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# Antisymmetrised $2p$-forms generalising curvature 2-forms and a corresponding $p$-hierarchy of Schwarzschild type metrics in dimensions $d > 2p + 1$

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**Abstract**

Starting with the curvature 2-form a recursive construction of totally antisymmetrised $2p$-forms is introduced, to which we refer as $p$-Riemann tensors. Contraction of indices permits a corresponding generalisation of the Ricci tensor. Static, spherically symmetric "$p$-Ricci flat" Schwarzschild like metrics are constructed in this context for $d > 2p+1$, $d$ being the spacetime dimension. The existence of de Sitter type solutions is pointed out. Our $2p$-forms vanish for $d < 2p$ and the limiting cases $d = 2p$ and $d = 2p + 1$ exhibit special features which are discussed briefly. It is shown that for $d = 4p$ our class of solutions correspond to double-selfdual Riemann $2p$-form (or $p$-Riemann tensor). Topological aspects of such generalised gravitational instantons and those of associated (through spin connections) generalised Yang-Mills instantons are briefly mentioned. The possibility of a study of surface deformations at the horizons of our class of "$p$-black holes" leading to Virasoro algebras with a $p$-dependent hierarchy of central charges is commented on. Remarks in conclusion indicate directions for further study and situate our formalism in a broader context.

# 1 Introduction

The basic aim of this work is to present Schwarzschild like solutions to a hierarchy of gravitational systems that we studied sometime ago [8, 15], which generalise the Einstein-Hilbert system in a very natural way. In this framework the Lagrangian is higher order in the Riemann tensor, but in such a way that only the quadratic power of the velocity fields (namely derivatives of the metric or the *vielbein*) appears in it. This formalism opens up broader possibilities of constructing gravitational instantons in suitable higher dimensions. Here we present another remarkable aspect, which is one more evidence of the aptitude of our generalisation. The spherically symmetric, static black hole (or the Schwarzschild metric) which is *unique* in the Einstein-Hilbert framework, is the first member of a hierarchy in our context. De Sitter type soluions also exist, rather systematically.

In the following sections we will deal with totally antisymmetrised $2p$-forms constructed from the Riemann tensor, to which we refer as Riemann $2p$-forms or $p$-Riemann tensors. The explicit recursive prescription is given below. Throughout we will consider spacetimes of arbitrary dimension $d$, with $(d-1)$ spatial dimensions. Our $2p$-forms vanish for $d < 2p$ and the dynamics leads to very special, degenerate cases for $d = 2p$ and $d = 2p+1$. The generic situation arises for $d > 2p+1$, $(p = 1, 2, 3, \dots)$.

Contracting indices of the Riemann $2p$-forms one obtains the standard Ricci tensor for $p = 1$ and a generalised hierarchy of "$p$-Ricci" tensors to be defined below. We show that for spherical symmetry one obtains as "$p$-Ricci flat" solutions a remarkable hierarchy of "$p$-black holes", generalising the Schwarzschild metric in $d$ dimensions as follows

$$\mathrm{d}s^2 = -N_{(p)}\mathrm{d}t^2 + N_{(p)}^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{(d-2)} \,,$$

with

$$N_{(p)} = 1 - \frac{2C}{r^{\frac{d-2p-1}{p}}} \,, \tag{1}$$

$\mathrm{d}\Omega^2_{(d-2)}$ being the line element on the unit $(d-2)$-sphere. For $p = 1$ one recovers the standard lapse function in $d$ dimensions [14],

$$N_{(1)} \equiv N = 1 - \frac{2C}{r^{d-3}} \,. \tag{2}$$

The derivation of the solution (1) and the study of various aspects of interest will be given below.

Let us present the construction of the $p$-Ricci tensors. A detailed account can be found in [15] where many sources are cited. We present here the essential steps using throughout differential forms in the frame-vector (*vielbein*) basis which will be most useful for our purposes.

In the following sections we will be concerned with a particular, simple, class of metrics. Generally, for some metric, a suitable set of tangent frame vectors are the 1-forms

$$e^a = e^a_\mu \, \mathrm{d}x^\mu \,,$$

where as usual $a, b, \ldots$ denote frame indices and $\mu, \nu, \ldots$ space-time ones.

The torsionless antisymmetric spin-connection 1-forms

$$\omega^{ab} = \omega^{ab}_{\mu} \mathrm{d}x^{\mu} = -\omega^{ba}$$

satisfy

$$\mathrm{d}e^a + \omega^{ab} \wedge e_b = 0. \tag{3}$$

The curvature tensor is now given by the antisymmetric 2-form

$$R^{ab} = \mathrm{d}\omega^{ab} + \omega^{a}{}_{c} \wedge \omega^{cb} = -R^{ba} , \tag{4}$$

which can be expressed in the "$e$-basis" as

$$R^{ab} = R^{ab}_{a'b'}\, e^{a'} \wedge e^{b'}. \tag{5}$$

(5) is the Riemann 2-form corresponding to $p = 1$. Now we introduce the following recursive construction of totally antisymmetric Riemann $2p$-forms.

For $p = 2$ we define

$$R^{abcd} = R^{ab} \wedge R^{cd} + R^{ad} \wedge R^{bc} + R^{ac} \wedge R^{db} , \tag{6}$$

having cyclically permuted $(b, c, d)$. Using the antisymmetry of $R^{ab}$ and, the fact that $R^{ab} \wedge R^{cd} = R^{cd} \wedge R^{ab}$ since 2-forms commute, one readily verifies that $R^{abcd}$ is totally antisymmetric.

For $p = 3$ we define

$$R^{abcdef} = R^{ab} \wedge R^{cdef} + R^{af} \wedge R^{bcde} + R^{ae} \wedge R^{fbcd} + R^{ad} \wedge R^{efbc} + R^{ac} \wedge R^{defb} \tag{7}$$

having cyclically permuted $(b, c, d, e, f)$. There are 15 terms of the type $R^{ab} \wedge R^{cd} \wedge R^{ef}$ in (7) and again it can be checked readily that $R^{abcdef}$ is totally antisymmetric.

The general recursion scheme is

$$R^{a_1 a_2 \ldots a_{2p}} = R^{a_1 a_2} \wedge R^{a_3 a_4 \ldots a_{2p}} + \text{ cyclic permutations of } (a_2, a_3, \ldots, a_{2p}) , \tag{8}$$

which consists of 3.5...$(2p-3)(2p-1)$ terms of the type $R^{a_1a_2} \wedge R^{a_3a_4} \wedge \ldots \wedge R^{a_{2p-1}a_{2p}}$, and is totally antisymmetric. In the $e$-basis

$$R^{a_1a_2\ldots a_{2p}} = R^{a_1a_2\ldots a_{2p}}_{b_1b_2\ldots b_{2p}}\, e^{b_1} \wedge e^{b_2} \ldots \wedge e^{b_{2p}}. \tag{9}$$

For $p = 1$ the Ricci tensor is given by

$$R_{(1)}{}^a_b = \sum_c R^{ac}_{bc} = R^a_b\ . \tag{10}$$

We define the $p$-Ricci tensor to be

$$R_{(p)}{}^{a_1}_{b_1} = \frac{1}{(2p-1)!} \sum_{(a_2,\ldots,a_{2p})} R^{a_1a_2\ldots a_{2p}}_{b_1a_2\ldots a_{2p}}\ , \tag{11}$$

such that the $p$-Ricci scalar pertaining to it is

$$R_{(p)} = \sum_a R_{(p)}{}^a_a\ . \tag{12}$$

The overall constant $(2p-1)!$ in (11) takes acount of the number of permutaions of the indices summed over subject to the total antisymmetry of both upper and lower indices. This normalisation factor, though convenient, is however not essential for our purposes.

The corresponding generalisations of the Einstein-Hilbert Lagrangian and the Einstein tensor are given in [15]. But, apart from pointing out in passing how de Sitter type solutions arise very simply and evidently, our concern here will be limited to $p$-Ricci flat solutions.

In the absence of matter and cosmological constant the Lagrangian is $\sqrt{|g|}R_{(p)}$ and the variational minima are obtained (in the $e$-basis) [15] from

$$R_{(p)}{}^a_b - \frac{1}{2p} g^a_b R_{(p)} = 0. \tag{13}$$

For the $p$-Ricci flat case, consistently with $R_{(p)} = 0$, the solutions satisfy

$$R_{(p)}{}^a_b = 0\ . \tag{14}$$

One computes (11) for a given metric introduced as an ansatz and involving one (or possibly more) unknown function(s) to be consistently determined to satisfy the constraints (14).

In this paper we study the *spherically symmetric* solutions, in which case it is natural to start (since we have the hierarchy (1) in mind) with

$$\mathrm{d}s^2 = -N\mathrm{d}t^2 + N^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{(d-2)} , \tag{15}$$

and see if one can obtain an $N$ satisfying all the constraints (14). In the event, we will start with the Kerr-Schild (K-S) form of the metric given by

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + 2Cl_\mu l_\nu \\ g^{\mu\nu} &= \eta^{\mu\nu} - 2Cl^\mu l^\nu \end{aligned} \tag{16}$$

where $\eta_{00} = -1, \eta_{ij} = \delta_{ij}$, (i=1,2, ... ,d-1), and

$$l^\mu l_\mu = \eta^{\mu\nu} l_\nu l_\mu = g^{\mu\nu} l_\nu l_\mu = 0 . \tag{17}$$

For black hole type solutions with a horizon, one assumes $C > 0$.

For spherical symmetry one may set

$$l_i = l_0 \frac{x_i}{r} = l_0 \hat{x}_i , \tag{18}$$

with $r = \sqrt{|x_i|^2}$ and $|\hat{x}_i|^2 = 1$, and $l_0 = l_0(r)$ a function of $r$ only.

Now (15) and (16) are related through

$$x_0 = t + \int \frac{\mathrm{d}r}{N} - r , \qquad L \equiv l_0^2 = \frac{1}{2C}(1 - N) . \tag{19}$$

The K-S form is non-diagonal and thus loses some simple properties of (15). But we consider its introduction worthwhile for the following two reasons.

- In (15) there are *two* distinguished coordinates $(t, r)$ while in (16) there is only *one*, $x_0$, all the space-coordinates being treated on the same footing. As will be shown below, this leads to a smaller number of groups of terms to be summed over. (In fact, for spherical coordinates each angular one, $\theta_i$, also contributes differently through $\sin\theta_i$ to the line element. But, as will be seen, this feature gets absorbed in the $e$-basis for $R^{ab}$.)

- The second, and more important reason is, that the K-S form is more efficient for generalisation to axial symmetry [14, 7]. In exploring such possibilities, faced with the complexities of the combinatorics involved, it would be helpful to to have ready the results of the spherically symmetric limit to fall back on as checks.

The second of these points concerns however a future project. As will be seen in the following sections we will also fully exploit the special advantages of (15).

# 2 Construction of $p$-Ricci tensors for spherical symmetry and solutions

For the K-S metric (16) the frame vectors are

$$\begin{aligned} e^a_\mu &= \eta^a_\mu + C\, l_\mu l^a \\ e^\mu_a &= \eta^\mu_a - C\, l^\mu l_a \ , \end{aligned} \tag{20}$$

and $e^a = e^a_\mu \mathrm{d}x^\mu$ with $e_a = \eta_{ab} e^b$.

The spin-connections are

$$\omega^{ab}_\mu = \eta^{b\nu}\partial_\nu e^a_\mu - \eta^{a\nu}\partial_\nu e^b_\mu \tag{21}$$

and $\omega^{ab} = \omega^{ab}_\mu \, \mathrm{d}x^\mu = -\omega^{ba}$.

Since (18) holds for spherical symmetry it is convenient to introduce the notations

$$\mathrm{d}r = \hat{x}_i \mathrm{d}x^i \ , \qquad e^r = \hat{x}_i e^i \ , \tag{22}$$

giving

$$\begin{aligned} e^0 &= \mathrm{d}x^0 - C\, L\, (\mathrm{d}x^0 + \mathrm{d}r) \\ e^i &= \mathrm{d}x^i + C\, L\, \hat{x}_i\, (\mathrm{d}x^0 + \mathrm{d}r) \\ e^r &= \mathrm{d}r + C\, L\, (\mathrm{d}x^0 + \mathrm{d}r) \end{aligned} \tag{23}$$

in which $L = l_0^2$.

Using now (4) and (21) one obtains, after straightforward simplifications,

$$\begin{aligned} R^{0i} &= A\, e^0 \wedge e^i + B\, \hat{x}_i\, e^0 \wedge e^r \\ R^{ij} &= F\, e^i \wedge e^j + H\, e^r \wedge (\hat{x}_j e^i - \hat{x}_i e^j) \end{aligned} \tag{24}$$

where, with $L' = \frac{dL}{dr}$ and $L'' = \frac{d^2L}{dr^2}$,

$$\begin{aligned} A &= \frac{CL'}{r}\,, \qquad B = C\left(L'' - \frac{L'}{r}\right) = (CL'' - A) \\ F &= \frac{2CL}{r^2}\,, \qquad H = C\left(\frac{2L}{r^2} - \frac{L'}{r}\right) = (F - A). \end{aligned} \tag{25}$$

Note that the de Sitter solution in $d$ dimensions is obtained by setting

$$L = r^2\,, \qquad A = F = 2C\,, \qquad B = H = 0\,, \tag{26}$$

yielding

$$R^{ab} = 2C\, e^a \wedge e^b\,. \tag{27}$$

For higher $p$ members of the hierarchy, one continues to obtain similar features. Starting with $p = 2$, where

$$R^{abcd} = 3(2C)^2 e^a \wedge e^b \wedge e^c \wedge e^d. \tag{28}$$

The situation does not change essentially as $p$ increases, giving at each stage a constant $R_{(p)}$. Having pointed this out, we consider henceforth only $p$-flat solutions without cosmological constant.

As a check one may verify, to start with, that for

$$L = r^{-(d-3)}$$

one indeed obtains the Schwarzschild solutions in $d$ dimensions [14] satisfying

$$R_{(1)}{}^a_b \equiv R^a_b = R^{ac}_{bc} = 0. \tag{29}$$

This will of course also emerge as the simplest particular ($p = 1$) case of our general solution given below. It is very instructive to study in detail the cases $p = 2$ and 3. For low values of $d$ one can even write all terms in the summations explicitly. The general structure is seen to emerge more clearly at each successive stage. For brevity however, we present directly the general case. From (8) and (24) one obtains the following results.

*No summations are involved* in the following particular cases of

$$R^{a_1 a_2 \dots a_{2p}}_{b_1 b_2 \dots b_{2p}}$$

necessary for the passage to $R_{(p)}{}^a_b$. One has

$$R^{0 i_1 \dots i_{2p-1}}_{i i_1 \dots i_{2p-1}} = 0 \tag{30}$$

$$R^{i i_1 \dots i_{2p-1}}_{0 i_1 \dots i_{2p-1}} = 0 \tag{31}$$

$$\begin{aligned} R^{0 i_1 \dots i_{2p-1}}_{0 i_1 \dots i_{2p-1}} &= (1.3.5. \dots (2p-3)) F^{p-2} \Big[ (2p-1) AF \\ &\quad + (BF - 2(p-1)AH)(\hat{x}^2_{i_1} + \cdots + \hat{x}^2_{i_{2p-1}}) \Big] \end{aligned} \tag{32}$$

$$R^{i_1 i_2 \dots i_{2p}}_{i_1 i_2 \dots i_{2p}} = (1.3.5. \dots (2p-1)) F^{p-1} \left[ F - H(\hat{x}^2_{i_1} + \dots + \hat{x}^2_{i_{2p}}) \right] \tag{33}$$

$$R^{i_1 i_2 \dots i_{2p}}_{j_1 i_2 \dots i_{2p}} = -(1.3.5. \dots (2p-1)) F^{p-1} H \hat{x}_{i_1} \hat{x}_{j_1} \,, \qquad i_1 \neq j_1 \tag{34}$$

$$R^{i_1 0 i_2 \dots i_{2p-1}}_{j_1 0 i_2 \dots i_{2p-1}} = (1.3.5. \dots (2p-3)) F^{p-2} \left[ (BF - 2(p-1)AH) \, \hat{x}_{i_1} \hat{x}_{j_1} \right] . \tag{35}$$

For obtaining $R_{(p)}{}^a_b$ one has to sum over the appropriate indices in each case. To start with one notes, trivially, from (30) that

$$R_{(p)}{}^0_i = \frac{1}{(2p-1)!} \sum_{i_1, \dots, i_{2p-1}} R^{0 i_1 \dots i_{2p-1}}_{i i_1 \dots i_{2p-1}} = 0 \,. \tag{36}$$

Similarly, from (31)

$$R_{(p)}{}^i_0 = 0 \,. \tag{37}$$

Concerning the other cases it is helpful to note the following points,

$$\sum_{i_1, \dots, i_{2p-1}} (\hat{x}^2_{i_1} + \cdots + \hat{x}^2_{i_{2p-1}}) = \begin{pmatrix} d-2 \\ 2p-2 \end{pmatrix} \tag{38}$$

and

$$\sum_{i_2, \dots, i_{2p-1}} (\hat{x}^2_{i_2} + \cdots + \hat{x}^2_{i_{2p-1}}) = \begin{pmatrix} d-3 \\ 2p-3 \end{pmatrix} (1 - \hat{x}^2_{i_1}) \,, \tag{39}$$

where we have used the usual notation

$$\left( \begin{array}{c} a \\ b \end{array} \right) = \frac{a!}{b!\,(a-b)!} \ .$$

For (38) one notes that in summing over the $(2p-1)$-tuples, *each* $\hat{x}_i$ occurs as many times as the possibility of selecting $(2p-2)$-tuples (of distinct $\hat{x}_j$'s) among the coordinates after fixing 0 and $i$, namely among $(d-2)$. Then one uses

$$\sum_{i=1}^{d-1} \hat{x}_i^2 = 1 \ .$$

For (39) one uses analogous arguments, and

$$\sum_{j \neq i} \hat{x}_j^2 = 1 - \hat{x}_i^2 \ .$$

We use such results and consider the normalisation factor in (11) to be absorbed implicitly by exhibiting only contributions of distinct numbers of combinations, as in (38) and (39). For convenience we also introduce the notations

$$\begin{aligned} X &\equiv L^{p-2}\left[p(rLL') + (d-2p-1)L^2\right] \\ Y &\equiv L^{p-2}\left[r^2(LL'' + (p-1)L'^2) + (d-2p)(rLL')\right] \ . \end{aligned} \tag{40}$$

Using all these results, combining terms and simplifying, completing the necessary summations and using (24), one obtains finally,

$$\begin{aligned} R_{(p)}{}_0^0 &= (1.3.5...(2p-3))F^{p-2}\left[(2p-1)AF \left( \begin{array}{c} d-1 \\ 2p-1 \end{array} \right) \right. \\ &\quad \left. +(BF - 2(p-1)AH) \left( \begin{array}{c} d-2 \\ 2p-2 \end{array} \right)\right] \\ &= C^p(1.3.5...(2p-3)) \left( \begin{array}{c} d-2 \\ 2p-2 \end{array} \right) \frac{2^{p-1}}{r^{2p}} Y \end{aligned} \tag{41}$$

$$
\begin{aligned}
R_{(p)}{}^i_j &= (1.3.5...(2p-3))F^{p-2}\left[(BF-2(p-1)AH)\begin{pmatrix} d-3 \\ 2p-2 \end{pmatrix}\right. \\
&\quad \left. -(2p-1)FH\begin{pmatrix} d-3 \\ 2p-1 \end{pmatrix}\right]\hat{x}_i\hat{x}_j \\
&= C^p(1.3.5...(2p-3))\begin{pmatrix} d-3 \\ 2p-2 \end{pmatrix}\frac{2^{p-1}}{r^{2p}}(Y-2X)\hat{x}_i\hat{x}_j, \quad i\neq j
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
R_{(p)}{}^i_i &= (1.3.5...(2p-3))F^{p-2}\left[(2p-1)AF\begin{pmatrix} d-2 \\ 2p-2 \end{pmatrix}\right. \\
&\quad + (BF-2(p-1)AH)\left(\begin{pmatrix} d-2 \\ 2p-2 \end{pmatrix}\hat{x}_i^2\right. \\
&\quad \left.\left. + \begin{pmatrix} d-3 \\ 2p-3 \end{pmatrix}(1-\hat{x}_i^2)\right)\right] \\
&\quad + (1.3.5...(2p-1))F^{p-1}\left[F\begin{pmatrix} d-2 \\ 2p-1 \end{pmatrix} - H\left(\begin{pmatrix} d-2 \\ 2p-1 \end{pmatrix}\hat{x}_i^2\right.\right. \\
&\quad \left.\left. + \begin{pmatrix} d-3 \\ 2p-2 \end{pmatrix}(1-\hat{x}_i^2)\right)\right] \\
&= C^p(1.3.5...(2p-3))\frac{(d-3)!}{(2p-2)!(d-2p)!}\frac{2^{p-1}}{r^{2p}}\left[(p-1)Y\right. \\
&\quad \left. + (d-2p)X + (d-2p)(Y-2X)\hat{x}_i^2\right].
\end{aligned} \tag{43}
$$

From (36), (37), (41), (42) and (43) we see that for *all* $(a,b)$,

$$R_{(p)}{}^a_b = 0$$

provided that

$$
\begin{aligned}
X &= L^{p-2}\left[p(rLL') + (d-2p-1)L^2\right] = 0 \\
Y &= L^{p-2}\left[r^2(LL'' + (p-1)L'^2) + (d-2p)(rLL')\right] = 0\,.
\end{aligned} \tag{44}
$$

Since a constant factor of $L$ can be absorbed in $C$, the *single* necessary and sufficient constraint satisfying (44), and hence the variational equations (41)-(43), turns out to be

$$L^p = r^{-(d-2p-1)}\,. \tag{45}$$

Differentiating (45) once yields

$$X = 0$$

and differentiating a second time,

$$Y = 0\,.$$

For $p = 1$ this reduces to the standard Schwarzschild metric in $d$ dimensions [14] with

$$L = l_0^2 = r^{-(d-3)}\,. \tag{46}$$

Let us now indicate briefly certain complementary features arising when one uses the diagonal metric (15). Now one has

$$\mathrm{d}s^2 = -N\mathrm{d}t^2 + N^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{(d-2)}\,,$$

where

$$\mathrm{d}\Omega^2_{(d-2)} = \mathrm{d}\theta_1^2 + \sin^2\theta_1\mathrm{d}\theta_2^2 + \cdots + \left(\prod_{n=1}^{d-3}\sin\theta_n\right)^2 \mathrm{d}\theta^2_{d-2}\,. \tag{47}$$

For a diagonal metric one can set (no summation being involved)

$$\begin{aligned} e^a &= \sqrt{g_{aa}}\mathrm{d}x^a\,, \qquad x^a = t, r, \theta_1, \ldots, \theta_{d-2} \\ \omega^{ab} &= \frac{1}{\sqrt{g_{aa}g_{bb}}}\left[(\partial_b\sqrt{g_{aa}})e^a - (\partial_a\sqrt{g_{bb}})e^b\right]\,. \end{aligned} \tag{48}$$

Thus, in particular,

$$\begin{aligned} \omega^{\theta_i\theta_j} \equiv \omega^{ij} &= -(\cos\theta_i\sin\theta_{i+1}\ldots\sin\theta_{j-1})\mathrm{d}\theta_j \\ &= -\frac{1}{r}\cos\theta_i\left(\prod_{n=1}^{i}\sin\theta_n\right)^{-1} e^j = -\omega^{ji}\,, \qquad i < j\,. \end{aligned} \tag{49}$$

These can be shown to satisfy the remarkable relation

$$\mathrm{d}\omega^{ij} + \sum_k \omega^{ik}\wedge\omega^{kj} = \frac{1}{r^2}e^i\wedge e^j\,, \qquad 1 \le i, j, k \le d-2\,, \tag{50}$$

independently of $d$.

By virtue of (50) as well as other more evident relations one obtains finally (with index $i$ standing for $\theta_i$)

$$\begin{aligned} R^{tr} &= -\frac{1}{2}N''\,e^t\wedge e^r\,, \qquad R^{ti} = -\frac{1}{2r}N'\,e^t\wedge e^i \\ R^{ri} &= -\frac{1}{2r}N'\,e^r\wedge e^i\,, \qquad R^{ij} = \frac{1}{r^2}(1-N)\,e^i\wedge e^j \end{aligned} \tag{51}$$

with $i, j = 1, 2, \ldots (d-2)$.

Thus $R^{ab}$ is "diagonalised" leading to dramatic simplifications in certain respects. But as pointed out in section **I**, there are now more distinct blocks to be summed over in evaluating $R_{(p)}{}^{a}_{b}$. Let us illustrate this for the simplest nontrivial case, $p = 2$. Now for $R_{(2)}{}^{i}_{i} (\equiv R_{(2)}{}^{\theta_i}_{\theta_i})$ one has *four* distinct classes of terms as compared to *two* in the previous case. Thus

$$R_{(2)}{}^{i}_{i} = \sum_{j} R^{itrj}_{itrj} + \sum_{jk} R^{itjk}_{itjk} + \sum_{jk} R^{irjk}_{irjk} + \sum_{jkl} R^{ijkl}_{ijkl} \,. \tag{52}$$

Implementing (51) in (6) one obtains finally, on the right hand side

$$\frac{d-3}{2r^4}\Big[ - r^2(1-N)N'' + r^2(-N')^2 - (d-4)r(1-N)N' \\ + (d-4)(1-N)\left(-2rN' + (d-5)(1-N)\right)\Big] \,. \tag{53}$$

The intermediate, suppressed, steps are straightforward.

Consistently with our previous derivation, this expression vanishes for

$$N = N_{(2)} = 1 - 2CL_{(2)} = 1 - 2Cr^{-\frac{1}{2}(d-5)} \,. \tag{54}$$

One can verify that for the general case one indeed obtains

$$N_{(p)} = 1 - 2CL_{(p)} = 1 - 2Cr^{-\frac{d-2p-1}{p}} \,. \tag{55}$$

We will not rederive here this result. But important uses of the diagonal metric are presented in the following sections.

# 3 Maximal extensions and $p$-dependent periodicity for Euclidean signature

The limiting cases

$$d = 2p \qquad \text{and} \qquad d = 2p+1$$

will be briefly discussed in the following section. For

$$d > 2p + 1$$

one has a Schwarzschild like event horizon in

$$\mathrm{d}s^2 = -N_{(p)}\mathrm{d}t^2 + N_{(p)}^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{(d-2)}^2 \,, \tag{56}$$

with

$$N_{(p)} = 1 - \frac{2C}{r^{\frac{d-2p-1}{p}}} \equiv 1 - \left(\frac{K}{r}\right)^{\frac{d-2p-1}{p}} \,,$$

at

$$r = K \,.$$

One can introduce Kruskal type coordinates, generalised for $p > 1$, to desingularise the horizon. We will be particularly interested in the Euclidean section and the $p$-dependence of the time-period which becomes necessary for consistency. For the standard case ($d$=4, $p = 1$) this period is well known to be inversely proportional to the Hawking temperature of the black hole. Here we generalise the treatment (for $d \geq 4$ , $p = 1$) presented in [5]. For $d = 4$ a detailed study citing basic sources is to be found in [2].

We start directly with the Euclidean continuation of (56),

$$\mathrm{d}s^2 = N_{(p)}\mathrm{d}t^2 + N_{(p)}^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{(d-2)}^2 \,. \tag{57}$$

Defining

$$r^\star = \int \frac{\mathrm{d}r}{N_{(p)}} \,, \tag{58}$$

we introduce the coordinates $(\eta, \zeta)$ satisfying (for some constant $\lambda$ to be fixed later)

$$\mathrm{e}^{2\lambda r^\star} = \frac{1}{4}(\eta^2 + \zeta^2) \,, \qquad \mathrm{e}^{i\lambda t} = \left(\frac{\eta - i\zeta}{\eta + \zeta}\right)^{\frac{1}{2}} \tag{59}$$

leading to

$$\mathrm{d}s^2 = N_{(p)}(4\lambda^2\mathrm{e}^{2\lambda r^\star})^{-1}(\mathrm{d}\zeta^2 + \mathrm{d}\eta^2) + r^2\mathrm{d}\Omega_{(d-2)}^2 \,. \tag{60}$$

Setting

$$r = K\rho^p \quad \text{and} \quad n = d - 2p - 1\,, \qquad r \geq K\,, \rho \geq 1 \tag{61}$$

$$\begin{aligned} r^\star &= \int \frac{\mathrm{d}r}{1-(\frac{K}{r})^{\frac{n}{p}}} = Kp \int \frac{\rho^{n+p-1}}{\rho^n - 1}\mathrm{d}\rho \\ &= Kp\left(\int \frac{\mathrm{d}\rho}{\rho^n-1} + \int \frac{\rho^{n+p-1}-1}{\rho^n-1}\mathrm{d}\rho\right)\,. \end{aligned} \tag{62}$$

It is not necessary for our purposes to evaluate the integrals, though that is possible. As in [5] we use

$$\frac{1}{x^n-1} = \frac{1}{n}\left(\frac{1}{x-1} - \frac{x^{n-2}+2x^{n-3}+\cdots+(n-2)x+(n-1)}{x^{n-1}+x^{n-2}+\cdots+x+1}\right) \tag{63}$$

resulting in

$$r^\star = \frac{Kp}{n}\int \frac{\mathrm{d}\rho}{\rho - 1} + f(\rho) \tag{64}$$

where the function $f(\rho)$ which we do not evaluate, does not contribute to the singularity at $\rho = 1$ or $r = K$. Hence

$$\mathrm{e}^{-2\lambda r^\star} = \left(\left(\frac{r}{K}\right)^{\frac{1}{p}} - 1\right)^{-(\frac{2\lambda Kp}{n})} \mathrm{e}^{-2\lambda h(r)} \tag{65}$$

where the function $h(r)$ creates no singularity at the horizon. Choosing

$$\lambda = -\left(\frac{n}{2Kp}\right) = -\left(\frac{d-2p-1}{2Kp}\right) \tag{66}$$

one obtains the required desingularisation at the horizon as a direct generalisation of the standard case for $p = 1$, namely of

$$\lambda = -\left(\frac{d-3}{2K}\right)\,. \tag{67}$$

It should be noted that for Lorentz signature there is already an essential singularity at $r = 0$ and for Euclidean signature the domain of real values of $(\eta,\ \zeta)$ corresponds to $r \geq K$. *Hence in both cases a fractional power of $r$ in $N_{(p)}$ does not introduce a crucial supplementary complication.* One may also note that though the power of $r$ in $N_{(p)}$ is in general fractional, for any $p$ and $d$ satisfying

$$d = (n' + 2)p + 1 \ , \qquad n' = 1, 2, 3, \ldots$$

it becomes an integer. Examples of such integers for $p > 1$ are

$$(d\ ,\ p\ ;\ n') = (10\ ,\ 3\ ;1),\ (11\ ,\ 2\ ;\ 3), \ldots$$

From (59) and (66) one obtains for $t$ a period

$$P_{(p)} = \frac{2\pi}{|\lambda|} = \frac{4\pi K p}{d - 2p - 1} \ , \qquad d > 2p + 1 \ . \tag{68}$$

For $d = 4$, $p = 1$, $K = 2M$ one gets back to the well-known result for the Schwarzschild metric

$$P \equiv P_{(1)} = 8\pi M \ . \tag{69}$$

For fields on the Euclidean section, periods and temperatures are related inversely through the Boltzmann constant. Here one sees that, for given $d$, $P_{(p)}$ increases and hence the temperature decreases as $p$ becomes larger.

# 4 Special cases

We briefly indicate certain features of the special cases $d = 2p$, $d = 2p + 1$, and $d = 4p$.

## 4.1 $d = 2p$

It is easily verified that when $d = 2p$ the Lagrangian $R_{(p)}$ is a total divergence and hence can possess topological rather than dynamical properties. In fact one obtains the Euler number which, if our solution is implemented, is constrained to be zero – a new possibility arising for

$p > 1$. Here we express the single surviving $2p$-form in terms of total differentials and note the relation with our general solution.

The simplest example is provided by

$$d = 4 \, , \quad p = 2 \, .$$

From (51) and (6)

$$\begin{aligned} R^{tr\theta_1\theta_2} &= R^{tr} \wedge R^{\theta_1\theta_2} + R^{t\theta_2} \wedge R^{r\theta_1} + R^{t\theta_1} \wedge R^{\theta_2 r} \\ &= \frac{1}{4r^2} \left[ -(1-N)N'' + (N')^2 \right] e^t \wedge e^r \wedge e^{\theta_1} \wedge e^{\theta_2} \, . \end{aligned} \tag{70}$$

In conventional notations, $\theta_1 = \theta$, $\theta_2 = \frac{\pi}{2} - \phi$, and using $N = 1 - 2CL$,

$$R^{tr\theta\phi} = \left( \frac{1}{2} C^2 \right) \mathrm{d}t \wedge \frac{\mathrm{d}}{\mathrm{d}r} \left( \frac{\mathrm{d}}{\mathrm{d}r} L^2 \right) \mathrm{d}r \wedge \mathrm{d}\cos\theta \wedge \mathrm{d}\phi \, . \tag{71}$$

For the general case

$$d = 2p \, , \qquad p = 1, 2, 3, \ldots$$

the angular factors can again be expressed as total differentials in an evident fashion while, crucially, the coefficient of $\mathrm{d}r$ can be shown to become

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}(L^p). \tag{72}$$

The computation is straightforward but will not be presented here. We just note that this coefficient vanishes for

$$L^p = c_1 + c_2 r \tag{73}$$

with $c_1$ and $c_2$ constants. Our generic solution

$$L^p = r^{-(d-2p-1)} = r \tag{74}$$

for $d = 2p$ is included in (73) for $c_1$=0 and $c_2 = 1$, giving

$$N_{(p)} = 1 - 2Cr^{\frac{1}{p}} = 1 - 2Cr^{\frac{2}{d}} \, . \tag{75}$$

For $d = 4$ this gives

$$N_{(2)} = 1 - 2Cr^{\frac{1}{2}} \, .$$

In section 3 we generalised Kruskal coordinates to *negative* fractional power of $r$ in $N_{(p)}$. Let us just mention that one can formally generalise Gibbons-Hawking coordinates [2, 10] for such a "cosmological" horizon involving a fractional power $\frac{1}{p}$.

Flat space is of course, as always, a solution $(c_1 = c_2 = 0)$. But for $c_2 = 0$ one obtains a constant $L^p$. This situation is discussed below since it arises generically for $d = 2p + 1$.

### 4.2 $d = 2p + 1$

In this case our generic solution itself gives

$$L^p = r^{-(d-2p-1)} = 1 \,. \tag{76}$$

This corresponds to the situation where the $L^2$ term is absent in $X$ given by (40).

For $2C < 1$, now

$$N = 1 - 2C = \lambda^2 < 1 \tag{77}$$

and one has still Lorentz signature with

$$\mathrm{d}s^2 = -\lambda^2 \mathrm{d}t^2 + \lambda^{-2}\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{(d-2)} \,. \tag{78}$$

Setting $r = \rho^\lambda$ one can obtain an "isotropic" form. We do not propose to discuss further here this degenerate case. Of much more interest is the following one.

### 4.3 $d = 4p$ : double selfduality

The general formulation of double selfduality and the generalised Yang-Mills instantons constructed, for Euclidean signature, in terms of the spin-connections was the major theme in [15]. For $d = 8$, de Sitter and Fubini-Study metrics were obtained by the present authors, imposing double selfduality [8]. Here we will verify that our Schwarzschild type ($p$-Schwarzschild) solutions also satisfy double selfduality for $d = 4p$.

To start with we note that (51) along with (8) ensures rhat $R^{a_1 a_2 \ldots a_{2p}}$ has *only one* non-zero component

$$R^{a_1 a_2 \ldots a_{2p}}_{a_1 a_2 \ldots a_{2p}} \, .$$

This "diagonalisation" simplifies dramatically the situation. In the general formulation, in frame indices, the double self-duality constraint

$$R^{a_1 a_2 \ldots a_{2p}}_{b_1 b_2 \ldots b_{2p}} = \frac{1}{(2p!)^2} \varepsilon^{a_1 a_2 \ldots a_{2p} d_1 d_2 \ldots d_{2p}} \varepsilon_{b_1 b_2 \ldots b_{2p} c_1 c_2 \ldots c_{2p}} R^{c_1 c_2 \ldots c_{2p}}_{d_1 d_2 \ldots d_{2p}} \tag{79}$$

reduces to

$$R^{a_1 a_2 \ldots a_{2p}}_{a_1 a_2 \ldots a_{2p}} = R^{a_{2p+1} a_{2p+2} \ldots a_{4p}}_{a_{2p+1} a_{2p+2} \ldots a_{4p}} \tag{80}$$

for *all* choices of complementary sets of the $4p$ indices.

For $d = 4, \; p = 1$

$$1 - N = \frac{2C}{r}$$

and the well-known double selfduality of the Schwarzschild metric is assured through the relations

$$-\frac{1}{2} N'' = \frac{2C}{r^3} = \frac{1}{r^2}(1 - N) \tag{81}$$

since from (51)

$$R^{tr}_{tr} = -\frac{1}{2} N'' \, , \qquad R^{12}_{12} = \frac{1}{r^2}(1 - N) \, . \tag{82}$$

Even more directly one has

$$R^{t2}_{t2} = R^{r1}_{r1} \, , \qquad R^{t1}_{t1} = R^{r2}_{r2} \tag{83}$$

each one being just

$$-\frac{N'}{2r} \, .$$

Thus one has a gravitational instanton on the Euclidean section with nontrivial topological indices (Euler number and Hirzbruch signature).

The spin-connections lead to a selfdual Yang-Mills field [4]. It is a general fact that such a construction provides a solution for the combined gravitational-YM system. This is because the metric is not affected by a **back-reaction** of the YM field, since the stress energy-momentum tensor of the selfdual YM field vanishes.

For $d = 8$ , $, p = 2$, we have

$$1 - N = 2C\, r^{-\frac{1}{2}(8-4-1)} = 2C\, r^{-\frac{3}{2}} \,. \tag{84}$$

This assures in the crucial equality between

$$R^{tr12}_{tr12} = \left(-\frac{1}{2}N''\right)\left(\frac{1}{r^2}(1-N)\right) + 2\left(-\frac{N'}{2r}\right)^2$$

and

$$R^{3456}_{3456} = 3\left(\frac{1}{r^2}(1-N)\right)^2$$

and hence the double selfduality constraint

$$R^{tr12}_{tr12} = R^{3456}_{3456} \,. \tag{85}$$

Here, as before, the indices $1, 2, \ldots, 6$ refer to $\theta_1, \theta_2, \ldots, \theta_6$ respectively. Other constraints due to (80) can als be shown to be satisfied systematically.

For the general case with $d = 4p$

$$1 - N = 2C\, r^{-\frac{d-2p-1}{p}} = 2C\, r^{-\frac{2p-1}{p}} \,. \tag{86}$$

This satisfies the crucial relation

$$\left(-\frac{1}{2}N''\right)\left(\frac{1}{r^2}(1-N)\right) + (2p-2)\left(-\frac{N'}{2r}\right)^2$$
$$= (2p-1)\left(\frac{1}{r^2}(1-N)\right)^2 \,, \tag{87}$$

which assures the double selfduality constraint

$$R^{tr12\ldots(2p-2)}_{tr12\ldots(2p-2)} = R^{(2p-1)(2p)\ldots(4p-2)}_{(2p-1)(2p)\ldots(4p-2)} \,. \tag{88}$$

Other constraints due to (80) can als be shown to be satisfied systematically.

A systematic study of the topological properties of this class of gravitational instantons and those of the associated YM ones (both for the general $p$ case in the sense of [15]) will be presented elsewhere.

# 5 A comment on surface deformations and near-horizon symmetry

A number of recent papers explore "gauge" algebra of surface deformations restricted to event or cosmological horizons [6, 13]. The references [6, 13] deal with any dimension $d$ , but of course for $p=1$. Other important sources, mostly concerning specific low dimensions, are cited in these papers. From both types of horizons, the $(r,t)$ plane playing a crucial role, a Virasoro subalgebra emerges from the study of surface deformations with a central charge

$$C=\frac{3A\beta}{2\pi GT} \tag{89}$$

where $A$ is the area of the horizon, $(8\pi G)^{-1}$ is the overall factor of the boundary terms in the full generator of surface deformations, $T$ is the period of rotational perturbations considered and lastly $\beta$ is obtained as a coefficient on developing the lapse function $N$ near the horizon. Note that $C$ in (89) denotes the central charge and should not be confused with the constant $C$ in the ansatz (20). For a Schwarzschild black-hole, for example, $\beta$ turns out to be the inverse of the Hawking temperature [6].

For our case, starting with (56), i.e.

$$N_{(p)}=1-\left(\frac{K}{r}\right)^{\frac{d-2p-1}{p}}$$

hence

$$\frac{\mathrm{d}N_{(p)}}{\mathrm{d}r}|_{r=K}=\left(\frac{d-2p-1}{pK}\right)\equiv\frac{2\pi}{\beta_{(p)}}$$

when

$$\beta_{(p)}=\frac{(2\pi K)\,p}{d-2p-1}\,. \tag{90}$$

This is, of course, directly proportional to the period $P_{(p)}$ derived before (see (68)) and gives for the Schwarzschild case with

$$d=4\ ,\ \ p=1\ \ ,\ K=2M\ ,$$

$$\beta = 4\pi M \ . \tag{91}$$

We do not intend to undertake a study of surface deformations for our horizons. We just point out that, quite plausibly, a $p$-dependent hierarchy of central charges will emerge in our context.

# 6 General remarks

To construct our metrics as variational minima of the traces of higher order terms, we have gone beyond Einsten theory generalised to higher dimensions. Higher order gravitational terms are nowadays familiar (the so called $(R^2)^2$ and $R^4$ terms) in string theory effective actions [11, 12]. But we consider separately the members of one particular hierarchy of higher order terms, that in which only velocity-square terms appear, generalising the Einsten-Hilbert term directly, and not perturbative contributions, starting with the standard formalism. In certain constructions of topological field theories [1] higher order terms are introduced directly at the start. One may keep such aspects in mind. But the real motivation for presenting our formal structure is the discovery of the remarkable class of solutions with their simple, suggestive and beautiful properties discussed above. Thus, for example, rich topological possibilities involving generalised gravitational and YM instantons enter in the wake of our solutions merely as a particular case ($d = 4p$). More generally, topological aspects in higher dimensions [1, 9] deserve study specifically in the context of our formulation.

An adequate formulation of surface deformation algebras, alluded to in Sec. **V**, might be of interest. We intend to explore elsewhere possibilities of axially symmetric, stationary solutions in our context. All these aspects along with a deeper understanding of the role of higher order forms provide an interesting program.

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