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On The Dynamics of Einstein's Equations in The Ashtekar Formulation*

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Abstract

We study the dynamics of Einstein's equations in Ashtekar's variables from the point of view of the theory of hyperbolic systems of evolution equations. We extend previous results and show that by a suitable modification of the Hamiltonian vector flow outside the sub-manifold of real and constrained solutions, a symmetric hyperbolic system is obtained for any fixed choice of lapse-shift pair, without assuming the solution to be a priori real. We notice that the evolution system is block diagonal in the pair (σ^a, A_b), and provide explicit and very simple formulae for the eigenvector-eigenvalue pairs in terms of an orthonormal tetrad with one of its components pointing along the propagation direction. We also analyze the constraint equations and find that when viewed as functions of the extended phase space they form a symmetric hyperbolic system on their own. We also provide simple formulae for its eigenvectors-eigenvalues pairs.

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1 Introduction

A substantial effort has been undertaken in the last decade to understand what a canonical quantization of space-time would mean in terms of self-dual variables and the corresponding loop representation it implies. Most of the effort has been geared to the study of the constraint equations and their algebra, while little has been done about the evolution equations, in the understanding that, since the Hamiltonian of the theory is a linear combination of the constraints, the information contained in their algebra is all what is needed.

Nevertheless we believe that the study of the evolution equations as such should be of relevance for a better understanding of the constraint algebra, for it might assert that it has very important properties which might otherwise be very difficult to recognize, namely that of giving rise to a well posed initial value formulation, that is, a theory in which one could predict the future based on knowledge gathered at an initial Cauchy surface.

It is not clear a priori that a classical well posed initial value formulation is essential for quantization. However all physically interesting quantum theories we know of correspond to well posed classical systems, and this fact is implicitly used in most approximation schemes.

The well posedness of the initial value problem for the classical theory in the usual tensorial variables was asserted in the fifties by Choquet-Bruhat[1] using a particular reduction of the equations in the harmonic gauge. Since then, a number of other formulations of the theory as symmetric hyperbolic systems have appeared (see [1, 2, 3, 4, 5, 6, 7] and for an updated review with a more complete reference list see [8]), mainly aiming to gain some advantages in dealing with certain specific problems, such as asymptotic studies, global existence, the Newtonian limit and numerical simulations of fully general relativistic configurations.

It is not clear that the study of the dynamics of Einstein's equations in Ashtekar's variables as a symmetric hyperbolic system would have an important impact on applications such as those mentioned above, but its intrinsic beauty, simplicity, and economy can not be in vane.

In dealing with first order quasi-linear systems of evolution equations, –as Einstein's equations naturally are in Ashtekar's variables– a sufficient condition for well posedness is symmetric hyperbolicity, that is, if the system can be written in the form:

$$M^0(u)u_t := M^a(u)\nabla_a u + B(u)u \tag{1}$$

where u is a "vector" and M^0, M^a and B are matrices which have smooth dependence on u and, in addition M^0 and M^a are symmetric and M^0 is positive definite [1]. This condition is the one of more immediate use, in particular for numerical simulations, for it enables one to estimate the growth of the solution in terms of conventional norms; therefore it is always worth checking whether or not it holds. If a system does not have this property it does not mean that its associated initial value formulation is not well posed, for the necessary condition for well posedness is a weaker condition, namely strong hyperbolicity [2], that is, that the eigenvalues of the principal symbol, $iM^a(u)k_a$ be purely imaginary and that it has a complete set of eigenvectors, all of them smooth in their dependence on u and k_a. It turns out that the Ashtekar's system can be extended outside the constraint and reality sub-manifold of the phase space in such a way that both hyperbolicity conditions hold.

In determining whether a formulation of general relativity is well posed or not there usually arises the problem of constraints, since we can not locally separate the fields into free and into dependent ones with respect to the constraints. We have to think on the evolution equations not only as giving the dynamics at the constraint sub-manifolds, but also as evolution equations for the complete fields, that is, as equations also valid in a neighborhood of the constraint sub-manifold. But this makes those equations non-uniquely determined; all dynamics which at the constraint sub-manifold give the same evolution vector field are valid, and therefore equivalent. Thus, if we modify the equations, by adding to them terms proportional to the constraints, we obtain equivalent systems of equations. Thus, in order to show well posedness one only has to search for one of these equivalent systems of equations in which the above conditions hold.

In the Ashtekar formulation of general relativity a further complication arises because besides the constraints there are reality conditions. Along evolution the soldering forms are in general not anti-hermitian, even if they started that way at the initial surface, so without assuming very stringent evolution gauge conditions or imposing elliptic gauges, one can only require the metric reality conditions along evolution and since these conditions are non-linear in terms of the basic dynamical variables, it becomes necessary to deal with them as if they were a new set of constraints.

In a recent paper [9], the authors have answered the question of

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[1] For a short introduction into the topic see [8] and references therein.

[2] The name strong comes because it is stronger than the primitive concept of hyperbolicity, namely that the eigenvalues of the associated symbol be purely imaginary, but it is a weaker condition than symmetric hyperbolicity.

hyperbolicity by finding a representative within the equivalent class of evolution equations and showing that it is a symmetric hyperbolic system. In that paper it was assumed that the soldering form was anti-hermitian, restricting unnecessarily the set of gauge conditions (choices of lapse-shift pair and of the $\mathbf{SU}(2)$ rotation gauge) for which symmetric hyperbolicity holds. That restriction can be easily lifted to allow for all soldering forms to be $\mathbf{SL}(2,\mathbf{C})$ rotated to an anti-hermitian one, that is, for all soldering forms satisfying the metric reality condition. We do this in the next section, (§2).

Unfortunately this lifting is not enough to grant the well posedness of the system, because a small departure from the reality sub-manifold would imply an immediate and strong instability for certain perturbations. Thus, the usual methods of constructing the solution via a contraction map of successive approximations can not be used, nor can one use these equations for numerical simulations. To overcome this problem a further modification to the evolution equations is needed, this time outside the reality condition sub-manifold, which makes the system symmetric-hyperbolic in a whole neighborhood of that sub-manifold. We study this modification in §3. In order to distinguish it from the prior modifications above mentioned, we shall call it *flow regularization*.

We then study, in §4, the diagonalization problem for the principal symbol of the system, that is, the issue of strong hyperbolicity and of the propagation speeds of perturbations. In particular we display remarkably simple expressions for the different eigenvalues-eigenvectors of the system. We conclude that the system can not be extended outside the reality condition sub-manifold in order to make it strongly hyperbolic there, keeping at the same time unchanged the simple dependence of the characteristic eigenvectors on the solution. On the other hand one can modify the equation into a strongly hyperbolic pseudo-differential system, the advantage here being that the characteristics of the system remain very simple.

In §5 we study the issue of the constraint and reality conditions propagation, showing that the system stays on those sub-manifolds if initially so. We show in particular that the constraints propagate also via a symmetric hyperbolic system. The diagonalization of this system is also remarkably simple.

2 Symmetric Hyperbolicity

2.1 Earlier Results

In a recent paper [9], the authors have shown that the Einstein's equations in Ashtekar's variables constitute a symmetric hyperbolic system. In that paper it was assumed that the soldering form was anti-hermitian [3], restricting unnecessarily the set of gauge conditions for which symmetric hyperbolicity holds.

In order to obtain a symmetric hyperbolic system for any given lapse-shift pair, we suitably extended the field equations outside the constraint sub-manifold. Since this extension only involves the addition of terms proportional to the constraints, we know that the evolution vector field is tangent to the constraint sub-manifold, and so the physically relevant evolution, that is, the dynamics inside this sub-manifold, remains unchanged.

For completeness, we repeat here part of the calculations made in [9], showing in addition that the above extension is unique.

It is known ([10], see appendix 6 for a brief overview) that, using the Hamiltonian formulation to determine the dynamics of the pair $(\tilde{\sigma}^a, A_b)$, Einstein's equations can be written as

$$\begin{array}{rcl} \mathcal{L}_t\tilde{\sigma}^b &=& \mathcal{L}_{\vec{N}}\tilde{\sigma}^b - \frac{i}{\sqrt{2}}\mathcal{D}_a(\underset{\sim}{N}[\tilde{\sigma}^a, \tilde{\sigma}^b]), \\ \mathcal{L}_t A_b &=& \mathcal{L}_{\vec{N}} A_b + \frac{i}{\sqrt{2}}\underset{\sim}{N}[\tilde{\sigma}^a, F_{ab}], \\ C(\tilde{\sigma}, A) &:=& \mathrm{tr}(\tilde{\sigma}^a\tilde{\sigma}^b F_{ab}) = 0, \\ C_a(\tilde{\sigma}, A) &:=& \mathrm{tr}(\tilde{\sigma}^b F_{ab}) = 0, \\ \tilde{C}_A{}^B(\tilde{\sigma}, A) &:=& \mathcal{D}_a\tilde{\sigma}^a{}_A{}^B = 0, \end{array} \tag{2}$$

Since the evolution equations have a block diagonal principal part, that is, in the time evolution for $\tilde{\sigma}^a$ (A_a) there only appear space derivatives of $\tilde{\sigma}^a$ (respectively A_a), the desired modification is of the following form: [4]

[3]Since we are using intensely the matrix algebra of the soldering forms we refer to them as anti-hermitian in the matrix sense, in the abstract sense, as defined in [10] they are actually hermitian.

[4]One can always add to these equations *given* $\mathbf{SU}(2, \mathbf{C})$ gauge terms in the usual way without affecting hyperbolicity, for if they do not depend on $\tilde{\sigma}^a$, nor on A_a, they do not enter the principal part.

$$
\begin{aligned}
\mathcal{L}_t \tilde{\sigma}^b &= \mathcal{L}_{\vec{N}} \tilde{\sigma}^b - \frac{i}{\sqrt{2}} \mathcal{D}_a(\underset{\sim}{N}[\tilde{\sigma}^a, \tilde{\sigma}^b]) + \alpha[\tilde{C}, \tilde{\sigma}^b], && (3)\\
\mathcal{L}_t A_b &= \mathcal{L}_{\vec{N}} A_b + \frac{i}{\sqrt{2}}\, \underset{\sim}{N}[\tilde{\sigma}^a, F_{ab}] + \beta\, \tilde{\sigma}_b C + \gamma\, \epsilon_b{}^{dc} \tilde{\sigma}_c C_d. && (4)
\end{aligned}
$$

where α, β and γ are complex functions to be determined so that the system becomes symmetric-hyperbolic, meaning that the principal symbol becomes anti-hermitian with respect to the canonical inner product

$$
\begin{aligned}
\langle u_2, u_1 \rangle &\equiv \langle (\tilde{\sigma}_2, A^2), (\tilde{\sigma}_1, A^1) \rangle \\
&\equiv \operatorname{tr}(\tilde{\sigma}_2^{\dagger a} \tilde{\sigma}_1^b)\, q_{ab} + \operatorname{tr}(A^{2\dagger}{}_a A^1{}_b)\, q^{ab},
\end{aligned}
$$

and considering the soldering forms, $\tilde{\sigma}^a$ anti-hermitian.

Recall that the principal symbol of a quasi-linear evolution equation system

$$\dot{u}^l = B^l{}_j{}^a(u) \nabla_a u^j + M^l(u)$$

is given by $P(u, ik_a) = i\; B^l{}_j{}^a(u) k_a$. In our case u denotes $u = (\sigma^a, A_b)$. Thus we need to prove that

$$P_{12} + P^\dagger{}_{12} \equiv \langle u_2, P u_1 \rangle + \langle u_2, P^\dagger u_1 \rangle = 0.$$

Using the fact that any complex matrix can be written as $u = -\frac{1}{\sigma^2} \operatorname{tr}(u \tilde{\sigma}_e) \tilde{\sigma}^e$ and that $[\tilde{\sigma}^a, \tilde{\sigma}^b] = \sigma\sqrt{2} \varepsilon^{ab}{}_c \tilde{\sigma}^c$, the principal symbol can be written as

$$
\begin{aligned}
P_\sigma(\tilde{\sigma}, ik_a) \tilde{\sigma}_1{}^b &= -i \frac{k_a N^a}{\sigma^2} \operatorname{tr}(\tilde{\sigma}_1{}^b \tilde{\sigma}_e) \tilde{\sigma}^e - \frac{k_a}{\sigma}\, \underset{\sim}{N} \operatorname{tr}(\tilde{\sigma}_1{}^b \tilde{\sigma}^e) \varepsilon^a{}_{ec} \tilde{\sigma}^c \\
&\quad - \frac{i\sqrt{2}}{\sigma} \left(\alpha - \frac{i}{\sqrt{2}} \underset{\sim}{N} \right) k_a \operatorname{tr}(\tilde{\sigma}_1{}^a \tilde{\sigma}_e) \varepsilon^{eb}{}_d \tilde{\sigma}^d, \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
P_A(\tilde{\sigma}, ik_c)\, A^1{}_b &= -i \frac{k_a N^a}{\sigma^2} \operatorname{tr}(A_{1b} \tilde{\sigma}_e) \tilde{\sigma}^e + \frac{2}{\sigma} \underset{\sim}{N} k_{[a} \operatorname{tr}(A^1_{b]} \tilde{\sigma}_e) \varepsilon^{ae}{}_c \tilde{\sigma}^c \\
&\quad + i\; \beta \sqrt{2} \sigma \varepsilon^{cde} k_c \tilde{\sigma}_b\, \operatorname{tr}(\tilde{\sigma}_e\; A^1_d) + i2\; \gamma \epsilon_b{}^{dc} \tilde{\sigma}_c\, \operatorname{tr}(\tilde{\sigma}^a\; k_{[d} A^1_{a]}) \qquad (6)
\end{aligned}
$$

A very simple calculus shows, using the inner product above and the anti-hermiticity of the background soldering form, that

$$P_{\sigma 12} + P^\dagger{}_{\sigma 12} = -i \frac{k_a N^a}{\sigma^2} \operatorname{tr}(\tilde{\sigma}_1{}^b \tilde{\sigma}_e) \operatorname{tr}(\tilde{\sigma}^\dagger_{2b} \tilde{\sigma}^e) + i \frac{k_a N^a}{\sigma^2} \operatorname{tr}(\tilde{\sigma}_2{}^{\dagger b} \tilde{\sigma}_e) \operatorname{tr}(\tilde{\sigma}_{1b} \tilde{\sigma}^e)$$

$$-\frac{1}{\sigma}\underset{\sim}{N}\ k_a\ \mathrm{tr}(\tilde{\sigma}_1{}^b\tilde{\sigma}_e)\varepsilon^{ae}{}_c\ \mathrm{tr}(\tilde{\sigma}^\dagger_{2b}\tilde{\sigma}^c) - \frac{1}{\sigma}\underset{\sim}{N}\ k_a\ \mathrm{tr}(\tilde{\sigma}_{1b}\tilde{\sigma}^e)\varepsilon^{ac}{}_e$$
$$\times\mathrm{tr}(\tilde{\sigma}_2{}^{\dagger b}\tilde{\sigma}_c) - \frac{i\sqrt{2}}{\sigma}(\alpha - \frac{i}{\sqrt{2}}\underset{\sim}{N})\ k_a\ \mathrm{tr}(\tilde{\sigma}_1{}^a\tilde{\sigma}_e)\varepsilon^{eb}{}_d\ \mathrm{tr}(\tilde{\sigma}^\dagger_{2b}\tilde{\sigma}^d)$$
$$+\frac{i\sqrt{2}}{\sigma}(\bar{\alpha} + \frac{i}{\sqrt{2}}\underset{\sim}{N})\ k_a\ \mathrm{tr}(\tilde{\sigma}_{1b}\tilde{\sigma}^d)\varepsilon^{eb}{}_d\ \mathrm{tr}(\tilde{\sigma}_2{}^{\dagger a}\tilde{\sigma}_e). \tag{7}$$

The sum of the first and the second term vanishes because of the symmetry of the background metric used in the inner product, the sum of the third and the fourth term vanishes because of the anti-symmetry of ε^{abc}, finally the last two terms should vanish in order to get the desired result, but it is impossible for the sum to vanish because there are different contractions on the indices and $\tilde{\sigma}_1$ is independent of $\tilde{\sigma}_2$, thus, the only way for these terms to vanish is that each one of them do so separately. This is obtained requiring

$$\alpha = \frac{i}{\sqrt{2}}\underset{\sim}{N}.$$

Next we calculate

$$P_{A12} + P^\dagger{}_{A12} = -i\frac{k_aN^a}{\sigma^2}\mathrm{tr}(A^1{}_b\tilde{\sigma}^e)\mathrm{tr}(A^{\dagger 2b}\tilde{\sigma}_e) + i\frac{k_aN^a}{\sigma^2}\mathrm{tr}(A^{1b}\tilde{\sigma}^e)$$
$$\times\mathrm{tr}(A^{\dagger 2}{}_b\tilde{\sigma}_e) + \underset{\sim}{N}\ \frac{k_a}{\sigma}\varepsilon^{aem}\mathrm{tr}(A^1{}_b\tilde{\sigma}_e)\mathrm{tr}(A^{\dagger 2b}\tilde{\sigma}_m) + \underset{\sim}{N}\ \frac{k_a}{\sigma}\varepsilon^{aem}\mathrm{tr}(A^{1b}\tilde{\sigma}_m)$$
$$\mathrm{tr}(A^{\dagger 2}{}_b\tilde{\sigma}_e) + i\beta\sqrt{2}\sigma\varepsilon^{cde}\ \mathrm{tr}(A^{\dagger 2b}\tilde{\sigma}_b)k_c\ \mathrm{tr}(A^1{}_d\tilde{\sigma}_e) - i\bar{\gamma}k_d\varepsilon^{dc}{}_b\ \mathrm{tr}(A^{\dagger 2a}\tilde{\sigma}_a)$$
$$k_c\mathrm{tr}(A^{1b}\tilde{\sigma}_c) + i\gamma\varepsilon^{dc}{}_bk_d\ \mathrm{tr}(A^{\dagger 2b}\tilde{\sigma}_c)\ \mathrm{tr}(A^{1a}\tilde{\sigma}_a) - i\bar{\beta}\sqrt{2}\sigma\varepsilon^{cde}k_c\ \mathrm{tr}(A^{\dagger 2}{}_d\tilde{\sigma}_e)$$
$$\mathrm{tr}(A^{1b}\tilde{\sigma}_b) - \underset{\sim}{N}\ \frac{k_b}{\sigma}\varepsilon^{apn}\ \mathrm{tr}(A^{\dagger 2b}\tilde{\sigma}_n)\ \mathrm{tr}(A^1{}_a\tilde{\sigma}_p) - i\gamma\varepsilon_a{}^{dp}k_b\ \mathrm{tr}(A^{\dagger 2a}\tilde{\sigma}_p)$$
$$\mathrm{tr}(A^1{}_d\tilde{\sigma}^p) - \underset{\sim}{N}\ \frac{k_b}{\sigma}\varepsilon^{apn}\ \mathrm{tr}(A^{\dagger 2}{}_a\tilde{\sigma}_p)\ \mathrm{tr}(A^{1b}\tilde{\sigma}_n) + i\bar{\gamma}\varepsilon_b{}^{dc}k_a\ \mathrm{tr}(A^{\dagger 2}{}_d\tilde{\sigma}^a)$$
$$\mathrm{tr}(A^{1b}\tilde{\sigma}_c). \tag{8}$$

The sum of the first and the second term vanishes because of the symmetry of the metric, the third and fourth terms vanish because of the anti-symmetry of ε^{abc}. The next four terms give rise to

$$i\ \mathrm{tr}(A^{2\dagger b}\tilde{\sigma}_b)\ \mathrm{tr}(A^1{}_d\tilde{\sigma}_e)\varepsilon^{cde}k_c(\beta\sqrt{2}\sigma + \bar{\gamma}) + i\ \mathrm{tr}(A^{2\dagger}{}_d\tilde{\sigma}_e)\ \mathrm{tr}(A^1{}_a\tilde{\sigma}^a)\varepsilon^{dce}k_c$$
$$\times(\bar{\beta}\sqrt{2}\sigma + \gamma)$$

and there are no other terms of the same kind, so they must vanish. The only way (since A^1 and A^2 are independent) this can happens is if

$$\beta\sqrt{2}\sigma = -\bar{\gamma}.$$

Finally the last four terms vanish if we use

$$2A^{[a}\tilde{\sigma}^{b]} = \varepsilon^{abe}\varepsilon_{dme}A^d\tilde{\sigma}^m,$$

and choose

$$\gamma = -\frac{i}{\sigma}\underset{\sim}{\mathrm{N}}, \qquad \text{and consequently} \qquad \beta = -\frac{i}{\sigma^2\sqrt{2}}\underset{\sim}{\mathrm{N}}.$$

2.2 Lifting the Anti-Hermiticity Condition

In the above calculations it was assumed that $\tilde{\sigma}^a$ was anti-hermitian. A simple modification of the scalar product into a "rotated" one, allows to conclude that the equations are still symmetric hyperbolic (w.r.t. the new scalar product) even when the soldering forms are not anti-hermitian, as long as they can be transformed into ones that are so by a $\mathbf{SL}(2, \mathbf{C})$ transformation, that is, as long as the metric they generate is real and positive definite.

Indeed, given $\tilde{\sigma}^a$ such that the metric it produces is real, there exist $\mathbf{SL}(2, \mathbf{C})$ transformations $L(\tilde{\sigma}^a)$ such that $\hat{\sigma}^a := L^{-1}\tilde{\sigma}^a L$ is anti-hermitian. (See Appendix 6 for a procedure to construct one.) Using any one of these transformations we define the following scalar product:

$$\begin{aligned} \langle u_2, u_1\rangle_{H(u)} &\equiv \langle H^{-1}(\tilde{\sigma})(\tilde{\sigma}_2, A^2), H(\tilde{\sigma})(\tilde{\sigma}_1, A^1)\rangle \qquad (9)\\ &\equiv \mathrm{tr}((L^{-1}\tilde{\sigma}_2^a L)^\dagger\, L^{-1}\tilde{\sigma}_1^b L)\, q_{ab} + \mathrm{tr}((L^{-1}A_a^2 L)^\dagger\, L^{-1}A_b^1 L)\, q^{ab}, \qquad (10) \end{aligned}$$

where the linear operator $H(u)$ is defined through L as in the second step. Thus, when computing

$$P_{12H} + P^\dagger{}_{12H} \equiv \mathrm{tr}((L^{-1}u_2L)^\dagger L^{-1}Pu_1L) + \mathrm{tr}((L^{-1}Pu_2L)^\dagger L^{-1}u_1L),$$

we obtain terms with the structure,[5]

$$\begin{aligned} &\mathrm{tr}((L^{-1}u_2L)^\dagger L^{-1}(L\hat{\sigma}L^{-1})L\mathrm{tr}(u_1L\hat{\sigma}L^{-1})) = \\ &\mathrm{tr}((L^{-1}u_2L)^\dagger\hat{\sigma})\mathrm{tr}(L^{-1}u_1L\hat{\sigma}), \end{aligned}$$

that is, with the same structure as the one in the calculation in [9], and above, if one substitutes u_1, and u_2 by their rotated versions, $L^{-1}u_1L$, and $L^{-1}u_2L$, and correspondingly $\tilde{\sigma}^c$ by $\hat{\sigma}^c$. Since $\hat{\sigma}^c$ is by construction anti-hermitian, symmetric-hyperbolicity follows for the extended system from the same calculations as in the previous subsection.

As argued below, symmetric hyperbolicity in the reality condition submanifold does not seems to be enough for the usual proof of well posedness

[5]We consider the case $N^a = 0$, for the part proportional to it is diagonal, hence symmetric, without any conditions on $\tilde{\sigma}^a$.

to work. Thus, a further modification of the equations is needed to obtain a set of equations which are symmetric hyperbolic in a whole neighborhood of the reality conditions sub-manifold. We do this in the next section.

3 Flow Regularization

Although, as we shall prove in section 5, the evolution respects the reality conditions (and therefore solutions whose initial data sets satisfy them, have a metric which stays real along evolution), this property is not sufficient to conclude that the system is well posed. Indeed, all schemes used to prove existence of solutions rely upon sequences of intermediate equations and their corresponding solutions, which are shown to form a contractive map, and hence to converge to the exact solution. These intermediate equations are in general linear versions of the original equations, but with the principal part evaluated at earlier approximated solutions. The problem is that this intermediate solutions do not propagate correctly the reality conditions and so one has to solve approximated systems in a neighborhood of the reality condition sub-manifold, but there the system is not symmetric hyperbolic, and so its eigenvalues have non-zero real part. Thus, the intermediate equations unstable, and so if intermediate solutions exist at all their norm can not be properly estimated.

This problem is overcome by further modifying the evolution equations outside the reality condition in such a way to obtain a symmetric hyperbolic system even outside of it. Then standard contracting maps schemes for proving existence of solutions can then be applied to obtain local well posedness.

Here we present a detailed description of the required extension to the flow in order to obtain a symmetric hyperbolicity in a whole neighborhood of the reality condition sub-manifold.

If the metric is real and positive definite, that is, if it lies in the reality condition sub-manifold, then to prove symmetric-hyperbolicity one can simply use the fact that there is an $\mathbf{SL}(2, \mathbf{C})$ transformation which makes σ anti-hermitian. Outside the reality sub-manifold no extension of the transformation would give an anti-hermitian soldering form, so the system there is not symmetric-hyperbolic without a further modification.

The extra modification of the evolution equations consists first in extending the $\mathbf{SL}(2, \mathbf{C})$ transformation outside the reality condition sub-manifold, with properties which ensure that when the reality conditions are satisfied the resulting $\tilde{\sigma^a}$ would be anti-hermitian; we present in detail one way to achieve this in Appendix 6 (the procedure is not unique). The second step is to change everywhere in the principal symbol of the system $\tilde{\sigma^a}$ by $L\hat{\sigma}^a L^{-1} := \frac{1}{2}(L(L^{-1}\tilde{\sigma}^a L - (L^{-1}\tilde{\sigma}^a L)^\dagger)L^{-1})$. At the reality sub-manifold we re-obtain $\tilde{\sigma}^a$ but, outside of it, the equations differ. Notice also that $\hat{\sigma}^a$ is, by construction, anti-hermitian. Therefore we claim that the system is everywhere symmetrizable, with symmetrizer $H(\tilde{\sigma}) := (L^{-1})^\dagger L^{-1}$ to the left and H^{-1} to the right (notice that H is symmetric and positive definite everywhere, since $L \in \mathbf{SL}(2,\mathbf{C})$), that is, with respect to the following scalar product:

$$\begin{aligned}\langle u_2, u_1\rangle_{H(u)} &\equiv \langle H^{-1}(\tilde{\sigma})(\tilde{\sigma}_2, A^2), H(\tilde{\sigma})(\tilde{\sigma}_1, A^1)\rangle \\ &\equiv \mathrm{tr}((L^{-1}\tilde{\sigma}_2^a L)^\dagger\, L^{-1}\tilde{\sigma}_1^b L)\,\hat{q}_{ab} + \mathrm{tr}((L^{-1}A_a^2 L)^\dagger\, L^{-1}A_b^1 L)\,\hat{q}^{ab},\end{aligned}$$

where $\hat{q}^{ab}$ is the metric constructed out of $\hat{\sigma}^a$, and therefore real. We shall denote with a ($\hat{}$) tensors which are constructed with $\hat{\sigma}$. Since they are $\mathbf{SL}(2,\mathbf{C})$ scalars they are also the ones constructed with $L\hat{\sigma}^a L^{-1}$. On the extended equations we shall raise and lower indices with $\hat{q}^{ab}$ and its inverse.

Again, we consider just the case $N^a = 0$, for the part proportional to it is diagonal, hence symmetric, without any conditions on $\tilde{\sigma}^a$. Recalling the expressions for the principal symbols,

$$P_\sigma(\tilde{\sigma}, ik_a)\tilde{\sigma}_1{}^b = -\frac{k_a}{\sigma}\,\underset{\sim}{N}\mathrm{tr}(\tilde{\sigma}_1{}^b\tilde{\sigma}^e)\varepsilon^a{}_{ec}\tilde{\sigma}^c, \tag{11}$$

$$\begin{aligned}P_A(\tilde{\sigma}, ik_c)\, A^1{}_b &= \frac{2}{\sigma}\underset{\sim}{N}k_{[a}\mathrm{tr}(A^1_{b]}\tilde{\sigma}_e)\varepsilon^{ae}{}_c\tilde{\sigma}^c \\ &\quad + \underset{\sim}{N}\varepsilon^{cde}\frac{k_c}{\sigma}\tilde{\sigma}_b\,\mathrm{tr}(\tilde{\sigma}_e\, A^1_d) + \frac{2}{\sigma}\underset{\sim}{N}\,\varepsilon_b{}^{dc}\tilde{\sigma}_c\,\mathrm{tr}(\tilde{\sigma}^a\, k_{[d}A^1_{a]}).\end{aligned} \tag{12}$$

Once modified, they become

$$\hat{P}_\sigma(\tilde{\sigma}, ik_a)\tilde{\sigma}_1{}^b = -\frac{k_a}{\hat{\sigma}^4}\,\underset{\sim}{N}\mathrm{tr}(\tilde{\sigma}_1^b L\hat{\sigma}^e L^{-1})\hat{\varepsilon}^a{}_{ec}L\hat{\sigma}^c L^{-1}, \tag{13}$$

and

$$\begin{aligned}\hat{P}_A(\tilde{\sigma}, ik_c)\, A^1{}_b &= \frac{2}{\hat{\sigma}}\underset{\sim}{N}k_{[a}\mathrm{tr}(A^1_{b]}L\hat{\sigma}_e L^{-1})\hat{\varepsilon}^{ae}{}_c L\hat{\sigma}^c L^{-1} \\ &\quad + \underset{\sim}{N}\hat{\varepsilon}^{cde}\frac{k_c}{\hat{\sigma}}L\hat{\sigma}_b L^{-1}\,\mathrm{tr}(L\hat{\sigma}_e L^{-1}\, A^1_d) + \frac{2}{\hat{\sigma}}\underset{\sim}{N}\,\hat{\varepsilon}_b{}^{dc}L\hat{\sigma}_c L^{-1} \\ &\quad \times\mathrm{tr}(L\hat{\sigma}^a L^{-1}\, k_{[d}A^1_{a]}).\end{aligned} \tag{14}$$

Which is identical to the one used in the previous section, but now $\hat{\sigma}^c$ is by construction anti-hermitian even outside the reality condition sub-manifold, therefore $L\hat{\sigma}^a L^{-1}$ can be trivially rotated into an anti-hermitian one, and so symmetric-hyperbolicity follows as before. With this modification the standard proof of well posedness of symmetric-hyperbolic systems apply.

Main Result: *If we assume that data, (σ_0^a, A_a^0), is given at a initial surface Σ_0 such that it belongs (locally) to the Sobolev space $H^s(\Sigma_0) \times H^{s-1}(\Sigma_0)$, $s \geq 3$, and such that $\tilde{q}^{ab}$ and its first time derivative at Σ_0 are very close to be real, (that is, we start with initial data in a sufficiently small neighborhood of the reality sub-manifold) then a local solution exists and it stays in these spaces along the generated foliation.*

4 Characteristic Directions and Strong Hyperbolicity

It is of interest to see whether there are simpler modifications to the equation system outside the constraint sub-manifold which would give us well posedness (strong-hyperbolicity) but with a system not necessarily symmetric-hyperbolic. In doing this we shall calculate the complete set of eigenvectors with their respective eigenvalues.

Recalling the expressions for the principal symbols of the system, (11), (12), we see that it is convenient to use at each point an orthogonal triad $\{k_a, m_a^+, m_a^-\}$ since we want to extend the results to a neighborhood of the reality condition sub-manifold. Let k^a –the wave vector– be real, with $|k_a k^a| = 1$ and let the other two vectors are taken to be null and orthogonal to k_a. They are normalized such that, $m_a^+ m_b^- q^{ab} := k^{-1} := (\sqrt{k_a k^a})^{-1}$, and chosen so that when the metric becomes real $\overline{m_a^+} = m_a^-$ (complex null vectors). Note in particular that we have the relation:

$$\varepsilon_{abc} = i6k_{[a}m_b^+ m_{c]}^-$$

Thus the symbol can be diagonalized in blocks with the following eigenvalues

$$\lambda_0 = i\, k_a N^a \qquad \lambda_+ = i(\, k_a N^a + \underset{\sim}{N} k \underset{\sim}{q}) \qquad \lambda_- = i(\, k_a N^a - \underset{\sim}{N} k \underset{\sim}{q})$$

and the subspaces associated with the above eigenvalues are

$$E_\sigma{}^0 \;=\; \mathrm{Span}\{k_a\, k_d \tilde{\underset{\sim}{q}}^d, m_a^+\, k_d \tilde{\underset{\sim}{q}}^d, m_a^-\, k_d \tilde{\underset{\sim}{q}}^d\},$$

$$
\begin{aligned}
E_\sigma{}^{+} &= \mathrm{Span}\{k_a\, m^-_d\, \underset{\sim}{\tilde{\sigma}}^d, m^+_a\, m^-_d\, \underset{\sim}{\tilde{\sigma}}^d, m^-_a\, m^-_d\, \underset{\sim}{\tilde{\sigma}}^d\},\\
E_\sigma{}^{-} &= \mathrm{Span}\{k_a\, m^+_d\, \underset{\sim}{\tilde{\sigma}}^d, m^+_a\, m^+_d\, \underset{\sim}{\tilde{\sigma}}^d, m^-_a\, m^+_d\, \underset{\sim}{\tilde{\sigma}}^d\},\\
E_A{}^{0} &= \mathrm{Span}\{k_a\, k_d \underset{\sim}{\tilde{\sigma}}^d, k_a\, m^+_d\, \underset{\sim}{\tilde{\sigma}}^d, k_a\, m^-_d\, \underset{\sim}{\tilde{\sigma}}^d\},\\
E_A{}^{+} &= \mathrm{Span}\{m^+_a\, k_d \underset{\sim}{\tilde{\sigma}}^d, m^+_a\, m^+_d\, \underset{\sim}{\tilde{\sigma}}^d, m^+_a\, m^-_d\, \underset{\sim}{\tilde{\sigma}}^d\},\\
E_A{}^{-} &= \mathrm{Span}\{m^-_a\, k_d \underset{\sim}{\tilde{\sigma}}^d, m^-_a\, m^+_d\, \underset{\sim}{\tilde{\sigma}}^d, m^-_a\, m^-_d\, \underset{\sim}{\tilde{\sigma}}^d\},
\end{aligned}
$$

In order to clarify how we have computed these eigenvectors and eigenvalues, let us check this result for $A^1_b = m^+_b l_d k^d \ \in E^+_A$, with $l_a \in \{k_a, m^+_a, m^-_a\}$. Consider the case $N^a = 0$, for the part proportional to it is diagonal, then

$$
\begin{aligned}
& P_A(\tilde{\sigma}, ik_c)\ A^1{}_b = \frac{2}{\sigma}\underset{\sim}{N} k_{[a}\mathrm{tr}(A^1_{b]}\tilde{\sigma}_e)\varepsilon^{ae}{}_c\tilde{\sigma}^c \\
& +\underset{\sim}{N}\varepsilon^{cde}\frac{k_c}{\sigma}\tilde{\sigma}_b\ \mathrm{tr}(\tilde{\sigma}_e\ A^1_d) + \frac{2}{\sigma}\underset{\sim}{N}\ \varepsilon_b{}^{dc}\tilde{\sigma}_c\ \mathrm{tr}(\tilde{\sigma}^a\ k_{[d}A^1_{a]}) \\
& = \sigma\underset{\sim}{N}\ \left(2k_{[b}m^+_{a]}l_e\varepsilon^{ae}{}_c\tilde{\sigma}^c + 2k_{[a}m^+_{d]}l^d\varepsilon^a{}_{bc}\tilde{\sigma}^c - k_a m^+_d l_e\varepsilon^{ade}\tilde{\sigma}_b\right) \\
& = -\frac{ik}{2}\sigma\underset{\sim}{N}\ (2\epsilon_{ban}m^{+n}\epsilon^{ae}{}_c l_e\tilde{\sigma}^c + 2\epsilon_{adn}m^{+n}\epsilon^a{}_{bc}l^d\tilde{\sigma}^c \\
& -\epsilon_{adn}m^{+n}\epsilon^{ade}l_e\tilde{\sigma}_b) \\
& = ik\sigma\underset{\sim}{N}\ m^+_b l_d k^d.
\end{aligned} \tag{15}
$$

Note that, assuming the metric to be real and choosing the space-like foliation such that k_a becomes the normal to a time-like hypersurface and the shift vector tangent to it, then we see that for each $\tilde{\sigma}^a$ and A_a, there are three incoming characteristics, the same number of outgoing (as should be the case for gauges respecting time-direction symmetry) and three characteristics which move along the boundary.

Note that the eigenvectors span the whole space on the solution manifold, furthermore they are smooth functions on all arguments for $k_a \neq 0$. Furthermore, if the metric is real, that is, if we are in the reality condition sub-manifold, then the eigenvalues are purely imaginary, thus we have a strongly-hyperbolic system. Can one make a simpler flow regularization and get strongly hyperbolicity (a weaker condition than symmetric-hyperbolicity)? That is, can we get away with the L transformation? From the form of the eigenvalues it is clear that this is the case. Indeed, one can modify the system outside the reality condition sub-manifold in such a way that the eigenvalues are purely imaginary also there, but since in general k is complex, and depends on k_a, the needed modification transforms the differential equation system into a pseudo-differential one. We do not pursue this further because such a modification would not be practical for most applications one envisions.

5 The evolution of constraints and reality conditions

Once a suitable extended evolution system for general relativity is shown to be symmetric hyperbolic in a whole neighborhood of the constraint and reality conditions sub-manifold, then standard results can be applied and different results on well posedness follow, in particular local evolution is granted if the initial data is smooth enough. For instance, for the specific system at hand, if the initial data $(\tilde{\sigma}^a, A_a)$ is in $H^s(\Sigma_0) \times H^{s-1}(\Sigma_0)$, $s \geq 3$, then the solution remains for a finite time in the corresponding spaces along the generated foliation Σ_t. [6]

Given a solution in a foliation, we can identify via the lapse-shift pair any point on Σ_t for any t with a point at the initial surface, namely the points that lying in the integral curve of the four dimensional vector field, $t^a = Nn^a + N^a$ and so we can pull back to the initial surface the pair $(\tilde{\sigma}^a, A_a)$, thus in the initial surface a solution can be seen as a one parameter family of fields in $H^s(\Sigma_0) \times H^{s-1}(\Sigma_0)$.

The following geometrical picture of evolution emerges: we have an infinite dimensional manifold, K, of pairs of soldering forms and connections in a three dimensional manifold, Σ_0 belonging to $H^s(\Sigma_0) \times H^{s-1}(\Sigma_0)$. Given any lapse-shift pair we can generate an evolution, that after the pull back is just a one parameter family of pairs in K, that is, a curve on that manifold. The tangent vector to that integral curve is our twice modified evolution equation system. Of course not all these integral curves are solutions to Einstein equations, for they would not generally satisfy the constraint nor the reality conditions, which in fact form a sub-manifold of K, denoted P.

The relevant question is whether the integral curves which start at P remain on P, for they conform the true solutions to Einstein's equations. This would happens if and only if the tangent vector fields to the integral curves are themselves tangent to the sub-manifold P. In order to see that one can proceed as follows:

We let RE, and CE denote the reality conditions *quantities*, namely $\Im q^{ab}$, $\Im \pi_{ab}$, and the constraint equations *quantities*, namely $\tilde{C}$, C, and C^a respectively. [7] We smear these expressions out with smooth tensors and so

[6] For the local problem one takes (through the choice of lapse-shift pairs) the foliation to be such that all the constant time surfaces coincide at their boundaries, that is, a *lens shaped* domain of evolution.

[7] This is basically a coordinatization of a neighborhood of P in K.

obtain maps from the manifold K into the complex numbers,

$$RE_{f_{ab}}(\tilde{\sigma}, A) := \int_\Sigma f_{ab} \Im q^{ab} \, \mathrm{d}x^3 \tag{16}$$

$$RE_{f^{ab}}(\tilde{\sigma}, A) := \int_\Sigma f^{ab} \Im \pi_{ab} \, \mathrm{d}x^3 \tag{17}$$

$$CE_{\tilde{f}}(\tilde{\sigma}, A) := -i\sqrt{2} \int_\Sigma \mathrm{tr}(\tilde{f}\tilde{C}) \, \mathrm{d}x^3 \tag{18}$$

$$CE_{f}(\tilde{\sigma}, A) := -i\sqrt{2} \int_\Sigma f C \, \mathrm{d}x^3 \tag{19}$$

$$CE_{f_a}(\tilde{\sigma}, A) := -i\sqrt{2} \int_\Sigma f_a C^a \, \mathrm{d}x^3, \tag{20}$$

We thus see that the reality-conditions-constraint sub-manifold P can be defined as the intersection of zero level set of each of this infinite set of maps. If these maps are sufficiently differentiable so that their gradients are well defined, it is clear that the tangency condition of the evolution vector field is just the requirement that when contracted with the differentials of the above defined maps the result should vanish at points of P. But these contractions are just the smeared out version of the time derivatives of the reality and constraint equations. Thus, provided we have enough differentiability,[8] we only have to check that this time derivatives vanish at P.

Since these calculations need only be done at points of P, it only involves the original evolution vectors field, and so the standard results can be used, namely that the time derivative of the constraint equations, and the time derivative of the reality conditions vanish at P. The first result (see appendix2) follows from the constraint algebra calculated in [10]. The second one, basically, follows from the fact that the Ashtekar system is equivalent, up to terms proportional to the constraints, to Einstein's evolution equations. Since in that equivalence one does not use any hermiticity nor reality condition explicitly, Ashtekar's equations are in fact equivalent to *complexified* gravity that is to equations identical to Einstein's, but where the metric and the second fundamental form on each slice can be complex. Thus, at points where both the metric and the second fundamental form are real, and the constraints are satisfied, that is, at P, and provided that the three metric is invertible, the imaginary part of these tensors clearly vanishes. We have also verified this directly (see appendix 2, and see also previous works, [11] and [12]), obtaining

$$\begin{aligned} \mathcal{L}_t q_{ab} &= \mathcal{L}_{\vec{N}} q_{ab} - 2N\pi_{(ab)} \\ \mathcal{L}_t \pi_{ab} &= \mathcal{L}_{\vec{N}} \pi_{ab} + N\pi\pi_{ab} - 2N\pi_a{}^e \pi_{eb} - NR_{ab} - D_a D_b N \end{aligned}$$

[8]Note that in this manipulations no surface term arises, for in treating the local problem the lapse-shift pair vanishes at the boundary

$$-\frac{N}{2}q_{ab}\left(R+\pi^2-\pi_{dc}\pi^{cd}\right)-iN\epsilon_{ab}{}^{d}D^{c}\left(\pi_{dc}-\pi q_{dc}\right)$$
$$+2N\left(\pi^{d}{}_{a}\pi_{[db]}-\pi\pi_{[ab]}+\pi_{[ad]}\pi^{d}{}_{b}\right)-iN\epsilon_{b}{}^{dm}D_{a}\pi_{[dm]}$$
$$-\frac{i}{2}q_{ab}\epsilon^{cmd}\left(D_{c}N\pi_{[md]}-ND_{c}\pi_{[md]}\right) \tag{21}$$

where

$$\pi_{ab}=-\mathrm{tr}(\pi_a\sigma_b) \quad \text{and} \quad \pi_{[ab]}=-\frac{i\epsilon_{abc}}{2}\,\mathrm{tr}(\sigma^c\mathcal{D}_d\sigma^d).$$

Since the terms, which could give an imaginary contribution to the expression when q_{ab} and π_{ab} are real, are proportional to the constraints, we see that at points of P the evolution of the imaginary parts q_{ab}, and π_{ab} vanishes. Note that the real part of π_{ab} corresponds to the extrinsic curvature, i.e. $\pi_{(ab)}=K_{ab}$. We conclude that,

Main Result: *Initial data satisfying both the reality conditions, and the constraint equations have an evolution which stays inside P, and so are solutions to the real and complete set of Einstein's equations.* [9]

We will conclude with a brief discussion of some issues relevant to numerical relativity.

What does this mean when one does not solve Einstein's equations exactly, but just approximate them via numerical simulations or other means? If the approximations were a contraction map as the one often used to prove existence of exact solutions of symmetric hyperbolic systems, then it would follow that the approximate solution must approach the manifold P as it is refined and made closer to the exact solution. In practice the approximation schemes are not contractive as required. Even after refinement, if the method yields convergence to the manifold P, this convergence could be very slow.

To explore this problem, and in analogy with a similar work of Fritelli [13], we have looked at the evolution equations that the constraints satisfy when the flow is extended , but assuming that the reality conditions hold. From this study we have found that the constraint quantities satisfy by themselves a symmetric hyperbolic system of equations. Thus, initial data sets which satisfy exactly the reality conditions, but are just near the constraint sub-manifold, if evolved by a scheme that respects the reality conditions then would stay "near" the constraint sub-manifold, in the sense that their departure would be bounded by a constant that depends only on the Sobolev

[9] The above argument is only valid for establishing local well posedness. For initial-boundary-value problems, one must be aware of boundary terms and impose conditions for them to also vanish.

norm of the initial data set. Unfortunately symmetric hyperbolicity by itself does not allow a finer control on the bound, which in principle could, even for linear equations, grow exponentially with the evolution time. Thus, this crude bound is not enough for controlling numerical simulations, but its absence would certainly make simulations very hard to implement.

The evolution equations for the constraints have been calculated in appendix 2. In order to prove the symmetric hyperbolicity we just need to consider the principal part of this set

$$\begin{aligned} \dot{\tilde{C}} &= N^a \partial_a \tilde{C} - \frac{i}{\sqrt{2}} \underset{\sim}{N} [\tilde{\sigma}^a, \partial_a \tilde{C}], \\ \dot{C} &= N^a \partial_a C - i\sqrt{2} \underset{\sim}{N} \sigma^2 \partial^a C_a, \\ \dot{C}_a &= N^b \partial_b C_a + \frac{i}{\sigma^2} \underset{\sim}{N} \tilde{\epsilon}_a{}^{bc} \partial_b C_c + \frac{i}{\sqrt{2}} \underset{\sim}{N} \partial_a C. \end{aligned} \tag{22}$$

Then we prove the following,

Lemma III.1: *The equation system (22) for any fixed, but arbitrary lapse and shift fields is a symmetric hyperbolic system in the reality condition submanifold.*

Proof:

The eigenvectors and the eigenvalues can now be easily calculated, note that the principal symbol in this system

$$P(\underset{\sim}{u}, ik_a) = P_{\tilde{C}}(\underset{\sim}{u}, ik_a) \oplus P_{\vec{C}}(\underset{\sim}{u}, ik_a), \tag{23}$$

here $\underset{\sim}{u} = (\underset{\sim}{\tilde{q}}, \underset{\sim}{A})$, and $\vec{C} = (C, C_b)$.

As above we use the orthogonal triad $\{k^a, m^a, \overline{m}^a\}$ being m^a and $\overline{m}^a$ null vectors. Thus the symbol can be diagonalized in blocks with the following eigenvalues

$$\lambda_0 = i\, k_a N^a \qquad \lambda_- = i(\, k_a N^a - \underset{\sim}{N} \underset{\sim}{\tilde{q}}) \qquad \lambda_+ = i(\, k_a N^a + \underset{\sim}{N} \underset{\sim}{\tilde{q}}),$$

and the subspaces associated with the above eigenvalues are

$$\begin{aligned} E_{\tilde{C}}{}^0 &= \mathrm{Span}\{k_d \underset{\sim}{\tilde{q}}^d\}, \qquad E_{\tilde{C}}{}^- = \mathrm{Span}\{\, m_d \underset{\sim}{\tilde{q}}^d\}, \qquad E_{\tilde{C}}{}^+ = \mathrm{Span}\{\overline{m}_d \underset{\sim}{\tilde{q}}^d\}, \\ E_{\vec{C}}{}^+ &= \mathrm{Span}\{(0, m_a), (\tilde{q}, \frac{i}{\sqrt{2}} k_a)\}, \\ E_{\vec{C}}{}^- &= \mathrm{Span}\{(0, \overline{m}_a), (\tilde{q}, -\frac{i}{\sqrt{2}} k_a)\}. \end{aligned}$$

This set is a complete orthonormal set of eigenvectors with respect to the inner product

$$\begin{aligned}\langle u^2, u^1 \rangle &\equiv \langle (\tilde{C}^2, \vec{C}^2), (\tilde{C}^1, \vec{C}^1) \rangle \\ &\equiv \mathrm{tr}(\tilde{C}^{\dagger 2}\tilde{C}^1) + \frac{1}{2\tilde{q}^2} C^{2\dagger} C^1 + q^{ab} C_a^{2\dagger} C_b^1,\end{aligned}$$

and since the eigenvalues are purely imaginary the principal symbol is anti-hermitian. This concludes the proof of the Lemma.♠

6 Conclusions

We have studied several aspects of Einstein's evolution equations as given in Ashtekar's formalism. We first have shown that when the reality conditions are satisfied, that is, when there is a $\mathbf{SL}(2, \mathbf{C})$ transformation that rotates the soldering form into a anti-hermitian one, the system is symmetric hyperbolic.

Contrary to what one might expect at first , this condition is not enough to conclude that there are solutions, even local ones, to the evolution equations. The problem being that, since the reality conditions are not linear conditions on the variables on which the problem is formulated, the usual contractive map of successive approximations does not respect them, and so evolution occurs in this approximations along paths where the system is not symmetric-hyperbolic, meaning that they can not be appropriately bounded by the usual methods.

To remedy this problem we have proposed two further modifications. One of the modifications, called the *regularized flows* is a standard trick commonly used to grant symmetric hyperbolicity, suitably extended to the case under consideration, that is, to account for the intermediate $\mathbf{SL}(2, \mathbf{C})$ rotation mentioned above. In that case one obtains a symmetric hyperbolic system of equations *even outside the reality condition sub-manifold.* This allows then to establish the local well posedness of the problem by standard procedures.

The second modification has the advantage of keeping unmodified the eigenvalues-eigenvectors structure of the principal part, which have a simple, and therefore probably useful, expression in terms of the solution, at the expense of transforming the differential equations into non local ones, namely into pseudo-differential equations. This modification can be implemented in numerical schemes –using fast Fourier transform– but only for a limited type of boundary conditions.

After the regularization of the flow and the subsequent implication of the existence of solutions of given differentiability, we discuss the problem of making sure that the solution, whose local existence we are asserting, does satisfy the whole set of Einstein's equations. In other words, whether integral curves to the evolution vector fields starting at the reality-constraint sub-manifold stay there. Since we have already shown that the solutions are unique (if sufficiently differentiability is assumed) then it is enough to show tangency of the vector field with respect to that sub-manifold. Provided some differentiability conditions are satisfied, this is equivalent to standard results which were nevertheless reviewed.

We concluded section 5 with some considerations about the problem of numerical simulations, or approximations in general. Since errors are unavoidable in numerical algorithms one can at most consider evolution in a hopefully small neighborhood of the reality-condition-constraint sub-manifold. Thus, even when –in the best of the cases– knowing that eventually, by refinements of the approximation scheme, the solution would converge to that sub-manifold, one would like to have a priori estimates of the deviation as a function of the initial error.

We have shown, following the ideas of Fritelli for the ADM formulation, that the constraint system –while at the reality sub-manifold– satisfies by itself a set of symmetric hyperbolic equations. Thus such an a priori bound follows directly. We consider this a preliminary result, for the bounds that follow from symmetric hyperbolicity are too crude for numerical purposes, and so a further refinement, this time using information about the full system of equations (contrary to symmetric-hyperbolicity which only uses the principal part structure of them), seems to be needed.

We will conclude with a remark. A substantial improvement on the handling of these equations would be the possibility of imposing reality conditions in a linear way, that is, by requiring the hermiticity (or anti-hermiticity if considered as matrices) of the soldering forms. Unfortunately with the present scheme of modifying the equations outside the constraint sub-manifold, the hermiticity condition restricts unnecessarily the possible evolutions, for it basically fixes a unique lapse. Otherwise the system can not be made symmetric hyperbolic in an straightforward way. This restriction is unnatural, for one would expect that this condition could be enforced fixing the $\mathbf{SU}(2)$ gauge freedom, and not the one associated with diffeomorphisms. But all attempts to fix the $\mathbf{SU}(2)$ gauge appropriately seems to involve (at best) elliptic conditions on the "time component" of the connection. Is there an alternative avenue to handle the linear reality conditions?

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APPENDIX A: THE CONSTRUCTION OF THE $\mathbf{SL}(2, \mathbf{C})$ TRANSFORMATION.

Given a soldering form, σ^a, we shall define here a procedure to construct a $\mathbf{SL}(2, \mathbf{C})$ transformation such that, when the metric is real the transformed soldering form is anti-hermitian. Given a transformation L making $\tilde{\sigma}^a$ anti-hermitian, there is a whole set of them with the same property, namely one can left-right multiply the new one by a $\mathbf{SU}(2)$ rotation and obtain a new one with the same anti-hermiticity property. To remove this arbitrariness in our construction we fix a real basis (not orthonormal), $\{e_i^a\}$ $i = 1, 2, 3$ of vector fields. For simplicity we shall work here with frames,$\{E_i^a\}$ $i = 1, 2, 3$ so that $\sigma^a = E_i^a \tau^i$ where τ^i is an anti-hermitian Pauli set of matrices. We then have $E_i^a E_j^b \delta^{ij} = q^{ab}$, and so $E_i^a E_j^b q_{ab} = \delta_{ij}$. We define L as the transformation such that the transformed frame $\{\hat{E}_i^a\}$ $i = 1, 2, 3$ satisfies:

$$\begin{aligned} \hat{E}_1^a &= b_1^1 e_1^a \\ \hat{E}_2^a &= b_2^1 e_1^a + b_2^2 e_2^a + b_2^3 e_3^a \quad \text{with } \frac{b_2^1}{b_2^2}, \frac{b_2^3}{b_2^2} \text{ real.} \end{aligned}$$

The ortho-normality condition then fixes the third vector, and so the complete transformation. Notice that if the initial frame can be rotated to a real one, that is, if the metric is real, then the rotated frame we are constructing will be real.

We let $\hat{E}_i^a = C_i{}^j e_j^a$ and $\hat{E}_i^a = L_i{}^j E_j^a = L_i{}^j C_j{}^k e_k^a = \mathcal{L}_i{}^j e_k^a$. Since $C_i{}^j$ is invertible, knowing $\mathcal{L}$ is equivalent to knowing L, we shall construct $\mathcal{L}$.

We have, $\mathcal{L}_1{}^1 = b_1^1$, $\mathcal{L}_1{}^2 = \mathcal{L}_1{}^3 = 0$, b_1^1 is fixed by the ortho-normality condition: $1 = (b_1^1)^2 e_1 \cdot e_1$, we choose the root with positive real part.

Next we have $\mathcal{L}_2{}^1 = b_2^1$ $\mathcal{L}_2{}^2 = b_2^2$ $\mathcal{L}_2{}^3 = b_2^3$, and two conditions, the normality condition,

$$1 = (b_2^1)^2 e_1 \cdot e_1 + (b_2^2)^2 e_2 \cdot e_2 + (b_2^3)^2 e_3 \cdot e_3 + 2b_2^1 b_2^2 e_1 \cdot e_2 + 2b_2^2 b_2^3 e_2 \cdot e_3 + 2b_2^1 b_2^3 e_1 \cdot e_3.$$

and the orthogonality one,

$$0 = b_1^1 (b_2^1 e_1 \cdot e_1 + b_2^2 e_1 \cdot e_2 + b_2^3 e_1 \cdot e_3)$$

We require that $\Im(\frac{e_1 \cdot e_3}{e_1 \cdot e_1}) \neq 0$, otherwise we impose the reality condition to b_2^1/b_2^3 and take $b_2^2 = 0$. We multiply the orthogonality relation by a phase $e^{i\theta}$ such that $\Im(e^{i\theta} e_1 \cdot e_2) = 0$, we then divide by b_2^2 and assume the quotient of the b's to be real. Taking real and imaginary part we get two real linear equations:

$$\frac{b_2^1}{b_2^2}\Im(e^{i\theta}e_1 \cdot e_1) + \frac{b_2^3}{b_2^2}\Im(e^{i\theta}e_1 \cdot e_3) = 0$$

$$\frac{b_2^1}{b_2^2}\Re(e^{i\theta}e_1 \cdot e_1) + \frac{b_2^3}{b_2^2}\Re(e^{i\theta}e_1 \cdot e_3) = -|e_1 \cdot e_2|,$$

which, because of the requirement above, have a non-vanishing determinant. Thus, it has a real solution for the quotients. We then solve the normality condition for b_2^2, choosing, as before, the root with the positive real part. The remaining coefficients of $\mathcal{L}$ are completely determined by the remaining ortho-normality conditions (three equations) and the condition that the real part of b_3^3 be positive.

APPENDIX B: BRIEF OVERVIEW OF THE HAMILTONIAN FORMULATION IN ASHTEKAR'S VARIABLES

In this section we shall present a brief overview of the Hamiltonian formulation in spinorial variables in order to get Einstein field equations as a system of evolution and constraints equations and thereby to study the evolution of the constraints (we follow [10] and [11]).

1 The Lagrangian Framework

Let us consider a four-manifold M which has topology $\Sigma \times \mathbf{R}$, for some three-manifold Σ, with a four dimensional $\mathbf{SL}(2, \mathbf{C})$ soldering form, $\sigma^a{}_A{}^{A'}$, and a connection ${}^4A_{aA}{}^B$ which acts on the unprimed spinor indices. We restrict the soldering form to be anti-Hermitian so that it defines a real space-time metric via $g^{ab} = \sigma^a_{AA'}\sigma^{bAA'}$ with signature $(-+++)$ and a unique torsion-free derivative operator ∇ compatible with $\sigma^a{}_A{}^{A'}$. Finally, the derivative operator ${}^4\mathcal{D}_a$ defined by ${}^4A_{aA}{}^B$ via ${}^4\mathcal{D}_a\lambda_A = \partial_a\lambda_A + {}^4A_{aA}{}^B\lambda_B$ acts on unprimed spinors.

The gravitational part of the Lagrangian density of weight 1 is:

$$\mathcal{L} = ({}^4\sigma)\sigma^a{}_A{}^{A'}\sigma^b{}_{BA'}{}^4F_{ab}{}^{AB},$$

where $({}^4\sigma)$ is the determinant of the inverse soldering form and ${}^4F_{abA}{}^B$ is the curvature tensor of ${}^4A_{aA}{}^B$.

It is useful to perform a $3+1$ decomposition of the action, this shall be needed latter in order to pass on to the Hamiltonian framework.

Let us introduce on M a smooth function t whose gradient is nowhere vanishing and whose level surfaces Σ_t are each diffeomorphic to Σ. Let t^a be a smooth vector field on M with affine parameter t and on each level surface, let n^a be the future-directed, unit,time-like vector field orthogonal to Σ_t. Denote the induced positive-definite metric on Σ_t by $q_{ab} = g_{ab} + n_a n_b$ and obtain the lapse and shift fields N and N^a by projecting t^a into and orthogonal to Σ_t , i.e. $t^a = Nn^a + N^a$.

Identifying unprimed $\mathbf{SL}(2,\mathbf{C})$ spinors on M with $\mathbf{SU}(2)$ on Σ_t, we introduce the soldering form $\sigma^a{}_A{}^B$ on $\mathbf{SU}(2)$ spinors. It defines the three metric on Σ_t as $q^{ab} = -\mathrm{tr}\ (\sigma^a \sigma^b\ )$. Here a matrix notation is employed, and shall be used in the following, for unprimed spinor indices in which adjacent summed indices go from upper left to lower right, e.g., $(\sigma^a\sigma^b)_A{}^B = \sigma^a{}_A{}^C\sigma^b_C{}^B$.

Let $A_{aA}{}^B$, $F_{abA}{}^B$ and $\mathcal{D}_a$ (the *Sen connection*) be the pull-backs to Σ_t of ${}^4A_{aA}{}^B$, ${}^4F_{abA}{}^B$ and ${}^4\mathcal{D}_b$ respectively. Then $\mathcal{D}_a\lambda_A = \partial_a\lambda_A + A_{aA}{}^B\lambda_B$ and $F_{ab} = 2\partial_{[a}A_{b]} + [A_a, A_b]$.

Finally, there is a natural (canonical) spinorial connection associated with the three-metric such that $D_a\sigma^b = 0$. It relates to the Sen connection via the extrinsic curvature K_{ab}on Σ_t as follows

$$\begin{aligned}
\mathcal{D}_a\lambda_A &= D_a\lambda_A + \frac{i}{\sqrt{2}}K_{aA}{}^B\lambda_B \\
&= \partial_a\lambda_A + \Gamma_{aA}{}^B\lambda_B + \frac{i}{\sqrt{2}}K_{aA}{}^B\lambda_B,
\end{aligned}$$

where $K_{aA}{}^B = K_{ab}\sigma^a{}_A{}^B$ and $\Gamma_{aA}{}^B$ is the spin connection 1-form of D. Then $A_a = \Gamma_a + \frac{i}{\sqrt{2}}K_a$ and using the fact that the derivative D is compatible with σ^a, i.e.

$$D_a\sigma_b = \partial_a\sigma_b - \Gamma_{ab}{}^c\sigma_c + [\Gamma_a, \sigma_b] = 0$$

with $\Gamma_{ab}{}^c$ denoting the Christoffel symbols; we calculate

$$\Gamma_a = \frac{\epsilon^{bcd}\sigma_a}{2\sqrt{2}}\mathrm{tr}\ (\sigma_b\partial_c\sigma_d) - \frac{\epsilon^{bcd}\sigma_b}{\sqrt{2}}\mathrm{tr}\ (\sigma_a\partial_c\sigma_d). \tag{24}$$

where the orientation three-form on Σ_t is written as $\epsilon^{abc} = -\sqrt{2}\mathrm{tr}\ (\sigma^a\sigma^b\sigma^c)$.

The action [10] can be expressed in terms of only of three-dimensional fields:

$$S = \int \mathrm{dt} \int \mathrm{d}^3x \ \mathrm{tr}\Big(\sqrt{2}\, i\tilde{\sigma}^b(\mathcal{L}_t A_b - \mathcal{D}_b({}^4A\cdot t)) - \underset{\sim}{N}\ \tilde{\sigma}^a\tilde{\sigma}^b F_{ab} - \sqrt{2}\, iN^a\tilde{\sigma}^b F_{ab}\Big).$$

where $\tilde{\sigma}^a{}_{AB} = (\sigma)\ \sigma^a{}_{AB}$, $\underset{\sim}{N} = (\sigma)^{-1}\ N$ and the Lie derivatives treat internal indices as scalars.

The action depends on five variables $\underset{\sim}{N}$, N^a, ${}^4A \cdot t$, $A_{aA}{}^B$ and $\tilde{\sigma}^a{}_A{}^B$. The first three variables play the role of the Lagrange multipliers, only the last two are dynamical variables. Varying the action with respect to the Lagrange multipliers we obtain the constraint equations:

$$\begin{aligned} C(\tilde{\sigma}, A) &:= \mathrm{tr}(\tilde{\sigma}^a\tilde{\sigma}^b F_{ab}) = 0, \\ C_a(\tilde{\sigma}, A) &:= \mathrm{tr}(\tilde{\sigma}^b F_{ab}) = 0, \\ \tilde{C}_A{}^B(\tilde{\sigma}, A) &:= \mathcal{D}_a\tilde{\sigma}^a{}_A{}^B = 0, \end{aligned} \tag{25}$$

and varying with respect to the dynamical variables, yields the evolution equations:

$$\begin{aligned} \mathcal{L}_t\tilde{\sigma}^b &= -[{}^4A\cdot t, \tilde{\sigma}^b] + 2\mathcal{D}_a(N^{[a}\tilde{\sigma}^{b]}) - \frac{i}{\sqrt{2}}\mathcal{D}_a(\underset{\sim}{N}[\tilde{\sigma}^a, \tilde{\sigma}^b]), \\ \mathcal{L}_t A_b &= \mathcal{D}_b({}^4A\cdot t) + N^a F_{ab} + \frac{i}{\sqrt{2}}\ \underset{\sim}{N}[\tilde{\sigma}^a, F_{ab}]. \end{aligned} \tag{26}$$

2 The Hamiltonian Framework

Recall that the dynamics of a mechanical system can be achieved having what is called a *symplectic manifold*, i.e a pair $(\Gamma, \Omega_{\alpha\beta})$, where Γ is an even-dimensional manifold, and $\Omega_{\alpha\beta}$ a *symplectic form*, i.e a 2-form which is closed and nondegenerate. Given any function $f : \Gamma \to \mathbf{R}$, the *Hamiltonian vector field* of f is defined by $X_f{}^\alpha = \Omega^{\beta\alpha}\nabla_\beta f$. Given any vector field $v^\alpha \in T_x\Gamma$, we say that v^α is a *symmetry* of the symplectic manifold if it leaves the symplectic form invariant, i.e if $\mathcal{L}_v\Omega_{\alpha\beta} = 0$, in which case the diffeomorfisms generated by v^α are called *canonical transformations.*

Given two functions $f, g : \Gamma \to \mathbf{R}$, their *Poisson bracket* is defined by

$$\{f, g\} := \Omega^{\alpha\beta}\nabla_\alpha f\nabla_\beta g \equiv \mathcal{L}_{X_f}g \equiv -\mathcal{L}_{X_g}f.$$

[10]The surface terms are not included here.

Thus the dynamics of a physical system is given by assigning a *phase space* $(\Gamma, \Omega_{\alpha\beta}, H)$ on which the evolution is generated by the Hamiltonian H from the initial state. Hence for any observable f

$$\dot{f} = \mathcal{L}_{X_H} f = \{H, f\}.$$

When the system is constrained, i.e. when there are points in the phase space that can not be reached by the physical system, the system remains in a sub-manifold called *constraint sub-manifold*, that can be specified by the vanishing of a set of functions $\bar{\Gamma} = \{p \in \Gamma / C_i(p) = 0, \text{for } i = 1 \cdots m\}$. The constrained system is said to be *first class* if there exist functions $f_{ij}{}^k, i, j, k = 1, \cdots, m$, called *structure functions*, such that $\{C_i, C_j\} = f_{ij}{}^k C_k$.

In the formulation of General relativity in Ashtekar variables, the configuration space $\mathcal{C}$ is the space of all weighted soldering forms $\tilde{\sigma}^a$, and the phase space Γ is the cotangent bundle over $\mathcal{C}$. The phase space is represented by the pairs $(\tilde{\sigma}^a, A_b)$ (these variables, apart from a numerical factor, the new canonically conjugate pair,). The action of a cotangent vector A_a on any tangent vector $(\delta\sigma)^a$ at a point $\tilde{\sigma}^a$ of $\mathcal{C}$ is given by

$$A(\delta\sigma) = -\int_\Sigma \mathrm{d}^3x \, \mathrm{tr}(A_a \delta\sigma^a).$$

We will not discuss boundary conditions here, but just remark that one possible choice would be to require that the canonically conjugate fields should admit a smooth extension to the point at spatial infinity if the three-surface Σ is made into the three sphere by the one point compactification, for an extensive discussion of fall-off properties of the fields see [10].

Thus in order to obtain the constraint sub-manifold where the physical gravitational states take place, we need to construct functionals from the constraints (25), i.e. we need to smear out these constraints with a function $\underset{\sim}{N}$, a vector field N^a and an anti-hermitian traceless $N_A{}^B$ test fields. We define

$$\begin{aligned}
C_N(\tilde{\sigma}, A) &:= -i\sqrt{2} \int_\Sigma \mathrm{d}^3x \quad \mathrm{tr}(N \mathcal{D}_a \tilde{\sigma}^a), \\
C_{\underset{\sim}{N}}(\tilde{\sigma}, A) &:= -i\sqrt{2} \int_\Sigma \mathrm{d}^3x \underset{\sim}{N} \; \mathrm{tr}(\tilde{\sigma}^a \tilde{\sigma}^b F_{ab}), \qquad (27)\\
C_{\vec{N}}(\tilde{\sigma}, A) &:= -i\sqrt{2} \int_\Sigma \mathrm{d}^3x \; N^a \, \mathrm{tr}(\tilde{\sigma}^b F_{ab} - A_a \mathcal{D}_b \tilde{\sigma}^b),
\end{aligned}$$

then we should ask if these constraint functions are *first class* in the above terminology, to answer this question we have to calculate their Poisson brackets. For that purpose, we introduce the symplectic form

$$\Omega|_{(\sigma,A)}\Big((\delta\sigma, \delta A), (\delta\sigma', \delta A')\Big) := \frac{i}{\sqrt{2}} \int_\Sigma \mathrm{d}^3x \, \mathrm{tr}(\delta A_a \delta\sigma'^a - \delta A'_a \delta\sigma^a),$$

where $(\delta\sigma, \delta A)$ and $(\delta\sigma', \delta A')$ are any two tangent vectors at the point (σ, A). Thus given two functionals f and g their Poisson bracket is

$$\{f, g\} = \frac{i}{\sqrt{2}} \int_\Sigma \mathrm{d}^3x \ \mathrm{tr}\left(\frac{\delta f}{\delta A_a}\frac{\delta g}{\delta \sigma^a} - \frac{\delta f}{\delta \sigma^a}\frac{\delta g}{\delta A_a}\right).$$

The Poisson bracket between the constraint functionals and our fundamental variables are:

$$\begin{aligned}
\{\tilde{\sigma}^a{}_{MN}, A_b{}^{AB}\} &= -\frac{i}{\sqrt{2}}\delta(x,y)\delta_M{}^{(A}\delta_N{}^{B)}, \\
\{C_N, \tilde{\sigma}^a\} &= [\tilde{\sigma}^a, N], \\
\{C_N, A_a\} &= \mathcal{D}_a N, \\
\{C_{\underset{\sim}{N}}, \tilde{\sigma}^a\} &= 2\,\mathcal{D}_b\,(\underset{\sim}{N}\tilde{\sigma}^{[a}\tilde{\sigma}^{b]}), \\
\{C_{\underset{\sim}{N}}, A_a\} &= \underset{\sim}{N}\,[\tilde{\sigma}^b, F_{ba}], \\
\{C_{\vec{N}}, \tilde{\sigma}^a\} &= -\mathcal{L}_{\vec{N}}\tilde{\sigma}^a, \\
\{C_{\vec{N}}, A_a\} &= -\mathcal{L}_{\vec{N}}A_a,
\end{aligned} \tag{28}$$

and the constraints algebra becomes

$$\begin{aligned}
\{C_N, C_M\} &= C_{[N,M]}, \\
\{C_{\vec{N}}, C_M\} &= C_{\mathcal{L}_{\vec{N}}M}, \\
\{C_{\vec{N}}, C_{\vec{M}}\} &= C_{[\vec{N},\vec{M}]}, \\
\{C_N, C_{\underset{\sim}{M}}\} &= 0, \\
\{C_{\vec{N}}, C_{\underset{\sim}{M}}\} &= C_{\mathcal{L}_{\vec{N}}\underset{\sim}{M}} \\
\{C_{\underset{\sim}{N}}, C_{\underset{\sim}{M}}\} &= C_{\vec{K}} + C_{A_m K^m},
\end{aligned} \tag{29}$$

where $K^a = 2\sigma^2\ (\underset{\sim}{M}\mathcal{D}^a\underset{\sim}{N} - \underset{\sim}{N}\mathcal{D}^a\underset{\sim}{M})$.

Using the canonical variables (σ^a, A_a) and the action, the resulting Hamiltonian writes:

$$\begin{aligned}
H &= \int_\Sigma \mathrm{d}^3x \left(i\,\sqrt{2}\mathrm{tr}(N^a\tilde{\sigma}^b F_{ab} - N^a A_a \mathcal{D}_b\tilde{\sigma}^b) + \underset{\sim}{N}\ \mathrm{tr}(\,\tilde{\sigma}^a\tilde{\sigma}^b F_{ab})\right), \\
&= -C_{\vec{N}} + \frac{i}{\sqrt{2}}C_{\underset{\sim}{N}},
\end{aligned} \tag{30}$$

where the Lagrange multiplier ${}^4A_a t^a$ has been chosen as $A_a N^a$.

Therefore, the Hamiltonian evolution of the dynamical variables, yields:

$$
\begin{aligned}
\mathcal{L}_t\tilde{\sigma}^b &= \{H,\tilde{\sigma}^b\} \\
&= \mathcal{L}_{\vec{N}}\tilde{\sigma}^b - \frac{i}{\sqrt{2}}\mathcal{D}_a(\underset{\sim}{N}[\tilde{\sigma}^a,\tilde{\sigma}^b]) \\
&= -[A_aN^a,\tilde{\sigma}^b] + 2\mathcal{D}_a(N^{[a}\tilde{\sigma}^{b]}) + N^b\mathcal{D}_a\tilde{\sigma}^a - \frac{i}{\sqrt{2}}\mathcal{D}_a(\underset{\sim}{N}[\tilde{\sigma}^a,\tilde{\sigma}^b]) \qquad (31) \\
\mathcal{L}_tA_b &= \{H,A_b\} \\
&= \mathcal{L}_{\vec{N}}A_b + \frac{i}{\sqrt{2}}\,\underset{\sim}{N}[\tilde{\sigma}^a,F_{ab}] \\
&= \mathcal{D}_b(A_aN^a) + N^aF_{ab} + \frac{i}{\sqrt{2}}\,\underset{\sim}{N}[\tilde{\sigma}^a,F_{ab}]. \qquad (32)
\end{aligned}
$$

Note that the evolution obtained from the Lagrangian differs from the evolution obtained from the Hamiltonian in a "constraint term".

APPENDIX C: THE CONSTRAINTS EVOLUTION

To compute the evolution equations for the constraints with the extended flow let us first calculate the Hamiltonian evolution of the constraints $(\tilde{C}_A{}^B, C, C_a)$. Using the Hamiltonian given by (30), the constraint algebra (29) and integrating by parts, we obtain

$$
\begin{aligned}
\dot{\tilde{C}} &= \{H,\tilde{C}\} = \mathcal{L}_{\vec{N}}\tilde{C}, \\
\dot{C} &= \{H,C\} = \mathcal{L}_{\vec{N}}C + i\sqrt{2}\,\sigma^2\underset{\sim}{N}\,\mathcal{D}^aC_a + i\,2\sqrt{2}\sigma^2C^a\mathcal{D}_a\underset{\sim}{N}. \qquad (33) \\
\dot{C}_a &= \{H,C_a\} = \mathcal{L}_{\vec{N}}C_a - \frac{i}{\sqrt{2}}\underset{\sim}{N}\,\mathcal{D}_aC - i\sqrt{2}C\mathcal{D}_a\underset{\sim}{N} + \frac{i}{\sqrt{2}}\underset{\sim}{N}\,\mathrm{tr}([\tilde{\sigma}^b,F_{ba}]\tilde{C}).
\end{aligned}
$$

The most complicated calculation is the last one (the evolution of C_a), in order to do this, we define the functional

$$\hat{C}_{\vec{M}} = -i\sqrt{2}\int_\Sigma \mathrm{d}^3x\; M^a\;\mathrm{tr}(\tilde{\sigma}^bF_{ab}),$$

then

$$\{H,\hat{C}_{\vec{M}}\} = -\{C_{\vec{N}},\hat{C}_{\vec{M}}\} + \frac{i}{\sqrt{2}}\{C_{\underset{\sim}{N}},\hat{C}_{\vec{M}}\}. \qquad (34)$$

We calculate each one of the terms above as follows

$$\{C_{\vec{N}},\hat{C}_{\vec{M}}\} = \frac{i}{\sqrt{2}}\int_\Sigma \mathrm{d}^3x\;\mathrm{tr}\left(\frac{\delta C_{\vec{N}}}{\delta A_a}\frac{\delta\hat{C}_{\vec{M}}}{\delta\sigma^a} - \frac{\delta C_{\vec{N}}}{\delta\sigma^a}\frac{\delta\hat{C}_{\vec{M}}}{\delta A_a}\right),$$

$$
\begin{aligned}
&= \{C_{\vec{N}}, C_{\vec{M}}\} + \{C_{\vec{N}}, C_P\}|_{P=M^a A_a} \\
&\quad + i\sqrt{2}\int_\Sigma \mathrm{d}^3x\, \mathcal{L}_{\vec{N}} A_a\, M^a C, \\
&= C_{[\vec{N},\vec{M}]} + C_{\mathcal{L}_{\vec{N}}(M^a A_a)} +, C_{M^a \mathcal{L}_{\vec{N}} A_a} \\
&= C_{[\vec{N},\vec{M}]} + C_{[\vec{N},\vec{M}]^a A_a}, \\
&= \hat{C}_{[\vec{N},\vec{M}]}, \qquad (35)
\end{aligned}
$$

and

$$
\begin{aligned}
\{C_{\underset{\sim}{N}}, \hat{C}_{\vec{M}}\} &= \{C_{\underset{\sim}{N}}, C_{\vec{M}}\} + \{C_{\underset{\sim}{N}}, C_P\}|_{P=M^a A_a} \\
&\quad - i\sqrt{2}\int_\Sigma \mathrm{d}^3x \underset{\sim}{N}\, \mathrm{tr}([\tilde{\sigma}^b, F_{ab}] M^a \tilde{C}), \\
&= -C_{\mathcal{L}_{\vec{M}}\underset{\sim}{N}} + C_{\underset{\sim}{N}[\tilde{\sigma}^b, F_{ab}]M^a}. \qquad (36)
\end{aligned}
$$

Inserting the Poisson brackets (35) and (36) in (34), using that the Lie derivative of a function with weight minus one is $\mathcal{L}_{\vec{M}}\underset{\sim}{N} = M^a \partial_a \underset{\sim}{N} - \underset{\sim}{N}\, \partial_a M^a$ and integrating by parts we get

$$
\{H, C_a\} = \mathcal{L}_{\vec{N}} C_a - \frac{i}{\sqrt{2}}\underset{\sim}{N}\, \mathcal{D}_a C - i\sqrt{2} C \mathcal{D}_a \underset{\sim}{N} + \frac{i}{\sqrt{2}}\underset{\sim}{N}\, \mathrm{tr}([\tilde{\sigma}^b, F_{ba}]\tilde{C}).
$$

Since we have changed the evolution of the dynamical variables outside the constraint sub-manifold, the calculus of the time derivative of the constraint equations must be done using equations (3) and (4). If $f : \Gamma \to \mathbf{C}$ then

$$
\dot{f}(\tilde{\sigma}, A) = \int_\Sigma d^3x \quad \mathrm{tr}\left(\frac{\delta f}{\delta \tilde{\sigma}^a}\dot{\tilde{\sigma}}^a + \frac{\delta f}{\delta A_a}\dot{A}_a\right);
$$

and since the perturbation we have made on the equations (3) and (4) are linear in the constraints, we obtain

$$
\begin{aligned}
&\dot{f}(\tilde{\sigma}, A) = \{H, f\} + \int_\Sigma d^3x \\
&\times \mathrm{tr}\left(\frac{\delta f}{\delta \tilde{\sigma}^b}\left(\frac{i}{\sqrt{2}}\, \underset{\sim}{N}[\tilde{C}, \tilde{\sigma}^b]\right) - \frac{\delta f}{\delta A_b}\left(\frac{i}{\sigma^2\sqrt{2}}\, \underset{\sim}{N}\tilde{\sigma}_b C + \frac{i}{\sigma^4}\, \underset{\sim}{N}\tilde{\epsilon}_b{}^{dc}\tilde{\sigma}_c C_d\right)\right) (37)
\end{aligned}
$$

Then, using (33) in (37), we have the constraints evolution

$$
\begin{aligned}
\dot{\tilde{C}} &= N^a \partial_a \tilde{C} - \frac{i}{\sqrt{2}}\underset{\sim}{N}\,[\tilde{\sigma}^a, \mathcal{D}_a \tilde{C}] - \frac{i}{\sqrt{2}}\, \mathcal{D}_a \underset{\sim}{N}\,[\tilde{\sigma}^a, \tilde{C}] - i2\sqrt{2}\underset{\sim}{N}\, C_d \tilde{\sigma}^d, \\
\dot{C} &= N^a D_a C - i\sqrt{2}\underset{\sim}{N}\sigma^2 D^a C_a - i\sqrt{2} C^d \underset{\sim}{N}\, \mathrm{tr}\left(\tilde{C}\tilde{\sigma}_d\right)
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{i}{\sqrt{2}}\underset{\sim}{N}\,\mathrm{tr}\left([\tilde{C},\tilde{\sigma}^a][\tilde{\sigma}^b,F_{ab}]\right)+\frac{i}{\sigma^4}\underset{\sim}{N}\,C\tilde{\epsilon}^{abe}\mathrm{tr}\left((\mathcal{D}_b\tilde{\sigma}_a)\,\tilde{\sigma}_e\right), \qquad (38)\\
\dot{C}_a \;=\;& N^bD_bC_a+\frac{i}{\sigma^2}\underset{\sim}{N}\,\tilde{\epsilon}_a{}^{bd}D_bC_d+\frac{i}{\sqrt{2}}\underset{\sim}{N}\,D_aC+C_bD_aN^b\\
& -\frac{i}{\sigma^2}C_dD_b\underset{\sim}{N}\,\tilde{\epsilon}_a{}^{db}-\frac{2i}{\sigma^4}\underset{\sim}{N}\,\mathrm{tr}\left(\left(\mathcal{D}_{[a}\tilde{\sigma}_{|c|}\right)\tilde{\sigma}^b\right)\tilde{\epsilon}_{b]}{}^{dc}C_d.
\end{aligned}
$$

APPENDIX D: THE REALITY CONDITIONS

Consider the variables $(\tilde{\sigma}^a, A_a)$. We want to prove that if they define a real metric on an initial surface, then this metric remains real under the Hamiltonian evolution. In order to do this, we shall calculate the evolution of the variables $q_{ab} = -\mathrm{tr}\ (\sigma_a\sigma_b)$ and $\pi_{ab} = -\mathrm{tr}\ (\pi_a\sigma_b)$, where π_a is defined by

$$\mathcal{D}_a\lambda_A = D_a\lambda_A + \frac{i}{\sqrt{2}}\pi_{aA}{}^B\lambda_B.$$

We write the evolution of $\tilde{\sigma}_a$ in terms of π_{ab} as follows

$$
\begin{aligned}
\mathcal{L}_t\tilde{\sigma}^b \;&=\; \mathcal{L}_{\vec{N}}\tilde{\sigma}^b-\frac{i}{\sqrt{2}}\mathcal{D}_a\left(\underset{\sim}{N}[\tilde{\sigma}^a,\tilde{\sigma}^b]\right)+\frac{i}{\sqrt{2}}\underset{\sim}{N}[\mathcal{D}_a\tilde{\sigma}^a,\tilde{\sigma}^b]\\
&=\; \mathcal{L}_{\vec{N}}\tilde{\sigma}^b-i\,\mathcal{D}_c\underset{\sim}{N}\,\sigma\,\epsilon^{cbe}\,\tilde{\sigma}_e+\sigma\underset{\sim}{N}\left(\pi^b{}_e\tilde{\sigma}^e-\pi\tilde{\sigma}^b\right), \qquad (39)
\end{aligned}
$$

then we calculate

$$
\begin{aligned}
\mathcal{L}_t\tilde{q}^{ab} \;&=\; -2\mathrm{tr}\left(\mathcal{L}_t\tilde{\sigma}^{(a}\tilde{\sigma}^{b)}\right)\\
&=\; \mathcal{L}_{\vec{N}}\tilde{q}^{ab}+2\underset{\sim}{N}\sigma^3\left(\pi^{(ab)}-\pi q^{ab}\right). \qquad (40)
\end{aligned}
$$

The evolution of q_{ab} follows from the evolution of $\sigma^2 \equiv \det(q_{ab})$, thus we calculate

$$\mathcal{L}_t\sigma^2 = 2\sigma^2D_aN^a - 2\sigma^3\pi\underset{\sim}{N}. \qquad (41)$$

Finally, we get

$$\mathcal{L}_tq_{ab} = \mathcal{L}_{\vec{N}}q_{ab} - 2N\pi_{(ab)}. \qquad (42)$$

Thus in order to ensure the reality of q_{ab} we need to know the reality of π_{ab}. The evolution of π_{ab} is given by

$$\mathcal{L}_t\pi_{ab} = -\mathrm{tr}\left(\pi_a\mathcal{L}_t\sigma_b\right) + i\sqrt{2}\mathrm{tr}\left(\sigma_b\mathcal{L}_tA_a\right) - i\sqrt{2}\mathrm{tr}\left(\sigma_b\mathcal{L}_t\Gamma_a\right).$$

Hence in order to calculate it, we need to rewrite the evolution of A_b and σ_a in terms of π_{ab}. Using

$$F_{ab} = R_{ab} - \pi_{[a}\pi_{b]} + i\sqrt{2}D_{[a}\pi_{b]}$$

where R_{ab} is the spinorial curvature, and redefining $C = \mathrm{tr}\left(\sigma^a \sigma^b F_{ab}\right)$ and $C_b = \mathrm{tr}\left(\sigma^a F_{ab}\right)$, we compute

$$\begin{aligned}
\mathcal{L}_t \sigma_b &= \mathcal{L}_{\vec{N}} \sigma_b - i\, D_c N\, \sigma\, \epsilon^c{}_{be}\, \sigma^e - N \pi_{eb} \sigma^e && (43)\\
\mathcal{L}_t A_b &= \mathcal{L}_{\vec{N}} A_b + \frac{i}{\sqrt{2}} N [\sigma^a, F_{ab}] - \frac{i}{\sqrt{2}} N \sigma_b C - i N \epsilon_b{}^{dc} \sigma_c C_d \\
&= \mathcal{L}_{\vec{N}} A_b + \frac{i}{\sqrt{2}} N [\sigma^a, R_{ab}] \\
&\quad - \frac{i}{\sqrt{2}} N \left(\pi_{an} \pi_b{}^a - \pi \pi_{bn}\right) \sigma^n - N [\sigma^a, D_{[a} \pi_{b]}] \\
&\quad - \frac{i}{\sqrt{2}} N \sigma_b C - i N \epsilon_b{}^{dc} \sigma_c C_d . && (44)
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{L}_t \pi_{ab} &= \mathcal{L}_{\vec{N}} \pi_{ab} + N \pi \pi_{ab} - 2 N \pi_{eb} \pi_a{}^e - N R_{ab} - q_{ab} N C + N \sqrt{2} \epsilon_{ab}{}^d C_d \\
&\quad - i 2 N \epsilon_b{}^{de} \mathrm{tr}\left(D_{[d} \pi_{a]} \sigma_e\right) + i D_c N \epsilon^{ce}{}_b \pi_{ae} \\
&\quad + i \sqrt{2} \mathrm{tr}\left(\left(\mathcal{L}_{\vec{N}} \Gamma_a - \mathcal{L}_t \Gamma_a\right) \sigma_b\right). && (45)
\end{aligned}$$

Using the fact that $D_a \sigma_b = 0$ and the formula 24 for Γ_a, the last three terms can be set as

$$\begin{aligned}
&- i 2 N \epsilon_b{}^{de} \mathrm{tr}\left(D_{[d} \pi_{a]} \sigma_e\right) + i D_c N \epsilon^{ce}{}_b \pi_{ae} + i \sqrt{2} \mathrm{tr}\left(\left(\mathcal{L}_{\vec{N}} \Gamma_a - \mathcal{L}_t \Gamma_a\right) \sigma_b\right) \\
= &- D_a D_b N - i N \epsilon_b{}^{dm} D_a \pi_{dm} - \frac{i}{2} q_{ab} \epsilon^{cmd} \left(D_c N \pi_{md} + N D_c \pi_{md}\right),
\end{aligned}$$

yielding

$$\begin{aligned}
\mathcal{L}_t \pi_{ab} &= \mathcal{L}_{\vec{N}} \pi_{ab} + N \pi \pi_{ab} - 2 N \pi_a{}^e \pi_{eb} - N R_{ab} - D_a D_b N \\
&\quad - \frac{N}{2} q_{ab} \left(R + \pi^2 - \pi_{dc} \pi^{cd}\right) - i N \epsilon_{ab}{}^d D^c \left(\pi_{dc} - \pi q_{dc}\right) + \\
&\quad 2 N \left(\pi^d{}_a \pi_{[db]} - \pi \pi_{[ab]} + \pi_{[ad]} \pi^d{}_b\right) \\
&\quad - i N \epsilon_b{}^{dm} D_a \pi_{[dm]} - \frac{i}{2} q_{ab} \epsilon^{cmd} \left(D_c N \pi_{[md]} - N D_c \pi_{[md]}\right). && (46)
\end{aligned}$$

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