which carry the spinor representation $\Delta_{1,1}$. This gives a class with tachyon field that generates the KR-group of the new transverse space $\mathbb{R}^{n+1,m+1}$. This class represents a $p-2$-brane of the Type II orientifold by $\Omega \cdot \mathcal{I}_{m+1}$. From this mathematical fact one deduces a new descent relation for Type II orientifold theories, whereby a $p-2$-brane localized at an $\mathrm{O}(8-m)$-plane in a Type II $\Omega \cdot \mathcal{I}_{m+1}$ orientifold is constructed as the tachyonic soliton of a bound state of a p-$\overline{p}$ pair located on top of an $\mathrm{O}(9-m)$-plane in a Type II $\Omega \cdot \mathcal{I}_m$ orientifold. This realizes the branes of a Type II orientifold as equivariant magnetic monopoles in the worldvolumes of brane-antibrane pairs of an orientifold with fixed point planes of one higher dimension. The former orientifold has 2^m $\mathrm{O}(8-m)$-planes each carrying RR charge -2^{3-m}, while the latter one has 2^{m+1} $\mathrm{O}(9-m)$-planes of charge -2^{4-m}. In the process of tachyon condensation the number of fixed point planes is doubled while their charges are lowered by a factor of 2 via a combined operation of charge transfer (via the equivariant monopole) and dimensional reduction through the orientifold planes. An example is the non-BPS state consisting of a D5-brane on top of an orientifold 5-plane in the Type IIB theory [7], which may be constructed via a tachyon condensate from a pair of D7-D$\overline{7}$ branes on an orientifold 6-plane in the Type IIA theory. The 8 O6-planes each carrying charge -2 are transfered to the 16 O5-planes of charge -1.

6.3 Type I$'$ D-Branes and $\mathrm{KR}^{-1}(X)$

Type I$'$ superstring theory is the T-dual of the Type I theory, which may be obtained as the orientifold of Type IIA string theory of the form $X/\Omega \cdot \mathcal{I}_1$. This theory contains unstable spacetime-filling 9-branes, whose configurations up to creation and annihilation of elementary 9-branes classify all D-brane charges. In terms of K-theory this corresponds to the group $\mathrm{KR}^{-1}(X)$, or the group of equivalence classes

$[(E,\alpha)]$ where E is a Real bundle with an involution that commutes with the orientifold group and α is an automorphism of E that also preserves the orientifold group action. In Type I$'$ string theory, E is identified with the Chan-Paton bundle on the worldvolume of the spacetime-filling 9-branes. At the orientifold planes, the gauge symmetry is reduced from $U(N)$ to $O(N)$. Each individual lower-dimensional brane is represented as a bound state of a certain number of unstable Type I$'$ 9-branes. The tachyon condensate is required to respect the $\mathbb{Z}_2$ orientifold symmetry (as in (6.11)), corresponding to a $\mathbb{Z}_2$-equivariant monopole. We will discuss this latter property in more detail in the next subsection.

6.4 The Bound State Construction for Type II Orientifolds

A T-duality transformation of the Type I theory on an m-torus $\mathbf{T}^m$ gives a Type II orientifold on $\mathbf{T}^m/\Omega \cdot \mathcal{I}_m$. In this section we shall describe some aspects of these orientifold theories using K-theoretic properties, thereby extending the discussion of the last subsection. In particular, we will demonstrate how the formalism allows one to naturally deduce the complete set of vacuum manifolds for tachyon condensation in the T-dual theories of the Type I theory (see table 7), and hence the worldvolume field contents of D-branes in these models. (Superstring compactifications will be discussed in more generality in section 7). The rich structure that now arises, in contrast to the two unique vacuum manifolds (3.18) and (4.16) for tachyon condensation in the ordinary Type II theories, is a consequence of the 8-fold periodicity of the KO- and KR-functors. In terms of iterated loop spaces, $\Omega^n BO(k)$ is of the same homotopy type as $\Omega^{n+8}BO(k)$, while $\Omega^m O(k)$ for $0 \leq m \leq 7$ are of the same homotopy types as the loop spaces of the Lie groups given in the fourth column of table 7 [56]. The vacuum manifolds of the Type II orientifolds are thereby very natural consequences of the homotopy properties of KO-theory. Indeed, the identification of these worldvolume gauge symmetries is a genuinely new prediction made solely by K-theory. Moreover, the periodicity of 8 is in agreement with the fact that the cycle of distinguishing properties and dualities of Type II orientifolds starts over again on the compactification torus $\mathbf{T}^8$. (A concise overview of the properties of Type II orientifolds and

m	Real Spin Module	Dimension	Vacuum Manifold	Dual Theory
0	—	—	$O(N)$	Type I
1	Δ_1	1	$\frac{O(2N)}{O(N)\times O(N)}$	Type I$'$
2	$\Delta_2^+ \oplus \Delta_2^-$	2	$\frac{U(2N)}{O(2N)}$	IIB on $\mathbf{T}^2/\Omega\cdot\mathcal{I}_2$
3	$\Delta_3 \oplus \Delta_3$	4	$\frac{Sp(2N)}{U(2N)}$	IIA on $\mathbf{T}^3/\Omega\cdot\mathcal{I}_3$
4	$\Delta_4^\pm \oplus \Delta_4^\pm$	4	$Sp(2N)$	IIA on $K3$
5	$\Delta_5 \oplus \Delta_5$	8	$\frac{Sp(4N)}{Sp(2N)\times Sp(2N)}$	IIB on $K3\times\mathbf{S}^1$
6	$\Delta_6^+ \oplus \Delta_6^-$	8	$\frac{U(8N)}{Sp(4N)}$	IIB on $\mathbf{T}^6/\Omega\cdot\mathcal{I}_6$
7	Δ_7	8	$\frac{O(16N)}{U(8N)}$	IIA on $\mathbf{T}^7/\Omega\cdot\mathcal{I}_7$
8	$\Delta_8^+ \oplus \Delta_8^-$	8	$O(16N)$	IIA on $\mathbf{T}^8/\mathbb{Z}_2$

Table 7: Type II orientifold theories on spacetimes $X = Y \times \mathbf{T}^{1,m}$ whose D-brane charges are classified by the group $\mathrm{KR}^{-m}(X)$. The general dual orbifold model in each case is listed (column 5) along with the corresponding vacuum manifold for tachyon condensation in the worldvolume of $2^{[m/2]+1}N$ spacetime filling 9-branes (column 4) whose stable homotopy group coincides with $\mathrm{KR}^{-m}(X)$. The second column lists the appropriate real spinor module which is used to map each KO-theory class of the Type I theory into the corresponding KR-theory class of the orientifold. Their dimensions (column 3) determine the appropriate increase in the number of 9-branes needed for the bound state construction as required by K-theoretic stabilization.

their moduli spaces may be found in [74].) In the rest of this subsection we shall give some physical interpretations to the appearence of these stable homotopy properties. More details can be found in [19].

After a T-duality transformation on $\mathbf{T}^1 = \mathbf{S}^1$, the superstring theory is, as mentioned in the last subsection, the Type I$'$ theory. The mapping between Type I and Type I$'$ is similar to the mapping in Type II superstring theory where T-duality maps Type IIB into Type IIA. The induced charge takes values in the higher KO-group $\widetilde{\mathrm{KO}}^{-1}(\mathbf{S}^{l+1})$, from which we identify the vacuum manifold in the second line of table 7. Next, consider the toroidal compactification of the Type I theory which is T-dual to Type IIB superstring theory on the $\mathbf{T}^2/\mathbb{Z}_2$ orientifold. There are N 7-$\overline{7}$ brane pairs that are described in terms of $2N$ 9-$\overline{9}$ brane pairs which were used in the bound state construction of D-branes in the original Type I theory. In the dual orientifold model, lower-dimensional branes may then be constructed out of the 7-branes using the "descent" procedures described earlier. The appearance of the unitary group $U(2N)$ in the third line of table 7 is then due to the following facts. Recall from section 2.8 that the chiral spinor modules $\Delta_2^\pm$ are complex, so that, in order to preserve the reality properties of the Type I theory, the desired map which takes us via the cup product between the K-groups of the two Type I theories must be taken with respect to the real spinor module $\Delta_2^+ \oplus \Delta_2^-$, as in (3.24). The overall number of 9-branes required for the bound state construction is given by multiplying the original number of 9-branes by the dimension of the spinor representation, given in the third column of table 7. The relevant homotopy is therefore defined with respect to a unitary symmetric space. Physically, the appearance of a unitary gauge symmetry can again be understood from (5.24), which leads to an inconsistency on 7-branes that are therefore quantized using the unprojected unitary gauge bundles. Thus, while the naive gauge group on the spacetime filling 9-branes is $O(2N) \times O(2N) \subset O(4N)$, the inconsistent Ω-projection on IIB 7-branes enhances the symmetry to $U(2N)$. The requisite tachyon field $T(x)$ is required to be $\mathbb{Z}_2$-equivariant with respect to the orientifold projection (in order that the resulting lower dimensional brane configurations be invariant under the $\mathbb{Z}_2$-action), i.e., it transforms under the orientifold group as in (6.11). As shown in [17], the tachyon vertex operator for a p-$\overline{p}$ brane pair acquires the phase $(\pm i)^{7-p}$ under the action of Ω^2. For the 7-branes this operator is even under Ω^2, and so the eigenvalues of the vacuum expectation value T_0 are real. Thus the tachyon field breaks the $U(2N)$ gauge symmetry down to its orthogonal subgroup $O(2N)$, and the induced brane charge is labelled by the winding numbers around the vacuum manifold $U(2N)/O(2N)$ of the IIB orientifold on $\mathbf{T}^2/\mathbb{Z}_2$. Note that, in general, the Type II orientifold on $X = Y \times \mathbf{T}^{1,m}$ is described by the KR-group $\mathrm{KR}^{-m}(X)$. The explicit relation between the KO- and KR-groups which implements the T-duality between the Type I and orientifold theories will be described in section 7.4.

For $m = 3$ we obtain the Type I′ theory on $\mathbf{T}^3$ which is T-dual to the $\mathbf{T}^3/\mathbb{Z}_2$ orientifold of Type IIA superstring theory. The appearance of a symplectic gauge group in table 7 follows from the mathematical fact that the complex spinor module Δ_3 is the restriction of a quaternionic Clifford module, so that the appropriate augmentation of the spin bundles on the 9-branes is taken with respect to the rank 4 real representation $\Delta_3 \oplus \Delta_3$. This means that there are now $4N$ unstable 9-branes which have an $Sp(2N)$ worldvolume gauge symmetry. This enhanced $Sp(2N)$ symmetry comes from the intermediate representation of a given Type I′ $p-3$-brane in terms of 6-$\overline{6}$ brane pairs [19] and is easily understood in terms of 5-branes, as we discussed in section 5.7. Again by $\mathbb{Z}_2$-equivariance the tachyon field breaks this gauge group to its complex subgroup $U(2N)$, so that the vacuum manifold is $Sp(2N)/U(2N)$. The rest of table 7 can be deduced from similar arguments. Note that the change of structure of the spinor modules and of the vacuum manifolds after the $m = 4$ compactification is in agreement with the property that the orientifold planes then begin acquiring fractional RR charges, leading to very different moduli spaces for these string theories [74].

In the last column of table 7 we have also indicated the appropriate dual superstring compactifications to the given toroidal compactification of the Type I theory (see [74] and references therein). For the cases $m = 4, 5$ and 8 we see that the moduli space of the Type I theory (or of the corresponding Type II orientifold) is actually non-perturbatively dual to a conventional *orbifold* of Type II superstring theory. The corresponding $\mathbb{Z}_2$-equivariant K-groups have been calculated in [19] using the product formulas (2.35) and (2.36), and the six-term exact sequence (6.9) (see also the computation at the end of section 6.6 to follow). This gives a heuristic way to check the given duality. However, the duality operations involve an intermediate S-duality transformation of the

Type II theory, whose description within the framework of K-theory is not yet known (see again the discussion in section 6.6 to follow). Thus one does not obtain isomorphisms of the corresponding K-groups, as would naively be expected. T-duality transformations of K-groups will be described in section 7.4.

Having identified the vacuum manifolds of the Type II orientifold models, we shall now describe the field content, where one must be careful about identifying the appropriate homotopy of the relevant vacuum manifolds. The classifying spaces for Real vector bundles are described in [75]. Consider an orientifold of the Type IIB theory, and a set of brane-antibrane pairs with worldvolume gauge symmetry $U(N)\times U(N)$. The $U(N)$ gauge group is endowed with its Hermitian conjugation involution, such that the fixed point set is the real subgroup $O(N)$. The tachyon field T is equivariant with respect to the orientifold group, so that

$$T(x,-y) = T(x,y)^* \tag{6.33}$$

where $(x,y) \in \mathbb{R}^n \oplus \mathbb{R}^m$ are coordinates of the transverse space to the induced lower dimensional brane configuration. It breaks the worldvolume gauge symmetry down to $U(N)_{\rm diag}$. The relevant homotopy group generated by (6.33) comes from decomposing the one-point compactification of $\mathbb{R}^{n,m}$ into upper and lower hemispheres as described in section 2.7, such that the tachyon field is the transition function on the overlap. The D-brane charges thereby reside in the KR-group of the transverse space which is given by

$$\widetilde{\rm KR}(\mathbb{R}^{n,m}) = \pi_{n,m}\Big(U(N)\Big)_{\rm R} \tag{6.34}$$

where the homotopy group is defined by the maps $\mathbf{S}^{n,m} \to U(N)$ which obey the Real equivariance condition (6.33). The refined Bott periodicity theorem for stable homotopy in KR-theory then reads

$$\pi_{n,m}\Big(U(\infty)\Big)_{\rm R} = \pi_{n+1,m+1}\Big(U(\infty)\Big)_{\rm R}. \tag{6.35}$$

In a similar way one may relate the Real K-groups $\widetilde{\rm KR}^{-1}(\mathbb{R}^{n,m}) = \widetilde{\rm KR}(\mathbb{R}^{n+1,m})$ to the stable equivariant homotopy of the complex Grassmannian manifold $U(2N)/[U(N)\times U(N)]$. Note that the gauge fields living on the brane worldvolumes in these cases must also satisfy an equivariance condition like (6.33). These remarks clarify the meaning of the term "equivariant soliton" in the bound state constructions for orbifold and orientifold theories.

6.5 Type $\widetilde{\mathrm{I}}$ D-Branes and KSC(X)

Type I$'$ superstring theory has two orientifold O8^- planes which each carry -8 units of RR charge. There is a natural extension of Type I$'$, which involves replacing one of its O8^- planes with an O8^+ plane that carries RR charge $+8$ and is quantized using symplectic gauge bundles (i.e., with $\Omega^2 = -1$). This theory requires no D8-branes to make the supersymmetric vacuum neutral, so it has no gauge group, yet it still contains interesting stable non-BPS D-branes in its spectrum. For the classification of D-brane charges, it is easier to start with the T-dual of this theory, which has been worked out in [76]. The theory is obtained by gauging a $\mathbb{Z}_2$-symmetry of Type IIB on a circle, which is realized by the composition of the worldsheet parity Ω with a half-circumference shift along the circle. This theory is usually called Type $\widetilde{\mathrm{I}}$. The natural K-group of Type $\widetilde{\mathrm{I}}$ D-brane charges is thus $\mathrm{KR}(X \times \mathbf{S}^{0,2})$ which, with a trivial involution action on X, is known to be isomorphic to the K-group KSC(X) of self-conjugate bundles on X [73]. This latter group can be defined as follows. Let X be a compact Real manifold with involution τ. A *self-conjugate* bundle over X is a complex vector bundle E together with an isomorphism $\alpha : E \xrightarrow{\approx} \overline{(\tau^* E)}$. Self-conjugate K-theory KSC(X) is then defined as the Grothendieck group generated by the category of self-conjugate bundles.

We will first prove that

$$\mathrm{KR}(X \times \mathbf{S}^{0,2}) = \mathrm{KSC}(X). \tag{6.36}$$

Consider the space $X \times \mathbf{S}^{0,2}$ and decompose the circle $\mathbf{S}^{0,2}$ into two halves $\mathbf{S}^{0,2}_\pm$ with $\mathbf{S}^{0,2}_+ \cap \mathbf{S}^{0,2}_- = \{\pm 1\}$. As usual, a Real vector bundle E over $X \times \mathbf{S}^{0,2}$ is equivalent to the specification of a complex vector bundle E_+ over $X \times \mathbf{S}^{0,2}_+$ (the corresponding restriction of E) together with an isomorphism

$$\psi : E|_{X\times\{+1\}} \xrightarrow{\approx} \tau^*(\overline{E}|_{X\times\{-1\}}). \tag{6.37}$$

D-brane	D8	D7	D6	D5	D4	D3	D2	D1	D0	D(–1)
Transverse space	$\mathbf{S}^1$	$\mathbf{S}^2$	$\mathbf{S}^3$	$\mathbf{S}^4$	$\mathbf{S}^5$	$\mathbf{S}^6$	$\mathbf{S}^7$	$\mathbf{S}^8$	$\mathbf{S}^9$	$\mathbf{S}^{10}$
$\widetilde{\mathrm{KSC}}(\mathbf{S}^n)$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$

Table 8: D-brane spectrum in Type $\widetilde{I}$ superstring theory from $\widetilde{\mathrm{KSC}}(\mathbf{S}^n)$.

Since $X \times \{+1\}$ is a deformation retract of $X \times \mathbf{S}^{0,2}_+$, we actually have an isomorphism $E_+|_{X\times\{-1\}} \xrightarrow{\approx} E_+|_{X\times\{+1\}}$ which is unique up to homotopy. This means that the specification of ψ is equivalent, up to homotopy, to giving an isomorphism $\alpha : E \xrightarrow{\approx} \overline{(\tau^* E)}$. In other words, isomorphism classes of Real bundles over $X \times \mathbf{S}^{0,2}$ are in one-to-one correspondence with homotopy classes of self-conjugate bundles over X. Taking τ to be trivial, we obtain the desired correspondence between $\mathrm{KR}(X \times \mathbf{S}^{0,2})$ and the K-theory of vector bundles E over a compact manifold X equipped with an antilinear automorphism $\alpha : E \xrightarrow{\approx} E$.

Using this equivalence, the D-brane charge spectrum of Type $\widetilde{\mathrm{I}}$ superstring theory can be computed using the results of [77], and is summarized in table 8. This demonstrates that K-theory predicts an interesting spectrum of BPS and non-BPS states in the Type $\widetilde{\mathrm{I}}$ theory. Upon analyzing the corresponding groups $\widetilde{\mathrm{KO}}(\mathbf{S}^n)$ and $\widetilde{\mathrm{KSp}}(\mathbf{S}^n)$ [32], one correctly accounts for the stable BPS D-branes whose charges are spread out over the two types of O8-planes. On the other hand, non-BPS $\mathbb{Z}_2$-charged D-branes which are locally stable near one kind of singular plane can become unstable due to the other singularities in the complete spacetime [32]. For example, analyzing $\widetilde{\mathrm{KO}}(\mathbf{S}^n)$ shows that there is a non-BPS D6-brane which is locally stable near the $\mathrm{O8}^-$-plane, because the orientifold projection removes the tachyonic mode present in the D6-brane mirror $\overline{\mathrm{D6}}$-brane system. However, the orientifold projection is different at the $\mathrm{O8}^+$-plane, so that the tachyon is no longer removed and the non-BPS D6-brane is no longer stable in the global theory. The $\mathbb{Z}_2$-valued charges in table 8 are precisely those non-BPS states which are globally stable.

The classifying space $BSC(k)$ for self-conjugate vector bundles is described in [77], so that KSC-groups are related to homotopy theory by

$$\mathrm{KSC}(X) = \Big[X\,,\,BSC(\infty)\Big]. \tag{6.38}$$

Alternatively, the connection with homotopy theory may be deduced from the KR-theory representation, from which we can identify the relevant bound state constructions for D-branes in the Type $\widetilde{\mathrm{I}}$ theory. From (6.25) it follows that Bott periodicity of self-conjugate K-theory is 4. Recall that the group $\mathrm{KR}^{-4}(X \times \mathbf{S}^{0,2})$ associates a symplectic projection to Ω. The 4-fold periodicity of KSC-theory is thereby the indication that the dual of Type $\widetilde{\mathrm{I}}$ has both $\mathrm{O8}^-$ and $\mathrm{O8}^+$ planes, since it means that orthogonal and symplectic gauge groups appear on equal footing in this model. Generally, self-conjugate K-theory is intimately tied to complex, real and quaternionic K-theories through the following long exact sequences [78]:

$$\cdots \longrightarrow \mathrm{K}^{-n-1}(X) \longrightarrow \mathrm{K}^{-n-1}(X) \longrightarrow \mathrm{KSC}^{-n}(X) \longrightarrow \mathrm{K}^{-n}(X) \longrightarrow \\ \longrightarrow \mathrm{K}^{-n}(X) \longrightarrow \cdots \tag{6.39}$$

$$\cdots \longrightarrow \mathrm{K}^{-n-1}(X) \longrightarrow \mathrm{KO}^{-n}(X) \oplus \mathrm{KSp}^{-n}(X) \longrightarrow \mathrm{K}^{-n}(X) \longrightarrow \\ \longrightarrow \mathrm{KSC}^{-n-2}(X) \longrightarrow \cdots \tag{6.40}$$

$$\cdots \longrightarrow \mathrm{KSC}^{-n-1}(X) \longrightarrow \mathrm{K}^{-n}(X) \longrightarrow \mathrm{KO}^{-n}(X) \oplus \mathrm{KSp}^{-n}(X) \longrightarrow \\ \longrightarrow \mathrm{KSC}^{-n}(X) \longrightarrow \cdots \tag{6.41}$$

which can be established from the KR version of the Barratt-Puppe exact sequence (2.54) and the excision theorem (2.48) applied to the pairs $(X \times \mathbf{S}^{0,p}, X \times \mathbf{S}^{0,q})$ for $(p,q) = (2,1)$, $(3,1)$ and $(3,2)$, respectively. These sequences illustrate how the symmetries of D-brane configurations whose charges are classified by a given KSC-group are related to webs of gauge symmetries that appear in the K-theories of Type I and Type II strings. These interrelationships could prove useful in extending the above analysis to other Type I models without vector structure [76].

6.6 The Hopkins Groups $\mathrm{K}_\pm(X)$

In this subsection we will study orientifolds of Type IIB superstring theory obtained via the quotient by the involution $\tau \cdot (-1)^{F_\mathrm{L}}$, where F_L is the left-moving spacetime fermion number operator. The operator

$(-1)^{F_L}$ changes the sign of all spacetime fields in the RR sector, and therefore the RR charge of a BPS D-brane changes sign and it gets mapped to its antibrane under $(-1)^{F_L}$. In this case, D-brane configurations on $X/\tau \cdot (-1)^{F_L}$ are related to those on X whose K-theory class is odd under the $\mathbb{Z}_2$ action. This means that τ^* maps the pair (E, F) to (F, E), i.e., there are isomorphisms $\psi : (E, F) \xrightarrow{\approx} (\tau^*(F), \tau^*(E))$ with $(\psi\tau^*)^2 = \mathrm{Id}$. A trivial pair is (H, H) with $H \cong \tau^*(H)$. The corresponding Grothendieck group is called the Hopkins group and is denoted by $\mathrm{K}_\pm(X)$ [17, 29].

It can be shown that the group $\mathrm{K}_\pm(X)$ may be computed in terms of conventional equivariant K-theory as

$$\widetilde{\mathrm{K}}_\pm(X) = \mathrm{K}^{-1}_{\mathbb{Z}_2}(X \times \mathbb{R}^{0,1}), \tag{6.42}$$

where the cyclic group $G = \mathbb{Z}_2$ acts on $X \times \mathbb{R}^{0,1}$ as the product of the action of τ on X and an orientation-reversing symmetry of $\mathbb{R}^{0,1}$. The validity of the formula (6.42) may be argued by defining $\mathrm{K}_\pm(X)$ as a (generalized) cohomology theory that satisfies the exact sequence

$$\ldots \longrightarrow \mathrm{K}^{-n}_{\mathbb{Z}_2}(X) \longrightarrow \mathrm{K}^{-n}(X) \longrightarrow \mathrm{K}^{-n}_\pm(X) \longrightarrow \ldots . \tag{6.43}$$

Comparing (6.43) with the six-term exact sequence (6.9) for the pair $(M, A) = (X \times \mathbb{R}^{0,1}, X \times (\mathbb{R}^{0,1} - \mathrm{pt}))$ gives the pair of exact sequences:

$$\begin{array}{ccccccccc}
\mathrm{K}^{-n}_{\mathbb{Z}_2}(X) & \to & \mathrm{K}^{-n}(X) & \to & \mathrm{K}^{-n}_\pm(X) & \to & \mathrm{K}^{-n-1}_{\mathbb{Z}_2}(X) & \to & \mathrm{K}^{-n-1}(X) \\
\| & & \| & & \downarrow & & \| & & \| \\
\mathrm{K}^{-n}_{\mathbb{Z}_2}(A) & \to & \mathrm{K}^{-n-1}_{\mathbb{Z}_2}(M, A) & \to & \mathrm{K}^{-n-1}_{\mathbb{Z}_2}(X \times \mathbb{R}^{0,1}) & \to & \mathrm{K}^{-n-1}_{\mathbb{Z}_2}(A) & \to & \mathrm{K}^{-n}_{\mathbb{Z}_2}(M, A).
\end{array} \tag{6.44}$$

Applying the five-lemma to (6.44), i.e., that the four isomorphisms between the two exact sequences in (6.44) imply that the remaining middle vertical mapping is also an isomorphism [40], we arrive at (6.42). An independent, algebraic argument using automorphism groups of the corresponding Clifford algebras may also be given [29].

For an orientifold of the type $X = \mathbb{R}^{d+1} \times (\mathbb{R}^{9-d}/(-1)^{F_L} \cdot \mathcal{I}_{9-d})$, the corresponding D$p$-brane charge over an orientifold d-plane takes values in $\widetilde{\mathrm{K}}_\pm(\mathbb{R}^{d-p,9-d})$. Since the right-hand side of (6.42) represents an equivariant functor on the category of complex vector bundles, we may use the suspension isomorphism with multiplication by $\mathbb{C}$ or $\mathbb{C}/\mathbb{Z}_2$ to derive the periodicities

$$\widetilde{\mathrm{K}}_\pm(\mathbb{R}^{p,q}) = \widetilde{\mathrm{K}}_\pm(\mathbb{R}^{p,q+2}), \quad \widetilde{\mathrm{K}}_\pm(\mathbb{R}^{p,q}) = \widetilde{\mathrm{K}}_\pm(\mathbb{R}^{p+2,q}). \tag{6.45}$$

This implies that $\widetilde{\mathrm{K}}_\pm(\mathbb{R}^{d-p,9-d})$ depends only on the parity of p and d. Suppose first that d is an even integer. Then using (6.42) and (6.45) we may compute

$$\begin{aligned}\widetilde{\mathrm{K}}_\pm(\mathbb{R}^{d-p,9-d}) &= \mathrm{K}^{-1}_{\mathbb{Z}_2}(\mathbb{R}^{d-p,10-d}) \\ &= \mathrm{K}^{p-1}_{\mathbb{Z}_2}(\mathrm{pt}) = \begin{cases}\mathrm{R}[\mathbb{Z}_2], & p \text{ odd} \\ 0, & p \text{ even},\end{cases}\end{aligned} \tag{6.46}$$

where $\mathrm{R}[\mathbb{Z}_2] = \mathbb{Z} \oplus \mathbb{Z}$ is the representation ring of the cyclic group $\mathbb{Z}_2$. Thus, when d is even, we obtain the standard spectrum of BPS Dp-brane charges for p odd localized over orientifold planes of odd dimension (the representation ring $\mathrm{R}[\mathbb{Z}_2]$ accounts for the mirror image brane charges induced by the given involution). The situation for d odd is a bit more involved. For this, we apply the six-term exact sequence (6.9) to the pair $(\mathbf{B}^{d-p,9-d}, \mathbf{S}^{d-p,9-d})$ to get

$$\begin{aligned}\dots \xrightarrow{\partial^*} \mathrm{K}^{-p}_{\mathbb{Z}_2}(\mathbf{B}^{0,9-d+1}, \mathbf{S}^{0,9-d+1}) &\longrightarrow \mathrm{K}^{-p}_{\mathbb{Z}_2}(\mathbf{B}^{0,9-d+1}) \\ &\xrightarrow{i^*} \mathrm{K}^{-p}(\mathbb{R}P^{9-d}) \xrightarrow{\partial^*} \dots\end{aligned} \tag{6.47}$$

where we have used the suspension isomorphism and $\mathbb{R}P^{9-d} = \mathbf{S}^{0,9-d+1}/\mathbb{Z}_2$ is the real projective space. The first K-group in (6.47) is isomorphic to the Hopkins group $\widetilde{\mathrm{K}}_\pm(\mathbb{R}^{d-p,9-d})$ that we are interested in. For the second K-group, we use the fact that the ball $\mathbf{B}^{0,9-d+1}$ is equivariantly contractible to get $\mathrm{K}^{-p}_{\mathbb{Z}_2}(\mathbf{B}^{0,9-d+1}) = \mathrm{K}^{-p}_{\mathbb{Z}_2}(\mathrm{pt}) = \delta^{p,\mathrm{even}}\,\mathrm{R}[\mathbb{Z}_2]$. The exact sequence (6.47) thereby relates the K-groups of interest to the cohomology of the real projective space [51]:

$$\mathrm{K}^{-p}(\mathbb{R}P^{9-d}) = \delta^{p,\mathrm{even}}\,\mathbb{Z} \oplus \mathbb{Z}_{2^r}, \tag{6.48}$$

where $r = \left[\frac{9-d}{2}\right]$. A careful analysis of the ring structure shows that the epimorphism i^* in (6.47) maps both of the generators of $\mathrm{K}^{-p}_{\mathbb{Z}_2}(\mathbf{B}^{0,9-d+1})$ into the generator of $\mathrm{K}^{-p}(\mathbb{R}P^{9-d})$, i.e., i^* is a surjective mapping of the free parts of the K-groups. The exactness of the sequence (6.47) then implies that

$$\mathrm{K}^{-p}_{\mathbb{Z}_2}(\mathbf{B}^{0,9-d+1}) = \mathrm{K}^{-p}(\mathbb{R}P^{9-d}) \,/\, \mathrm{K}^{-p}_{\mathbb{Z}_2}(\mathbf{B}^{0,9-d+1}, \mathbf{S}^{0,9-d+1}) \oplus \mathbb{Z}_{2^r}, \tag{6.49}$$

from which we arrive finally at

$$\widetilde{\mathrm{K}}_{\pm}(\mathbb{R}^{d-p,9-d}) = \begin{cases} \mathbb{Z}, & p \text{ even} \\ 0, & p \text{ odd}, \end{cases} \tag{6.50}$$

for d an odd integer.

As an example, we see that the D-particle over an O5-plane carries an integer-valued charge. This configuration is S-dual to the stable non-BPS D-particle on the O5-plane of the corresponding $\Omega{\cdot}\mathcal{I}_4$ orientifold [8, 10]. The apparent contradiction that arises here owes to the usual fact that K-theory only classifies the charges of topologically stable objects at weak string coupling, as mentioned in section 6.4. It is an open problem as of yet to determine how K-theory correctly incorporates the S-duality symmetry of Type IIB superstring theory. Note that the coincidence of the brane charges (6.50) with those of Type IIA superstring theory can be traced back to the IIB orientifold boundary states in the case at hand, which are of the form [8, 10]

$$|\widetilde{Dp}\rangle = \frac{1}{2}\Big(|Up,+\rangle_{\rm NS} - |Up,-\rangle_{\rm NS}\Big) + \frac{1}{2}\Big(|Tp,+\rangle_{\rm R} + |Tp,-\rangle_{\rm R}\Big), \tag{6.51}$$

where T and U label the twisted and untwisted sectors of the closed string Hilbert space under the $(-1)^{F_{\rm L}}$ orientifold projection. The boundary state (6.51) has precisely the same form as that of the ordinary Type IIA Dp-brane.

The relationship with the Type IIA theory can also be seen by taking $d = 0$ in the above construction. In this case we are simply quotienting the IIB theory by the operator $(-1)^{F_{\rm L}}$, which is known to map it into Type IIA superstring theory. In general, the operation of modding out the Type II spectrum m times by $(-1)^{F_{\rm L}}$ determines a mapping [19]

$$\widetilde{\mathrm{K}}^{-n}(X) \longrightarrow \widetilde{\mathrm{K}}_{\mathbb{Z}_2}^{-n-1}(X \times \mathbb{R}^{0,m}), \tag{6.52}$$

where now the $\mathbb{Z}_2$ acts only as a reflection on $\mathbb{R}^{0,m}$. The right-hand side of (6.52) may be evaluated using the six-term exact sequence (6.9). For example, for $m = 1$ we consider in (6.9) the pair $(X \times \mathbb{R}^{0,1}, X \times \{0\})$. Then the quotient space $X \times \mathbb{R}^{0,1}/X \times \{0\}$ is homotopic to two copies of $X \times \mathbb{R}$ which are exchanged by the involution. Since the $\mathbb{Z}_2$ action on this quotient is free, the equivariant K-groups may be computed by using the homotopy invariance of the K-functor and the suspension isomorphism to get

$$\mathrm{K}_{\mathbb{Z}_2}^{-n-1}\Big((X\times\mathbb{R})\amalg(X\times\mathbb{R})\Big)=\mathrm{K}^{-n-1}(X\times\mathbb{R})=\mathrm{K}^{-n}(X). \quad (6.53)$$

On $X\times\{0\}$ the $\mathbb{Z}_2$ action is trivial, so that

$$\mathrm{K}_{\mathbb{Z}_2}^{-n-1}\Big(X\times\{0\}\Big)=\mathrm{K}^{-n-1}(X\times\mathrm{pt})\otimes\mathrm{R}[\mathbb{Z}_2]. \quad (6.54)$$

Finally, since in this case $X\times\{0\}$ is an equivariant retract of $X\times\mathbb{R}^{0,1}$, we have $\ker\partial^*=\mathrm{K}_{\mathbb{Z}_2}^{-n-1}(X\times\{0\})$ and so the horizontal exact sequences in (6.9) split. The general result is then

$$\widetilde{\mathrm{K}}_{\mathbb{Z}_2}^{-n-1}(X\times\mathbb{R}^{0,m})=\Big(\widetilde{\mathrm{K}}^{-n-1}(X)\otimes\mathrm{R}[\mathbb{Z}_2]\Big)\oplus\widetilde{\mathrm{K}}^{-n-m-1}(X). \quad (6.55)$$

The group $\widetilde{\mathrm{K}}^{-n-1}(X)$ in (6.55) comes from the trivial part of the $\mathbb{Z}_2$ action and as such represents the untwisted brane charges. The other part $\widetilde{\mathrm{K}}^{-n-m-1}(X)$ comes from the free part of the $\mathbb{Z}_2$ action and represents the twisted sector.

The case $n=0$, $m=1$ represents the result of quotienting the IIB theory by $(-1)^{F_{\rm L}}$ [19]. The projection onto the first factor in (6.55) thereby represents the condensation of the quotiented IIB brane configuration onto the corresponding IIA D-brane (along with the mirror images under the $(-1)^{F_{\rm L}}$ projection). The second direct summand in (6.55) represents the twisted sector of the $(-1)^{F_{\rm L}}$-quotient which should be properly projected out in the mapping onto the Type IIA theory. The further quotient by $(-1)^{F_{\rm L}}$ corresponds to taking $n=1$, $m=2$ in (6.55), which maps back into the IIB theory with the same set of twisted charges projected out. More details about the explicit construction of these maps in terms of K-theory classes can be found in [19]. This K-theory construction agrees with the boundary state description in [20] and also the analysis of the open string spectrum of a Type II $p-\overline{p}$-brane configuration in [11]. In the former case it was shown that the result of quotienting the closed superstring Hilbert space by the operator $(-1)^{F_{\rm L}}$ projects onto the NS-NS part of all IIB p-brane boundary states, with no contributions from the twisted sector. The result is then a boundary state of the form (4.10), which, as discussed in section 4, via tachyon condensation can decay into a stable IIA D$(p-1)$ configuration. On the other hand, the superposition of a p-brane with a $\overline{p}$-brane can be described by the boundary state (c.f. eqs. (3.5) and (3.6))

$$|Dp\rangle + |D\overline{p}\rangle = |Dp,+\rangle_{\rm NS} - |Dp,-\rangle_{\rm NS}, \tag{6.56}$$

which thereby produces the same configuration as that obtained above. In these cases, the K-theory construction shows that the $(-1)^{F_{\rm L}}$ quotient on the spacetime-filling Type IIB 9-branes leaves an equal number of (identical) 9-branes and $\overline{9}$-branes which are used in the bound state construction of the $p - \overline{p}$ brane pair [19]. Again this is in complete agreement with the Type IIA $p-1$-brane configuration that is eventually reached by tachyon condensation. The naturality of the $(-1)^{F_{\rm L}}$ mapping as a canonical projection on K-theory groups is simply an indication of the fact that $(-1)^{F_{\rm L}}$ acts as a genuine non-perturbative symmetry of Type II superstring theory, as discussed in [20].

7 Global Aspects

This previous section concludes our general analysis of the ways of classifying D-branes using topological K-theory. There are many more exotic theories that one would like to study at this stage, for example orientifolds arising from quotients by the operator $\Omega \cdot (-1)^{F_{\rm L}} \cdot \mathcal{I}_m$. However, the corresponding (equivariant) K-groups for such involutions are not well understood (see [29, 19] for some discussion), and such an analysis must await further developments in the mathematics literature. Let us note that these latter orientifolds are also important for a more thorough description of the Type II orientifolds of sections 6.2–6.4 above, in that the $\Omega \cdot \mathcal{I}_m$ orientifold projection should strictly speaking be accompanied by the action of the operator $(-1)^{\frac{1}{2}(9-p)(8-p)F_{\rm L}}$ on Dp-brane states in order to preserve the $\mathbb{Z}_2$-equivariant structures. It is possible that there are approaches based on *algebraic* K-theory which could also be used to incorporate S-duality, and also the construction of M-branes, as has been recently discussed in [33]. We shall not pursue such matters here, which are still at best at a very preliminary stage. Instead, in this final section we shall proceed to analyze the interesting D-brane configurations that arise when one accounts for the global topology of the (possibly non-trivial) spacetime X and the associated brane worldvolume embeddings.

7.1 The Chern Character

Before proceeding to describe the global aspects of D-branes and their associated bound state constructions, which we will start in section 7.3, we shall first need some more mathematical preliminaries. In dealing with global properties of a space, we shall be forced to consider the cohomology of the manifolds, in addition to the K-theory of the relevant Chan-Paton bundles. One of the features of K-theory which makes it so useful in a variety of applications is the existence of the Chern character homomorphism, which provides a link between K-theory and ordinary cohomology theory by relating the ring $\mathrm{K}^{\#}(X)$ to the usual cohomology ring $H^{\#}(X)$ (here we shall deal mostly with Čech cohomology). In this subsection we will describe the construction of the Chern character in topological K-theory.

Let E be a complex vector bundle of rank k over a compact topological space X. We can associate to E certain cohomology classes $c_n(E) \in H^{2n}(X,\mathbb{Z})$ called the *Chern characteristic classes* of E which measure the twisting of the vector bundle and which are defined as follows. As in section 2.7, we consider the universal bundle $Q(k,\infty;\mathbb{C})$ over the classifying space $BU(k)$, whose pullbacks generate vector bundles such as E, i.e., $E = f^*Q(k,\infty;\mathbb{C})$ for a certain map $f : X \to BU(k)$. The cohomology ring $H^{\#}(BU(k),\mathbb{Z})$ of the classifying space has even-degree generators $c_n(Q(k,\infty;\mathbb{C}))$ whose pullbacks under f are precisely the characteristic classes of E:

$$c_n(E) \equiv f^*c_n\Big(Q(k,\infty;\mathbb{C})\Big) \in H^{2n}(X,\mathbb{Z}). \tag{7.1}$$

The basic properties of these characteristic classes are as follows:

(i) $c_0(E) = 1 \in H^0(X,\mathbb{Z})$.

(ii) For all $l \geq 0$, $c_l(E \oplus F) = \sum_{n+m=l} c_n(E) \wedge c_m(F)$.

(iii) (Naturality) If $f : Y \to X$ is a continuous map, then $c_n(f^*E) = f^*c_n(E)$.

For a rank k bundle E, the *total Chern class* is defined as

$$c(E) = 1 + c_1(E) + \ldots + c_k(E), \tag{7.2}$$

and from property (ii) above it follows that $c(E)$ is multiplicative,

$$c(E \oplus F) = c(E) \wedge c(F), \tag{7.3}$$

under Whitney sums. In particular, we may invoke the *splitting principle* which states that E is always a Whitney sum of complex line bundles $\mathcal{L}_n$ (more precisely, E is the pullback of some other vector bundle which is a sum of line bundles over another space) [51], and take

$$E = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_k. \tag{7.4}$$

We then have

$$c(E) = \prod_{n=1}^{k} c(\mathcal{L}_n) = \prod_{n=1}^{k} (1 + \lambda_n), \tag{7.5}$$

where we have defined $\lambda_n \equiv c_1(\mathcal{L}_n)$. This yields explicit expressions for the Chern classes of E in terms of elementary symmetric functions of the two-cocycles λ_n:

$$\begin{aligned}
c_1(E) &= \sum_n \lambda_n \\
c_2(E) &= \sum_{n<m} \lambda_n \wedge \lambda_m \\
&\cdots \\
c_m(E) &= \sum_{n_1<n_2<\cdots<n_m} \lambda_{n_1} \wedge \lambda_{n_2} \wedge \cdots \wedge \lambda_{n_m} \\
&\cdots \\
c_k(E) &= \lambda_1 \wedge \lambda_2 \wedge \cdots \wedge \lambda_k.
\end{aligned} \tag{7.6}$$

The *Chern character* of the vector bundle E is now defined by

$$\mathrm{ch}(E) = \sum_{n=1}^{k} \mathrm{e}^{\lambda_n} \in H^{\#}(X, \mathbb{Q}), \tag{7.7}$$

which can be thought of as a generating function for the characteristic classes. Note that it takes values in rational cohomology $H^{\#}(X,\mathbb{Q}) = H^{\#}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$, so that $\mathrm{ch}(E)$ cannot detect any torsion subgroups of the cohomology. Using (7.6), the degree $2m$ part $\mathrm{ch}_m(E)$ of the inhomogeneous cocycle (7.7) can be written in terms of the characteristic classes of E. For example,

$$\mathrm{ch}(E) \equiv \sum_{m\geq 0} \mathrm{ch}_m(E) = k + c_1(E) + \frac{1}{2}\Big(c_1(E) \wedge c_1(E) - 2c_2(E)\Big) + \dots. \tag{7.8}$$

The definition of the classes $c_m(E)$ (and hence also of the Chern character) can be generalized to bundles whose rank is not necessarily constant. For this, one partitions X into open subsets X_i such that the rank of $E|_{X_i}$ is constant, and then defines $c_m(E)$ as the unique cohomology class with $c_m(E)|_{X_i} = c_m(E|_{X_i})$.

The Chern character enjoys the following properties:

(i) $\mathrm{ch}_0(E) = \mathrm{rk}(E) \in H^0(X, \mathbb{Z})$.

(ii) $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$.

(iii) $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \wedge \mathrm{ch}(F)$.

(iv) (Naturality) $\mathrm{ch}(f^*E) = f^*\,\mathrm{ch}(E)$ for any continuous map $f : Y \to X$.

These properties imply that the Chern character respects the semi-ring structure on the category of vector bundles. Notice that property (i) makes an explicit connection with the rank function defined in (2.17), i.e., the virtual dimension defines a characteristic class in degree 0. In fact, we can use the Chern character to provide a complete map between $\mathrm{K}(X)$ and the cohomology ring $H^\#(X)$. Namely, for a virtual bundle $[(E,F)] \in \mathrm{K}(X)$ we define the homomorphism

$$\begin{aligned} \mathrm{ch} : \mathrm{K}(X) &\longrightarrow H^\#(X, \mathbb{Q}) \\ \mathrm{ch}\Big([E] - [F]\Big) &= \mathrm{ch}(E) - \mathrm{ch}(F). \end{aligned} \tag{7.9}$$

This map is well-defined provided that $[(E,F)] = [(G,H)]$ in $\mathrm{K}(X)$ implies $\mathrm{ch}(E) - \mathrm{ch}(F) = \mathrm{ch}(G) - \mathrm{ch}(H)$. That this is indeed true is a consequence of the behaviour (ii) of the Chern character under Whitney sums. For the particular case where $X = \mathbf{S}^{2n}$, the map ch is an isomorphism onto $H^{\#}(\mathbf{S}^{2n},\mathbb{Z})$. More generally, it can be shown [79] that the associated map

$$\mathrm{ch}\,:\,\mathrm{K}(X)\otimes_{\mathbb{Z}}\mathbb{Q}\longrightarrow H^{\rm even}(X,\mathbb{Q})\equiv\bigoplus_{n\geq 0}H^{2n}(X,\mathbb{Q}) \tag{7.10}$$

is an isomorphism, and moreover that this map extends to a ring isomorphism

$$\mathrm{ch}\,:\,\mathrm{K}^{\#}(X)\otimes_{\mathbb{Z}}\mathbb{Q}\xrightarrow{\approx}H^{\#}(X,\mathbb{Q}) \tag{7.11}$$

which maps $\mathrm{K}^{-1}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ onto $H^{\rm odd}(X,\mathbb{Q})$.

In the case where X is a smooth manifold, there is a useful explicit description of the Chern character. We assume that E is a smooth vector bundle equipped with a Hermitian connection ∇_E, whose curvature is ∇_E^2. The Chern character $\mathrm{ch}(E)\in H^{\#}(X,\mathbb{R})$ can then be represented by the closed inhomogeneous differential form:

$$\mathrm{ch}(E)=\mathrm{tr}\,\exp\left(\nabla_E^2/2\pi i\right). \tag{7.12}$$

In this case the λ_n's which appear above are the skew-eigenvalues of the two-form $\nabla_E^2/2\pi i$. To obtain numerical invariants of X, we consider a closed deRham current δ_Y which is a delta-function supported representative of the cohomology class of the Poincaré dual to an embedded submanifold $Y\overset{i}{\hookrightarrow}X$. Then we can associate to Y a map $I_Y\,:\,\mathrm{K}^{\#}(X)\to\mathbb{C}$ defined by the natural bilinear pairing on deRham cohomology:

$$I_Y(E)=\Big\langle\delta_Y\,,\,\mathrm{ch}(E)\Big\rangle_{\rm DR}\equiv\int\limits_X\delta_Y\wedge\mathrm{ch}(E)=\int\limits_Y i^*\,\mathrm{ch}(E). \tag{7.13}$$

7.2 The Thom Isomorphism

In this subsection we will describe the Thom isomorphism which relates the K-theory of a manifold X to the K-theory of the total spaces of complex vector bundles over X. In general, this enables one to compute the K-groups of some relatively complicated spaces in terms of much simpler base spaces. For example, the K-groups (6.48) of real projective spaces may be determined by the K-theory of a suitable total space over the base $X = \mathrm{pt}$. In this way the complete set of K-groups for projective spaces may be determined (see [51] for the details of such calculations). We shall begin with a description of the map at the level of cohomology, and then turn to the K-theoretical description. The Thom isomorphism will play an important role in our discussion of brane anomalies in section 7.5.

Let X be an oriented manifold of dimension n, and let $H^\#(X)$ be its cohomology ring (it will suffice to consider the cohomology ring with compact support). A well-known result of differential topology is *Poincaré duality*, which gives a canonical isomorphism

$$\mathcal{D}_X : H^p(X) \xrightarrow{\approx} H_{n-p}(X), \tag{7.14}$$

for all $p = 0, 1, \ldots, n$. Now consider another manifold Y of dimension m and let $f : Y \to X$ be continuous. Then for all $p \geq m - n$ there is a linear map, called the *Gysin homomorphism*:

$$f_! : H^p(Y) \longrightarrow H^{p-(m-n)}(X), \tag{7.15}$$

which is defined such that the diagram

$$\begin{array}{ccc} H^p(Y) & \xrightarrow{\mathcal{D}_Y} & H_{m-p}(Y) \\ f_! \downarrow & & \downarrow f_* \\ H^{p-(m-n)}(X) & \xleftarrow{\mathcal{D}_X^{-1}} & H_{m-p}(X) \end{array} \tag{7.16}$$

is commutative, i.e., such that $f_! = \mathcal{D}_X^{-1} f_* \mathcal{D}_Y$. Here f_* is the natural push-forward map acting on homology. An important example to which this construction applies is the case that Y is an oriented vector bundle E over X of fiber dimension k. Then we consider the canonical projection map $\pi : E \to X$ and the inclusion $i : X \to E$ of the zero section. They induce maps on homology with $\pi_* i_* = \mathrm{Id}$, so that

$$\pi_! : H^{p+k}(E) \xrightarrow{\approx} H^p(X)\ , \tag{7.17}$$

$$i_! : H^p(X) \xrightarrow{\approx} H^{p+k}(E)\ , \tag{7.18}$$

are isomorphisms for all p. The Gysin map $\pi_!$ can be thought of as integration over the fibers of $E \to X$. It is easy to see that $\pi_! i_! = \mathrm{Id}$, so that $\pi_! = (i_!)^{-1}$. The map (7.18) is called the *Thom isomorphism* of the oriented vector bundle E.

An important special instance of the Thom isomorphism (7.18) is the case $p = 0$. This defines a map $H^0(X) \to H^k(E)$, and the image of $1 \in H^0(X)$ thereby determines a cohomology class

$$\Phi[E] = i_!(1) \in H^k(E), \tag{7.19}$$

which is called the *Thom class* of E. The Thom isomorphism (7.18) is then generated by taking the cup product with this class:

$$i_!(\omega) = \pi^*(\omega) \wedge \Phi(E). \tag{7.20}$$

This cohomology class will play a central role in section 7.5. It is related to the *Euler class* $\chi(E)$ of the (even dimensional) real vector bundle $E \to X$ of rank $k = 2m$, which is a characteristic class of the bundle taking values in $H^{2m}(X)$. It can be defined as the pullback of the Thom class by the zero section:

$$\chi(E) = i^* \, \Phi[E]. \tag{7.21}$$

When E is a *complex* vector bundle of rank k, then the Euler class of E is defined as the Euler class of its underlying real bundle E_r (of real rank $2k$): $\chi(E) \equiv \chi(E_r)$. Moreover, in this case the Euler class of E can be shown to coincide with the top Chern class:

$$\chi(E) = c_k(E) = \prod_{n=1}^{k} \lambda_n. \tag{7.22}$$

If the (real) rank of the vector bundle E coincides with the dimension of X, then one can also introduce the *Euler number* $e(E)$, which is defined as the Euler class evaluated on the homology cycle $[X]$:

$$e(X) = \chi(E)[X] = \int_X \chi(E). \tag{7.23}$$

Furthermore, if X is compact, then for all $\phi \in H^{\#}(X)$ we have the identity [52]

$$i^* \, i_!(\phi) = \chi(E) \wedge \phi, \tag{7.24}$$

which follows from the fact that the Euler class is given as $\chi(E) = i^* \, i_!(1)$. Another important property of these cohomology classes is that if $s : X \to E$ is *any* section of E, then $s^*\Phi[E]$ is a closed form whose cohomology class coincides with the Euler class. From this fact one may also deduce that $s^*\Phi[E] = \delta_{\mathcal{Z}(s)}$, where $\mathcal{Z}(s) \overset{i}{\hookrightarrow} X$ is the zero locus of the section s, so that

$$\int\limits_{\mathcal{Z}(s)} i^*\,\omega = \int\limits_X s^*\,\Phi[E] \wedge \omega. \tag{7.25}$$

Let us now describe the Thom isomorphism in K-theory which, using the Chern character, can be related to the cohomological Thom isomorphism above. Let $E \to X$ be a complex vector bundle over X. Then $\mathrm{K}^\#(E)$ is naturally a $\mathrm{K}^\#(X)$-module, with an associative and distributive module multiplication,

$$\mathrm{K}^\#(X) \otimes_{\mathbb{Z}} \mathrm{K}^\#(E) \longrightarrow \mathrm{K}^\#(E), \tag{7.26}$$

defined according to the sequence of homomorphisms

$$\mathrm{K}^\#(X) \otimes_{\mathbb{Z}} \mathrm{K}^\#(E) \longrightarrow \mathrm{K}^\#(X \times E) \longrightarrow \mathrm{K}^\#(E). \tag{7.27}$$

Here the first map is induced by the cup product and the second map is the pullback on K-theory of the map $\pi \times \mathrm{Id}$. An important example is the case when $E = I^m = X \times \mathbb{C}^m$ is the trivial complex vector bundle over X. Define $\omega \in \mathrm{K}(E)$ to be the class

$$\omega = \Big[\pi^*\Lambda^{\rm even}E\,,\,\pi^*\Lambda^{\rm odd}E\,;\,\mu\Big], \tag{7.28}$$

where $\pi^*\Lambda E$ is the trivial m-plane bundle over E and $\Lambda^{\rm even,odd}E$ denote the even and odd degree exterior product bundles corresponding to E. The isomorphism μ is defined by

$$\mu_{x,v}(\phi) = v \wedge \phi - v^\dagger \neg\, \phi, \tag{7.29}$$

for $(x,v) \in X \times \mathbb{C}^m$ and $\phi \in \pi^*\Lambda^{\rm even}E$. Using the identification $\mathbb{R}^{2m} \cong \mathbb{C}^m$ and choosing the canonical orientation, this element can be written as

$$\omega = [\mathcal{S}^+, \mathcal{S}^-; \mu], \tag{7.30}$$

where $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ is the irreducible complex graded $C\ell_{2m}$-module (extended trivially over X), so that, according to (2.78), $\mu_{x,v}(\phi) = v \cdot \phi$ coincides with the usual Clifford multiplication. The fundamental Bott periodicity theorem then implies that $\mathrm{K}^\#(E)$ is a free $\mathrm{K}^\#(X)$-module of rank 1 with generator ω, so that ω gives a K-theory orientation for the bundle E.

Now consider a general (possibly non-trivial) complex vector bundle over X. We say that $\omega \in \mathrm{K}(E)$ is a *Bott class* if ω determines a K-theory orientation in any local trivialization of E over a closed subset $C \subset X$, i.e., $\mathrm{K}^\#(E|_C)$ is a free $\mathrm{K}^\#(C)$-module generated by ω whenever $E|_C$ is trivial. It can be shown [51, 52] that any Bott class is a K-theory orientation for E. In particular, if $E \to X$ is a complex Hermitian vector bundle over a compact space X, then the class

$$\Lambda_{-1}(E) = \Big[\pi^*\Lambda^{\rm even}E\,,\,\pi^*\Lambda^{\rm odd}E\,;\,\mu\Big] \in \mathrm{K}(E), \tag{7.31}$$

with $\mu_v(\phi) = v \wedge \phi - v^\dagger \neg\, \phi$, defines a K-theory orientation for E. This follows from the example above which showed that $\Lambda_{-1}(E)$ is a Bott class. The K-group element (7.31) is the K-theoretic Thom class of the vector bundle E, which is natural and multiplicative:

$$\Lambda_{-1}(E \oplus F) = \Lambda_{-1}(E) \otimes \Lambda_{-1}(F). \tag{7.32}$$

By taking cup products with it, it follows that the map $i_! : \mathrm{K}(X) \to \mathrm{K}(E)$ defined by

$$i_!(\alpha) = \pi^*(\alpha) \otimes \Lambda_{-1}(E), \quad \alpha \in \mathrm{K}(X), \tag{7.33}$$

is an isomorphism. This is the Thom isomorphism in complex K-theory. When $X = \mathrm{pt}$ is the space consisting of a single point, and $E = \mathbb{C}^n$ is the trivial bundle over X, then the Thom isomorphism is just the statement of Bott periodicity in the form $\widetilde{\mathrm{K}}(\mathbf{S}^{2n}) = \mathbb{Z}$. This follows from the fact that $\mathrm{K}(X) = \mathrm{K}(\mathrm{pt}) = \mathbb{Z}$ and $\mathrm{K}(E) = \mathrm{K}(\mathbb{C}^n) = \widetilde{\mathrm{K}}(\mathbf{S}^{2n})$. More generally, taking $E = I^m = X \times \mathbb{C}^m$ and using $\mathbb{R}^{2m} \cong \mathbb{C}^m$, the Thom isomorphism is just the statement of Bott periodicity in the form of the suspension isomorphism (2.22). For some more examples and applications, as well as the description of the Thom isomorphism in KO-theory and KR-theory, see [51, 52].

The relationship between the K-theoretic and cohomological Thom isomorphisms may be described as follows. Let $E \to X$ be a complex vector bundle of rank m, and let $i_!^{\mathrm{K}} : \mathrm{K}(X) \to \mathrm{K}(E)$ and $i_!^H : H^\#(X,\mathbb{Q}) \to H^\#(E,\mathbb{Q})$ be the Thom isomorphisms in K-theory and cohomology, respectively. We introduce the natural, multiplicative *Todd* *class* $\mathrm{Td}(E) \in H^{\mathrm{even}}(X,\mathbb{Q})$ by

$$\begin{aligned}\mathrm{Td}(E) &= \prod_{n=1}^{m} \frac{\lambda_n}{1-\mathrm{e}^{-\lambda_n}} \\ &= 1 + \tfrac{1}{2}\,c_1(E) + \tfrac{1}{12}\Big(c_1(E)\wedge c_1(E) + c_2(E)\Big) + \cdots . \end{aligned} \tag{7.34}$$

Then for each class $\omega \in \mathrm{K}(X)$, we have the formula:

$$\mathrm{ch}\Big(i_!^{\mathrm{K}}(\omega)\Big) = i_!^{H}\Big(\mathrm{ch}(\omega)\wedge \mathrm{Td}(E)\Big). \tag{7.35}$$

The Thom isomorphism also enables the construction of a K-theoretic Gysin map which will be a crucial ingredient in the global bound state construction that will be presented in the next subsection. Consider an embedding $f : Y \hookrightarrow X$ of a submanifold Y of even codimension $2k$ in X. (The restriction to embeddings is not necessary but is assumed for simplicity.) The normal bundle $N(Y,X)$ of Y in X can be defined through the exact sequence of vector bundles:

$$0 \longrightarrow TY \xrightarrow{f_*} TX \longrightarrow N(Y,X) \longrightarrow 0 \tag{7.36}$$

which decomposes the tangent bundle TX of X as $TX = TY \oplus N(Y,X)$. This identifies the normal bundle with a tubular neighbourhood of Y in X (this means that one chooses a suitable metric on X and defines $N(Y,X)$ to be the set of all points of distance $< \epsilon$ from Y in X, for some small ϵ), and also with the bundle $f^*(TX)/TY$ over Y. The vector bundle $N(Y,X)$ has structure group $SO(2k)$, which we assume is extendable globally to $\mathrm{Spin}(2k)$, i.e., $N(Y,X)$ admits a spin structure. (Again this requirement can be relaxed, but we will defer this discussion to the next subsection). Given the Thom isomorphism $i_! : \mathrm{K}(Y) \to \mathrm{K}(N(Y,X))$, we then define the Gysin homomorphism by

$$f_* = j_* \circ i_! : \mathrm{K}(Y) \longrightarrow \mathrm{K}(X) \tag{7.37}$$

where j_* is induced by the morphism $N(Y,X) \overset{j}{\hookrightarrow} X$ of locally compact spaces. The map f_* is independent of the choice of tubular neighbourhood and it depends only on the homotopy class of f. Its basic properties are as follows. First of all, if $f : Y \to X$ and $g : Z \to Y$ are two embeddings, then $(f\circ g)_* = f_* \circ g_*$. Furthermore, there are the identities

$$f_*(\omega \otimes f^*\alpha) = f_*(\omega)\otimes\alpha, \quad \forall\omega \in \mathrm{K}(Y), \alpha \in \mathrm{K}(X) \tag{7.38}$$

and, if X is compact,

$$f^* \circ f_*(\omega) = \chi\Big(N(Y,X)\Big) \otimes \omega \tag{7.39}$$

where the K-theoretic Euler class is defined as the restriction of the corresponding K-theoretic Thom class to the zero section. In the same way, one may construct the Gysin homomorphism for KO-theory, with the further requirements that $\dim X - \dim Y \equiv 0 \bmod 8$ and that $N(Y,X)$ admits a spin structure.

7.3 Global Version of the Bound State Construction

The bound state constructions that we have described thus far only apply locally in the spacetime X. In this subsection we will discuss the features that arise when global topology is taken into account. We shall describe the details only for Type IIB superstring theory, as then the generalization to other string theories will be evident. For this, we must be careful about the topology of the (non-trivial) normal bundle of the D-brane worldvolume in X, which must thereby be treated more carefully using the Thom isomorphism and the Gysin map discussed in the previous subsection. Actually, the mapping (3.23) is a local version of the Thom isomorphism, with the transverse space $\mathbf{S}^{2k}$ identified with the normal bundle of Y in X and $Y \times \mathbf{B}^{2k}$ with a small neighbourhood of Y in spacetime. Globally then, the Thom isomorphism $f_! : \mathrm{K}(Y) \xrightarrow{\approx} \mathrm{K}(X)$ applied to the normal bundle $N(Y,X) \xrightarrow{\pi} Y$ yields the mapping

$$[E] \longmapsto f_![E] = \pi^*\Big([E]\Big) \otimes \Lambda_{-1}\Big(N(Y,X)\Big). \tag{7.40}$$

A representative of the Thom class of the normal bundle is then given by the ABS construction, as described above. However, to achieve the map (7.40) one needs to extend the bundle $\pi^* E$ to the whole of X, which requires some special care and treatment of the normal bundle topology that we shall now discuss. The main idea is that the global obstructions which prevent the ABS class $[\mathcal{S}^+, \mathcal{S}^-; \mu]$ from producing a K-theory class of $\mathrm{K}(X)$ can be typically eliminated by nucleating extra 9-branes and $\overline{9}$-branes. In certain cases (to be described below) one has to stabilize (in the K-theory sense) the original configuration of

9-branes and $\overline{9}$-branes by pair creating extra $9-\overline{9}$-brane configurations and thus yield a configuration of $9-\overline{9}$-brane pairs with K-theory class $[\mathcal{S}^+ \oplus H, \mathcal{S}^- \oplus H; \mu \oplus \mathrm{Id}]$. This construction will then demonstrate that, globally, brane charges in a spacetime X can always be described by a configuration of 9-branes and $\overline{9}$-branes [17] and are therefore classified by $\mathrm{K}(X)$.

Let us start with the case of codimension 2. Recall that in the case of flat brane worldvolumes, in order to build a p-brane we need a $p+2$-brane-antibrane pair wrapping a submanifold $\mathbb{R}^{p+3}$ of the spacetime X which gives rise to a $U(1) \times U(1)$ gauge field and a tachyon field T of charges $(1,-1)$. T vanishes on a codimension 2 subspace that is identified with the worldvolume of the p-brane and breaks the gauge symmetry from $U(1) \times U(1)$ to $U(1)$. For the global construction, let $Y \subset Z$ be the worldvolume manifold of the p-brane embedded in the $p+3$ dimensional submanifold Z of spacetime. To build such a p-brane we consider a $p+2$-brane-antibrane pair on Z. Let $\mathcal{L}$ be a complex line bundle over Z and μ a section of $\mathcal{L}$ that vanishes on Y. By placing a $U(1)$ gauge field on the $p+2$-brane, with the same p-brane charge as that of a p-brane on Y, and a trivial $U(1)$ gauge field on the $\overline{p+2}$-brane, the system can be interpreted as a p-brane wrapping Y.

However, a p-brane wrapping Y also has in general lower-dimensional brane charges $p-2, p-4, \ldots$ which depend on the choice of a line bundle $\mathcal{K}$ on Y. If the line bundle $\mathcal{K}$ extends over Z then a p-brane wrapping Y is described by taking the bundle $\mathcal{L} \otimes \mathcal{K}$ on the $p+2$ brane and the bundle $\mathcal{K}$ on the $\overline{p+2}$ brane. If $\mathcal{K}$ does not extend over Z, then one uses the following classic K-theory construction [57]. Let Y' be a tubular neighborhood of Y in Z, whose closure we denote by $\overline{Y}$ and whose boundary is ∂Y. If E and F are bundles over Y of the same rank then they determine an element of $\widetilde{\mathrm{K}}(Y)$. The inclusion $i : Y \hookrightarrow \overline{Y}$ then induces a map on K-theory such that (E, F) also defines a unique element of $\widetilde{\mathrm{K}}(\overline{Y})$. The tachyon field is a map $T : E \longrightarrow F$, which is an isomorphism of vector bundles outside an open set $U \subset X$ whose closure $\overline{U}$ is compact. Now suppose that T is also a tachyon field on $\overline{Y}$ which is an isomorphism on ∂Y. In that case one can construct a natural map $\widetilde{\mathrm{K}}(\overline{Y}) \hookrightarrow \widetilde{\mathrm{K}}(Z)$, showing that D-branes wrapping Y are classified by $\mathrm{K}(Z)$, as desired. This map can be described as follows. Let $Z' = Z - Y'$. If we can extend the bundle F from ∂Y to all of Z' then F would be defined over all of Z. Since E and F are isomorphic (under the map T) on ∂Y, in that case E can also be extended over Z', so that (E,F) would define an element of $\widetilde{\mathrm{K}}(Z)$. If F does not extend over Z, then we may use Swan's theorem to construct a bundle H over Y such that $F\oplus H$ is trivial (assuming Y is compact) and therefore also trivial on $\overline{Y}$. Now we replace $E\to E\oplus H$, $F\to F\oplus H$ and $T\to T\oplus\mathrm{Id}$. Then we can extend $F\oplus H$ over Z and also extend $E\oplus H$ by setting it equal to $F\oplus H$ over Z', so that $(E\oplus H, F\oplus H)$ defines an element of $\widetilde{\mathrm{K}}(Z)$. In summary, if $\mathcal{K}$ does not extend over Z, then one instead finds a bundle H over Y such that $\mathcal{K}\oplus H$ is trivial. Then the bundle $\mathcal{L}\otimes\mathcal{K}\oplus H$ can be extended over Z. If we now consider a collection of $p+2$-branes on Z with gauge bundle $\mathcal{L}\otimes\mathcal{K}\oplus H$ and a collection of $\overline{p+2}$-branes on Z with bundle $\mathcal{K}\oplus H$, along with a tachyon field which equals $T\oplus\mathrm{Id}$ near Y and is in the gauge orbit of the vacuum outside Y', then this system describes a p-brane on Y with gauge bundle $\mathcal{K}$.

In the case that Y is of codimension greater than 2 in X, one proceeds as follows. Let Y be of codimension $2k$ in X. Its normal bundle $N(Y,X)$ in X then has structure group $SO(2k)$. Suppose first that $N(Y,X)$ is a spin manifold, so that its second Stiefel-Whitney class vanishes in $H^2(N(Y,X),\mathbb{Z}_2)$, $w_2(N(Y,X))=0$. Then associated with the 2^{k-1} $9-\overline{9}$-brane pairs we get a pair of spinor bundles $\mathcal{S}^\pm$ which are identified with the gauge bundles on the 9-branes. As usual, the tachyon field is a map $T:\,\mathcal{S}^-\to\mathcal{S}^+$ with

$$T(x)=\sum_{i=1}^{2k}\Gamma_i\,x^i,\quad x\in Y' \tag{7.41}$$

and the system describes a p-brane wrapped on Y. This configuration can be extended over X if $\mathcal{S}^-$ extends. Otherwise one can find a bundle H such that $\mathcal{S}^-\oplus H$ extends and then replace $(\mathcal{S}^+,\mathcal{S}^-)\to(\mathcal{S}^+\oplus H,\mathcal{S}^-\oplus H)$ and also $T\to T\oplus\mathrm{Id}$. Similarly, for a p-brane with line bundle $\mathcal{K}$, we start from the pair of bundles $\mathcal{K}\otimes\mathcal{S}^\pm$ and use the same construction just presented.

Let us now relax the requirement that $N(Y,X)$ be a spin manifold. According to the analysis of [80], for Type II compactifications with vanishing cosmological constant, the normal bundle to a D-brane wrapping a supersymmetric cycle always admits a spin^c structure. This means that instead of being extendable to a principal $\mathrm{Spin}(2k)$ bundle over Y, the structure group of the normal bundle extends to $\mathrm{Spin}^c(2k)$, where $\mathrm{Spin}^c(n)=\mathrm{Spin}(n)\times_{\mathbb{Z}_2}U(1)$ is the quotient of the product group

$\mathrm{Spin}(n) \times U(1)$ by the equivalence relation $(p, z) \sim (-p, -z)$ and it covers the rotation group $SO(n)$ according to the split exact sequence

$$1 \longrightarrow U(1) \longrightarrow \mathrm{Spin}^c(n) \longrightarrow SO(n) \longrightarrow 1. \tag{7.42}$$

The criteria for the existence of a spin^c structure can be formulated as follows. Consider the split exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0, \tag{7.43}$$

where the third map is reduction modulo 2. This sequence gives rise to a long exact sequence in cohomology

$$\begin{aligned} \ldots \longrightarrow H^n(X, \mathbb{Z}) \xrightarrow{\times 2} H^n(X, \mathbb{Z}) \longrightarrow H^n(X, \mathbb{Z}_2) \\ \xrightarrow{\beta} H^{n+1}(X, \mathbb{Z}) \longrightarrow \ldots \end{aligned} \tag{7.44}$$

where the map β is called the *Bockstein homomorphism*. The kernel of β is the set of classes in $H^{\#}(X, \mathbb{Z}_2)$ which are modulo 2 reductions of integral cohomology classes. If $w_n \in H^n(X, \mathbb{Z}_2)$ denotes the n-th Stiefel-Whitney class of X, then $W_n \equiv \beta(w_{n-1})$ measures whether or not the $(n-1)$-th Stiefel-Whitney class is the modulo 2 reduction of an integral class. The normal bundle $N(Y, X)$ admits a spin^c structure if and only if $W_3(N(Y, X)) = 0$ (so that in particular any spin manifold is canonically a spin^c manifold). Since X is a spin manifold, $w_1(X) = w_2(X) = 0$, and Y is orientable, $w_1(Y) = 0$, one can easily show using multiplicativity of the total Stiefel-Whitney class [17] that $w_2(N(Y, X)) = w_2(Y)$ and therefore also that $W_3(N(Y, X)) = W_3(Y)$. Thus $N(Y, X)$ admits a spin^c structure only if the p-brane worldvolume manifold Y does.

The existence of a spin^c structure on $N(Y, X)$ implies the following features for the bound state construction. Let U_i be an open covering of X. The transition functions g_{ij} of $\mathcal{S}^+$ on $U_i \cap U_j$ are then maps $g_{ij} : U_i \cap U_j \to \mathrm{Spin}(2k)$. The existence of a spin structure is equivalent to the vanishing of the two-cocycle

$$\varphi_{ijk} \equiv g_{ij} g_{jk} g_{ki} : U_i \cap U_j \cap U_k \longrightarrow \mathbb{Z}_2, \tag{7.45}$$

in $H^2(X, \mathbb{Z}_2)$. This defines a cohomology class $[\varphi] \in H^2(X, \mathbb{Z}_2)$, which vanishes precisely when $N(Y, X)$ is a spin manifold and $N(Y, X)$ admits a spin^c structure if $[\varphi]$ is the modulo 2 reduction of an integral class in $H^2(X,\mathbb{Z})$. Let $\mathcal{L}$ be the complex line bundle corresponding to this integral class (i.e., $c_1(\mathcal{L})$ is equal to this element in $H^2(X,\mathbb{Z})$), and let $\gamma_{ij} : U_i \cap U_j \to \mathbf{S}^1$ be the transition functions for $\mathcal{L}$. Suppose we want to find a square root of $\mathcal{L}$, i.e., a line bundle $\mathcal{L}^{1/2}$ with $\mathcal{L}^{1/2} \otimes \mathcal{L}^{1/2} = \mathcal{L}$. Then since $U_i \cap U_j$ is contractible we can define a square root $\tilde{\gamma}_{ij} \equiv \pm\sqrt{\gamma_{ij}} : U_i \cap U_j \to \mathbf{S}^1$. The obstruction to the existence of a consistent set of transition functions $\tilde{\gamma}_{ij}$ is the two-cocycle

$$\varphi'_{ijk} \equiv \tilde{\gamma}_{ij}\tilde{\gamma}_{jk}\tilde{\gamma}_{ki} : U_i \cap U_j \cap U_k \longrightarrow \mathbb{Z}_2 = \ker\sigma \tag{7.46}$$

where σ is the map $\sigma(z) \equiv z^2$ corresponding to the split exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbf{S}^1 \xrightarrow{\sigma} \mathbf{S}^1 \longrightarrow 0. \tag{7.47}$$

The class $[\varphi'] \in H^2(X,\mathbb{Z}_2)$ is the coboundary of $[\tilde{\gamma}] \in H^1(X,\mathbf{S}^1)$ under the associated long exact sequence in cohomology. In fact, consider the following commutative diagram:

$$\begin{array}{ccccc} H^1(X,\mathbf{S}^1) & \xrightarrow{\sigma} & H^1(X,\mathbf{S}^1) & \xrightarrow{\varphi'} & H^2(X,\mathbb{Z}_2) \\ \downarrow\approx & & \downarrow\approx & & \| \\ H^2(X,\mathbb{Z}) & \xrightarrow{\times 2} & H^2(X,\mathbb{Z}) & \xrightarrow{\rho} & H^2(X,\mathbb{Z}_2)\,. \end{array} \tag{7.48}$$

It follows that $[\varphi'] = \rho(c_1(\mathcal{L})) = [\varphi]$ and therefore $[\varphi'] + [\varphi] = 0$, or equivalently

$$c_1(\mathcal{L}) \equiv w_2\Big(N(Y,X)\Big) \mod 2. \tag{7.49}$$

This means that while we cannot construct the spinor bundles and we cannot construct the complex line bundle $\mathcal{L}^{1/2}$ globally, we can construct their tensor product. Thus, the existence of a spin^c structure means that $\mathcal{L}^{1/2} \otimes \mathcal{S}^{\pm}$ exist as vector bundles even though $\mathcal{L}^{1/2}$ and $\mathcal{S}^{\pm}$ do not. This in turn means that if $N(Y,X)$ is a spin^c bundle then we can proceed as in the case of spin bundles with the pair $(\mathcal{L}^{1/2}\otimes\mathcal{S}^+, \mathcal{L}^{1/2}\otimes\mathcal{S}^-)$ determining an element of $\mathrm{K}(X)$ and representing a D-brane wrapped on Y.

7.4 Compactifications and T-Duality

A T-duality transformation maps Type IIA superstring theory to Type IIB superstring theory, under which a Dp-brane is mapped to a D$(p+1)$-brane if the transformation is done in a direction transverse to the brane worldvolume. Since Type IIB branes are classified by $\mathrm{K}(X)$ and Type IIA branes by $\mathrm{K}^{-1}(X)$, it is natural to study the action of T-duality at the level of K-groups [32, 30, 19]. For this, we shall need to understand how to measure D-brane charge on spacetime compactifications in terms of K-theory and how to achieve natural isomorphisms of the corresponding K-groups.

We first need to explain an intimate connection between the index theory of Fredholm operators and topological K-theory, which will also be used in the next subsection. A *Fredholm operator* $\mathcal{T}$, acting on a separable Hilbert space $\mathcal{H}$, is a bounded linear operator whose kernel and cokernel are finite dimensional subspaces of $\mathcal{H}$. Such operators therefore have a well-defined *index*:

$$\operatorname{index}\mathcal{T} = \dim\ker\mathcal{T} - \dim\operatorname{coker}\mathcal{T}\ , \tag{7.50}$$

which is invariant under perturbations by any compact operator $\mathcal{A}$,

$$\operatorname{index}(\mathcal{T}+\mathcal{A}) = \operatorname{index}\mathcal{T}. \tag{7.51}$$

Moreover, if $\mathcal{S}$ is a bounded operator that is sufficiently close in the operator norm to $\mathcal{T}$, then $\mathcal{S}$ is also a Fredholm operator and $\operatorname{index}\mathcal{T} = \operatorname{index}\mathcal{S}$.

The importance of these properties stems from the fact that one can also describe the group $\mathrm{K}(X)$ in terms of Fredholm operators. For this, let $\mathcal{F}$ be the space of Fredholm operators on $\mathcal{H}$ with the operator norm topology. Then (7.50) defines a continuous map

$$\operatorname{index} : \mathcal{F} \longrightarrow \mathbb{Z}, \tag{7.52}$$

which can be shown to induce a bijection

$$\pi_0(\mathcal{F}) \longrightarrow \mathbb{Z} \tag{7.53}$$

between the set of connected components of $\mathcal{F}$ and the integers. More generally, let X be a compact topological space and consider the set $[X,\mathcal{F}]$ of homotopy classes of maps from X to $\mathcal{F}$. Since the product of two Fredholm operators is again a Fredholm operator, $[X,\mathcal{F}]$ is a monoid. It can be shown that there is an isomorphism:

$$[X,\mathcal{F}] \xrightarrow{\approx} \mathrm{K}(X), \tag{7.54}$$

which may be described as follows. Let $\mathcal{T}_x$ be a continuous family of Fredholm operators labelled by the parameter $x \in X$. Then the family of vector spaces $\ker \mathcal{T}_x$ forms a vector bundle $\ker \mathcal{T}$ over X. This statement is also true for the cokernel of $\mathcal{T}$, so that we can define the index of a family of operators $\mathcal{T}_x$ as the class

$$\operatorname{Index} \mathcal{T} \equiv \Big[(\ker \mathcal{T}\,,\,\operatorname{coker} \mathcal{T})\Big] \in \mathrm{K}(X). \tag{7.55}$$

Note that this is similar to the correspondence that was made in (2.20). With this correspondence, the composition of operators in $\mathcal{F}$ corresponds to the addition in $\mathrm{K}(X)$, while adjoints correspond to inversion. In particular, in the case where X is a point (so that $\mathrm{K}(X) = \mathbb{Z}$) the isomorphism (7.54) is just the index map (7.53). In other words, the virtual dimension of the K-theory class (7.55) coincides with the index defined in (7.50):

$$\mathrm{ch}_0(\operatorname{Index} \mathcal{T}) = \operatorname{index} \mathcal{T}. \tag{7.56}$$

Moreover, the set of homotopy classes of Fredholm operators defines the K-homology group $\mathrm{K}_0(X)$. The duality with K-theory is provided by the natural bilinear pairing

$$\Big([E]\,,\,[\mathcal{F}]\Big) \longmapsto \operatorname{index} \mathcal{F}_E \in \mathbb{Z} \tag{7.57}$$

where $[E] \in \mathrm{K}(X)$ and $\mathcal{F}_E = \mathcal{F}_{E,E}$ denotes the action of the Fredholm operator $\mathcal{F}$ on the Hilbert space $\mathcal{H} = L^2(\Gamma(X,E))$ of square-integrable sections of the vector bundle $E \to X$ as $\mathcal{F} : \Gamma(X,E) \to \Gamma(X,E)$.

For the present purposes we shall be interested in applying these ideas to a special class of operators, namely the Dirac operators associated to vector bundles over a spin manifold X. Dirac operators are examples of pseudo-differential elliptic operators, which are Fredholm operators when viewed as operators on a Hilbert space. To this end, we consider the case $\mathcal{F} = i\not{D} : \Gamma(X, \mathcal{S}_E^+) \to \Gamma(X, \mathcal{S}_E^-)$, where $E \to X$ is a real spin bundle (of even rank) and $\mathcal{S}_E^\pm$ are the corresponding twisted chiral spinor bundles lifted from E. The Chern character (7.9) (along with a version of the Gysin map introduced at the end of section 7.2) then allows one to map the analytical index of $i\not{D}$ defined in terms of K-theory classes into a topological index which can be expressed in terms of cohomological characteristic classes. The result is the celebrated

Atiyah-Singer index theorem [81]:

$$\operatorname{index} i\not{D} = -\int_X \operatorname{ch}(E) \wedge \widehat{A}(TX) \tag{7.58}$$

where the Dirac genus of the vector bundle E is defined by

$$\begin{aligned}\widehat{A}(E) &= \prod_n \frac{\lambda_n/2}{\sinh(\lambda_n/2)} \\ &= 1 - \frac{1}{24}p_1(E) + \frac{1}{5760}\Big(7p_1(E)\wedge p_1(E) - 4p_2(E)\Big) + \dots\end{aligned} \tag{7.59}$$

and $p_n(E) = (-1)^n c_{2n}(E \otimes_{\mathbb{R}} \mathbb{C})$ is the n-th Pontryagin class of E. An important special instance of this index formula is obtained by taking $E = TX$ to be the tangent bundle of the manifold X. Then the Euler number (7.23) can be expressed in terms of the Euler-Poincaré characteristic of X:

$$e(X) = \dim \mathrm{K}(X) \otimes_{\mathbb{Z}} \mathbb{Q} - \dim \mathrm{K}^{-1}(X) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{7.60}$$

We can apply these ideas to give an index-theoretical interpretation of T-duality acting on K-theory classes in various superstring theories [30]. The basic motivation for this analysis is the expression for the transformation of RR tensor fields under T-duality [82]. It can be shown that the RR fields on spacetimes of the form $\mathbf{T}^n \times M$ and those of the T-dual theory on $\widehat{\mathbf{T}}^n \times M$ (where $\widehat{\mathbf{T}}^n$ is the dual torus of $\mathbf{T}^n$) are related according to (in the absence of a Neveu-Schwarz B-field)

$$\widehat{H} = \int_{\mathbf{T}^n} \operatorname{ch}(\mathcal{P}) \wedge H, \tag{7.61}$$

where $H = \sum_p H^{(p+2)}$ is the gauge-invariant, total RR form field strength. Here

$$\operatorname{ch}(\mathcal{P}) = \exp\left(\sum_{i=1}^n d\widehat{y}_i \wedge dy^i\right) \tag{7.62}$$

is the Chern character of the Poincaré (complex line) bundle $\mathcal{P}$ over $\widehat{\mathbf{T}}^n \times \mathbf{T}^n$, with y^i and $\widehat{y}_i$ dual coordinates on $\mathbf{T}^n$ and $\widehat{\mathbf{T}}^n$. The Poincaré bundle is defined as the quotient of the trivial bundle $\mathbf{T}^n \times (\mathbb{R}^n)^* \times \mathbb{C}$ by the action of the rank n lattice $2\pi\Lambda^*$ (where $\mathbf{T}^n = \mathbb{R}^n/2\pi\Lambda$) defined by $(x, x^*, z) \mapsto (x, x^* + m^*, e^{im_i^* x^i} z)$. The relationship (7.61) is reminescent of a formula that arises in the family index theory [81] for a family of Dirac operators on $\mathbf{T}^n$ parametrized by $\widehat{\mathbf{T}}^n$ which is carried by the bundle $\mathcal{P}$ over $\widehat{\mathbf{T}}^n \times \mathbf{T}^n$. This motivates the search for a relatively simple explanation of the transformation property (7.61) in terms of K-theory which provides the analogous transformation rule for D-branes (which are sources for the RR fields).

For illustration, let us consider the case of D-branes in Type IIB superstring theory compactified on a circle $\mathbf{S}^1$. Spacetime is then $\mathbf{S}^1 \times M$, where M is a nine-dimensional manifold, and the dual geometry is $\widehat{\mathbf{S}}^1 \times M$. As usual, a Type IIB D-brane is constructed as a bound state of $9 - \overline{9}$-branes with Chan-Paton bundles $\mathcal{S}^\pm$, gauge connections $A^\pm$ and a tachyon field $T : \mathcal{S}^+ \to \mathcal{S}^-$. We probe this system with a D1-brane wrapped on $\mathbf{S}^1$, so that the dual system is a D0-brane moving in $\widehat{\mathbf{S}}^1 \times M$. The mass matrix of the fermionic modes coming from the $1 - 9$ and $1 - \overline{9}$ strings is given by the Dirac operator

$$i\not{D} = \begin{pmatrix} D_+ & -T^\dagger \\ T & -D_- \end{pmatrix} = \begin{pmatrix} \partial_y - ia & 0 \\ 0 & -\partial_y + ia \end{pmatrix} + \mathcal{A}. \qquad (7.63)$$

Here $D_\pm = \partial_y + A_y^\pm - ia$ is the Dirac operator on $\mathbf{S}^1$ coupled to the connection $A_y^\pm - ia$ on the bundle $\mathcal{S}^\pm \otimes \mathcal{P}$, where $\mathcal{P}$ is the Poincare bundle over $\mathbf{S}^1 \times \widehat{\mathbf{S}}^1$ with curvature $-ida \wedge dy$. The operator (7.63) can be interpreted as the usual Dirac operator twisted by the superconnection (3.16) on the $9 - \overline{9}$-branes coupled to the probe D-strings. It can also be interpreted as the tachyon field of the unstable Type IIA 9-branes of the T-dualized system [30], whereby the Wilson line on a Dp-brane is mapped onto the position of a D$(p-1)$-brane (c.f. eq. (4.21)).

Since $i\not{D}$ is a skew-adjoint operator its index vanishes identically as an element of $\mathrm{K}(\mathbf{S}^1 \times M)$. Rather, it can be shown [81] that the index takes values in the higher K-group $\mathrm{K}^{-1}(\widehat{\mathbf{S}}^1 \times M)$ of the parameter space for the family through the following construction. Given the family $i\not{D}(x)$ of skew-adjoint Fredholm operators labelled by $x \in W = \widehat{\mathbf{S}}^1 \times M$, one can define a family over $[-\frac{\pi}{2}, \frac{\pi}{2}] \times W$ by

$$i\widetilde{\not{D}}(t, x) = -\sin t + i\not{D}(x)\cos t\ . \qquad (7.64)$$

This is no longer a skew-adjoint operator and therefore it can have distinct kernel and cokernel. Furthermore, since $i\widetilde{\not{D}}(-\frac{\pi}{2}, x) = -i\widetilde{\not{D}}(\frac{\pi}{2}, x) =$

1, its kernel and cokernel are isomorphic at $t = \pm\frac{\pi}{2}$ and therefore $\mathrm{Index}\, i\widetilde{\not{D}} \in \mathrm{K}([-\frac{\pi}{2}, \frac{\pi}{2}] \times W, \partial[-\frac{\pi}{2}, \frac{\pi}{2}] \times W) = \mathrm{K}^{-1}(W)$. It follows then that T-duality determines a map:

$$\mathrm{K}(\mathbf{S}^1 \times M) \longrightarrow \mathrm{K}^{-1}(\widehat{\mathbf{S}}^1 \times M) \tag{7.65}$$

which can be identified as the sequence of homomorphisms:

$$\mathrm{K}(\mathbf{S}^1 \times M) \xrightarrow{\otimes[\mathcal{P}]} \mathrm{K}(\mathbf{S}^1 \times \widehat{\mathbf{S}}^1 \times M) \xrightarrow{\mathrm{Index}\, i\not{D}} \mathrm{K}^{-1}(\widehat{\mathbf{S}}^1 \times M), \tag{7.66}$$

where the last map is defined by $[(E, F)] \mapsto \mathrm{Index}\, i\not{D}_{E,F}$. In an analogous way one may construct the inverse map, so that the transformation (7.66) is actually an isomorphism of K-groups. To compare the transformation (7.66) with (7.61), we compute the index using the family index theorem to get

$$\mathrm{ch}(\mathrm{Index}\, i\not{D}_{E\otimes\mathcal{P}}) = \int\limits_{\mathbf{S}^1} \mathrm{ch}(E \otimes \mathcal{P}) \wedge \widehat{A}(T\mathbf{S}^1) \tag{7.67}$$

with $\widehat{A}(T\mathbf{S}^1) = 1$ and $\mathrm{ch}(E \otimes \mathcal{P}) = \mathrm{ch}(E) \wedge \mathrm{ch}(\mathcal{P})$. Since the K-groups of $\mathbf{S}^1$ are torsion free, the Chern character (7.9) is an isomorphism onto the subring $H^{\mathrm{even}}(\mathbf{S}^1, \mathbb{Z})$ of $H^{\mathrm{even}}(\mathbf{S}^1, \mathbb{Q})$, which makes the connection with the formula (7.61). This construction can be generalized to compactifications on the n-torus $\mathbf{T}^n$, thus defining maps $\mathrm{K}^{-m}(\mathbf{T}^n \times M) \longrightarrow \mathrm{K}^{-m-1}(\widehat{\mathbf{T}}^n \times M)$. Similar arguments can be applied to Real vector bundles [81], yielding the corresponding maps on KR-groups appropriate for the Type I and Type II orientifold theories.

There is another way to see the T-duality isomorphism in terms of *relative* K-theory [32]. Consider a general compactification manifold Z of dimension d. We want to determine all D-brane charges of codimension m in the non-compact space $\mathbb{R}^{9-d}$. These charges arise from D-branes which wrap non-trivial cycles of Z and from D-branes located at particular points in Z. As usual, one considers configurations of finite energy and therefore only those which are equivalent to the vacuum asymptotically in the transverse space $\mathbb{R}^m$. So $\mathbb{R}^m$ is replaced by its one-point compactification $\mathbf{S}^m$ by the addition of a copy of the compactification manifold Z at infinity. This corresponds precisely to considering charges which take values in the relative K-group (2.47) (with $Y = Z$ and $X = \mathbf{S}^m \times Z$). Thus, for example, for compactifications of Type IIB superstring theory on a submanifold Z, D-brane charges are classified by $\mathrm{K}(\mathbf{S}^m \times Z, Z)$, and by $\mathrm{K}^{-1}(\mathbf{S}^m \times Z, Z)$ for Type IIA compactifications. For instance, consider the compactification of Type II on an n-torus $\mathbf{T}^n$. By iterating the relations (2.35), (2.36) and (2.55), one may easily derive the natural isomorphisms

$$\begin{aligned} \mathrm{K}(M \times \mathbf{T}^n, \mathbf{T}^n) &= \bigoplus_{k=0}^{n} \widetilde{\mathrm{K}}^{-k}(M)^{\oplus\binom{n}{k}} = \widetilde{\mathrm{K}}(M)^{\oplus 2^{n-1}} \oplus \mathrm{K}^{-1}(M)^{\oplus 2^{n-1}} \\ &\cong \mathrm{K}^{-1}(M \times \mathbf{T}^n, \mathbf{T}^n). \end{aligned} \tag{7.68}$$

From this point of view, Type II T-duality is then a consequence of the periodicity of 2 of complex K-theory. Furthermore, from (2.40) we see that under the isomorphism (7.68) of K-groups for $n = 1$, $\widetilde{\mathrm{K}}(M) \otimes_{\mathbb{Z}} \mathrm{K}(\mathbf{S}^1)$ maps to $\widetilde{\mathrm{K}}(M) \otimes_{\mathbb{Z}} \mathrm{K}^{-1}(\mathbf{S}^1)$ with the summands $\mathrm{K}(\mathbf{S}^1)$ and $\mathrm{K}^{-1}(\mathbf{S}^1)$ interchanged. From this it follows that T-duality exchanges wrapped and unwrapped D-brane configurations. For $n > 1$, the decomposition (7.68) gives the anticipated degeneracies 2^{n-1} of brane charges arising from the higher supersymmetric branes wrapped on various cycles of the torus $\mathbf{T}^n$. This may be attributed to the fact that the T-duality mapping generates the spinor representation of the target space duality group $O(n, n, \mathbb{Z})$, in agreement with the fact that $O(n, n, \mathbb{Z})$ acts on the IIA and IIB RR potentials in the positive and negative chirality spinor representations, respectively. The complete agreement with the predictions of cohomology theory is once again a consequence of the Chern isomorphism of the integer K-groups of $\mathbf{T}^n$ with the corresponding integer cohomology ring.

This analysis generalizes to other string theories as well. For instance, we can write down the explicit T-duality isomorphism between D-brane charges of Type I compactified on a torus and those of the corresponding Type II orientifold compactification. Using the analog of the decomposition (2.55) for KO-theory and (5.12), we may iteratively compute the relevant group for the compactification of the Type I theory,

$$\mathrm{KO}(M \times \mathbf{T}^n, \mathbf{T}^n) = \bigoplus_{k=0}^{n} \widetilde{\mathrm{KO}}^{-k}(M)^{\oplus\binom{n}{k}} \tag{7.69}$$

whereas for the corresponding T-dual orientifold theory we may use

(6.27) to get

$$\begin{aligned}\mathrm{KR}^{-n}(M \times \mathbf{T}^{1,n}, \mathbf{T}^{1,n}) &= \bigoplus_{k=0}^{n} \widetilde{\mathrm{KR}}^{n,k}(M)^{\oplus\binom{n}{k}} = \bigoplus_{k=0}^{n} \widetilde{\mathrm{KO}}^{k-n}(M)^{\oplus\binom{n}{k}} \\ &\cong \mathrm{KO}(M \times \mathbf{T}^{n}, \mathbf{T}^{n}). \end{aligned} \tag{7.70}$$

where we have used the fact that the KR-involution acts trivially on M. The corresponding spectrum of BPS and $\mathbb{Z}_2$ non-BPS D-branes agrees again with the degeneracies of the various wrapped branes. The complexity of the decomposition (7.70) as compared to the Type II case owes to the periodicity of 8 of the KO and KR-groups, as discussed in section 6.4. In these cases, the precise bookkeeping of D-brane charges requires the concept of "D-brane transfer", whereby a D-brane which is located over an orientifold plane is "transfered" via a wrapped D-brane of one higher dimension to another orientifold plane. This is required to compensate for the apparently absent $\mathbb{Z}_2$ charges in the K-theory spectrum (7.69) (see [32] for more details).

Other K-theoretic interpretations of the T-duality isomorphism may also be given. In [31] it was discussed how to describe Type II D-branes wrapped on complex submanifolds of complex varieties using a holomorphic version of K-theory (more precisely, the Grothendieck groups of coherent and locally free sheaves), which further encodes a choice of connection on the brane worldvolume, and how the action of T-duality can be understood in terms of Fourier-Mukai transformations (see also [30]). In [19], T-duality was interpreted as being a consequence of the weak Bott periodicity sequence for the stable homotopy groups of the finite-dimensional vacuum manifolds for the Type II and Type I theories (c.f. section 6.4).

7.5 D-Brane Anomalies

In the final part of this review we will derive an explicit formula for the charge of a Dp-brane when it wraps a submanifold Y of the spacetime X. Locally, this formula has its origin in ordinary cohomology theory, but as we shall demonstrate, when global topology is taken into account the expression involves quantities which are most naturally understood in terms of K-theory [34] in exactly the same spirit as our previous discussions of D-brane charge. The basic idea comes from the fact that a Weyl fermion on an even-dimensional manifold always yields an anomalous variation of its action given by the well-known descent formula [83]. This formula determines the anomaly in terms of the representation of the gauge group carried by the fermions, and the corresponding Yang-Mills and gravitational connections. The same phenomenon occurs whenever a D-brane wraps around a non-trivial supersymmetric cycle of a curved manifold, because the twisting of its normal bundle can induce chiral asymmetry in its worldvolume field theory. The form of these chiral anomalies can be deduced by considering the field content on the intersection of two branes, which contains chiral fermions. The anomaly term then comes from the tensor product of the spinor bundles with the Chan-Paton vector bundles over the two D-branes. The anomalous zero modes on the intersection of the branes come from the massless excitation spectrum of the worldvolume field theory which consists of Weyl fermions in the mixed sector $\mathbf{N}_1 \otimes \overline{\mathbf{N}}_2$ and $\overline{\mathbf{N}}_1 \otimes \mathbf{N}_2$ representations of the gauge group $U(N_1) \times U(N_2)$ on the intersecting brane worldvolume. To render the theory anomaly-free thereby requires the addition of Wess-Zumino terms to the D-brane action. These induced terms imply that topological defects (such as instantons or monopoles) on the D-branes carry their own RR charge determined by their topological quantum numbers [27, 68].

Let $f : Y \hookrightarrow X$ be the embedding of a $p+1$ dimensional brane worldvolume Y into the spacetime manifold X of Type IIB superstring theory. The anomalous D-brane coupling takes the form of a Wess-Zumino type action,

$$S_Y = \int\limits_Y f^* C \wedge \mathcal{Y}(\nabla_E^2, g), \tag{7.71}$$

where $C = \sum_p C^{(p+1)}$ is the total RR form potential and $\mathcal{Y}(\nabla_E^2, g)$ is the D-brane source field which is an invariant polynomial of the Yang-Mills field strength and gravitational curvature on Y. Here ∇_E is the Hermitian curvature of a $U(N)$ gauge bundle E on the brane, while g is the restriction of the spacetime metric to Y. The anomalies on the D-brane result from the chiral asymmetry of their massless fermionic modes which are in one-to-one correspondence with the ground states of the relevant open string Ramond sectors. Open string quantization requires the Ramond ground states to be sections of the spinor bundle lifted from the spacetime tangent bundle TX tensored with a vector bundle in the adjoint $\mathbf{N} \otimes \overline{\mathbf{N}}$ representation of the brane gauge group $U(N)$, as dictated by the incorporation of the usual Chan-Paton factors. The GSO projection restricts the fermions to have a definite $SO(9,1)$ chirality. When the normal bundle of Y in X is trivial, so that $TX = TY$, a standard index-theoretical calculation gives

$$\mathcal{Y}_0(\nabla_E^2, g) = \mathrm{ch}(E) \wedge f^* \sqrt{\widehat{A}(TX)} \ . \tag{7.72}$$

However, the cohomology class (7.72) needs to be refined in the case that the normal bundle is non-trivial and, as we will demonstrate, this refinement leads to a formula for the D-brane RR charge which is most naturally understood in terms of K-theory classes, rather than cohomology classes. Assuming that $N(Y,X)$ admits a spin structure, one can determine the fermion quantum numbers of the spinor bundle associated with $N(Y,X)$. When $N(Y,X) \neq \emptyset$, the fermions have quantum numbers $(+,+) \oplus (-,-)$ under the worldvolume Lorentz group $\mathrm{Spin}(1,p)$ and the spacetime Lorentz group $\mathrm{Spin}(9-p)$ restricted to $N(Y,X)$. If the normal bundle is flat, then left- and right-moving fermions in the worldvolume field theory are treated equally and the theory is non-chiral. However, when $N(Y,X)$ has a non-vanishing curvature, chiral asymmetry is induced on the brane worldvolume and a distinction arises between the $(+,+)$ and $(-,-)$ quantum numbers.

It is well-known that the index of the Dirac operator on an even-dimensional manifold X gives the perturbative chiral gauge anomaly of a Dirac spinor on X. The positive and negative chirality spinor bundles $\mathcal{S}^\pm_{TX}$ corresponding to the tangent bundle of X can be decomposed in terms of the positive and negative chirality spin bundles $\mathcal{S}^\pm_{TY}$ and $\mathcal{S}^\pm_{N(Y,X)}$ lifted from the tangent and normal bundles to Y in X:

$$\mathcal{S}^\pm_{TX} = \left[\mathcal{S}^\pm_{TY} \otimes \mathcal{S}^+_{N(Y,X)}\right] \oplus \left[\mathcal{S}^\mp_{TY} \otimes \mathcal{S}^-_{N(Y,X)}\right] \tag{7.73}$$

The Dirac operator for the charged and reduced fermions acts on sections of the bundles (7.73) via the two-term complex

$$i\not{D} : \Gamma(Y, E^+) \longrightarrow \Gamma(Y, E^-), \tag{7.74}$$

where

$$E^\pm = \left(\left[\mathcal{S}^\pm_{TY} \otimes \mathcal{S}^+_{N(Y,X)}\right] \oplus \left[\mathcal{S}^\mp_{TY} \otimes \mathcal{S}^-_{N(Y,X)}\right]\right) \otimes E. \tag{7.75}$$

The standard index theorem applied to the two-term complex (7.74, 7.75) yields

$$\begin{aligned}\mathrm{index}\, i\not{D} &= (-1)^{\frac{(p+1)(p+2)}{2}} \int\limits_Y \mathrm{ch}(E) \wedge \left[\mathrm{ch}(S^+_{TY}) - \mathrm{ch}(S^-_{TY})\right] \\ &\qquad \wedge \left[\mathrm{ch}(S^+_{N(Y,X)}) - \mathrm{ch}(S^-_{N(Y,X)})\right] \wedge \frac{\mathrm{Td}(TY)}{\chi(TY)} \qquad (7.76)\end{aligned}$$

with $\sqrt{\mathrm{Td}(TY \otimes_{\mathbb{R}} \mathbb{C})} = \widehat{A}(TY)$ and $\mathrm{ch}(S^\pm_E) = \prod_n \mathrm{e}^{\pm\lambda_n/2}$. Using the identity

$$\mathrm{ch}(S^+_E) - \mathrm{ch}(S^-_E) = \frac{\chi(E)}{\widehat{A}(E)} \qquad (7.77)$$

which holds for any orientable, real spin bundle E, we see that the appropriate modification of (7.72) due to the normal bundle topology is

$$\mathcal{Y}(\nabla^2_E, g) = \mathcal{Y}_0(\nabla^2_E, g) \wedge \left[\widehat{A}\Big(N(Y,X)\Big)\right]^{-1}. \qquad (7.78)$$

In arriving at (7.78) we have re-written (7.71) as an integral over X using the appropriate deRham current δ_Y, and used the discussion of section 7.2 (c.f. eq. (7.25)) to write

$$\delta_Y \wedge \chi\Big(N(Y,X)\Big) = \delta_Y \wedge \delta_Y. \qquad (7.79)$$

Finally, as with the total Chern class, the Dirac genus is a multiplicative characteristic class, so that $\widehat{A}(TX) = \widehat{A}(TY) \wedge \widehat{A}(N(Y,X))$ and eq. (7.78) can be written as

$$\mathcal{Y}(\nabla^2_E, g) = \mathrm{ch}(E) \wedge \sqrt{\frac{\widehat{A}(TY)}{\widehat{A}\Big(N(Y,X)\Big)}}. \qquad (7.80)$$

Now we will describe how the anomalous coupling affects brane charges in the language of K-theory. To obtain the D-brane charge, we study the RR equations of motion and Bianchi identity coming from the complete action for the RR tensor fields:

$$S = -\frac{1}{4} \int\limits_X H(C) \wedge {}^*H(C) - \frac{\mu_{(p)}}{2} \int\limits_X \delta_Y \wedge f^*C \wedge \mathcal{Y}(\nabla^2_E, g), \qquad (7.81)$$

where $H(C)$ is the curvature of C. Then the equations of motion and Bianchi identity for a given $(p+1-m)$-form potential are

$$\begin{aligned} d^*H(C) &= \mu_{(p)}\,\delta_Y \wedge \mathcal{Y}(\nabla_E^2, g) \\ dH(C) &= -\mu_{(p)}\,\delta_Y \wedge \overline{\mathcal{Y}}(\nabla_E^2, g), \end{aligned} \tag{7.82}$$

where $\overline{\mathcal{Y}}$ is obtained from $\mathcal{Y}$ by complex conjugation of the Chan-Paton gauge group representation (note that $c_n(\overline{E}) = (-1)^n c_n(E)$, so that the Chern classes are torsion cohomology classes in the case that $E \cong \overline{E}$ and n is odd). From eq. (7.80) it follows that the formula for the charge vector $Q \in H^{\#}(X)$ defined on the right-hand side of (7.82) is

$$Q = f_! \left(\mathrm{ch}(E) \wedge \widehat{A}(TY) \wedge \frac{1}{f^* \sqrt{\widehat{A}(TX)}} \right), \tag{7.83}$$

where $f_! : H^n(Y, \mathbb{Z}) \to H^{n+9-p}(X, \mathbb{Z})$ is the (push-forward) Gysin map acting on cohomology as defined in (7.15). From the point of view of the worldvolume field theory on the D-brane, the characteristic class $\mathcal{Y}(\nabla_E^2, g)$ on the right-hand side of (7.83) measures the topological charge of a gravitational/Yang-Mills "instanton". From eq. (7.82) we see that $\delta_Y \wedge \mathcal{Y}$ can be thought of as the brane current for a "fat" D$(p-m)$-brane bound to and spread out over the Dp-brane. When the instanton shrinks to zero size, $\mathcal{Y}$ acquires a delta-function singularity, so that the quantity $\delta_Y \wedge \mathcal{Y}$ behaves just like a brane current. For some specific examples wherein the twisting of the normal bundle $N(Y, X)$ modifies the induced charge, see [35].

To write the class (7.83) in a more suggestive form, we make use of the Thom isomorphism for cohomology in the form of eq. (7.24) and the identity

$$f_! f^* \phi = \mathcal{D}_0 \wedge \phi, \tag{7.84}$$

where $\phi \in H^{\#}(X, \mathbb{Z})$ and $\mathcal{D}_0$ is the Poincaré dual of the zero section. Then we have

$$Q = f_! \Big(\mathrm{ch}(E) \wedge \widehat{A}(TY) \Big) \wedge \frac{1}{\sqrt{\widehat{A}(TX)}}. \tag{7.85}$$

Now we apply the Atiyah-Hirzebruch version of the Riemann-Roch theorem [51] which gives (see eq. (7.35))

$$f_! \Big(\mathrm{ch}(E) \wedge \widehat{A}(TY) \Big) = \mathrm{ch}(f_! E) \wedge \widehat{A}(TX), \tag{7.86}$$

where $f_![E] \in \mathrm{K}(X)$ is defined using the Thom isomorphism (7.40). From (7.86) it follows that, as an element of $H^{\#}(X)$, the RR charge associated to a D-brane wrapping a supersymmetric cycle in spacetime $f : Y \hookrightarrow X$ with Chan-Paton bundle $E \to Y$ is given by

$$Q = \mathrm{ch}(f_! E) \wedge \sqrt{\widehat{A}(TX)}. \tag{7.87}$$

The result (7.87) has a very natural K-theory interpretation using the Chern isomorphism (7.10). The cohomology rings $\mathrm{K}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $H^{\mathrm{even}}(X, \mathbb{Q})$ both have natural inner products defined on them. On $H^{\mathrm{even}}(X, \mathbb{Q})$, the bilinear form is given as in eq. (7.13), while the pairing on $\mathrm{K}(X)$ is given by the index of the Dirac operator (c.f. eq. (7.57)):

$$\Big\langle [E]\,,\,[F]\Big\rangle_{\mathrm{K}} = \mathrm{index}\, i\not{D}_{E\otimes F} \tag{7.88}$$

which using the Atiyah-Singer index theorem (7.58) may be written in terms of the deRham inner product as

$$\Big\langle [E]\,,\,[F]\Big\rangle_{\mathrm{K}} = \Big\langle \mathrm{ch}(E) \wedge \sqrt{\widehat{A}(TX)}\,,\, \mathrm{ch}(F) \wedge \sqrt{\widehat{A}(TX)}\Big\rangle_{\mathrm{DR}} . \tag{7.89}$$

This implies that the modified Chern isomorphism

$$[E] \longmapsto \mathrm{ch}(E) \wedge \sqrt{\widehat{A}(TX)} \tag{7.90}$$

is an isometry with respect to the natural inner products on $\mathrm{K}(X)$ and $H^{\#}(X)$. Thus, the result (7.87) is in complete agreement with the fact that D-brane charge is given by $f_![E] \in \mathrm{K}(X)$, and it moreover gives an explicit formula for the brane charges in terms of the Chern character homomorphism on K-theory. Integrating (7.87) over suitable cycles of the spacetime manifold X, as in (7.13), one thereby obtains the various p'-brane charges of the Dp-brane.

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References

[1] M.J. Duff, *M Theory (The Theory Formerly Known as Strings)*, Int. J. Mod. Phys., **A11** (1996), 5623, `hep-th/9608117`;
J.H. Schwarz, *Lectures on Superstring and M-Theory Dualities*, Nucl. Phys. Proc. Suppl. **55B** (1997), 1, `hep-th/9607201`.

[2] E. Witten, *String Theory Dynamics in Various Dimensions*, Nucl. Phys., **B443** (1995), 85, `hep-th/9503124`;
J.H. Schwarz, *The Power of M-Theory*, Phys. Lett., **B367** (1996), 97, `hep-th/9510086`.

[3] J.H. Schwarz, *An $SL(2,\mathbb{Z})$ Multiplet of Type IIB Superstrings*, Phys. Lett., **B360** (1995), 13, `hep-th/9508143`.

[4] A. Dabholkar, *Ten Dimensional Heterotic String as a Soliton*, Phys. Lett., **B357** (1995), 307, `hep-th/9506160`;
C.M. Hull, *String-String Duality in Ten Dimensions*, Phys. Lett., **B357** (1995), 545, `hep-th/9506194`;
J. Polchinski and E. Witten, *Evidence for Heterotic-Type I String Duality*, Nucl. Phys., **B460** (1996), 525, `hep-th/9510169`.

[5] J. Polchinski, *Dirichlet Branes and Ramond-Ramond Charges*, Phys. Rev. Lett., **75** (1995), 4724, `hep-th/9510017`; *TASI Lectures on D-branes*, in 'Fields, Strings and Duality', Proc. TASI 96, eds. C. Efthimiou and B.R. Greene (World Scientific, Singapore, 1997), p. 293, `hep-th/9611050`.

[6] M.J. Duff, R.R. Khuri, and J.X. Lu, *String Solitons*, Phys. Rep., **259** (1995), 213, `hep-th/9412184`.

[7] A. Sen, *Stable Non-BPS States in String Theory*, J. High Energy Phys., **9806** (1998), 007, `hep-th/9803194`.

[8] A. Sen, *Stable Non-BPS Bound States of BPS D-branes*, J. High Energy Phys., **9808** (1998), 010, hep-th/9805019; *Tachyon Condensation on the Brane-Antibrane System*, J. High Energy Phys., **9808** (1998), 012, hep-th/9805170.

[9] A. Sen, *SO(32) Spinors of Type I and other Solitons on Brane-Antibrane Pair*, J. High Energy Phys., **9809** (1998), 023, hep-th/9808141.

[10] A. Sen, *Type I D-Particle and its Interactions*, J. High Energy Phys., **9810** (1998), 021, hep-th/9809111;
O. Bergman and M.R. Gaberdiel, *Stable Non-BPS D-Particles*, Phys. Lett., **B441** (1998), 133, hep-th/9806155.

[11] A. Sen, *BPS D-Branes on Non-Supersymmetric Cycles*, J. High Energy Phys., **9812** (1998), 021, hep-th/9812031.

[12] O. Bergman and M.R. Gaberdiel, *Non-BPS States in Heterotic-Type IIA Duality*, J. High Energy Phys., **9903** (1999), 013, hep-th/9901014.

[13] A. Sen, *Non-BPS States and Branes in String Theory*, hep-th/9904207;
A. Lerda and R. Russo, *Stable Non-BPS States in String Theory: A Pedagogical Review*, Int. J. Mod. Phys., **A15** (2000), 771, hep-th/9905006.

[14] M.B. Green, *Point-like States for Type IIB Superstrings*, Phys. Lett., **B329** (1994), 435, hep-th/9403040;
T. Banks and L. Susskind, *Brane-Antibrane Forces*, hep-th/9511194;
V. Periwal, *Antibranes and Crossing Symmetry*, hep-th/9612215.

[15] M.B. Green and M. Gutperle, *Light-Cone Supersymmetry and D-Branes*, Nucl. Phys., **B476** (1996), hep-th/9604091;
G. Lifshytz, *Comparing D-Branes to Black-Branes*, Phys. Lett., **B388** (1996), 720, hep-th/9604156;
E. Gava, K.S. Narain and M.H. Sarmadi, *On the Bound States of p- and $(p+2)$-Branes*, Nucl. Phys., **B504** (1997), 214, hep-th/9704006;
M. Srednicki, *IIB or not IIB*, J. High Energy Phys., **9806** (1998), 005, hep-th/9807138.

[16] I. Pesando, *On the Effective Potential of the Dp-Anti-Dp System in Type II Theories*, Mod. Phys. Lett., **A14** (1999), 1545, `hep-th/9902181`;
Y. Matsuo, *Fate of Unoriented Bosonic String after Tachyon Condensation*, `hep-th/9905044`;
S.P. de Alwis, *Tachyon Condensation in Rotated Brane Configurations*, Phys. Lett., **B461** (1999), 329, `hep-th/9905080`.

[17] E. Witten, *D-Branes and K-Theory*, J. High Energy Phys., **9812** (1998), 019, `hep-th/9810188`.

[18] P. Hořava, *Type-IIA D-branes, K-Theory and Matrix Theory*, Adv. Theor. Math. Phys., **2** (1998), 1373, `hep-th/9812135`.

[19] K. Olsen and R.J. Szabo, *Brane Descent Relations in K-Theory*, Nucl. Phys., **B566** (2000), 562, `hep-th/9904153`.

[20] K. Dasgupta and J. Park, *Brane Mapping under* $(-1)^{F_L}$, Phys. Lett., **B457** (1999), 47, `hep-th/9903125`.

[21] P. Hořava, *Strings on Worldsheet Orbifolds*, Nucl. Phys., **B327** (1989), 461; *Background Duality of Open String Models*, Phys. Lett., **B231** (1989), 251;
J. Dai, R.G. Leigh, and J. Polchinski, *New Connections between String Theories*, Mod. Phys. Lett., **A4** (1989), 2073.

[22] M. Li, *Boundary States of D-Branes and Dy-Strings*, Nucl. Phys.. **B460** (1996), 351, `hep-th/9510161`;
M. Bershadsky, V. Sadov, and C. Vafa, *D-Branes and Topological Field Theories*, Nucl. Phys., **B463** (1996), 420, `hep-th/9511222`;
M.B. Green, J.A. Harvey, and G. Moore, *I-Brane Inflow and Anomalous Couplings on D-Branes*, Class. Quant. Grav., **14** (1997), 47, `hep-th/9605033`.

[23] M.R. Douglas, *Branes within Branes*, in 'Strings, Branes and Dualities', Proc. Cargèse 1997, eds. L. Baulieu, P. Di Francesco, M.R. Douglas, V.A. Kazakov, M. Picco, and P. Windey (Dordrecht, Kluwer Academic, 1999), p. 267, `hep-th/9512077`.

[24] E. Witten, *Bound States of Strings and p-Branes*, Nucl. Phys., **B460** (1996), 33, `hep-th/9510135`.

[25] A. Giveon and D. Kutasov, *Brane Dynamics and Gauge Theory*, Rev. Mod. Phys., **71** (1999), 983, hep-th/9802067.

[26] A. Strominger, *Open p-Branes*, Phys. Lett., **B383** (1996), 44, hep-th/9512059.

[27] G.W. Semenoff and K. Zarembo, *Solitons on Branes*, hep-th/9903140, Nucl. Phys., **B556** (1999), 247.

[28] H. García-Compéan, *D-Branes in Orbifold Singularities and Equivariant K-Theory*, Nucl. Phys., **B557** (1999), 480, hep-th/9812226.

[29] S. Gukov, *K-Theory, Reality and Orientifolds*, Comm. Math. Phys., **210** (2000), 621, hep-th/9901042.

[30] K. Hori, *D-Branes, T-Duality and Index Theory*, hep-th/9902102, to appear in Adv. Theor. Math. Phys.

[31] E. Sharpe, *D-Branes, Derived Categories and Grothendieck Groups*, Nucl. Phys., **B561** (1999), 433, hep-th/9902116.

[32] O. Bergman, E.G. Gimon, and P. Hořava, *Brane Transfer Operations and T-Duality of Non-BPS States*, J. High Energy Phys., **9904** (1999), 010, hep-th/9902160.

[33] I.V. Vancea, *On the Algebraic K-Theory of the Massive D8 and M9 Branes*, hep-th/9905034.

[34] R. Minasian and G. Moore, *K-Theory and Ramond-Ramond Charge*, J. High Energy Phys., **9711** (1997), 002, hep-th/9710230.

[35] Y.-K.E. Cheung and Z. Yin, *Anomalies, Branes and Currents*, Nucl. Phys., **B517** (1998), 69, hep-th/9710206.

[36] P. Hořava, *Equivariant Topological Sigma Models*, Nucl. Phys., **B418** (1994), 571, hep-th/9309124;
D.R. Morrison, *The Geometry Underlying Mirror Symmetry*, in 'Proc. European Algebraic Geometry Conference' (Warwick, 1996), alg-geom/9608006;
J.A. Harvey and G. Moore, *On the Algebra of BPS States*, Comm. Math. Phys., **197** (1998), 489, hep-th/9609017.

[37] M.R. Douglas and G. Moore, *D-Branes, Quivers and ALE Instantons*, hep-th/9603167.

[38] M.J. Duff and J.X. Lu, *Strings from Fivebranes*, Phys. Rev. Lett., **66** (1991), 1402;
J.A. Dixon, M.J. Duff, and J.C. Plefka, *Putting String/Fivebrane Duality to the Test*, Phys. Rev. Lett., **69** (1992), 3009, hep-th/9208055.

[39] M. Frau, L. Gallot, A. Lerda, and P. Strigazzi, *Stable Non-BPS D-Branes in Type I String Theory*, Nucl. Phys., **B564** (2000), 60, hep-th/9903123.

[40] E.H. Spanier, *Algebraic Topology* (Springer-Verlag, 1966);
M.J. Greenberg, *Lectures on Algebraic Topology*, (Benjamin, New York, 1967).

[41] T. Eguchi, P.B. Gilkey, and A.J. Hanson, *Gravitation, Gauge Theories and Differential Geometry*, Phys. Rep., **66** (1980), 213.

[42] M.B. Green, J.H. Schwarz, and E. Witten, *Superstring Theory*, (Cambridge University Press, 1987);
J. Polchinski, *String Theory*, (Cambridge University Press, 1998).

[43] P. Yi, *Membranes from Five-Branes and Fundamental Strings from Dp-Branes*, Nucl. Phys., **B550** (1999), 214, hep-th/9901159.

[44] J. Kalkkinen, *Gerbes and Massive Type II Configurations*, J. High Energy Phys., **9907** (1999), 002, hep-th/9905018.

[45] H. Awata, S. Hirano and Y. Hyakutake, *Tachyon Condensation and Graviton Production in Matrix Theory*, hep-th/9902158;
N. Kim, S.-J. Rey, and J.-T. Yee, *Stable Non-BPS Membranes on Matrix Orientifold*, J. High Energy Phys., **9904** (1999), 003, hep-th/9903129;
M. Massor and J. Troost, *The Longitudinal Fivebrane and Tachyon Condensation in Matrix Theory*, Nucl. Phys., **B569** (2000), 417, hep-th/9907128.

[46] A. Connes, M.R. Douglas, and A.S. Schwartz, *Noncommutative Geometry and Matrix Theory: Compactification on Tori*, J. High Energy Phys., **9802** (1998), 003, hep-th/9711162;

H. García-Compéan, *On the Deformation Quantization Description of Matrix Compactifications*, Nucl. Phys., **B541** (1999), 651; M. Anazawa, *D0-Branes in an H-Field Background and Noncommutative Geometry*, Nucl. Phys., **B569** (2000), 680, `hep-th/9905055`.

[47] A. Borel and J.P. Serre, *Le Théorème de Riemann-Roch (d'après Grothendieck)*, Bull. Soc. Math. France, **86** (1958), 97.

[48] M.F. Atiyah and F. Hirzebruch, *Vector Bundles and Homogeneous Spaces*, Proc. Symp. Pure Math., Amer. Math. Soc., **3** (1961), 7.

[49] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Geometry* (Princeton University Press, 1952).

[50] M.F. Atiyah, *K-Theory* (Benjamin, 1964).

[51] M. Karoubi, *K-Theory* (Springer-Verlag, 1978).

[52] H.B. Lawson, Jr. and M.-L. Michelsohn, *Spin Geometry* (Princeton University Press, 1989).

[53] D. Husemoller, *Fibre Bundles* (Springer-Verlag, 1994).

[54] M.F. Atiyah, *Vector Bundles and the Künneth Formula*, Topology, **1** (1962), 245.

[55] A. Trautman, *Solutions of the Maxwell and Yang-Mills Equations associated with Hopf Fibrings*, Int. J. Theor. Phys., **16** (1977) 561; J. Nowakowski and A. Trautman, *Natural Connections on Stiefel Bundles are Sourceless Gauge Fields*, J. Math. Phys., **19** (1978), 1100.

[56] R. Bott, *The Stable Homotopy of the Classical Groups*, Ann. Math., **70** (1959), 313.

[57] M.F. Atiyah, R. Bott, and A. Shapiro, *Clifford Modules*, Topology, **3** (1964), 3.

[58] P. Di Vecchia, M. Frau, A. Lerda, I. Pesando, R. Russo, and S. Sciuto, *Classical p-branes from the Boundary State*, Nucl. Phys., **B507** (1997), 259, `hep-th/9707068`;
M. Billò, P. Di Vecchia, M. Frau, A. Lerda, I. Pesando, R. Russo, and S. Sciuto, *Microscopic String Analysis of the D0-D8*

Brane System and Dual RR States, Nucl. Phys., **B526** (1998) 199, `hep-th/9802088`.

[59] D. Quillen, *Superconnections and the Chern Character*, Topology, **24** (1985), 89.

[60] C. Kennedy and A. Wilkins, *Ramond-Ramond Couplings on Brane-Antibrane Systems*, Phys. Lett., **B464** (1999), 206, `hep-th/9905195`.

[61] M. Billò, B. Craps, and F. Roose, *Ramond-Ramond Couplings of Non-BPS D-Branes*, J. High Energy Phys., **9906** (1999), 033, `hep-th/9905157`.

[62] B.A. Dubrovin, A.T. Fomenko, and S.P. Novikov, *Modern Geometry* (Springer-Verlag, 1990).

[63] T.T. Wu and C.N. Yang, *Concept of Non-integrable Phase Factors and Global Formulation of Gauge Fields*, Phys. Rev., **D12** (1975), 3845.

[64] G. 't Hooft, *Magnetic Monopoles in Unified Gauge Theories*, Nucl. Phys., **B79** (1974), 276;
A.M. Polyakov, *Particle Spectrum in Quantum Field Theory*, JETP Lett., **20** (1974), 194.

[65] A. Sen, *D0 Branes on* $\mathbf{T}^n$ *and Matrix Theory*, Adv. Theor. Math. Phys., **2** (1998), 51, `hep-th/9709220`;
N. Seiberg, *Why is the Matrix Model Correct?*, Phys. Rev. Lett., **79** (1997), 3577, `hep-th/9710009`.

[66] B. Grossman, T.W. Kephart, and J.D. Stasheff, *Solutions to Yang-Mills Field Equations in Eight-dimensions and the Last Hopf Map*, Comm. Math. Phys., **96** (1984), 431; *Solutions of Gauge Field Equations in Eight-dimensions: Conformal Invariance and the Last Hopf Map*, Phys. Lett., **B220** (1989), 431;
G. Landi, *The Natural Spinor Connection on* $\mathbf{S}^8$ *is a Gauge Field*, Lett. Math. Phys., **11** (1986), 171;
L. Dabrowski and A. Trautman, *Spinor Structures in Spheres and Projective Spaces*, J. Math. Phys., **27** (1986), 2022.

[67] E.G. Gimon and J. Polchinski, *Consistency Conditions for Orientifolds and D-Manifolds*, Phys. Rev., **D54** (1996), 1667, `hep-th/9601038`.

[68] E. Witten, *Small Instantons in String Theory*, Nucl. Phys., **B460** (1996), 541, `hep-th/9511030`.

[69] G.B. Segal, *Equivariant K-Theory*, Publ. Math. IHES (Paris), **34** (1968), 129.

[70] G.B. Segal, *The Representation Ring of a Compact Lie Group*, Publ. Math. IHES (Paris), **34** (1968), 113.

[71] A. Lawrence, N. Nekrasov, and C. Vafa, *On Conformal Theories in Four Dimensions*, Nucl. Phys., **B533** (1998), 199, `hep-th/9803015`.

[72] T. Kambe, *The Structure of* K_Λ*-Rings of the Lens Space and their Applications*, J. Math. Soc. Japan, **18** (1966), 135.

[73] M.F. Atiyah, *K-Theory and Reality*, Quart. J. Math. Oxford, **17** (1966), 367.

[74] S. Mukhi, *Orientifolds: The Unique Personality of each Spacetime Dimension*, in 'Frontiers of Field Theory, Quantum Gravity and Strings', eds. R.K. Kaul, J. Maharana, S. Mukhi and S. Kalyana Rama (Nova Science, 1999), `hep-th/9710004`.

[75] A.L. Edelson, *Real Vector Bundles and Spaces with Free Involutions*, Trans. Amer. Math. Soc., **157** (1971), 179.

[76] E. Witten, *Toroidal Compactification without Vector Structure*, J. High Energy Phys., **9802** (1998), 006, `hep-th/9712028`.

[77] P.S. Green, *A Cohomology Theory based upon Self-Conjugacies of Complex Vector Bundles*, Bull. Amer. Math. Soc., **70** (1964), 522.

[78] D.W. Anderson, *The Real K-Theory of Classifying Spaces*, Proc. Nat. Acad. Sci., **51** (1964), 634.

[79] M.F. Atiyah and F. Hirzebruch, *Analytical Cycles on Complex Manifolds*, Topology, **1** (1962), 25.

[80] R.L. Bryant and E. Sharpe, *D-Branes and Spin^c Structures*, Phys. Lett., **B450** (1999), 353, `hep-th/9812084`.

[81] M.F. Atiyah and I.M. Singer, *The Index of Elliptic Operators I–III*, Ann. Math., **87** (1968), 484, 531, 546; *The Index of Elliptic Operators IV, V*, Ann. Math., **93** (1971), 119, 139; *Index Theory for Skew-Adjoint Fredholm Operators*, Publ. Math. IHES (Paris), **37** (1969), 305.

[82] E. Bergshoeff, C.M. Hull, and T. Ortin, *Duality in the Type II Superstring Effective Action*, Nucl. Phys., **B451** (1995), 547, `hep-th/9504081`;
E. Eyras, B. Janssen and Y. Lozano, *5-Branes, KK-Monopoles and T-Duality*, Nucl. Phys., **B531** (1998), 275, `hep-th/9806169`.

[83] M.F. Atiyah and I.M. Singer, *Dirac Operators coupled to Vector Bundles*, Proc. Natl. Acad. Sci., **81** (1984), 2597.