The Heterotic String,
The Tangent Bundle
and Derived Categories

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Abstract

We consider the compactification of the $E_8 \times E_8$ heterotic string on a K3 surface with "the spin connection embedded in the gauge group" and the dual picture in the type IIA string (or F-theory) on a Calabi–Yau threefold X. It turns out that the same X arises also as dual to a heterotic compactification on 24 point-like instantons. X is necessarily singular, and we see that this singularity allows the Ramond-Ramond moduli on X to split into distinct components, one containing the (dual of the heterotic) tangent bundle, while another component contains the point-like instantons. As a practical application we derive the result that a heterotic string compactified on the tangent bundle of a K3 with ADE singularities acquires nonperturbatively enhanced gauge symmetry in just the same fashion as a type IIA string on a singular K3 surface. On a more philosophical level we discuss how it appears to be natural to say that the heterotic string is compactified using an object in the derived category of coherent sheaves. This is necessary to properly extend the notion of T-duality to the heterotic string on a K3 surface.

1   Introduction

Back in the neolithic period of string theory [1] it was realized that one of the simplest methods of achieving a fairly realistic model from string theory is to compactify the heterotic string on a Calabi–Yau manifold and "embed the spin connection in the gauge group". Such a compactification consists of using the tangent bundle of the Calabi–Yau manifold as the vector bundle on which the gauge degrees of freedom of the heterotic string are compactified. When this is done, certain nonperturbative aspects of the world-sheet field theory tend to simplify [2].

Not long after this proposal, in such papers as [3], it was realized that perhaps the tangent bundle did not play a particularly distinguished rôle in the compactification of heterotic strings. Having said that, the underlying conformal field theory does have more supersymmetry in the case of the tangent bundle than it would in the generic case and for that reason alone one should expect it to have somewhat special properties.

More recently we have learnt that duality is a very powerful way of probing all the nonperturbative effects that may come into play when considering a compactification. In this context, the duality between the heterotic string and the type IIA string (or F-theory) should be a good way to analyze such compactifications.

At present the analysis of moduli for the heterotic string on a Calabi–Yau threefold and of its duality with F-theory is a technically formidable subject, although some progress has been made (see, for example, [4–9]). In this paper we will tackle the simpler question concerning the $E_8 \times E_8$ heterotic string compactified on the tangent bundle of a K3 surface. We embed the SU(2) of the spin connection into one of the E_8's leaving an unbroken $E_7 \times E_8$ generically. First, we wish to pose the following question: On what Calabi–Yau threefold X should one compactify F-theory so that the theory is dual to the $E_8 \times E_8$ heterotic string compactified on the tangent bundle of a K3 surface S? Equivalently one may find X such that the type IIA string compactified on X is dual to the $E_8 \times E_8$ heterotic string compactified on the tangent bundle of the product of a K3 surface and a 2-torus.

Although the study of the heterotic string on a K3 surface should be much easier than for the heterotic string on a Calabi–Yau threefold, it is still far from completely understood. The initial conjectures regarding its duality to the type IIA string were presented in [10] and then progress was made in [11–13].

If we go to part of the moduli space of theories where both the type IIA string and the heterotic string are free from nonperturbative corrections then one may define a systematic map between the two theories by using "stable degenerations" as used in [4, 9, 14, 15]. This is the method by which we will attack the tangent bundle problem.

The problem of finding the geometry of X is not particularly difficult but it gives rise to many questions. In particular we will find that the tangent bundle appears to be almost identical to point-like instantons. This raises a puzzle as the physics of the heterotic string on the tangent bundle should be quite different than that of point-like instantons.

We are led to a careful analysis of the Ramond-Ramond degrees of freedom of the type IIA string, as it is these which distinguish the two cases. This problem has also been studied for the case of the tangent bundle in [16]. We will discover that it is necessary to make some refinements of the usual rules of F-theory in predicting gauge symmetries and counting of tensor moduli in six dimensions.

The understanding of the tangent bundle can be pursued in various directions. One of the things one may do is to build on the work of papers such as [14, 17] in cataloging the possibilities of six dimensional physics for compactification of the heterotic string on a K3 surface. We do this in section 4 and discover that the heterotic string may acquire enhanced gauge symmetries on A-D-E singularities in a way very similar to the type IIA string.

Instead, on a more fundamental level, we may try to analyze global questions about the moduli space. As mentioned above, the usual heterotic/F-theory duality arguments are confined to a large codimension boundary of the moduli space where both theories are "weakly-coupled". We do not currently have a description of string theory which is in any way exact as we move into the open interior of the hypermultiplet moduli space — both theories suffer from quantum corrections [15]. By restricting to the tangent bundle of the heterotic string we will actually be able to penetrate deeply into the moduli space along a closed subspace while retaining exactness. This will lead us to some new claims about the definition of the data specifying the heterotic string. We discuss this in section 5.

These two goals will require a fairly sophisticated analysis of the geometry of sheaves on a K3 surface and this comprises a large part of this paper. In particular in section 2 we will review the mathematics of sheaves which we will require. Of particular interest are the Mukai vector and the notion of the derived category of coherent sheaves. In section 3 we discuss how

the heterotic string compactified on a sheaf is interpreted in the type IIA or F-theory language. We discuss how F-theory rules for determining the massless spectrum are affected by RR moduli. We also propose that a point-like instanton should be identified with the ideal sheaf of a point. To some extent the contents of section 2 are important for section 5 and the contents of section 3 are important for section 4. Having said this, these two subjects are somewhat interconnected and it would be awkward to disentangle them.

2   The Fourier-Mukai Functor

2.1   Some Sheaf Generalities

To do the analysis required in this paper it will be much easier to use the language of sheaves rather than vector bundles. This will allow us to handle vector bundles on the K3 surface, vector bundles over curves within this surface, and "skyscraper" sheaves all at once. Consider a vector bundle V over some manifold M. We will assume that the vector bundle is a holomorphic vector bundle, and in particular that the structure group of the bundle is $SU(N)$ for some N. This assumption is valid for what we require in this paper but is certainly not sufficient for a general analysis of the heterotic string. See [6, 7, 9, 15] for methods in the general case. The sheaf version of this holomorphic vector bundle is to associate to each open neighbourhood $U \subset M$, the group of holomorphic sections of V over U. A sheaf obtained in this way is called "locally free". There are sheaves which are not locally free however. Consider a submanifold $B \subset M$. One may consider a sheaf over M which is defined by a vector bundle $W \to B$ as follows. For every open neighbourhood $U \subset M$ associate the group of holomorphic sections of W restricted to $U \cap B$. If $U \cap B$ is empty, take this group to be 0.

Thus we can have a sheaf over M defined by a vector bundle over some subspace of M. In a sense, the rank of this vector bundle has some fixed value over B but is zero elsewhere. Indeed in this way a sheaf can represent a bundle whose rank varies over M. In the example where the groups associated to the sheaf vanish unless $U \cap B$ is nonempty, then the sheaf is said to be "supported" over B. The extreme example of this is the "skyscraper sheaf" which is supported only at a single point, where the "fibre" is $\mathbb{C}$.

Sheaves are very natural objects in the context of string duality. They have been used for example in [18, 19]. It seems clear that one cannot hope to gain a full understanding of the moduli space of string theories when restricting one's attention to only locally-free sheaves. We will also see further

evidence in this paper to this effect. The main difference between vector bundles and sheaves is that the former objects can be described in terms of differential geometry, the latter objects are creatures very much of an algebro-geometrical upbringing. For this reason, much of the mathematics involved tends to be of an algebraic nature.

If we have some algebraic variety X, the most basic sheaf over X is the structure sheaf $\mathscr{O}_X$. This is the sheaf built from sections of the trivial holomorphic line bundle over X if X is smooth. One can view $\mathscr{O}_X$ as a ring. It is then natural to generalize the concept of a holomorphic vector bundle to an $\mathscr{O}_X$-module. A free $\mathscr{O}_X$-module will correspond to a trivial vector bundle. We will assume that we are always dealing with a particular type of $\mathscr{O}_X$-module — namely a *coherent* sheaf.[1]

We will be concerned with the way that one may build sheaves from others using the algebraic notion of "extensions". Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be two sheaves over some fixed variety X. To be pedantic, we consider $\mathscr{E}_1$ and $\mathscr{E}_2$ to be $\mathscr{O}_X$-modules. One may then build a third sheaf, $\mathscr{F}$, from the exact sequence of $\mathscr{O}_X$-modules:

$$0 \to \mathscr{E}_1 \xrightarrow{a} \mathscr{F} \to \mathscr{E}_2 \to 0. \tag{1}$$

Here, $\mathscr{F}$ is not defined uniquely. Algebraically, in the above sequence $\mathscr{F}$ is defined by an element of $\mathrm{Ext}_{\mathscr{O}_X}(\mathscr{E}_2, \mathscr{E}_1)$. As a sheaf however, two elements of $\mathrm{Ext}_{\mathscr{O}_X}(\mathscr{E}_2, \mathscr{E}_1)$ which differ by a nonzero constant give rise to two isomorphic $\mathscr{F}$'s. This is because we may multiply the map a in (1) by a nonzero constant, thus changing the extension class without changing the sheaf $\mathscr{F}$.

One must pay special attention to rather peculiar things which may happen when considering the moduli space of sheaves (or indeed vector bundles). Of particular concern is the notion of "S-equivalence". Let us consider the problem of building a family of sheaves of the above form over some fixed variety X. Let $A = \mathrm{Ext}_{\mathscr{O}_X}(\mathscr{E}_2, \mathscr{E}_1)$. We may then build the "universal extension", $\mathscr{F}_A$ over $A \times X$ by

$$0 \to p_2^* \mathscr{E}_1 \to \mathscr{F}_A \to p_2^* \mathscr{E}_2 \to 0, \tag{2}$$

where $p_2 : A \times X \to X$ is the natural projection. The restriction of $\mathscr{F}_A$ to a point $a \in A$ will then specify a sheaf $\mathscr{F}_a$ over X.

Consider a complex line through the origin in A and restrict $\mathscr{F}_A$ to it. From what we have said above $\mathscr{F}_a$ will be everywhere the same except where

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[1] If one believes that string theory has an algebraic origin then (quasi-)coherent sheaves are a natural choice. A (quasi-)coherent sheaf is defined as a sheaf which may be constructed from the same underlying algebraic structure as the scheme on which it lives. See [20] for an exact statement of this.

$a = 0$. That is we have a family of sheaves which is constant everywhere except for one member where it "jumps" to something inequivalent. This is precisely the kind of thing one wants to avoid when building a nice moduli space!

The simplest approach is to define our way out of the above problem by saying that we wish to build the moduli space of S-equivalence classes of sheaves over X. We then say that two sheaves $\mathscr{F}_1$ and $\mathscr{F}_2$ are "S-equivalent" if we may build a family of sheaves which are all isomorphic to $\mathscr{F}_1$ except for one member which is isomorphic to $\mathscr{F}_2$.

While such an approach is fine if one is concerned with the isolated problem of building a nice-looking moduli space, we are supposed to be describing moduli spaces of string theories in this paper. Presumably string theory "knows" whether it really wants S-equivalent but not isomorphic sheaves to be considered the same or not.

An example which will be important to us is the following. Let $X = E$, an elliptic curve, and let $\mathscr{E}_1$ and $\mathscr{E}_2$ be isomorphic to $\mathscr{O}_E$. Thus we have

$$0 \to \mathscr{O}_E \to \mathscr{F} \to \mathscr{O}_E \to 0, \tag{3}$$

where $\mathscr{F}$ is a rank 2 sheaf. Now $\text{Ext}_{\mathscr{O}_E}(\mathscr{O}_E, \mathscr{O}_E) \cong H^1(E, \mathscr{O}_E) \cong \mathbb{C}$ (see for example page 234 of [20]) and so (3) defines two possibilities for $\mathscr{F}$. Either we have the trivial extension $\mathscr{F} \cong \mathscr{O}_E \oplus \mathscr{O}_E$, or we have the unique nontrivial extension.

Consider now building the moduli space of degree zero rank 2 (semi-stable) sheaves over E. The moduli space of S-equivalence classes will be nice but the true moduli space of sheaves will essentially have two points, representing the two possibilities above, at one of the locations of moduli space — the moduli space is not separated. This is discussed in some detail in [7].

Now if we were to consider the moduli space of heterotic strings on a 2-torus, the above system should fit nicely into it. The moduli space of the heterotic string on a 2-torus is well-understood and has no pathologies. Thus in this case, we need to use the moduli space of S-equivalence classes to get agreement with string theory. This can be explained because in this case the moduli space in question can be written in terms of the moduli space of solutions to Yang–Mill's equations. The relationship between Yang–Mill's bundles and stability is well-understood (see for example [21]). As we shall see however, it is not true that we may always use S-equivalence for all problems in string theory. The distinction becomes especially important when we consider families of sheaves: the behavior of the family away from

the central fiber may force us to choose one of several S-equivalent objects over the special fiber. In section 3.2 we will in fact want to distinguish between the two inequivalent extensions of (3).

2.2 The Fourier-Mukai Transform of a Sheaf

In this section we wish to analyze the moduli space of sheaves on an elliptic K3 surface. We will review the usual method [4,5,7] of first analyzing sheaves on a single elliptic curve and then extending this idea to elliptic surfaces.

Let E be a 2-torus with a complex structure and a distinguished point $0 \in E$. That is, E is an elliptic curve. The moduli space of flat line bundles on E may be viewed as the space of flat connections which in turn can be viewed as the space $\hat{E} = H^1(E, \mathbb{R}/\mathbb{Z})$. $\hat{E}$ is dual to E. We may also view $\hat{E}$ as the Jacobian of E. Actually $\hat{E}$ has the same complex structure as E and so is in some sense isomorphic to E. Indeed we may define a degree zero line bundle on E by fixing a meromorphic section which has a pole at 0 and a zero at some other point $x \in E$. The moduli space of line bundles on E of degree zero may then be identified with the moduli space of x — which is E.

The Poincaré bundle, $\mathscr{P}$, is a bundle over $E \times \hat{E}$ with the following properties. It is "universal" in the sense that when restricted to $E \times x$, where $x \in \hat{E}$, it gives the bundle over E defined by the point in the moduli space $x \in \hat{E}$. We also demand that $\mathscr{P}$ restricted to $0 \times \hat{E}$ is trivial. We mainly view $\mathscr{P}$ as the associated sheaf to this vector bundle. Formally we may define $\mathscr{P}$ as

$$\mathscr{P} = \mathscr{O}_{E \times \hat{E}}(\Delta - 0 \times \hat{E} - E \times \hat{0}), \qquad (4)$$

where Δ is the diagonal divisor representing the graph of the isomorphism $E \to \hat{E}$.

Let Z be a surface and let $\pi : Z \to B$ be an elliptic fibration. We may replace every fibre by its Jacobian to obtain another elliptically fibred surface $\hat{\pi} : \hat{Z} \to B$. Some care is needed in analysis of this process over the bad fibres (see, for example, [22]). If the fibration $\pi : Z \to B$ has a section, which we assume is the case from now on, then $\hat{Z}$ is actually isomorphic, as a complex variety, to Z. Nevertheless we will often find it convenient to distinguish between Z and $\hat{Z}$, as was done in [5].

We may extend $\mathscr{P}$ to be a sheaf over $Z \times_B \hat{Z}$ in the obvious way (except again for subtleties at bad fibres). Recall that $Z \times_B \hat{Z}$ is the fibration

$Z \times_B \hat{Z} \to B$ with fibre $\pi^{-1}(b) \times \hat{\pi}^{-1}(b)$ for any point $b \in B$. We may also define projection maps $p : Z \times_B \hat{Z} \to Z$ and $\hat{p} : Z \times_B \hat{Z} \to \hat{Z}$.

Now take a nice smooth locally-free sheaf, $\mathscr{F}$, associated to some smooth $SU(N)$-bundle over Z. Consider the following sheaf:

$$F^1(\mathscr{F}) = R^1\hat{p}_*(\mathscr{P} \otimes p^*\mathscr{F}). \tag{5}$$

It is common to call $F^1(\mathscr{F})$, which is a sheaf over $\hat{Z}$, the "Fourier-Mukai transform" of $\mathscr{F}$ after [23]. We are working in the "relative" setting. That is we are applying the transformation to each elliptic fibre of a fibration. This was analyzed in [24]. We refer to this paper for more details of what follows. We now explain (5) in detail.

For any sheaf $\mathscr{E}$ over $Z \times_B \hat{Z}$, one may loosely take $R^1\hat{p}_*\mathscr{E}$ to be the sheaf whose "fibre" (or stalk) over a point $z \in \hat{Z}$ is the group $H^1(E_z, \mathscr{E}|_{E_z})$, where $E_z \subset Z$ is the elliptic fibre $\pi^{-1}\hat{\pi}(z)$. Let $\mathscr{E} = \mathscr{P} \otimes p^*\mathscr{F}$. Since we have an $SU(N)$-bundle (with zero c_1) the sheaf $\mathscr{E}|_{E_z}$ will have degree zero. Riemann-Roch then tells us that $\dim H^0(E_z, \mathscr{E}|_{E_z}) = \dim H^1(E_z, \mathscr{E}|_{E_z})$. Now a generic degree zero sheaf over E_z will have no global sections and so $H^1(E_z, \mathscr{E}|_{E_z}) = 0$ generically. This shows that $F^1(\mathscr{F})$ is supported over some proper subset of $\hat{Z}$.

The sheaf $F^1(\mathscr{F})$ will be supported at points where $\mathscr{E}|_{E_z}$ contains a trivial summand. This will generically happen over N points of the elliptic curve $\hat{\pi}^{-1}(b)$ where the Poincaré bundle nicely cancels out one of the summands of $\mathscr{F}$. Therefore $F^1(\mathscr{F})$ is supported on a curve which is an N-fold cover of B. This is the "spectral curve" $C_S \subset \hat{Z}$. Generically only one summand will be trivialized at a time and so $F^1(\mathscr{F})$ is rank one over C_S.

As an example let us consider the simplest case of $\mathscr{O}_Z$. That is, we begin with the sheaf corresponding to the trivial line bundle over Z. Now clearly $F^1(\mathscr{O}_Z) = R^1\hat{p}_*\mathscr{P}$. This will be a sheaf supported along the section of the fibration $\pi : Z \to B$. Actually, it follows from either relative duality or from Grothendieck-Riemann-Roch (see, for example Theorem 2.8 of [24]) that this sheaf corresponds to the canonical bundle of B, i.e., a degree -2 line bundle over $B \cong \mathbb{P}^1$.

It is natural to consider a closely-related transform:

$$F^0(\mathscr{F}) = \hat{p}_*(\mathscr{P} \otimes p^*\mathscr{F}). \tag{6}$$

In our above argument this would amount to considering a sheaf whose stalks were given by $H^0(E_z, \mathscr{E}|_{E_z})$ rather than $H^1(E_z, \mathscr{E}|_{E_z})$. One might believe therefore that it is equivalent to $F^1(\mathscr{F})$ in some sense. Actually, due to

sloppiness in the above argument this is not the case. One really should be very careful when building these sheaves to treat the notion of a stalk of a sheaf correctly. If $\mathscr{F}$ is a locally-free sheaf as above then $F^0(\mathscr{F}) = 0$. This follows from the fact that sections of $F^0(\mathscr{F})$ over an open set $U \subset \hat{Z}$ are the same as sections of $\mathscr{P} \otimes p^*\mathscr{F}$ over $\hat{p}^{-1}(U)$. Since fibres of the latter vanish at most points of $\hat{p}^{-1}(U)$, the sections must all vanish identically.

The reasons for the difference between F^0 and F^1 are important but rather complicated and we do not discuss them here. We refer the reader to section III.12 of [20] for a full treatment.

Now let $\mathscr{E}_0$ be a sheaf on Z supported over the zero section where it is the trivial line bundle, of degree zero. It is not hard to see that $F^0(\mathscr{E}_0) \cong \mathscr{O}_Z$. Thus, except for the mismatch in degrees, F^0 looks like the inverse of F^1 when we act on the structure sheaf $\mathscr{O}_Z$.

To cure this we introduce a new transform:

$$T(\mathscr{F}) = \mathscr{F} \otimes \mathscr{O}_Z(E), \tag{7}$$

where $\mathscr{O}_Z(E)$ is the sheaf corresponding to the line bundle whose c_1 is equivalent to a generic elliptic fibre of Z. We also define:[2]

$$S^i(\mathscr{F}) = TF^i(\mathscr{F}) = R^i\hat{p}_*(\mathscr{P} \otimes p^*\mathscr{F}) \otimes \mathscr{O}_{\hat{Z}}(E). \tag{8}$$

The reader may like to check that for a locally-free sheaf $\mathscr{F}$ with $c_1 = 0$, $S^0 S^1(\mathscr{F})$ is very nearly the same thing as $\mathscr{F}$. That is S^0 acts rather like the inverse of S^1. The slight mismatch is that $S^0 S^1$ actually acts as -1 on every elliptic fibre of $\pi : Z \to B$. Since S^0 takes a locally free sheaf to 0, then it cannot be generally true that S^0 and S^1 are inverses to each other for any sheaf. In order to make a cleaner statement about S^0 and S^1 we are required to go to the "derived category".

2.3 The Derived Category and the Mukai Vector

Our goal is to find some kind of transformation we can do to a sheaf to turn it into another sheaf such that the resulting sheaf is "equivalent" to the first sheaf as far as string theory is concerned.

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[2]Note that our use of S and T differs from [24]. Our notation is supposed to be reminiscent of $SL(2, \mathbb{Z})$ in that S satisfies $S^2 = -1$ while T is of infinite order. But actually our S and T sit in the two distinct copies of $SL(2, \mathbb{Z})$ (inside $O(2, 2; \mathbb{Z})$), so they commute, and generate an abelian subgroup $\mathbb{Z} \times \mathbb{Z}_4$.)

The S^1 generator above looks promising in as much as it gives us a sheaf supported along the spectral curve when applied to a locally free sheaf. It is somewhat unsatisfactory that S^1, when applied to this resulting sheaf on the spectral curve, gives us 0. Clearly S^1 does not generally map within any kind of nontrivial equivalence class. We had to use S^0 instead to return to our original locally-free sheaf. It would be useful if we could somehow combine S^0 and S^1 into a single operator $\mathbf{S}$. (We also combine F^0 and F^1 into $\mathbf{F}$.)

In order to do this we need to change the objects we use. It turns out that rather than thinking about sheaves, we should think of complexes of sheaves. In categorical language we will use the "derived" category of coherent sheaves. We will denote the (bounded) derived category of coherent sheaves on Z by $\mathbf{D}(Z)$. The notion of derived categories is rather complicated and the details are beyond what we require for this paper. We refer the reader to [25] for an account of the original motivation of their construction or [26] for a detailed definition.

The general idea is this. An object of the derived category is no longer a sheaf $\mathscr{F}$, but rather a complex of sheaves $\mathbf{C}(\mathscr{F})$:

$$\ldots \to C^2(\mathscr{F}) \to C^1(\mathscr{F}) \to C^0(\mathscr{F}) \to \ldots , \tag{9}$$

taken up to certain equivalences. For example, if one begins with a sheaf $\mathscr{F}$, then one may associate to this the complex $\mathbf{S}(\mathscr{F})$:

$$\ldots \to 0 \to S^1(\mathscr{F}) \xrightarrow{0} S^0(\mathscr{F}) \to 0 \to \ldots \tag{10}$$

Often only one of the terms in this complex will be nontrivial and we may use this sheaf to "represent" $\mathbf{S}(\mathscr{F})$. When this is the case, the functor $\mathbf{S}$ looks like it should satisfy our requirements. In general however $\mathbf{S}$ is a functor acting on the *derived* category of sheaves and not the category of sheaves itself.

If we identify $\hat{Z}$ with Z we may then state the following relative version of Mukai's result.

Theorem 1 *The functor* $\mathbf{S}$ *satisfies*

$$\mathbf{S}^2 = (-1)_E[-1], \tag{11}$$

where $(-1)_E$ *is the inversion of each elliptic fibre and* $[-1]$ *denotes a shift of the complex of one to the right.*

We refer to [23, 24] for a proof.

The other construction due to Mukai that we will also find of great use in this paper is the "Mukai Vector" [27]. Let Z be a K3 surface and let us represent an element of $H^*(Z, \mathbb{Z}) = H^0(Z, \mathbb{Z}) \oplus H^2(Z, \mathbb{Z}) \oplus H^4(Z, \mathbb{Z})$ by the triple (a, b, c). Here $a \in H^0(Z, \mathbb{Z})$, $b \in H^2(Z, \mathbb{Z})$, and $c \in H^4(Z, \mathbb{Z})$. We may take a and c to be integers since $H^0(Z, \mathbb{Z}) \cong H^4(Z, \mathbb{Z}) \cong \mathbb{Z}$.

If $\mathscr{F}$ is a coherent sheaf we then define the Mukai vector $\boldsymbol{\mu}(\mathscr{F}) \in H^*(Z, \mathbb{Z})$ by

$$
\begin{aligned}
\boldsymbol{\mu}(\mathscr{F}) &= \operatorname{ch}(\mathscr{F}).\sqrt{\operatorname{td}(Z)} \\
&= \left(r, c_1(\mathscr{F}), r + \tfrac{1}{2}(c_1^2(\mathscr{F}) - 2c_2(\mathscr{F})) \right),
\end{aligned}
\tag{12}
$$

where r is the rank of $\mathscr{F}$. The Mukai vector has already made appearances in the string literature. It has appeared in the context of D-branes and anomalies in [28] and has been used subsequently in work such as [19]. Some interesting observations about the nature of the Mukai vector in string theory were also made in [29]. Given that it takes values in $H^*(Z, \mathbb{Z})$ which is a very natural object in string theory [30], it is not at all surprising that it should also make an appearance in the analysis of the heterotic string. Note that $\boldsymbol{\mu}$ has the following property induced from the Chern character:

$$
\boldsymbol{\mu}(\mathscr{E} \oplus \mathscr{F}) = \boldsymbol{\mu}(\mathscr{E}) + \boldsymbol{\mu}(\mathscr{F}).
\tag{13}
$$

We may also extend the definition of $\boldsymbol{\mu}$ to $\mathbf{D}(Z)$ by

$$
\operatorname{ch}(\mathbf{C}(\mathscr{F})) = \sum_i (-1)^i \operatorname{ch}(C^i(\mathscr{F})).
\tag{14}
$$

By doing this we can specify the induced action of $\mathbf{S}$ on $H^*(Z, \mathbb{Z})$. By using (11) we can immediately see the induced action of $\mathbf{S}^2$. Since $(-1)_E$ acts trivially on our even homology cycles and $[-1]$ reverses the sign because of (14), we see that $\mathbf{S}^2$ simply acts as -1 on $H^*(Z, \mathbb{Z})$.

We will consider the case where Z is a generic elliptic K3 surface with a section. This means that the Picard lattice of Z is a two dimensional lattice in the form of a hyperbolic plane U. Let v_0 represent the class of the generic elliptic fibre and let v_1 represent the sum of the class of the section and v_0. It can be easily seen that the intersection form on Z gives rise to the product

$$
\begin{aligned}
v_0.v_0 &= v_1.v_1 = 0 \\
v_0.v_1 &= 1.
\end{aligned}
\tag{15}
$$

We also define an inner product on $H^*(Z, \mathbb{Z})$ by

$$(a_1, b_1, c_1).(a_2, b_2, c_2) = b_1.b_2 - a_1 c_2 - c_1 a_2. \tag{16}$$

Before we proceed we need one more fact. Consider a line bundle of degree d on some curve $C \subset Z$. Let $\mathscr{F}$ be the sheaf on Z but supported only on C corresponding to this bundle. One may show that

$$\begin{aligned} c_1(\mathscr{F}) &= C \\ c_2(\mathscr{F}) &= C.C - d, \end{aligned} \tag{17}$$

where $C.C$ is the self-intersection of C.

We now specify the action of **S** on some elements of $H^*(Z, \mathbb{Z})$.

- $\mu = (1, 0, 1)$. This is the structure sheaf $\mathscr{O}_Z$. As mentioned above, F^1 transforms this to a sheaf of degree -2 supported along the section $v_1 - v_0$. Applying T will change the degree to -1. It follows that

$$\mathbf{S}(1, 0, 1) = (0, v_0 - v_1, 0). \tag{18}$$

- $\mu = (0, v_0, 0)$. This is a sheaf supported along a single elliptic fibre of Z. Over this fibre, the sheaf has rank one and degree 0. When S^1 is applied, we obtain the skyscraper sheaf $\mathscr{O}_z$ supported at the single point z. Applying T does nothing. One may show that $c_2(\mathscr{O}_z) = -1$ and therefore

$$\mathbf{S}(0, v_0, 0) = (0, 0, -1). \tag{19}$$

- $\mu = (0, v_1 - v_0, 0)$. The corresponding sheaf with this Mukai vector is $\mathscr{O}(-1)$ on the section. Its F^0 is $\mathscr{O}_Z(-E)$, so its S^0 is $\mathscr{O}_Z$, and we have:

$$\mathbf{S}(0, v_1 - v_0, 0) = (1, 0, 1). \tag{20}$$

- $\mu = (0, 0, 1)$. This is the skyscraper sheaf $\mathscr{O}_z$. Applying S^0 to this we obtain the sheaf supported along a single elliptic fibre. Again T does nothing.

$$\mathbf{S}(0, 0, 1) = (0, v_0, 0). \tag{21}$$

To review (and using linearity) we obtain

$$\begin{aligned} \mathbf{S}(1, 0, 0) &= (0, -v_1, 0) \\ \mathbf{S}(0, v_0, 0) &= (0, 0, -1) \\ \mathbf{S}(0, v_1, 0) &= (1, 0, 0) \\ \mathbf{S}(0, 0, 1) &= (0, v_0, 0) \end{aligned} \tag{22}$$

Note that $\mathbf{S}^2 = -1$ as required by theorem 1 and that, up to signs, $\mathbf{S}$ simply exchanges the hyperbolic plane generated by (v_0, v_1) with the hyperbolic plane generated by $H^0(Z)$ and $H^4(Z)$.

As an application let us consider a locally-free sheaf, $\mathscr{F}_k$, corresponding to an $SU(N)$-bundle with $c_2 = k$ over a K3 surface Z. The above shows that

$$\boldsymbol{\mu}(\mathscr{F}_k) = (N, 0, N - k)$$
$$\boldsymbol{\mu}(\mathbf{F}(\mathscr{F}_k)) = (0, (N - k)v_0 - Nv_1, N) \tag{23}$$
$$\boldsymbol{\mu}(\mathbf{S}(\mathscr{F}_k)) = (0, (N - k)v_0 - Nv_1, 0).$$

This is consistent with the fact that $F^1(\mathscr{F}_k)$ is a sheaf supported on a curve in the class $(k - N)v_0 + Nv_1$ — the spectral curve C_S. This curve has self-intersection $2N(k-N)$ and thus has genus $g = N(k-N)+1$. It follows from the above calculations that the degree of the line bundle on C_S producing the required sheaf is $N(k - N - 1) = g - 1 - N$.[3] This result can also be obtained by noting that the direct image $\hat{\pi}_*$ preserves Euler characteristics: in our case, the image $\hat{\pi}_*(F^1(\mathscr{F}_k))$ is a rank N vector bundle on $B = \mathbb{P}^1$, which is easily seen to be the canonical bundle $\mathscr{O}_{\mathbb{P}^1}(-2)$ tensored with the restriction of $\mathscr{F}_k$. Its Euler characteristic therefore equals $-N$. This agrees with the Euler characteristic of a line bundle of degree $g - 1 - N$ on C_S.

We end this section by noting that one of the uses of the Mukai vector is to calculate the (complex) dimension of the moduli space of sheaves. To do this we restrict to the case where $\mathscr{F}$ is "simple" — that is the only automorphisms of $\mathscr{F}$ are global rescalings. In this case the dimension of the moduli space is given by [27]

$$\dim \operatorname{Ext}_{\mathscr{O}_Z}(\mathscr{F}, \mathscr{F}) = \boldsymbol{\mu}(\mathscr{F}).\boldsymbol{\mu}(\mathscr{F}) + 2. \tag{24}$$

This restriction to simple sheaves is fairly severe but will work for the above locally free sheaf $\mathscr{F}_k$. In this case the dimension of the moduli space is $2g = 2N(k - N) + 2$ in agreement with standard methods.

3   The heterotic string on the tangent bundle

3.1   Review

The general system we wish to analyze is an $E_8 \times E_8$ heterotic string compactified on a product of a K3 surface, Z, and a 2-torus. This is supposedly

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[3]There appears to be a typo in [5] for this formula.

dual to the type IIA string compactified on a Calabi–Yau threefold, X. We refer the reader to [15,31,32] for a general description of this system together with much of the notation we will be using.

We will study the moduli space coming from the heterotic string corresponding to deformations of Z together with a sheaf on Z. This maps by duality to deformations of the complex structure of X together with "RR"-parameters which, when X is smooth, take values in $H^3(X, \mathbb{R}/\mathbb{Z})$ (and the dilaton-axion). This part of the moduli space has a quaternionic Kähler structure and is the moduli space of hypermultiplets. The heterotic string's 2-torus has nothing to do with this moduli space and can be ignored. Thus we may view the physics as derived from a six-dimensional theory obtained by compactifying the heterotic string on Z. This is the F-theory picture.

Let us fix notation. We will assume that X is a Calabi–Yau threefold which has the structure both of an elliptic fibration $\pi_F : X \to \Sigma$ and a K3 fibration $p_F : X \to B$. Here Σ is a Hirzebruch surface $\mathbb{F}_n$ and $B \cong \mathbb{P}^1$. Z is an elliptic fibration $\pi : Z \to B$ as above. All the fibrations have sections. See [31] for a discussion of these assumptions.

Let C_0 be the section of Σ with self-intersection $-n$. Thus, C_0 is the isolated section of Σ if $n > 0$. Let f be a generic $\mathbb{P}^1$-fibre of Σ. Let σ_0 be the section of $\pi : Z \to B$ and let E refer to a generic elliptic fibre of either $\pi : Z \to B$ or $\pi_F : X \to \Sigma$. Connecting to section 2.3 we see that $v_0 = [E]$ and $v_1 = [\sigma_0] + [E]$.

One of the key points is that neither the heterotic nor type IIA string description of the moduli space is exact. Both suffer from quantum corrections. In the type IIA string case, we may switch the corrections off by setting the dilaton equal to $-\infty$. In terms of the heterotic string we may take this limit by letting the area of σ_0 go to infinity.

Once this is done we may remove the quantum corrections from the heterotic side by making the generic fibre of this elliptic fibration very large. As we will see later, this is not a necessary condition to remove the corrections but it is sufficient. In terms of X, this deformation corresponds to a change in complex structure which takes X to a stable degeneration. This degeneration is of the form of two threefolds, X_1 and X_2 which intersect along a K3 surface isomorphic to Z. We identify Z with this intersection from now on.

Essentially string duality identifies the geometry of X_1 with one of the E_8's of the heterotic string and X_2 with the other E_8. From now on we just focus on X_1 — all things stated also being true for X_2. We now have a fibration $p^1_F : X_1 \to B$ with fibre a rational elliptic surface and an elliptic
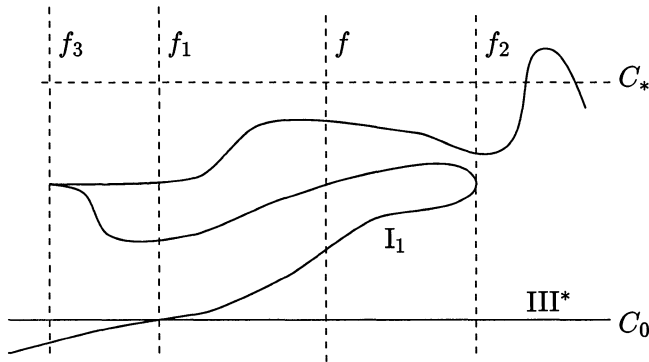
Figure 1: The discriminant for an SU(2)-bundle.

fibration $\pi_F^1 : X_1 \to \Sigma$ where Σ is still the Hirzebruch surface $\mathbb{F}_n$ [14]. We let C_* denote the section of Σ disjoint from C_0 where X_1 intersects X_2. Thus C_* is really the same thing as σ_0. Note that C_* is isolated if $n < 0$ (as will happen the case of interest).

The elliptic fibration $\pi_F^1 : X_1 \to \Sigma$ degenerates over a discriminant curve $\Delta^1 \subset \Sigma$. The geometry of X_1 also determines a spectral curve $C_S^1 \subset Z$. Indeed the moduli of Δ^1 determine the moduli of C_S^1. We refer to [15] for a description of this.

Under the stable degeneration the deformations of complex structure and RR-parameters on X divide into three pieces [15]. The complex structure of X defines the complex structure of Z and the complex structures of C_S^1 and C_S^2, together with the embeddings, $C_S^1 \subset Z$, $C_S^2 \subset Z$. The RR-parameters of X describe a B-field in $H^2(Z, \mathbb{R}/\mathbb{Z})$ and a line bundle (of a fixed degree) on each of the spectral curves by specifying an element of the Jacobian $H^1(C_S, \mathbb{R}/\mathbb{Z})$.

Suppose now we have a bundle with structure group $SU(N) \subset E_8$ used to compactify one of the E_8's of the heterotic string. In the language of the last section, the data for the spectral curve, C_S, and its line bundle is simply obtained by applying F^1 (or S^1) to the sheaf corresponding to this vector bundle.

Of particular interest to us is the case of SU(2)-bundles with $c_2 = k$. The geometry of Σ is then fixed by $n = 12 - k$ [11]. The geometry of this case was explicitly constructed in section 3.2 of [15]. We show the geometry of $\Delta^1 \subset \Sigma$ in figure 1. Δ^1 is a reducible curve having a component along C_0 above which the elliptic fibres are in Kodaira class III* and another

component above which the class is I$_1$. Here S_C is a double cover of B branched at $4k - 4$ points. Mapping Σ to B, these branch points occur at $k-4$ places where the I$_1$ component of Δ^1 collides with C_0 (marked in figure 1 as f_1) and the $3k$ points of tangency of Δ^1 with the f direction (marked as f_2).

Generically, X_1 has a curve of E_7 singularities (from the III* fibres) which suggests an E_7 gauge symmetry [12]. This arises since SU(2) $\subset E_8$ centralizes E_7. In some special cases this need not be the exact gauge symmetry as we discuss later.

3.2   Two special cases

From above we see that for a nice smooth SU(2)-bundle, or the associated sheaf $\mathscr{F}_k$, we obtain a spectral curve $C_S^1 \subset Z$ in the class

$$[C_S^1] = 2[\sigma_0] + k[E]. \tag{25}$$

Over this curve we have a line bundle of degree $2k - 6$. This represents the sheaf $F^1 \mathscr{F}_k$. The moduli of the curve C_S^1 are fixed, in the type IIA language, by the moduli of X_1. The moduli of the line bundle are fixed by the RR-fields in the type IIA string.

If C_S^1 is smooth then the moduli space of line bundles is a torus (abelian variety) $H^1(C_S^1, \mathbb{R}/\mathbb{Z})$ parametrized in the type IIA language by elements of $H^3(X_1, \mathbb{R}/\mathbb{Z})$. We will be concerned with the case where C_S^1 and X_1 are *not* smooth. In this case, the Jacobian of C_S^1 need not be a torus and so the RR-fields will not parameterize a torus either.

In the genus one case, this degeneration of the Jacobian is familiar from Kodaira's classification. We are doing a similar construction here except that the spectral curve has a large genus. Thus, we want to know how an abelian variety of a large complex dimension can degenerate in a family. This is in general a very hard question. In our case we will see that the Jacobian becomes reducible, so the RR moduli can take their values in either of several distinct components. Quite generally, this happens whenever the spectral curve becomes non-reduced. A detailed analysis of these components is given in [33] in the case where the spectral cover becomes everywhere non-reduced $C = 2C'$. In our present situation there is one additional feature, namely that only part of the spectral curve becomes non-reduced: it becomes (twice) the zero-section σ_0, plus (once) each of 24 fibres. However, the distinction we will establish between the two F-theory compactifications will be entirely due to the contribution of the non-reduced locus as seen in [33].

We will focus on two specific examples with $k = 24$. First we will consider degeneration of the SU(2)-bundle to that of 24 "point-like instantons". That is, this bundle becomes everywhere flat except at 24 points. We will denote the associated sheaf by $\mathscr{X}_{24}$. The precise mathematical meaning of this will become apparent. Second we will consider the tangent sheaf $\mathscr{T}_Z$.

In order to compute the spectral curves we need to compute F^1 of each of these sheaves. To do that we first consider a Fourier-Mukai transformation at a generic elliptic fibre, E, of Z.

First consider the point-like instanton sheaf $\mathscr{X}_{24}$. Clearly it restricts as

$$\mathscr{X}_{24}|_E \cong \mathscr{O}_E \oplus \mathscr{O}_E, \qquad (26)$$

so long as we are not at one of the 24 points where the instantons are located.

Now for the tangent sheaf case we have the exact sequence

$$0 \to \mathscr{T}_E \to \mathscr{T}_Z|_E \to \mathscr{N}_{E/Z} \to 0, \qquad (27)$$

where $\mathscr{N}_{E/Z}$ is the normal sheaf of $E \subset Z$. Now since $\mathscr{T}_E \cong \mathscr{N}_{E/Z} \cong \mathscr{O}_E$, we see that $\mathscr{T}_Z|_E$ is an extension of $\mathscr{O}_E$ by $\mathscr{O}_E$. In particular *on a generic fibre, the tangent sheaf is S-equivalent to the point-like instanton sheaf.*

To a string theorist it should come as a nasty surprise that the tangent sheaf and the point-like instanton sheaf should be so similar. There is certainly no hope that these two theories could somehow be dual to each other. The heterotic string on the tangent bundle of a smooth K3 surface should be a very well-behaved theory with no extra massless particles. The point-like instantons, on the other hand, are known to leave an unbroken E_8 gauge symmetry and generate massless tensor particles in six dimensions [11, 34].

One might hope that the full form of the spectral curve will get us out of this difficulty but we will see that this is not the case. Recall that we saw earlier that $F^1(\mathscr{O}_Z)$ produced a sheaf in Z supported along σ_0. Thus in each case we see that the generic elliptic fibres produce a double copy (in a way to be made precise) of σ_0 in the spectral curve. Thus, from (25) the non-generic contribution must be in the class $24[E]$. The only divisor of Z in this class is 24 copies of E. In both cases therefore C_S is a reducible curve containing σ_0 doubly and 24 fibres. The only question is which 24 fibres.

In the case of $\mathscr{X}_{24}$ the answer is clear — we use the 24 elliptic fibres containing the 24 instantons. In the case of $\mathscr{T}_Z$ for a generic K3 surface the answer is also obvious — we use the 24 I_1 fibres where the smooth elliptic fibres degenerate.[4] These are the only locations where the above arguments

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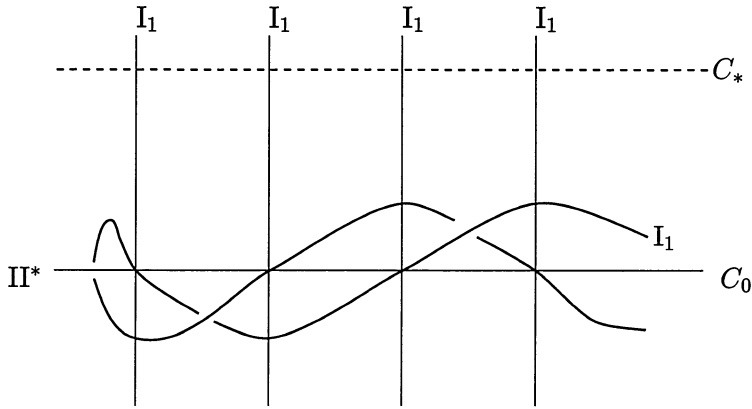[4]This was shown explicitly using toric geometry in [8].

Figure 2: The discriminant for the tangent bundle and point-like instantons.

fail.

In general we see that the spectral curve for $\mathscr{X}_{24}$ will be different than that for the tangent sheaf. Generically $F^1(\mathscr{X}_{24})$ contains 24 smooth fibres whereas the 24 fibres in $\mathscr{T}_Z$ are singular. We can remove this discrepancy however by specializing to the case of $\mathscr{X}_{24}$ where all 24 points are located on singular fibres of $\pi : Z \to B$. Now the spectral curves for $\mathscr{X}_{24}$ and $\mathscr{T}_Z$ look identical!

We thus obtain

Proposition 1 *Let X_T be the Calabi–Yau threefold which represents the F-theory dual of the $E_8 \times E_8$ heterotic string compactified on the tangent bundle of a K3 surface. Let X_I be the Calabi–Yau threefold which represents the F-theory dual of the same heterotic string theory compactified on the same K3 surface except with one point-like instanton located at a point on each of the 24 I_1 fibres of the elliptic fibration $\pi : Z \to B$. Then X_T is isomorphic to X_I.*

We draw the form of the discriminant locus $\Delta_1 \subset \Sigma$ for X_1 for this case in figure 2. This is also discussed in [15].

Applying the usual rules of F-theory to figure 2 we see that we have a curve of type II* fibres along C_0 which implies an E_8 gauge symmetry. We also have 24 I_1-II* collisions each of which must be blown up to resolve X. This implies 24 massless tensors. The result of an E_8 gauge symmetry and 24 massless tensors is exactly what we would expect for the 24 point-like instantons. It is not at all what we want for the tangent bundle. Clearly

we have to think a little more carefully about how the rules for F-theory should be applied and, more importantly, what precisely should distinguish the tangent bundle from the point-like instanton case.

3.3   Ramond-Ramond moduli and the degenerate Jacobian

The only thing in the type IIA language on X that remains that can possibly distinguish the two cases are the RR moduli. We know that these parameterize the exact form of the spectral sheaf supported on the spectral curve C_S. If C_S were smooth, we would expect the RR moduli to live on the torus representing the Jacobian of C_S. In order for duality to work, the same thing should be true when C_S degenerates except that this time the Jacobian will generally not be a torus. As we have seen before, the genus of the spectral curve C_S is $2k - 3 = 45$. So in the smooth case the Jacobian is a 45 (complex) dimensional torus, while in the singular case we expect the Jacobian to be more complicated, but still 45 dimensional.

We thus expect $\mathscr{I}_Z$ and $\mathscr{L}_{24}$ to be distinguished by the fact that their transforms under F^1 will differ as sheaves even though the spectral curve, the support of this sheaf, may look the same in both cases.

We begin by computing $F^1(\mathscr{I}_Z)$. This was also discussed in section 6.2 of [16]. Let us analyze the Fourier-Mukai transform acting on a single generic elliptic fibre, E, of X again — this time with a little more care. As mentioned earlier, since the restriction $\mathscr{I}_Z|_E$ is an extension of $\mathscr{O}_E$ by $\mathscr{O}_E$ we expect this to map, under F^1, to "twice the origin" of $\hat{E}$. To avoid cluttering notation we will omit the restriction to E for a while. Restricting our earlier definition of the Fourier-Mukai transformation to just a single elliptic fibre we have

$$F^1(\mathscr{I}_Z) = R^1\hat{p}_*(\mathscr{P} \otimes p^* \mathscr{I}_Z). \tag{28}$$

where now $p : E \times \hat{E} \to E$ and $\hat{p} : E \times \hat{E} \to \hat{E}$. This is a sheaf over $\hat{E}$ which has a trivial stalk except at the origin. Let us further restrict this bundle to the origin, $\hat{0}$. Now

$$\begin{aligned} F^1(\mathscr{I}_Z)|_{\hat{0}} &= \Big(R^1\hat{p}_*(\mathscr{P} \otimes p^* \mathscr{I}_Z)\Big)\big|_{\hat{0}} \\ &= H^1(E, \mathscr{P}|_{E \times \hat{0}} \otimes \mathscr{I}_Z) \\ &= H^1(E, \mathscr{I}_Z|_E), \end{aligned} \tag{29}$$

where we have used the "base change" theorem of [20] and the definition of the Poincaré bundle we stated earlier.

Now the exact sequence (27) gives

$$0 \to H^0(E, \mathscr{T}_E) \to H^0(E, \mathscr{T}_Z|_E) \to H^0(E, \mathscr{N}_{E/Z}) \xrightarrow{r} H^1(E, \mathscr{T}_E) \to$$
$$H^1(E, \mathscr{T}_Z|_E) \to H^1(E, \mathscr{N}_{E/Z}) \to 0. \quad (30)$$

Since $\mathscr{T}_E \cong \mathscr{N}_{E/Z} \cong \mathscr{O}_E$ we see that

$$\dim H^1(E, \mathscr{T}_Z|_E) = 2 - \mathrm{rank}(r). \quad (31)$$

It is also true that $\mathrm{Hom}(H^0(E, \mathscr{N}_{E/Z}), H^1(E, \mathscr{T}_E)) = \mathrm{Ext}(\mathscr{N}_{E/Z}, \mathscr{T}_E)$ and so $\mathrm{rank}(r) = 0$ if (27) splits and $\mathrm{rank}(r) = 1$ otherwise.

The map r is given by the Kodaira-Spencer theory of deformations. It shows how for a family of elliptic curves $\pi : Z \to B$, the tangent directions in the base $\mathscr{N}_{E/Z} \cong \mathscr{T}_B$ map into deformations of E. Generically, our elliptic K3 surface will not have constant fibre and so this map is not zero. Thus $\mathrm{rank}(r) = 1$ and $\dim H^1(E, \mathscr{T}_Z|_E) = 1$ for a generic E.

We have thus concluded that $F^1(\mathscr{T}_Z)$ restricted to $\hat{E}$ is a rather peculiar sheaf. It is of "length" two — that is it has two complex degrees of freedom, since $\mathscr{T}_Z$ is rank two, and yet it is supported only at a single point $\hat{0} \in \hat{E}$ whereupon it restricts to a sheaf of rank one. How can this be?

The answer is that it is the skyscraper sheaf on a "fat" point. Let $\mathscr{I}_{\hat{0}}^2$ be the sheaf of functions which vanish at $\hat{0}$ and whose derivative also vanishes at $\hat{0}$. The sheaf we require is $\mathscr{O}_{\hat{E}}/\mathscr{I}_{\hat{0}}^2$.

The upshot of all this is that we may state more clearly what we meant by saying that the spectral curve C_S of $F^1(\mathscr{T}_Z)$ contains σ_0 "doubly". C_S really contains a "fat" copy of the line σ_0. The sheaf $F^1(\mathscr{T}_Z)$ may be viewed as a sheaf supported on this fat σ_0 plus 24 I_1 fibres. Over this support the sheaf can be thought of as having rank one.

Let us contrast this to the transform of the point-like instantons. All of the analysis above may be repeated except that this time the sequence defining $\mathscr{Z}_{24}$ *is* generically split and so $r = 0$. This means that $\dim H^1(E, \mathscr{Z}_{24}|_E) = 2$ and so over a generic $\hat{E}$, $F^1(\mathscr{Z}_{24})$ looks like two copies of the skyscraper sheaf, $\mathscr{O}_{\hat{0}} \oplus \mathscr{O}_{\hat{0}}$.

We see that the sheaf $F^1(\mathscr{Z}_{24})$ is a sheaf whose support contains the reduced (non-fat) σ_0 but it has rank two over this curve.

The two sheaves $F^1(\mathscr{T}_Z)$ and $F^1(\mathscr{Z}_{24})$ are therefore qualitatively different objects even though naïvely their support looks the same. Fixing this spectral curve, each of these sheaves may be varied in a 45 parameter family.

This is what we would expect for the (complex) dimension of the RR moduli space. However, since these two families are quite different we must have at least two *components* to the Jacobian in which the RR moduli live. We thus resolve our problem with

Proposition 2 *When $X \cong X_1 \cong X_T$, the moduli space of RR fields has at least two components. The two points corresponding to the tangent bundle and the point-like instantons lie in different components.*

In particular, the moduli space of RR fields need not be a simple torus when X is singular. Its torus part is, in general, only 24 complex dimensional: it is the Jacobian of the reduced part of the spectral curve C_S, i.e. the product of the 24 elliptic fibers. (In the case we are actually considering, all 24 fibers happen to be singular, so this 24 dimensional part of the Jacobian is no longer compact, but rather the product of 24 copies of $\mathbb{C}^*$.) The remaining 21 dimensions come from the non-reduced structure along σ_0. It is this part which is reducible, with one component corresponding to deformations of the tangent bundle and another corresponding to deformations of the point-like instantons. We will see a more detailed description of these 21 parameters for deforming the point-like instanton in the next subsection.

3.4   Obstructed extremal transitions

Having sorted out why the tangent bundle is not the same thing as 24 point-like instantons as far as string duality is concerned, we need to tidy up the rules of F-theory. The heterotic string on a tangent bundle should not have massless tensors or totally unbroken gauge symmetry even though figure 2 suggests it should.

Let us consider the general picture of an extremal transition. Begin with a smooth three-dimensional Calabi–Yau manifold Y. Deform the complex structure of Y to produce a singular variety $Y^{\natural}$. In some cases $Y^{\natural}$ can be resolved by blowing up to produce a smooth Calabi–Yau threefold Y' which is not topologically (or birationally) equivalent to Y. It is not hard to see that $h^{2,1}(Y') < h^{2,1}(Y)$ and $h^{1,1}(Y') > h^{1,1}(Y)$.

Now when we put such an extremal transition in the context of type IIA compactifications we need to worry about the RR fields. Since $h^{2,1}(Y') < h^{2,1}(Y)$ it must be that the type IIA string compactified on Y' must have fewer RR degrees of freedom than Y. Thus, in order to pass through the extremal transition some of the RR parameters *must* be tuned to a fixed value — which we call zero.

Now we wish to claim that an enhancement of gauge symmetry corresponds to an extremal transition. To see this we think again in the 4-dimensional type IIA picture corresponding to the heterotic string compactified on Z times a 2-torus. When the structure group of the bundle decreases, the gauge symmetry is enhanced and we may switch on more vector moduli (which break the enhanced gauge symmetry back down to its Cartan subgroup). This corresponds to blowing up the fibre in the elliptic fibration of X [11] (see also [31]).

This potentially explains why the tangent sheaf doesn't give rise to the E_8 gauge symmetry. The blow-up in the fibre which raised the gauge symmetry from E_7 to E_8 completes an extremal transition which kills some of the RR fields which are not zero for the tangent sheaf. We will now try to justify this claim by arguing that these RR fields *are* zero for the point-like instantons.

First we wish to make a change to our concept of the point-like instanton. Until now we have really considered it to be an object that lives on the boundary of the moduli space of SU(2)-bundles. While this is true, the point-like instanton is really no more an SU(2)-object than an SU(3)-object or an object tied to any nontrivial group. Indeed, as the holonomy of a point-like instanton is trivial it would be more natural to associate it to a trivial structure group.

We will claim the following:

Proposition 3 *A point-like instanton is the ideal sheaf of a point, $\mathscr{I}_z$.*

In particular we claim that

$$\mathscr{Z}_{24} \cong \mathscr{O}_Z \oplus \mathscr{I}_{z_1,z_2,\dots,z_{24}}, \qquad (32)$$

where $\mathscr{I}_{z_1,z_2,\dots,z_{24}}$ is the ideal sheaf of functions vanishing at $z_1, z_2, \dots, z_{24}$ which are the locations of the point-like instantons. We no longer insist that these points lie in the I_1 fibres.

Clearly this claim is perfectly reasonably away from the instantons since both sides of (32) are then equal to $\mathscr{O}_Z \oplus \mathscr{O}_Z$. It follows that the spectral curve is a copy of σ_0, over which the Fourier-Mukai transformed sheaf has rank 2, and the 24 elliptic fibres containing the z_i's. The above argument implies that they live in the same component of the RR moduli space for this fixed spectral curve.

Let us now consider how we might parameterize this 45 dimensional component of the RR moduli space. Clearly 24 directions are given by

moving the locations of the z_i up and down the fibres. Locally near each instanton we may view the sheaf as $\mathscr{O}_Z \oplus \mathscr{I}_{z_i}$. However there is no reason why we need to globally insist that the sheaf decomposes as a sum over the whole of Z. At each of the 24 points z_i choose a direction $\mathbb{C} \subset \mathbb{C}^2$ (i.e., a point on $\mathbb{P}^1$) which specifies how we locally decompose the rank 2 sheaf into $\mathscr{O}_Z \oplus \mathscr{I}_{z_i}$. By an $SL(2,\mathbb{C})$ symmetry action on this $\mathbb{C}^2$ fibre, this gives us $24 - 3 = 21$ more parameters.

These numbers work perfectly for comparison with the E_8 transition. When the E_8 appears we do the blow-up in the fibre, $h^{2,1}$ decreases by 21 and we lose 21 complex parameters from the RR moduli. That is, in order to switch on the E_8 transition, the sheaf must *globally* decompose as $\mathscr{O}_Z \oplus \mathscr{I}_{z_i}$ in agreement with (32).

In many ways this gives the ideal sheaf the interpretation of an "SU(1)-bundle". Consider the moduli space of the sheaf, $\mathscr{F}(N)$, corresponding to an SU(N) bundle. There is a locus within this moduli space where the sheaf decomposes $\mathscr{O}_Z \oplus \mathscr{F}(N-1)$. This corresponds to perturbative enhanced gauge symmetry for the heterotic string. For example an SU(3)-bundle will leave an E_6 gauge symmetry unbroken but if we tune the bundle moduli we may decrease its structure group to SU(2) and obtain an unbroken E_7 gauge symmetry. Given this behaviour, it is natural to interpret the ideal sheaf, $\mathscr{I}_{z_1,z_2,\ldots,z_{24}}$, as a degenerate SU(1) bundle with $c_2 = 24$. The only feature particular to point-like instantons is that the extremal transition is more complicated and leads to massless tensors in addition to more gauge symmetry.

We hope the reader agrees that this is reasonable evidence to assert that the point-like instanton case really is the same thing as the ideal sheaf and that this is where we identify the RR-fields as having value "zero" allowing the extremal transition.

Having accepted this, it follows that the tangent bundle does *not* have the correctly-tuned RR fields to allow the extremal transitions giving rise to the massless tensor or the E_8 gauge symmetry. Indeed it is quite apparent why this is so. One cannot possibly try to decompose the sheaf $F^1(\mathscr{T}_Z)$ as it has this strange part looking like a rank one sheaf over a "fat" non-reduced curve. The two degrees of freedom we expect from this part of the sheaf are inextricably wound together. This can be contrasted to the point-like instantons where we have the sum of two copies of the structure sheaf over the same curve.

4   The tangent sheaf of an orbifold K3

As an application of the technology we have pursued above let us consider the six-dimensional physics of the heterotic string compactified on the tangent bundle of a K3 surface when we go to an orbifold limit.

Given that we know explicitly how to identify the K3 moduli in the F-theory language [4,14,15], it is straight-forward to identify the exact form of the required Calabi–Yau threefold, X, or more precisely its stable degeneration, $X_1 \cup X_2$, required to make the K3 surface have a particular complex structure compatible with the elliptic fibration. We will consider the case where the complex structure puts an orbifold point in Z.

Let s and t be affine coordinates for a patch on $\Sigma \cong \mathbb{F}_n$ (where $n = -12$ to obtain the correct c_2) as in [14] and let our elliptic fibration, $\pi^1_F : X_1 \to \Sigma$ be in standard Weierstrass form

$$y^2 = x^3 + a(s,t)\, x + b(s,t). \tag{33}$$

This has a discriminant equal to $4a^3 + 27b^2$. Restricting this fibration to C_* we obtain the Weierstrass form for the K3 surface Z. We may let t be an affine coordinate on C_*. It is not difficult to show

Theorem 2 *Consider the heterotic string compactified on the tangent sheaf of the K3 surface Z which has Weierstrass form*

$$y^2 = x^3 + \alpha(t)\, x + \beta(t). \tag{34}$$

This is dual to F-theory compactified on the Calabi–Yau threefold X which, in the stable degeneration, gives X_1 in the form (33) with

$$
\begin{aligned}
a(s,t) &= \alpha(t)s^4 \\
b(s,t) &= \beta(t)s^6 + \left(4\alpha(t)^3 + 27\beta(t)^2\right)s^5.
\end{aligned}
\tag{35}
$$

This may be proven by imposing the condition that the fibration of X_1 reduces to that of Z when restricted to C_* (where $s = \infty$) and by analyzing the form of the discriminant to obtain the required factorization shown in figure 2.

We show the discriminant locus for X_1 in figure 3 where Z acquires a $\mathbb{C}^2/\mathbb{Z}_2$ singularity. Locally this may be modeled by putting $\alpha = -3$ and $\beta = 2 + t^2$. The K3 surface has a $\mathbb{C}^2/\mathbb{Z}_2$ singularity because the I_2 fibre intersects C_* which is where Z sits in the elliptic fibration.
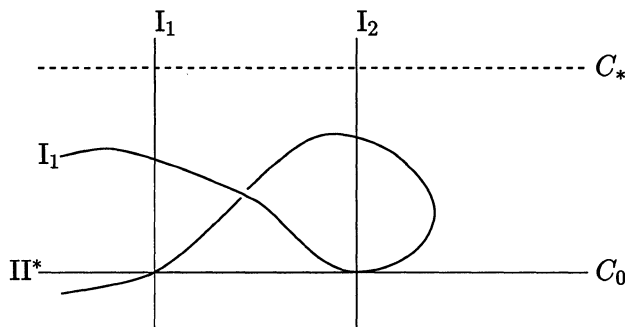
Figure 3: The discriminant for the tangent sheaf on a $\mathbb{C}^2/\mathbb{Z}_2$ singularity.

Applying the usual rules of F-theory to figure 3 would require us to blow-up the collision between the curves of II* and I_2 fibres twice to resolve the threefold. This would imply a local contribution of

- A nonperturbative SU(2) gauge symmetry in addition to the full unbroken perturbative E_8 gauge symmetry.

- Two massless tensors.

- Four hypermultiplets in the **2** representation of the above SU(2).

Now this would seem to be somewhat excessive! The problem is of course that we need to worry about the RR moduli again and the fact that they may block various extremal transitions giving rise to the above spectrum. It seems reasonable to assume that the above massless spectrum actually corresponds to the point-like instanton case. That is, two point-like instantons coalesce at the quotient singularity. This was also asserted in [14] (without worrying about RR moduli).

Now in many ways figure 3 is the same thing as figure 2 except that two of the vertical I_1 lines have coalesced. Therefore much of the discussion in section 3.3 applies. This can be used to argue that for the tangent sheaf case we do *not* have any massless tensors and the E_8 in not fully restored — we only have an E_7 part of it as for any generic SU(2)-bundle.

An interesting question is whether we actually have the nonperturbative SU(2) gauge symmetry and its associated hypermultiplets. We now argue that we do.

The problem we need to address is how to identify which RR moduli are lost when we go through the extremal transition associated to this SU(2)

gauge symmetry enhancement. As mentioned earlier, in order to see the extremal transition we need to go to four-dimensional physics given by the type IIA string compactified on X (where the fibres of the F-theory elliptic fibration can be given nonzero size).

If we really have an SU(2) gauge symmetry in the four dimensional theory then we should be able to break it to U(1) by switching on a vector multiplet. In terms of X_1, this is the deformation of the Kähler form which blows up all the I_2 fibres along the vertical line in figure 3. As this line passes through C_* it corresponds to blowing up to produce an algebraic $\mathbb{P}^1 \subset Z$.

Recall that in [15] some of the RR modes were identified with 3 cycles which intersected Z along transcendental 2-cycles. As we have produced an algebraic 2-cycle in Z in this extremal transition we must have lost a transcendental 2-cycle. This lost 2-cycle is naturally the one which lay in the 3-cycle corresponding to the lost RR mode.

The analysis in [15] argued that the RR fields associated to 3-cycles which intersect Z in 2-cycles are interpreted as B-fields in the heterotic language. That is, *the RR mode which potentially blocks the appearance of the nonperturbative* SU(2) *gauge symmetry is a B-field on the K3 surface.* (We expect the masslessness of the four **2**'s of SU(2) to be blocked in the same way since there are no other moduli which could further affect these modes.)

The conclusion is therefore that in order to get enhanced gauge symmetry for the tangent sheaf of an orbifold, the corresponding component of the B-field must be set to zero. This is highly reminiscent of the type IIA string on a K3 surface as described in [35, 36] where it was also argued that the B-field must also be set to zero to see the enhanced gauge symmetry from an orbifold point.

It is not difficult to generalize this discussion to that of a general A-D-E singularity. We find

Proposition 4 *If the $E_8 \times E_8$ heterotic string is compactified on the tangent sheaf of a K3 surface with an A-D-E singularity then one acquires the corresponding A-D-E gauge group nonperturbatively if and only if the B-field associated to the shrunken 2-cycle(s) is set to zero.*

For the cyclic quotient singularity we expect an SU(N) gauge symmetry to come with $2N$ hypermultiplets in the fundamental representation. This is implied by the F-theory geometry and and can also be seen to correctly cancel anomalies. Similarly F-theory suggests $2N - 8$ hypermultiplets in the

vector representation of $SO(2N)$ and no hypermultiplets in the E_6, E_7, and E_8 case.

It is also perhaps worth mentioning a local description of the tangent sheaf of a quotient singularity in terms of previously analyzed point-like instantons. In [17] methods were described which analyze a point-like instanton with a $\mathbb{Z}_2$ holonomy which broke E_8 to $(E_7 \times SU(2))/\mathbb{Z}_2$. Such an instanton is forced to sit on a $\mathbb{C}^2/\mathbb{Z}_2$ quotient singularity (or worse). By using methods similar to section 4.1 of [17] one may show that this instanton has $c_2 = \frac{1}{2}$. Figure 3 is precisely the F-theory picture for what happens when one allows such an instanton to coalesce with a normal free point-like instanton in the $E_8 \times E_8$ heterotic string. Therefore up to changes in the RR-moduli, the tangent sheaf of $\mathbb{C}^2/\mathbb{Z}_2$ looks locally like a sum of a "half" instanton with a full instanton. It therefore has a local charge of $c_2 = \frac{3}{2}$. This is consistent with the construction of a K3 surface as a quotient $T^4/\mathbb{Z}_2$. This has 16 orbifold points of the above form and the Euler characteristic is 24 — so each orbifold point "contributes" $\frac{3}{2}$ to the Euler characteristic.

5 T-Duality

Finally we will study some global issues about the moduli space of the heterotic string compactified on the tangent bundle of a K3 surface. We will discover that the mathematical abstraction of derived categories is pretty well forced upon us if we are to make sense of conventional ideas such as T-duality.

Consider first the non-linear σ-model of a superstring with a K3 target space. When the metric on this K3 surface is Ricci-flat, we have a conformal field theory which has $N = (4, 4)$ supersymmetry. It was argued in [37–39] that the moduli space of such conformal field theories receives no quantum corrections. It was then deduced in [30, 40] that the moduli space must be exactly of the form

$$\mathscr{M} \cong O(\Gamma_{4,20})\backslash O(4, 20)/(O(4) \times O(20)), \qquad (36)$$

where $\Gamma_{4,20}$ is the even unimodular lattice of signature $(4, 20)$ and $O(\Gamma_{4,20})$ is its discrete isometry group (often called, more loosely, $O(4, 20; \mathbb{Z})$).

The best way to view the action of $O(\Gamma_{4,20})$ is as follows. Consider $H^*(Z, \mathbb{Z}) = H^0(Z, \mathbb{Z}) \oplus H^2(Z, \mathbb{Z}) \oplus H^4(Z, \mathbb{Z})$ and let $w \in H^*(Z, \mathbb{Z}) \cong \Gamma_{4,20}$ be a primitive null vector which denotes the generator of $H^4(Z, \mathbb{Z})$. Given a point in the moduli space and a choice of $w \in \Gamma_{4,20}$, we may define a Ricci-flat metric on Z and a choice of B-field [30,31]. The modular group $O(\Gamma_{4,20})$

acts transitively on the set of possible choices for w within the lattice $\Gamma_{4,20}$ and so generates the possible target space interpretations of a single point in the moduli space. These identifications include such notions as mirror symmetry and $R \leftrightarrow 1/R$ symmetries.

The $N = (4,4)$ conformal field theory has no preference for which direction in $H^*(Z, \mathbb{Z})$ is $H^4(Z, \mathbb{Z})$ — this is what the modular group $O(\Gamma_{4,20})$ expresses. In order to impose a geometric interpretation on the theory we are forced to choose a w, breaking this modular group.

Any string theory compactified in a conventional way should have the moduli space of corresponding conformal field theories as part of its moduli space — supplemented by more parameters such as the dilaton, RR fields etc. The conformal field theory moduli space may receive corrections which are quantum with respect to the string coupling (as distinct from the σ-model coupling). Given enough supersymmetry, such quantum corrections may vanish leaving the conformal field theory moduli space intact.

An example of this is the type IIA string compactified on a K3 surface. The well-known duality of this to the heterotic string on a 4-torus requires that the moduli space (36) be exact.

We can also expect to rid ourselves of such quantum corrections if the dilaton somehow lives in another sector of the theory. This happens when the heterotic string is compactified on a K3 surface — the dilaton lives in a tensor multiplet whereas the conformal field moduli space is composed of hypermultiplet moduli. When we "embed the spin connection in the gauge group" for the heterotic string, we obtain an $N = (4,4)$ underlying conformal field theory and so the space (36) *must* be an exact subspace of the moduli space of heterotic strings on a K3 surface.

Note that when we deform the bundle of the heterotic string away from the tangent bundle, we generically break the world-sheet supersymmetry to $N = (4,0)$ and then the moduli space may well pick up quantum corrections in the non-linear σ-model. This was discussed in [15].

The question we want to address is, can we extend the T-duality for conformal field theories to T-duality for heterotic strings? The answer is strikingly clear. The Mukai vector lives in $H^*(Z, \mathbb{Z})$, We must therefore let the T-duality group act on this too.

It is worth pointing out that Mukai's work of [27] precedes the notion of mirror symmetry and that the appearance of $H^*(Z, \mathbb{Z})$ in both contexts appeared to be somewhat unrelated at first. One connection between these

objects has already been pointed out by Morrison [41] based on the work [42]. Here we have another relation — perhaps more unavoidable because of the way that the heterotic string combines vector bundles with conformal field theories.

As explained in section 2.3, the group $O(\Gamma_{4,20})$ does not act in a natural way on the space of vector bundles or even the space of sheaves. Instead we are forced to use the derived category of coherent sheaves. We can then make the following

Proposition 5 *Let Z_1 be a K3 surface and consider the heterotic string compactified on the tangent bundle of Z_1. This is mapped by T-duality to an element, W, of the derived category of coherent sheaves over Z_2. Z_2 is related to Z_1 by the usual action of $O(\Gamma_{4,20})$ on the moduli space of conformal field theories. The Mukai vector of W is obtained by the action of $O(\Gamma_{4,20})$ on the Mukai vector of the tangent sheaf of Z_1.*

It is worth pointing out that the way one might generate $O(\Gamma_{4,20})$ in terms of conventional T-dualities on a K3 surface and how one might generate it from actions on the Mukai vector are remarkably similar. It was shown in [30] how $O(\Gamma_{4,20})$ could be generated from three sets of elements:

1. The classical diffeomorphisms of K3 generating
   $O^+(H^2(Z, \mathbb{Z})) \cong O^+(\Gamma_{3,19})$.

2. Mirror Symmetry.

3. Shifts in the B-field by an element of $H^2(Z, \mathbb{Z})$.

It is not hard to see that the map **S** of section 2.3 is similar to mirror symmetry[5] and the map **T** is the analogue of a shift by the class of the elliptic fibre.

We can now try to resolve an important question as to how, in general, the heterotic string can be viewed. In particular what kind of data should be specified in order to compactify it? The conventional point of view was that a vector bundle over some Calabi–Yau manifold was the correct picture. It was seen in [18] that this is not sufficient to fill out the moduli space in

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[5]This similarity can be a little confusing. Whether or not one really identifies it with mirror symmetry comes down to a question of the actual definition of mirror symmetry for a K3 surface. In [41] an identification with mirror symmetry was made. Having said that, **S** is not mirror symmetry in the sense of mirror symmetry between families of algebraic K3's as discussed in [43] for example.

the case of compactification over Calabi–Yau threefolds. Some phases of the moduli space correspond to non-locally-free sheaves. In this case the sheaves were "reflexive", i.e., they were locally free except over codimension 3 (that is, points).

One might therefore suspect that reflexive sheaves might be the right choice. If one believes that string theory really has an algebraic underpinning then one might also propose that coherent sheaves are the natural choice.

Instead we propose that one must go even further and specify an element of the derived category of coherent sheaves as the heterotic string data. One must do this to get T-duality.

Let us spell out a little more clearly what we mean by this. Suppose one takes a large smooth K3 surface with its tangent bundle. This bundle is a locally-free sheaf. We now shrink the K3 surface to a size well below the α' scale. We know that we may use T-duality to turn such a K3 surface back into a large K3 surface. To do this however we are required to reinterpret the way that we divided $\Gamma_{4,20} \cong H^*(Z, \mathbb{Z})$ into $H^0(Z, \mathbb{Z}) \oplus H^2(Z, \mathbb{Z}) \oplus H^4(Z, \mathbb{Z})$. For example, a good $R \leftrightarrow 1/R$ symmetry is one which exchanges the rôle of H^0 and H^4 [31]. When we do this however we are also forced to reinterpret the bundle data. A vector bundle has Chern classes for example which live in $H^*(Z, \mathbb{Z})$ and so these must be changed in accordance with T-duality. The natural way to do this remapping is to use the Mukai vector and the derived category $\mathbf{D}(Z)$.

Once we have accepted this claim for the tangent sheaf, we are required to extend it to many other, if not all, possible sheaves on which we may compactify the heterotic string. This is because we may deform the tangent bundle into other SU(2)-bundles and then we may connect via extremal transitions to many other cases. It would appear that the Mukai vector and $\mathbf{D}(Z)$ provides a good general setting for analysis of the heterotic string.

We close with the observation that this is not the first time that the derived category of coherent sheaves has appeared in string theory. It was used by Kontsevich in a conjectural description of mirror symmetry [44], which has recently been proved for the one-dimensional case in [45]. While such abstract objects as derived categories are not the kind of things a physicist would normally wish to consider, this clearly deserves to be studied further!

Acknowledgements

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