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Holomorphic Anomaly Equation and BPS State Counting of Rational Elliptic Surface

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Abstract

We consider the generating function (prepotential) for Gromov-Witten invariants of rational elliptic surface. We apply the local mirror principle to calculate the prepotential and prove a certain recursion relation, holomorphic anomaly equation, for genus 0 and 1. We propose the holomorphic anomaly equation for all genera and apply it to determine higher genus Gromov-Witten invariants and also the BPS states on the surface. Generalizing Göttsche's formula for the Hilbert scheme of g points on a surface, we find precise agreement of our results with the proposal recently made by Gopakumar and Vafa[11].

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1 Introduction

Since the pioneering work by Candelas et al. in 1991[6], the theory of the Gromov-Witten invariants has been one of the central topics in mathematical physics related to string theory. Due to many contributions on this subject we have now well-developed mathematical theory[16][4] of the invariants as well as the concrete methods to calculate them applying the mirror symmetry of Calabi-Yau manifolds. However up to very recently our concrete methods have been restricted to the genus zero or genus one Gromov-Witten invariants. Although we have mathematical definition of the higher genus Gromov-Witten invariants, little was known about how to determine them explicitly for a given Calabi-Yau manifold. Regarding this a breakthrough has been made recently in [18] for a special class of Calabi-Yau manifolds which have a K3 fibration and have a dual description in the heterotic string. Independently Gopakumar and Vafa[10][11] have derived a general form of the prepotential for the higher genus Gromov-Witten invariants, which includes several interesting mathematical predictions on the Gromov-Witten invariants.

In this paper we will propose a recursion relation *holomorphic anomaly equation* as a basic equation for the higher genus Gromov-Witten invariants of rational elliptic surface, and will make explicit predictions for them. We find exquisite agreement of our results with those by Gopakumar and Vafa.

To state main results of this paper let us consider a generic rational elliptic surface obtained by blowing up nine base points of two generic cubics in $\mathbf{P}^2$. Under the assumption for the cubics the surface S has an elliptic fibration over $\mathbf{P}^1$ with exactly twelve singular fibers of Kodaira I_1 type. We consider a situation in which the generic rational elliptic surface S appears as a divisor in a Calabi-Yau 3-fold X. Since the normal bundle $\mathcal{N}_{X/S}$ is given by the canonical bundle K_S we can extract the genus g Gromov-Witten invariants $N_g(\beta)$ of class $\beta \in H_2(S,\mathbf{Z})$ taking a suitable limit of the prepotential of the Calabi-Yau 3-fold X, which is called *local mirror principle.* Since even for genus zero invariants the determination of $N_{g=0}(\beta)$ is technically tedious, in what follows, we will mainly be concerned with the following sum of the invariants

$$N_{g;d,n} := \sum_{(\beta,H)=d,\ (\beta,F)=n} N_g(\beta)$$

where H and F represent the pull back of the hyperplane class of $\mathbf{P}^2$ and the fiber class, respectively. Associated to these invariants we define generating functions;

$$Z_{g;n}(q) := \sum_{d=0}^{\infty} N_{g;d,n} q^d \quad , \quad F_g(q,p) := \sum_{n=0}^{\infty} Z_{g;n} p^n \quad .$$

The latter is the genus g prepotential in topological string theory. For $g = 0$ and $g = 1$ we determine it via the local mirror principle applying to X, and find a recursion relation satisfied by $Z_{g;n}$ $(g = 0, 1; n = 1, 2, \cdots)$ which we generalize for arbitrary g as follows:

Conjecture 1.1 (Holomorphic anomaly equation for all g) *The generating function $Z_{g;n}(q)$ has the form*

$$P_{2g+2n-2}(\phi, E_2, E_4, E_6)\,(Z_{0;1}(q))^n \quad , \tag{1.1}$$

with some 'quasi-modular form' for the modular subgroup $\Gamma(3)$ of weight $2g + 2n - 2$. (In the special cases of $g = 0$ and $n = 1$, it simplifies to $P_{2n-2}(E_2, E_4, E_6)$ and $P_{2g}(E_2(q^3), E_4(q^3), E_6(q^3))$, i.e., exactly the quasi-modular forms of weight $2n - 2$ and $2g$, respectively). And it satisfies the recursion relation

$$\frac{\partial Z_{g;n}}{\partial E_2} = \frac{1}{72} \sum_{g'+g''=g} \sum_{s=1}^{n-1} s(n-s) Z_{g';s} Z_{g'';n-s} + \frac{n(n+1)}{72} Z_{g-1;n} \quad . \tag{1.2}$$

We may 'integrate' our holomorphic anomaly equation under certain vanishing conditions. In this paper we focus mainly on the special case of $n = 1$ in which the equation simplifies to

$$\frac{\partial Z_{g;1}}{\partial E_2} = \frac{1}{36} Z_{g-1;1} \quad . \tag{1.3}$$

We integrate 1.3 with the vanishing conditions and the initial data

$$Z_{0;1}(q) = q^{\frac{3}{2}} \frac{\Theta_{E_8}(3t, t\gamma)}{\eta(q^3)^{12}}$$

which has been found in [15][14]. The E_8 theta function comes from the Mordell-Weil group of the rational elliptic surface and the eta functions in the denominator come from the twelve singular fibers. See [14] for the details and notations. Then we find that the solutions $Z_{g;1}$ may be arranged into an all genus partition function of the topological string theory:

Proposition 1.2 (Topological string partition function on S)

$$q^{\frac{3}{2}} \frac{\Theta_{E_8}(3t, t\gamma)}{\eta(q^3)^{12}} \prod_{n \geq 1} \frac{(1-q^{3n})^4}{(1-t_L q^{3n})^2 (1-\frac{1}{t_L} q^{3n})^2} = \sum_{g \geq 0} Z_{g;1}(q) \lambda^{2g-2} (2\sin\frac{\lambda}{2})^2 \quad , \tag{1.4}$$

where λ represents the string coupling and $t_L = e^{i\lambda}$.

We derive the same result following the proposal made in [11] for the BPS state counting of the families of genus g curves. From this viewpoint our result 1.4 comes from the following generalization of Göttsche's formula[9] for the Hilbert scheme $S^{[g]}$ of g points on a surface S:

Proposition 1.3 (Göttsche's formula with $SL(2,\mathbf{C})_L \times SL(2,\mathbf{C})_R$ Lefschetz action) *For the Hilbert scheme $S^{[g]}$ of g points on a surface S with a fibration structure we can decompose the Lefschetz $SL(2,\mathbf{C})$ action on $H^*(S^{[g]})$ into the product $SL(2,\mathbf{C})_L \times SL(2,\mathbf{C})_R$, one for the natural fiber space of $S^{[g]}$ and the other for the base space. If we write the Poincaré polynomial by*

$$P_{t_L,t_R}(S^{[g]}) = (t_L t_R)^g \mathrm{Tr}_{H^*(S^{[g]})}(t_L^{2j_{3,L}} t_R^{2j_{3,R}}) \ ,$$

then the generating function $G(t_L,t_R,q) = \sum_{g\geq 0} P_{t_L,t_R}(S^{[g]})q^g$, for the surface with $b_1(S)=0$, is given by

$$G(t_L,t_R,q) = \prod_{n\geq 1}\left\{\frac{1}{(1-(t_Lt_R)^{n-1}q^n)(1-(t_Lt_R)^{n+1}q^n)}\right.$$
$$\times \left.\frac{1}{(1-t_L^2(t_Lt_R)^{n-1}q^n)(1-t_R^2(t_Lt_R)^{n-1}q^n)(1-(t_Lt_R)^nq^n)^{b_2(S)-2}}\right\} . \tag{1.5}$$

We explain our result 1.4 in terms of the above generalization of Göttsche's formula by

$$\Theta_{E_8}(3t,t\gamma)G(-t_L,-1,\frac{q^3}{t_L}) = \sum_{g\geq 0} Z_{g;1}(q)\lambda^{2g-2}(2\sin\frac{\lambda}{2})^2 \ . \tag{1.6}$$

This implies that the genus g curves $\mathcal{C}_g$ in S satisfying $(\mathcal{C}_g,F)=1$ split into irreducible parts, one coming from the Mordell-Weil group and the others from elliptic curves (with possible nodal singularities) in the fiber direction.

The readers who are not interested in the derivation and the proofs of the holomorphic anomaly equation may omit the following two sections and may start from the section 4 for our main results.

The organization of this paper is as follows: In section 2, we will introduce a Calabi-Yau hypersurface and its mirror, and introduce the hypergeometric series representing the prepotential $F_0(t)$ for the Calabi-Yau hypersurface.

In section 3, we will take a limit to reduce the prepotential to the one relevant to the rational elliptic surface (, *local mirror principle*). We will analyze the reduced prepotential and the mirror maps in detail, and will prove the recursion relation, holomorphic anomaly equation at $g = 0$. Using the formula in [1] for F_1, we will also prove the recursion relation at $g = 1$. In section 4, we will *propose* our recursion relation for all genera, and solve the recursion relation with some vanishing conditions. There we also discuss about the Gromov-Witten invariants coming from $Z_{g;1}(q)$. In the final section, we discuss some relations to the recent developments on the counting problem of the BPS states in topological string theory[11][15]. There we will find a generalization of Göttsche's formula for the Poincaré polynomials of $S^{[g]}$.

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After our submission of this paper to the e-print archive, hep-th, we are informed by A. Klemm that he is testing the higher genus prepotentials F_g for several surfaces other than our rational elliptic surface (work to appear). We would like to thank him for sending us his preliminary draft prior to publication.

In References [15] and [19], a different base $F + e_9$, instead of our H, is used to define $Z_{g=0;n}$. In this case we consider our generating function $Z_{g;n}$ for the invariants $N_{g;d,n} = \sum N_g(\beta)$ summed over β with $(\beta, F + e_9) = d, (\beta, F) = n$. Then our Conjecture 1.1 should be read as follows:

The generating function $Z_{g;n}(q)$ has the form

$$P_{2g+6n-2}(E_2, E_4, E_6)\frac{q^{\frac{n}{2}}}{\eta(q)^{12n}}$$

with a quasi-modular form $P_{2g+6n-2}(E_2, E_4, E_6)$ of weight $2g + 6n - 2$, and satisfies the same holomorphic anomaly equation as 1.2 replaced by the prefactors $\frac{1}{72}$ and $\frac{1}{24}$.

It is worthwhile remarking here that in this case the integration constants may be determined *consistently* for all g and n by simply requiring the vanishing conditions for the first few terms in the q-expansion of $\tilde{Z}_{g;n}(q)$, which is defined by 4.5 (3.18 and 3.27) with $D(g,h,k) = C_h(g-h,1)k^{2g-3}(0 \le h \le g)$[11].

2 Mirror symmetry of a Calabi-Yau hypersurface in $\mathbf{P}^2 \times \mathbf{F}_1$

In this section we will consider a Calabi-Yau hypersurface which contains a generic rational elliptic surface as a divisor. We collect necessary formulas for the (genus zero) prepotential.

Let us start with the Hirzebruch surface $\mathbf{F}_1$ which is defined by the quotient $(\mathbf{C}^4 \setminus \mathcal{Z})/\sim$ with

$$\mathcal{Z} = \{(u_1,u_2,u_3,u_4) \in \mathbf{C}^4 \mid u_1 = u_2 = 0 \text{ or } u_3 = u_4 = 0\}$$

and $\mathbf{C}^*$-actions

$$(u_1,u_2,u_3,u_4) \sim (\lambda_1 u_1, \lambda_1 u_2, 1/\lambda_1 u_3, u_4) \sim (u_1,u_2,\lambda_2 u_3, \lambda_2 u_4) \quad , \quad (\lambda_1,\lambda_2 \in \mathbf{C}^*) \quad .$$

We may consider a generic hypersurface in the product of the surface $\mathbf{F}_1$ with $\mathbf{P}^2$ given by the data

$$X = \left(\begin{matrix}\mathbf{P}^2 \\ \mathbf{F}_1\end{matrix} \middle\| \begin{matrix} 3 \\ (1,2)\end{matrix}\right)^{3,75}$$

where $(1,2)$ refers to the homogeneous degrees with respect to the first scaling by λ_1 and the second one by λ_2. The defining equation may be written explicitly as

$$g_{3,3}(z_1,z_2,z_3,u_1,u_2)u_3^2 + f_{3,1}(z_1,z_2,z_3,u_1,u_2)u_4^2 = 0 \ , \qquad (2.1)$$

where z_1, z_2, z_3 represents the homogeneous coordinate of $\mathbf{P}^2$ and $g_{3,3}$ $(, f_{3,1},)$ refers to a generic homogeneous polynomial with bi-degree $(3,3)$ (, and $(3,1)$, respectively,) for the coordinates z_1, z_2, z_3 and u_1, u_2. This is an elliptic Calabi-Yau hypersurface over $\mathbf{F}_1$ with the Hodge numbers $h^{1,1} = 3$ and $h^{2,1} = 75$. Two of the three elements in $H^{1,1}(X)$ come from the base $\mathbf{F}_1$ and the other comes from the fiber elliptic curve. We may find in X a rational elliptic surface S with its defining equation of bi-degree $(3,1)$ in $\mathbf{P}^2 \times \mathbf{P}^1$. It appears as a divisor $u_3 = 0$, which is the cubics in $\mathbf{P}^2$ over the (-1) curve in $\mathbf{F}_1$.

The positive classes in $H^2(X,\mathbf{Z})$ are generated by the three integral elements in $H^{1,1}(X)$ corresponding to the divisors

$$H=(z_1=0)\cap X \quad , \quad F=(u_1=0)\cap X \quad , \quad D=(u_4=0)\cap X \quad .$$

We sometimes denote the corresponding forms by J_1, J_2 and J_3, respectively. It is straightforward to determine the non-zero intersection numbers $K^{top}_{abc} = \int_X J_a\wedge J_b\wedge J_c$ and $c_2J_a=\int_X c_2(X)\wedge J_a$ with the second Chern class $c_2(X)$ to be

$$\begin{gathered} K^{top}_{112}=2 \;\; , \;\; K^{top}_{113}=3 \;\; , \;\; K^{top}_{123}=3 \;\; , \;\; K^{top}_{133}=3 \;\; , \\ c_2J_1=36 \;\; , \;\; c_2J_2=24 \;\; , \;\; c_2J_3=36 \;\; . \end{gathered}$$

The ambient space $\mathbf{P}^2\times\mathbf{F}_1$ is so-called the toric Fano manifold, and thus we can easily construct the mirror Calabi-Yau hypersurface $X^\vee$ based on Batyrev's toric method[Bat]. Furthermore the prepotential of the mirror Calabi-Yau hypersurface $X^\vee$ is determined by the general formula obtained in [12][13].

Here we collect necessary formulas to determine the prepotential. We start with a hypergeometric series representing a period integral for a deformation family of $X^\vee$ parameterized locally by x,y,z;

$$w_0(\vec{x})=\sum_{n,m,k\geq 0} c(n,m,k)x^ny^mz^k \quad ,$$

with

$$c(n,m,k)=\frac{\Gamma(1+3n+m+2k)}{\Gamma(1+n)^3\Gamma(1+m)^2\Gamma(1+k-m)\Gamma(1+k)} \quad .$$

The local parameters (x,y,z) has been chosen so that its origin represents the celebrated boundary point where the monodromy is maximally degenerated[21][22]. The series $w_0(\vec{x})$ represents the period integral for the invariant cycle about this degeneration point and satisfies Picard-Fuchs differential equation (see Appendix). As a complete set of the solutions of the Picard-Fuchs equation, we have $\{w_0(\vec{x})\ ,\ w^{(1)}_a(\vec{x})\ ,\ w^{(2)}_b(\vec{x})\ ,\ w^{(3)}(\vec{x})\}$ $(a,b=1,2,3)$ where

$$\begin{aligned} w^{(1)}_a(\vec{x}) &= \tfrac{1}{2\pi i}\tfrac{\partial}{\partial\rho_a}w_0(\vec{x},\vec{\rho})|_{\vec{\rho}=0} \ , \\ w^{(2)}_b(\vec{x}) &= \tfrac{1}{(2\pi i)^2}\tfrac{1}{2!}\textstyle\sum_{c,d}K^{top}_{bcd}\tfrac{\partial}{\partial\rho_c}\tfrac{\partial}{\partial\rho_d}w_0(\vec{x},\vec{\rho})|_{\vec{\rho}=0} \ , \\ w^{(3)}(\vec{x}) &= -\tfrac{1}{(2\pi i)^3}\tfrac{1}{3!}\textstyle\sum_{a,b,c}K^{top}_{abc}\tfrac{\partial}{\partial\rho_a}\tfrac{\partial}{\partial\rho_b}\tfrac{\partial}{\partial\rho_c}w_0(\vec{x},\vec{\rho})|_{\vec{\rho}=0} \end{aligned}$$

with $w_0(\vec{x},\vec{\rho}) = \sum_{n,m,k\geq 0} c(n+\rho_1, m+\rho_2, k+\rho_3) x^{n+\rho_1} y^{m+\rho_2} z^{k+\rho_3}$. In terms of the solutions of the Picard-Fuchs equation, the mirror map is defined by the relation

$$t_a = \frac{w_a^{(1)}(\vec{x})}{w_0(\vec{x})} \quad (a = 1,2,3), \tag{2.2}$$

which connects the deformation parameters (x,y,z) to those (t_1,t_2,t_3) parameterizing the complexified Kähler moduli of X at the large radius $(\mathrm{Im}(t_a) \to \infty)$. Now the prepotential of the mirror $X^\vee$ is defined to be

$$F(\vec{x}) = \frac{1}{2}\frac{1}{w_0(\vec{x})^2}\left(w_0(w^{(3)} - \sum_b \frac{c_2 J_b}{12} w_b^{(1)}) + \sum_a w_a^{(1)} w_a^{(2)}\right) .$$

Then the mirror symmetry conjecture asserts that the prepotential $F(\vec{x})$ of the mirror $X^\vee$ combined with the mirror map provides, up to terms of classical topological invariants, the generating function $F_0(t)$ of the Gromov-Witten invariants of X;

$$\begin{aligned} F(t) &= \frac{1}{3!}\sum_{a,b,c} K^{top}_{abc} t_a t_b t_c - \sum \frac{(c_2 J_b) t_b}{24} + \frac{\zeta(3)\chi(X)}{2(2\pi i)^3} \\ &\quad + \frac{1}{(2\pi i)^3} \sum_{0\neq\beta\in H_2(X,\mathbf{Z})} N(\beta) e^{2\pi i (\beta, \sum J_c t_c)}, \end{aligned}$$

where we substitute the inverse relation of 2.2 into $F(\vec{x})$.

3 Holomorphic anomaly equations for $g = 0, 1$

Let us consider the following limit

$$F(q,p) := F(t) - (\textit{topological terms})|_{\mathrm{Imt}_3 \to \infty}$$

with $q = e^{2\pi i t_1}$ and $p = e^{2\pi i t_2}$. Since the class J_3 measures the volume of the fiber $\mathbf{P}^1$ of $\mathbf{F}_1$ parameterized by u_3, u_4 and the volume of the curve contained in the divisor $S = (u_3 = 0) \cap X$ are measured to be zero by this class, the limit $\mathrm{Im} t_3 \to \infty$ throw away all the Gromov-Witten invariants except those of the curves contained in the rational elliptic surface S. Thus we may expect that the reduced prepotential $F(q,p)$ coincides with the generating function defined in Section 1. This is so-called the local mirror principle, and somehow generalize the arguments done for the isolated $(-1,-1)$-curves in Calabi-Yau manifolds[17][8].

In our case of the elliptic Calabi-Yau hypersurface X in $\mathbf{P}^2 \times \mathbf{F}_1$, the limit $\mathrm{Im} t_3 \to \infty$ which translates to the limit $z \to 0$ in the mirror $X^\vee$ greatly simplifies the period integral $w_0(\vec{x})$;

$$w_0(x,y,z)|_{z=0} \quad = \textstyle\sum_{n\geq 0} \frac{\Gamma(1+3n)}{\Gamma(1+n)^3} x^n =: \phi(x) \ .$$

We note that the series $\phi(x)$ is nothing but a solution of the Picard-Fuchs equation of the fiber elliptic curve of S:

$$\{\theta_x^2 - 3x(3\theta_x+2)(3\theta_x+1)\}\phi(x) = 0 \ . \tag{3.1}$$

Another solution of 3.1 about $x=0$ may be given by

$$\tilde{\phi}(x) = \log(x)\phi(x) + \sum_{n\geq 0} \frac{\Gamma(1+3n)}{\Gamma(1+n)^3} \left(\psi(1+3n) - 3\psi(1+n)\right) x^n \ ,$$

where $\psi(z) = \frac{d}{dz}\log\Gamma(z)$.

Now looking at the relations between the 'periods' of X via the prepotential and the period integrals of $X^\vee$:

$$\left(1, t_a, \frac{\partial F}{\partial t_b}, 2F - \sum_c t_c \frac{\partial F}{\partial t_c}\right) = \frac{1}{w^{(0)}}\left(w^{(0)}, w_a^{(1)}, w_b^{(2)}, w^{(3)} - \frac{1}{12}\sum_c (c_2 J_c) w_c^{(1)}\right) \ ,$$

it is straightforword, although involved technically, to derive the following consice form for the derivative $\frac{\partial}{\partial t_p}F(p,q)$ $(t_p := t_2)$ under the limit $\mathrm{Im} t_3 \to \infty$ (cf. Sect.7 of [14]).

Proposition 3.1

$$\frac{\partial}{\partial t_p}F(q,p) = \sum_{n\geq 1} \frac{f_n(x)}{\phi(x)^2} y^n \ ,$$

where we define

$$f_n(x) = -\frac{1}{3}\{\phi(x)\mathcal{L}_n\tilde{\phi}(x) - \tilde{\phi}(x)\mathcal{L}_n\phi(x)\} \ ,$$

with a linear operator

$$\mathcal{L}_n = \frac{(-1)^n}{n \times n!}\prod_{k=1}^{n}(3\theta_x + k) \ .$$

Remark 3.2 By constructing Barnes integral representation of the series $\phi(x)$, it is an easy exercise to make an integral symplectic basis of the Picard-Fuchs equation 3.1. It turns out that our bases $\phi(x)$ and $\tilde{\phi}(x)$ in fact constitute an integral symplectic basis about $x = 0$. Therefore we may write the holomorphic one form of the elliptic curve by $\Omega_E = \phi_0 A + \phi_1 B$, where A and B are symplectic bases of the elliptic curve. Then the function $f_1(x)$ may be written by $-3\int_E \Omega_E \wedge \theta_x \Omega_E$. This is so-called the classical Yukawa coupling [6] of the elliptic curve, and may be determined from the Picard-Fuchs equation 3.1 to be $f_1(x) = \frac{9}{1-27x}$. For the other functions $f_m(x)(m \geq 2)$ we will find a powerful recursion relation.

The relation 2.2 is also simplified in the limit $z \to 0$ due to the following relations

$$\begin{aligned} w_0(x,y,0) &= \phi(x), \ w_1^{(1)}(x,y,0) = \tilde{\phi}(x), \\ w_2^{(1)}(x,y,0) &= \xi(x) + \sum_{m \geq 1} \mathcal{L}_m \phi(x) y^m, \end{aligned}$$

where $\xi(x) = \sum_{n \geq 0} \frac{(3n)!}{(n!)^3} \left(\psi(3n+1) - \psi(1)\right) x^n$. As the inverse relations of 2.2, we have $x = x(q,p)$ and $y = y(q,p)$ with $q = \mathrm{e}^{2\pi i \frac{w_1^{(1)}(x,y,0)}{w_0(x,y,0)}}$ and $p = \mathrm{e}^{2\pi i \frac{w_2^{(1)}(x,y,0)}{w_0(x,y,0)}}$.

Proposition 3.3 *Under the limit $z \to 0$, we find that:*

1. *The inverse series $x(q,p)$ does not depend on p and is given by the level three modular function;*

$$x(q,p) = \frac{1}{t_{3B}(q)} \ , \tag{3.2}$$

where $t_{3B}(q) = \frac{\eta^{12}(q)}{\eta^{12}(q^3)}$ is the Thompson series in the notation of [7].

2. *The inverse series $y(q,p)$ is determined iteratively as a power series of p through the relation*

$$y(q,p) = p\psi(x)\mathrm{e}^{-\sum_{m \geq 1} c_m(x) y^m} \ , \tag{3.3}$$

where $\psi(x) = \mathrm{e}^{-\frac{\xi(x)}{\phi(x)}}$ and $c_m(x) = \frac{\mathcal{L}_m \phi(x)}{\phi(x)}$.

Remark 3.4 The function $\psi(x)$ with $x = \frac{1}{t_{3B}(q)}$ has first appeared in [14], and has been determined in terms of the modular functions of level three[28];

$$\psi(x(q)) = q^{\frac{1}{6}} \left(t_{3A}(q)\right)^{-\frac{1}{2}} \left(t_{3B}(q)\right)^{\frac{2}{3}} ,$$

where $t_{3A}(q) = \frac{1}{x(1-27x)}$. Also the following relations are standard results coming from the Gauss-Schwarz theory for the Picard-Fuchs equation 3.1;

$$\phi(x(q)) = \theta_3(q)\theta_3(q^3) + \theta_2(q)\theta_2(q^3) , \tag{3.4}$$

$$\phi(x)^{12} x(1-27x)^3 = \eta(q)^{24} , \quad \frac{1}{2\pi i}\frac{dx}{dt} = \phi(x)^2 x(1-27x) , \tag{3.5}$$

where $\theta_2(q) = \sum_{m\in\mathbf{Z}} q^{(m+\frac{1}{2})^2}$ and $\theta_3(q) = \sum_{m\in\mathbf{Z}} q^{m^2}$.

The following lemma may be derived directly from the definition $c_m(x)$ and the relation

$$\frac{\theta_x \phi(x)}{\phi(x)} = -\frac{1}{3}\left(1 - \frac{f_1}{12} - \frac{f_1}{36}\frac{E_2(q)}{\phi(x)^2}\right) ,$$

which follows from 3.5;

Lemma 3.5 *Under the relation* $x = \frac{1}{t_{3B}(q)}$, *the function* $c_m(x) = \frac{\mathcal{L}_m \phi(x)}{\phi(x)}$ *may be written by*

$$c_m(x) = B_m(f_1)\frac{E_2(q)}{\phi(x)^2} + D_m(f_1) , \tag{3.6}$$

where B_m *and* D_m *are some polynomials of* f_1 *determined by the following recursion relation;*

$$\begin{aligned} B_{m+1} &= -\tfrac{m}{(m+1)^2}\left\{(3\theta_x + m + 2 - \tfrac{f_1}{12})B_m + \tfrac{f_1}{36}D_m\right\} , \\ D_{m+1} &= -\tfrac{m}{(m+1)^2}\left\{-\tfrac{1}{4}(f_1 - 8)B_m + (3\theta_x + m + \tfrac{f_1}{12})D_m\right\} , \end{aligned} \tag{3.7}$$

with initial values $B_1 = -\frac{f_1}{36}$ *and* $D_1 = -\frac{f_1}{12}$.

We present here the first few terms coming from the recursion relation 3.7;

$$B_2 = \frac{1}{432}f_1^2, B_3 = \frac{7}{5832}\left(f_1^2 - \frac{2}{7}f_1^3\right), B_4 = \frac{1}{3888}\left(f_1^2 - \frac{5}{3}f_1^3 + \frac{1}{4}f_1^4\right), \cdots \tag{3.8}$$

$$D_2 = -\frac{1}{36}\left(f_1 - \tfrac{f_1^2}{4}\right) \ , \ D_3 = -\tfrac{1}{162}\left(f_1 - \tfrac{5f_1^2}{4} + \tfrac{f_1^3}{6}\right) \ ,$$
$$D_4 = \tfrac{17}{3888}\left(f_1^2 - \tfrac{8f_1^3}{17} + \tfrac{3f_1^4}{68}\right) \ , \cdots \tag{3.9}$$

We can verify directly using the Picard-Fuchs equation 3.1 that the formal solution of the recursion relation may be written in terms of the functions $f_m(x)$ in Proposition 3.1 as

$$B_m = -\frac{f_m}{36} \ , \ D_m = \frac{1}{f_1}\left\{\frac{(m+1)^2}{m} f_{m+1} + (3\theta_x + m + 2 - \frac{f_1}{12}) f_m\right\} \ . \tag{3.10}$$

As a result we see that the functions $f_m(x)$'s may be determined in terms of the recursion relation 3.7.

Since both the B_m and D_m are polynomials of $f_1(x) = \frac{9}{1-27x} = 9\frac{t_{3A}(q)}{t_{3B}(q)}$, they have nice behavior under the level three modular subgroup $\Gamma(3)$. Therefore the *modular anomaly* comes from the E_2-term in c_m. We may express this anomalous behavior via the partial derivative of c_m;

$$\frac{\partial c_m(x)}{\partial E_2} = -\frac{1}{36}\frac{f_m(x)}{\phi(x)^2} \ , \tag{3.11}$$

which plays a central role in the following derivations of the holomorphic anomaly equations.

3.1 Holomorphic anomaly equation at $g = 0$

Now we are ready to prove the recursion relation for $Z_{0;n}(q)$'s, which come from the mirror symmetry conjecture through the expansion

$$\frac{\partial}{\partial t_p} F(q,p) = \sum_{n\geq 1} \frac{f_n(x)}{\phi(x)^2} y^n = \sum_{n\geq 1} n Z_{0;n}(q) p^n \ . \tag{3.12}$$

Theorem 3.6 (Holomorphic anomaly equation at $g = 0$ (c.f.[19])) *The function $Z_{0;n}$ satisfies the recursion relation*

$$\frac{\partial Z_{0;n}}{\partial E_2} = \frac{1}{72}\sum_{s=1}^{n-1}(n-s)s Z_{0;n-s} Z_{0;s} \quad (n \geq 1). \tag{3.13}$$

(Proof) From Proposition 3.1 and Proposition 3.3 , we have

$$\frac{\partial}{\partial t_p}F(q,p)=\sum_{n=1}^{\infty}\frac{f_n(x)}{\phi(x)^2}y^n \quad , \quad y=p\psi(x)\mathrm{e}^{-\sum_{m\geq 1}c_m y^m} \quad ,$$

where the quasi-modular property (anomalous modular property) appears in c_m through 3.11. Now we first note

$$\frac{\partial}{\partial t_p}y=\frac{y}{1+\sum_{m=1}^{\infty}mc_m y^m} \quad .$$

Using this and the relation 3.11, we have

$$\frac{\partial y}{\partial E_2}=\frac{-y\sum_{m=1}^{\infty}\frac{\partial c_m}{\partial E_2}y^m}{1+\sum_{m=1}^{\infty}mc_m y^m}=\frac{1}{36}(\frac{\partial}{\partial t_p}y)(\frac{\partial}{\partial t_p}F) \quad . \tag{3.14}$$

Now we have

$$\begin{aligned}\frac{\partial}{\partial E_2}(\frac{\partial}{\partial t_p}F) &= \frac{1}{\phi^2}\sum_{m\geq 1}f_m(x)my^{m-1}\frac{\partial y}{\partial E_2}\\ &= \frac{1}{\phi^2}\sum_{m\geq 1}f_m(x)my^{m-1}\frac{1}{36}(\frac{\partial}{\partial t_p}y)(\frac{\partial}{\partial t_p}F)\\ &= \frac{1}{36}((\frac{\partial}{\partial t_p})^2F)(\frac{\partial}{\partial t_p}F) \ ,\end{aligned}$$

which says, up to constant terms for p, that

$$\frac{\partial F(q,p)}{\partial E_2}=\frac{1}{72}\left(\frac{\partial}{\partial t_p}F(q,p)\right)^2 \quad .$$

This proves the recursion relation. []

Now we determine the explicit forms $Z_{0;n}$ for lower n from the formula 3.12. After coming to a conjecture about the form of $Z_{0;n}$, we remark that the holomorphic anomaly equation 3.13 with some obvious inputs for the Gromov-Witten invariants suffices to determine the form $Z_{0;n}$ for all n.

Since our formula 3.12 is written in terms of the known functions $f_1(x), \psi(x)$ with $x=\frac{1}{t_{3B}(q)}$ and $E_2(q)$ for each order of p, and also the order of p coincides with the order of $\psi(x)$, it is easy to deduce that $Z_{0;n}$ has the following form in general,

$$Z_{0;n}=G_{0;n}(f_1,\frac{E_2}{\phi^2})\phi^{2n}\left(\frac{f_1\psi}{\phi^2}\right)^n \quad , \tag{3.15}$$

where we have factored the form of $Z_{0;1} = \frac{f_1 \psi}{\phi^2}$. Looking into the detail of the expansion of 3.12, we see in general that $\phi^2 \times G_{0;n}(f_1, \frac{E_2}{\phi^2})$ is a polynomial of $\frac{E_2}{\phi^2}$ with coefficients being polynomials of $1/f_1$ over $\mathbf{Q}$. Here we present the first few of them,

$$\begin{aligned} G_{0;2} &= \frac{1}{72} E_2 \phi^{-4} \ , \ G_{0;3} = \frac{5}{7776} \left((1 - \frac{8}{f_1}) \phi^4 + \frac{3}{5} E_2^2 \right) \phi^{-6} \\ G_{0;4} &= -\frac{1}{31104} \left((1 - \frac{12}{f_1} + \frac{24}{f_1^2}) \phi^6 - \frac{5}{3} (1 - \frac{8}{f_1}) \phi^4 E_2 - \frac{4}{9} E_2^3 \right) \phi^{-8} \\ G_{0;5} &= \frac{269}{62208000} \Big((1 - \frac{16}{f_1} + \frac{64}{f_1^2}) \phi^8 + \frac{6250}{7263} (1 - \frac{8}{f_1}) E_2^2 \phi^4 \\ & \quad -\frac{2000}{2421} (1 - \frac{12}{f_1} + \frac{24}{f_1^2}) E_2 \phi^2 + \frac{3125}{21789} E_2^4 \Big) \phi^{-10} \ . \end{aligned}$$

Now we note the following relations for the polynomials of $1/f_1$;

$$\begin{aligned} E_4 = \ & 9 \left(8 - \tfrac{8}{f_1} \right) \ , \ E_6 = -27 \left(1 - \tfrac{12}{f_1} + \tfrac{24}{f_1^2} \right) \ , \\ & 27\phi^8 - 18 E_4 \phi^4 - E_4^2 - 8 E_6 \phi^2 = 0 \ . \end{aligned} \tag{3.16}$$

Using these relations we find that

$$\begin{aligned} G_{0;3} = \ & \tfrac{5}{69984} \left(E_4 + \tfrac{27}{5} E_2^2 \right) \phi^{-6} \ , \ G_{0;4} = \tfrac{1}{839808} \left(E_6 + 5 E_4 E_2 + 12 E_2^3 \right) \phi^{-8}, \\ & G_{0;5} = \tfrac{1}{40310784} \left(\tfrac{269}{125} E_4^2 + \tfrac{16}{3} E_6 E_2 + \tfrac{50}{3} E_4 E_2^2 + 25 E_2^4 \right) \phi^{-10} \ . \end{aligned}$$

Here we observe explicitly that ϕ disappears nontrivially in the final expression of $Z_{0;n}$ for lower n's. We do not have general proof about this but may state it as follows;

Conjecture 3.7 *The function $Z_{0;n}(q)$ in 3.15 takes the form*

$$Z_{0;n}(q) = P_{2n-2}(E_2, E_4, E_6) \left(Z_{0;1}(q) \right)^n \ , \tag{3.17}$$

where P_{2n-2} is a quasi-modular form of weight $2n-2$.

Remark 3.8 The function $Z_{0;n}$ contains the multiple cover contributions. We may subtract these contributions considering

$$\tilde{Z}_{0;n}(q) := Z_{0;n}(q) - \sum_{k|n, k \neq 1} \frac{1}{k^3} \tilde{Z}_{0;n/k}(q^k) \ . \tag{3.18}$$

The q series coefficients of $\tilde{Z}_{0;n}$ "count" the numbers of rational curves $\mathcal{C}$ in our rational elliptic surface S satisfying $(\mathcal{C}, H) = d$ and $(\mathcal{C}, F) = n$. The homology classes of curves in S, in general, have the form $[\mathcal{C}] = dH - a_1e_1 - a_2e_2 - \cdots - a_9e_9$ with $a_1, a_2, \cdots, a_9 \geq 0$. Therefore we have $3d - a_1 - a_2 - \cdots - a_9 = n$ for $(\mathcal{C}, F) = n$, which implies $3d \geq n$. In other words, we should have

$$\tilde{N}_{0;d,n} = 0 \ \text{ for } \ d < \frac{n}{3} \tag{3.19}$$

for the coefficients of $\tilde{Z}_{0;n}$. From a simple counting of the dimensionality of the quasi-modular forms of weight $2n - 2$, we see that the vanishing condition 3.19 together with the above Conjecture 3.7 provides sufficient data to determine the integration constants for the recursion 3.13, and determine completely $Z_{0;n}$ for all n (, see Section 4).

3.2 Holomorphic anomaly equation at $g = 1$

According to [1], we have an explicit expression for the genus one prepotential $F_1^{BCOV}(t)$ in terms of the discriminant of the hypersurface X;

$$\begin{aligned} F_1^{BCOV}(t) &= \log\Big\{ \Big(\frac{1}{w_0(\vec{x})}\Big)^{3+h^{1,1}(x)-\frac{\chi(X)}{12}} \\ &\quad \times dis(x,y,z)^{-\frac{1}{6}} x^{-4} y^{-3} z^{-4} \det\Big(\frac{\partial x_a}{\partial t_b}\Big) \Big\}, \end{aligned} \tag{3.20}$$

where the discriminant may be determined from the characteristic variety of the Picard-Fuchs equation presented in the Appendix. Several exponents in 3.20 have been fixed by the requirements of the asymptotics of F_1^{BCOV} when $\mathrm{Im}t_a \to \infty$. The explicit form of the discriminant is a complicated polynomial of x, y and z, however, in the limit $z \to 0$, it simplifies to

$$dis(x,y,z)|_{z=0} = (1-27x)^3\{1 - 27x + (1+y)^3 - 1\} \ . \tag{3.21}$$

Also under this limit, it is easy to show from the definition that the mirror map $z(q,p,r)$ $(r = e^{2\pi i t_3})$ simplifies to

$$z(q,p,r) = r e^{-\frac{2\xi(x)}{\phi(x)} + \sum_{m\geq 1} c_m(x) y^m} = r\psi(x)^3 \frac{p}{y(q,p)} \ . \tag{3.22}$$

Now using the relations 3.5 it is straightforward to derive;

Proposition 3.9 *When $r \to 0$, up to the topological term $-\frac{1}{12}\sum_c (c_2 J_c) t_c = -3\log q - 2\log p - 3\log r$, we have;*

$$\begin{aligned} F_1^{BCOV}(q,p) &= \log\{\phi^4(1-27x)^{\frac{3}{2}} q^{\frac{5}{3}} \eta(q)^{-40}\} \\ &\quad + \log\{\{(1-27x)+(1+y)^3-1\}^{-\frac{1}{6}} \frac{e^{-\sum_{m\geq 1} c_m(x) y^m}}{1+\sum_{m\geq 1} m c_m(x) y^m}\} . \end{aligned} \quad (3.23)$$

Remark 3.10

1. For the p-independent term of $F_1^{BCOV}(q,p)$ the local mirror symmetry does not apply. This is because these curves are parallel to the fiber elliptic curves of X and therefore can move outside of the rational elliptic surface. In fact we see that

$$\tilde{N}_{g=1;d,0} = \begin{cases} 4 & d=3 \\ 0 & d \neq 3 \end{cases}$$

after subtracting the genus zero contributions $\tilde{N}_{g=0;d,0} = 168$ ($d \equiv 1,2 \bmod 3$), 144 ($d \equiv 0 \bmod 3$). The number 4 should be regarded as the Euler number of the base $\mathbf{F}_1$ for the elliptic fibration.

2. There is a difference in the normalization of the prepotentials between F_1^{BCOV} and our $F_{g=1}$ in the introduction. These are related by the factor 2 coming from the orientation of curves as

$$F_1(q,p) = \frac{1}{2} F_1^{BCOV}(q,p) \ . \quad (3.24)$$

Now we define the generating function $Z_{1;n}$ through

$$F_1(q,p) = \sum_{p\geq 1} Z_{1;n}(q) p^n \ ,$$

and prove the holomorphic anomaly equation at $g=1$.

Theorem 3.11 (Holomorphic anomaly equation at $g=1$) *The function $Z_{1;n}(q)$ satisfies the following recursion relation;*

$$\frac{\partial Z_{1;n}}{\partial E_2} = \frac{1}{36}\sum_{s=1}^{n-1}(n-s)s Z_{1;n-s} Z_{0;s} + \frac{n(n+1)}{72} Z_{0;n} \ \ (n \geq 1). \quad (3.25)$$

(Proof) Since we have already shown the relations

$$\frac{\partial y}{\partial E_2}=\frac{1}{36}(\frac{\partial}{\partial t_p}y)(\frac{\partial}{\partial t_p}F_0)\ ,\quad \frac{\partial c_m}{\partial E_2}=-\frac{1}{36}\frac{f_m}{\phi^2}\ ,$$

and

$$\frac{\partial}{\partial t_p}y=\frac{y}{1+\sum_{m=1}^{\infty}mc_my^m}\ ,$$

it is straightforward to derive

$$\begin{aligned}\frac{\partial F_1^{BCOV}}{\partial E_2} &= \frac{\partial}{\partial y}\log\{\{(1-27x)+(1+y)^3-1\}^{-\frac{1}{6}}\\ &\qquad\times\frac{e^{-\sum_{m\geq 1}c_m(x)y^m}}{1+\sum_{m\geq 1}mc_m(x)y^m}\}\frac{\partial y}{\partial E_2}\\ &\quad -\sum_{m\geq 1}\frac{\partial c_m}{\partial E_2}y^m-\frac{\sum_{m\geq 1}m\frac{\partial c_m}{\partial E_2}y^m}{(1+\sum_{m\geq 1}mc_my^m)}\\ &= \frac{\partial}{\partial t_p}\log\{\{(1-27x)+(1+y)^3-1\}^{-\frac{1}{6}}\\ &\qquad\times\frac{e^{-\sum_{m\geq 1}c_m(x)y^m}}{1+\sum_{m\geq 1}mc_m(x)y^m}\}\frac{1}{36}(\frac{\partial}{\partial t_p}F_0)\\ &\quad +\frac{1}{36}(\frac{\partial}{\partial t_p}F_0)+\frac{1}{36}\sum_{m\geq 1}m\frac{f_m}{\phi^2}y^{m-1}(\frac{\partial}{\partial t_p}y)\\ &= \frac{1}{36}(\frac{\partial}{\partial t_p}\log\{(1-27x)+(1+y)^3-1\}^{-\frac{1}{6}}\\ &\qquad\times\frac{e^{-\sum_{m\geq 1}c_m(x)y^m}}{1+\sum_{m\geq 1}mc_m(x)y^m})\times(\frac{\partial}{\partial t_p}F_0)\\ &\quad +\frac{1}{36}\frac{\partial}{\partial t_p}F_0+\frac{1}{36}(\frac{\partial}{\partial t_p})^2F_0\\ &= \frac{1}{36}(\frac{\partial}{\partial t_p}F_1^{BCOV})(\frac{\partial}{\partial t_p}F_0)+\frac{1}{36}\frac{\partial}{\partial t_p}(\frac{\partial}{\partial t_p}+1)F_0\end{aligned}$$

Taking into account the difference of the normalization 3.24, we conclude the recursion relation. □

Now we may determine the generating function $Z_{1;n}(q)$ explicitly from 3.23 under the relation $F_1(q,p)=\frac{1}{2}F_1^{BCOV}(q,p)$. As in the cese of genus zero, we may represent $Z_{1;n}$ in terms of $f_1(x), \psi(x)$ and $E_2(q)$. Corresponding to 3.15, we have

$$Z_{1;n} = G_{1;n}(f_1, \frac{E_2}{\phi^2})\phi^{2n} \left(\frac{f_1\psi}{\phi^2}\right)^n = G_{1;n}(f_1, \frac{E_2}{\phi^2})\phi^{2n} \left(Z_{0;1}\right)^n \quad . \qquad (3.26)$$

After straightforward evaluation of 3.23, we obtain for the first few of $G_{1;n}$'s;

$$\begin{aligned}
G_{1;1} &= \frac{1}{18}\left(\phi^2 + \frac{1}{2}E_2\right)\phi^{-2} \ , \\
G_{1;2} &= \frac{1}{1728}\left(\left(1+\frac{24}{f_1}\right)\phi^4 + \frac{8}{3}E_2\phi^2 + \frac{5}{3}E_2^2\right)\phi^{-4} \ , \\
G_{1;3} &= \frac{\phi^{-6}}{15552}\left(\left(1+\frac{48}{f_1^2}\right)\phi^6 + \frac{13}{6}\left(1-\frac{8}{13f_1}\right)E_2\phi^4 + E_2^2\phi^2 + \frac{13}{18}E_2^3\right) \ .
\end{aligned}$$

If we use the relations 3.16 for the polynomials of $1/f_1$, we find

$$\begin{aligned}
G_{1;1} &= \frac{1}{18}\left(\phi^2 + \frac{1}{2}E_2\right)\phi^{-2} \ , \\
G_{1;2} &= \frac{1}{432}\left(\phi^4 - \frac{1}{12}E_4 + \frac{2}{3}\phi^2 E_2 + \frac{5}{12}E_2^2\right)\phi^{-4} \ , \\
G_{1;3} &= \frac{1}{7776}\left(\phi^6 - \frac{1}{6}\phi^2 E_4 - \frac{1}{27}E_6 + E_2\left(\phi^4 + \frac{1}{108}E_4\right)\right. \\
&\qquad \left. + \frac{1}{2}\phi^2 E_2^2 + \frac{13}{36}E_2^3\right)\phi^{-6} \ .
\end{aligned}$$

Contrary to the case of $g = 0$, the ϕ-dependence remains in $Z_{1;n}(q)$ after the elimination of the polynomials of $1/f_1$. Thus we arrive at the follwing weaker statements about $Z_{1;n}(q)$;

Proposition 3.12 *The generation function $Z_{1,n}(q)$ in 3.26 takes the form*

$$Z_{1;n}(q) = P_{2n}(\phi, E_2, E_4, E_6)\left(Z_{0;1}(q)\right)^n \quad ,$$

where P_{2n} is a 'quasi-modular form' of weight $2n$ for the modular subgroup $\Gamma(3)$.

Remark 3.13

1. The form of the polynomial P_{2n} is not unique because of the relation 3.16 among ϕ, E_4 and E_6.

2. The function $Z_{1;n}$ contains the contribution from the genus zero curves (i.e., the degenerated instanton [1][8],) as well as the contribution from the multiple covers. We may separate these in $Z_{1;n}$ as follows;

$$Z_{1;n}(q) = \tilde{Z}_{1;n}(q) + \sum_{k|n,k\neq 1} \sigma_{-1}(k)\tilde{Z}_{1;n/k}(q^k) + \frac{1}{12}\sum_{k|n}\frac{1}{k}\tilde{Z}_{0;n/k}(q^k) \ , \tag{3.27}$$

where $\sigma_{-1}(k) = \sum_{m|k}\frac{1}{m}$. The function $\tilde{Z}_{1;n}(q)$ is expected to 'count' the numbers of the elliptic curves C with $(C,F) = n$ in S. As is the case of $g = 0$, we have certain vanishing conditions for the elliptic curves, which is useful to determine the integration constants for our holomorphic anomaly equation 3.25. However as we will argue in the next section the appearance ϕ in the polynomial P_{2n} increases unknown parameters in the integration constants. From this reason holomorphic anomaly equation become less powerful than the case of $g = 0$ to determine $Z_{1;n}$.

4 Predictions for Gromow-Witten invariants of higher genera

4.1 General considerations

From the analysis for $g = 0$ and $g = 1$, we naturally come to the following conjecture about the holomorphic anomaly equation for all genera.

Conjecture 4.1 (Holomorphic anomaly equation for all g) *The generating function $Z_{g;n}(q)$ has the form*

$$P_{2g+2n-2}(\phi, E_2, E_4, E_6)\,(Z_{0;1}(q))^n \tag{4.1}$$

with some 'quasi-modular form' for $\Gamma(3)$ of weight $2g+2n-2$. (In the special case of $g = 0$, it simplifies to $P_{2n-2}(E_2, E_4, E_6)$, i.e., exactly the quasi-modular form of weight $2n-2$). And it satisfies the recursion relation

$$\frac{\partial Z_{g;n}}{\partial E_2} = \frac{1}{72}\sum_{g'+g''=g}\sum_{s=1}^{n-1} s(n-s)Z_{g';s}Z_{g'';n-s} + \frac{n(n+1)}{72}Z_{g-1;n} \ . \tag{4.2}$$

In the following we will consider the solutions of the holomorphic anomaly equation. For this purpose, first of all, we need to have some data to fix the 'integration constants' for our recursion relation 4.2. Let us suppose that a curve $\mathcal{C}_g$ in the rational elliptic surface is in a homology class $[\mathcal{C}_g] = dH - a_1e_1 - a_2e_2 \cdots - a_9e_9$, where e_i's refer to the -1 curves from the blowing ups. Then, since $Z_{g;n}$ counts the Gromov-Witten invariants of genus g curves with $([\mathcal{C}_g], F) = n$, we should have

$$([\mathcal{C}_g], F) = 3d - \sum_{i=1}^{9} a_i = n \ , \tag{4.3}$$

and for the arithmetic genus

$$g_a(\mathcal{C}_g) = \frac{(d-1)(d-2)}{2} - \sum_{i=1}^{9} \frac{a_i(a_i-1)}{2} = g \ . \tag{4.4}$$

If the curve $\mathcal{C}_g$ is singular, the arithmetic genus might be different from the genus of the normalization of $\mathcal{C}_g$. We will come to this point later, however for the moment we will ignore this difference. Then it is easy to see that the above two constrains provide us several vanishing conditions on the numbers of curves. In Table 1 we have presented the lowest degree $d = ([\mathcal{C}_g], H)$ for which a curve $\mathcal{C}_g$ may exist for given g and n.

$n \backslash g$	0	1	2	3	4	5	6	7	8	9
1	0	3	6	9	12	15	18	21	24	27
2	1	3	4	6	7	9	10	12	13	15
3	1	3	4	4	6	7	7	9	10	10
4	2	3	4	4	5	6	7	7	8	9
5	2	3	4	4	5	5	6	7	7	8

Table 1. Each number shows the lowest degree $d = ([\mathcal{C}_g], H)$ for curves of given g and n.

The homology classes of curves in the table may be written explicitly for given g and n. For example, for $n = 1$ they are simply given by $[\mathcal{C}_g] = e_i + gF \quad (i = 1, \cdots, 9)$. The data in the table provides us vanishing conditions for the first few terms in the q-expansion of $\tilde{Z}_{g;n}(q)$, where tilde represents the subtraction of the *degenerated instantons* from the Gromov-Witten invariants. We may relate $Z_{g;n}(q)$ $(g \geq 2)$ to the subtracted functions by

$$Z_{g;n}(q) = \tilde{Z}_{g;n}(q) + \sum_{k|n, k\neq 1} D(g,g,k)\tilde{Z}_{g;n/k}(q^k) + \sum_{h=0}^{g-1}\sum_{k|n} D(g,h,k)\tilde{Z}_{h;n/k}(q^k) \ , \tag{4.5}$$

with some rational numbers $D(g,h,k)$. Therefore all we need to fix from the vanishing conditions are the 'integration constants' together with the rational numbers $D(g,h,k)$. For higher $n>4$ it turns out that the vanishing conditions in Table 1 are not sufficient to determine completely both the integration constants of the recursion 4.2 and the form of the degenerated instanton $D(g,h,k)$. However we will see based on a simple counting arguments that for lower $n\leq 3$ they suffices at least to fix the integration constants for our recursion relation. Especially for the case $n=1$ they determine both the integration constants and the form of the degenerated instanton.

As an extreme case let us first consider the genus zero generating functions $Z_{0;n}(q)$ $(n=1,2,\cdots)$. As we have already considered in Remark 3.8, the only undetermined in this case are the integration constants in the polynomial P_{2n-2} in 3.17. We see that these constants can be fixed by the vanishing conditions in Table 1 comparing the first column of the table with the dimensionality of the modular form of a given weight, which is given by the series

$$\frac{1}{(1-t^4)(1-t^6)}=1+t^4+t^6+t^8+t^{10}+2\,t^{12}+t^{14}+2\,t^{16}+2\,t^{18}+2\,t^{20}+\cdots\ . \tag{4.6}$$

Now let us look at the functions $Z_{g;n}(q)$ for all g and lower $n(\leq 3)$. In this case we only have the weaker assumption on $P_{2g+2n-2}$ in 4.1. Taking into account the relation among ϕ, E_4 and E_6 in the second line of 3.16, we may estimate the relevant dimensionality of the modular form $P_{2g+2n-2}|_{E_2=0}$ (, integration constants of the holomorphic anomaly equation,) by

$$\begin{aligned}\frac{(1+t^2+t^4+t^6)}{(1-t^4)(1-t^6)} &= 1+t^2+2\,t^4+3\,t^6+3\,t^8+4\,t^{10}+5\,t^{12}+5\,t^{14}+6\,t^{16}\\ &\quad +7\,t^{18}+7\,t^{20}+\cdots\ .\end{aligned} \tag{4.7}$$

The growth of the dimensions should be compared with each line of the table 1 under a suitable shift. From these comparisons of the numbers of the 'integration constants' and the numbers of the vanishing conditions, we may deduce that for $n\leq 3$ the vanishing conditions suffices to determine the integration constants while leaving some of $D(g,h,k)$ undetermined. To go beyond this rather unsatisfactory situation, we need to know more details about the numbers of genus g curves of a given homology classes or the form of the degenerated instanton $D(g,h,k)$. Though our simple vanishing conditions are insufficient to determine all the unknowns, we see from the first line of the table they are restrictive enough to fix the form of $Z_{g;1}(q)$ and $D(g,h,1)$ completely. In the next subsection we will present a detailed analysis of $Z_{g;1}(q)$ for all g.

4.2 Gromov-Witten invariants $Z_{g;1}$ and degenerated instantons

To get some intuition about the curves $\mathcal{C}_g$ in S, let us first recall the form of $Z_{0;1}(q)$ obtained in [14][28]

$$Z_{0;1}(q) = \frac{q^{\frac{3}{2}}\Theta_{E_8}(3t,t\gamma)}{\eta(q^3)^{12}} \quad \left(= 9\frac{q^{\frac{1}{6}}}{\eta(q)^4} \right) \tag{4.8}$$

where $\gamma = (1,\cdots,1,-1)$. The appearance of the E_8 theta function originates from the well-established fact that the sections of the rational elliptic surface form additive group called Mordell-Weil group, and it becomes a lattice isomorphic to the E_8 lattice endowed with a positive definite bilinear form[25][14][24]. The eta functions in the denominator have been explained by introducing *pseudo-section* which is a composite of a section with some of the twelve singular fibers with its homology class $\sigma + kF$. Thus the function $Z_{0;1}$ counts the numbers of pseudo-sections in S which are not irreducible but naturally come in the theory of the stable maps[5].

Now in our general case of genus g curves with $(\mathcal{C}_g, F) = 1$ the function $Z_{g;1}$ counts the genus g sections of the elliptic fibration of S. Since the generic fiber spaces are elliptic curves, the genus g section are composite of two components, one is a pseudo-section and the other consists of g fiber elliptic curves. The genus g sections of the lowest degree are those with their homology classes given by

$$[\mathcal{C}_g] = e_i + gF \quad (i = 1,\cdots,9)\ . \tag{4.9}$$

Thus the expansion of $Z_{g;1}(q)$ start from q^{3g} with its coefficient 'counting' the number of the genus g sections of class 4.9. (This is the vanishing condition we have listed in Table 1 for $n = 1$ and g.) The g elliptic curves in general avoid the twelve singular fibers and make a g-dimensional family parameterized by $\mathrm{Sym}^g(\mathbf{P}^1)$.

As we have already remarked, the vanishing conditions for the 'numbers' of curves grow much faster than the dimensionality of the integration constants 4.7 plus the numbers of the unknowns $D(g,h,1)$ in

$$Z_{g;1}(q) = \tilde{Z}_{g;1}(q) + \sum_{h=0}^{g-1} D(g,h,1)\tilde{Z}_{h,1}(q)\ . \tag{4.10}$$

Owing to this nice property, we can integrate our holomorphic anomaly equation for $n = 1$,

$$\frac{\partial Z_{g;1}}{\partial E_2} = \frac{1}{36} Z_{g-1;1}\ , \tag{4.11}$$

with the results listed in Table 2 for $\tilde{Z}_{g;1}(q)$'s. For the degenerated instantons $D(g,h,1)$ we find

$$\begin{aligned}
Z_{1;1} &= \tilde{Z}_{1;1} + \frac{1}{12}\tilde{Z}_{0;1} \\
Z_{2;1} &= \tilde{Z}_{2;1} + \chi(M_2)\tilde{Z}_{0;1} \\
Z_{3;1} &= \tilde{Z}_{3;1} + \frac{1}{3!}\chi(M_3)\tilde{Z}_{0;1} - \frac{1}{12}\tilde{Z}_{1;1} \\
Z_{4;1} &= \tilde{Z}_{4;1} + \frac{1}{5!}\chi(M_4)\tilde{Z}_{0;1} + \frac{1}{360}\tilde{Z}_{2;1} - \frac{1}{6}\tilde{Z}_{3;1} \\
Z_{5;1} &= \tilde{Z}_{5;1} + \frac{1}{7!}\chi(M_5)\tilde{Z}_{0;1} - \frac{1}{20160}\tilde{Z}_{2;1} + \frac{1}{80}\tilde{Z}_{3;1} - \frac{1}{4}\tilde{Z}_{4,1} \ ,
\end{aligned} \tag{4.12}$$

with the orbifold Euler number of the moduli space of genus g stable curves, $\chi(M_g) = \frac{|B_{2g}|}{2g(2g-2)!}(g \geq 2)$. In fact these forms of the degenerated instantons $D(g,h,1)$ $(0 \leq h < g)$ coincide with recently established results in [11][23], where we have

$$Z_{g;1}(q) = \sum_{h=0}^{g} C_h(g-h,1)\tilde{Z}_{h;1}(q) \ , \tag{4.13}$$

with the coefficients determined by

$$\left(\frac{\sin(t/2)}{t/2}\right)^{2g-2} = \sum_{h=0}^{\infty} C_g(h,1)t^{2h} \ . \tag{4.14}$$

$\tilde{Z}_{0;1} = 9 + 36q + 126q^2 + 360q^3 + 945q^4 + 2268q^5 + 5166q^6 + 11160q^7 + \cdots$
$\tilde{Z}_{1;1} = -18q^3 - 72q^4 - 252q^5 - 774q^6 - 2106q^7 - 5292q^8 - 12564q^9 + \cdots$
$\tilde{Z}_{2;1} = 27q^6 + 108q^7 + 378q^8 + 1224q^9 + 3411q^{10} + 8820q^{11} + 21663q^{12} + \cdots$
$\tilde{Z}_{3;1} = -36q^9 - 144q^{10} - 504q^{11} - 1710q^{12} - 4860q^{13} - 12852q^{14} + \cdots$
$\tilde{Z}_{4;1} = 45q^{12} + 180q^{13} + 630q^{14} + 2232q^{15} + 6453q^{16} + 17388q^{17} + \cdots$
$\tilde{Z}_{5;1} = -54q^{15} - 216q^{16} - 756q^{17} - 2790q^{18} - 8190q^{19} - 22428q^{20} + \cdots$

Table 2. Solutions of the holomorphic anomaly equation 4.11. These are related to $Z_{g;1}$ by 4.12.

We note in Table 2 that for the first three terms in the expansion we have

$$\tilde{Z}_{g;1}(q) = (-1)^g \chi(\mathrm{Sym}^g(\mathbf{P}^1))(9q^g + 36q^{g+1} + 126q^{g+2} + \cdots) \tag{4.15}$$

where $\chi(\mathrm{Sym}^g(\mathbf{P}^1))$ represents the Euler number of $\mathrm{Sym}^g(\mathbf{P}^1) = \mathbf{P}^g$. This is also in agreement with the argument in [11] for counting curves with moduli. From the fourth term in the series expansion 4.15 contributions from the singular fibers come in, which somehow generalize the situation we encountered in the case of $g = 0$.

For the first few of $Z_{g;1}(q)$'s we have determined explicitly the forms of the polynomials $P_{2g}(\phi, E_2, E_4, E_6)$. We find that if we use the following relations

$$\begin{aligned} 3E_2(q^3) &= 2\phi(q)^2 + E_2(q) \ , \ 9E_4(q^3) = 10\phi(q)^4 - E_4(q) \ , \\ 27E_6(q^3) &= 35\phi(q)^6 - 7E_4(q)\phi(q)^2 - E_6(q) \ , \end{aligned} \tag{4.16}$$

they summarize into concise forms;

$$\begin{aligned} Z_{0;1} &= \frac{q^{\frac{3}{2}}\Theta_{E_8}(3t, t\gamma)}{\eta(q^3)^{12}} \ , \qquad Z_{1;1} = \frac{1}{12}E_2(q^3)Z_{0;1} \\ Z_{2;1} &= \frac{1}{1440}(5\,E_2(q^3)^2 + E_4(q^3))Z_{0;1} \\ Z_{3;1} &= \frac{1}{362880}(35E_2(q^3)^3 + 21E_2(q^3)E_4(q^3) + 4E_6(q^3))Z_{0;1} \ . \end{aligned} \tag{4.17}$$

Proposition 4.2 *The solutions of the holomorphic anomaly equation 4.11 take the following general form*

$$Z_{g;1}(q) = P_{2g}(E_2(q^3), E_4(q^3), E_6(q^3))\ Z_{0;1}(q) \ ,$$

where P_{2g} is a quasi-modular form of weight $2g$.

The reason of this simplification will be explained in the next section. Here for later use we define $\mathcal{G}_{g;1}$ by

$$\tilde{Z}_{g;1}(q) = \Theta_{E_8}(3t, t\gamma)\ \mathcal{G}_{g;1}(q^3) \ , \tag{4.18}$$

which should count the genus g pseudo-sections made from a section, say the zero section. We verify directly that the functions $\mathcal{G}_{g;1}$'s depend on q through q^3 and have the following expansions;

$$\begin{aligned} \mathcal{G}_{0;1} &= 1 + 12q^3 + 90\,q^6 + 520\,q^9 + 2535\,q^{12} + 10908\,q^{15} + \cdots \\ \mathcal{G}_{1;1} &= -2q^3 - 30\,q^6 - 260\,q^9 - 1690\,q^{12} - 9090\,q^{15} - 42614\,q^{18} - \cdots \\ \mathcal{G}_{2;1} &= 3q^6 + 52\,q^9 + 507\,q^{12} + 3636\,q^{15} + 21307\,q^{18} + 107772\,q^{21} + \cdots \\ \mathcal{G}_{3;1} &= -4q^9 - 78\,q^{12} - 840\,q^{15} - 6570\,q^{18} - 41580\,q^{21} - 225432\,q^{24} - \cdots \\ \mathcal{G}_{4;1} &= 5q^{12} + 108\,q^{15} + 1271\,q^{18} + 10756\,q^{21} + 73083\,q^{24} + \cdots \\ \mathcal{G}_{5;1} &= -6q^{15} - 142\,q^{18} - 1812\,q^{21} - 16494\,q^{24} - 119770\,q^{27} - \cdots \end{aligned} \tag{4.19}$$

In the next section we will discuss geometric interpretation of the numbers in the above expansions.

5 Discussions

5.1 Counting BPS states

In this section we would like to discuss relations of our results to the very interesting proposals made in the recent works by Gopakumar and Vafa [10][11].

It is known in physics that the genus g prepotential F_g has a meaning as the genus g topological partition function of the twisted Calabi-Yau sigma model in the type IIA string theory and it does not receive the string perturbative corrections due to the fact that the dilaton belongs to the hypermultiplet in the type IIA string. Since the heterotic/type II string duality connects the heterotic dilaton field to one of the vectormultiplet moduli in the type IIA side, we may expect to extract the non-perturbative properties in the type IIA side from the perturbation theory in the heterotic string. In our case of the topological amplitude F_g, Gopakumar and Vafa [10][11] have found that it can be derived from the one loop integral in the heterotic side. They found that the Schwinger one-loop calculation for a particle in a constant background electro-magnetic filed applied to the BPS states with spin (j_1, j_2) under $SO(4) = SU(2)_L \times SU(2)_R$ determines the higher genus F_g. According to [11] the contribution of each BPS state with spin (j_1, j_2) to F_g is determined by the following decomposition with respect to $SU(2)_L \subset SU(2)_L \times SU(2)_R$;

$$(j_1, j_2) = \sum_{r=0}^{2j_1} \alpha_r [(\frac{1}{2}) \oplus 2(0)]^{\otimes r} \quad , \tag{5.1}$$

where we allow α_r formally to be *negative* integer. Then the topological partition function has the following expression

$$\sum_{g=0}^{\infty} \lambda^{2g-2} F_g = \sum_{\Gamma:BPS} \sum_{r=0}^{2j_1(\Gamma)} \sum_{m\in\mathbf{Z}} \alpha_r(\Gamma) \int_{\epsilon}^{\infty} \frac{ds}{s} \left(2\sin\frac{s}{2}\right)^{2r-2} \mathrm{e}^{-2\pi\frac{s}{\lambda}(A(\Gamma)+im)} \quad , \tag{5.2}$$

where A measures the central charge of the BPS state Γ and the summation over m is explained as the central charge of the fifth dimension which originates in the M-theory. In the type IIA theory the central charge A is measured by the Kähler classes of the corresponding Calabi-Yau manifolds, and after the integration we arrive at

$$\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t) = \sum_{0\neq\eta\in H_2(X,\mathbf{Z})} \sum_{r\geq 0} \sum_{k>0} \alpha_r^{\eta} \frac{1}{k} \left(2\sin\frac{k\lambda}{2}\right)^{2r-2} q^{k\eta} \quad , \tag{5.3}$$

where α_r^η represents the summation of $\alpha_r(\Gamma)$ over the BPS states Γ with charge η, and $q^{k\eta} = \exp(-2\pi(k\eta, K))$ with $K = t_1 J_1 + \cdots + t_{h^{1,1}} J_{h^{1,1}}$ the Kähler class. Expanding 5.3 with respect to λ we obtain

$$F_g(q) = \sum_{0 \neq \eta \in H_2(X,\mathbf{Z})} \sum_{h=0}^{g} \sum_{k|\eta} \alpha_h^{\eta/k} k^{2g-3} C_h(g-h,1) q^\eta \ , \tag{5.4}$$

which is the general form proposed in [11] and we have reproduced in our special case (, see 4.12 and 4.14).

In the type IIA picture the BPS state appears as the D2 brane with the flat $U(1)$ connection. Let us suppose that a genus g curve in a Calabi-Yau manifold comes with the moduli of the deformation $\mathcal{M}$. Together with the Jacobian of each curve we have the fiber space $\hat{\mathcal{M}} \to \mathcal{M}$ for the BPS states. Gopakumar and Vafa argued based on the M-theory that in this case the counting BPS states in 5.2 is interpreted by the Lefschetz $SL(2,\mathbf{C})$ decomposition of the cohomology of the moduli space $\hat{\mathcal{M}}$. They propose the Lefschetz decomposition with respect to the fiber $SL(2,\mathbf{C})_L$ and the base $SL(2,\mathbf{C})_R$ of the cohomology $H^*(\hat{\mathcal{M}})$. Once we assume existence of these $SL(2,\mathbf{C})$ actions[1] , we may arrange this decomposition as

$$I_g \otimes R_g + I_{g-1} \otimes R_{g-1} + \cdots + I_0 \otimes R_0 \ , \tag{5.5}$$

where $I_k := [(\frac{1}{2}) \oplus 2(0)]^{\otimes k}$. Identifying this decomposition with that in 5.1, they propose

$$\alpha_k = \chi(R_k) \quad (k = 0, \cdots, g) \ , \tag{5.6}$$

where $\chi(R_k)$ is the dimensions of the $SL(2,\mathbf{C})_R$ representations in R_k weighted with $(-1)^{2j_R}$. As argued in [11] we can determine α_g and α_0 easily by the geometric Euler numbers;

$$\alpha_g = (-1)^d \chi(\mathcal{M}) \ , \ \alpha_0 = (-1)^{\hat{d}} \chi(\hat{\mathcal{M}}) \ , \tag{5.7}$$

where $d = \dim\mathcal{M}$ and $\hat{d} = \dim\hat{\mathcal{M}}$.

Now let us consider our genus g pseudo-sections with a fixed section, say the zero section. Counting BPS states for these pseudo-sections are summarized in the generating function $\mathcal{G}_{g;1}$ in 4.19. As we have seen in the last section the genus g pseudo-sections come with the moduli $\mathcal{M}(g) := \mathrm{Sym}^g(\mathbf{P}^1) = \mathbf{P}^g$. Furthermore the data of the Jacobian may be specified by

[1] One of the Lefschetz $SL(2,\mathbf{C})$ actions is the multiplication of the Kähler form k. Since our moduli spaces have a natural fibration structure $\pi : \hat{\mathcal{M}}(g) \to \mathcal{M}(g)$ with a section ι, we may decompose the Kähler class k into $k_L = (k - \pi^*(\iota^*(k)))$ and $k_R = \iota^*(k)$. This decomposition defines the $SL(2,\mathbf{C})_L \times SL(2,\mathbf{C})_R$ actions. We would like to thank Y. Shimizu for pointing this out to us.

g points on S, one for each elliptic fiber. Therefore we naturally come to the space

$$\hat{\mathcal{M}}(g) = \widetilde{\mathrm{Sym}}^g(S) \ , \tag{5.8}$$

where ˜ represents the resolution of the orbifold singularities via the Hilbert scheme of g points on S, which we denote $S^{[g]}$. The construction of the Lefschetz actions on this space and making the decomposition 5.5 would be interesting problem, however we already have predictions for the decomposition;

$$I_g \times \chi(R_g) + I_{g-1} \times \chi(R_{g-1}) + \cdots + I_0 \times \chi(R_0) \ . \tag{5.9}$$

Namely we can read 5.9 in our expansion 4.19:

$$\begin{array}{ll} g=1 & -2\,I_1 + 12\,I_0 \\ g=2 & \ \ 3\,I_2 - 30\,I_1 + 90\,I_0 \\ g=3 & -4\,I_3 + 52\,I_2 - 260\,I_1 + 520\,I_0 \\ g=4 & \ \ 5\,I_4 - 78\,I_3 + 507\,I_2 - 1690\,I_1 + 2535\,I_0 \\ g=5 & -6\,I_5 + 108\,I_4 - 840\,I_3 + 3636\,I_2 - 9090\,I_1 + 10908\,I_0 \end{array} \tag{5.10}$$

In the next subsection we propose a natural generalization of Göttsche's formula for the Poincaré polynomials of $S^{[g]}$ and reproduce the above predictions. Our generalization of Göttsche's formula suffices to determine the decomposition 5.5.

5.2 Göttsche's formula with $SL(2,\mathbf{C})_L \times SL(2,\mathbf{C})_R$

Göttsche's formula describes the generating function for the Poincaré polynomials of the Hilbert scheme of g points on a surface S. In our case it appears as a natural resolution $S^{[n]}$ of the symmetric product $\mathrm{Sym}^g(S)$ for the rational elliptic surface. If we assume the existence of the Lefschetz actions $SL(2,\mathbf{C})_L \times SL(2,\mathbf{C})_R$, then the Poincaré polynomial can be written

$$P_t(S^{[g]}) = (t^{2g}) \ \mathrm{Tr}_{H^*(S^{[g]})} t^{2(j_{3,L}+j_{3,R})} \tag{5.11}$$

in terms of the diagonal $SL(2,\mathbf{C})$ action. Then the problem is to recover both left and right charges in the above formula, namely, $P_{t_L,t_R}(S^{[g]}) = (t_L^g t_R^g)\mathrm{Tr} t_L^{2j_{3,L}} t_R^{2j_{3,R}}$. In the case of $g=1$, we note that the decomposition is unique as follows

$$(\frac{1}{2},\frac{1}{2})_{L,R} \oplus 8(0,0)_{L,R} = (1)_{L+R} \oplus 9(0)_{L+R} \tag{5.12}$$

Now let us recall Göttsche's formula[9]

$$G(t,q) = \prod_{n\geq 1} \frac{1}{(1-t^{2n-2}q^n)\,(1-t^{2n}q^n)^{10}\,(1-t^{2n+2}q^n)} \ . \tag{5.13}$$

As has been interpreted in [26] there is a close relation between (co)homology elements and the bosonic oscillators associated to each elements in $H^*(S)$. Under this correspondence the classical cohomology is represented by the lowest modes, say $a_k(-1)$ $(k=1,\cdots,12)$, and generate the symmetric product of $H^*(S)$. The higher mode excitations $a_k(-m)$ come from the singular strata of the point configurations. Here it is natural to assume that the higher mode excitation $a_k(-m)$ have the same spin as the lowest mode $a_k(-1)$, whose spin contents are uniquely determined in 5.12. Under this assumption it is easy to recover the $SL(2,\mathbf{C})_L \times SL(2,\mathbf{C})_R$ spin weights in Göttsche's formula;

$$\begin{aligned} G(t_L,t_R,q) &= \prod_{n\geq 1} \left\{ \frac{1}{(1-(t_Lt_R)^{n-1}q^n)\,(1-(t_Lt_R)^{n+1}q^n)} \right. \\ &\times \left. \frac{1}{(1-t_L^2(t_Lt_R)^{n-1}q^n)\,(1-t_R^2(t_Lt_R)^{n-1}q^n)\,(1-(t_Lt_R)^nq^n)^8} \right\}, \end{aligned} \tag{5.14}$$

which provides us the Poincaré polynomial $P_{t_L,t_R}(S^{[g]})$ as the coefficient of q^g. Then our predictions (5.10) for the Lefschetz decomposition should be verified in the formula

$$\begin{aligned} \frac{1}{(t_Lt_R)^g} P_{t_L,t_R}(S^{[g]})\Big|_{t_R=-1} &= \chi(R_g)\left(t_L+\frac{1}{t_L}+2\right)^g \\ &+\chi(R_{g-1})\left(t_L+\frac{1}{t_L}+2\right)^{g-1} + \cdots + \chi(R_0) \ . \end{aligned} \tag{5.15}$$

For lower g, we have found complete agreements of our predictions from the generating functions $\mathcal{G}_{g;1}$ with those coming from the above formula. For example, for $g=3$ we obtain from 5.15

$$-4\left(t_L+\frac{1}{t_L}+2\right)^3 + 52\left(t_L+\frac{1}{t_L}+2\right)^2 - 260\left(t_L+\frac{1}{t_L}+2\right) + 520 \ ,$$

which should be compared with 5.10. We may summarize the above results into a formula relating Göttsche's formula with our generating functions 4.18;

$$G(t_L,-1,-\frac{q^3}{t_L}) = \sum_{g\geq 0} \mathcal{G}_{g;1}(q^3)\left(t_L+\frac{1}{t_L}+2\right)^g \ . \tag{5.16}$$

5.3 Topological string partition function

So far we have fixed a section to discuss the moduli space of the curves $\mathcal{C}_g$. To recover the contributions from the Mordell-Weil lattice, we simply need to multiply the E_8 theta function to the functions we have discussed. Therefore for the BPS state counting on the rational elliptic surface S we consider the function $\Theta_{E_8}(3t, t\gamma)G(-t_L, -1, \frac{q^3}{t_L})$ in terms of the generalized Göttsche's formula. Now it is easy to deduce the following relation;

$$\begin{aligned}\Theta_{E_8}(3t, t\gamma)G(-t_L, -1, \frac{q^3}{t_L}) &= \sum_{g\geq 0} \tilde{Z}_{g;1}(q)(-t_L - \frac{1}{t_L} + 2)^g \\ &= \sum_{g\geq 0} Z_{g;1}(q)\lambda^{2g-2}\,(2\mathrm{sin}\frac{\lambda}{2})^2\ , \qquad (5.17)\end{aligned}$$

where $t_L = \mathrm{e}^{i\lambda}$ and we have used the form of the degenerated instantons given by 4.13. This implies the function $\Theta_{E_8}(3t, t\gamma)G(-t_L, -1, \frac{q^3}{t_L})$ provides us the all genus topological partition function. We may write 5.17 explicitly by

$$q^{\frac{3}{2}}\frac{\Theta_{E_8}(3t, t\gamma)}{\eta(q^3)^{12}} \prod_{n>1} \frac{(1-q^{3n})^4}{(1-t_L q^{3n})^2(1-\frac{1}{t_L}q^{3n})^2} = \sum_{g\geq 0} Z_{g;1}(q)\lambda^{2g-2}(2\mathrm{sin}\frac{\lambda}{2})^2\ . \qquad (5.18)$$

Here we recognize the helicity generating function in the left hand side. Thus the topological partition function has a simple but suggestive form that the genus zero function $Z_{0;1}$ multiplied by the helicity generating function, which is actually the starting point of the analysis done in [11].

Here for completeness we prove the compatibility of the above result 5.17 with our holomorphic anomaly equation 4.11. For this purpose we note the following identity which can be proved in a straightforward way;

$$\left(\frac{\lambda/2}{\mathrm{sin}\lambda/2}\right)^2 \prod_{n\geq 1} \frac{(1-q^n)^4}{(1-\mathrm{e}^{i\lambda}q^n)^2(1-\mathrm{e}^{-i\lambda}q^n)^2} = \exp\left(2\sum_{k\geq 1}\frac{\zeta(2k)}{k}E_{2k}(q)\left(\frac{\lambda}{2\pi}\right)^{2k}\right) \qquad (5.19)$$

The compatibility may be easily verified if we use $3E_2(q^3) = 2\phi(q)^2 + E_2(q)$ found in 4.16 and the value $\zeta(2) = \frac{\pi^2}{6}$, since we have $\frac{\lambda^2}{36}$ for the both side of 5.18 after the differentiation with respect to $E_2(q)$. Also the formula 5.19 explains the simplification we have encountered in Proposition 4.2. Namely we have

$$\sum_{g\geq 0} Z_{g;1}(q)\lambda^{2g} = Z_{0;1}(q)\exp\left(2\sum_{k\geq 1}\frac{\zeta(2k)}{k}E_{2k}(q^3)\left(\frac{\lambda}{2\pi}\right)^{2k}\right)\ . \qquad (5.20)$$

Appendix. Picard-Fuchs equations of the mirror $X^\vee$

Following [12][13] the Picard-Fuchs differential operators about the large complex structure limit are determined to be

$$
\begin{aligned}
\mathcal{D}_1 &= 9\theta_x^2 - 3\theta_x\theta_y - 6\theta_x\theta_z + 24\theta_z\theta_y - 16\theta_z^2 - 27x(3\theta_x + \theta_y + 2\theta_z + 2) \\
&\quad \times(3\theta_x + \theta_y + 2\theta_z + 1) - 3y(\theta_x - 8\theta_z)(\theta_y - \theta_z) \\
&\quad + z(60\theta_x + 32\theta_z + 32)(3\theta_x + \theta_y + 2\theta_z + 1) \\
\mathcal{D}_2 &= \theta_y^2 + y(3\theta_x + \theta_y + 2\theta_z + 1)(\theta_y - \theta_z) \\
\mathcal{D}_3 &= (\theta_z - \theta_y)\theta_z - z(3\theta_x + \theta_y + 2\theta_z + 2)(3\theta_x + \theta_y + 2\theta_z + 1)
\end{aligned}
$$

where $\theta_x = x\frac{\partial}{\partial x}$, etc. Looking at the characteristic variety of this system we have determined the discriminant 3.21.

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