Gravitational Anomaly Cancellation
for
M-Theory Fivebranes

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Abstract

We study gravitational anomalies for fivebranes in M theory. We show that an apparent anomaly in diffeomorphisms acting on the normal bundle is cancelled by a careful treatment of the M theory Chern-Simons coupling in the presence of fivebranes. One interesting aspect of our treatment is the way in which a magnetic object (the fivebrane) is smoothed out through coupling to gravity and the resulting relation between antisymmetric tensor gauge transformations and diffeomorphisms in the presence of a fivebrane.

1   Introduction

M theory is believed to be a consistent theory of quantum gravity which at low energies reduces to the unique supergravity theory in $D = 11$ spacetime dimensions. M theory contains two types of extended objects, membranes and fivebranes. Membranes have odd-dimensional worldvolumes and so there are no continous anomalies associated with the zero modes of a membrane. The cancellation of global membrane anomalies has been demonstrated in [1]. Fivebranes on the other hand have even dimensional worldvolumes and chiral zero modes so a computation is needed to see if there are continous anomalies in the presence of fivebranes. We will not investigate global anomalies for fivebranes in this paper.

A fivebrane of M theory with worldvolume W_6 embedded into eleven-dimensional spacetime M_{11} breaks the Lorentz symmetry from $SO(10,1)$ to $SO(5,1) \times SO(5)$. [2] If we believe that M theory is a well defined theory then diffeomorphisms or equivalently local Lorentz transformations which map the fivebrane to itself should be symmetries of the theory.

Diffeomorphisms preserving the fivebrane worldvolume $W_6 \to W_6$ are generated by vector fields acting either as diffeomorphisms of the fivebrane worldvolume W_6 or as $SO(5)$ gauge transformations on the connection on the normal bundle. Using the metric the normal bundle may be regarded as a bundle with metric and connection and structure group $SO(5)$. The potential anomalies in worldvolume diffeomorphisms and $SO(5)$ gauge transformations have two obvious sources. The first is the presence of chiral zero modes on the fivebrane worldvolume. For a charge one fivebrane the zero modes consist of a tensor multiplet of $(2,0)$, $D = 6$ supersymmetry. The chiral fields in this multiplet consist of a chiral fermion transforming in the spinor representation of $SO(5)$ and a two-form potential with anti-self-dual field strength which is a singlet under $SO(5)$. The anomaly due to these zero modes can be computed from the standard descent formalism [2–5] and is determined by descent on an eight-form I_8^{zm}. That is, we have $I_8 = dI_7^{(0)}$ and $\delta I_7^{(0)} = dI_6^{(1)}$ and the anomaly is given by

$$2\pi \int_{W_6} I_6^{zm(1)}. \tag{1.1}$$

The second source of anomalies comes from the presence in supergravity of a coupling

$$\Delta S = \int_{M_{11}} C_3 \wedge I_8^b(R) \tag{1.2}$$

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[2] Since we will be considering fermions we should really be discussing the covering groups $Spin(n)$, this distinction will not be important in what follows

with I_8^b a specific eight-form constructed out of the curvature on M_{11}. Integrating by parts and taking the variation of this term gives

$$\delta\Delta S = \int_{M_{11}} dG_4 \wedge I_6^{b(1)}. \tag{1.3}$$

The fivebrane of M theory acts as a magnetic source for the three-form potential C_3 of M theory. With $G_4 = dC_3$ the corresponding field strength this means, roughly speaking, that

$$dG_4 = 2\pi\delta_5 \tag{1.4}$$

where δ_5 is a five-form which integrates to one in the directions transverse to the fivebrane and has delta function support on the fivebrane[3]. We thus have for the total gravitational anomaly

$$2\pi \int_{W_6} \left(I_6^{zm} + I_6^b\right)^{(1)} = 2\pi \int_{W_6} \left(\frac{p_2(N)}{24}\right)^{(1)} \tag{1.5}$$

with $p_2(N)$ the second Pontrjagin class of the normal bundle [6]. The fact that anomalies in diffeomorphisms of the tangent bundle cancel between these two sources was pointed out in [7]. If we believe that M theory exists and that the fivebrane of M theory is a well defined object then there must be some additional mechanism which cancels the anomaly in diffeomorphisms of the normal bundle. The cancellation of the normal bundle anomaly has been investigated in [8, 9, 10], but a completely satisfactory answer has not yet emerged.

In field theory there are many examples where a smooth soliton solution of the field theory has chiral zero modes with an anomaly which is cancelled by inflow from the bulk [11]. This cancellation is inevitable in field theory since if the original theory was consistent then the effective action must make sense for any background fields including those of a smooth soliton. In theories including gravity the situation is more problematic. The extremal fivebrane of M theory is non-singular, but becomes singular when perturbed [12, 13]. In order to study anomalies it is necessary to study not just a particular fivebrane configuration but families of fivebrane configurations.

There are two related ways to understand the need for families of fivebranes. First, the problem of anomalies is the problem of defining the effective action $e^{iS_{eff}}$ as a function of the fields of the theory. Thus the effective action is a section of a line bundle over field space. The anomaly vanishes if we can trivialize this bundle. Studying this question involves studying families of field configurations. From a local point of view anomalies involve a lack of conservation of a current or of the energy-momentum tensor in the

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[3]We will soon give a much more precise definition of δ_5

presence of background fields. In order to study this conservation we have to turn on gravitational fields in addition to the background fields of the fivebrane. Since the fivebrane becomes singular when we vary the metric it is far from clear that the anomalies must in fact vanish. A direct approach would involve evaluation of the Rarita-Schwinger operator in backgrounds with a horizon and singularity and would be problematic if not impossible.

In this paper we will not try to study the fivebrane directly as a solution of $D = 11$ supergravity. Rather we will study the fivebrane as a magnetic source for the three-form potential of M theory and will divide the fields up into bulk fields and zero mode fields which are localized on the fivebrane. We will show that a careful treatment in this framework allows us to understand the cancellation of all anomalies. We leave to the future the very interesting question of the relation of this approach to that based on a direct study of solutions to supergravity.

We conclude this introduction with a brief comment about the descent formalism. In the physics literature on anomalies one commonly writes Chern-Weil forms such as I_8 above as differentials of Chern-Simons forms $I_7^{(0)}$. This is valid globally if we fix a reference trivial connection, but in general the Chern-Simons forms only exist locally. In the supergravity theory discussed here we do not want to impose unnecessary global restrictions on the spacetime and the fivebrane, so the descent equations are only valid locally. We make some brief comments about the global structure in section 4. A more complete treatment, together with an exposition of the anomaly cancellation in that global framework, will appear in [14].

In this paper we will follow the conventions and normalizations of [6]. In order to suppress many factors of 2π in various formulae we define $\mathcal{C}_3 = C_3/2\pi$ and $\mathcal{G}_4 = G_4/2\pi$ with C_3 the three form potential of M theory and G_4 its field strength.

2   The Fivebrane Source

Consider a fivebrane of M theory located at $y^a = 0$, $a = 1, 2, \ldots 5$ and with longitudinal coordinates x^μ, $\mu = 0, 1, \ldots 5$. The most naive expression for the Bianchi identity in the presence of the fivebrane is

$$d\mathcal{G}_4 = \delta(y^1) \cdots \delta(y^5) dy^1 \wedge \cdots \wedge dy^5. \qquad (2.1)$$

The quantity on the right hand side is a five-form with integral one over the transverse space and delta function support on the fivebrane. However as discussed above, we need to consider families of metrics so the above expression could at most be correct locally. While a delta function source is sufficient for computations where C_3 enters linearly, as in (1.2), we will soon encounter a Chern-Simons term which is cubic in C_3. In order to have a

completely well defined and non-singular prescription in such cases we need to smooth out the delta function source. Having done this we will see that in the presence of a non-zero $SO(5)$ connection on the normal bundle we will have to modify the right hand side of (2.1) in order that it transform covariantly under $SO(5)$ gauge transformations.

In order to define the fivebrane more carefully we first use the metric to define a radial direction away from the fivebrane and we cut out a disc of radius ϵ around the fivebrane. That is, we remove a tubular neighborhood of the fivebrane of radius ϵ. Let $D_\epsilon(W_6)$ denote the total space of the resulting disc bundle with base W_6 and fibers the discs of radius ϵ. We will define all bulk integrals as limits as ϵ goes to zero of integrals over $M_{11} - D_\epsilon(W_6)$:

$$\int_{M_{11}} \mathcal{L} \equiv \lim_{\epsilon \to 0} \int_{M_{11} - D_\epsilon(W_6)} \mathcal{L}. \tag{2.2}$$

We will later integrate by parts and use the fact the 10-dimensional boundary of $M_{11} - D_\epsilon(W_6)$ is the total space of the S^4- sphere bundle over W_6 of radius ϵ, whose total space we denote $S_\epsilon(W_6)$.

In order to smooth out the fivebrane source we choose a smooth function of the radial direction with transverse compact support near the fivebrane, $\rho(r)$, with $\rho(r) = -1$ for sufficiently small r and $\rho(r) = 0$ for sufficiently large r. The bump form $d\rho$ then has integral one in the radial direction. The smoothed form of (2.1) should then read

$$d\mathcal{G}_4 = d\rho \wedge e_4/2, \tag{2.3}$$

where $de_4 = 0$, e_4 is gauge invariant under $SO(5)$ transformations of the normal bundle, $e_4/2$ has integral one over the fibers of S_ϵ, and $d\rho \wedge e_4/2$ should reduce to the naive expression on the r.h.s of (2.1) for a flat infinite fivebrane when $d\rho$ approaches a delta function. Physically what we are doing is smoothing out the magnetic charge of the fivebrane to a sphere of magnetic charge linking the horizon.

The construction of the smoothed out source involves standard mathematics [15]. The right hand side of (2.3) involves differential forms which arise in a geometric construction of the Thom class of an oriented vector bundle, in this case the normal bundle to W_6 in M_{11}. As described in [15] we may identify the total space of the normal bundle with a tubular neighborhood of W_6 in M_{11}. With this identification the differential form $d\rho \wedge e_4/2$ represents the Thom class of the normal bundle and $e_4/2$ is the global angular form. Although the properties of e_4 follow from general principles, an explicit local formula is useful for constructing explicit objects which will later appear in the M theory action.

We have $E \to W_6$ a rank 5 real vector bundle with metric and connection. Let $P \to W_6$ be the principal $SO(5)$ bundle associated to the rank 5 bundle

E. Following [16] we work on $P \times S^4$ and construct a basic form which descends to the sphere bundle $S(E)$. Think of $S^4 \subset \mathbb{R}^5$, choose coordinates y^a for $\mathbb{R}^5$ and let $\hat{y}^a \equiv y^a/r$. Of course, $\hat{y}^a$ is defined only outside of $0 \in \mathbb{R}^5$, which corresponds to the complement of the zero section of E. On that complement the pullback of E has a tautological line subbundle, and a perpendicular oriented 4-plane bundle which we call F. Readers with less tolerance for mathematics can pick a gauge and with little harm done simply think of the $\hat{y}^a$ as isotropic coordinates on the S^4 fibres of $S(E)$.

The $SO(5)$ bundle is equipped with a globally defined connection $\Theta^{ab} = -\Theta^{ba}$. (We identify $\mathbf{so(5)} \cong \Lambda^2 \mathbb{R}^5$.) The Lie algebra $\mathbf{so(5)}$ acts on $P \times S^4$ in the standard way and we have horizontal forms:

$$(D\hat{y})^a \equiv d\hat{y}^a - \Theta^{ab}\hat{y}^b$$
$$F^{ab} = d\Theta^{ab} - \Theta^{ac} \wedge \Theta^{cb}. \tag{2.4}$$

We now consider the forms:

$$\epsilon_{a_1 \cdots a_5}(D\hat{y})^{a_1} \cdots (D\hat{y})^{a_4}\hat{y}^{a_5} \tag{2.5}$$

$$\epsilon_{a_1 \cdots a_5}F^{a_1 a_2} \wedge (D\hat{y})^{a_3}(D\hat{y})^{a_4}\hat{y}^{a_5} \tag{2.6}$$

and

$$\epsilon_{a_1 \cdots a_5}F^{a_1 a_2} \wedge F^{a_3 a_4}\hat{y}^{a_5}. \tag{2.7}$$

These forms are all annihilated by $\iota(X), \mathcal{L}(X)$, for $X \in \mathbf{so(5)}$, where $\iota(X)$ is the contraction and $\mathcal{L}(X)$ is the Lie derivative with respect to the vector field X. It follows that these forms are basic and descend to $S(E)$. Moreover, (2.5) restricts to the volume form on the S^4 fiber and thus reduces to the naive expression in (2.1) in the appropriate limit. However, (2.5) is not closed. One can use the identities

$$d\hat{y}^a = \Theta^{ab}\hat{y}^b + (D\hat{y})^a$$
$$d(D\hat{y})^a = \Theta^{ab}(D\hat{y})^b - F^{ab}\hat{y}^b \tag{2.8}$$
$$dF^{ab} = -F^{ac}\Theta^{cb} + F^{bc}\Theta^{ca}$$

plus the rotational invariance of the forms and the fact that $\hat{y}^a(D\hat{y})^a = 0$ to show that up to an overall scale there is a unique closed linear combination of (2.5) - (2.7).

Equivalently, the curvature of the oriented 4-plane bundle F defined above is the restriction to F of the curvature of E minus a second fundamental form term. The Pfaffian of this curvature is represented by the basic

4-form

$$
e_4(\Theta) = \frac{1}{64\pi^2} \left(\epsilon_{a_1 \cdots a_5} (D\hat{y})^{a_1} (D\hat{y})^{a_2} (D\hat{y})^{a_3} (D\hat{y})^{a_4} \hat{y}^{a_5} \right.
$$

$$
\left. - 2\epsilon_{a_1 \cdots a_5} F^{a_1 a_2} \wedge (D\hat{y})^{a_3} \wedge (D\hat{y})^{a_4} \hat{y}^{a_5} + \epsilon_{a_1 \cdots a_5} F^{a_1 a_2} \wedge F^{a_3 a_4} \hat{y}^{a_5} \right). \quad (2.9)
$$

One can apply the standard descent formalism to expressions of the form (2.9). For example, assuming that the normal bundle is trivial and choosing $\Theta = 0$ as a basepoint reference connection we have

$$
e_3^{(0)}(\Theta, \hat{y}) = \frac{1}{32\pi^2} \epsilon_{a_1 \cdots a_5} \left(\Theta^{a_1 a_2} d\Theta^{a_3 a_4} \hat{y}^{a_5} \right.
$$

$$
\left. - \frac{1}{2} \Theta^{a_1 a_2} \Theta^{a_3 a_4} d\hat{y}^{a_5} - 2\Theta^{a_1 a_2} d\hat{y}^{a_3} d\hat{y}^{a_4} \hat{y}^{a_5} \right). \quad (2.10)
$$

More generally one can write such formulae for the difference of two Chern-Simons forms for two connections Θ_1, Θ_2 on E.

The gauge transformations $\delta\Theta^{a_1 a_2} = (D\varepsilon)^{a_1 a_2}$ and $\delta\hat{y}^a = \varepsilon^{aa'} \hat{y}^{a'}$ give

$$
e_2^{(1)}(\varepsilon, \Theta, \hat{y}) = \frac{1}{16\pi^2} \epsilon_{a_1 \cdots a_5} \left(\varepsilon^{a_1 a_2} d\hat{y}^{a_3} d\hat{y}^{a_4} \hat{y}^{a_5} - \varepsilon^{a_1 a_2} \Theta^{a_3 a_4} d\hat{y}^{a_5} \right). \quad (2.11)
$$

The above expressions have natural generalizations to all real oriented bundles of odd rank. We give the general formulae in the appendix.

While we do not see a direct connection between the analysis presented here and the discussion in [6] concerning the normal bundle anomaly, it is interesting to note that the last term in (2.9) is very close to the expressions which appear in the discussion there. We expect that upon dimensional reduction our mechanism becomes equivalent to the anomaly cancellation mechanism for the IIA fivebrane described in [6], but we have not worked out the details of this.

3   Connection to Anomalies

In giving a precise definition of the fivebrane source we encountered the global angular form $e_4/2$. It is clear from (2.9) that the global angular form depends on the connection on the normal bundle and is closed and gauge invariant under $SO(5)$ gauge transformations acting on the normal bundle. As described above we can thus apply descent:

$$
e_4 = de_3^{(0)}, \qquad \delta e_3^{(0)} = de_2^{(1)}. \quad (3.1)
$$

We will now express the uncanceled anomaly (1.5) in terms of (3.1) using a result of Bott and Cattaneo [16]. Consider a real vector bundle $N \to M$ of odd rank $2n + 1$, and for convenience fix a metric. Let $\pi : S(N) \to M$ denote the unit sphere bundle in N. Then the lift $\pi^* N$ has a tautological line subbundle L, and in [16] it is shown that the Euler class of the orthogonal complement $L^\perp$ satisfies $\pi_*[e_{2n}(L^\perp)^3] = 2p_n(N)$. The factor of 2 is the Euler characteristic of the even dimensional sphere. At the level of cohomology, this formula follows from a simple argument using the splitting principle. If N has an orthogonal connection, then we represent real characteristic classes as differential forms using Chern-Weil representatives. The formula also holds at the level of differential forms, since both sides are gauge invariant and depend on only a finite number of derivatives of the connection. Applying the result of [16] to our case, and applying the descent formalism we have:

$$\frac{1}{6} \int_{S_\epsilon(W_6)} \frac{e_4}{2} \wedge \frac{e_4}{2} \wedge \frac{e_2^{(1)}}{2} = \int_{W_6} \frac{(p_2(N))^{(1)}}{24}. \qquad (3.2)$$

The $D = 11$ supergravity which describes the low-energy limit of M theory contains a Chern-Simons term

$$S_{CS} = -\frac{2\pi}{6} \int_{M_{11}} \not{C}_3 \wedge d\not{C}_3 \wedge d\not{C}_3 = \frac{2\pi}{6} \int_{M_{12}} d\not{C}_3 \wedge d\not{C}_3 \wedge d\not{C}_3, \qquad (3.3)$$

where M_{12} is a twelve-manifold with boundary M_{11}. In the absence of five-branes we have $G_4 = dC_3$ and $dG_4 = 0$. In the presence of fivebranes we have argued above that this equation should be modified to $d\not{G}_4 = d\rho \wedge e_4/2$. This requires that we modify the relation between G_4 and C_3. The modified Bianchi identity is satisfied with

$$\not{G}_4 = d\not{C}_3 + A\rho e_4/2 - B d\rho \wedge e_3^{(0)}/2, \qquad (3.4)$$

where locally C_3 can be viewed as a small fluctuation field about the five-brane[4] and $A + B = 1$. Since have smoothed out the fivebrane source we expect on physical grounds that C_3 and G_4 should be smooth on the five-brane, and in fact in the treatment of [6] it is important that C_3 be well defined on the fivebrane. Since ρe_4 is singular at the fivebrane this requires that we take $A = 0$ and hence $B = 1$.

We thus have

$$\not{G}_4 = d\not{C}_3 - d\rho \wedge e_3^{(0)}/2. \qquad (3.5)$$

This relation is quite analogous to the relation $H_3 = dB_2 - \omega_3$ which occurs in $D = 10$, $N = 1$ supergravity coupled to gauge theory and is central to the

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[4]The global definition of C_3 is given in the following section.

Green-Schwarz anomaly cancellation mechanism. In particular, the relation (3.5) implies that C_3 must have an anomalous variation under $SO(5)$ gauge transformations in order that G_4 be gauge invariant,

$$\delta \mathcal{C}_3 = -d\rho \wedge e_2^{(1)}/2. \tag{3.6}$$

Given the modified relation between G_4 and dC_3 we must ask how the Chern-Simons term should be modified. These modifications will involve higher derivative metric interactions. Actually, describing these terms as higher derivative terms is slightly misleading since it presupposes a local description of the physics. However σ_3 is not local in the metric, since we use the exponential map to transfer forms from the total space of the normal bundle to a neighborhood of the fivebrane. We expect that an eventual microscopic derivation will explain this nonlocality or replace it with a local description. With this caveat in mind, there are many higher derivative terms one could add to the supergravity action. These are constrained by physical principles such as supersymmetry and gauge invariance. Here we will only examine the constraints of gauge invariance under diffeomorphism and 3-form gauge transformations. We introduce the expression σ_3 defined by:

$$\mathcal{G}_4 - \rho e_4/2 = d(\mathcal{C}_3 - \rho e_3^{(0)}/2) \equiv d(\mathcal{C}_3 - \sigma_3). \tag{3.7}$$

(Note that G_4 is not exact.) A natural set of higher order terms relevant to the anomaly cancellation problem is obtained by replacing C_3 by σ_3 or dC_3 by $G_4, \rho e_4$, or $d\sigma_3$. One finds in this way twelve linearly independent higher derivative metric interactions. One combination of higher derivative terms which maintains the Chern-Simons structure of the original interaction is

$$S'_{CS} = \lim_{\epsilon \to 0} -\frac{2\pi}{6} \int_{M_{11}-D_\epsilon(W_6)} (\mathcal{C}_3 - \sigma_3) \wedge d(\mathcal{C}_3 - \sigma_3) \wedge d(\mathcal{C}_3 - \sigma_3) \tag{3.8}$$

and we will take (3.8) as the modified Chern-Simons term. It includes higher derivative interactions involving up to eleven derivatives of the metric. Moreover, it is not gauge invariant by itself under diffeomorphisms. Under diffeomorphisms ($SO(5)$-gauge transformations of the normal bundle) the variation of C_3 leads to a variation of (3.8). Indeed, it follows from (3.6) that

$$\delta(\mathcal{C}_3 - \sigma_3) = -d(\rho e_2^{(1)}/2). \tag{3.9}$$

Computing the variation we have

$$\delta S'_{CS} = \lim_{\epsilon \to 0} \frac{2\pi}{6} \int_{M_{11}-D_\epsilon(W_6)} d(\rho e_2^{(1)}/2) \wedge d(\mathcal{C}_3 - \sigma_3) \wedge d(\mathcal{C}_3 - \sigma_3). \tag{3.10}$$

Integrating by parts and taking the limit and using the fact that G_4 and C_3 are smooth near the fivebrane we obtain

$$\delta S'_{CS} = -\frac{2\pi}{6} \int_{S_\epsilon(W_6)} \frac{e_4}{2} \wedge \frac{e_4}{2} \wedge \frac{e_2^{(1)}}{2},\tag{3.11}$$

which by (3.2) cancels the remaining anomaly in diffeomorphisms of the normal bundle. In [6] the cancellation of antisymmetric tensor gauge transformation of C_3 was also studied. It is not hard to see that the modification we have made to the Chern-Simons coupling preserves the cancellation found in [6].

4   Global Structure

Far from the fivebrane $d\rho$ vanishes and we have locally $G_4 = dC_3$. On the other hand we have by the definition of a fivebrane that

$$\int_{S_\infty^4} \slashed{G}_4 = 1\tag{4.1}$$

so C_3 cannot be globally well defined. Rather, we must define C_3^i in patches of an open cover $\mathcal{U}_i$ and relate the C_3^i across patches by antisymmetric tensor gauge transformations, $C_3^i - C_3^j = d\Lambda^{ij}$. The appropriate machinery for this construction is the Čech de Rham complex. A very readable account aimed towards physicists can be found in [17]. The situation here is more complicated due to the mixture of tensor gauge transformations and diffeomorphisms required by the Green-Schwarz like structure. Because of the way we have smoothed out the fivebrane source the quantity with constant linking number through an S^4 surrounding the fivebrane is $\slashed{G}_4 - \rho e_4/2$. In particular

$$\int_{S_r^4} \slashed{G}_4\tag{4.2}$$

varies from 1 to zero as r decreases from large r to small r. To explain the consequences for C_3 we first describe the patching conditions on C_3 near the fivebrane. Then we summarize briefly a global description; see [14] for a more leisurely exposition.

Choose an open cover $\mathcal{V}_\alpha$ for the fivebrane W_6. Then in the transverse space which we take to be $\mathbb{R}^5$ for simplicity we choose radial coordinates and split $S^4 = S_+^4 \cup S_-^4$ into northern and southern hemispheres.

We then have patches

$$\begin{aligned} &\mathcal{V}_\alpha \times S_+^4 \times [r>0]\\ &\mathcal{V}_\alpha \times S_-^4 \times [r>0] \end{aligned}\tag{4.3}$$

The antiderivative $e_3^{(0)}(\Theta, \hat{y})$ transforms across the overlap region $S^3 \times I \cong S_+^4 \cap S_-^4$ by an $SO(5)$ gauge transformation $g_\pm$.

Using the fact that C_3 is well defined on W_6 and the gauge invariance of G_4 we therefore take:

$$\mathcal{C}_3^+ - \mathcal{C}_3^- = \frac{1}{2}(1+\rho)de_2^{(1)}(g_\pm, \Theta_+, \hat{y}_+), \tag{4.4}$$

where $e_2^{(1)}$ is the integrated form of the cocycle $e_2^{(1)}$ given in (2.11). Note that this is *not* a an antisymmetric tensor gauge transformation. But we can use the fact that C_3 has picked up a diffeomorphism variation to write this as:

$$\mathcal{C}_3^+ - \mathcal{C}_3^- = d\left[\frac{(1+\rho)}{2}e_{(2)}^1(g_\pm, \Theta_+, \hat{y}_+)\right] - \frac{1}{2}d\rho \wedge e_{(2)}^1(g_\pm, \Theta_+, \hat{y}_+) \tag{4.5}$$

which is a sum of an antisymmetric tensor gauge transformation and a diffeomorphism gauge transformation. We thus find that C_3 is well defined near W_6. As we move away from the fivebrane C_3 requires non-trivial transition functions between patches which are a combination of antisymmetric tensor gauge transformations and diffeomorphisms. At infinity these reduce to pure antisymmetric tensor gauge transformations.

A global discussion may be framed in terms of a general "Γ-calculus," which is an extension of the usual calculus of differential forms. We describe it in terms of an open cover $\{U_i\}$ of M which is *good* in the sense that all intersections

$$U_{i_0...i_p} = U_{i_0} \cap \cdots \cap U_{i_p} \tag{4.6}$$

are contractible. The set of good covers is contractible in a suitable sense, so the particular choice of good cover does not affect the result of any computation. Let $(\Gamma^\bullet(M), D)$ be the total complex of the modified Čech-de Rham complex

$$
\begin{array}{c|cccc}
\vdots & & & & \\
\Omega^2(U) & & & & \\
 & d\uparrow & & & \\
\Omega^1(U) & & \vec{\delta} & & \\
 & & & & \\
C^\infty(U,T) & & & & \\
\hline
 & U_i & U_{ij} & U_{ijk} & \cdots
\end{array}
\qquad (4.7)
$$

Here δ is the usual Čech differential, the columns form the de Rham complex modified by replacing 0-forms by circle-valued functions, and $D = \delta \pm d$ is the total differential. There is a subspace $\Theta^p(M) \subset \Gamma^p(M)$ of "connection-like" elements:

$$
\Theta^p(M) = \{\omega \in \Gamma^p(M) : D\omega \in \Omega^{p+1}(M)\}. \qquad (4.8)
$$

For example, an element $\omega = \{g_{ij}, \alpha_i\} \in \Theta^1(M)$ has the form

$$
\begin{array}{c|ccc}
2 & \Omega & & \\
 & \uparrow & & \\
1 & \alpha_i & \rightarrow & 0 \\
 & & \uparrow & \\
0 & & g_{ij} & \rightarrow & 0 \\
\hline
 & U_i & U_{ij} & U_{ijk}
\end{array}
\qquad (4.9)
$$

It represents a circle bundle with connection: g_{ij} are the transition functions of some local trivializations, α_i are the local connection forms, and $\Omega = D\omega$ is the curvature. Intuitively, $\omega \in \Theta^p(M)$ for $p > 1$ is a higher degree version of a connection on a circle bundle. An element $\sigma \in \Gamma^{p-1}(M)$ is a trivialization of the trivial bundle $D\sigma \in \Theta^p(M)$.

Given the fivebrane $W_6 \subset M_{11}$ we first define a Poincaré dual form

$$
\Omega(g) \in \Omega^5(M) \qquad (4.10)
$$

by the right hand side of (2.3). It depends "functorially" on the metric g on M: the construction is invariant under diffeomorphisms. There is also a diffeomorphism-invariant antiderivative

$$
\mu(g) \in \Omega^4(M \backslash W) \qquad (4.11)
$$

on the complement of the fivebrane; it was denoted "$\rho e_4/2$" previously. By working universally we can also choose a diffeomorphism-invariant connection-like object

$$\omega(g) \in \Theta^4(M) \qquad (4.12)$$

which is an antiderivative globally: it has curvature $D\omega = \Omega$. Furthermore, there is a trivialization

$$\sigma(g) \in \Gamma^3(M\backslash W) \qquad (4.13)$$

off of the fivebrane with "covariant derivative" $\mu(g)$: we write $D\sigma = \mu - \omega$. The quartet $(\omega, \sigma, \Omega, \mu)$ is our global description of the smeared-out fivebrane. It depends on the metric g and the fixed cutoff function ρ.

In the absence of any fivebranes the 3-form field $\not{C}$ is globally an element of $\Theta^3(M)$ with curvature $\not{C}_4 = D\not{C}_3$. In the presence of the fivebrane, $\not{C}$ is a global trivialization

$$\not{C}_3 \in \Gamma^3(M) \qquad (4.14)$$

of $\omega(g)$ with covariant derivative

$$\not{C}_4 \in \Omega^4(M), \qquad (4.15)$$

i.e., $D\not{C}_3 = \not{C}_4 - \omega(g)$. With these definitions the modified Chern-Simons term (3.8) makes sense globally and leads to the anomaly cancellation computed above.

5 Reduction on a Calabi-Yau 3-fold

One of the closest relatives of M-theory is $\mathcal{N} = 1$ supergravity in five dimensions. This theory has Chern-Simons interactions and chiral strings, which can lead to both gauge and gravitational anomalies [18]. A natural way to produce such models is via compactification of M-theory on a Calabi-Yau manifold X. In this case there are $h^{1,1}(X)$ independent vector fields, including one graviphoton and $h^{1,1}(X) - 1$ vectormultiplets.

Suppose a 5-brane W wraps a four-cycle P with homology class $[P] = p^A[\Sigma_A]$, where $[\Sigma_A]$ is an integral basis for $H_4(X, \mathbb{Z})$. At long distances the noncompact part of W is a chiral string in the $\mathbb{R}^{1,4}$ supergravity. The number of left and right-moving bosonic zero modes $N^B_{L,R}$ and right-moving fermionic zero modes N^F_R on the string are given by [19]:

$$N^B_L = 6D + c_2 \cdot P$$
$$N^B_R + \frac{1}{2}N^F_R = 6D + \frac{1}{2}c_2 \cdot P \qquad (5.1)$$

where

$$c_2 \cdot P = \int_P c_2(TX), \tag{5.2}$$

and the self-intersection is given by

$$D = \frac{1}{6} \int_X \hat{P}^3 = D_{ABC} p^A p^B p^C, \tag{5.3}$$

where $\hat{P}$ is the Poincaré dual class in $H^2(X, Z)$.

The BPS string is magnetically charged under the gauge field obtained from the Kaluza-Klein reduction of C_3 on X dual to P, namely, $C_3 = C_1 \wedge p^A \theta_A$ where $C_1 \in \Omega^1(\mathbb{R}^{1,4})$ and θ_A is an integral basis of harmonic two-forms on X. Defining the field strength associated to C_1 in the presence of a string requires a treatment very similar to what we presented in the fivebrane case. In particular, the normal bundle is now an $SO(3)$ bundle with connection Θ and smoothing out the string source requires that we modify the relation between G_2 and dC_1 to

$$d\cancel{C}_1 = \cancel{G}_2 + d\rho \wedge e_1^{(0)}(\Theta)/2. \tag{5.4}$$

The zero mode spectrum is anomalous and the anomaly is given by descent from

$$\widetilde{I}_4(TW, N) = \frac{1}{48}\left(c_2 \cdot P\left(p_1(TW) + p_1(N)\right) + 12 D p_1(N)\right). \tag{5.5}$$

The cancellation of anomalies in diffeomorphisms of the tangent bundle by inflow requires a bulk coupling

$$\Delta S^5 = \int_{M_5 - D_\epsilon(W)} C_1 \wedge I_4(TM), \tag{5.6}$$

where now $I_4(TM) = \frac{1}{48}(c_2 \cdot P)p_1(TM)$. This coupling can be obtained by reduction on X of the eleven-dimensional bulk term (1.2). It is easy to see that

$$\iota^*\left(I_4(TM)\right) = \widetilde{I}_4(TW, N) - \frac{D}{4}p_1(N), \tag{5.7}$$

where ι is the inclusion map. Again there is a part of the anomaly involving gauge transformations of the normal bundle which is uncancelled by the inflow. It is interesting to note that the anomaly of the normal bundle arises only when the self-intersection is non-zero which is precisely the condition for having a Chern-Simons interaction in five dimensions. Thus the mechanism

for cancellation of the normal bundle anomaly should be the same as in M theory.

The modified Chern-Simons coupling is

$$S'^5_{CS} = \lim_{\epsilon \to 0} -12D\pi \int_{M_5 - D_\epsilon(W)} (\mathcal{C}_1 - \sigma_1) \wedge d(\mathcal{C}_1 - \sigma_1) \wedge d(\mathcal{C}_1 - \sigma_1). \quad (5.8)$$

The anomaly in pure C_1 gauge transformations is compensated by a phase factor coming from the coupling of C_1 to the string worldsheet in a way completely analogous to the discussion of C_3 antisymmetric tensor gauge transformations in [6].

The cancellation of the anomaly in gauge transformations of the normal bundle follows as in the fivebrane case upon application of the relevant version of the Bott-Catteneo formula:

$$\int_{S_\epsilon(W_2)} e_2 \wedge e_2 \wedge e_0^{(1)} = 2 \int_{W_2} (p_1(N))^{(1)} \quad (5.9)$$

It would be interesting to explore whether there are implications of this discussion for the black hole entropy following the discussion in [19].

Acknowledgments

Appendix A: Volume form for all odd rank bundles

The formulae used in the text for $e_4, e_3^{(0)}$ etc. have natural extensions to the $SO(2n + 1)$ case. The global angular form can be given, as above, and as in [16] as a basic form on $P \times S^{2n}$. We find:

$$e_{2n} = \frac{1}{2(4\pi)^n n!} \sum_{j=0}^{n} (-1)^j \frac{n!}{j!(n-j)!} \epsilon(F)^j (D\hat{y})^{2n-2j} \hat{y}, \quad (A.10)$$

where

$$\epsilon(F)^j (D\hat{y})^{2n-2j}\hat{y} \equiv \epsilon_{a_1 \cdots a_{2n+1}} F^{a_1 a_2} \cdots F^{a_{2j-1} a_{2j}} (D\hat{y})^{a_{2j+1}} \cdots (D\hat{y})^{a_{2n}} \hat{y}^{a_{2n+1}}$$

(A.11)

and the normalization is fixed by noting that the volume form defined by $d^{2n+1}y = r^{2n}dr\Omega_{2n}$ has $\int_{S^{2n}} \Omega_{2n} = 2\pi^{n+1/2}/\Gamma(n+1/2)$.

Similarly, the Chern-Simons form for the general $SO(2n+1)$ case is given by:

$$e^0_{2n-1}(\Theta_1) - e^0_{2n-1}(\Theta_0)$$

$$= -\frac{\epsilon}{2(4\pi)^n n!} \int_0^1 dt \sum_{j=0}^{n-1} (-1)^j \frac{n!}{j!(n-j-1)!} \Theta(F_t)^j (D_t\hat{y})^{2n-2j-2}\hat{y},$$

(A.12)

where $\Theta_t = t\Theta_1 + (1-t)\Theta_0$, and $F_t = d\Theta_t - \Theta_t^2$ and $D_t = (d - \Theta_t)$. The equations simplify for the case that the normal bundle is topologically trivial. In that case there is a canonical choice of basepoint connection $\Theta = 0$ for which we may take an antiderivative of the volume form of the sphere S^{2n}.

It is also worth noting that one can give e_{2n} a Mathai-Quillen-like representation. We can introduce $2n + 1$ orthonormal antighost zeromodes and write, up to a constant, the angular form for the odd rank case as:

$$e_{2n}(g) = \frac{1}{2(2\pi)^n} \int \prod_{a=1}^{2n+1} d\rho^a \exp\left[\rho^a \rho^b F^{ab} - (D\hat{y})^a \rho^a + \rho^a \hat{y}^a\right]. \quad \text{(A.13)}$$

The MQ representative of a rapid-decrease Thom class of odd-rank bundles is therefore:

$$\Phi^{\text{universal}} = \kappa_n e^{-(y,y)} \int \prod_{a=1}^{2n+1} \frac{d\rho^a}{\sqrt{2\pi}} \exp\left[\rho^a \rho^b \phi^{ab} - (Dy)^a \rho^a + \rho^a y^a\right], \quad \text{(A.14)}$$

where κ_n is a normalization constant and ϕ is in the Weil algebra. The expression (A.14) closely resembles the Mathai-Quillen representative of the universal Thom form of even rank bundles, which is a starting point for the development of topological field theory (see, for example, [20]). It would be interesting to see if the above expressions could also be used to develop new topological field theories.

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