Three-Point Functions of Chiral Operators in $D = 4$, $\mathcal{N} = 4$ SYM at Large N

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Abstract

We study all three-point functions of normalized chiral operators in $D = 4$, $\mathcal{N} = 4$, $U(N)$ supersymmetric Yang-Mills theory in the large N limit. We compute them for small 't Hooft coupling $\lambda = g_{YM}^2 N \ll 1$ using free field theory and at strong coupling $\lambda = g_{YM}^2 N \gg 1$ using the AdS/CFT correspondence. Surprisingly, we find the same answers in the two limits. We conjecture that at least for large N the exact answers are independent of λ.

1   Introduction

The conjectured duality [1] (for earlier related references see [2, 3, 4, 5]) between string/M theory on Anti-de Sitter space (AdS) times a compact manifold, and conformal field theory (CFT) living on the boundary of AdS has attracted much attention. According to this proposal, Type IIB string theory on $AdS_5 \times S^5$ is dual to $D = 4$, $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM$_4$).

In [6, 7] a detailed dictionary relating S-matrix elements of the string theory to Green's functions of the CFT was proposed. The operators of the CFT are mapped to on-shell bulk fields on AdS. The CFT operators interact with the boundary values of these bulk fields through an interaction action S_{int}. The partition function of the string theory with fixed boundary values of fields is then identified with the partition function of the CFT with external sources coupled to the corresponding operators.

Using this dictionary, two point functions of CFT operators corresponding to massive scalars [6, 7, 8, 9], vectors [7, 9], the graviton [10], and spinors [11] have been computed.

In a series of recent papers, the 3-point functions of operators in a CFT$_4$ corresponding to massive minimally coupled scalars [8, 9], or scalars and spinors [12], or vectors and spinors [13] on the AdS_5 with certain generic, arbitrarily prescribed, interactions have been computed.

Certain computations of correlation functions of operators in actual SYM$_4$ have also been performed. Using a proposed form of S_{int}, the 2-point functions of the stress energy tensor and $\mathbf{Tr}(F_{\mu\nu}F^{\mu\nu})$ were computed in [6]. Using the model independent coupling of gauge fields to currents, the 3-point functions of the R-symmetry currents of SYM$_4$ were computed in [9, 14]. Similarly, 3-point functions of the dilaton and the stress energy tensor were computed in [10].

Local operators in SYM$_4$ are organized into infinite dimensional families, each of which is an irreducible representation of the $D = 4$, $\mathcal{N} = 4$, superconformal algebra. Each family (or module) contains special operators of lowest scaling dimension in an $SU(4)$ representation. We will call them primary operators (PO) (strictly, only the operator with the highest $SU(4)$ weight is primary). SYM$_4$ contains a set of special short families that contain fewer operators than the generic module. Such families include primary operators which are chiral under an $\mathcal{N} = 1$ subalgebra; the scaling dimension of operators in these families is determined by the superconformal algebra

[15, 16] in terms of their $SU(4)$ R-symmetry representation. We will loosely refer to all the lowest dimension operators in such a representation as chiral primary operators (CPO). Under a given $\mathcal{N} = 1$ subalgebra, the $\mathcal{N} = 4$ chiral primaries include $\mathcal{N} = 1$ chiral operators, $\mathcal{N} = 1$ anti-chiral operators and non-chiral operators.

It should be stressed that unlike the situation in $\mathcal{N} = 1$, these chiral primary fields do not form a ring. The product of two $\mathcal{N} = 4$ chiral operators includes a product of an $\mathcal{N} = 1$ chiral operator with an $\mathcal{N} = 1$ anti-chiral operator and even two $\mathcal{N} = 1$ non-chiral operators, which are singular. Because of such singularities the $\mathcal{N} = 4$ chiral operators do not form a ring.

In this paper, we study the 3-point functions of all CPOs, in the large N limit of SYM$_4$. We first compute them in the limit of weak 't Hooft coupling $\lambda = g_{YM}^2 N \ll 1$ using free field theory. We then study them in the limit of large 't Hooft coupling $\lambda = g_{YM}^2 N \gg 1$ using Type IIB supergravity (SUGRA). Surprisingly, we find the same answers. Clearly, this agreement for the primary fields guarantees similar agreements for all their descendants.

Banks and Green [17] showed that for infinite N the leading order result at large λ is not corrected at the next order. Given that we found that the leading order result agrees with the weak coupling answer, we are led to conjecture that *the 3-point functions of all chiral primary operators at large N is independent of $\lambda = g_{YM}^2 N$.*

Since R-symmetry currents and the stress energy tensor are descendents of CPOs, our results include all previous results on 3-point functions [9, 10, 14], as special cases. Also, the discussion of [18, 19] shows that some of these 3-point functions are independent of the coupling even for finite N.

We point out that a similar result cannot be true for the 4-point function of these chiral operators. Unlike the 3-point functions, the 4-point functions depend on λ at the next to leading order [17].

It might be that even a stronger claim is true, and these 3-point functions are independent of g_{YM} even for finite N. (For some of the 3-point functions this was proven in [19].) From the weak coupling side it is clear that the 3-point functions depend on N (even the spectrum of chiral primary operators depends on N). Therefore, if this stronger claim is true, then at strong coupling, on the AdS side, the coupling of three gravitons depends on N; i.e. it is corrected by quantum stringy effects. It is well known that such corrections are absent around flat space. This result is a consequence of the large amount of supersymmetry in the flat space theory. Since the $AdS_5 \times S^5$ background preserves the same number of supersymmetries as the flat space

background, one might guess that here too the scattering of three gravitons is not affected by quantum corrections. This guess cannot be simultaneously correct with the claim that the 3-point functions are not corrected at finite[1] N.

This paper is organized as follows. In section 2, we compute the correlation functions in the weak coupling limit. In section 3, we identify fields on *AdS* which represent the modes corresponding to the chiral operators and construct their effective action to cubic order in the fields. In section 4, we use this action to obtain the 3-point functions of normalized CPOs of the SYM_4. We compare this result with the free field calculation of section 2 and find precise agreement. In Appendix A, we explain our notations and conventions. Appendix B is devoted to spherical harmonics on S^5; we define scalar, vector and tensor spherical harmonics in arbitrary dimensions, and obtain several formulae needed for the calculation in section 3.

2   Correlation Functions at Weak Coupling

CPOs of SYM_4 are operators of the form

$$\mathcal{O}^I = \mathcal{C}^I_{i_1..i_k} \mathrm{Tr}(\phi^{i_1}..\phi^{i_k}),$$

where $i_1, \cdots, i_k$ are $SO(6)$ vector indices and ϕ^i are six $N \times N$ matrices transforming in the adjoint of $U(N)$. The trace in the formula above is over $U(N)$ indices. $\mathcal{C}^I$ is a totally symmetric traceless rank k tensor of $SO(6)$. We can choose an orthonormal basis on the vector space $\{\mathcal{C}^I\}$ such that $\langle \mathcal{C}^{I_1} \mathcal{C}^{I_2} \rangle \equiv \mathcal{C}^{I_1}_{i_1 \cdots i_k} \mathcal{C}^{I_2 \, i_1 \cdots i_k} = \delta^{I_1 I_2}$. We normalize our action as

$$S = -\int \frac{1}{2g^2_{YM}} \mathrm{Tr} F^2 + \cdots = -\int \frac{1}{4g^2_{YM}} F^a_{\mu\nu} F^{a\mu\nu} + \cdots.$$

In this normalization the Yang-Mills coupling and the string coupling are related by $g^2_{YM} = 4\pi g_s$. The propagators of interest are

$$\langle \phi^i_a(x) \phi^j_b(y) \rangle = \frac{g^2_{YM} \delta_{ab} \delta^{ij}}{(2\pi)^2 |x-y|^2},$$

where $a, b, \ldots$ are $U(N)$ color indices.

The 2-point function of two CPOs specified by tensors $\mathcal{C}^{I_1}_{i_1 \ldots i_{k_1}}$ and $\mathcal{C}^{I_2}_{j_1 \ldots j_{k_2}}$, is computed in free field theory by contracting all the ϕs pair-wise and is

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[1] We thank T. Banks for a useful discussion on this point.

nonzero only if $k_1 = k_2 = k$. Consider

$$g(x,y) = \langle \mathbf{Tr}(\phi^{i_1}(x)..\phi^{i_k}(x))\mathbf{Tr}(\phi^{j_1}(y)...\phi^{j_k}(y))\rangle.$$

In the large N limit only planar diagrams contribute. Planar diagrams correspond to contracting $i's$ and $j's$ in the same cyclic order in which they appear in $g(x,y)$. One finds

$$g(x,y) = \frac{N^k g_{YM}^{2k}(\delta^{i_1 j_1}\delta^{i_2 j_2}..\delta^{i_k j_k} + \text{cyclic})}{(2\pi)^{2k}|x-y|^{2k}}.$$

Using the orthonormality of the C coefficients one thus deduces that (the $\delta^{I_1 I_2}$ term in the equation below is replaced by $\langle C^{I_1} C^{I_2}\rangle$ when considering the 2-point function of arbitrary CPOs which are not necessarily orthogonal)

$$\langle \mathcal{O}^{I_1}(x)\mathcal{O}^{I_2}(y)\rangle = \lambda^k \frac{k}{(2\pi)^{2k}|x-y|^{2k}}\delta^{I_1 I_2}. \tag{2.1}$$

In a similar fashion one may compute the 3-point function of CPOs specified by $C^{I_1}_{i_1..i_{k_1}}, C^{I_2}_{j_1..j_{k_2}}, C^{I_3}_{l_1..l_{k_3}}$. To ensure that all ϕs are contracted, $\alpha_3 = \frac{k_1 + k_2 - k_3}{2}$ of the ϕs must contract between the first and second of these operators and similarly for other pairs. In the large N limit, one finds

$$\langle \mathcal{O}^{I_1}\mathcal{O}^{I_2}\mathcal{O}^{I_3}\rangle = \frac{\lambda^{\Sigma/2}}{N}\frac{k_1 k_2 k_3}{(2\pi)^{\Sigma}|x-y|^{2\alpha_3}|y-z|^{2\alpha_1}|z-x|^{2\alpha_2}}\langle C^{I_1}C^{I_2}C^{I_3}\rangle, \tag{2.2}$$

where $\Sigma = k_1 + k_2 + k_3$ and $\langle C^{I_1}C^{I_2}C^{I_3}\rangle$ represents the unique $SO(6)$ invariant that can be formed from $C^{I_1}, C^{I_2}, C^{I_3}$ (by contracting α_1 indices between C^{I_2} and C^{I_3}; α_2 indices between C^{I_3} and C^{I_1} and α_3 indices between C^{I_1} and C^{I_2})

We rescale the CPOs $O^I = \mathcal{O}^I \frac{(2\pi)^k}{\lambda^{k/2}\sqrt{k}}$ such that they have normalized 2-point functions $i.e.$,

$$\langle O^{I_1}O^{I_2}\rangle = \frac{\delta^{I_1 I_2}}{|x-y|^{2k}}. \tag{2.3}$$

Their 3-point function is

$$\langle O^{I_1}(x)O^{I_2}(y)O^{I_3}(z)\rangle = \frac{1}{N}\frac{\sqrt{k_1 k_2 k_3}\langle C^{I_1}C^{I_2}C^{I_3}\rangle}{|x-y|^{2\alpha_3}|y-z|^{2\alpha_1}|z-x|^{2\alpha_2}}. \tag{2.4}$$

This result is correct only at large N and receives nonzero corrections at $O(\frac{1}{N^2})$ from non-planar diagrams.

Finally note that the contraction of two or three $C's$ may be related to the integrals of two or three spherical harmonics over the sphere, by the formulae given in Appendix B.

3   Equations of Motion and Actions

3.1   Foreword to the Calculation

The particle spectrum of Type IIB SUGRA has been worked out in [20]. The particles are grouped into supermultiplets [21]. It turns out that the supermultiplets present in the theory correspond to representations of the superconformal algebra labeled by $SU(4)$ weight $(0, k, 0)$, $SO(4)$ $j_1 = j_2 = 0$ and scaling dimension $\epsilon_0 = k$ [22] . According to the results of [15, 16] these are short representations. These supermultiplets of particles must correspond to CPOs (and their descendents) in SYM$_4$ with the same $SO(6)$, $SO(4)$ and scaling dimension labels. These are the operators discussed at the beginning of section 2. The AdS fields that correspond to CPOs are particles in the $SU(4)$ representation with weight $(0, k, 0)$, $SO(4)$ representation with $j_1 = j_2 = 0$ and mass $m^2 = \epsilon_0(\epsilon_0 - 4) = k(k - 4)$ [7] . Studying [20] (table III in particular), we conclude that the required fields s^I are mixtures of the trace of the graviton on the sphere, and the five form field strength on the sphere.

Before identifying these fields and starting the calculation we make a few comments.

1. Since gravity is a gauge theory, not all fields in the IIB SUGRA action are physical. We need to choose a gauge and then solve the Gauss law constraints to identify the physical fields. Only these correspond to operators of the SYM$_4$.

2. Because of the absence of a simple covariant action for IIB SUGRA, we choose to work with equations of motion rather than an action. In order to compute the action for the fields s^I to cubic order, we compute their equations of motion to quadratic order, and then produce an action that leads to these equations of motion. The action thus produced is of uncertain normalization; we fix this ambiguity by comparison with the correctly normalized action proposed in [23], at quadratic order.

3. We need to identify the SUGRA fields that couple to various operators only at linear order in fluctuations about the $AdS_5 \times S^5$ background. Nonlinear higher order corrections modify the computed correlation functions of the corresponding operators only by contact terms. This translates in spacetime to the fact that we compute only S matrix elements which are not modified by field redefinitions. We use this freedom to simplify our analysis.

With the cubic action in hand we then use the procedure of [8, 9] to obtain the correlation functions of interest.

3.2 The Setting

The IIB SUGRA equations of motion of the graviton and the 5-form field strength are

$$R_{mn} = \frac{4}{4!} F_{mijkl} F_n{}^{ijkl} \tag{3.1}$$

$$F_{m_1 m_2 m_3 m_4 m_5} = \frac{1}{5!} \epsilon_{m_1 m_2 m_3 m_4 m_5 n_1 n_2 n_3 n_4 n_5} F^{n_1 n_2 n_3 n_4 n_5}. \tag{3.2}$$

We use units in which the scale $R_0 = (\lambda \alpha'^2)^{1/4}$ of the AdS_5 and S^5 is set to be unity. See Appendix A for other conventions.

The $AdS_5 \times S^5$ background solution is

$$
\begin{aligned}
ds^2 &= \frac{1}{z^2}(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dz^2) + d\Omega_5^2, \\
R_{\mu\lambda\nu\sigma} &= -(g_{\mu\nu}g_{\lambda\sigma} - g_{\mu\sigma}g_{\lambda\nu}); \quad R_{\mu\nu} = -4g_{\mu\nu}; \quad R_1 = -20, \\
R_{\alpha\gamma\beta\delta} &= (g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\delta}g_{\gamma\beta}); \quad R_{\alpha\beta} = 4g_{\alpha\beta}; \quad R_2 = 20, \\
\bar{F}_{\mu_1\mu_2\mu_3\mu_4\mu_5} &= \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}, \quad \bar{F}_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = \epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}.
\end{aligned} \tag{3.3}
$$

Bulk fields of interest are fluctuations about this background. Following [20], we set

$$
\begin{aligned}
G_{mn} &= g_{mn} + h_{mn}, \\
h_{(\alpha\beta)} &= h_{(\alpha\beta)} + \frac{h_2}{5}; \quad g^{\alpha\beta}h_{(\alpha\beta)} = 0, \\
h_{\mu\nu} &= h'_{\mu\nu} - \frac{h_2}{3}g_{\mu\nu}, \quad h'_{\mu\nu} = h'_{(\mu\nu)} + \frac{h'}{5}g_{\mu\nu}; \quad g^{\mu\nu}h'_{(\mu\nu)} = 0, \\
F &= \bar{F} + \delta F, \quad \delta F_{ijklm} = \nabla_i a_{jklm} + 4 \ \text{terms} = 5\nabla_{[i}a_{jklm]}.
\end{aligned} \tag{3.4}
$$

We choose to (almost completely) fix diffeomorphic and 4-form gauge invariance by choosing the de Donder gauge $\nabla^\alpha h_{\alpha\beta} = \nabla^\alpha h_{\mu\alpha} = \nabla^\alpha a_{\alpha m_1 m_2 m_3 m_4} = 0$. With this choice the most general expansion of these functions about the sphere is given by [11] (see Appendix B for information on spherical harmonics). For our purposes, it suffices to note that

$$
\begin{aligned}
h'_{\mu\nu} &= \sum Y^I h'^I_{\mu\nu}, \\
h_2 &= \sum Y^I h_2^I, \\
a_{\alpha_1\alpha_2\alpha_3\alpha_4} &= \sum \nabla^\alpha Y^I \epsilon_{\alpha\alpha_1\alpha_2\alpha_3\alpha_4} b^I, \\
a_{\mu_1\mu_2\mu_3\mu_4} &= \sum Y^I a^I_{\mu_1\mu_2\mu_3\mu_4}.
\end{aligned} \tag{3.5}
$$

3.3   Linear Constraints and Equations of Motion

The Einstein and self-duality equations about this background have been written out to linear order in [20]. Of interest to us are the three constraint equations (E3.2), (E2.2) and (M2.2) in that paper,

$$\left(\frac{1}{2}h'^I - \frac{8}{15}h_2^I\right)\nabla_{(\alpha}\nabla_{\beta)}Y^I = 0, \tag{3.6}$$

$$\left[\nabla_\mu h'^{\mu\nu I} - \nabla^\nu\left(h'^I - \frac{8}{15}h_2^I + 8b^I\right) - \frac{8}{4!}\epsilon^{\nu\mu_1\mu_2\mu_3\mu_4}a_{\mu_1\mu_2\mu_3\mu_4}^I\right]\nabla_\alpha Y^I = 0, \tag{3.7}$$

$$(a_{\mu_1\mu_2\mu_3\mu_4}^I + \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}\nabla^{\mu_5}b^I)\nabla_\alpha Y^I = 0, \tag{3.8}$$

and the dynamical equations for b and h_2 (Eq.(2.31) and (2.32) of [20]),

$$\left[\nabla_m\nabla^m b^I + \left(\frac{1}{2}h'^I - \frac{4}{3}h_2^I\right)\right]Y^I = 0, \tag{3.9}$$

$$\left[(\nabla_m\nabla^m - 32)h_2^I + 80\nabla_\alpha\nabla^\alpha b^I + \nabla_\alpha\nabla^\alpha\left(h'^I - \frac{16}{15}h_2^I\right)\right]Y^I = 0. \tag{3.10}$$

We are interested in modes with $k \geq 2$ only. For such modes the constraint (3.6) may be used to eliminate h'^I from (3.9) and (3.10) to yield

$$\nabla_m\nabla^m b - \frac{4}{5}h_2 = 0, \tag{3.11}$$

$$(\nabla_m\nabla^m - 32)h_2 + 80\nabla_\alpha\nabla^\alpha b = 0. \tag{3.12}$$

These two equations may now be diagonalized. Using the fact that $\nabla_\alpha\nabla^\alpha Y^I = -k(k+4)Y^I$ as shown in Appendix B, we find that the diagonal linear combinations (We choose the normalization such that the inverse relations are simple: $h_2^I = 10ks^I + 10(k+4)t^I$, $b^I = -s^I + t^I$.),

$$s^I = \frac{1}{20(k+2)}[h_2^I - 10(k+4)b^I], \tag{3.13}$$

$$t^I = \frac{1}{20(k+2)}[h_2^I + 10kb^I]$$

obey the equations of motion

$$\begin{aligned}
\nabla_\mu\nabla^\mu s^I &= k(k-4)s^I, \\
\nabla_\mu\nabla^\mu t^I &= (k+4)(k+8)t^I.
\end{aligned} \tag{3.14}$$

To linear order, s^I corresponds to CPOs in SYM$_4$, and it will be the focus of our attention through the rest of the paper.

The scalars t^I, on the other hand, correspond to descendents of CPOs; specifically they map to the operator $\phi^{(6)}$ in Table 1 of [11] . The expansion of t proportional to the kth spherical harmonic, corresponds to an operator formed by acting with 4 Qs and 4 $\bar{Q}s$ on the trace of $k + 4$ ϕ operators. The 3-point functions of these operators are determined in terms of those of CPOs by the superconformal algebra, and so we will not compute them directly. Henceforth we set $t^I = 0$.

We now construct an action whose variations leads to the equations of motion of s^I.

$$S = \int \sum \frac{A_I}{2} [-(\nabla_\mu s^I)^2 - k(k-4)(s^I)^2] \qquad (3.15)$$

with A_I undetermined constants which depend on k.

3.4   Normalization of the Quadratic Action

The normalization coefficients A_I may be determined by comparison of (3.15) with the full 'actual' action of IIB SUGRA [23]

$$S = \frac{1}{2\kappa^2} \bar{S} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} \left\{ R - \frac{8}{4!} \frac{\nabla^m a \nabla_n a}{(\nabla a)^2} (F - \widetilde{F})_{mijkl} \widetilde{F}^{nijkl} \right\},$$
$$(3.16)$$

where $\widetilde{F}$ is defined by the right-hand-side (RHS) of (3.2), and a is an auxiliary field. In our units $\frac{1}{2\kappa^2} = \frac{4N^2}{(2\pi)^5}$.

In order to obtain A_I from (3.16) we work at quadratic order, choose a gauge, solve for all constrained fields in terms of physical fields, and then set all physical fields except s^I to zero.

Firstly we eliminate the auxiliary field a in (3.16) . As shown in [23], we are free to fix a gauge by choosing an arbitrary function for a. We will set $a = x^4$, which amounts to removing the components of the 4-form potential of the form A_{ijk4}.

Having done this use (3.6), (3.7), (but not yet (3.8)) in (3.16) and set all unconstrained fields other than b and h_2 and $h'_{\mu\nu}$ to zero. The action we obtain at the end of this process is ($z(k)$ is defined in Appendix B equation (B.4))

$$\bar{S} = \sum_I z(k) \int d^5x \sqrt{-g_1} \left\{ L_1^I + L_2^I + L_3^I \right\}. \tag{3.17}$$

L_1^I contains terms from the Einstein part of the action except those involving $h'^I_{(\mu\nu)}$,

$$L_1^I = -\frac{2}{15} \left\{ (\nabla h_2^I)^2 + k(k+4)(h_2^I)^2 \right\} + \frac{32}{1875} \left\{ 8(\nabla h_2^I)^2 + 5(k^2 + 4k - 9)(h_2^I)^2 \right\}, \tag{3.18}$$

where the first group comes from the h_2 kinetic and mass terms while the second group was obtained by inserting (3.6) into the h' kinetic and mass terms. L_2^I contains terms from the F^2 part except $h'^I_{(\mu\nu)}$ terms,

$$L_2^I = -8k(k+4) \left\{ (\nabla b^I)^2 + k(k+4)(b^I)^2 + \frac{8}{5} h_2^I b^I \right\} - \frac{352}{125}(h_2^I)^2 \tag{3.19}$$

L_3^I is the part of (3.16) quadratic in $h'^I_{(\mu\nu)}$:

$$\begin{aligned} L_3^I &= -\frac{1}{4} \nabla_\lambda h'^I_{(\mu\nu)} \nabla^\lambda h'^{I(\mu\nu)} + \frac{1}{2} \nabla^\mu h'^I_{(\mu\nu)} \nabla_\lambda h'^{I(\lambda\nu)} - \frac{8}{25} \nabla^\mu h_2^I \nabla^\nu h'^I_{(\mu\nu)} \\ &\quad -\frac{1}{4}(k^2 + 4k - 2) h'^I_{(\mu\nu)} h'^{I(\mu\nu)}. \end{aligned} \tag{3.20}$$

We now attempt to use (3.7) to obtain the quadratic dependence of L_3^I on b and h_2. On eliminating h'^I and $a^I_{\mu_1\mu_2\mu_3\mu_4}$ from (3.7) and separating out the trace explicitly we obtain

$$\nabla^\mu h'^I_{(\mu\nu)} = \nabla_\nu \left\{ \frac{8}{25} h_2^I + 16 b^I \right\}. \tag{3.21}$$

We can solve the equation by setting

$$h'^I_{(\mu\nu)} = H^I_{(\mu\nu)} + \nabla_{(\mu} \nabla_{\nu)} K^I,$$

where $H^I_{(\mu\nu)}$ obeys $\nabla^\mu H^I_{(\mu\nu)} = 0$ and K^I satisfy $(\nabla^2 - 5)K^I = \frac{2}{5} h_2^I + 20 b^I$. Note that unlike $h'^I_{(\mu\nu)}$, $H^I_{(\mu\nu)}$ may consistently be set to zero for arbitrary h_2 and b. Substituting this into L_3^I leads unfortunately to an action non-local in b and h_2.

To avoid undue complications, we notice that it is sufficient for us to compute (3.17) on shell in order to obtain A_I. In that case

$$K^I = \frac{2}{5(k+1)(k+3)}(h_2^I - 30 b^I)$$

We substitute $h_2^I = 10ks^I$, $b^I = -s^I$ in (3.15) to find

$$
\begin{aligned}
L[s^I] &= -\left\{\frac{64}{5}k^2 + 32k - 128\right\}(\nabla s^I)^2 - \frac{32}{5}k^2(2k+1)(k-4)(s^I)^2 \\
&\quad - \frac{4(k^2+4k-2)}{(k+1)^2}(\nabla_{(\mu}\nabla_{\nu)}s^I)^2 - \frac{4}{(k+1)^2}(\nabla_\lambda\nabla_{(\mu}\nabla_{\nu)}s^I)^2 \quad (3.22)
\end{aligned}
$$

(3.17) vanishes on shell in the bulk (as every quadratic action does), but is nonzero as a function of boundary values due to surface terms. We now compute each of (3.15) and (3.17) as a function of boundary values of s^I, and compare the two results to read off the value of A_I. The result is

$$
A_I = 32\frac{k(k-1)(k+2)}{k+1}z(k). \tag{3.23}
$$

3.5   Cubic Couplings

To study the 3-point functions of the field s^I, we need the cubic terms in the action (3.15). To compute these we need quadratic corrections to Eqs. (3.6), (3.8), (3.9) and (3.10). We define

$$
h' = \frac{16}{15}h_2 + 10Q_1, \tag{3.24}
$$

$$
a_{\mu_1\mu_2\mu_3\mu_4} = -\epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}(\nabla^{\mu_5}b + Q_3^{\mu_5}),
$$

$$
(\nabla_m\nabla^m - 32)h_2 + 80\nabla_\alpha\nabla^\alpha b + \nabla_\alpha\nabla^\alpha(h' - \frac{16}{15}h_2) = 10Q_2, \tag{3.25}
$$

$$
5\nabla_{[\mu_1}a_{\mu_2\mu_3\mu_4\mu_5]} = \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}\left[\nabla_\alpha\nabla^\alpha b + \frac{1}{2}h' - \frac{4}{3}h_2 + Q_4\right].
$$

Substituting (3.24) into (3.25), we obtain,

$$
(\nabla_m\nabla^m - 32)h_2 + 80\nabla_\alpha\nabla^\alpha b + 10(\nabla_\alpha\nabla^\alpha Q_1 - Q_2) = 0, \tag{3.26}
$$

$$
\nabla_m\nabla^m b - \frac{4}{5}h_2 + 5Q_1 + \nabla_\mu Q_3^\mu + Q_4 = 0.
$$

The corrected equation of motion for s is a linear combination of the two above:

$$
\begin{aligned}
[\nabla_\mu\nabla^\mu - k(k-4)]s^I &= \frac{1}{2(k+2)} \\
&\quad \times \{(k+4)(k+5)Q_1 + Q_2 + (k+4)(\nabla_\mu Q_3^\mu + Q_4)\}^I.
\end{aligned} \tag{3.27}
$$

To calculate the $Q_i's$ we use the methods outlined in [11]. The first lines of (3.24) and (3.25) are the coefficients of $\nabla_{(\alpha}\nabla_{\beta)}Y^I$ and $Y^I g_{\alpha\beta}$ respectively, in the equation $R_{\alpha\beta} = \frac{4}{4!}F_{\alpha_1 i_2 i_3 i_4 i_5}F_{\beta_1}^{i_1 i_2 i_3 i_4 i_5}$. To compute Q_1 and Q_2, we must therefore compute R_{mn} and $F_{mijkl}F_n{}^{ijkl}$ to second order in s [24]. Since we are only interested in the s dependence of these quantities, we substitute

$$
\begin{aligned}
h_{\mu\nu}^I &= -\frac{3}{25}h_2^I g_{\mu\nu} + \frac{2}{5(k+1)(k+3)}\nabla_{(\mu}\nabla_{\nu)}(h_2^I - 30b^I) \\
&= U(k)s^I g_{\mu\nu} + W(k)\nabla_{(\mu}\nabla_{\nu)}s^I, \qquad (3.28)
\end{aligned}
$$

$$
\begin{aligned}
h_{\alpha\beta}^I &= \frac{h_2^I}{5}g_{\alpha\beta} = V(k)s^I g_{\alpha\beta}, \\
b^I &= X(k)s^I, \quad h_{\alpha\mu} = 0, \\
V(k) &= -\frac{5}{3}U(k) = 2k, \quad W(k) = \frac{4}{(k+1)}, \quad X(k) = -1. \qquad (3.29)
\end{aligned}
$$

to find

$$
\begin{aligned}
R_{\alpha\beta} &= \frac{1}{2}(Y + \frac{1}{10}Z_\gamma^\gamma)g_{\alpha\beta} + \frac{1}{4}Z_{(\alpha\beta)}, \\
Y &\equiv V_1 V_2 \nabla^\gamma(s_1\nabla_\gamma s_2) + U_1 V_2 \nabla^\mu(s_1\nabla_\mu s_2) \\
&\quad + W_1 V_2 \nabla_\mu(\nabla^{(\mu}\nabla^{\nu)}s_1\nabla_\nu s_2), \\
Z_{\alpha\beta} &\equiv (3V_1 V_2 + 5U_1 U_2)(\nabla_\alpha s_1\nabla_\beta s_2 + 2s_1\nabla_\alpha\nabla_\beta s_2) \\
&\quad + W_1 W_2(\nabla_\alpha\nabla^{(\mu}\nabla^{\nu)}s_1\nabla_\beta\nabla_{(\mu}\nabla_{\nu)}s_2 \\
&\quad + 2\nabla^{(\mu}\nabla^{\nu)}s_1\nabla_\alpha\nabla_\beta\nabla_{(\mu}\nabla_{\nu)}s_2) \\
\frac{4}{4!}F_{\alpha ijkl}F_\beta{}^{ijkl} &= 4g_{\alpha\beta}\{X_1 X_2(\nabla^\gamma\nabla_\gamma s_1\nabla^\delta\nabla_\delta s_2 + \nabla^\mu\nabla^\gamma s_1\nabla_\mu\nabla_\gamma s_2) \\
&\quad - 8V_1 X_2 s_1\nabla^\gamma\nabla_\gamma s_2 + 10V_1 V_2 s_1 s_2\} \\
&\quad - 8X_1 X_2\nabla_\alpha\nabla_\mu s_1\nabla_\beta\nabla^\mu s_2. \qquad (3.30)
\end{aligned}
$$

In the equations above, the symbol s_i is used as shorthand for $s^{I_i}Y^{I_i}$ and $U_i, \cdots, X_i$ as shorthand for $U(k_i), \cdots, X(k_i)$, respectively. Summation over I_1 and I_2 is assumed.

Projection of these quantities onto $\nabla_{(\alpha}\nabla_{\beta)}Y^I$ yields

$$
Q_1^{I_1} = \frac{1}{20q(k_1)z(k_1)}\sum_{2,3}\left\{(c_{123} + d_{231} + d_{321})T^{23} + 32X_2 X_3 c_{123}\nabla_\mu s_2\nabla^\mu s_3\right\},
$$
$$
(3.31)
$$

$$
T_{23} \equiv (3V_2 V_3 + 5U_2 U_3)s_2 s_3 + W_2 W_3\nabla^{(\mu}\nabla^{\nu)}s_2\nabla_{(\mu}\nabla_{\nu)}s_3,
$$

where c_{123}, *etc.*, are used as shorthand for $c(k_1, k_2, k_3)$, *etc.* defined in Appendix B (dropping an overall factor of $\langle \mathcal{C}^{I_1} \mathcal{C}^{I_2} \mathcal{C}^{I_3} \rangle$ from the equations which will be reinstate later) and s_i as shorthand for s^{I_i}.

Projection onto $g_{\alpha\beta} Y^I$ yields

$$
\begin{aligned}
Q_2^{I_1} &= \frac{1}{20 z(k_1)} \sum_{2,3} 10 S_{123} + T_{23}(b_{123} - 2 f_3 a_{123}) \\
&\quad + 32 X_2 X_3 \nabla_\mu s_2 \nabla^\mu s_3 b_{123}, \\
S_{123} &\equiv -V_2 V_3 b_{213} s_2 s_3 + V_3 U_2 a_{123} \nabla^\mu (s_2 \nabla_\mu s_3) \\
&\quad + W_2 V_3 a_{123} \nabla_\mu (\nabla^{(\mu} \nabla^{\nu)} s_2 \nabla_\nu s_3) \\
&\quad - 8 X_2 X_3 (a_{123} f_2 f_3 s_2 s_3 + b_{123} \nabla^\mu s_2 \nabla_\mu s_3) \\
&\quad - a_{123}(64 V_2 X_3 f_3 + 80 V_2 V_3) s_2 s_3.
\end{aligned}
\tag{3.32}
$$

Expansion of the self-duality equations to quadratic order and projection onto appropriate spherical harmonics yields Q_3 and Q_4. Q_3 arises as the coefficient of $\nabla_\alpha Y^I$ and Q_4 from the coefficient of Y^I in the self duality equation (3.2). The answers are

$$
\begin{aligned}
Q_3^{\mu I_1} &= -\frac{1}{f(k_1) z(k_1)} \sum_{2,3} (U_2 + 3 V_2) X_3 s_2 \nabla^\mu s_3 \\
&\quad + W_2 X_3 \nabla^{(\mu} \nabla^{\nu)} s_2 \nabla_\nu s_3 b_{213},
\end{aligned}
\tag{3.33}
$$

$$
Q_4^{I_1} = -\frac{1}{4 z(k_1)} \sum_{2,3} \left\{ T^{23} - (16 V_2 X_3 f_3 + 40 V_2 V_3) s_2 s_3 \right\} a_{123}, \tag{3.34}
$$

where $f(k) \equiv k(k+4)$ and T^{23} is the same as in (3.31).

This completes the evaluation of the RHS of the equation of motion (3.27) which now takes the form

$$
\begin{aligned}
(\nabla_\mu \nabla^\mu - m_{I_1}^2) s^{I_1} &= \sum_{I_2, I_3} D_{I_1 I_2 I_3} s^{I_2} s^{I_3} + E_{I_1 I_2 I_3} \nabla_\mu s^{I_2} \nabla^\mu s^{I_3} \\
&\quad F_{I_1 I_2 I_3} \nabla^{(\mu} \nabla^{\nu)} s^{I_2} \nabla_{(\mu} \nabla_{\nu)} s^{I_3}.
\end{aligned}
\tag{3.35}
$$

where D, E and F are computed by substituting (3.31),(3.32),(3.33) and (3.34) into (3.27). We can remove the derivative terms on the RHS of (3.35)

by a field redefinition

$$s^{I_1} = s'^{I_1} + \sum_{I_2, I_3} \left\{ J_{I_1 I_2 I_3} s'^{I_2} s'^{I_3} + L_{I_1 I_2 I_3} \nabla^\mu s'^{I_2} \nabla_\mu s'^{I_3} \right\}, \qquad (3.36)$$

where

$$L_{I_1 I_2 I_3} = \frac{1}{2} F_{I_1 I_2 I_3}, \quad J_{I_1 I_2 I_3} = \frac{1}{2} E_{I_1 I_2 I_3} + \frac{1}{4} F_{I_1 I_2 I_3} (m_{I_1}^2 - m_{I_2}^2 - m_{I_3}^2 + 8),$$

such that (3.35) becomes (we henceforth omit the primes on redefined fields)

$$(\nabla_\mu \nabla^\mu - m_{I_1}^2) s^{I_1} = \sum_{I_2, I_3} \lambda_{I_1 I_2 I_3} s^{I_2} s^{I_3}, \qquad (3.37)$$

where

$$\lambda_{I_1 I_2 I_3} = D_{I_1 I_2 I_3} - (m_{I_2}^2 + m_{I_3}^2 - m_{I_1}^2) J_{I_1 I_2 I_3} - \frac{2}{5} L_{I_1 I_2 I_3} m_{I_2}^2 m_{I_3}^2. \quad (3.38)$$

Putting together the values of $Q_i's$ and reintroducing the factors of $\langle \mathcal{C}^{I_1} \mathcal{C}^{I_2} \mathcal{C}^{I_3} \rangle$ that we suppressed for notational convenience (the definition of Σ and α_i are as in Section 2) we obtain

$$\begin{aligned} \lambda_{I_1 I_2 I_3} &= \frac{a(k_1, k_2, k_3)}{2z(k_1)(k_1 + 2)} \frac{8\Sigma\{(\frac{1}{2}\Sigma)^2 - 1\}\{(\frac{1}{2}\Sigma)^2 - 4\}\alpha_1\alpha_2\alpha_3}{(k_1 - 1)k_1(k_2 + 1)(k_3 + 1)} \\ &\quad \times \langle \mathcal{C}^{I_1} \mathcal{C}^{I_2} \mathcal{C}^{I_3} \rangle. \end{aligned} \qquad (3.39)$$

Taking into account the normalization of the quadratic action (3.23), the cubic coupling constant is

$$\begin{aligned} \mathcal{G}_{I_1 I_2 I_3} &= A_{I_1} \lambda_{I_1 I_2 I_3} \\ &= a(k_1, k_2, k_3) \frac{128\Sigma\{(\frac{1}{2}\Sigma)^2 - 1\}\{(\frac{1}{2}\Sigma)^2 - 4\}\alpha_1\alpha_2\alpha_3}{(k_1 + 1)(k_2 + 1)(k_3 + 1)} \langle \mathcal{C}^{I_1} \mathcal{C}^{I_2} \mathcal{C}^{I_3} \rangle. \end{aligned}$$
$$(3.40)$$

Note that $\mathcal{G}_{I_1 I_2 I_3}$ is totally symmetric, which ensures that the equations of motion can be derived from an action.

4   The Strong Coupling Limit of The Three-point Function

The cubic equations of motion for the fields s^I may be derived from the action

$$S = \frac{4N^2}{(2\pi)^5} \int d^5x \sqrt{-g_1} \sum_I \frac{A_I}{2} [-(\nabla s_I)^2 - k(k-4)(s_I)^2] \quad (4.1)$$

$$+ \sum_{I_1,I_2,I_3} \frac{1}{3} \mathcal{G}_{I_1 I_2 I_3} s^{I_1} s^{I_2} s^{I_3}.$$

There is an ambiguity in the use of this action due to our lack of knowledge of S_{int}. We know only that the field that couples to the primary operator of interest is proportional to s^I. The unknown proportionality constant may be a function of N, λ and k. Let $\tilde{s}^I$ be the field that couples to CPOs via $S_{int} = \int \tilde{s}^I \mathcal{O}^I$ and $s^I = w^I \tilde{s}^I$ for some function w^I. The action written in terms of $\tilde{s}$ is

$$S = \frac{4N^2}{(2\pi)^5} \int d^5x \sqrt{-g_1}$$

$$\times -\sum_I \frac{A_I (w^I)^2}{2} [(\nabla \tilde{s}^I)^2 + k(k-4)(\tilde{s}^I)^2]$$

$$+ \sum_{I_1 I_2 I_3} \frac{w^{I_1} w^{I_2} w^{I_3} \mathcal{G}_{I_1 I_2 I_3}}{3} \tilde{s}^{I_1} \tilde{s}^{I_2} \tilde{s}^{I_3}. \quad (4.2)$$

To compute the 2- and 3-point functions of CPOs from (4.2) we apply the formulae derived, for instance, in [9]. From Eq. 17 and the correction factor in Eq. 95 of [9], we derive that in the large N limit of SYM$_4$,

$$\langle \mathcal{O}^{I_1}(x) \mathcal{O}^{I_2}(y) \rangle = \frac{4N^2}{(2\pi)^5} \frac{1}{\pi^2} \frac{\Gamma(k+1)}{\Gamma(k-2)} \frac{2k-4}{k} \frac{A_I (w^I)^2 \delta^{I_1 I_2}}{|x-y|^{2k}}. \quad (4.3)$$

From Eq. 25 of the same paper we derive that

$$\langle \mathcal{O}^{I_1}(x) \mathcal{O}^{I_2}(y) \mathcal{O}^{I_3}(z) \rangle = -\frac{4N^2}{(2\pi)^5} \frac{1}{\pi^4} \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\frac{1}{2}\Sigma - 2)}{\Gamma(k_1 - 2)\Gamma(k_2 - 2)\Gamma(k_3 - 2)}$$

$$\times \frac{w^{I_1} w^{I_2} w^{I_3} \mathcal{G}_{I_1 I_2 I_3}}{|x-y|^{\alpha_3} |y-z|^{\alpha_1} |z-x|^{\alpha_2}}. \quad (4.4)$$

Using (3.23) and the formula (B.1), we can simplify (4.3) to read

$$\langle \mathcal{O}^{I_1}(x) \mathcal{O}^{I_2}(y) \rangle = \frac{4N^2}{(2\pi)^5} \frac{\pi}{2^{k-7}} \frac{k(k-1)^2(k-2)^2}{(k+1)^2} \frac{(w^I)^2 \delta^{I_1 I_2}}{|x-y|^{2k}}. \quad (4.5)$$

Similarly using (3.40) and (B.2) we have,

$$\langle \mathcal{O}^{I_1}(x)\mathcal{O}^{I_2}(y)\mathcal{O}^{I_3}(z)\rangle = -\frac{4N^2}{(2\pi)^5}\frac{1}{\pi 2^{\frac{1}{2}\Sigma-9}}\frac{w^{I_1}w^{I_2}w^{I_3}\langle\mathcal{C}^{I_1}\mathcal{C}^{I_2}\mathcal{C}^{I_3}\rangle}{|x-y|^{\alpha_1}|y-z|^{\alpha_2}|z-x|^{\alpha_3}}$$
$$\times\frac{k_1(k_1-1)(k_1-2)}{(k_1+1)}\frac{k_2(k_2-1)(k_2-2)}{(k_2+1)}$$
$$\times\frac{k_3(k_3-1)(k_3-2)}{(k_3+1)}. \qquad (4.6)$$

Finally, we obtain the 3-point functions of normalized CPOs,

$$\langle O^{I_1}(x)O^{I_2}(y)O^{I_3}(z)\rangle = \frac{1}{N}\frac{\sqrt{k_1 k_2 k_3}\langle\mathcal{C}^{I_1}\mathcal{C}^{I_2}\mathcal{C}^{I_3}\rangle}{|x-y|^{2\alpha_3}|y-z|^{2\alpha_1}|z-x|^{2\alpha_2}}, \qquad (4.7)$$

which agree exactly with the weak coupling result (2.4) in Section 2. Note that all the numerical factors as well as the unknown function w^I present in (4.5) and (4.6) have been canceled.

The action (4.2) was obtained up to an overall normalization merely from the equations of motion. To obtain this overall normalization, we had to make assumptions about the 'true' action (including surface terms) for IIB SUGRA. Changing the normalization of (4.2) by a factor η scales the result in (4.7) by $\frac{1}{\sqrt{\eta}}$, i.e a factor independent of k. We present here a further argument that the 3-point functions in (4.7) are correctly normalized.

R-symmetry currents are descendents of Tr $(\phi^i\phi^j)$ (after subtracting the trace). Specifically $J^b_{\alpha\dot\beta a} \propto \epsilon^{ijk(b}(Q_{\alpha i}\bar{Q}^l_{\dot\beta} - \frac{1}{4}Q_{\alpha m}\bar{Q}^m_{\dot\beta}\delta^l_i)$ $\times$Tr$(\phi_{a)j}\phi_{kl} - \frac{1}{4!}\phi_{mn}\phi_{pq}\epsilon^{mnpq}\epsilon_{a)jkl})$. Here $a,b,\cdots = 1,\cdots,4$ label the **4** or $\bar{\mathbf{4}}$ of $SU(4)$, brackets indicate trace removal, α is a chiral spinor index and $\dot\beta$ is an anti-chiral spinor index. Therefore, the correlation functions of R-symmetry currents are determined in terms of those of $Tr(\phi^i\phi^j)$. However, the 2- and 3-point functions of R-symmetry currents are known to be given exactly by the free field value ([9], and references cited therein). This is sufficient to ensure that the overall normalization in (4.7) agrees with the free field result at least for $k = 2$, and hence for all k.

Acknowledgment

Appendix A. Notations and Conventions

Consider the manifold $AdS_5 \times S^5$. We use Latin indices $i, j, k, l, m, \ldots$ for the whole 10-dimensional manifold. Indices $\mu, \nu, \lambda, \ldots$ are AdS_5 indices and run from 0 to 4. Indices $\alpha, \beta, \gamma, \ldots$ are S^5 indices and run from 5 to 9. Our choice of the signature of the metric is $(- + \ldots +)$. We use G_{mn} for the metric and g_{mn} for its background value. The conventions for metric connection, curvature tensor, Ricci tensor and the scalar curvature are

$$
\begin{aligned}
\Gamma^i_{jk} &= \frac{1}{2} G^{il}(\partial_k G_{lj} + \partial_j G_{lk} - \partial_l G_{jk}), \\
R^i{}_{jkl} &= \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{km} \Gamma^m_{lj} - \Gamma^i_{lm} \Gamma^m_{kj}, \\
R_{jl} &= R^i{}_{jil}, \quad R = G^{jl} R_{jl}.
\end{aligned}
\tag{A.1}
$$

For comparison, we note that Ref. [20] uses the same convention as ours except that they define $R_{mn} = R^k_{mnk} = -R^k_{mkn}$. The determinant of the metric is denoted by G. The determinants of the AdS metric $g_{\mu\nu}$ and the S^5 metric $g_{\alpha\beta}$ are denoted by G_1 and G_2, respectively. The completely antisymmetric $\epsilon_{m_0 \cdots m_9}$ symbol is defined to be a tensor of rank 10, such that $\epsilon_{0123456789} = \sqrt{-G}$ and $\epsilon^{0123456789} = -1/\sqrt{-G}$.

Appendix B. Spherical Harmonics

B.1 Scalar Spherical Harmonics

The set of scalar functions on S^D form a vector space which is an infinite dimensional reducible representation of $SO(D+1)$. Scalar spherical harmonics (SSH) form a complete basis on this space.

It is convenient to regard a function on S^D as a restriction of functions on the R^{D+1} in which S^D is embedded. An arbitrary C^∞ function on R^{D+1} may be expanded in polynomials in the Cartesian coordinates x^i, so it is sufficient to consider separately functions on R^{D+1} homogeneous in x^i of degree k. Not all such functions are independent when restricted to a sphere. Consider, for example, $r^2 x^{i_1} \ldots x^{i_k}$. This is a function of degree $k+2$ but when restricted to the sphere, it is identical to $x^{i_1} \ldots x^{i_k}$, a function of degree k. If at each degree we wish to restrict ourselves to functions linearly independent of those of lower degree, we must consider only functions $C_{i_1 \ldots i_k} x^{i_1} \ldots x^{i_k}$ such that $C_{i_1 \ldots i_k} \delta^{i_m i_n} = 0$ for any $1 \le m, n \le k$. With no loss of generality, we may demand that $C_{i_1 \ldots i_k}$ be symmetric in $i_1 \ldots i_k$.

Thus we have shown that each independent component of a totally symmetric traceless tensor of rank k defines a spherical harmonic by $Y^I =$

$C^I_{i_1 \ldots i_k} x^{i_1} \ldots x^{i_k}$. This construction clearly shows which representation of $SO(D+1)$ the spherical harmonics transform in. The Gelfand-Zetlin indices for the representation are $(h_1, h_2, h_3, \ldots) = (k, 0, 0, \ldots)$. The degeneracy of the harmonics is the number of symmetric polynomials of degree k minus the number of symmetric polynomials of degree $k - 2$, *i.e.* $\binom{D+k}{D} - \binom{D+k-2}{D}$.

Since $M_{\alpha\beta} = (-i)(x_\alpha \nabla_\beta - x_\beta \nabla_\alpha)$, the Casimir of this representation, $L^2 = \frac{1}{2} M_{\alpha\beta} M^{\alpha\beta} = k(k + D - 1)$, is simply the value of $-r^2 \nabla^2$ on the sphere. Therefore we deduce that the degree k spherical harmonics are eigenvectors of ∇^2 on the sphere, with eigenvalues $-k(k + D - 1)$.

The eigenvalue of ∇^2 may be obtained in an alternative fashion. Note that the harmonics as polynomials in R^{D+1} obey $(\nabla^2)_{D+1} f_k = 0$. However, $(\nabla^2)_{D+1} = \frac{1}{r^D} \partial_r r^D \partial_r + \frac{1}{r^2} (\nabla^2)_{S^D}$. Since our functions behave as r^k, $(\nabla^2)_{D+1} f_k = 0$ implies $(\nabla^2)_{S^D} f_k = -k(k + D - 1) f_k$ as above.

B.2   Scalar Harmonic Contractions on S^5

We need to evaluate the integral of the product of two or three scalar spherical harmonics over S^5. Let $Y^I = C^I_{i_1 \ldots i_k} x^{i_1} \ldots x^{i_k}$ be the spherical harmonics. The results of the integration are

$$\frac{1}{\omega_5} \int_{S^5} Y^{I_1} Y^{I_2} = \frac{\delta^{I_1 I_2}}{2^{k-1}(k+1)(k+2)}, \qquad \text{(B.1)}$$

$$\frac{1}{\omega_5} \int_{S^5} Y^{I_1} Y^{I_2} Y^{I_3} = \frac{1}{(\frac{1}{2}\Sigma + 2)! 2^{\frac{1}{2}(\Sigma - 2)}} \frac{k_1! k_2! k_3!}{\alpha_1! \alpha_2! \alpha_3!} \langle C^{I_1} C^{I_2} C^{I_3} \rangle, \quad \text{(B.2)}$$

where $\alpha_1 = \frac{1}{2}(k_2 + k_3 - k_1)$, $\Sigma = k_1 + k_2 + k_3$ and $\omega_5 = \pi^3$ is the area of a unit 5-sphere. Both of the above equations can be derived by using the following general formula:

$$\frac{1}{\omega_5} \int_{S^5} x^{i_1} \cdots x^{i_{2m}} = \frac{2^{1-m}}{(m+2)!} \times (\text{All possible contractions}), \qquad \text{(B.3)}$$

where "All possible contractions" means $\delta^{i_1 i_2}$ for $m = 1$, $\delta^{i_1 i_2} \delta^{i_3 i_4} + \delta^{i_1 i_3} \delta^{i_2 i_4} + \delta^{i_1 i_4} \delta^{i_2 i_3}$ for $m = 2$ and analogous objects for higher m. This formula can be proved by starting from

$$\int_{S^5} x^{i_1} \cdots x^{i_{2m}} = \frac{\partial^{2m}}{\partial J_{i_1} \cdots \partial J_{i_{2m}}} \int_{S^5} e^{J \cdot x}.$$

The following integrals occur naturally when one considers projection of the equations in section 4 onto appropriate spherical harmonics.

$$\int Y^{I_1} Y^{I_2} = z(k)\delta^{I_1 I_2},$$

$$\int \nabla_{(\alpha}\nabla_{\beta)} Y^{I_1} \nabla^{(\alpha}\nabla^{\beta)} Y^{I_2} = q(k)z(k)\delta^{I_1 I_2},$$

$$\int Y^{I_1} Y^{I_2} Y^{I_3} = a(k_1, k_2, k_3)\langle C^{I_1}C^{I_2}C^{I_3}\rangle, \qquad \text{(B.4)}$$

$$\int Y^{I_1} \nabla_\alpha Y^{I_2} \nabla^\alpha Y^{I_3} = b(k_1, k_2, k_3)\langle C^{I_1}C^{I_2}C^{I_3}\rangle,$$

$$\int \nabla^{(\alpha}\nabla^{\beta)} Y^{I_1} \nabla_\alpha Y^{I_2} \nabla_\beta Y^{I_3} = c(k_1, k_2, k_3)\langle C^{I_1}C^{I_2}C^{I_3}\rangle,$$

$$\int Y^{I_1} \nabla^{(\alpha}\nabla^{\beta)} Y^{I_2} \nabla_\alpha \nabla_\beta Y^{I_3} = d(k_1, k_2, k_3)\langle C^{I_1}C^{I_2}C^{I_3}\rangle.$$

The functions q, a, b, c, d can be evaluated by integrating by parts and using the fact that $\nabla_\alpha \nabla^\alpha Y^I = -k(k+4)Y^I$.

B.3   Vector and Tensor Spherical Harmonics

One may now ask for a basis in the space of vector functions on the sphere. To find such a basis, one again considers vectors of the form $e_a Y^a = e_a C^a_{i_1 \dots i_k} x^{i_1} \dots x^{i_k}$ in R^{D+1}, where e_a is a unit vector in the ath Cartesian direction. This is a complete set of vector functions on R^{D+1} but is over-complete on S^D for two reasons. The first is the same as that for SSH and may be fixed in the same fashion. The second reason is that some of these vectors have no projection onto the tangent space of the sphere.

The vector function $e_a C^a_{i_1 \dots i_k} x^{i_1} \dots x^{i_k}$ transforms in the product of the vector representation and $(k, 0, \dots, 0)$ under $SO(D+1)$. For the rest of this subsection we assume that D is odd as it is in our paper. That product has 3 irreducible representations, $(k-1, 0, 0, \dots, 0)$, $(k+1, 0, 0, \dots, 0)$ and $(k, 1, 0, \dots, 0)$. The first corresponds to a vector of the form $Y^a = x^a Y(k-1)$, where $Y(k-1)$ is a SSH of degree $k-1$. It has no projection onto the tangent space of S^D. The second corresponds to a vector of the form $Y^a = \partial^a Y(k+1)$. It is a derivative of a SSH of one higher degree. Projected onto S^D, this becomes $\nabla_\alpha Y(k+1)$. The last corresponds to vector functions that obey $x_a Y^a = \partial_a Y^a = 0$, which implies $\nabla_\alpha Y^\alpha = 0$ on S^D. This function is called a vector spherical harmonic.

In summary, an arbitrary vector function on S^D is a linear combination of the gradients of SSH and vector spherical harmonics introduced above.

The story is very similar for symmetric tensors. Any symmetric tensor on the sphere can be decomposed into a sum of the form

$$S_{\alpha\beta} = \sum_I g_{\alpha\beta} A^I Y^I + B^I \nabla_{(\alpha} \nabla_{\beta)} Y^I + C^I \nabla_{(\alpha} Y^I_{\beta)} + D^I Y^I_{(\alpha\beta)},$$

where A, B, C, D are constants. The Y^I_β and $Y^I_{(\alpha\beta)}$ are vector and symmetric tensor spherical harmonics. Symmetric tensor spherical harmonics of degree k are a new set of functions. They transform in the $(k, 1, 1, 0, ..0)$ representation of $SO(D + 1)$ and obey

$$\nabla^\beta Y^I_{(\alpha\beta)} = g^{\alpha\beta} Y^I_{(\alpha\beta)} = 0$$

These properties, and the orthogonality of SSHs on the sphere imply that

$$A^I = \frac{1}{D} \frac{\int S_{\alpha\beta} g^{\alpha\beta} Y^I}{\int Y^I Y^I}, \quad B^I = \frac{\int S_{\alpha\beta} \nabla^{(\alpha} \nabla^{\beta)} Y^I}{\int \nabla_{(\alpha} \nabla_{\beta)} Y^I \nabla^{(\alpha} \nabla^{\beta)} Y^I}$$

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