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Midi-Superspace Quantization of Non-Compact Toroidally Symmetric Gravity

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Abstract

We consider the quantization of the midi-superspace associated with a class of spacetimes with toroidal isometries, but without the compact spatial hypersurfaces of the well-known Gowdy models. By a symmetry reduction, the phase space for the system at the classical level can be identified with that of a free massless scalar field on a fixed background spacetime, thereby providing a simple route to quantization. We are then able to study certain non-perturbative features of the quantum gravitational system. In particular, we examine the quantum geometry of the asymptotic regions of the spacetimes involved and find some surprisingly large dispersive effects of quantum gravity.

1 Introduction

Any quantum theory of gravity must deal with two sets of difficulties inherent in classical general relativity. The first is the diffeomorphism invariance of the theory, from which it follows that there is generally no fixed spacetime geometry on which quantization can be performed. Indeed, the geometry itself is the dynamical quantity we study. The second set of difficulties arise because the field equations of general relativity are highly nonlinear

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and difficult to solve. These features raise both technical and conceptual problems with the formulation of general relativity as a physical theory in the conventional sense. To understand the nature of these problems, and perhaps their solutions, it is useful to consider simpler models which also exhibit them. One class of such models can be found by requiring the full theory of general relativity to satisfy various symmetry properties, thereby reducing the number of possible spacetimes while leaving at least the diffeomorphism invariance essentially intact.

The idea of using symmetry reduction to simplify the solution of Einstein's equations is nearly as old as general relativity itself. In fact, all the known solutions to date have been found under the assumption of one kind of symmetry or another. The application of such simplifications to the problem of quantum gravity, however, seems to have begun with the so-called mini-superspace systems. These systems consist of general relativity, perhaps coupled with some matter fields, together with the requirement of so much symmetry that the number of independent degrees of freedom becomes finite. Since mini-superspace systems derive from general relativity, they do not depend on any fixed background spacetime structure. On the other hand, since they retain only a finite number of degrees of freedom, they do not capture the non-linear *field*-theoretic complexity of a complete gravitational theory. To incorporate both of these features, we need to consider some less restrictive (i.e., less symmetric) models.

One of the simplest examples of a symmetry reduced gravitational system arises from the dimensional reduction of 3+1-dimensional general relativity with respect to a (spacelike) hypersurface orthogonal Killing vector field. It is well known that such a system is mathematically equivalent to 2+1-dimensional general relativity coupled with a free, massless scalar field [1,2]. Although the degrees of freedom of 2+1-dimensional gravity are topological in nature and therefore finite in number, the reduced model still possesses an infinite number of degrees of freedom which are characterized by the excitations of the scalar field. Unfortunately, despite the comparative simplicity of these dimensionally reduced systems, they remain somewhat difficult to study in practice. The problem lies in the question of global existence of solutions to the 2+1-dimensional field theory in the presence of gravity. To resolve this issue, it is sufficient to posit the existence of a second, independent, hypersurface orthogonal Killing field on the original 3+1-dimensional spacetime. One can then show that the resulting symmetry of the 2+1-dimensional spacetime effectively *decouples* the dynamics of the scalar field from that of the gravitational field in the sense that one can solve for the scalar field without any reference to the *physical* spacetime geometry. Rather, it becomes possible to solve for the scalar field on a fictitious, fixed background spacetime and then, afterwards, solve for the gravitational degrees of freedom in terms of a given scalar field configuration. This procedure demonstrates explicitly the global existence of solutions to the reduced theory. Thus, the assumption of two Killing vectors removes most of the technical difficulty associated with the diffeomorphism invariance of general relativity while leaving its field-theoretic properties intact. One might hope, therefore, to gain some further insight into the nature of quantum gravity through the study of such systems.

The most familiar examples of 3+1-dimensional spacetimes with a pair of independent hypersurface orthogonal Killing fields are those of Einstein-Rosen waves and the Gowdy models. The first of these describes asymptotically flat[1] metrics on R^4 which are cylindrically symmetric. The Gowdy models describe 3+1-dimensional spacetimes with compact spatial topology — T^3, $S^2 \times S^1$ or S^3 — which have a toroidal symmetry group, $U(1) \times U(1)$. The quantization of these systems is not a new area of research. The quantum Einstein-Rosen model, for example, was described by Kuchař [5] as early as 1971. More recently, both of these models have been analyzed [6,7] from a more rigorous standpoint, with careful attention paid to the definition of the phase space and to certain subtleties of the quantum theory. The present paper performs a similar analysis for a third class of spacetimes which has only recently been introduced by Schmidt [8]. The spacetimes in question have topology $\mathrm{R}^2 \times T^2$ and their Killing fields are assumed to have compact, toroidal orbits. Although the toroidal symmetry which the Schmidt model shares with the Gowdy models will give rise to certain similarities between the two, the global structures of the spacetimes involved are very different. In particular, the spacetimes considered here do not possess compact spatial hypersurfaces as in the Gowdy models [7]. Furthermore, they are also not asymptotically flat as in the Einstein-Rosen model [3]. In the previous cases, these characteristics are used in part to specify the phase space of the system and are therefore closely connected with the final theory. In our case, however, we will see that it is possible to specify the phase space completely without initially imposing such restrictions on the spacetime geometry. Instead, the structure of the phase space will be motivated only by the spacetime description of the model and by certain analytical requirements.

Since the reduced Schmidt model can be identified with a free scalar field on a fixed, 2+1-dimensional background spacetime, its quantization is relatively straightforward. Once that quantization is accomplished, however, one can express the original gravitational variables of the system in terms

[1]Note that due to the translational symmetry of the Einstein-Rosen spacetimes, they cannot be asymptotically flat in the usual 3+1-dimensional sense. Rather, the manifold of orbits of the translational Killing field is required to be flat in an appropriate 2+1-dimensional sense [2–4].

of the quantum scalar field and thereby obtain a non-perturbative quantum gravitational theory. It is also possible to regard the reduced model of the scalar field as a quantum field theory in its own right by neglecting the coupling of matter to gravity. Therefore, by comparing these two theories, one might reasonably hope to gain some insight into how the introduction of gravity will affect our current understanding of quantum field theory. For example, it is often suggested that a quantum field theory which incorporates gravity will come equipped with a natural cut-off at roughly the Planck scale which will alter the dynamics of field excitations with trans-Planckian energies. In this model, we will have a concrete example of a such a theory. We will see that, although there is no natural cut-off, there are some surprising quantum effects in the presence of high-frequency excitations of the scalar field. These effects suggest that the classical, metric structure of spacetime, which is a good low-energy approximation, breaks down at high field energies. The dynamics of the quantum system do remain well defined in that regime, but the notion of a classical spacetime metric — whether fixed or dynamical — becomes a poor approximation to the full theory.

The outline of the paper is as follows. In the second section, we describe the Hamiltonian formulation of the Schmidt model. We pay special attention to the boundary conditions which we impose on the physical fields and to the deparameterization process that isolates the true degrees of freedom of the theory. This procedure yields an unconstrained reduced phase space for the system which can be identified with that of a free scalar field on a certain background spacetime. In the third section, we define a quantum analog of this reduced system using a standard Kähler quantization scheme. We are then able to use the model to examine certain geometrical questions and develop some intuition about quantum gravity in general. In the fourth and final section, we summarize the results of the previous two and raise some questions for future investigation.

Lastly, we should explain some of our notational conventions. Throughout this paper, the speed of light will be taken to be unity: $c = 1$. However, since we will be interested in comparing situations in which different physical effects are taken into account, we will not do the same with the gravitational constant G, or with Planck's constant $\hbar$.

2 Hamiltonian Formulation

2.1 The Midi-Superspace

Let us begin with a precise definition of the system we will study. The spacetimes of the Schmidt model [8] are topologically $\mathrm{R}^2 \times T^2$ and are required to support a pair of independent spacelike, hypersurface-orthogonal Killing vector fields. We will assume here that the orbits of the Killing fields are 2-tori, so the manifold of orbits will be topologically R^2. It then follows from these symmetry conditions that the metric ${}^4g_{ab}$ may be written as a sum of two pieces

$$ {}^4g_{ab} = {}^2g_{ab} + \tau\,\sigma_{ab}\,. \tag{2.1}$$

Here, σ_{ab} is a flat metric with unit total volume on a toroidal orbit of the symmetry group, while ${}^2g_{ab}$ is the metric on the 2-manifolds orthogonal to the orbits. We use τ to denote the scale factor for the metric on the toroidal orbits in anticipation of its eventual role as the time parameter of the reduced theory. It is possible to use τ as a time parameter since, as in the Gowdy model [9], the symmetry properties imply that τ must have a timelike gradient.

The analysis presented below begins by considering the quotients of these spacetimes by only one of their Killing fields. It was shown in [2] that, after a conformal rescaling of its metric by τ, any one of these quotient manifolds will define a solution of 2+1-dimensional general relativity coupled with a zero rest-mass scalar field. We may therefore consider a quotient *spacetime*, M, which is topologically $\mathrm{R}^2 \times S^1$ and supports a (nowhere-vanishing) spacelike, hypersurface-orthogonal Killing field with closed orbits which we denote by σ^a. To simplify the following discussion, we will work entirely within the context of this 2+1-dimensional theory. We will also assume, for convenience, that both the manifold structure and all the fields we consider here are smooth (C^∞).

Due to the hypersurface-orthogonality of σ^a, we can write the 2+1-dimensional metric as

$$g_{ab} = h_{ab} + \tau^2\,\nabla_a\sigma\,\nabla_b\sigma. \tag{2.2}$$

Here, $h_{ab} := \tau\,{}^2g_{ab}$ is the metric on the 2-manifolds orthogonal to σ^a, and σ is the angular coordinate conjugate to σ^a (i.e., $\nabla_a\sigma = \tau^{-2}\,g_{ab}\sigma^b$). Because we assumed above that the Killing fields of the original 3+1-dimensional spacetimes had toroidal orbits, it follows that the space O of orbits of σ^a will be topologically R^2 and will inherit a manifold structure from M. Furthermore, since σ^a is a spacelike Killing field, h_{ab} will give rise to a metric of signature $(-,+)$ on O which we will also denote by h_{ab}. Finally, the function τ and the scalar field ψ must be Lie-dragged by σ^a and will therefore also restrict to O.

To begin the Hamiltonian analysis of this system, we need to introduce a time structure. It is simplest to do this directly on the manifold O of orbits of σ^a. Thus, we choose a foliation of this 2-manifold by spacelike lines of constant t, and pick a transverse dynamical vector field $t^a = Nn^a + N^z\hat{z}^a$.

Here, n^a is the unit, future-pointing, timelike normal to the foliation, and $\hat{z}^a$ is a unit-vector field on each of its leaves. If we now pick a coordinate z on a single leaf of the foliation, we can carry it to all of the others using t^a. In this way, we arrive at the usual dynamical decomposition of the metric

$$h_{ab} = \left(-N^2 + (N^z)^2\right) \nabla_a t \nabla_b t + 2N^z \nabla_{(a} t \nabla_{b)} z + e^\gamma \nabla_a z \nabla_b z. \tag{2.3}$$

The quantities N, N^z and γ are all functions of t and z. Similarly, the functions τ and ψ can now be expressed as functions of the (t, z)-coordinates on O.

The midi-superspace we have built therefore includes five real-valued functions on R^2: the lapse N, the (norm of the) shift N^z, the metric functions γ and τ, and the scalar field ψ. These functions will be required to satisfy the Einstein-Klein-Gordon field equations[2]

$$G_{ab} = T_{ab} \quad \text{and} \quad g^{ab} \nabla_a \nabla_b \psi = 0, \tag{2.4}$$

where G_{ab} is the Einstein tensor of g_{ab} and T_{ab} is the usual stress-energy tensor of the massless scalar field ψ

$$T_{ab} = \nabla_a \psi \nabla_b \psi - \tfrac{1}{2} g_{ab} \left(g^{cd} \nabla_c \psi \nabla_d \psi\right). \tag{2.5}$$

To make sense of these as differential equations on $(N, N^z, \gamma, \tau, \psi)$, we need only observe that we can now construct a global coordinate system (t, z, σ) on M.

To finish the construction of our midi-superspace, we still need to specify the boundary conditions for the fields it comprises. We are, however, in a somewhat unusual situation in this regard. The spacetimes we construct have two disjoint asymptotic regions in which we must specify the fall-off conditions for the fields. In particular, they do not have compact spacelike hypersurfaces. There is also no appropriate sense of asymptotic flatness in either of the asymptotic regions. We will therefore need to use some other criteria to decide what the asymptotic values of the fields should be and how quickly they should approach them. The criteria we will choose are very closely tied to the phase space formulation of the theory to be discussed in the next subsection. Nevertheless, for the sake of completeness, we will specify here the fall-off conditions for the fields introduced so far. As $z \to \pm\infty$, we require

$$\begin{array}{ll} \gamma \to \gamma_\pm(t) + \mathcal{O}(z^{-1}) & N \to N_\pm(t) + \mathcal{O}(z^{-1}) \\ \tau \to \tau_\pm(t) + \mathcal{O}(z^{-1}) & N^z \to \mathcal{O}(z^{-1}) \\ \psi \to \mathcal{O}(z^{-1}). & \end{array} \tag{2.6}$$

[2]Note that as in [6] we have chosen the normalization of ψ which is most consistent with the reduction from 3+1 to 2+1 dimensions. This accounts for the unusual normalization of Einstein's equation in eq. 2.4. Strictly speaking, the physical scalar field is given by $\Phi := \psi/\sqrt{8\pi G}$, where G is Newton's constant in 2+1 dimensions.

The notation $f_\pm(t)$ indicates that the asymptotic values of f can take any value, and are not a priori fixed quantities. Also, the expression $\mathcal{O}(z^{-n})$ denotes any function $f(z,t)$ such that $z^n f(z,t)$, $z^{n+1}f'(z,t)$ and $z^{n+2}f''(z,t)$ all have finite limits as z becomes infinite at fixed t.

The fall-off conditions in eq. 2.6 for γ, τ and N are about as weak as possible; they are only required to approach their limits in a reasonably uniform manner. The conditions on ψ and N^z are somewhat more restrictive in that they require these functions to approach *specific* limits, namely zero. In the case of ψ, this fall-off condition implies that ψ is square integrable, and therefore that its Fourier transform will exist. This condition is usually imposed, even in ordinary Minkowski-space field theories, in order to avoid infrared divergences in the quantum field. This restriction is therefore justifiable on physical grounds. The reason for the seemingly undesirable restriction on the asymptotic value of N^z can best be seen within the phase space formulation to be discussed below. Because of this, we will reserve its discussion for the next subsection.

2.2 The Phase Space

The usual Einstein-Hilbert action for 2+1-dimensional general relativity coupled to a scalar field may be written as

$$S[g,\psi] = \frac{1}{16\pi G}\int_M d^3x\,\sqrt{g}\,\left(R - g^{ab}\,\nabla_a\psi\,\nabla_b\psi\right) + \frac{1}{8\pi G}\oint_{\partial M} d^2x\,\sqrt{h}\,K. \tag{2.7}$$

Here, R represents the scalar curvature of g_{ab}. The second integral is taken over the asymptotic boundary ∂M of the space time, and h and K are respectively the determinant of the induced metric and the trace of the extrinsic curvature on that surface. However, for the class of metrics we have chosen, it turns out that K vanishes identically on ∂M. We may therefore drop it from the action. From the phase space point of view, although the second term is usually needed to ensure the functional differentiability of the action, in this case the action is already differentiable in its absence. We will be able to see this below.

To pass to the Hamiltonian formulation of this theory, we make use of the dynamical decomposition of the metric described in eq. 2.3. Then, by direct computation, we can find the scalar curvature of the metric 2.2 and use it to perform the Legendre transformation to momentum phase space. As a result, the action takes the standard form

$$S = \frac{1}{16\pi G}\int dt\,\left(\int dz\,\left[p_\gamma\dot\gamma + p_\tau\dot\tau + p_\psi\dot\psi\right] - C[N] - C_z[N^z]\right), \tag{2.8}$$

where the functionals $C[N]$ and $C_z[N^z]$ are given by

$$\begin{aligned} C[N] &:= \int dz\, N\, e^{-\gamma/2} \left[2\tau'' - \gamma'\tau' - p_\gamma p_\tau + \tfrac{p_\psi^2}{4\tau} + \tau\psi'^2 \right] \\ C_z[N^z] &:= \int dz\, N^z\, e^{-\gamma/2} \left[p_\gamma\gamma' + p_\tau\tau' + p_\psi\psi' - 2p_\gamma' \right]. \end{aligned} \tag{2.9}$$

Note that, as one might have expected, the Hamiltonian of this system is written as a sum of constraints, and that the lapse and shift functions appear as Lagrange multipliers enforcing these constraints.

The phase space Γ for our system is coördinatized by $(\gamma, p_\gamma, \tau, p_\tau, \psi, p_\psi)$. To complete our description of Γ, we must specify the fall-off conditions on the canonical momenta. These conditions will be motivated in essence by the requirement that the symplectic structure

$$\Omega[\delta_1, \delta_2] := \frac{1}{16\pi G} \int dz\, \left(\delta_1 p_\gamma\, \delta_2\gamma + \delta_1 p_\tau\, \delta_2\tau + \delta_1 p_\psi\, \delta_2\psi - [1 \leftrightarrow 2] \right), \tag{2.10}$$

as well as the Hamiltonian vector fields of the constraint functionals be well defined. Furthermore, motivated by the spacetime treatment of this system in [8], we hope eventually to use τ as the time parameter for the system. We therefore expect that its time derivative should not vanish anywhere, even in the asymptotic regions. It follows from this, together with the definition of the momenta in terms of the time derivatives of the fields, that p_γ cannot vanish asymptotically. As stated in the previous subsection, however, we want to allow γ to approach *arbitrary* (non-zero) limits in the asymptotic regions. The only way to ensure the convergence of the integral in eq. 2.10 is to require that p_γ approach *fixed* values in the asymptotic regions. In this way, although p_γ will be finite asymptotically, we can guarantee that its *variation* approaches zero rapidly enough to make the integral converge. We therefore take the fall-off conditions for both the fields and their momenta to be

$$\begin{array}{ll} \gamma \to \gamma_\pm(t) + \mathcal{O}(z^{-1}) & p_\gamma \to -1 + \mathcal{O}(z^{-2}) \\ \tau \to \tau_\pm(t) + \mathcal{O}(z^{-1}) & p_\tau \to \mathcal{O}(z^{-2}) \\ \psi \to \mathcal{O}(z^{-1}) & p_\psi \to \mathcal{O}(z^{-1}) \\ N \to N_\pm(t) + \mathcal{O}(z^{-1}) & N^z \to \mathcal{O}(z^{-1}). \end{array} \tag{2.11}$$

The asymptotic value $p_\gamma \to -1$ is chosen here primarily to simplify the discussion of gauge fixing in the next subsection.

Let us return now to the discussion of the fall-off of the shift function begun in the previous subsection. In phase space terms, $C_z[N^z]$ represents a class of functions on phase space which we require to be differentiable, and the fall-off condition on N^z amounts to a restriction on this class of functions. The phase space functions corresponding to shifts which do not vanish asymptotically are not differentiable at *any* point of the phase space described above. To have them be differentiable, we would have to work on a different phase space wherein γ would also approach fixed values in the asymptotic regions. Since the class of N^z which vanish asymptotically already separates the points of a spatial slice (and can therefore enforce the local constraint *density*), there is no need to use the larger class of N^z which are free to take arbitrary asymptotic values. Thus, our choice of fall-off conditions on the shift function is not only sufficient to enforce the constraint, but also preferable since the constraint functional is then defined on a phase space which can describe many more spacetimes.

We now have a complete characterization of the phase space Γ and the constraint functionals $C[N]$ and $C_z[N^z]$. A straightforward calculation shows that these constraints are first class. We have seen in eq. 2.8 that the Hamiltonian for our system can be written as a sum of constraints and therefore vanishes identically on the constraint surface. Note that this is *not* the case in the canonical treatment of the Einstein-Rosen model in [6]. In that model, the Hamiltonian was equal to the surface term in the gravitational action, eq. 2.7. In the present model, this surface term vanishes due to the boundary conditions 2.11 we have chosen for the fields. As a result, there is no canonical separation of gauge transformations and true dynamical evolution in our system as there was in the Einstein-Rosen case. To accomplish this separation and isolate the true degrees of freedom of the theory, we are forced to break the space-time covariance in a more or less ad hoc way. We will do so in the following subsection by using intuition garnered from the spacetime picture to "deparameterize" the system [10].

2.3 Deparameterization

There are a number of ways to approach the phase space reduction of a generally covariant system (e.g., frozen time formalism [11, and references therein], presymplectic mechanics [12], etc.). The simplest approach in the present case seems to be that of deparameterization [10]. The idea of the deparameterization procedure is that a generally covariant system must contain its own time parameter. There is a (highly non-unique!) procedure to isolate this time parameter and reduce the phase space to a proper symplectic manifold.

We begin with a presymplectic space (Γ, Ω) on which there are a number of first class constraints. First, we seek to gauge-fix all but one of the first class constraints. The gauge-fixed constraint surface $\hat{\Gamma}$ will then be odd dimensional, and the pullback of the presymplectic form to this surface will have precisely one degenerate direction given by the Hamiltonian vector field

X_C of the remaining constraint. Second, we need to find a phase space function T which satisfies $X_C(T) = 1$. The level surfaces of T will be everywhere transverse to X_C, so the pullback $\overline{\Omega}$ of Ω to any one level surface $\overline{\Gamma}$ will be non-degenerate. It follows that $(\overline{\Gamma}, \overline{\Omega})$ is a true symplectic space. This is the phase space of the deparameterized theory. Note that, in the end, some of the observables we will want to consider may be time-dependent. Therefore, the observables of the reduced theory should be given by equivalence classes of functions which agree on $\hat{\Gamma}$, rather than just on $\overline{\Gamma}$. This asymmetry in the treatment of the constraints when defining observables seems to be where the most injustice is done to the original covariance of the system. On the other hand, the deparameterization procedure yields a concrete model in which it is possible to calculate physically interesting quantities. Obviously, this is a very strong argument in its favor.

Let us apply these ideas to the phase space described in the previous subsection. In this particular situation, we already know from the spacetime formulation of the theory that we would like to identify τ with the parameter time t of the system. Therefore, we choose the gauge fixing conditions to be

$$\tau' = 0 \quad \text{and} \quad p_\gamma + 1 = 0. \tag{2.12}$$

The first of these conditions guarantees that the prospective time parameter is constant on a spatial slice of the dynamical foliation of spacetime. The second essentially does the same for $\dot{\tau}$. Note that the second condition is actually equivalent to $p_\gamma' = 0$ since we required p_γ to approach -1 at its asymptotic limits. However, we made no such requirement of τ. Thus, although τ is constant on the spatial slice, its value may still vary in time. In other words, the infinite number of degrees of freedom represented by $\tau(z)$ have been reduced to just one, whereas those represented by p_γ have been eliminated altogether.

To check that eqs. 2.12 are admissible gauge fixing conditions for use in the deparameterization program, we need to compute their Poisson brackets with the constraints:

$$\begin{aligned} \{\tau', C[N]\} &\approx \left[N e^{-\gamma/2}\right]' & \{\tau', C_z[N^z]\} &\approx 0 \\ \{p_\gamma + 1, C[N]\} &\approx 0 & \{p_\gamma + 1, C_z[N^z]\} &\approx -\left[N^z e^{-\gamma/2}\right]'. \end{aligned} \tag{2.13}$$

Since the matrix so defined is invertible, the gauge-fixing conditions are indeed admissible except when

$$N = N_0 e^{\gamma/2} \quad \text{or} \quad N^z = N_0^z e^{\gamma/2}, \tag{2.14}$$

where N_0 and N_0^z are arbitrary constants. Note, however, that the second possibility here is ruled out by the fall-off conditions imposed on N^z. Thus, as required, there is exactly one first class constraint *function* C (up to scaling) which is not solved by these gauge fixing conditions. Observe that setting $N_0 = 1$ will give $X_C(\tau) = 1$. This choice will therefore allow us to identify τ with the parameter time t of the system as expected.

Let us now turn to the characterization of the phase space of the deparameterized theory. To describe the surface $\overline{\Gamma}$, we must first solve the set of second class constraints formed by the first class constraints, eqs. 2.9, together with their gauge-fixing conditions, eqs. 2.12. When we then identify τ with the parameter time t of the theory, we find[3]

$$\begin{aligned} \gamma &= \int_{-\infty}^{z} dz' \, p_\psi \psi' \qquad & p_\gamma &= -1 \\ \tau &= t \qquad & p_\tau &= -\tfrac{p_\psi^2}{4t} - t\psi'^2. \end{aligned} \tag{2.15}$$

These calculations show that $\overline{\Gamma}$ is coordinatized by the pair (ψ, p_ψ). All the other dynamical variables of the original system are redundant and should now be treated as observables of the reduced system. Thus, we see that all the true degrees of freedom of our theory now reside in the scalar field and its conjugate momentum. This agrees with the usual notion that in 2+1-dimensional gravity, all the local degrees of freedom should reside in the matter fields.

To finish the construction of the deparameterized phase space, we also have to specify the symplectic structure and the Hamiltonian function. Both of these can be found by restricting the action functional to $\overline{\Gamma}$. We find the reduced action to be

$$\overline{S} = \frac{1}{16\pi G} \int dt \left(\int dz \left[p_\psi \dot{\psi} \right] - \int dz \left[\tfrac{p_\psi^2}{4t} + t\psi'^2 \right] \right). \tag{2.16}$$

The first term of this action gives us a canonical 1-form on $\overline{\Gamma}$ whose exterior derivative is the reduced symplectic structure $\overline{\Omega}$. Clearly, we will just find that ψ and p_ψ are canonical coordinates on the reduced phase space. The second term in the action gives the Hamiltonian for the deparameterized theory. Remarkably, these agree exactly with the symplectic structure and Hamiltonian of a scalar field theory on a *fixed* background spacetime whose metric is given by

$$\mathring{g}_{ab} = -\nabla_a t \, \nabla_b t + \nabla_a z \, \nabla_b z + t^2 \nabla_a \sigma \, \nabla_b \sigma. \tag{2.17}$$

[3]Note that we have made an additional choice in the expression for γ. The solution of the constraint density of $C[N]$ actually only implies that $\gamma' = p_\psi \psi'$. To recover γ, we have to integrate this equality, leading to an undetermined constant of integration. This constant can be used to fix the value of γ at any one point of the spatial slice to be any value we like. We have chosen to take $\gamma \to 0$ as $z \to -\infty$. This choice can be thought of as the completion of the gauge fixing procedure outlined above.

This metric is again defined on a 3-manifold with topology $R^2 \times S^1$. In fact, this background spacetime corresponds to the point $\psi = p_\psi = 0$ of the reduced phase space.

The identification of the reduced phase space for our system with that of a scalar field on a fixed background is very important for the quantization which we will describe in the following section. This is due to the relative simplicity of the later description. Although other descriptions of the reduced phase space are certainly possible, they do not offer as simple a quantization scheme as the scalar field. For this reason, it is useful to restrict our attention to the scalar field theory to study the quantization of our system. We can then use the classical expressions derived above to find quantum observables describing geometric quantities of interest.

3 Quantum Theory

3.1 Preliminaries

In the previous section, we were primarily concerned with the geometrical significance of the scalar field in our theory. We therefore focussed on the dimensionless quantity ψ. The approach we will take to quantization, however, is based on the fully reduced phase space described above — that of a free scalar field. Therefore, in this section, it is convenient to rescale the field so that it acquires the proper dimension for a physical scalar field in 2+1 dimensions. Accordingly, we will now switch our attention to the quantity $\phi := \psi/\sqrt{8\pi G}$. The natural choice for the momentum conjugate to ϕ is $p_\phi := p_\psi/\sqrt{32\pi G}$. Using these variables, we can reexpress the reduced action of eq. 2.16 as

$$S = \int dt \left(\int dz \left[p_\phi \dot{\phi} \right] - \int dz \, \tfrac{1}{2} \left[t^{-1} p_\phi^2 + t\phi'^2 \right] \right). \tag{3.1}$$

This is just the usual action for a free scalar field on the background spacetime given by eq. 2.17. Thus, the phase space Γ has the structure of a real vector space and is coordinatized by the canonical pair (ϕ, p_ϕ). Note that we have changed notation slightly from the previous section: we have dropped the bars over both the (reduced) action and the (reduced) phase space. This is done to emphasize that the scalar filed theory is now regarded as fundamental. The other geometrical quantities discussed above will now be treated as secondary, derived observables.

The quantization of scalar fields on fixed background spacetimes is by now very well understood. In the present discussion, we will follow the standard constructions laid out, for example, in [13]. This procedure yields the Hilbert space of the quantum field theory by building a Fock space over a single-particle Hilbert space derived from the space $\mathcal{S}$ of solutions of the classical equations of motion. We should therefore begin with the description of this space.

The equations of motion for the field theory follow easily from the action 3.1. They are

$$\left\{ \begin{array}{l} \dot{\phi} = t^{-1} p_\phi \\ \dot{p}_\phi = t\phi'' \end{array} \right\} \quad \Rightarrow \quad \ddot{\Phi} + t^{-1}\dot{\Phi} - \Phi'' = 0. \tag{3.2}$$

Here, and below, the quantity $\Phi = \Phi(t, z)$ denotes a solution to the equations of motion, while $\phi = \phi(z)$ denotes the value of the field on a given spatial slice. Note that, as expected for a free field theory, the equations of motion are linear. It follows that the space $\mathcal{S}$ of their real solutions will be a real vector space. We may therefore expand the most general real-valued solution in terms of certain fundamental solutions as

$$\Phi(t, z) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \left[A(k) f_k^{(2)}(t, z) + \overline{A(k)} f_k^{(1)}(t, z) \right]. \tag{3.3}$$

with the fundamental solutions given by

$$f_k^{\binom{1}{2}}(t, z) := \frac{\sqrt{\pi}}{2}\, H_0^{\binom{1}{2}}(|k|\, t)\, e^{\mp ikz}, \tag{3.4}$$

where $H_0^{\binom{1}{2}}(\cdot)$ denotes the zeroth-order Hankel function of type 1 or 2. The wave profile $A(k)$ we have introduced here will generically be *complex*, and can therefore be viewed as a complex coordinate on the real vector space $\mathcal{S}$. It will also have to satisfy come requirements related to the fall-off conditions on ϕ and p_ϕ. By allowing some modifications to the fall-off conditions given previously in eq. 2.11, we can make these requirements precise. In particular, if we require ϕ and p_ϕ to be Schwartz functions (i.e., they fall off at infinity faster than any polynomial), we will find the corresponding wave profiles are also Schwartz. This change in fall-off conditions will not change any the results to follow owing to the completion of $\mathcal{S}$ to a Hilbert space which will take place during the quantization. The point of all this is that we can concretely identify the solution space $\mathcal{S}$ with the space of Schwartz functions of a single variable.

There is a natural map from the space $\mathcal{S}$ to the phase space Γ of our system. One defines this map by identifying a solution of the equations of motion with its initial data at some fixed initial time t_0. It is not difficult to show that this map is linear, invertible and (bi-)continuous when the fall-off conditions described above are applied. We can therefore use it to induce a symplectic structure on $\mathcal{S}$ using the given one on Γ. This symplectic structure can be written in terms of the wave profiles $A(k)$ as

$$\Omega(\Phi_1, \Phi_2) = i \int_{-\infty}^{\infty} dk \left[\overline{A_1(k)} A_2(k) - A_1(k)\overline{A_2(k)}\right] . \tag{3.5}$$

We have chosen to write the symplectic structure as an antisymmetric bilinear form on $\mathcal{S}$ rather than as a differential 2-form. There is no problem with this since $\mathcal{S}$ is linear and can therefore be identified with its tangent space. Moreover, the quantization procedure requires the symplectic structure be given as a bilinear form, rather than as a differential form, which explains why the procedure would not work for non-free field theories.

3.2 Quantization

The key step in the quantization of a free field theory is to pick a Kähler structure on the space $\mathcal{S}$ of solutions to its classical equations of motion. That is, we need to choose a complex structure $J : \mathcal{S} \to \mathcal{S}$ such that

$$\mu(\Phi_1, \Phi_2) := -\tfrac{1}{2}\,\Omega(\Phi_1, J \circ \Phi_2) \tag{3.6}$$

is a positive-definite real inner product on $\mathcal{S}$. There are always a number of possible ways to do this, and the various ways do not necessarily lead to equivalent quantum theories. In our case, however, we already have a complex coordinate $A(k)$ on $\mathcal{S}$, so there is a natural candidate for such a structure given by $J : A(k) \mapsto iA(k)$. This is indeed an acceptable choice since it yields the inner product

$$\mu(\Phi_1, \Phi_2) = \tfrac{1}{2} \int_{-\infty}^{\infty} dk \left[\overline{A_1(k)} A_2(k) + A_1(k)\overline{A_2(k)}\right] , \tag{3.7}$$

which is manifestly real and positive-definite.

With the choice of a complex structure, it becomes possible to view $\mathcal{S}$ as a *complex* vector space. Moreover, we can define a complex inner product on $\mathcal{S}$ by

$$\begin{aligned} \langle \Phi_1, \Phi_2 \rangle_1 &:= \tfrac{1}{\hbar}\,\mu(\Phi_1, \Phi_2) - \tfrac{i}{2\hbar}\,\Omega(\Phi_1, \Phi_2) \\ &= \hbar^{-1} \int_{-\infty}^{\infty} dk\, \overline{A_1(k)} A_2(k), \end{aligned} \tag{3.8}$$

where the factors of $\hbar$ are included at this point to render the inner product of two vectors dimensionless. This inner product will be Hermitian in the complex structure defined by J, and is again manifestly positive-definite. Thus, we have endowed $\mathcal{S}$ with the structure of a complex pre-Hilbert space. The Cauchy completion of this space in the inner product norm gives a complex Hilbert space $\mathcal{H}_1$ which, due to the simple form of the inner product

$\langle \cdot, \cdot \rangle_1$, can immediately be identified with $L^2(\mathrm{R})$. This is the single-particle state space for the quantum theory. The Hilbert space of the field theory is then given by the symmetric Fock space $\mathcal{H} := \mathcal{F}_S(\mathcal{H}_1)$ built on $\mathcal{H}_1$.

To complete the quantization of the theory, we have to introduce a set of operators on $\mathcal{H}$ corresponding to a "complete" set of classical observables and specify a Hamiltonian operator. It is simplest to do the first of these in the form of the quantum field itself

$$\hat{\Phi}(t,z) := \sqrt{\hbar} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \left[f_k^{(2)}(t,z)\, \hat{a}_k + f_k^{(1)}(t,z)\, \hat{a}_k^{\dagger} \right], \tag{3.9}$$

which we have expressed using the creation and annihilation operators provided by the Fock structure of the Hilbert space. As usual, this is actually an operator-valued distribution. The true observables of the theory are the smeared field operators $\hat{\phi}(t;g]$ and $\hat{p}_\phi(t;g]$ gotten by integrating this distribution (and the one corresponding to its canonical momentum) against a suitably well-behaved function $g(z)$. One can check that the smeared field operators do actually obey the proper canonical commutation relations.

The classical Hamiltonian of the system is given by the second term in the action 3.1. It can be promoted to a quantum observable on $\mathcal{H}$ which will take the standard form

$$\hat{H} = \int_{-\infty}^{\infty} dk\, \hbar\, |k|\, a_k^{\dagger} a_k. \tag{3.10}$$

We have chosen the conventional normal-ordering of the right side of this expression to subtract the (infinite) ground state energy and make $\hat{H}$ a well-defined operator on the Hilbert space $\mathcal{H}$. It does not require any further regularization.

We conclude this discussion with a remark. As we noted above, there is no canonical choice of complex structure on $\mathcal{S}$ and different choices can lead to inequivalent quantum theories. In the case of Minkowskian quantum field theories, the complex structure is fixed by the requirement of Poincaré invariance. In the present case, however, we have only a rotational invariance which unfortunately is insufficient to determine the complex structure uniquely. As a result, one should note that the quantization we have performed above is *not* unique and there are other realizations of the quantum system. Nevertheless, the complex structure we have chosen does have some nice properties which justify its use, even though they do not single it out. From the geometric point of view, the only interesting classical observable of the system is the metric function γ. This function is quadratic on the phase space of the system but unfortunately is not bounded from below. The complex structure we have used here does commute with the infinitesimal canonical transformation generated by γ on the classical phase space. Had

γ been bounded from below, this commutativity would have been sufficient to choose the complex structure uniquely [14]. As it stands, this fact only shows that our choice is one of a class of complex structures which are well adapted to the observables of interest for the system. From a practical point of view, however, this choice of complex structure has the property that it *does* allow us to complete the quantization of the model exactly.

3.3 Quantum Geometry

Since in the previous two subsections we have constructed an exact quantum theory of a gravitational system, we can now ask whether there are new physical insights to be drawn from the model. In this subsection, we will see that there are indeed some lessons we can learn. In [15], it was shown that surprisingly large quantum dispersions in the Coulombic modes of the spacetime metric would result from the presence of high-frequency excitations in the quantum Einstein-Rosen model of [6]. The Schmidt model is similar to the Einstein-Rosen model in certain respects, but has a decidedly different overall structure. It is therefore natural to ask whether these large quantum effects persist in this model. As we will see below, they do. This will constitute a first check on the robustness of the dispersion results.

The content of the spacetime geometry in the Schmidt model is encoded in the 2+1-dimensional metric g_{ab}. In the classical theory, after the gauge-fixing conditions have been applied, the metric of eqs. 2.2 and 2.3 becomes

$$g_{ab} = e^{\gamma(t,z)}(-\nabla_a t\, \nabla_b t + \nabla_a z\, \nabla_b z) + t^2 \nabla_a \sigma\, \nabla_b \sigma, \tag{3.11}$$

where $\gamma(t,z)$ is defined in terms of the scalar field by eq. 2.15. The only nontrivial component of this metric is clearly $g_{zz} = -g_{tt} = e^{\gamma(t,z)}$, and we will accordingly focus our attention here on the quantum analog $e^{\hat{\gamma}(z,t)}$ of this observable.

Let us begin by describing the operator $\hat{\gamma}(t,z)$ itself. We can use the classical expression of eq. 2.15 to express this in terms of the creation and annihilation operators as

$$\begin{aligned}\hat{\gamma}(t,z) = 16\pi G\hbar \int dk\, d\ell\, \Big[&\overline{\Delta_+(t,z;k,\ell)}\, \hat{a}_k^\dagger \hat{a}_\ell^\dagger + \Delta_+(t,z;k,\ell)\, \hat{a}_k \hat{a}_\ell + \\ &\Delta_-(t,z;k,\ell)\, \hat{a}_k^\dagger \hat{a}_\ell + \overline{\Delta_-(t,z;k,\ell)}\, \hat{a}_\ell^\dagger \hat{a}_k\Big]\end{aligned} \tag{3.12}$$

where the Green's functions $\Delta_\pm(t,z;k,\ell)$ are given by

$$\Delta_\pm(t,z;k,\ell) = \frac{i\,|k|\,\ell t}{8}\, H_1^{\binom{2}{1}}(|k|\,t)\, H_0^{(2)}(|\ell|\,t) \int_{-\infty}^{z} dz'\, e^{i(\ell\pm k)z'}. \tag{3.13}$$

Note that, as with the Hamiltonian above, we have chosen to normal-order the right hand side of eq. 3.12. This is done both to regularize the operator and to preserve the classical relationship between γ and the field momentum. Also note that the Green's functions $\Delta_{\pm}(t, z; k, \ell)$ which we use here are actually distributional due to the last term in eq. 3.13 (which is the Fourier transform of a step function). Therefore, eq. 3.12 does not define an operator on the Hilbert space $\mathcal{H}$, but rather another operator-valued distribution. To make sense of this as a proper observable, one must define an appropriate class of smearing functions to mitigate the singular nature of the integrals involved. Although this is certainly possible in principle, there are significant technical problems with the procedure; we will concentrate instead on the structure of the *asymptotic* metric operator. This is both technically simpler and physically more interesting.

All of the spacetimes we consider in this paper have two distinct asymptotic regions. In each of these regions, the expression for the operator $\hat{\gamma}$ of eq. 3.12 simplifies considerably. In the limit as $z \to -\infty$, we find that the asymptotic form of $\hat{\gamma}$ is identically zero. Therefore, the metric in this region will simply agree with the background metric introduced in eq. 2.17. In particular, it is a "c-number;" it has no dispersion. This is really a result only of the choices we made in the gauge fixing procedure. Recall that, classically, we were able to fix the value of γ to be any given quantity at any particular point z, and that we used this freedom to fix $\gamma \to 0$ as $z \to -\infty$. The quantum mechanical theory, then, is based on a space where other asymptotic values of γ simply do not occur. In light of these facts, the above result about the metric operator $\hat{\gamma}_-$ is not surprising. It is also not very interesting.

In the opposite asymptotic limit, as $z \to +\infty$, the distributional integrals in eq. 3.13 converge, in an appropriate sense, to delta functions. As a consequence, the asymptotic form of $\hat{\gamma}$ becomes

$$\hat{\gamma}_+ := \lim_{z\to\infty} \hat{\gamma}(t, z) = -16\pi G \int dk\, \hbar k\, a_k^\dagger a_k. \tag{3.14}$$

Unlike the generic expression for $\hat{\gamma}(t, z)$ given in eq. 3.12, this expression *does* define a proper self-adjoint operator on $\mathcal{H}$. Thus, we may define and study the geometrical operator $\hat{g}^+_{zz} = e^{\hat{\gamma}_+}$.

To illustrate the quantum gravitational effects in the system, we wish to concentrate on a class of semi-classical states which approximate classical spacetimes. Luckily, there already exists a well-known class of semi-classical states which approximate classical states of the scalar field, namely the coherent states

$$|A\rangle := e^{-\|A\|_1^2/2}\, e^{\hbar^{-1/2} \int dk\, A(k)\, \hat{a}_k^\dagger}\, |0\rangle\,. \tag{3.15}$$

The norm $\|\cdot\|_1^2$ which appears here is that associated with the one-particle Hilbert space inner product of eq. 3.8, and $|0\rangle$ denotes the (unique) ground state of the theory. As usual, each of these states is peaked at the classical configuration of the scalar field given by the wave profile $A(k)$ and minimizes the uncertainty in the value of the field. With the normalization factor given in eq. 3.15, this set of coherent states also reflects both the classical and quantum structure of the system:

$$\langle A_1 | A_2 \rangle = e^{-\|A_1 - A_2\|_1^2/2} \, e^{-i\Omega(A_1,A_2)/2\hbar}. \tag{3.16}$$

The classical system is reflected in the phase of the inner product which involves the symplectic structure of eq. 3.5. On the other hand, the amplitude of the inner product is given in terms of the single-particle Hilbert space norm. Although it has been shown [16] that there are other classes of semi-classical states which minimize the *combined* uncertainty in the field and the metric, we will continue to use these here because of the close integration with the reduced classical system.

The results on quantum geometry we will obtain shortly all follow from a simple proposition. Suppose we have an observable $\hat{O} = \int dk\, f(k)\, \hat{a}_k^\dagger \hat{a}_k$ which can be written as a sum of contributions from each particle present in the quantum state. The action of the exponential of this operator on a coherent state of the particle system is then given by

$$e^{\hat{O}} |A\rangle = \exp\left(\frac{1}{2\hbar} \int dk \, \left[e^{2f(k)} - 1 \right] \, |A(k)|^2 \right) \left| e^f A \right\rangle , \tag{3.17}$$

where $\left|e^f A\right\rangle$ denotes the coherent state associated with the wave profile $A'(k) = e^{f(k)} A(k)$. Note that since, as we have done above, the wave profiles $A(k)$ are typically taken to be Schwartz functions, $\left|e^f A\right\rangle$ is not necessarily a well-defined coherent state. If f diverges faster than logarithmically at infinity, the function $e^f A$ will not necessarily be Schwartz, and the operator $e^{\hat{O}}$ may not be defined. This is not too surprising since the operator in question is certainly unbounded and we expect therefore that we should have to restrict the domain of definition of the operator. Fortunately, there is a natural domain which we can always choose: the coherent states corresponding to wave profiles of compact support. It is not difficult to show that these can approximate the Schwartz function coherent states to arbitrarily close precision. Since the later are already (over-)complete, this shows the exponential operator can be densely defined. Thus, although the expression may initially appear suspicious, we should not be too concerned with the functional analytic subtleties of eq. 3.17.

Since, according to eq. 3.14, $\hat{\gamma}_+$ is of the required form, we can apply the above result to the study of the metric operator $\hat{g}^+_{zz}$. The expectation value of the metric operator and the relative uncertainty in its measurement may be expressed in closed form as

$$\langle \hat{g}^+_{zz} \rangle = \exp\left(\hbar^{-1} \int dk \left[e^{-16\pi G\hbar k} - 1\right] |A(k)|^2\right) \tag{3.18}$$

$$\frac{(\Delta g^+_{zz})^2}{\langle g^+_{zz} \rangle^2} = \exp\left(\hbar^{-1} \int dk \left[e^{-16\pi G\hbar k} - 1\right]^2 |A(k)|^2\right) - 1. \tag{3.19}$$

Meanwhile, the classical expression for the asymptotic metric is

$$\left(g^+_{zz}\right)_{\text{classical}} = \exp\left(-16\pi G \int dk\, k\, |A(k)|^2\right). \tag{3.20}$$

These expressions clearly show that there are non-trivial quantum effects present in our system, even when the scalar field is sharply peaked at a classical configuration. Since we now have a canonical frequency scale given by the Planck value $1/G\hbar$, we can gain some qualitative understanding of these effects by examining some of the limiting behavior of these expressions. In each case, we will consider a wave profile $A(k)$ which is sharply peaked at a certain characteristic wave number k_0. There are three distinct regimes we will discuss. In the discussion, it is useful to define $N := \|A\|_1^2$, which may be interpreted as the expected total number of particles in the coherent state associated with the wave profile $A(k)$.

1. *Low frequency* ($|G\hbar k_0| \ll 1$): In this case, we can simply expand the various expressions in powers of $G\hbar k_0$ to find the expectation value of the metric operator and its relative uncertainty. The results are:

$$\begin{aligned} \langle \hat{g}^+_{zz} \rangle &\approx \left(g^+_{zz}\right)_{\text{classical}} \left[1 + \tfrac{N}{2}(16\pi G\hbar k_0)^2 + \cdots\right] \\ \frac{(\Delta g^+_{zz})^2}{\langle g^+_{zz} \rangle^2} &\approx N(16\pi G\hbar k_0)^2 + \cdots . \end{aligned} \tag{3.21}$$

   This shows that in the low frequency limit, both the deviation of the quantum metric from the classical and its relative uncertainty are second order in the expansion parameter[4]. Thus, in this regime, the quantum system closely mimics the classical one.

2. *High frequency, forward direction* ($G\hbar k_0 \gg 1$): This situation occurs when the quantum state describes a number of high-frequency particles

[4]Note that we should also have $N(G\hbar k_0)^2 \ll 1$ to get these results. As a result, the number of particles expected in the quantum state cannot be too large. This limitation is somewhat surprising since the coherent states of the scalar field approximate their classical counterparts only when the expected particle number *is* large: $N \gg 1$. However, since the characteristic frequency of the particles, $G\hbar k_0$, is already very small, there is presumably some intermediate range wherein both approximations are reasonably accurate.

moving *toward* $z = +\infty$. In this limit, the expectation value of the metric and its relative uncertainty become

$$\langle \hat{g}^{+}_{zz} \rangle \approx e^{-N} \quad \text{and} \quad \frac{(\Delta g^{+}_{zz})^2}{\langle g^{+}_{zz} \rangle^2} \approx e^{N} - 1 \tag{3.22}$$

Meanwhile, the classical metric in this limit becomes $e^{-16N\pi G\hbar k_0}$, which is small, even when compared with the quantum expectation value. Furthermore, when there are more than a couple particles present, the uncertainty in the metric can be quite large compared to its expectation value.

3. *High frequency, backward direction* ($-G\hbar k_0 \gg 1$): This situation occurs when the quantum state describes a number of particles moving *away from* $z = +\infty$. In this limit, we can again find approximate values for the metric and its relative uncertainty

$$\langle \hat{g}^{+}_{zz} \rangle \approx \exp\left(Ne^{-16\pi G\hbar k_0}\right) \quad \text{and} \quad \frac{(\Delta g^{+}_{zz})^2}{\langle g^{+}_{zz} \rangle^2} \approx \exp\left(Ne^{-32\pi G\hbar k_0}\right). \tag{3.23}$$

The classical metric in this case is again approximately $e^{-16N\pi G\hbar k_0}$. This, too, is small compared to the quantum expectation value. In this case, even when there is only *one* particle expected in the quantum state, the uncertainty in the metric can be huge compared to its expected value.

These results seem to agree, on the whole, with the results found in [15] regarding the cylindrical wave case. There *do* exist semi-classical states within our system, but they belong to a very restrictive class. Specifically, states with too few or too many particles, as well as *any* state which contains particles of Planckian or trans-Planckian frequencies do not occur in the classical limit of the quantum theory. All of these states are narrowly peaked around a particular classical field configuration. However, in the case of the excluded states, not only is the uncertainty in the metric large, but the expected value of the metric is wildly different from the classical approximation. It has been shown in [16] that the uncertainty in the metric measurement is partially a consequence of the particular family of semi-classical states we used in our analysis; a different choice could decrease this uncertainty at the expense of introducing a larger uncertainty in the value of the field. However, even if some other set of states were chosen, the corresponding classical field configurations would remain poor approximations to the solutions of the full, interacting quantum theory.

4 Discussion

It has been known for some time that general relativity, under the assumption of two hypersurface-orthogonal Killing fields, can be described in terms of a free scalar field theory. In light of this fact, the equations governing the Schmidt model are not unexpected. However, in the phase space formulation and quantization, there *are* some unusual aspects of the construction which merit further discussion.

At the classical level, we considered a class of spacetimes with the somewhat unusual property that they are neither spatially compact nor asymptotically flat. These properties, one of which is usually assumed for one reason or another, often play a decisive role in formulating the canonical theory for the system under consideration. Nevertheless, we saw that it is possible to define a proper phase space for the Schmidt model by requiring only the convergence of the symplectic structure 2.10 and the differentiability of the constraint functions 2.9. However, the resulting classical system does not have any natural time structure and its Hamiltonian vanishes weakly. To isolate its true, independent degrees of freedom, we therefore found it necessary to "deparameterize" the theory. Fortunately, since we had previous knowledge of the spacetime treatment of the model, we had a natural candidate for the time parameter of the reduced system. Furthermore, after the reduction, the system took the expected form of a free scalar field on a fixed background. This was the key fact which enabled us to quantize the theory in a relatively straightforward way.

In the quantum theory, we saw that there is a natural class of semi-classical states of the quantum system which are built from its classical solutions. These states can be identified with the usual coherent states of the scalar field theory which describes the reduced model. As was pointed out in [16], there are ways of realizing other such sets of semi-classical states within the quantum theory. However, the set which we have chosen is particularly well adapted to a discussion of the modifications to the fixed-background scalar field theory which result from coupling it to a dynamical gravitational field. This is because these semi-classical states remain sharply peaked at a classical scalar field configuration even in the presence of the gravitational field. The other possible sets of coherent states do minimize a *combination* of the metric and scalar field uncertainties at the expense of introducing a larger dispersion in the value of the scalar field by itself.

Using the exact quantum model we had constructed, we were able to find some unexpectedly large dispersions in the (asymptotic) metric caused by the presence of high-frequency scalar field excitations. Surprisingly, the dispersions are independent of where the particle is located. Even a single high-frequency particle located at any point of space can cause very large dispersions in the *asymptotic* metric. This result can be interpreted at a couple of levels.

Firstly, one can consider the implications for quantum field theory in general. It is often hypothesized that the introduction of gravity will introduce a natural cut-off in quantum field theory that will obviate the need for renormalization. In our model, no such cut-off has emerged. However, the dispersions in the metric expectation suggest that a classical spacetime geometry simply fails to be a good approximation to the quantum state when high-frequency particles are present[5]. The coupling of the matter field to the gravitational field seems to cause the classical spacetime structure to break down in this regime. Conversely, the absence of metric dispersions when no high-frequency particles are present suggests that flat spacetime quantum field theory is a reasonable approximation to the true situation in that regime. These facts lend some concrete credence to the notion that quantum field theory is a low-energy limit of a larger theory which includes gravity. As such, infinities can and do arise when this limiting theory is pushed beyond its domain of validity.

The Schmidt model also provides an infinite number of examples of exact solutions to semi-classical gravity. Semi-classical gravity is the theory of quantum fields propagating on a *dynamical* classical spacetime. A state of this theory must therefore specify a Lorentzian manifold (M, g_{ab}), a quantum field $\hat{\phi}$ on that manifold, and a state $|\psi\rangle$ of the quantum field which satisfy the dynamical equation

$$G_{ab} = 8\pi \left\langle \hat{T}_{ab} \right\rangle_{\psi} . \tag{4.1}$$

In our model, the expectation value of the stress-energy operator in any coherent state is exactly equal to its value in the associated classical state. Consequently, eq. 4.1 will be satisfied for any coherent state by taking the metric to be that of the corresponding classical solution. However, as was pointed out in [15], the states of semi-classical gravity which include high-frequency particles should not be taken seriously. Again, these solutions do not approximate the full quantum gravitational theory at all closely.

Finally, we should discuss the limitations of this model. At a technical level, we have described only one possible quantization of the Schmidt

[5]There is another possible interpretation of this result: that the dispersions signal the breakdown of the rotational symmetry of the configuration rather than of the spacetime picture itself. That is, that symmetric high-frequency excitations of the fields are unstable and rapidly become asymmetric when quantum effects are taken into account. By itself, however, this would be an unforeseen, genuinely quantum mechanical effect. There are efforts under way [17] to examine this question by studying quantum gravity effects in configurations which are asymmetrically perturbed from the symmetric states described here.

model; it is not unique. There are two points at which one could make different choices to quantize the system. First, as was discussed above, it is possible to pick a different complex structure to use in the quantization procedure. Although different choices *can* lead to inequivalent quantum theories, this ambiguity is fairly tame for our purposes since the large quantum gravity effects which we have found in this paper will persist in alternate quantizations. Second, and more importantly, there are ambiguities in the gauge fixing procedure already at the classical level. To make sense of our theory for quantization, we had to isolate its true degrees of freedom using a deparameterization procedure. In this case, it is *not* clear that alternate gauge fixings would reproduce the qualitative content of our results. However, from a practical point of view, one should note that the quantization procedure given our choices *can* be completed and that we *can* recover concrete results from the theory. In general, this would probably not be the case.

At a more fundamental level, although we have caught some glimpses of quantum geometry in this model, the spacetimes involved are only 2+1-dimensional. The physically interesting case, of course, is that of 3+1-dimensional gravity, and there are fundamental differences between the two. Most notably, the 2+1-dimensional spacetimes do not allow the possibility of black holes (without permitting a non-zero cosmological constant). The present system models a different sector of the 3+1-dimensional theory: the radiative modes of the gravitational field which are not sufficiently strong to cause gravitational collapse. The simplest 3+1-dimensional analog of the systems discussed here and in [6, 7] describes the collapse of a spherically symmetric scalar field in a Schwarzschild spacetime. It would be quite illuminating to understand the quantization of that model. There are at least two reasons to expect this. First, it incorporates the black hole sector of gravity which the models considered until now have ignored. Second, it is a truly 3+1-dimensional system, so the radiative modes of the gravitational field are expected to die off asymptotically as the inverse of the radius rather than logarithmically. Consequently, the details of the dispersion effects due to high-frequency excitations may differ slightly from those described here. Nevertheless, since all of these models describe scalar fields propagating on spacetimes on which the gravitational field does not have its own local degrees of freedom, it is reasonable to hope that the calculations in the spherically symmetric case can be completed using intuition garnered from the lower-dimensional models.

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