

Prepotential, Mirror Map and F -Theory on $K3$

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Abstract

We compute certain one-loop corrections to F^4 couplings of the heterotic string compactified on T^2 , and show that they can be characterized by holomorphic prepotentials \mathcal{G} . We then discuss how some of these couplings can be obtained in F -theory, or more precisely from 7-brane geometry in type IIB language. We in particular study theories with $E_8 \times E_8$ and $SO(8)^4$ gauge symmetry, on certain one-dimensional sub-spaces of the moduli space that correspond to constant IIB coupling. For these theories, the relevant geometry can be mapped to Riemann surfaces. Physically, the computations amount to non-trivial tests of the basic F -theory – heterotic duality in eight dimensions. Mathematically, they mean to associate holomorphic 5-point couplings of the form $\partial_t^5 \mathcal{G} \sim \sum g_t \ell^5 \frac{q^t}{1-q^t}$ to $K3$ surfaces. This can be seen as a novel manifestation of the mirror map, acting here between open and closed string sectors.

1 Introduction

To support the various string duality conjectures, numerous tests have been successfully performed. So far lacking is a more quantitative test of the basic heterotic- F theory [1] duality in eight dimensions, i.e., a comparison of non-trivial terms in the effective action.

The terms we have in mind are certain corrections to the F^4 couplings. These couplings are related to the special class of “holomorphic” [2] or “BPS-saturated” [3, 4] amplitudes, which involve $n = 1, 2, 4$ external gauge bosons (in theories with 4, 8 or 16 supercharges, resp.), and which are directly related to the heterotic elliptic genus $\hat{\mathcal{A}}_{2n+2}$ [5] in $2n + 2$ dimensions. It turns out that supersymmetry relates parity even ($i\xi F^n$) and parity odd ($\frac{\theta}{2\pi} F \wedge F \wedge \dots F$) sectors, and therefore one can conveniently combine the theta-angle and the coupling constant ξ into one complex coupling, τ_{eff} . Specifically, functionally non-trivial one-loop corrections arise in T^2 compactifications of $2n + 2$ dimensional $N = 1$ supersymmetric heterotic strings generically as follows:

$$\text{Re}[\tau_{\text{eff}}(T, U)] F \wedge^n F \sim \int \frac{d^2\tau}{\tau_2} Z_{(2,2)}(T, U, q, \bar{q}) \hat{\mathcal{A}}_{2n+2}(F, q) \Big|_{2n\text{-form}} \quad (1.1)$$

Here, T, U are the Kähler and complex structure moduli and $Z_{(2,2)}$ is the partition function of the two-torus T^2 (we have switched off any Wilson lines inFrom analyticity, the imaginary part of τ_{eff} is determined essentially by the same expression.¹ For gauge theories with the indicated number of supersymmetries, it is known that perturbative contributions arise to one loop order only and have the form this formula)

$$\tau_{\text{eff}}(a) \equiv i\xi(a) + \frac{1}{2\pi}\theta(a) = \frac{1}{2\pi i} \ln[a/\Lambda], \quad (1.2)$$

where a is the Higgs field and Λ a cutoff scale; in string theory, there will be additional α' corrections. As is by now familiar, logarithmic monodromy shifts correspond to physically irrelevant shifts of the θ -angle.

In heterotic compactifications on $K3 \times T^2$ to four dimensions, such couplings multiplying F^2 are given by two derivatives of a holomorphic prepotential, $\tau_{\text{eff}}(t) = (\partial_t)^2 \mathcal{F}(t)$. They have been computed for a variety of heterotic compactifications [6, 7, 8] and reproduced in the dual type II string setup [9, 10]. Here one makes use of mirror symmetry [11] of suitable CY manifolds, to compute three-point couplings of the generic form $\mathcal{F}_{ttt} \equiv \partial_t^3 \mathcal{F} = \sum c_\ell \ell^3 \frac{q^\ell}{1-q^\ell}$, which in turn are equal to $\partial_t(\tau_{\text{eff}})(t)$.

¹The mild non-harmonicity that arises from massless states can easily be kept under control.

Analogous F^4 couplings in $d = 8$ have recently been analyzed in detail [3, 12, 4, 13], focusing on heterotic and type I string compactifications on T^2 and on gauge fields F belonging to $E_8 \times E_8$ or $SO(32)$. It was argued in [8] and shown in [13] that such four-point couplings can be expressed in terms of holomorphic prepotentials, similar as in four dimensions. This hints at the existence of a superspace representation of the effective lagrangians; however, to our knowledge, neither this formalism nor an analog of “special geometry” seems to have been developed.

In contrast to the couplings in $d = 4$, the heterotic couplings τ_{eff} in $d = 8$ are supposedly exact at one loop order [12]. Indeed, for theories with 16 supercharges, the dilaton sits in a gravitational multiplet and thus cannot mix with the vector moduli space. Therefore there is no room for non-perturbative corrections, and correspondingly the cutoff in (1.2) is given by $\Lambda = (\alpha')^{-1/2}$. The couplings τ_{eff} thus provide an ideal, quantitative testing ground for the heterotic- F theory duality.

In the present paper, we will concentrate on certain couplings $\tau_{\text{eff}}(T, U)$ arising in theories with $E_8 \times E_8$ and $SO(8)^4$ gauge symmetry. In section 2, we will compute these couplings from the heterotic string, which amounts to evaluating certain integrals that were not treated before in the literature; some more technical material is deferred to Appendices.

We will find that the couplings in the T, U sub-sector derive from an underlying prepotential \mathcal{G} ,

$$(\tau_{\text{eff}})_{ijkl}(T, U) \Big|_{\substack{\text{harmonic} \\ \text{part}}} \equiv \mathcal{G}_{ijkl}(T, U) = \partial_i \partial_j \partial_k \partial_l \mathcal{G}(T, U), \quad (1.3)$$

$$i, j, \dots \in \{T, U\},$$

and that they formally have a structure that is very similar to the well-known couplings in $4d$. That is, taking one more derivative, we have

$$\mathcal{G}_{ijklm}(T, U) \equiv \text{const} + \sum_{n_T, n_U} g(n_T \cdot n_U) n_i n_j n_k n_l n_m \frac{q_T^{n_T} q_U^{n_U}}{1 - q_T^{n_T} q_U^{n_U}}, \quad (1.4)$$

where $q_T \equiv e^{2\pi i T}$ etc., and where g are certain integer coefficients. From the heterotic string point of view, this is perhaps not too surprising, given that the structure of the modular integrals is similar in spirit (though much more complicated) as in four dimensions.

The situation is however quite non-trivial when considered from a “dual” viewpoint, which is F -theory [1]. It is well-known that heterotic compactifications on T^2 are dual to F -theory on $K3$, so it is natural to expect that the

above couplings should be computable entirely from $K3$ data, at a purely geometrical level. However, it is a priori not obvious how such holomorphic, quintic couplings could be canonically related to $K3$.²

Indeed, for a complex d -fold, the natural holomorphic couplings correspond to intersections of d two-forms, and this is why one can use mirror symmetry on 3-fold CY's to compute the three-point couplings \mathcal{F}_{ttt} . For $K3$ the natural holomorphic couplings are two-point functions (which are trivial, ie., constant [16, 17]), and not five-point functions \mathcal{G}_{ttttt} . The question thus arises what the geometrical principles are that lead to five-point functions of the requisite structure, and in particular what the meaning of the integral coefficients is. For three-point functions on 3-folds, the coefficients are known to count rational curves. What they precisely count in eq. (1.4) for a $K3$, may turn out to be an interesting mathematical question; see section 4 for a few remarks on curve counting on $K3$.

We have thus all reason to expect a novel mechanism of mirror symmetry to be at work here, and it is the purpose of the paper to make some first modest steps in uncovering it. We will not find a complete solution of the problem, but will at least show how to relate some couplings to the geometry of $K3$. This will be addressed in section 3, starting with a discussion of the 7-brane geometry for one of the two models that we consider. We then show how the problem can be simplified, by going to a subspace of the moduli space that corresponds to constant IIB coupling and to finite order monodromies in the z -plane. This allows to lift the geometry to Riemann surfaces, which happen to be nothing but well-known Seiberg-Witten curves. In this way we can find Greens functions with the requisite global properties, with which we are able to compute some F^4 gauge couplings from geometry. For the couplings of the $SO(8)^4$ model, we find perfect agreement with the corresponding heterotic string results. We will conclude with some remarks in section 4.

We intend to give a comprehensive treatment in a more detailed forthcoming analysis.

2 Heterotic Computation of Holomorphic Prepotentials in $d = 8$

²Related tree-level four-point functions for type IIA strings compactified on $K3$ have been considered in [14], but these couplings do not seem to be obviously holomorphic; see also [15].

Compactifications of the ten-dimensional heterotic string on a torus have in common s $N=1$ abelian vector multiplets, whose scalars are the T and U moduli of the torus T^2 , besides $s - 2 = 0, \dots, 16$ Wilson lines; for some canonical examples, see Table 1. The bosonic content of the supergravity multiplet in $d = 8$ contains two more vectors and one real scalar, which is the dilaton, in addition to the graviton and an anti-symmetric tensor field. The manifold of the scalar fields is described by the Kählerian coset space $SO(s, 2)/[SO(s) \times SO(2)] \times SO(1, 1)$. The tree-level action for $N=1$, $d = 8$ supergravity has been constructed up to the two-derivative level in [18] and may be obtained by performing a dimensional reduction of the ten-dimensional heterotic string action. Due to the number of fermionic zero modes, string one-loop corrections start at the four-derivative level. In the following we focus on the (four-derivative) couplings of the scalars of the vector multiplets, and derive their underlying prepotential.

gauge group	max. pert. enhanced	sublocus
$E_8^2 \times U(1)^2$	$E_8 \times E_8 \times SU(3)$	$U = T = \rho$
$E_7^2 \times SU(2)^2 \times U(1)^2$	$E_7 \times E_7 \times SO(8)$	$U = -1/T = i + 1$
$E_6^2 \times SU(3)^2 \times U(1)^2$	$E_6 \times E_6 \times E_6$	$U = -1/T = \rho - 1$
$SO(8)^4 \times U(1)^2$	$SO(8)^4 \times U(1)^2$	–

modular group	shift
$\Gamma_T \times \Gamma_U$	1
$\Gamma_0(2)_T \times \Gamma^0(2)_U$	\mathbb{Z}_2
$\Gamma_0(3)_T \times \Gamma^0(3)_U$	\mathbb{Z}_3
$\Gamma_0(4)_T \times \Gamma_U$	$\mathbb{Z}_2 \times \mathbb{Z}_2$

Table 1: Examples of heterotic gauge groups, possible maximal gauge symmetry enhancements, and their shifts in the Narain lattice $SO(18, 2)$. We also listed the duality groups in the T, U subsector that we find to be left unbroken by the Wilson lines.

In the following, we will mainly consider the first and the last of these examples, namely with A: $E_8 \times E_8$ and B: $SO(8)^4$ unbroken gauge symmetries.

2.1 Example A: $E_8 \times E_8$ Gauge Symmetry

The generic form of the (T, U) -dependent couplings has been sketched in eq.(1.1). It directly applies to couplings τ_{eff} multiplying E_8 gauge fields,

where Wilson lines are switched off; such couplings have been computed in [13]. We will consider here the couplings that are intrinsic to the T^2 sub-sector of the theory, ie., the ones that multiply powers of the gauge fields F_T, F_U . They are given by the following world-sheet modular integrals:

$$\begin{aligned} \Delta_{F_T^4} &= \frac{(U - \bar{U})^2}{(T - \bar{T})^2} \int \frac{d^2\tau}{\tau_2} \sum_{(P_L, P_R)} \bar{P}_R^4 q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}|P_R|^2} \frac{\bar{E}_4^2}{\bar{\eta}^{24}} \quad (2.1) \\ \Delta_{F_T^2 F_U^2} &= \int \frac{d^2\tau}{\tau_2} \sum_{(P_L, P_R)} \left[\left(|P_R|^2 - \frac{1}{\pi\tau_2} \right)^2 - \frac{1}{2\pi^2\tau_2^2} \right] q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}|P_R|^2} \frac{\bar{E}_4^2}{\bar{\eta}^{24}} \\ \Delta_{F_T^3 F_U} &= \frac{U - \bar{U}}{T - \bar{T}} \int \frac{d^2\tau}{\tau_2} \sum_{(P_L, P_R)} \left(\bar{P}_R^3 P_R - \frac{3}{2\pi\tau_2} \bar{P}_R^2 \right) q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}|P_R|^2} \frac{\bar{E}_4^2}{\bar{\eta}^{24}}, \end{aligned}$$

with $q = e^{2\pi i\tau}$, $\bar{q} = e^{-2\pi i\bar{\tau}}$ and the (complex) Narain momenta :

$$\begin{aligned} P_L &= \frac{1}{\sqrt{2T_2U_2}} (m_1 + m_2\bar{U} + n_1\bar{T} + n_2\bar{T}\bar{U}) \\ P_R &= \frac{1}{\sqrt{2T_2U_2}} (m_1 + m_2\bar{U} + n_1T + n_2T\bar{U}) \end{aligned}$$

The above integrals³, can be handled by using techniques developed in general in [19, 7] and specifically in [8]. The corrections may be understood as arising from the pieces $\frac{\widehat{E}_2^2 E_4^2}{\eta^{24}} (\text{Tr} \mathcal{R}^2)^2$ and $\frac{E_4^3}{\eta^{24}} \text{Tr}(\mathcal{R}^4)$ of the elliptic genus in ten dimensions [5], supplemented with the corresponding zero modes \bar{P}_R ; for more details of a similar situation, see [8]. A direct string amplitude computation will be presented in [20] in the light of heterotic-type I string duality. As result of lengthy calculations, we find

$$\begin{aligned} \Delta_{F_T^4} &= 16\pi i \left(\partial_T + \frac{2}{T - \bar{T}} \right) \partial_T \left(\partial_T - \frac{2}{T - \bar{T}} \right) \left(\partial_T - \frac{4}{T - \bar{T}} \right) \mathcal{G} \\ &\quad - 16\pi i \frac{(U - \bar{U})^4}{(T - \bar{T})^4} \left(\partial_{\bar{U}} - \frac{2}{U - \bar{U}} \right) \partial_{\bar{U}} \left(\partial_{\bar{U}} + \frac{2}{U - \bar{U}} \right) \\ &\quad \times \left(\partial_{\bar{U}} + \frac{4}{U - \bar{U}} \right) \bar{\mathcal{G}} \\ \Delta_{F_T^2 F_U^2} &= 16\pi i \left(\partial_U - \frac{2}{U - \bar{U}} \right) \left(\partial_U - \frac{4}{U - \bar{U}} \right) \left(\partial_T - \frac{2}{T - \bar{T}} \right) \end{aligned}$$

³In [13] four-point couplings involving the E_8 gauge bosons and the graviton were considered. Their corresponding τ -integrals show a quite different structure and are expressed in terms of three prepotentials. This is similar as for $N = 2, d = 4$ theories where all two-point couplings involving the E_8 gauge bosons and the graviton may be expressed by two functions [7].

$$\begin{aligned}
 & \times \left(\partial_T - \frac{4}{T - \bar{T}} \right) \mathcal{G} + hc. \tag{2.2} \\
 \Delta_{F_T^3 F_U} = & 16\pi i \partial_T \left(\partial_T - \frac{2}{T - \bar{T}} \right) \left(\partial_T - \frac{4}{T - \bar{T}} \right) \left(\partial_U - \frac{4}{U - \bar{U}} \right) \mathcal{G} \\
 & - 16\pi i \frac{(U - \bar{U})^2}{(T - \bar{T})^2} \partial_{\bar{U}} \left(\partial_{\bar{U}} + \frac{2}{U - \bar{U}} \right) \left(\partial_{\bar{U}} + \frac{4}{U - \bar{U}} \right) \\
 & \times \left(\partial_{\bar{T}} + \frac{4}{T - \bar{T}} \right) \bar{\mathcal{G}}.
 \end{aligned}$$

It seems remarkable, at least formally, that these couplings integrate to one and the same holomorphic prepotential \mathcal{G} . In the chamber $T_2 > U_2$, it is given by

$$\begin{aligned}
 \mathcal{G}(T, U) = & -\frac{1}{120} U^5 - \frac{i}{(2\pi)^5} \sum_{(k,l) > 0} g(kl) \mathcal{L}i_5[q T^k q U^l] + \mathcal{Q}(T, U) \\
 & - \frac{ic(0)\zeta(5)}{64\pi^5}, \tag{2.3}
 \end{aligned}$$

where the polylogarithm is defined by ($a \geq 1$):

$$\mathcal{L}i_a(z) = \sum_{p > 0} \frac{z^p}{p^a}, \quad \text{with } \left(z \frac{\partial}{\partial z} \right)^a \mathcal{L}i_a(z) = \frac{z}{1-z}. \tag{2.4}$$

The sum in (2.3) runs over the positive roots $k > 0, l \in \mathbb{Z} \wedge k = 0, l > 0$. The coefficients $g(n)$ are closely related to the chiral light-cone partition function in the Ramond-sector, $\mathcal{C}(q, \bar{q}) = \bar{\eta}^{-24} \text{Tr}_R[(-1)^F q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}]$, which for heterotic compactifications is given by $\mathcal{C}(q, \bar{q}) = Z_{(2,2)}(q, \bar{q}) \frac{E_4^2}{\bar{\eta}^{24}}$ with $Z_{(2,2)}(q, \bar{q}) = \sum_{(P_L, P_R)} q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}|P_R|^2}$. Explicitly, they are given by

$$\frac{E_4^2}{\eta^{24}} \equiv: \sum_{n \geq -1} g(n) q^n. \tag{2.5}$$

The prepotential \mathcal{G} has modular weights $(w_T, w_U) = (-4, -4)$ under T - and U -duality. It is determined up to a quartic polynomial $\mathcal{Q}(T, U)$ in $X^2 = T, X^3 = U$ and $X^4 = TU$. As we will see, for smaller gauge groups, \mathcal{C} takes a different form and the duality group $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U \times \mathbb{Z}_2^{T \leftrightarrow U}$ is broken down to a subgroup (c.f., Table 1).

The correction $\Delta_{F_T^4}$ represents a function of weights $(w_T, w_U) = (4, -4)$ and $(w_{\bar{T}}, w_{\bar{U}}) = (0, 0)$, respectively. Its holomorphicity is spoiled by non-harmonic terms arising from massless string states running in the loop.

A holomorphic, covariant quantity may be obtained via an additional T -modulus insertion in the four-point function, by considering

$$\mathcal{G}_{TTTTT} = -\frac{i}{8} \frac{(U - \bar{U})^2}{(T - \bar{T})^3} \int d^2\tau \sum_{(P_L, P_R)} P_L \bar{P}_R^5 q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}|P_R|^2} \frac{\bar{E}_4^2}{\eta^{24}}.$$

It is a non-trivial feature that this integral indeed yields a holomorphic covariant quantity:

$$\begin{aligned} \mathcal{G}_{TTTTT} &= \left(\partial_T + \frac{4}{T - \bar{T}}\right) \left(\partial_T + \frac{2}{T - \bar{T}}\right) \partial_T \left(\partial_T - \frac{2}{T - \bar{T}}\right) \left(\partial_T - \frac{4}{T - \bar{T}}\right) \mathcal{G} \\ &= \frac{E_6(T) E_4^2(U)}{[J(T) - J(U)] \eta^{24}(U)} \tag{2.6} \\ &= \sum_{(k,l) > 0} k^5 g(kl) \frac{q_T^k q_U^l}{1 - q_T^k q_U^l} \equiv (\partial_T)^5 \mathcal{G}. \end{aligned}$$

This is similar for the other couplings, for example we find

$$\mathcal{G}_{TTTUV} = -\frac{1}{2\pi i} \frac{J'(T)}{J(T) - J(U)} - \frac{1}{2\pi i} \partial_T \ln \Psi_0(T, U) \equiv (\partial_T)^3 (\partial_U)^2 \mathcal{G}, \tag{2.7}$$

where

$$\Psi_0(T, U) = q_T \prod_{(k,l) > 0} (1 - q_T^k q_U^l)^{d(kl)}. \tag{2.8}$$

Ψ_0 stays finite everywhere in the moduli space, i.e., $d(-1) = 0 = d(0)$, and thus represents a cusp form on moduli space. The exponents are generated by

$$\sum_{n > 0} d(n) q^n := \frac{1}{12} \frac{E_2^2 E_4^2}{\eta^{24}} + \frac{1}{3} \frac{E_2 E_4 E_6}{\eta^{24}} - \frac{5}{24} \frac{E_6^2}{\eta^{24}} - \frac{5}{24} \frac{E_4^3}{\eta^{24}}. \tag{2.9}$$

Note that \mathcal{G}_{TTTUV} (and thus Ψ_0) is not modular, in contrast to \mathcal{G}_{TTTTT} (this is familiar from the $4d$ coupling \mathcal{F}_{TTU} , which also is not modular).⁴

The relevant physical feature of the four-point couplings is that they have a logarithmic singularity of the form

$$\tau_{\text{eff}}(T, U) = \frac{1}{2\pi i} \ln[J(T) - J(U)] + \frac{1}{2\pi i} \ln[\text{cusp form}]. \tag{2.10}$$

⁴The appearance of the non-modular function E_2 in (2.9) implies that the theorem of Borcherds [21] concerning the modular weight of product functions Ψ does not apply here (at least not directly).

Similar to the analogous situation in four dimensions [22, 6], the J -functions encode the gauge enhancements pertaining to the compactification torus T^2 : $SU(2)$ for $T = U$, $SU(2) \times SU(2)$ at $T = U = i$ and $SU(3)$ at $T = U = \rho \equiv e^{2\pi i/3}$, and in particular reflect the charge multiplicities of the states becoming light near the singularities. Specifically, near the $SU(2)$ locus the prepotential looks:

$$\mathcal{L}i_5[q_T(qU)^{-1}] \sim a^4 \ln[\sqrt{\alpha'} a], \quad a \equiv \frac{1}{\sqrt{\alpha'}}(T - U), \quad (2.11)$$

and similar for the other gauge groups. This is indeed the expected behavior (c.f., (1.2) of the one-loop field theory effective action⁵ of $d = 8$ $N = 2$ SYM.

2.2 Example B: $SO(8)^4$ Gauge Symmetry

The general structure of the one-loop corrections is similar to (1.1). The elliptic genus encodes the information about the gauge group. Since we break the $E_8 \times E_8$ gauge group down to $SO(8)^4$ with Wilson lines, the Narain lattice $Z_{(2,2)}$ will be affected. More precisely, with the (discrete) Wilson line background

$$a_1^I = \frac{1}{2}(0, 0, 0, 0, 0, 0, 0, 0; 1, 1, 1, 1, 1, 1, 1, 1), \quad (2.12)$$

$$a_2^I = \frac{1}{2}(0, 0, 0, 0, 1, 1, 1, 1; 0, 0, 0, 0, 1, 1, 1, 1),$$

the internal part $Z_{(18,2)}$ of the partition function reads [24]

$$Z_{(18,2)}^{SO(8)^4}(q, \bar{q}) = \frac{T_2}{\tau_2} \sum_{(\mathbf{h}, \mathbf{g})} \sum_{\mathcal{A}(\mathbf{h}, \mathbf{g})} e^{-2\pi i T \det \mathcal{A}(\mathbf{h}, \mathbf{g})} e^{-\frac{\pi T_2}{\tau_2 \bar{\nu}_2} |(\frac{1}{U})^t \mathcal{A}(\mathbf{h}, \mathbf{g}) (\frac{1}{U})|^2} \mathcal{C}_{(\mathbf{h}, \mathbf{g})}(q), \quad (2.13)$$

with:

$$\mathcal{A}_{(\mathbf{h}, \mathbf{g})} = \begin{pmatrix} 2n^1 + h_1 & 2l^1 + g_1 \\ 2n^2 + h_2 & 2l^2 + g_2 \end{pmatrix}, \quad h_i, g_i = 0, 1. \quad (2.14)$$

In addition, we introduce the functions $\mathcal{C}_0 := \mathcal{C}_{(0,0;0,0)}$, $\mathcal{C}_1 := \mathcal{C}_{(0,0;0,1)} = \mathcal{C}_{(0,0;1,0)} = \mathcal{C}_{(0,0;1,1)}$ with

$$\mathcal{C}_0(q) = \frac{E_4^2}{\eta^{24}}, \quad \mathcal{C}_1(q) = \frac{1}{64} \frac{(G_4 + G_2^2)^2}{\eta^{24}} := \sum_{n \geq -1} c_1(n) q^n \quad (2.15)$$

⁵For a recent exposition including references, see eg. ref. [23].

(the $\Gamma_0(2)$ modular forms G_2, G_4 are defined in Appendix A). The remaining functions $\mathcal{C}_2 = \theta_2^8 \theta_3^8 / \eta^{24} := \sum_{n \geq 0} c_2(n) q^n$, $\mathcal{C}_3 = \theta_2^8 \theta_4^8 / \eta^{24}$ follow from modular invariance. Inspecting the partition function (2.13) shows that the model under consideration is equivalent to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ toroidal orbifold⁶, with Kähler modulus $\tilde{T} = 4T$ and the (freely acting) shift $\theta = \frac{1}{2}(0, 0, 1, 0)$, $\omega = \frac{1}{2}(0, 0, 0, 1)$ on $(P_L, P_R) \in \mathcal{N}_{2,2}$, accompanied with the shifts $\Theta = a_1$, $\Omega = a_2$ in the $SO(32)$ root lattice. Moreover, thanks to the relation $\mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 = \mathcal{C}_0$, the partition function (2.13) boils down to the “N=2 subsectors” $i = 1, 2, 3$ of a \mathbb{Z}_4 -orbifold that has appeared in [26]

$$Z_{(2,2)_i}(q, \bar{q}) = \nu_i \sum_{A_i} e^{2\pi i \tau(m_1 n^1 + m_2 n^2)} e^{-\frac{\pi \tau_2}{T_2 U_2} |\tilde{T} U n^2 + \tilde{T} n^1 - U m_1 + m_2|^2} \quad (2.16)$$

with Narain cosets A_i and the volume factor $\nu_i = vol(\mathcal{N}_{2,2}_i) = \{1, \frac{1}{4}, \frac{1}{4}\}$ (for example, $A_1 = \{m_j \in 2\mathbb{Z}, n^j \in \mathbb{Z}\}$ for the untwisted sector; for further details see [26]).

We are now prepared to evaluate (1.1) for the corrections $\Delta_{\alpha\beta} \equiv \Delta_{\text{Tr} F_{SO(8)_\alpha}^2 \text{Tr} F_{SO(8)_\beta}^2}$, involving two gauge bosons from one $SO(8)_\alpha$ and two from the same or another one, $SO(8)_\beta$ (traces are taken in the adjoint of $SO(8)$). In addition, we also consider $\Delta_\alpha \equiv \Delta_{\text{Tr} F_{SO(8)_\alpha}^4}$. All of these become a sum over three cosets that represent the orbifold sectors

$$\begin{aligned} \Delta_{\alpha\beta} &= \int \frac{d^2\tau}{\tau_2} \sum_{i=1,2,3} Z_{(2,2)_i}(q, \bar{q}) \left(Q_\alpha^2 - \frac{1}{4\pi\tau_2}\right) \left(Q_\beta^2 - \frac{1}{4\pi\tau_2}\right) \bar{\mathcal{C}}_i(\bar{q}) \quad (\alpha \neq \beta) \\ \Delta_{\alpha\alpha} &= \int \frac{d^2\tau}{\tau_2} \sum_{i=1,2,3} Z_{(2,2)_i}(q, \bar{q}) \left((Q_\alpha^2)^2 - \frac{1}{2\pi\tau_2} Q_\alpha^2 + \frac{1}{16\pi^2\tau_2^2}\right) \bar{\mathcal{C}}_i(\bar{q}) \quad (2.17) \\ \Delta_\alpha &= \int \frac{d^2\tau}{\tau_2} \sum_{i=1,2,3} Z_{(2,2)_i}(q, \bar{q}) Q_\alpha^4 \bar{\mathcal{C}}_i(\bar{q}) . \end{aligned}$$

The charge operators act on the relevant $U(1)$ factors, in complete analogy to considerations in refs. citeellis (e.g.: $Q_\alpha^4 \equiv E_4 - 9(8q \frac{d}{dq} \ln \theta_\alpha - \frac{1}{3} E_2)^2$.) They have the effect of derivatives acting on the corresponding $SO(8)$ factors, contributing $\mathcal{C}_{D_4} := \frac{1}{\sqrt{8}}(G_4 + G_2^2)^{\frac{1}{2}} = \theta_3^2 \theta_4^2$ in the partition function (2.15). In total we get

$$\Delta_{\alpha\beta} = \int \frac{d^2\tau}{\tau_2} \left\{ -b_{\alpha\beta} + Z_{(2,2)}(q, \bar{q}, \tilde{T}, U) x_{\alpha\beta} \right.$$

⁶Without torsion, i.e. twisted sectors like e.g. $(\theta, \theta\omega)$ do not appear [25].

$$\begin{aligned}
 & + \sum_{i=1,2,3} Z_{(2,2)_i}(q, \bar{q}, \tilde{T}, U)(z_{\alpha\beta} \widehat{\mathcal{B}}_i + y_{\alpha\beta}) \} \tag{2.18} \\
 \Delta_\alpha & = \frac{1}{2} \int \frac{d^2\tau}{\tau_2} \left\{ Z_{(2,2)}(q, \bar{q}, \tilde{T}, U) - \frac{2}{3} \sum_{i=1,2,3} Z_{(2,2)_i}(q, \bar{q}, \tilde{T}, U) \right\}
 \end{aligned}$$

with $x_{\alpha\alpha} = -4$, $y_{\alpha\alpha} = 4$, $z_{\alpha\alpha} = \frac{1}{4}$ and $b_{\alpha\alpha} = 4$, plus $x_{12} = 2$, $y_{12} = -4$, $z_{12} = \frac{1}{4}$ and $b_{12} = -2$. Above, the functions \mathcal{B}_i are defined by

$$\begin{aligned}
 \mathcal{B}_1 & = \frac{1}{\eta^{24}} \mathcal{C}_{D_4}^2 \left(q \frac{d}{dq} \mathcal{C}_{D_4} \right)^2 = \frac{1}{36 \cdot 64} \frac{(G_4 + G_2^2)^2}{\eta^{24}} [E_2 - \frac{1}{2}(\theta_3^4 + \theta_4^4)]^2 \\
 \mathcal{B}_2 & = \frac{\theta_2^4 \theta_3^4}{\eta^{24}} \left(q \frac{d}{dq} \theta_2^2 \theta_3^2 \right)^2 = \frac{1}{36} \frac{\theta_2^8 \theta_3^8}{\eta^{24}} [E_2 + \frac{1}{2}(\theta_2^4 + \theta_3^4)]^2, \tag{2.19} \\
 \mathcal{B}_3 & = \frac{\theta_2^4 \theta_4^4}{\eta^{24}} \left(q \frac{d}{dq} \theta_2^2 \theta_4^2 \right)^2 = \frac{1}{36} \frac{\theta_2^8 \theta_4^8}{\eta^{24}} [E_2 + \frac{1}{2}(\theta_4^4 - \theta_2^4)]^2.
 \end{aligned}$$

(For $\widehat{\mathcal{B}}_i$ we need to replace E_2 by $E_2 - \frac{3}{\pi\tau_2}$ in (2.20).) According to [26] the structure of the integrands in (2.19) and all related couplings is $[Z_1(\tau) + Z_1(-\frac{1}{4\tau}) + Z_1(-\frac{1}{4\tau-2})] \mathcal{B}_1(\tau)$. When introducing the orbits for the non-degenerate orbit and enlarging the integration region to the upper half-plane, we turn the integrands into a contribution of a single coset:

$$Z_1(-\frac{1}{2\tau}) [\mathcal{B}_2(2\tau) + \mathcal{B}_1(\frac{\tau}{2}) + \mathcal{B}_1(\frac{\tau}{2} + \frac{1}{2})] \tag{2.20}$$

The following remarkable identity⁷

$$\mathcal{B}_2(2\tau) + \mathcal{B}_1(\frac{\tau}{2}) + \mathcal{B}_1(\frac{\tau}{2} + \frac{1}{2}) = 16 \tag{2.21}$$

has the effect that the non-trivial q -dependence cancels out in the harmonic contributions. We thus eventually arrive at

$$\begin{aligned}
 \Delta_{\alpha\alpha} & = -4 \ln [T_2 U_2 |\eta(U)|^4] + 4 \left[\ln |\eta(4T)|^4 - 2 \ln |\eta(2T)|^4 \right] \\
 & \text{non-harm.} \tag{2.22}
 \end{aligned}$$

$$\Delta_\alpha + \frac{1}{8} \Delta_{\alpha\alpha} = -\frac{1}{2} \ln [T_2 U_2 |\eta(2T)|^4 |\eta(U)|^4] + \text{non-harm.}$$

and moreover:

$$\Delta_{12} = 2 \ln [T_2 U_2 |\eta(U)|^4] - \left[2 \ln |\eta(4T)|^4 - 4 \ln |\eta(2T)|^4 \right] + \text{non-harm.} \tag{2.23}$$

(We do not display here the non-harmonic piece, which is related to \widehat{E}_2 and which can be easily computed along the lines of [7].) According to the

⁷Analogous to the Atkin-Lehner cancellations of ref. [28].

choice (2.13), the coupling of $SO(8)_1$ to the remaining $SO(8)$'s are obtained by expanding around the other two cusps at $T = 0, \frac{1}{2}$, which gives:

$$\begin{aligned} \Delta_{13} &= 2 \ln[T_2 U_2 |\eta(U)|^4] + [2 \ln |\eta(T)|^4 - 2 \ln |\eta(2T)|^4 + 2 \ln |\eta(4T)|^4] \\ &\quad + \text{non-harm.} \end{aligned} \tag{2.24}$$

$$\Delta_{14} = 2 \ln[T_2 U_2 |\eta(U)|^4] + [-2 \ln |\eta(T)|^4 + 4 \ln |\eta(2T)|^4] + \text{non-harm.}$$

Note that the T, U sectors effectively decouple, which is a consequence of the cancellation (2.21). For sake of completeness, we have also computed the corrections $\Delta_{F_T^4}$ etc. in (2.2). Since for these the analog of 2.20 is

$$Z_1(-\frac{1}{2\tau})[2^{-4}C_2(2\tau) + C_1(\frac{\tau}{2}) + C_1(\frac{\tau}{2} + \frac{1}{2})] = 0, \tag{2.25}$$

these couplings vanish identically, and the underlying prepotential is trivial. The $U(1)^2$ gauge symmetry cannot be enhanced to a non-Abelian group, in accordance with a statement given in [29].

This may be also seen by looking at the one-loop correction $\Delta_{\mathcal{R}^4}$ to the gravitational coupling $\mathcal{R} \wedge \mathcal{R} \wedge \mathcal{R} \wedge \mathcal{R}$. The techniques described above may be applied to the integral $\Delta_{\mathcal{R}^4} = \int \frac{d^2\tau}{\tau_2} [\sum_{i=1,2,3} Z_i f_i - 360]$, where we have $f_1 = \Phi_2 + 256$. With a similar identity as (2.21), namely $f_2(2\tau) + f_1(\frac{\tau}{2}) + f_1(\frac{\tau}{2} + \frac{1}{2}) = 720$, we derive $\Delta_{\mathcal{R}^4} = -360 \ln T_2 U_2 |\eta(2T)|^4 |\eta(U)|^4 + \text{const.}$, which is obviously non-singular. Besides, we calculated the harmonic part of the one-loop correction to $\text{Tr} \mathcal{R}^2 \text{Tr} F_{SO(8)}^2$: $\Delta_{\mathcal{R}^2 F_{\alpha}^2} = -\int \frac{d^2\tau}{\tau_2} [\sum_{i=1,2,3} Z_i \mathcal{E}_i - 144] = 144 \ln T_2 U_2 |\eta(2T)|^4 |\eta(U)|^4 + \text{non-harm.}$, with $\mathcal{E}_1 = \eta^{-24} \theta_3^8 \theta_4^8 E_2 [E_2 - \frac{1}{2}(\theta_3^4 + \theta_4^4)]$. There is again no mixing between T and U in the harmonic part, thanks to $\mathcal{E}_2(2\tau) + \mathcal{E}_1(\frac{\tau}{2}) + \mathcal{E}_1(\frac{\tau}{2} + \frac{1}{2}) = 288$.

3 F-Theory Approach

3.1 7-Brane Interactions

Dual to the heterotic string on T^2 is F -theory on elliptic $K3$ surfaces [1], and thus the aim would be to reproduce the prepotentials \mathcal{G} from suitable $K3$'s. Such surfaces must be elliptic fibrations over \mathbb{P}^1 (here coordinatized by z), and F -theory compactified on them are defined to be type IIB string compactifications on the base \mathbb{P}^1 augmented with extra 7-branes. The positions of the 7-branes are precisely the locations $z = z_i^*$ in the base where the elliptic fiber \mathcal{E} degenerates.

Every 7-brane is characterized by some charges (p, q) , and encircling it in the z -plane leads to a monodromy

$$M_{(p,q)} = \begin{pmatrix} 1 + pq & q^2 \\ -p^2 & 1 - pq \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{3.1}$$

The monodromy transformations act on the type IIB coupling $\tau_s(z) \equiv C^{(0)} + iB^{(0)}$ as fractional linear transformations, and in the canonical way on the $SL(2, \mathbb{Z})$ doublets $(C^{(2)}, B^{(2)})$ and $(C^{(6)}, B^{(6)})$ (here, $B^{(p)}, C^{(p)}$ denote the NS-NS and R-R p -forms that couple to $(p - 1)$ branes). On the other hand, the self-dual 4-form $C^{(4)}$ remains invariant. In general, the 7-branes are mutually non-local with respect to each other, which means that $pq' - p'q \neq 0$ for two given branes.

Effective interactions in $8d$ space-time are generated by superimposing world-volume actions, and also by integrating out exchanges between the 7-branes. This will in general be very subtle, though, at least because of the generic mutual non-locality of the 7-branes. We will circumvent this problem by considering only certain couplings and by restricting to a favorable subspace of the theory, where the relevant branes are effectively local. We also will consider only the parity-odd (wedge-product) terms in the effective theory, as these are generated by simple exchanges of $(C^{(p)}, B^{(p)})$ multiplets between branes. Specifically, we will consider world-volume couplings of the form [30]:

$$I_{(0,1)D7\text{-brane}} = C^{(0)}F \wedge F \wedge F \wedge F + C^{(2)} \wedge F \wedge F \wedge F + C^{(4)} \wedge F \wedge F + C^{(6)} \wedge F + C^{(8)} \cdot 1 + \dots \tag{3.2}$$

For (p, q) branes we will have instead $C^{(2)} \rightarrow qC^{(2)} + pB^{(2)}$ etc.⁸

The antisymmetric tensor fields behave like scalar fields on the base \mathbb{P}^1 , and thus have Greens functions as follows:⁹

$$G^{(p)}(z_1, z_2) \equiv \langle C^{(p)}(z_1), C^{(8-p)}(z_2) \rangle = \ln|z_1 - z_2| + \text{non-singular in } (z_1, z_2) \tag{3.3}$$

and similar for $B^{(p)}$. For a single pair of local 7-branes, this leads to the following term in the effective action [31, 32, 33]:

$$I \sim \tau_{\text{eff}}(z_1, z_2) (F_1 - F_2)^4, \quad \text{where}$$

⁸We will actually consider certain non-local compounds of 7-branes, whose exact world-volume couplings are not known. However, for our purposes numerical coefficients are not important, and all we need is to assume the existence of couplings of the indicated generic form.

⁹In the following, we will often write only the holomorphic pieces of Greens functions, dropping the anti-holomorphic and non-harmonic pieces.

$$\tau_{\text{eff}}(z_1, z_2) \Big|_{\substack{\text{harmonic} \\ \text{part}}} = \frac{1}{2\pi i} \ln[z_1 - z_2] + \text{finite} . \quad (3.4)$$

As we will see, it is easy to match this logarithmic singularity, coming from tree-level exchange of massless antisymmetric tensor fields, with the logarithmic singularity in the heterotic 1-loop couplings. This boils essentially down to find the map between the locations $z_{1,2}$ of the relevant branes, and the heterotic moduli (T, U) . This is what we will discuss, as a warm-up, in the next section for the $E_8 \times E_8$ model (Example A).

However, an exact matching of the full (T, U) dependence is more challenging, since this also requires the non-singular terms in (3.4) to coincide. The problem includes in particular to find the non-singular terms in the Greens functions (3.3), which reflect the global structure of the z -plane, ie., the presence and monodromies of all the 7-branes. This is in general a hard problem, since we do not know suitable Greens functions on $K3$. However, we will be able to use insight gained from Example A and map the problem to a simpler one, which is to some extent tractable. We will then apply this to Example B (with $SO(8)^4$ symmetry), and finally reproduce some of the heterotic one-loop corrections from $K3$ geometry.

3.2 Example A: $E_8 \times E_8$ Gauge Symmetry.

Since we have unbroken $E_8 \times E_8$ gauge symmetry, the $K3$ surface necessarily has $E_8 \times E_8$ singularities. The Weierstraß form of such a surface has been presented in [1] and looks¹⁰

$$P_{K3} = y^2 + x^3 + z^5 w^7 + z^7 w^5 + \alpha x z^4 w^4 + \beta z^6 w^6 = 0 . \quad (3.5)$$

In the patch $w = 1$, this represents an elliptic fibration over the z -plane, with E_8 singularities at $z = 0$ and $z = \infty$. The exact dependence of the $K3$ moduli α, β on the heterotic moduli (T, U) was found in ref. [34] by indirect reasoning and is given by

$$(48\alpha)^3 = -J(T)J(U) , \quad (864\beta)^2 = (J(T)-1728)(J(U)-1728) . \quad (3.6)$$

We show in Appendix B that this, as expected, coincides with the mirror map, ie., the map from α, β to flat coordinates, which are indeed the natural coordinates T, U of the heterotic string moduli space.

The IIB coupling τ_s varies over the \mathbb{P}^1 base according to

$$J(\tau_s(z)) = 4\alpha^3 z^{12} (\Delta_{\mathcal{E}}(z, \alpha, \beta))^{-1} , \quad (3.7)$$

¹⁰Various aspects of this model have been investigated in refs. [34, 35]

where the discriminant of the elliptic fiber \mathcal{E} is

$$\begin{aligned} \Delta_{\mathcal{E}}(z, \alpha, \beta) &= z^{10}(4\alpha^3 z^2 + 27(1 + \beta z + z^2)^2) \\ &\equiv: z^{10} \prod_{i=1}^4 (z - z_i^*(\alpha, \beta)) . \end{aligned} \tag{3.8}$$

This means that four 7-branes are located at the zeros z_i^* of the discriminant where \mathcal{E} degenerates. Moreover, ten 7-branes are localized at the E_8 singularity at $z = 0$ and ten more at $z = \infty$, totalling 24.

It is known that the monodromy around each of the E_8 singularities is given by $ST \equiv \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, with $(ST)^3 = -1$. Thus only finite order (\mathbb{Z}_6) branch cuts emanate from the “ E_8 -planes” at $z = 0, \infty$.

The remaining four 7-branes, located at $z = z_i^*(\alpha, \beta)$, encode the relevant data about the two-torus on the heterotic side, the various open string trajectories between them reflecting the BPS winding states on the T^2 . By carefully tracing the locations of these branes, we have verified that two of them have (p, q) charges given by $(1, 1)$ and the other two have $(0, 1)$.¹¹ If two or more of these four 7-branes collide, there may be or may be not an enhanced gauge symmetry, depending on the mutual locality properties. Enhanced gauge symmetries only occur if the $K3$ discriminant¹²

$$\Delta_{K3} = (4\alpha^3 + 27(\beta + 2)^2)(4\alpha^3 + 27(\beta - 2)^2) \tag{3.9}$$

vanishes. Via (3.6), this precisely matches the known gauge enhancement loci in the heterotic (T, U) moduli space, ie., $\Delta_{K3} \sim (J(T) - J(U))^2$.

For the purpose of computing pieces of the effective action, we do not really know how to handle mutually non-local 7-branes. A related problem is how to distribute the field strengths F_T, F_U on the 7-branes; we have four 7-branes, each locally hosting a $U(1)$, but in total only a $U(1) \times U(1)$ gauge symmetry within the T, U sub-sector. The requisite reduction of independent $U(1)$ factors is supposedly due to global effects [1, 32], but the precise mechanism has not yet been well enough understood in order to be helpful for our purposes.

¹¹This non-invariant statement depends on the basis and choice of paths, but the only important point here is that the branes are mutually non-local.

¹²This is identical to the discriminant of the $SU(3)$ Seiberg-Witten curve [36], the two non-local Argyres-Douglas points [37] corresponding to $SU(3)$ gauge enhancement etc. This is because the $K3$ surface (3.5) can be brought by rescalings to the form $z + \frac{1}{z} + (x^3 + ax + b) + y^2 = 0$, and this differs from the SW curve only by a quadratic piece. SW curves will indeed reappear further below.

However, we can make life more easy by considering a certain distinguished sub-space of the moduli space. It is given by $\alpha = 0$, which amounts to e.g. $U = \rho \equiv e^{2\pi i/3}$, and from (3.7) to constant type IIB coupling, $\tau_s(z) \equiv \rho$. In this limit, pairs of (1, 1) and (0, 1) branes coincide at

$$z_{1,2}^*(J(T)) = \frac{1}{2}(-\beta(T) \mp \sqrt{\beta^2(T) - 4}) , \quad \beta(T) \equiv 2\sqrt{1 - J(T)/1728} , \quad (3.10)$$

to form what one may call “ H_0 -planes” (associated with Kodaira singularity type II ; we follow here the nomenclature of ref. [38]). Due to the non-locality, there is no gauge enhancement on the H_0 -planes. Similar to the E_8 -planes discussed above, no net logarithmic branch cuts emanate from the locations $z_{1,2}^*$ of the H_0 -planes: there are only a finite order monodromies given by $(ST)^{-1}$. Thus, in the $\alpha = 0$ subspace, there isn’t any object with logarithmic monodromy left, so we effectively deal with only (almost) local objects.

It is now easy to recover the singular behavior of the heterotic one-loop amplitudes at $U = \rho$. The only branes that can possibly give rise to the singularity are the two H_0 -planes. Their world-volume couplings are identical, and it is easy to see that if we consider only the coupling $\tau_{\text{eff}} = \mathcal{G}_{TTUU}$, then the relevant contributions come from

$$C^{(4)} \wedge F_T \wedge F_U \quad (3.11)$$

(which reflects the $K3$ intersection form, $\Omega_{TU} = \delta_{TU}$). Thus the singularity of this coupling is carried by $C^{(4)}$ exchange between the H_0 branes, and we immediately obtain

$$\begin{aligned} \tau_{\text{eff}}(T, U \equiv \rho) \Big|_{\substack{\text{harmonic} \\ \text{part}}} &= \frac{1}{2\pi i} \ln[z_1^* - z_2^*] + \text{finite} \\ &= \frac{1}{2\pi i} \ln[J(T)] + \text{finite} . \end{aligned} \quad (3.12)$$

Remembering that $J(\rho) = 0$, this exactly matches the heterotic result (2.10). To obtain the finite term is however much more complicated and will not be attempted here for Example A (but we will compute an exact result for Example B later).

Let us instead pause and see what can be learned by reinterpreting the result (3.12), which was obtained by tree-level closed string $C^{(4)}$ exchange, in terms of open strings.

3.3 The Mirror Map as a Map Between Open and Closed String Sectors

We focus on the logarithmic singularity at $J(T) = 0$, which reflects gauge symmetry enhancement from $U(1)^2$ to $SU(3)$. Expanding around $T = \rho$ by writing $a \equiv \frac{T-\rho}{T-\rho^2}$, we have $J(a) \sim a^3 + \mathcal{O}(a^6)$, which gives a contribution to τ_{eff} of $\frac{3}{2\pi i} \ln[a]$. The factor of 3 reflects the multiplicity of the gluons becoming massless at $a = 0$. These gluons correspond to open strings stretching between the H_0 -planes. Indeed, since $M_{(0,1)}$ and $M_{(1,1)}$ do not commute, there is an ambiguity as to what kind of “charge” one can attribute to an H_0 -plane. In effect, many types of strings can end on a H_0 -plane, and in particular a degenerate triplet of mutually non-local strings of charges $[0, 1]$, $[1, 1]$ and $[1, 0]$ (plus their negatives) can link the two H_0 -planes (their $[p, q]$ charges correspond to the roots of $SU(3)$); see Fig.1. These strings becomes tensionless if the two H_0 -planes collide (for $\beta = \pm 2$), which then form a single “ H_2 -plane”,¹³ associated with monodromy $(ST)^{-2}$.

More explicitly, in terms of open strings the logarithmic singularity in (3.13) can be viewed as arising from perturbative one-loop contributions of the form

$$\tau_{\text{eff}} \sim \sum_i Q_i^4 \ln[m_i], \tag{3.13}$$

where the sum runs over all relevant individual states separately. The mass of a single BPS string of type $[p, q]$, stretching between some locations z_1 and z_2 , is according to [39, 40] given by

$$m_{p,q} = |w_{p,q}(z_1) - w_{p,q}(z_2)|,$$

with $w_{p,q} = \int dw_{p,q}$, where the metric on the z -plane is

$$dw_{p,q}(z) = \frac{1}{\varpi_0} (p + \tau_s(z) q) \eta(\tau_s(z))^2 \prod_i (z - z_i^*)^{-1/12} dz. \tag{3.14}$$

¹³Note that the Kodaira elliptic singularity type of the H_2 -plane is not I_3 but IV [38]. This is why we get $SU(3)$ with four colliding, mutually non-local 7-branes instead, as usual, with three local ones.

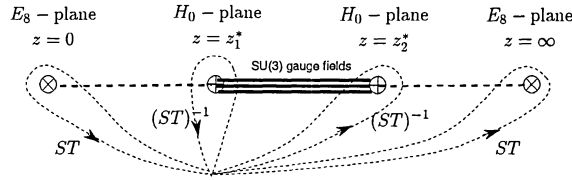


Figure 1: In the limit $\alpha = 0$, pairs of non-local 7-branes combine into H_0 -planes with finite order monodromy. The planes are connected by triplets of mutually non-local open strings. For $\beta = \pm 2$ the H_0 -planes further merge into a single H_2 -plane, the strings between them then giving rise to massless charged $SU(3)$ gauge fields. They reflect the intersecting vanishing cycles of an Argyres-Douglas point.

Here, z_i^* are the zeros of the elliptic discriminant $\Delta_{\mathcal{E}}$ and ϖ_0 denotes the fundamental period of the $K3$ surface.¹⁴ In our situation with $\alpha = 0$, where $\tau_s(z) \equiv \rho$ is constant, the τ_s -dependent terms are irrelevant and the integral can be done explicitly. We so find that the mass of a string stretched between the two H_0 -planes (located at $z_{1,2}^*$ (3.10)) is proportional to

$$\begin{aligned} & \int_{z_1^*(j)}^{z_2^*(j)} z^{-5/6} (z - z_1^*(j))^{-1/6} (z - z_2^*(j))^{-1/6} dz \\ &= (i\sqrt{j(1-j)} - j)^{5/6} (-j)^{-1/12} \\ & \quad \times {}_2F_1(5/6, 5/6, 5/3; 2j - 2i\sqrt{j(1-j)}) , \end{aligned}$$

where $j \equiv J(T)/1728$. This expression, via a non-trivial hypergeometric identity, turns out to coincide with the $K3$ period $f_2(J(T))$ defined in (B.6) in Appendix B. Dividing out ϖ_0 (B.7) we see that the mass of the stretched string is given by a certain combination of the geometrical $K3$ periods T, U (where $U = \rho$):

$$m_{p,q} = |(p + \rho q) a| , \quad a \equiv \frac{T - \rho}{T - \rho^2} .$$

The strings with p, q charges $[1, 0]$, $[1, 1]$ and $[0, 1]$ have thus exactly the same, degenerate masses as the corresponding non-abelian gauge bosons on the heterotic side. As $T \rightarrow \rho$, they indeed give the leading contribution of $\frac{3}{2\pi i} \ln[a]$ to (3.13).

In order to obtain the full expression (3.13) from (3.13), we would need to take contributions of infinitely many open string states into account. Their masses are given by distances in the w -plane and thus by linear combinations

¹⁴The prefactor ϖ_0 plays no rôle in the rigid limit discussed in [39], but must be there in order to provide the correct normalization of the BPS masses.

of the $K3$ periods $(1, T, U, TU)$. On the other hand, by simply considering distances in the z -plane (which governs closed string $C^{(p)}$ exchange), we had managed in the previous section to sum up these infinitely many contributions in a single stroke (signified by the appearance of a modular function, ie., $z_1^* - z_2^* \sim J(T)$).

In fact, it is known (see e.g. [33]) that interactions between 7-branes can be characterized by both open or closed string exchanges: open string exchanges being dominant at short distances, while closed string effects being important at large distances. In our example, this is reflected by the expansion $J(a) \sim a^3 + \mathcal{O}(a^6)$, the short distance variable a being naturally adapted to open strings, while $J(a)$ provides the analytic continuation to the closed string regime.¹⁵

In other words, the map between open and closed string sectors is provided by the period map between the w -plane and the z -plane, and thus is essentially nothing but the mirror map, $T \leftrightarrow J(T)$.

3.4 Constant Coupling τ_s , and Reduction to Curves

The above considerations can be canonically extended to the list of examples in Table 1, including Example B with $SO(8)^4$ symmetry. In fact the $SO(8)^4$ model, introduced by Sen [39], was the first example where the type IIB coupling τ_s is constant (and for this model arbitrary) over the z -plane. Later it was found that there exist also other branches of the moduli space where the coupling is constant, though frozen in value: $\tau_s = i$ (5 remaining moduli) or $\tau_s = \rho$ (9 remaining moduli) [41].¹⁶

We will restrict ourselves here to one-dimensional sub-spaces of the moduli space of such theories, given by the following one-parameter families of elliptic $K3$'s:

$$\begin{aligned}
 (E_8, H_0) : \quad & y^2 + x^3 + z^5(z-1)(z-z^*(\tau)) = 0 \\
 (E_7, H_1) : \quad & y^2 + x^3 + xz^3(z-1)(z-z^*(\tau)) = 0 \\
 (E_6, H_2) : \quad & y^2 + x^3 + z^4(z-1)^2(z-z^*(\tau))^2 = 0 \\
 (D_4, D_4) : \quad & y^2 + x^3 + z^3(z-1)^3(z-z^*(\tau))^3 = 0
 \end{aligned}
 \tag{3.15}$$

Each of these models has two pairs of singularities in the z -plane of the indicated types, in generalization of Fig.1. The first model is equivalent

¹⁵As we have already indicated, there are further corrections to $\ln J$, but for sake of easy reading we did not write them here.

¹⁶Various aspects of theories with constant coupling have been investigated in ref. [42].

to Example A ((3.5 restricted to $U = \rho$, i.e., $\alpha = 0$), and the last one to our Example B (strictly speaking restricted to $U = \rho$; however the z -plane geometry does not depend at all on U , so our considerations will be valid for any U .)

We have chosen to put 7-planes with symmetry $G_1 = D_4, E_6, E_7, E_8$ at $z = 0, \infty$, and planes of type $G_2 = D_4, H_2, H_1, H_0$, at $z = 1, z^*$, respectively. In fact, the Kodaira singularity types of these two sets are “dual” to each other, in that the monodromies of the G_1 planes and of the G_2 planes are inverses of each other; they belong to \mathbb{Z}_N , $N = 2, 3, 4, 6$, respectively.

In the one-dimensional moduli spaces, two interesting things can happen. First, a G_1 - and a G_2 -plane can collide, to yield an “ \widehat{E}_8 ” singularity of the local form $y^2 + x^3 + z^6 = 0$. This corresponds to the decompactification limit on the heterotic side, $T \rightarrow i\infty$ [35, 43]. Second, similar to what we discussed in the previous section, two G_2 -planes can collide to produce a 7-plane associated with some extra non-abelian gauge symmetry G_3 (for D_4 , where there is no further gauge enhancement, this also corresponds to the decompactification limit). In other words, the generic non-abelian gauge symmetry is $(G_1 \times G_2)^2$, which can be enhanced to $(G_1)^2 \times G_3$. All this information is summarized in Table 2. By computations completely analogous to those in Appendix B, we find directly from the $K3$'s in (3.16) that the $z^*(\tau)$ satisfy the Schwarzian differential equations

$$\frac{z^{*'''}}{z^{*'}} - \frac{3}{2} \left(\frac{z^{*''}}{z^{*'}} \right)^2 = -2Q(z^*) z^{*'}{}^2,$$

where

$$Q(z^*) = \frac{N^2(z^* - 1)^2 + 4(N - 1)z^*}{4N^2(z^* - 1)^2 z^{*2}}. \tag{3.16}$$

From these equations for $N = 2, 3, 4, 6$, we can easily infer what the modular subgroups are of which z^* is the Hauptmodul;¹⁷ those subgroups are listed in Table 2. It is reassuring that these precisely match the modular subgroups that arise on the heterotic side by switching on the corresponding Wilson lines (cf., Table 1). From (3.16) also follows [44] that the mirror map can be written as

$$\tau(z^*) = s(0, 0, 1 - 2/N; z^*), \tag{3.17}$$

which supposedly matches the modulus T on the heterotic side. Above, $s(a, b, c)$ denotes the triangle function where the entries a, b, c denote the angles of a fundamental region. This means that there are generically two cusps (corresponding to the decompactification limits $z^* \rightarrow 0, \infty$) and one

¹⁷Explicitly, we find $z^*(\tau) = -\frac{1}{A}[\eta(\tau)/\eta((6 - N)\tau)]^{24/(5 - N)}$ with $A = (16, 27, 64)$ for $N = 2, 3, 4$ and $z^*(\tau) = (\sqrt{-j} + \sqrt{1 - j})^2$ for $N = 6$.

point with gauge enhancement ($z^* \rightarrow 1$). However, for $N = 2$ there are three cusps, which reflects that two colliding D_4 singularities correspond to decompactification. Indeed, for $N = 2$ we have $z^*(\tau) = \lambda(\tau)$ (the standard $\Gamma[2]$ modular function up to an $SL(2, \mathbb{Z})$ transformation), which has three cusps only.

N	G_1	Kod	M	G_2	Kod	M	G_3	Kod
6	E_8	II^*	ST	H_0	II	$(ST)^{-1}$	H_2	IV
4	E_7	III^*	S	H_1	III	S^{-1}	D_4	I_0^*
3	E_6	IV^*	$(ST)^2$	H_2	IV	$(ST)^{-2}$	E_6	IV^*
2	D_4	I_0^*	S^2	D_4	I_0^*	S^2	\widehat{E}_8	–

N	mod. subgroup	τ_s
6	Γ^*	ρ
4	$\Gamma_0(2)$	i
3	$\Gamma_0(3)$	ρ
2	$\Gamma_0(4) \cong \Gamma(2)$	any

Table 2: Data of the two pairs of 7–planes in the four examples of Table 1: G_1 and G_2 characterize the singularities associated with the planes at $z = 0, \infty$ and $z = 1, z^*$, respectively (the H_n planes carry A_n gauge symmetry). G_3 is the symmetry that appears if two G_2 –planes collide. “Kod” denotes the Kodaira singularity type, and $M \in SL(2, \mathbb{Z})$ the associated monodromy. We also indicate the modular subgroup of which $z^*(\tau)$ is a modular function, and the value of the constant type IIB string coupling τ_s (which coincides with the value of U in Table 1 up to an $SL(2, \mathbb{Z})$ transformation; above, Γ^* denotes the modular group together with some extra involution).

Note that since the coupling τ_s is constant, the integrals over the open string metrics

$$\begin{aligned}
 dw &= dz \prod_{i=1}^{24} \frac{1}{(z - z_i^*)^{1/12}} \\
 &= \frac{dz}{z^{1-1/N}(z-1)^{1/N}(z-z^*)^{1/N}}
 \end{aligned}
 \tag{3.18}$$

can be done explicitly. The two basic cycles give the following hypergeometric functions:

$$\begin{aligned}
 \varpi_0 &= (-1)^{-2/N} \pi \csc(\pi/N) {}_2F_1(1/N, 1/N, 1; z^*) \\
 \varpi_1 &= z^{*-1/N} (-1)^{-2/N} \pi \csc(\pi/N) {}_2F_1(1/N, 1/N, 1; 1/z^*) ,
 \end{aligned}
 \tag{3.19}$$

which coincide up to factors with the relevant period integrals of the $K3$'s in (3.15). The flat coordinate is then alternatively given by $\tau = \varpi_1/\varpi_0$.

An important observation is now that the holomorphic differentials (3.18) can also be viewed as abelian differentials $dw = \frac{1}{x} \frac{dz}{z}$ of the following genus $g = N - 1$ Riemann surfaces:

$$\Sigma_{N-1} : x^N = z^{-1}(z - 1)(z - z^*) . \tag{3.20}$$

These are nothing but \mathbb{Z}_N symmetric variants [45] of the well-known Seiberg-Witten curves for $SU(N)$ [36]. Exactly these curves, as well as the differentials (3.18), had in the past played a rôle in computing tree-level correlation functions in \mathbb{Z}_N orbifolds [46, 47, 48]. That they appear now at this point is no accident, since we have in fact a similar situation here: the G_1 and G_2 planes effectively figure as \mathbb{Z}_N twist fields σ_N and $\bar{\sigma}_N$, respectively, with \mathbb{Z}_N branch cuts running between them in the z -plane.¹⁸

The \mathbb{Z}_N monodromies are in particular felt by the $SL(2, \mathbb{Z})$ doublets $(C^{(2)}, B^{(2)})$ and $(C^{(6)}, B^{(6)})$, and correspond to the monodromies inflicted on the scalar field $X(z)$ in a \mathbb{Z}_N orbifold compactification. We may thus view $X(z), \bar{X}(z)$ as given by appropriate complex linear combinations of these NS-NS and R-R tensor fields. On the other hand, the RR field $C^{(4)}$ is monodromy invariant and so corresponds to a real scalar in the untwisted sector. These observations then finally yield the following proposal for the requisite Greens functions $G(z_1, z_2)$ in (3.3): they should simply be given by the scalar Greens functions in the presence of background twist fields $\sigma_N, \bar{\sigma}_N$ (with singularities $X \rightarrow \sigma_N, \bar{\sigma}_N$ subtracted). According to [46, 47], such Greens functions can be described in terms of suitable covering spaces, and those are precisely the curves of eq. (3.20) – see Fig 2. We thus expect that the Greens functions should simply be given by appropriate Greens functions on the curves Σ_{N-1} . That is, the fields that are twisted in the z -plane would correspond to \mathbb{Z}_N -odd functions on Σ_{N-1} , while single-valued fields (like $C^{(4)}$) would correspond to the \mathbb{Z}_N -symmetric functions on Σ_{N-1} .

The Greens function of a scalar field on a Riemann surface is known to be given by the logarithm of the prime form [49]

$$G_{\Sigma}(z_1, z_2) = \ln \left| \frac{\theta_{\delta} \left[\int_{z_1}^{z_2} \vec{w} | \Omega \right]}{\sqrt{\xi(z_1)} \sqrt{\xi(z_2)}} \right| - \pi \left[\text{Im} \int_{z_1}^{z_2} \vec{w} \right] \cdot (\text{Im} \Omega)^{-1} \cdot \left[\text{Im} \int_{z_1}^{z_2} \vec{w} \right] , \tag{3.21}$$

¹⁸Note that the z -plane plays in this context the rôle of a tree-level world-sheet with twist field insertions, and not (for $N > 2$) of a target space T^2/\mathbb{Z}_N (c.f., [41]).

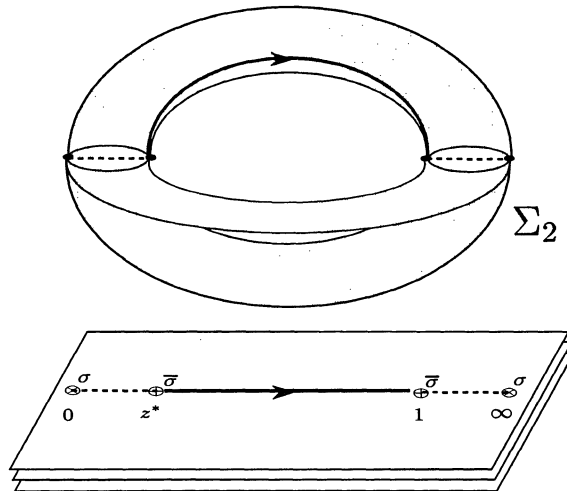


Figure 2: Lift of the z -plane to an N -fold cover, which is equivalent to the \mathbb{Z}_N symmetric $SU(N)$ Seiberg-Witten curve Σ_{N-1} . The four 7-planes correspond to \mathbb{Z}_N twist fields σ and their conjugates, $\bar{\sigma}$, and carry gauge symmetries G_1 and G_2 , respectively, as listed in Table 2. Shown is also an open string trajectory that corresponds to a half-period on Σ_{N-1} .

where $\sqrt{\xi(z)} \equiv \sqrt{\frac{\partial}{\partial z_i}(\theta_\delta[\vec{z}|\Omega]) \cdot w^i(z)}$ is a certain 1/2-differential that cancels spurious zeros of the numerator theta-function (where δ denotes an arbitrary odd characteristic). Indeed, the only singularity of the prime form is at coincident points, i.e., $G_\Sigma(z_1 \rightarrow z_2) \sim \ln[\frac{z_2 - z_1}{\sqrt{dz_1}\sqrt{dz_2}}] + \text{finite terms}$. By construction, the finite terms implement the requisite global properties of the Greens functions. Of course, due to the high degree of symmetry of our curves Σ_{N-1} , much of the information in (3.21) is redundant in our examples for $N > 2$ (e.g., the period matrix Ω is proportional to the A_{N-1} Cartan matrix and depends only on one multiplicative parameter given by τ); in practice, one may wish to express the Greens functions more concisely in terms of suitable, Weyl invariant Jacobi forms.¹⁹

3.5 Example B: $SO(8)^4$ Gauge Symmetry

We now consider certain couplings in Example B. Though considerably more complicated than Example A on the heterotic side, it is much simpler than Example A on the F -theory side, because just a genus one curve is involved (and not genus five). We will focus here on the simplest couplings, namely

¹⁹This might be related to the considerations in [50].

on those which involve two different $SO(8)$ gauge fields (denoted by $\Delta_{\alpha\beta} \equiv \Delta_{\text{Tr}F_{SO(8)\alpha}^2 \text{Tr}F_{SO(8)\beta}^2}$ in section 2.2). It is easy to see that the only contribution to these couplings can come from $C^{(4)}$ exchange between the two D_4 -planes, each with couplings

$$C^{(4)} \wedge F_{SO(8)\alpha} \wedge F_{SO(8)\alpha} .$$

As mentioned above, there is no gauge enhancement if two of such branes collide. Rather, the singularities of pairwise collisions of the form $\ln[z^*]$, $\ln[1 - z^*]$ and $\ln[1/z^*]$ correspond to decompactification limits on the heterotic side. Indeed, we know from the previous section that $z^* = \lambda(\tau)$ (the $\Gamma(2)$ modular invariant), and $\lambda = 0, 1, \infty$ are the three cusp points.

Thus all what we need is the Greens function that is appropriate to $C^{(4)}$. Since this is an untwisted field, we need to consider a \mathbb{Z}_2 even function on the torus Σ_1 . Up to an overall factor of two [46], this coincides with the ordinary Greens function on the torus:

$$G(z_\alpha, z_\beta) = \ln \left| \frac{\theta_1[\int_{z_\alpha}^{z_\beta} w|\tau]}{\theta_1'[\tau]/2\pi} \right| - \pi \frac{\text{Im}^2(\int_{z_\alpha}^{z_\beta} w)}{\text{Im}(\tau)} .$$

The integrals are trivial since they just give the half-periods,

$$\begin{aligned} \theta_1[\int_{z^*}^0 w] &= \theta_1[\tfrac{1}{2}|\tau] = \theta_2 \\ \theta_1[\int_{z^*}^\infty w] &= \theta_1[\tfrac{1}{2}(1 + \tau)|\tau] = e^{-i\pi\tau/4}\theta_3 \\ \theta_1[\int_{z^*}^1 w] &= \theta_1[\tfrac{1}{2}\tau|\tau] = ie^{-i\pi\tau/4}\theta_4 , \end{aligned}$$

and therefore we have up to irrelevant constants:

$$\begin{aligned} \Delta_{z^*,0} \Big|_{\text{harmonic part}} &= \ln \left[\frac{\theta_2[\tau]}{\theta_1'[\tau]} \right] \\ \Delta_{z^*,1} \Big|_{\text{harmonic part}} &= \ln \left[\frac{\theta_3[\tau]}{\theta_1'[\tau]} \right] \\ \Delta_{z^*,\infty} \Big|_{\text{harmonic part}} &= \ln \left[\frac{\theta_4[\tau]}{\theta_1'[\tau]} \right] . \end{aligned} \tag{3.22}$$

Identifying τ with $2T$, and using the identities $\theta_2/\theta_1'(\tau) = 2\eta(2\tau)^2/\eta(\tau)^4$, $\theta_3/\theta_1'(\tau) = \eta(\tau)^2/\eta(2\tau)^2/\eta(\tau/2)^2$ and $\theta_4/\theta_1'(\tau) = \eta(\tau/2)^2/\eta(\tau)^4$, we find perfect agreement between the heterotic and the F -theory results ! So indeed we have finally reproduced a piece of the effective action from the relevant $K3$ surface in (3.15).

Observe that the sum of the three contributions (3.22) yields $6 \ln[\eta(2T)]$, which coincides (up to a factor) with the result (2.22) that we got for the self-couplings, $\Delta_\alpha + \frac{1}{8}\Delta_{\alpha\alpha}$, $\alpha = 1, \dots, 4$. We may interpret this as resulting from

$C^{(0)} - C^{(8)}$ exchange between a given brane and all the others, contributing $\text{Tr}F^4 + \frac{1}{8}(\text{Tr}F^2)^2$ from the given brane and 1 from the three others (c.f., (3.2)). The same reasoning can be made for the gravitational couplings $\Delta_{\mathcal{R}^4}$ and $\Delta_{\mathcal{R}^2F^2}$, which indeed have the same functional form (see the end of section (2.2)) and so can be obtained too by summing up C -exchanges between all the 7-planes. Note that the gravitational couplings (just like the F^4 couplings) arise here in the geometric IIB string setup at tree level (and not at one-loop order, as one naively might have guessed.²⁰) Moreover recall that we have seen in section (2.2) that one-loop corrections of type $\Delta_{F_T^4}$ do not arise on the heterotic side, and this is reflected here in the geometry by the absence of any other 7-branes besides the $SO(8)$ -planes.

4 Conclusions

The successful derivation of heterotic one-loop couplings from classical $K3$ geometry represents one of the first quantitative tests of the basic heterotic–F theory duality in eight dimensions. It also reinforces the hope that mirror symmetry would correctly sum up all relevant perturbative and non-perturbative open string interactions, in terms of tree-level closed string geometry.

However, this is not yet that clearcut: for the model for which we have the strongest results, namely the $SO(8)^4$ model, the U dependence and thus the type IIB coupling constant dependence factors out. This means that the test we have made was not sensitive to non-perturbative effects. For the other three models, the type IIB coupling is fixed to a non-zero value and the U -dependence does not factor out. Therefore the successful reproduction of an F^4 coupling for one of those models would definitely be a non-perturbative test. Unfortunately, the Greens functions that we have proposed for these models are technically much harder to come by at higher genus, and we decided to leave this to future investigations. Moreover, the situation is much more complicated if we consider couplings of types other than $\Delta_{F_G^2 F_{G'}^2}$, for which only $C^{(4)}$ exchange was relevant. To fully reproduce more general couplings, we would need to work much harder, at least because for such couplings also twisted field exchange is relevant, and possibly higher than two point correlators. Also, bulk contributions cannot a priori be ruled out. Clearly, a more detailed study is necessary in order to have a final answer to the question whether or not classical F -theory geometry captures the

²⁰Note that these parity-odd gravitational \mathcal{R}^4 terms have an index structure different to that of the terms discussed in refs. [51].

relevant quantum effects for *all* types of F^4 couplings.

Another interesting point is the mathematical interpretation of the integer “instanton” coefficients g of the five-point couplings (1.4).²¹ In fact, a simple interpretation can be given for the expansion coefficients \tilde{g} of the complete prepotential where all 18 moduli are switched on. Specifically, noting that the factor E_4^2 in (2.5) can be attributed to the $E_8 \times E_8$ partition function, we are led to conclude that the complete, unique prepotential that contains all moduli is (up to a quintic polynomial, in a particular chamber) given by

$$\mathcal{G}(T, U, \vec{V}) = -\frac{i}{(2\pi)^5} \sum_{\substack{(k,l,\vec{r}) > 0 \\ \vec{r} \in \Lambda_{E_8 \times E_8}}} \tilde{g}(kl - \vec{r}^2/2) \times \mathcal{L}i_5[e^{2\pi i(kT+lU+\vec{r}\cdot\vec{V})}] \quad (4.1)$$

where \vec{V} are the $E_8 \times E_8$ Wilson lines and where \tilde{g} are the expansion coefficients of

$$\eta(q)^{-24} \equiv \frac{1}{q} \prod_{l \geq 1} (1 - q^l)^{-24} =: \sum_{n \geq -1} \tilde{g}(n) q^n. \quad (4.2)$$

The various prepotentials corresponding to different gauge groups (e.g. those of Table 1) can be obtained as specializations of (4.1), since they correspond to different expansions around special points in the moduli space; in particular, if we switch off all the $E_8 \times E_8$ Wilson lines, we recover the prepotential (2.3). That the η -function appears here as a counting function for $K3$ is perhaps not too surprising, in view of the fact that it counts 1/2-BPS states on $K3$ [52, 53, 54] (corresponding to singular rational curves that are holomorphic in a given, hyperkähler-compatible complex structure). Indeed it is known that the only contributions to the heterotic one-loop corrections to F^4 are due to 1/2-BPS states [3].

Clearly, we have followed in this paper a pragmatic physicist’s approach, but we are convinced that ultimately there should exist a much more direct route leading from $K3$ surfaces to the prepotential (4.1); this might be related to the considerations of ref. [55]. Understanding the map between open and closed string sectors more generally in terms of a mirror map may also be useful for the recently discovered relationships between gravity and gauge theory.

Acknowledgements

²¹Formally, these couplings look like the canonical holomorphic couplings associated with five-folds. It would be interesting to find out whether five-folds exist that reproduce these couplings, and if so, whether this would have any physical significance. A naive first guess would have been a fibration of $K3$ over \mathbb{P}^3 (in the limit of large \mathbb{P}^3), but, as we have checked, this does not work.

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Appendix A

Modular functions for $\Gamma_0(2)$ and generalized τ -integrals

A.1 Ring of Modular Forms

The fundamental region of $\Gamma_0(2)$ has two cusp points at $\tau = 0$ and $\tau = i\infty$. The ring of modular forms consists of two generators of weights 2 and 4, respectively:

$$G_2 = \theta_3^4 + \theta_4^4, \tag{A.1}$$

$$G_4 = (\theta_3^4 + \theta_4^4)^2 - 2\theta_2^8, \tag{A.2}$$

with the θ -functions:

$$\begin{aligned} \theta_2 &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2}, & \theta_3 &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2} \\ \theta_4 &= \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2}. \end{aligned} \tag{A.3}$$

The unique normalized Hauptmodul $\Phi_2(\tau)$ of $\Gamma_0(2)$ is given by

$$\Phi_2(\tau) = \left[\frac{\eta(\tau)}{\eta(2\tau)} \right]^{24} = q^{-1} - 24 + 276q - 2048q^2 + \dots \tag{A.4}$$

To derive (2.19), we have used the following identities:

$$\begin{aligned} q \frac{d}{dq} G_2 &= \frac{1}{6} E_2 G_2 + \frac{1}{24} G_2^2 - \frac{1}{8} G_4 \\ q \frac{d}{dq} G_4 &= \frac{1}{3} E_2 G_4 + \frac{1}{12} G_2 G_4 - \frac{1}{4} G_2^3. \end{aligned} \tag{A.5}$$

In addition, note

$$\begin{aligned} C_{D_4}^3 q \frac{d}{dq} \left(q \frac{d}{dq} C_{D_4} \right) &= \frac{1}{8 \cdot 36} \frac{1}{64} (G_4 + G_2^2)^2 \\ &\quad (G_2^2 - 12E_2 G_2 + 12E_2^2 + 3G_4 - 4E_4), \end{aligned} \tag{A.6}$$

as well as the relation between $(F_\alpha^2)^2$ - and $F_\alpha^2 F_\beta^2$ -charge insertions

$$3C_{D_4}^2 \left(q \frac{d}{dq} C_{D_4} \right)^2 = 2C_{D_4}^3 q \frac{d}{dq} \left(q \frac{d}{dq} C_{D_4} \right) + \frac{1}{4} \eta^{24} . \tag{A.7}$$

A.2. Generalized τ -integrals

In section 2.2 we had to deal with τ -integrals whose integrands are certain sums over Narain cosets as they appear from orbifold shifts; see e.g. the decomposition in (2.14) for a $\mathbb{Z}_2 \times \mathbb{Z}_2$ shifted lattice. After reducing the Narain lattice sum to (2.16), we could use techniques that were developed in [26] to perform the integrations. This method can be applied in general. Let us demonstrate this with the \mathbb{Z}_2 coset:

$$\mathcal{A}_{(h,g)} = \begin{pmatrix} n^1 + \frac{1}{2}h & l^1 + \frac{1}{2}g \\ n^2 & l^2 \end{pmatrix} , \quad h, g = 0, 1 . \tag{A.8}$$

Apart from the integral over the untwisted sector, which is of the kind treated in [7] (with the modular function $\mathcal{C}_0 - \mathcal{C}_1 - \mathcal{C}_2 - \mathcal{C}_3$), we have to do the integral²²

$$\int \frac{d^2 \tau}{\tau_2} \sum_{i=1,2,3} \nu_i \sum_{A_i} \overline{P_{R_i}}^{a-1} q^{\frac{1}{2}|P_{L_i}|^2} \overline{q}^{\frac{1}{2}|P_{R_i}|^2} \mathcal{C}_i(\tau) , \tag{A.9}$$

with $\nu_i = \text{vol}(\mathcal{N}_{2,2,i}) = \{1, \frac{1}{2}, \frac{1}{2}\}$, $A_1 = \{m_1 \in 2\mathbb{Z}, m_2, n^1, n^2 \in \mathbb{Z}\}$ and $A_2 = \{n_1 \in 2\mathbb{Z}, m_1, m_2, n^2 \in \mathbb{Z}\}$ etc. To perform the integration we may choose the orbits as in ref. [26]. Generically, in the non-degenerate orbit (and part of the degenerate orbit), the three sectors $i = 1, 2, 3$ sum up in the following combination:

$$\sum_{(k,l)>0} 2^{2-a} c_1(2kl) \mathcal{L}i_a(q_T^k q_U^{2l}) + 2^{2-a} c_2\left(\frac{kl}{2}\right) \mathcal{L}i_a(q_T^{k/2} q_U^l) + hc. . \tag{A.10}$$

E.g. for $a = 5$ this combination gives, up to a quintic polynomial, the prepotential for the model with $[E_7 \times SU(2)]^2$ gauge group (the second entry in Table 1). In this case $\mathcal{C}_1 = \frac{1}{32} \eta^{-24} (G_4 + G_2^2) G_2^2$, $T \rightarrow 2T$, $U \rightarrow U/2$ and again, $\mathcal{C}_0 = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3$. For $a = 1$, the trivial orbit provides in addition $\frac{\pi}{3} [c_1(0) + c_2(0) + c_3(0) - 24(c_1(-1) + c_2(-1) + c_3(-1))] \frac{T_2}{2}$, if \mathcal{C}_i contains no E_2 factor. The remaining part of the degenerate orbit may be easily calculated along ref. [7] with the orbits of ref. [26]. There are further non-harmonic contributions, if \mathcal{C}_i contain powers of E_2 . However, again those have the

²²These cosets appear in the \mathbb{Z}_8 -orbifold calculation of [26], where further details may be looked up.

structure (A.8) (with coefficients \tilde{c}_i and multiplied by powers T_2, U_2) and may be determined along refs. [7, 13] with the orbits of ref. [26].

Appendix B Mirror Map and $K3$ Periods for Example A

We like to determine the dependence of the $K3$ moduli α, β in (3.5) on the flat coordinates $s_i \equiv \{T, U\}$. For this, we closely follow the method detailed in [56], and introduce the period integrals

$$u_0 = \int \frac{q(s)}{P_{K3}} (\wedge dx_a)^4, \quad u_k = \int \frac{q(s)\phi_k}{P_{K3}^2} (\wedge dx_a)^4. \tag{B.1}$$

Here ϕ_k form a basis of the chiral ring (generated by the middle polynomials xz^4w^4 and z^6w^6), and $q(s)$ is a conformal rescaling factor to be determined. These periods satisfy the Gauß-Manin differential equation

$$\frac{\partial^2}{\partial s_i \partial s_j} u_0 = c_{ij}^k u_k + \Gamma_{ij}^k \frac{\partial}{\partial s_k} u_0. \tag{B.2}$$

In general, the conformal factor $q(s)$ is determined by requiring that (B.2) has no piece proportional to u_0 , while $\alpha(s), \beta(s)$ are determined by the vanishing of the connection Γ . More concretely, let us focus on the $\frac{\partial^2}{\partial^2 s_1}$ piece, and denote the denominators proportional to $(P_{K3})^{-\ell}$ on the RHS of (B.2.) by κ_ℓ :

$$\begin{aligned} \kappa_1 &= q''(s) \\ \kappa_2 &= -2q'(s)P_{K3}'(s, x_a) - q(s)P_{K3}''(s, x_a) \\ \kappa_3 &= 2q(s)(P_{K3}'(s, x_a))^2, \end{aligned} \tag{B.3}$$

where the prime denotes derivatives wrt. s_1 . As usual, the order of κ_3 as a polynomial in x_a can be reduced by expanding into ring elements modulo vanishing relations, and subsequently integrating by parts. The vanishing of the connection Γ then corresponds to the vanishing of the terms proportional to $1/(P_{K3})^2$, and this gives:

$$\begin{aligned} 0 &= 2\beta'(s)q'(s) + q(s)(-12\alpha(s)\beta(s)(\alpha'(s))^2 \\ &\quad + 36\beta(s)(\beta'(s))^2 + 108\beta''(s) + 4\alpha(s)^3\beta''(s) - 27\beta(s)^2\beta''(s)) \\ 0 &= 2\alpha'(s)q'(s) + q(s)(-6\alpha(s)^2(a'(s))^2 \\ &\quad + 18\alpha(s)(\beta'(s))^2 + 4\alpha(s)^3a''(s) - 27(\beta(s)^2 - 4)\alpha''(s)). \end{aligned}$$

Furthermore, the vanishing of the piece proportional to u_0 corresponds to the vanishing of the terms proportional to $1/(P_{K3})$, and this yields:

$$0 = 4(4\alpha(s)^3 - 27\beta(s)^2 + 108) q''(s) + (3\beta'(s)^2 - \alpha(s)\alpha'(s)^2) q(s) .$$

Similar equations are obtained by considering other second order derivatives on the LHS of eq. (B.2). One easily checks that the differential equations are solved by $q(s) = -\frac{1}{\eta^2(T)\eta^2(U)}$ and (3.6), indeed exactly as proposed in ref. [34].

Writing α, β in terms of J -functions (3.6) drastically simplifies the Picard-Fuchs equations associated with the $K3$ surface (3.5). These are given by

$$\begin{aligned} \mathcal{L}_1 &= \alpha + 48\alpha\beta\partial_\beta + 24\alpha^2\partial_\alpha + 48\alpha^2\beta\partial_\alpha\partial_\beta + 4(4\alpha^3 - 27\beta^2 + 108)\partial_\alpha^2 \\ \mathcal{L}_2 &= -3 - 144\beta\partial_\beta - 72\alpha\partial_\alpha - 144\alpha\beta\partial_\alpha\partial_b + 4(4\alpha^3 - 27\beta^2 + 108)\partial_\beta^2 . \end{aligned}$$

Then indeed, transforming to variables

$$\begin{aligned} J(T(\alpha, \beta)) &= -8 \left(4\alpha^3 + 27(-4 + \beta^2) \right) \\ &\quad - 8 \left(\sqrt{1728\alpha^3 + (4\alpha^3 + 27(-4 + \beta^2))^2} \right) \end{aligned} \tag{B.4}$$

$$\begin{aligned} J(U(\alpha, \beta)) &= 8 \left(108 - 4\alpha^3 - 27\beta^2 \right) \\ &\quad + 8 \left(\sqrt{1728\alpha^3 + (4\alpha^3 + 27(-4 + \beta^2))^2} \right) \end{aligned}$$

and rescaling the solutions, the PF system separates and has the following solutions:

$$\tilde{\omega}_i = J(T)^{1/12} J(U)^{1/12} \begin{pmatrix} f_1(J(T))f_1(J(U)) \\ f_2(J(T))f_1(J(U)) \\ f_1(J(T))f_2(J(U)) \\ f_2(J(T))f_2(J(U)) \end{pmatrix}, \quad i = 0, \dots, 3, \tag{B.5}$$

where

$$\begin{aligned} f_1(J) &= {}_2F_1(1/12, 1/12, 2/3, J/1728) \\ f_2(J) &= (J/1728)^{1/3} {}_2F_1(5/12, 5/12, 4/3, J/1728) . \end{aligned} \tag{B.6}$$

Dividing out $\tilde{\omega}_0$ these solutions give the geometrical integral $K3$ periods $\varpi_i = \{1, T, U, TU\}(\alpha, \beta)$ up to linear combinations. Specifically, in terms of analytically continued functions, we find for the fundamental geometric period

$$\begin{aligned} \omega_0 &= {}_2F_1(1/12, 5/12, 1, 1728/J(T)) \times {}_2F_1(1/12, 5/12, 1, 1728/J(U)) \\ &= E_4(T)^{1/4} E_4(U)^{1/4}, \end{aligned} \tag{B.7}$$

exactly as for the well-known $K3$ of type $X_{12}[1, 1, 4, 6]$ [57, 44]. Indeed the PF systems of the two $K3$'s can be transformed into each other, reflecting the universality of the T, U subsector of the moduli space. That this must be so also follows from the structure of the relevant toric polyhedra.

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