On N=1 Yang-Mills in Four Dimensions

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Abstract

Extending previous work on geometric engineering of $N = 1$ Yang-Mills in four dimensions for simply laced $(A_n, D_n, E_{6,7,8})$ gauge groups, we construct local models for all other gauge groups (B_n, C_n, F_4, G_2) in terms of F-theory. We compute the radius dependent superpotential upon further compactification on a circle to $d = 3$ in the dual M-theory and use it to show that the number of vacua in four dimensions for each group is given by its dual coxeter number, in accordance with expectations based on gaugino condensates.

1 Introduction

In this note we continue the previous work [1] on geometric engineering of pure $N = 1$ Yang-Mills in four dimensions and its further compactification on a circle to 3 dimensions, by extending it to include the non-simply laced gauge groups. In [1] the $N = 1$ simply laced $A_n, D_n, E_{6,7,8}$ gauge groups were considered. It was shown there how one can compute the inequivalent vacua of these theories in four dimensions using geometric engineering in terms of F-theory on an elliptic Calabi-Yau fourfold and its relation to the three dimensional description in terms of M-theory upon further compactification on a circle. The basic idea there was to note that even though the four dimensional theory is strongly coupled, upon a further compactification, the dynamics is weakened because the theory is generically abelianized (through

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vevs for Wilson lines around the circle) in which case the superpotential of the 3 dimensional theory can be computed using point like instantons, which are realized as Euclidean M-theory 5-brane instantons [2]. The critical points of the superpotential for large radius gives the number of vacua in the four dimensional limit. The number of vacua in four dimensions is expected to be given by $c_2(G)$ the dual coxeter number of the group, with the vev of gaugino bilinear $\langle \lambda^2 \rangle$ playing the role of the order parameter:

$$\langle \lambda^2 \rangle = \exp(\frac{-1}{c_2(G)g^2})\omega$$

where ω is a $c_2(G)$-th root of unity (we have set the scale $\Lambda = 1$). This expectation is based on one-instanton computations where there are $2c_2(G)$ gaugino zero modes, and the cluster decomposition property of QFT's (see [3] for a review).

In this note we review [1] and indicate the modification needed for the non-simply laced cases. Our approach indicates the power of geometric engineering in constructing and studying dynamics of gauge theories. A similar approach for $N = 4$ Yang-Mills in $d = 4$ [4] was used to show how Montonen-Olive duality for all gauge groups can be reduced to T-duality of type II strings. A similar approach was considered for $N = 2$ case in [5] [6] and led in [7] to the solution of Coulomb branch geometry for all asymptotically free gauge theories with SU gauge groups with arbitrary bifundamental matter between pairs of groups.

2 F-Theory and N=1 Yang-Mills in Four Dimensions

Let us briefly review [1]: If we consider F-theory compactification on elliptic Calabi-Yau fourfolds we obtain an $N = 1$ theory in $d = 4$. As discussed in [1] if we are interested in constructing an $N = 1$ gauge theory with no matter, this can be done by considering an elliptic fourfold which has an A-D-E singularity over a complex 2-manifold S which is "rigid" (with $h^{1,0} = h^{2,0} = 0$), such as $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$. The rigidity is necessary to avoid matter in the adjoint representation. This gives rise in four dimensions to $N = 1$ A-D-E Yang-Mills theory in $4d$ where the bare gauge coupling constant is given by

$$\frac{1}{g_4^2} = V_S \tag{2.1}$$

where V_S denotes the volume of S. If we compactify the $N = 1$ theory from $d = 4$ to $d = 3$ we obtain an $N = 2$ theory in $d = 3$. By the chain of duality

in [8] the compactification of F-theory on a circle is dual to M-theory on the same elliptic Calabi-Yau where the radius of the circle R is related to the Kähler class of the elliptic fiber $V_{T^2} = \frac{1}{R}$. Moreover there is a Weyl rescaling of the metric so that the volume of S in M-theory is given by

$$V_S^M = \frac{1}{g_3^2} = \frac{R}{g_4^2} = R\, V_S^F$$

In particular we have

$$\frac{V_S^M}{R} = \frac{1}{Rg_3^2} = V_S^F \qquad (2.2)$$

If we want to retain the R-dependence in the physical quantities, we have to note that the 4-fold is an elliptic one with a singularity over the surface S.

$N = 2$ in $d = 3$ has a Coulomb branch: The Wilson line of the four dimensional gauge field along the circle as well as the dual to the vector gauge field in $d = 3$ which is a scalar, form a complex scalar field ϕ with values in the Cartan of the gauge group. Going to non-zero value of ϕ (corresponding to Wilson lines) is realized geometrically by blowing the singularity of ADE type, and ϕ is identified with the blow up parameters. The complex part of ϕ, being dual to a $U(1)$ vector field, is a periodic variable. The periodicity is fixed by the integrality of $H_6(K, \mathbf{Z})$ where K is the fourfold. In particular if the real part of V denotes the volume of a generator of $H_6(K, \mathbf{Z})$, the periodicity of its complex part is such that the good variable is $\exp(-V)$.

For $N = 2$ in $d = 3$ Yang-Mills, one expects to obtain a superpotential $W(\phi)$ as was shown for $SU(2)$ gauge group in [9]. If this theory comes from a reduction of $N = 1$ in $d = 4$ on a circle of radius R where $1/g_3^2 = R/g_4^2$, the superpotential develops an R dependent piece (for the $SU(2)$ case this dependence was determined in [10]). The superpotential and its R-dependence in the present case was determined by using the identification of superpotential with point-like instantons corresponding to Euclidean 5-branes [2]. In particular it was shown in [2] that for each complex 3 dimensional manifold I which is a subspace of the 4-fold and which is rigid (i.e. where $h^{1,0} = h^{2,0} = h^{3,0} = 0$) one gets a term in the superpotential of the form $\exp(-V_I)$ where V_I denotes the superfield corresponding to the volume of I.

The geometry of blow up of A-D-E is such that over each point on S we obtain a collection of $r + 1$ spheres e_i, where r is the rank of the corresponding group. Moreover the spheres intersect each other according to the corresponding Affine Dynkin Diagram (spheres correspond 1-1 to the Dynkin nodes). Furthermore, the class of the elliptic fiber is given by

$$[T^2] = \sum_{i=1}^{r+1} a_i[e_i] \qquad (2.3)$$

where a_i correspond to Dynkin indices of the affine Dynkin diagram. For A_n, $a_i = 1$, for D_n, $a_i = 2$ except for the four boundary nodes of the affine Dynkin diagram where they are 1. For E-series they are given by

$$E_6 : \qquad 1, 1, 1, 2, 2, 2, 3$$
$$E_7 : \qquad 1, 1, 2, 2, 2, 3, 3, 4$$
$$E_8 : \qquad 1, 2, 2, 3, 3, 4, 4, 5, 6$$

Note that $\sum a_i = c_2(G)$ for all the A-D-E groups:

$$c_2(SU(N)) = N, \qquad c_2(SO(2N)) = 2N - 2, \qquad c_2(E_6) = 12,$$
$$c_2(E_7) = 18, \qquad c_2(E_8) = 30$$

Recall that the volume of T^2 is $1/R$, thus we learn from (2.3) that

$$\sum_{i=1}^{r+1} a_i \phi_i = \frac{1}{R} \tag{2.4}$$

where ϕ_i denotes the volume of the i-th sphere. Note that the volume of one of the nodes can be determined in terms of all the rest. In fact in the gauge theory description in three dimensions all the nodes are small except for the affine node which becomes large as $R \to 0$ and one solves (2.4) for the volume of the affine node which becomes non-dynamical in the 3d theory. Note that the Dynkin number for the affine node is 1, and so we can write its volume V_{r+1} as

$$V_{r+1} = \frac{1}{R} - \sum_{i=1}^{r} a_i \phi_i \tag{2.5}$$

The condition that we obtain gauge symmetry $A - D - E$ requires the existence of a "split" resolution, which means that each e_i sphere which locally is a $\mathbf{P}^1$ over S is globally a $\mathbf{P}^1$ bundle over S. Let us call the corresponding 3-dimensional complex manifold consisting of these $\mathbf{P}^1$ bundles over S by $\hat{e}_i$. As was argued in [1] $\hat{e}_i$ satisfy the condition $h^{i,0}(\hat{e}_i) = 0$ for $i \neq 0$ and so give rise to superpotential terms once they are wrapped by Euclidean 5-branes. We thus have $r + 1$ point instantons, one for each $\hat{e}_i$, i.e., one for each node of the affine Dynkin diagram. Thus

$$W = \sum_{i=1}^{r} \exp(\frac{-1}{g_3^2} \phi_i) + \exp[\frac{-1}{Rg_3^2} + \sum_{i=1}^{r} \frac{a_i \phi_i}{g_3^2}]$$

where we used the fact that $V_{\hat{e}_i} = V_{e_i} V_S^M$ and used $V_{e_i} = \phi_i$ for $i = 1, ..., r$ and also used (2.2) and (2.5). Let us define the good variables $x_i = \exp(\frac{-\phi_i}{g_3^2})$,

then we can write this as

$$W = \sum_{i=1}^{r} x_i + \gamma \prod_{i=1}^{r} x_i^{-a_i}$$

where $\gamma = \exp(\frac{-1}{g_4^2})$. We have rewritten the superpotential in terms of 4d coupling (in terms of γ). For large enough R we should have the same number of vacua as the 4d theory. This in particular is the number of critical points of W. Moreover the condition that W have isolated critical points would be expected if the 4d theory has mass gap, as is believed to be the case. Solving $dW = 0$ we find that there are $\sum_{i=1}^{r+1} a_i = c_2(G)$ isolated critical points given by

$$x_i = \text{const.} \frac{1}{a_i} \omega \cdot \exp(\frac{-1}{c_2(G)g_4^2})$$

where ω is a $c_2(G)$-th root of unity. This is as expected based on consideration of gaugino condensates in four dimensions. Certain aspects of these results have been further studied and elaborated in [11]. The case of $SU(n)$ was also studied from the field theoretic viewpoint in [12] [13].

3   Non-simply Laced Case

We now would like to extend this analysis to the non-simply laced gauge groups. The basic idea is that the non-simply laced gauge groups arise from simply laced groups. The description of non-simply laced groups as simply laced ones modulo the imposition of an outer automorphism is well known mathematically, and was used in physics in [14] [15] [4]. What this means in the present context is that the blow up spheres e_i are not "split". In other words the e_i preserve their identity as we move over S *only up to an outer automorphism of the Dynkin diagram*, which exchanges some of them. This permutation means that the actual group is the group modulo the outer automorphism. The non-simply laced groups are obtained as

$$SO(2n) \to SO(2n-1) \qquad \mathbf{Z_2}$$
$$SU(2n) \to SP(n) \qquad \mathbf{Z_2}$$
$$E_6 \to F_4 \qquad \mathbf{Z_2}$$
$$SO(8) \to G_2 \qquad \mathbf{Z_3}$$

where in the SO case the $\mathbf{Z_2}$ outer automorphism exchanges the two end nodes of the Dynkin diagram, in the SU case the $\mathbf{Z_2}$ acts as a reflection on the Dynkin diagram (fixing one node in the ordinary Dynkin diagram, and two nodes on the affine), for the E_6 it exchanges the two long ends of the

Dynkin diagram and for the $SO(8)$ case the Z_3 cyclically permutes the three end nodes of the ordinary Dynkin diagram (fixing the affine node).

Let us see how this modifies the analysis of the superpotential. For the spheres e_i which are not exchanged under this automorphism, we continue to have a well defined complex three manifold $\hat{e}_i$ which is a $\mathbf{P}^1$ bundle over S. For the other e_i which are permuted, or cyclically exchanged, the single $\hat{e}_i$ does not make sense. However there is a double (triple) cover of the 5-brane, in the case of the $\mathbf{Z_2}$ ($\mathbf{Z_3}$) outer automorphism which does make sense and consists of a bundle over S whose fiber is the union of e_i which form a single orbit under the outer automorphism. It is easy to see that these fivebranes are rigid and do satisfy the criterion of [2] for contributing to superpotential. The main novelty now is that the euclidean 5-brane volume in the case of $\mathbf{Z_2}$ ($\mathbf{Z_3}$) outer automorphism is twice (three times) bigger than what it used to be. We thus end up with the superpotential

$$W = \sum_{i=1}^{r'} x_i^{m_i} + \gamma \prod_{i=1}^{r'} x_i^{-m_i a_i}$$

where r' denotes the rank of the non-simply laced group and m_i denotes the number of nodes in the same orbit as e_i under the outer automorphism. However, we have to note that the good variables now are

$$y_i = x_i^m.$$

The reason for this, as explained before is that $H_6(K, \mathbf{Z})$ fixes the periodicity of the phase of the chiral fields, where K is the Calabi-Yau fourfold. In the non-simply laced case some of the generators of $H_6(K, \mathbf{Z})$ are m_i times bigger than what they used to be. This implies that y_i are now the correct variables, in terms of which we have

$$W = \sum_{i=1}^{r'} y_i + \gamma \prod_{i=1}^{r'} y_i^{-a_i}$$

This is now exactly of the same form as in the simply laced case and so finding the critical points of the superpotential in this case gives the number of vacua which is

$$1 + \sum_{i=1}^{r'} a_i = c_2(G')$$

where the sum is now over all the Dynkin indices of the simply laced group, one for each orbit of the outer automorphism. It is easy to check that this gives for the number of vacua the dual coxeter number of the group, i.e.,

$$c_2(SO(2n-1)) = 2n - 3, \quad c_2(SP(n)) = n + 1, \quad c_2(G_2) = 4, \quad c_2(F_4) = 9$$

(for the case of F_4 the inequivalent a_i are $1, 2, 2, 3$).

Acknowledgments

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